Conduction of Heat in Solids

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Introduction

Heat transfer is an area of thermal engineering that focuses on the transport, exchange, and redistribution of thermal energy. The three modes or ways that heat can be transferred have been termed conduction, convection, and radiation. In this chapter, the basic physics associated with conduction heat transfer will be presented, and it will be shown through examples how the tools and analysis typically used for conduction problems can be applied to design and analysis when fire occurs.

Conduction heat transfer only occurs in a medium. This is a distinction between conduction and radiation, which does not require a medium. The medium or state of matter in which conduction takes place can be a gas, liquid, or solid. The distinction between conduction and convection heat transfer is associated with whether the medium has some ordered flow or bulk motion. Heat transfer, when there is a mass averaged velocity, is termed convection. Heat transfer that takes place in a stationary frame of reference is called conduction. More details will be presented on the mechanisms that allow heat transfer to occur in a stationary medium as we proceed through this discussion. Solutions will be provided for selected configurations and scenarios. The treatise of Carslaw and Jaeger [1] covers most solutions for conduction phenomena. Other useful texts that discuss conduction phenomena are readily available [2, 3]. It is useful to build up this discussion by first identifying where conduction heat transfer ties into overall energy conservation and energy transfer.

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Energy Conservation

The fundamental laws that allow us to analyze and predict fire phenomena are often termed conservation laws. Conservation laws are essentially balance equations that allow us to model how variables that describe the physical world dynamically evolve. In fire systems, we typically model the physical world using mass conservation, momentum conservation, energy conservation, and chemical species conservation. For this chapter, we are interested in describing how heat is transferred in media that are not deforming (i.e., are in rigid body motion with no unbalanced forces) or reacting (fixed chemical species and mass). We do assume, however, that the medium can possibly have heat transferred to it either through interactions with its surrounding or through some other energy input into it. Also, we assume that the medium may have different amounts of thermal energy stored within it at different locations. To more precisely describe energy transfer processes, we rely on the first law of thermodynamics. The energy conservation principle is the basis for heat transfer.

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Thermodynamic Properties

The first law of thermodynamics is a statement of energy conservation [4, 5]. It states that the change in energy for an identifiable set of matter can only result from heat transferred across the material's boundary or work done either by or on the material. The thermodynamic property or energy function that best describes the molecular, atomic, electronic, and nuclear energy of a material is the internal energy. In terms of the internal energy, U, the first law is

$$\frac{dU}{dt} = \dot{Q}_{net,in} + \dot{W}_{net,in}$$

U is the internal energy, Q is heat added to the system and W is work done on the system.

The total internal energy U is a system integrated value that represents the total thermal energy of the material system of interest. We can describe the local internal energy in terms of a mass specific internal energy, u, that is simply the total internal energy, U, for a region of matter divided by the mass of that region. The internal energy, like any other thermodynamic variable can be defined in terms of other thermodynamic variables. There is an approximation used in thermodynamics that states that the internal energy for an incompressible material can be specified in terms of the temperature. The thermodynamic property specific heat capacity at constant volume, c_v, relates differential changes in the mass specific internal energy to differential changes in temperature.

$$c_v = \frac{du}{dT} \text{ and } mc_v\, dT = dU$$

The mass is defined as the product of density and volume. The control-mass statement of the first law for a case with no net work done becomes:

$$\rho V c \frac{dT}{dt} = \dot{Q}_{net,in}$$

This form of the first law neither provides information about spatial variations in energy within the medium nor describes how energy is transferred. Experience tells us that the heat transfer into some identifiable mass element likely depends on temperature differences. It will be necessary to define the heat transfer rate in terms of temperature differences. The empirical law defining the heat transfer rate to a body immersed in a fluid is called Newton's law of cooling. When Newton's law of cooling is used, the heat transfer rate to the body is $\dot{Q}_{net,in} = hA(T - T_\infty)$ If we apply Newton's law of cooling to the first law, we arrive at a result called the lumped thermal approximation in conduction analysis.

Lumped Thermal Analysis

Briefly, the lumped thermal approximation allows one to model the overall transient thermal response of a body at some initial temperature subjected to either a change to the external fluid temperature or as a result of some local heating within the object. The validity of this approximation will be discussed in more detail in later sections. For the purposes of this discussion, we will say that the approximation is valid when the time scales for internal energy transfer and subsequent homogenization of the temperature field within an object are much smaller than the time scales for energy transfer from the surface of the body to an external thermal reservoir. In short, the lumped thermal approximation is reasonable when temperature differences within a body are relatively small when compared to temperature differences between the surface of the body and a characteristic temperature of the exterior fluid. It can be shown that a nondimensional heat transfer parameter called the Biot number (Bi) which represents the ratio of the internal conductive resistance to the external convective resistance should be small for the lumped thermal approximation to be valid. A mathematical statement of the energy equation in the lumped approximation is (Fig. 2.1):

$$\frac{dT}{dt} = -\frac{h_c A}{\rho V c}(T - T_\infty)$$

This first order ordinary differential equation can be integrated and one form of the solution is:

$$\frac{T - T_e}{T_0 - T_e} = e^{-(h_c A/\rho V c)t} = e^{-t/t_c}$$

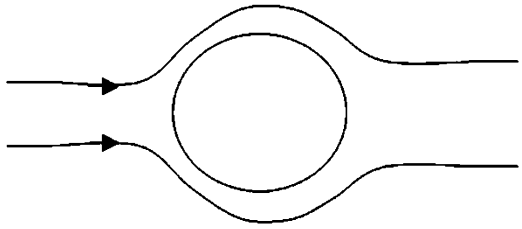


Fig. 2.1 Schematic showing convective flow over an object that will be analyzed using a lumped thermal approximation

In the above, T_0 is the initial temperature of the body and T_e is the external fluid temperature surrounding the object. There is a characteristic time in the problem defined as:

$$t_c = \frac{\rho V c}{hA}$$

The characteristic time provides an estimate of the time required for the nondimensional temperature to relax to its steady value. This relatively simple solution is useful in characterizing a large number of important problems in fire systems [6].

Example 1 The lumped thermal approximation is frequently used to analyze the response of a sprinkler head as it activates due to a change in the environment temperature because of a fire. A sprinkler head fuse can be modeled as a cylinder of diameter 4 mm and length 12 mm. The density can be approximated as being 1000 kg/m^3. The specific heat capacity is approximately 1 kJ/kgK. The heat transfer coefficient of the smoke gases is 20 W/m^{2}K. If the smoke gases are 200 °C and the fuse is initially at 20 °C, how long will it take for the fuse to open if the activation temperature is 80 °C?

The solution is arrived at from inverting:

$$\frac{T - T_e}{T_0 - T_e} = e^{-\left(\overline{h}_c A/\rho V c\right)t} = e^{-t/t_c}$$

$$t_{ACT} = \frac{\rho L c}{\overline{h}_c} \ln \frac{T_0 - T_e}{T_{ACT} - T_e}$$

For the values that we specified, we find that the fuse opens in 243 s. Chapter 3, shows that the heat transfer coefficient is proportional to the fluid velocity $h = Cu^{1/2}$. This results in:

$$\frac{t_{ACT} u^{1/2}}{\ln \dfrac{T_0 - T_e}{T_{ACT} - T_e}} = \frac{\rho L c}{C} = RTI$$

This combination of parameters is the well known response time index for sprinklers.

Fourier's Law of Conduction

As previously noted, the lumped approximation does not allow one to predict the spatial variation of temperature within a body. In some sense, it provides an average or lumped temperature response. To be able to predict the spatial variation of temperature, it is necessary to introduce another physical law that models how heat is transported when temperatures differences exist within a body. We expect heat to flow across a body in proportion to the temperature difference across the body, and perhaps inversely related to the distance across the body. Fourier's law states that the heat flux is proportional to the temperature gradient (the spatial derivative of the temperature). For a one dimensional homogeneous and isotropic object this reduces to the simple expression:

$$q'' = -k \frac{dT}{dx}$$

We use the notation q″ to indicate a heat transfer rate per unit area. The proportionality between the heat flux and the spatial derivative of temperature is the thermal conductivity.

Thermal Conductivity

For materials like air, water, glass, and copper, the thermal conductivity is isotropic (i.e., does not depend on orientation), but it has a temperature dependence. Under conditions in which the overall thermal conductivity difference across the body is small relative to the any particular value of the thermal conductivity in the body, we

can consider k to be essentially a constant. In fire applications, this is often not the case, but for the sake of analysis we will often use this approximation when generating analytical solutions.

There are materials for which the thermal conductivity depends both on the local temperature and also on the orientation. In contrast to isotropic materials for which there is no directional effect, anisotropic materials have this directional dependence. The most commonly encountered anisotropic material in fire applications is wood. The grain structure of wood is the source of the anisotropy. Practically, we would find that for the same temperature difference across a given thickness of wood, the heat transfer rate depends on whether this temperature difference is aligned with the grains or aligned perpendicular to the grains. Of course, as one heats wood, there are also chemical changes to the wood. So, the thermal conductivity depends on the temperature, composition, and orientation. For a simple analysis, the effects of decomposition are often neglected for the initial ignition process.

Again, Fourier's law states that the heat flux vector, $\vec{q}''$, is proportional to the temperature gradient, where the proportionality constant is the thermal conductivity, k. In general, k is a second order tensor and has different values depending on the face and orientation of a differential volume [1–3]. For a general anisotropic material

$$\vec{q}'' = -\begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \nabla T$$

This suggests that the component of the heat flux vector in the x-direction depends on all components of the temperature gradient.

$$q_x = -\left(k_{xx}\frac{\partial T}{\partial x} + k_{xy}\frac{\partial T}{\partial x} + k_{xz}\frac{\partial T}{\partial x} \right)$$

For some materials that are frequently dealt with in fire analyses, such as wood, there is some simplification in the dependence of thermal conductivity on orientation. Laminates like wood are said to be orthotropic. For an orthotropic material, the off-diagonal elements of the thermal conductivity tensor are zero and the diagonal elements are not equal to each other.

$$k = \begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{zz} \end{bmatrix}$$

For metals, many crystalline solids, many amorphous solids, liquids, and gases, the conduction process is considered to take place in an isotropic medium. For such materials, the thermal conductivity can vary spatially and with temperature, but does not have an orientation effect.

$$k = k(T(x,y)) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Systems

Most obvious in gases, it is known that random molecular motion transfers heat from hot molecules to cooler ones. For solids other wave like effects are important. There is a relatively simple theory that describes the physics of thermal conductivity. Conduction heat transfer can be thought of in terms of a carrier particle with a characteristic velocity and characteristic length scale over which it acts. The development of this perspective of thermal conductivity, based on the properties of notional particles is described by Kaviany [7, 8]. In some sense, this description is a simple generalization of the kinetic theory description of thermal conductivity for gases. For gases, we understand that the kinetic theory of gases describes how k varies in terms of a characteristic gas velocity, u, the number density of molecules, n, the mean free path, l, and the molecular internal energy described by the molecular mass and heat capacity (mc).

$$k \cong \frac{1}{3} mc_v nul$$

In the following table adapted from Kaviany [7, 8], the characteristic parameters for various types of conduction systems are provided (Table 2.1).

Examples of Homogenous Materials

In fire analysis, most solid materials are approximated as being homogeneous. Examples of homogeneous systems in fire applications are

Table 2.1 Characteristic quantities used in microscale carrier model of conduction

Microscale carrier	Fluid particle (random motion)	Phonon (quantal lattice vibration)	Electron
Regimes	Dilute gases	Acoustic phonon and optical phonon	Free electrons and valence electrons
Mean free path	Interparticle spacing	Lattice dimension	Lattice dimension
Carrier concentration	Fluid density	Solid density	Free electron density
Carrier speed	Thermal speed	Speed of sound	Electron drift velocity

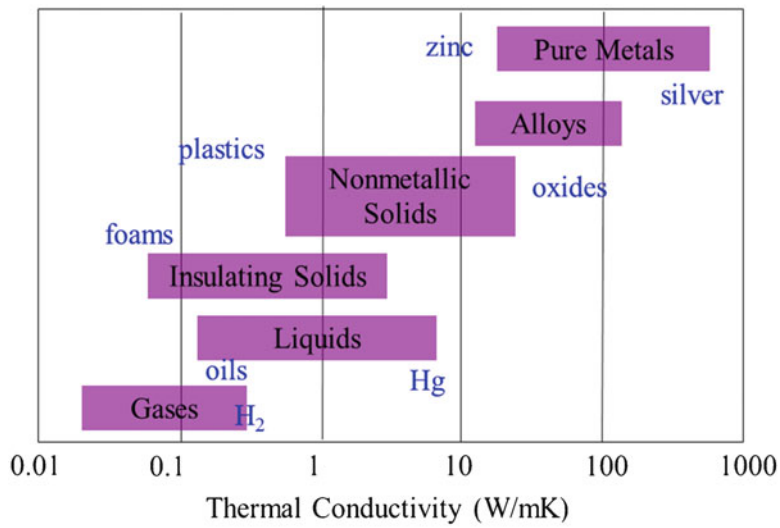


Fig. 2.2 Range of thermal conductivities for different materials (Adapted from [2])

simple polymeric materials, metals, and various types of insulating materials (Fig. 2.2).

Composite Systems

Treatment of composite material thermal conductivity is somewhat more complicated than the treatment for homogeneous materials. With increased use of composite materials like polymer impregnated concrete as structural components, it is useful to discuss how to construct an effective thermal conductivity for such materials. The key to constructing an effective thermal conductivity is to develop a meaningful way to average the thermal properties for the system. A representative averaging volume is the term used to describe the volume over which one can meaningfully average the properties of the composite in order to properly thermally characterize the material. The simplest treatments of composite media thermal conductivity use either series or parallel resistance models. For a mixed medium that is comprised of several different conducting elements, the parallel approximation provides an upper bound on an effective thermal conductivity, while the series approximation provides a lower bound.

Examples of Composite Materials

Examples of composite materials include many types of insulating materials in which at least two types of materials are mixed in various mass fractions. The mass or volume fractions of the constituents can then be used along with their individual conductivities to define an effective conductivity for the system. Various mixing rules have been developed for the effective thermal conductivity. Gebhart [9] discusses a general way of classifying the effective thermal conductivity of a binary system comprised of a matrix material a and added material b as:

$$\frac{k_e}{k_a} = f\left(\frac{k_b}{k_a}, \Phi_b, \frac{L_i}{L}, Bi\right).$$

Depending on the ratio of the thermal conductivities, the ratios of the characteristic lengths of the a and b segments within the medium, and the relative volumetric ratios, different correlations exist for the effective conductivity.

Kaviany [7] presents a correlation for the effective conductivity for random porous solids (e.g., continuous solid and fluid phases) as might occur for a wound insulation material,

$$\frac{\langle k \rangle}{k_f} = \left(\frac{k_s}{k_f}\right)^{0.280 - 0.757 \log(\varepsilon) - 0.057 \log\left(k_s/k_f\right)}$$

which is valid for fluid porosity (volume fraction) in the range of $0.2 < \varepsilon < 0.6$.

Heat Equation Formulations

The heat equation is the name given to the differential equation that models heat conduction in materials. The heat equation is most generally developed in a three dimensional, unsteady form. Depending on the scenario of interest, it is not always necessary to solve the full formulation of the heat equation. By formulating an appropriate reduced form of the heat equation, one can generally compute an accurate representation of the temperature profile and heat flux distribution in the material. In the following sections, several reduced model forms for the heat equation will be discussed.

Steady One Dimensional Models

Under conditions in which there is a primary heat transfer direction, it is appropriate to formulate a one-dimensional form of the heat equation. Further, when the time scale for changes in boundary conditions and sources are large relative to the time scale over which the thermal system equilibrates, the analysis can be treated as being steady. A discussion of how to define the time to equilibrate in conduction systems will follow in a later section.

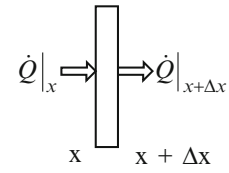


Fig. 2.3 Schematic of differential volume in which steady one dimensional heat equation is developed

To develop the one dimensional conduction model, we consider an elemental volume, ΔV, located between spatial locations x and x + Δx for a heat transfer process that is in steady state. We can apply the first law of thermodynamics to the elemental volume and consider a case in which there is no internal generation (Fig. 2.3).

$$\dot{Q}\big|_x = \dot{Q}\big|_{x+\Delta x} = \text{Constant}$$

$$\dot{Q} = q''A = -kA\frac{dT}{dx}$$

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) = 0$$

Application of Fourier's law leads to an energy equation specified in terms of temperature gradients defined within the solid. The solution can be found by simple integration.

If the thermal conductivity, k, is nearly constant over the temperature range of interest to the problem, then we see that a very simple relationship holds between the temperature difference across the solid, the thermal conductivity, and the thickness of the solid. It is apparent that an analogy holds between this form and Ohm's law, where the heat transfer rate is identified as a current, the temperature difference, ΔT is identified as a potential change, and L/kA is identified as a generalized resistance.

Cylindrical Shells

This same type of analysis can be formulated for cylindrical shells. The difference in the analysis is that the cylindrical shell has variable surface area (Fig. 2.4).

Applying Fourier's law over concentric cylindrical elements yields

$$\dot{Q} = Aq = 2\pi rL\left(-k\frac{dT}{dr}\right)$$

Similar to the development for the planar slab geometry, an effective resistance can be defined for the cylindrical system. Integrating the equation twice yields:

$$\dot{Q} = \frac{2\pi kL(T_1 - T_2)}{\ln(r_2/r_1)}$$

We can extract a resistance from this expression to be:

$$R = \frac{\ln(r_2/r_1)}{2\pi kL}$$

Fin Approximation

The fin approximation refers to one dimensional conduction analysis where heat transfer has a predominant direction, there is no transverse temperature gradient, and the heat transfer in the transverse direction is simply defined through Newton's law of cooling. The simplest example of use of the fin approximation is in the development of the pin fin model. A pin fin is slender rod of length L and diameter D with convective heat transfer taking place over most of the rod's surface. At least one end of the rod is assumed to be fixed at a temperature different from the environmental fluid temperature. For one dimensional heat transfer to be valid, the length of the fin divided by the diameter should be large and a Biot number Bi = hD/k for the fin should be small. In a fire scenario, a fully exposed beam might be modeled as being a fin [10]. Development of the pin fin equation begins with a power balance on a differential section of the fin, as shown below (Fig. 2.5).

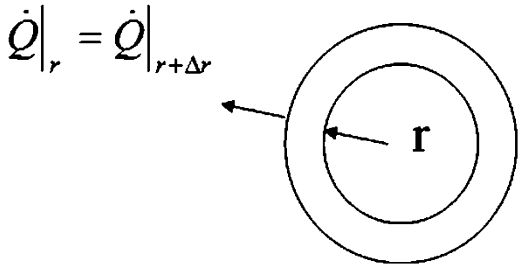


Fig. 2.4 Schematic of cylindrical shell in which one dimensional, steady cylindrical formulation of heat equation is developed

One dimensional analysis (radially lumped) is valid when d/L<<1 and Bi<<1.

The power balance on the differential element coupled with performing a limiting process as Δx approaches zero results in the pin fin equation shown below.

$$kA_c\frac{d^2T}{dx^2} - h_cP(T - T_\infty) = 0$$

This second order ordinary differential equation requires two boundary conditions (BCs). The base temperature is often specified as known (i.e., $T(x = 0)$). Typical BCs at $x = L$ are:

(a) Known tip temperature or Dirichlet condition (e.g., $T(L) = 20\ ^\circ C$)
(b) Known tip heat flux or Neumann condition (e.g., $q''(x = L) = 0$)
(c) Convection tip or Robin's condition (e.g., $-k\frac{dT}{dx}\big|_{x=L} = h(T(x = L) - T_\infty)$

Solutions and examples of use of the fin approximation will be provided in a later section. It is, however, useful to discuss one limiting case solution for the pin fin. A pin fin is said to be semi-infinite if the effect of the imposed temperature at the fin base does not affect the temperature distribution over the entire length of the fin.

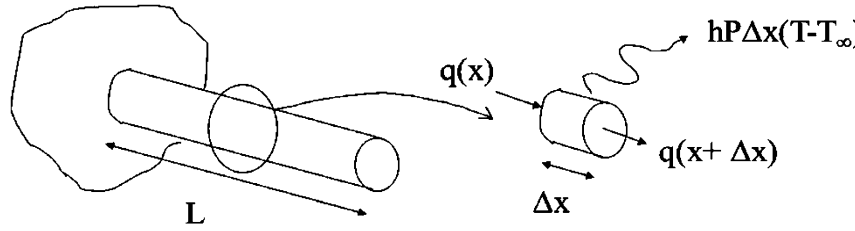


Fig. 2.5 Development of pin fin equation for cylindrical rod

Simple 1D Composite Systems

The simplest composite systems can be thought of in two limiting cases. One might imagine configurations with two different conductors either in a series or in a parallel configuration. For the simpler case, in which the conductors are in series, it is useful to discuss the nature of the interface between the two conductors. The idealized interface between surfaces is considered to be perfect contact in which the interface temperatures are the same on both surfaces. In contrast, an imperfect contact is said to have a contact resistance and there is a temperature discontinuity at the interface between the two surfaces (Fig. 2.6). In the equation shown below h_i is known as the interface conductance with the same units as the heat transfer coefficient (W/m^{2}K)

$$-k_A \left.\frac{dT}{dx}\right|_u = h_i(T_u - T_s) = -k_B \left.\frac{dT}{dx}\right|_s$$

Example 2 A composite wall for a furnace is made of two materials, an insulating material with thermal conductivity k_A and an exterior skin with thermal conductivity k_B. Within the furnace there is an internal heat transfer coefficient $h_{c,i}$ and internal temperature T_i. Outside of this wall there is a heat transfer coefficient hc,o and external temperature T_o. We can calculate the total heat flux (heat transfer rate per unit area) and also the intermediate temperatures (e.g., the interface temperature between the two materials) using a simple resistance analogy (Fig. 2.7).

By identifying a simple conductive resistance for the slabs we see that the heat transfer rate can be specified as

$$\dot{Q} = \frac{T_i - T_1}{1/h_{c,i}A} = \frac{T_1 - T_2}{L_A/k_A A} = \frac{T_2 - T_3}{L_B/k_B A} = \frac{T_3 - T_0}{1/h_{C,o}A}$$

The heat transfer rate can be thought of as being proportional to a potential difference (i.e., the

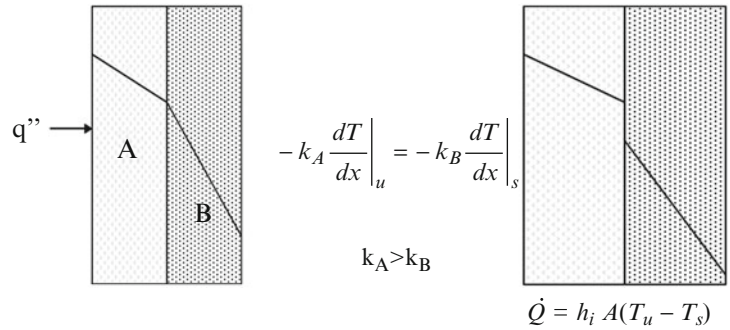


Fig. 2.6 Comparison of perfect and imperfect thermal contact showing contact resistance temperature jump for imperfect contact

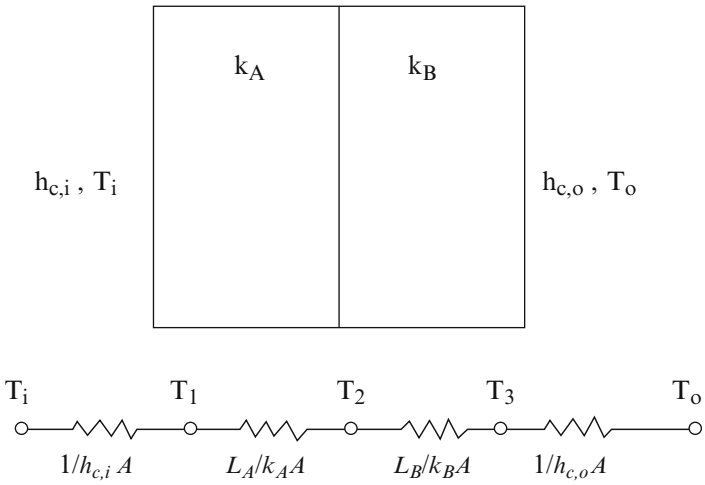


Fig. 2.7 Circuit analogy for conduction in two slabs

temperature drop) and inversely proportional to a resistance. We can use:

$$\dot{Q}R_i = \Delta T_i$$
$$\dot{Q}\sum R_i = \sum \Delta T_i$$

The sum of the individual potential drops equals the total potential difference across the system. Dividing by the sum of the resistances, we get:

$\dot{Q} = \frac{T_i - T_o}{\sum R_i}$ which on a per unit area basis becomes

$q'' = \frac{T_i - T_o}{\sum R_i''}$

For a case with $T_i = 1000\ ^\circ C$, $T_o = 300\ ^\circ C$, $h_{ci} = 30\ W/(m^2K)$, $h_{co} = 10\ W/(m^2K)$, $L_A = 2\ cm$, $L_B = 0.2\ cm$, $k_A = 1\ W/(mK)$, $k_B = 20\ W/(mK)$ we find that the equivalent resistance

$$R_{EQ}'' = \sum R_i'' = 0.153\frac{Km^2}{W}$$

The heat flux is $q'' = \frac{1000-300}{0.153} = 4.56 kW/m^2$. Check the magnitudes of all the resistance elements. If any of these is particularly small compared to the others, we can sometimes neglect that effect in our analysis.

Steady-Multidimensional Models

For problems in which the heat transfer processes internally equilibrate on time scales that are shorter than the times over which external time dependent processes and/or thermal forcing occur, the problem can be formulated as being steady. There are conditions in which there is not a dominant or preferred direction for heat flow for which a one dimensional formulation is inappropriate. For such problems a multidimensional model is required. The governing equation in this formulation is simply:

$$\nabla^2 T = \dot{Q}_v''' \Big/ k$$

Where $\dot{Q}_v'''$ is a volumetric source term. Problems can be set up in Cartesian or cylindrical or spherical coordinate systems depending on the particular type of problem of interest.

Boundary Condition Approximations and Assumptions

Boundary conditions are one of the three types that were previously identified in the fin discussion: prescribed temperature also known as Dirichlet conditions, prescribed flux also known as Neumann conditions, or mixed flux and temperature also known as Robins conditions. For Neumann type conditions, a so-called compatibility condition must hold for a steady formulation to be valid. In practice, this means that the next heat flux into the object must be zero to ensure that a steady solution exists. The three boundary conditions are stated mathematically below.

$$T|_{x=L} = T_s$$
$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = q_s$$
$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h_c\left(T|_{x=L} - T_e\right)$$

Transient One Dimensional Models

For formulations in which transient effects must be included, there are conditions in which there is a predominant heat transfer direction. When there is a preferred direction for heat flow, the thermal forcing can either be from one of the boundaries or from an internal source. For most fire problems, the forcing will occur from a boundary or face. While all physical problems have some finite characteristic length, we will discuss an approximation that specifies the extent of the domain to be semi-infinite.

Thermally Thick and Thin Approximations

We can summarize studies of one dimensional unsteady conduction using a map of solution approximations in a Fourier number (Fo) and Biot number (Bi) space. The Fourier number

physically represents the thermal penetration thickness, δ, divided by the geometric dimension of the system, L.

$$Fo = \frac{\alpha t}{L^2} = \left(\frac{\delta}{L}\right)^2$$

As previously noted, the Biot number can be interpreted as a ratio of the conductive resistance to the convective resistance or as the ratio of the temperature differences in the solid to the driving convective temperature difference. Recall that a lumped thermal approximation is valid when Bi<<1.

The Biot number defined in terms of the true length scale of the object can be considered to be the global Biot number for the scenario. A local Biot number can be constructed using the thermal diffusion length as the characteristic length. In the following equation, it can be shown that the local Biot number is specified as the product of the global Biot number and the square root of the Fourier number [11].

$$Bi = \frac{hR}{k} = \frac{h\delta}{k}\frac{R}{\delta} = \frac{h\delta}{k} Fo^{-1/2}$$

$$\frac{h\delta}{k} = Bi\ \ Fo^{1/2} = \frac{T_w - T_o}{T_\infty - T_w}$$

This local Biot number can then be interpreted as an estimate of the temperature difference between the wall and interior temperatures relative to difference between the wall and fluid reservoir temperatures. Because it is a local Biot number and defined in terms of the thermal penetration depth, it also provides insight into what might be considered to be a semi-infinite domain. For small Fourier numbers, the thermal diffusion distance δ is smaller than the global or geometric length of the object. The map below can be used to characterize the various domains for which different types of approximations are valid. The diagonal line in the upper left hand quadrant represents the demarcation between a convective boundary condition and fixed surface temperature on a semi-infinite body (Fig. 2.8).

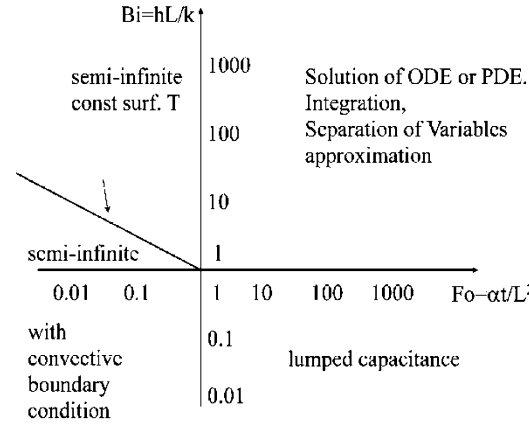


Fig. 2.8 Map identifying different conduction solution regimes as described by Biot and Fourier numbers (Adapted from [12])

Transient Multidimensional Models

For transient multidimensional cases, we rely on the full solution of the heat equation. For simple geometries, we can use analytical methods to construct the solutions. With increased geometrical complexity, computational methods are required. In later sections, both analytical and computational techniques for solving these problems will be detailed.

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{Q}_v'''$$

Analytical Solutions and Examples

Despite the increased accessibility and power of computing devices, analytical models continue to be important in analyzing and characterizing conduction phenomena in fire systems. Analytical solutions are essential in the verification of computational models and are often used to provide back-of-the-envelope engineering guidance. In this section, solutions will be provided for some of conduction formulations that have been discussed.

Steady-One Dimensional Examples

Critical Thickness of Insulation

For the cylindrical system, we noted that the one dimensional steady form of the heat equation is:

$$\frac{d}{dr}\left(rk\frac{dT}{dr}\right)=0$$

Integrating twice, we derive the temperature solution.

$$\frac{T_1-T}{T_1-T_2}=\frac{\ln(r/r_1)}{\ln(r_2/r_1)}$$

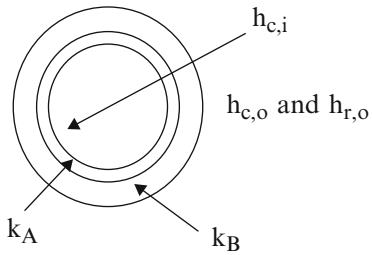
The heat transfer rate is given by:

$$\dot{Q}=\frac{2\pi kL(T_1-T_2)}{\ln(r_2/r_1)}$$

A resistance can be defined as:

$$R=\frac{\ln(r_2/r_1)}{2\pi kL}$$

Often in fire scenarios there is a desire to minimize heat transfer to a pipe or rod from the high temperature environment. Insulating the pipe or rod is one approach to minimize the heat transfer rate. The effect of insulating a pipe is explored in this discussion. A schematic of the system is shown below:



Consider that an internal heat transfer coefficient exists because of the fluid flow within a pipe. The pipe is made of material A and has a thermal conductivity, k_A. Insulation wrapped around the pipe has thermal conductivity, k_B. The fluid flowing in the pipe is at a temperature T_i. External to the insulated pipe is an environment at temperature T_o that interacts with the insulated pipe through convection and radiation. There is a convective heat transfer coefficient $h_{c,o}$ and the radiative transfer process is approximated using a radiation heat transfer coefficient. We can define an overall conductance for the system using the convective resistances and also the conductive resistance that we just developed for cylindrical systems.

$$\dot{Q}=UA(T_i-T_o)=\frac{T_i-T_o}{1/UA}$$

The overall conductance is modeled as $\frac{1}{UA}=\frac{1}{2\pi r_1 L h_{c,i}}+\frac{\ln(r_2/r_1)}{2\pi k_A L}+\frac{\ln(r_3/r_2)}{2\pi k_B L}+\frac{1}{2\pi r_3 L\left(h_{c,o}+h_{r,o}\right)}$ where r_1 is the inner radius of the pipe, r_2 is the outer radius of the pipe, and r_3 is the outer radius of the insulation. We can neglect the internal convection resistance and the conduction resistance of the pipe relative to the external convection resistance and the conduction resistance of the insulation when:

$$\frac{h_{c,o}r_3}{h_{c,i}r_1}<<1 \quad \text{and} \quad \frac{h_{c,o}r_3\ln(r_2/r_1)}{k_A}<<\frac{h_{c,o}r_3\ln(r_3/r_2)}{k_B}$$

For such a case, we find that the heat transfer rate is related to the driving temperature difference as:

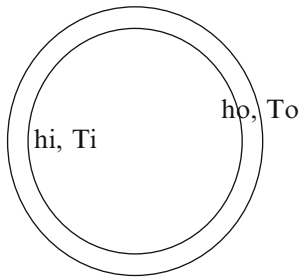
$$\dot{Q}=\frac{T_i-T_o}{R}=\frac{T_i-T_o}{\ln(r_o/r_i)/2\pi Lk+1/2\pi Lr_o h_o}$$

While we might assume that the addition of an insulating layer will always result in a decrease in the heat transfer rate, we can easily see that the denominator has a minimum value when there has been addition of insulation. This minimum value of resistance means that the heat transfer rate increases with the addition of some insulation. Taking the derivative of the denominator and identifying the extremum shows that when the insulating layer has a radius of

$$r_0=r_{cr}=\frac{k}{h_o}$$

the heat transfer has a maximum value.

Example 3 An exposed water sprinkler line is within a compartment on fire. The fire products are at a temperature of 1000 °C. We can assume that the heat transfer coefficient between the fire products and the pipe is 30 W/(m^{2}K). The pipe has a diameter of 5 cm, a thickness of 5 mm, and a thermal conductivity of 30 W/(mK). At one section of the pipe, the water has a mean temperature of 30 °C and an internal heat transfer coefficient of 100 W/(m^{2}K). We can calculate the heat flux at this section of pipe using a thermal resistance circuit.



$$q_i'' A_i = \frac{(T_i - T_o)A_i}{\sum R_i''} = \frac{T_i - T_o}{\frac{1}{h_i A_i} + \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi k L} + \frac{1}{h_o A_o}}$$

Note that the radial location of the heat flux must be specified as the heat flux is different at the outer edge of the pipe from its value at the inner edge. The total heat transfer rate is 2.7 kW/m of pipe. The heat flux at the inner wall is 9.7 kW/m^2.

For an insulator with thermal conductivity of 1 W/(mK) and the same external heat transfer coefficient of 30 W/(m^{2}K), the critical radius $r_{cr} = \frac{k}{h_o}$ is 0.03 m or 3 cm. Thus, adding 1 cm of an insulator with a thermal conductivity of 1 W/(mK) to the pipe increases the heat transfer rate. We would have:

$$q_i'' A_i = \frac{(T_i - T_o)A_i}{\sum R_i''} = \frac{T_i - T_o}{\frac{1}{h_i A_i} + \frac{\ln\left(\frac{r_p}{r_i}\right)}{2\pi k L} + \frac{\ln\left(\frac{r_o}{r_p}\right)}{2\pi k L} + \frac{1}{h_o A_o}}$$

The total heat transfer rate increases to 2.9 kW/m. As previously noted, while we increased the conductive resistance by the insulation, we decreased the external convective resistance by increasing the area. The total resistance has decreased from 0.354 Km/W to 0.33 Km/W. The choice of insulator matters. The same thickness insulation, but with a thermal conductivity of 0.1 W/(mK) increases the equivalent resistance to 0.59 Km/W and decreases the total heat transfer rate to 1.6 kW/m.

Fin Model of a Beam Extending Between Two Walls

Frequently, engineers must evaluate the thermal response of beams and columns that are affected by high temperature gases associated with a fire [10]. Imagine that two walls of a compartment are at temperatures T_w and a high temperature gas flows around a fully exposed beam. We can evaluate the thermal response of the beam using a fin model (Fig. 2.9).

Earlier, we derived the constant cross-sectional area fin equation:

$$kA_c \frac{d^2T}{dx^2} - h_c P(T - T_\infty) = 0$$

Solutions to this equation are either in terms of exponential functions or hyperbolic sine and cosine. It is convenient to define an excess temperature in terms of the difference between the fin temperature and the fluid temperature (Fig. 2.10).

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$
$$\theta = T - T_\infty$$
$$m^2 = h_c P / kA_c$$

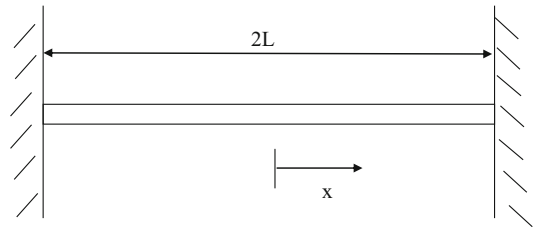


Fig. 2.9 Schematic of beam convectively heated between two walls

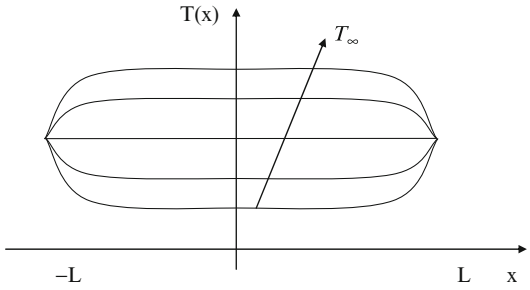


Fig. 2.10 Temperature profiles for suspended beam example showing effect of the gas to wall temperature difference on the temperature profile

For this case in which the walls are at the same temperature we arrive at a solution:

$$T = T_\infty + (T_w - T_\infty)\frac{\cosh(mx)}{\cosh(mL)}$$

If we had defined the origin for the x coordinate as being one of the walls, and if the parameter mL>>1, the exponential solution is appropriate and decays to zero at large x values.

$$\frac{\theta}{\theta_b} = e^{-mx}$$

Example 4 An unprotected round steel bar extends across two compartments. The bar has a diameter of 5 cm. One compartment is fully involved in fire with gas temperatures of 600 °C around the bar. The heat transfer coefficient is 30 W/(m^{2}K). The compartment temperature of the room that the bar extends into is at 20 °C. For convenience, we assume that the void space also has a air temperature of 20 °C and that the heat transfer coefficients are also 30 W/(m^{2}K) in the void space and in the nonfire compartment. We can estimate the temperature of the bar within the void space to see if it might pose an ignition hazard using fin analysis.

For a sufficiently long bar we can use the semi-infinite assumption which states that the temperature distribution in the fin has an exponential variation (Fig. 2.11).

$$\frac{\theta_C(x)}{\theta_{C,B}} = \frac{T(x^+) - T_{C,\infty}}{T_I - T_{C,\infty}} = e^{-mx^+} \text{ and } \frac{\theta_F(x)}{\theta_{F,B}} = \frac{T(x^-) - T_{F,\infty}}{T_I - T_{F,\infty}} = e^{-mx^+}$$

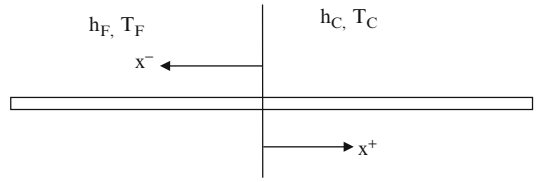


Fig. 2.11 Schematic diagram of a bar extending between two domains with different temperatures and different heat transfer coefficients

At the interface of the fire flow and the cold flow, the bar temperature is continuous and can be specified to be an interface temperature T_I. Also, the heat flux is continuous at the interface. Since the thermal conductivity does not change, we simply write this as:

$$\frac{\partial T}{\partial x^+} = -\frac{\partial T}{\partial x^-}$$

$$-(T_I - T_{C,\infty})\left(\frac{h_A P}{kA_c}\right)^{1/2} = (T_I - T_{F,\infty})\left(\frac{h_F P}{kA_c}\right)^{1/2}$$

$$T_I = \frac{m_F T_{F,\infty} + m_A T_{C,\infty}}{m_F + m_A}$$

For a case in which the heat transfer coefficient is the same on both sides of the interface, the interface temperature is 310 °C and is a simple average of the fire side gas temperature and the cold compartment air temperature. If the heat transfer coefficients had been different, the interface temperature is the weighted average of the two temperatures as shown above.

Flame Temperature Thermocouple Measurement

Consider a thermocouple that is modeled as a rod of diameter D, thermal conductivity k, and length L with half of the thermocouple length in air and the other half inserted into a pool fire flame. It is useful to determine if there is a conduction error in the thermocouple. There are several possible fin models that can be used to illustrate this effect. Here, we take an idealized scenario in which the thermocouple wire is modeled as being a fin with insulated ends in the hot fluid and also in the cold fluid. In reality, there is a convective end condition in the hot region (i.e., at the thermocouple junction) and the termination point in the cold region is often very far away

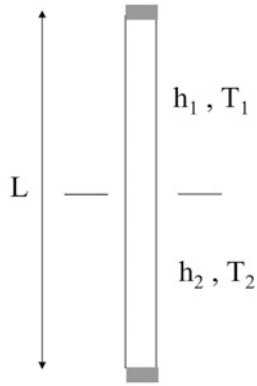


Fig. 2.12 Fin suspended across free surface in fluids with different temperatures and heat transfer coefficients

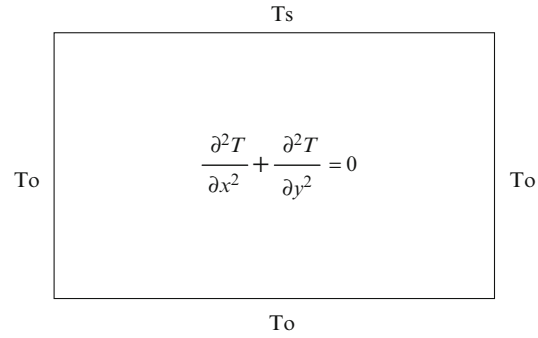


Fig. 2.13 Plate in which Laplace equation is to be solved to determine 2D temperature profile

from the point where the thermocouple is inserted into the flame. The question posed here is whether the temperature at the end placed into the hot fluid (i.e., the flame) reaches the hot fluid temperature (in which case, the thermocouple is measuring the correct temperature) or whether the conduction losses to the cold side are affecting the measurement. The insulated end solution for a rod of diameter D, thermal conductivity k, and length L suspended between two fluids/flows is

$$\frac{\theta_A}{\theta_{A0}}=\frac{T_A-T_{A\infty}}{T_{A0}-T_{A\infty}}=\frac{\cosh[m_A(L_A-x_A)]}{\cosh(m_A L_A)}$$

If the fin is not semi-infinite (i.e., mL is not large relative to unity), we need to use the full solution to get the interface temperature. The interface temperature (Fig. 2.12) is

$$T_{A0}=\frac{T_{A\infty}m_A\tanh(m_A L_A)+T_{B\infty}m_B\tanh(m_B L_B)}{[m_A\tanh(m_A L_A)+m_B\tanh(m_B L_B)]}$$

As we saw earlier, if the heat transfer coefficient is the same on the hot and cold sides, we get an interface temperature that is the simple mean value of the hot and cold temperatures. The error in the measured end temperature

$$\frac{T_A(L)-T_{A\infty}}{T_{A\infty}}=\frac{1}{2}\left(\frac{T_{B\infty}}{T_{A\infty}}-1\right)\left(\frac{1}{\cosh(m_A L_A)}\right)$$

For a case with an air temperature of 300 K and a flame temperature of 2000 K, if the thermocouple has a diameter of 2 mm, a heat transfer coefficient of 30 W/m^{2}K, thermal conductivity of 60 W/(mK), and a length of 6 cm equally split between the hot and cold fluids, we find that the interface temperature is 1150 K and that the tip temperature is 1751 K. This represents an approximately 12 % error in the predicted freestream temperature.

Steady Multidimensional Example

There are many heat transfer systems for which there is a need to generate multidimensional solutions. Often it is appropriate to model a two dimensional temperature variation for geometries in which two characteristic lengths are of comparable magnitude and the third characteristic length is significantly longer. For such geometries, if one is interested in specifying the temperature variation at a cross-section of the geometry at lengths far from the boundaries of the long direction, a two dimensional approximation often proves to be valid. In this section, we discuss the separation of variables approach to solving such problems.

Separation of Variables Applied to Two Dimensional Fin

Example 5 A very long rectangular bar has three sides maintained at temperature 20 °C and one side at temperature 120 °C (Fig. 2.13). Because the bar is very long, we can neglect the axial heat transfer problem and focus our attention on the heat transfer processes at some intermediate slice

within the bar. We are simplifying this three dimensional problem into a two dimensional problem. Further, we can define a new temperature, sometimes called the excess temperature, defined as the temperature subtracting off some reference value. For this problem, it is convenient to think of the 20 °C temperature as a reference value.

We define an excess temperature to be $\theta(x, y) = T(x, y) - 20$. In terms of the excess temperature, we have the following boundary conditions:

$$\begin{aligned} x &= 0, \;\; 0 < y < b; \;\; \theta = 0 \\ y &= 0, \;\; 0 < x < a; \;\; \theta = 0 \\ x &= a, \;\; 0 < y < b; \;\; \theta = 0 \\ y &= b, \;\; 0 < x < a; \;\; \theta = 100 \end{aligned}$$

We use a standard technique for the solution of finite domain partial differential equations called separation of variables to solve for the temperature distribution in the plate section. Separation of variables relies on expanding the dependent function θ(x,y) in terms of an appropriate set of basis functions. For this Cartesian coordinates example, the basis functions turn out to be sine and cosine functions. The members of the family of functions that constitute the set of basis functions are said to be orthogonal to each other in a weighted integral sense. There is a deep relationship between the process in separation of variables and the theory of Fourier series as well as many computational techniques for solving differential equations. One simple starting point for separation of variables solution is to assume a solution form of

$$\theta(x, y) = X(x)Y(y)$$

Upon substituting this solution form into the partial differential equation, one obtains two separable ordinary differential equations: $-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2$

The solution of the X(x) equation yields the trigonometric functions, while the solution of the Y(y) equation yields hyperbolic trigonometric functions sinh and cosh. Substitution of this form of solution into the PDE results in unique choices for the parameters λ and also presents an opportunity to define the so-called Fourier coefficients for the problem (Fig. 2.14).

$$\theta(x, y) = 100 \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi \sinh(n\pi b/a)} \times \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

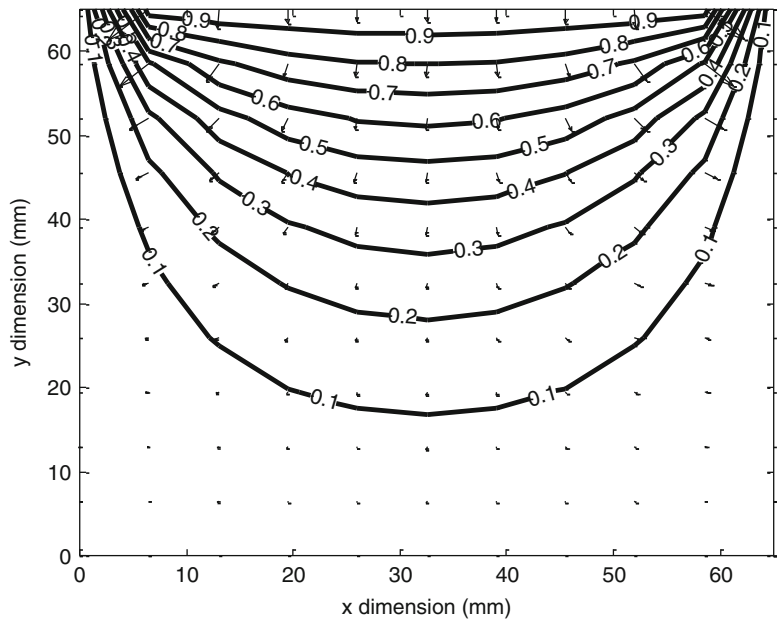


Fig. 2.14 Separation of variables solution ($\theta(x, y)$/100) for 2D plate conduction problem

Transient Lumped Examples with Time Dependent Forcing

Often the transient response of a thermal system must be evaluated. The lumped thermal approximation is valid when the internal conduction resistance is small relative to the external convective resistance.

$$\frac{\text{Internal conduction resistance}}{\text{External convection resistance}} \cong \frac{L/k_s A}{1/\overline{h}_c A} = \frac{\overline{h}_c L}{k_s}$$

Imagine a cold spherical heat sensor emersed in an initially cold gas for which the gas temperature is a linearly increasing function of time. It is useful to understand how the heat sensor's temperature will vary with time. Assume that Bi = 0.1 and Fo ~ 10. T(r,t = 0) = To.

The model used to characterize the temperature variation of the sphere is the same lumped thermal model previously discussed. The difference is that because the reservoir temperature is time varying, the temperature solution is no longer a simple exponential function (Fig. 2.15).

$$\rho V c \frac{dT}{dt} = -\overline{h}_c A (T - T_\infty)$$

$$\frac{dT}{dt} = -\frac{\overline{h}_c A}{\rho V c}(T - T_\infty) \quad \text{with} \quad T_\infty(t) = T_o + at$$

The solution (Fig. 2.16) with $\theta(t) = T - T_o$ is

$$\theta(t) = a\tau\left[\exp\left(\frac{-t}{\tau}\right) - 1\right] + at$$

$$= a\left[t - \tau\left(1 - \exp\left(\frac{-t}{\tau}\right)\right)\right]$$

We see in Fig. 2.16 that there is a time lag between the increasing temperature of the fluid and the increasing temperature of the detector. Additionally, we see that the detector temperature at any given time is always lower than the fluid temperature.

Laplace Transform Methods

The Laplace transform is one of several integral transform methods that can be used in conduction analysis [12]. For time dependent functions, the Laplace transform maps the time dependent derivative terms into algebraic terms that are

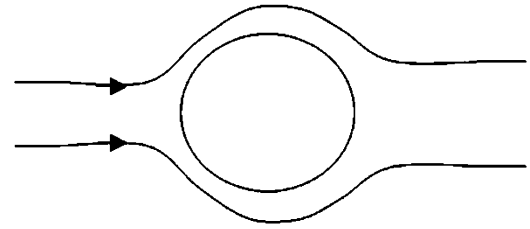


Fig. 2.15 Schematic showing convective flow over an object that will be analyzed using a lumped thermal approximation

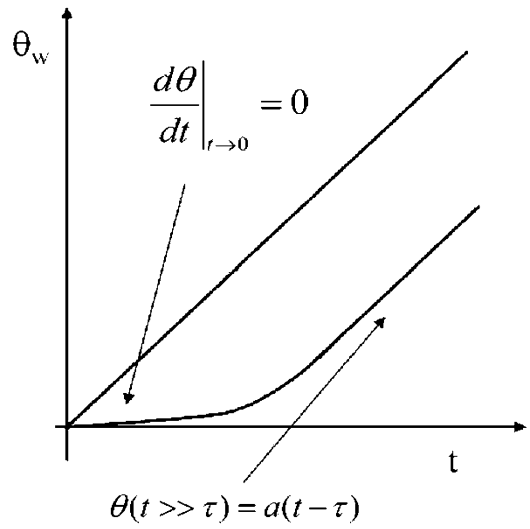


Fig. 2.16 Solution for lumped approximation problem in which fluid temperature is a linearly increasing function of time

parameterized by a new independent variable. For an ordinary differential equation (e.g., a thermally lumped system analysis), application of the Laplace transform yields an algebraic equation that can be solved for the transformed dependent variable. The time dependent form is retrieved using an inversion integral called the Bromwich integral. In the equation below, the Laplace transform integral operator is applied to a general temperature function that depends on time and spatial location. The definition of the Laplace transform is shown below.

$$\mathcal{L}(T) = \hat{T}(x,s) = \int_0^\infty T(x,t) e^{-st} dt$$

An example of the use of the Laplace transform is provided below. Consider the lumped analysis system that we have been using

$$\frac{dT}{dt} = -\frac{\overline{h}_c A}{\rho V c}(T - T_\infty)$$

Straightforward application of the transform yields:

$$s\hat{T}(s) - T(0) = -\frac{\overline{h}_c A}{\rho V c}\left(\hat{T}(s) - \frac{T_\infty}{s}\right)$$

Because the initial condition shows up in this transformation, it is often convenient to use superposition to force the initial condition to be zero. For such a case we have,

$$s\hat{\theta}(s) = -\frac{\overline{h}_c A}{\rho V c}\left(\hat{\theta}(s) - \frac{\theta_\infty}{s}\right)$$

Solving for the transformed temperature variable yields:

$$\hat{\theta}(s) = \frac{\theta_\infty}{\tau_c s\left(s + \tau_c^{-1}\right)}$$

Using any one of a number of inversion tables or online calculators, we retrieve the inverse of this function.

$$\theta(t) = \theta_\infty\left(1 - \exp\left(\frac{-t}{\tau_c}\right)\right)$$

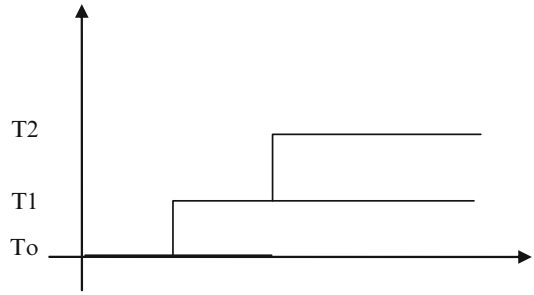
The result above is equivalent to the result shown earlier. The power of the Laplace transform method is that it can be applied to arbitrarily complex differential equations. The challenge of the method has typically been generating the inverse transform. With increasing accessibility of symbolic mathematical software tools, with some freely available online such as Wolfram Alpha, this particular challenge is no longer quite as severe.

Duhamel Integral Methods

The Duhamel integral is a method for generating a solution for an arbitrarily complex time dependent forcing of a conduction system using the solution for a step change. A simple case to present the ideas of this concept is the problem in which there is a step change in the external temperature of a reservoir in contact with a lumped thermal system. We know the solution from earlier sections.

$$\frac{dT}{dt} = -\frac{\overline{h}_c A}{\rho V c}(T - T_\infty)$$

Graphically, a series of external temperature step changes can be looked at as follows:



If we focus on the initial step change, this is no different from the original problem that we considered. The offset in time can be addressed using an offset in time. Simply by creating a new time variable, e.g., $s = t - t_1$, we create an offset time variable. To explore what is meant by a step change in the external temperature, we formally formulate the mathematical equation using the Heaviside step function.

$$\frac{dT}{dt} = -\frac{\overline{h}_c A}{\rho V c}(T - T_0 - (T_1 - T_0)H_s(t - t_1) - (T_2 - T_1)H_s(t - t_2))$$

If we now define a relative temperature $T - T_0$, we arrive at the equation

$$\frac{d\theta}{dt} = -\frac{\overline{h}_c A}{\rho V c}(\theta - \Delta\theta_1 H_s(t - t_1) - \Delta\theta_2 H_s(t - t_2))$$

To solve this equation, it is convenient to define a series of linear transformations and use superposition to solve this individual equations.

$$\theta = F + G$$

$$\frac{dF}{dt} + \frac{dG}{dt} = -\frac{\overline{h}_c A}{\rho V c}(F + G - \Delta\theta_1 H_s(t - t_1) - \Delta\theta_2 H_s(t - t_2))$$

$$\frac{dF}{dt} = -\frac{\overline{h}_c A}{\rho V c}(F - \Delta\theta_1 H_s(t - t_1))$$

$$\frac{dG}{dt} = -\frac{\overline{h}_c A}{\rho V c}(G - \Delta\theta_2 H_s(t - t_2))$$

Because $\theta(0) = 0$, we can enforce that both F and G are also equal to zero and have solutions of form:

$$F(t) = \Delta\theta_1\left(1 - e^{-(t-t_1)/t_c}\right)$$

The solution for the excess temperature is then:

$$\theta(t) = \Delta\theta_1\left(1 - e^{-(t-t_1)/t_c}\right) + \Delta\theta_2\left(1 - e^{-(t-t_2)/t_c}\right)$$

For a series of step changes, we can generalize this superposition of solutions to generate:

$$\theta(t) = \sum \frac{\Delta\theta_i}{\Delta\tau_i}\left(1 - e^{-(t-\tau_i)/t_c}\right)\Delta\tau_i = \int_0^t \frac{d\theta}{d\tau}\left(1 - e^{-(t-\tau)/t_c}\right)d\tau$$

We immediately see that for the case in which the freestream temperature is changing as a linear function of time $T_\infty(t) \quad = T_o + at$

We get

$$\theta(t) = \int_0^t a\left(1 - e^{-(t-\tau)/t_c}\right)d\tau$$

$$= at - ae^{-t/t_c}t_c\left(e^{t/t_c} - 1\right)$$

$$= at - at_c\left(1 - e^{-t/t_c}\right) = a\left[t - t_c\left(1 - e^{-t/t_c}\right)\right]$$

which is the same result that we arrived at using a much simpler analysis in an earlier section. The power of the Duhamel formulation is evident in problems like the semi-infinite slab problem for which the solution is expressed in terms of the error function. One interesting application of the Duhamel method that we will use in the section on inverse analysis is the determination of the heat flux from a temperature measurement. The generalization of the Duhamel form states that the time dependent variation of temperature can be defined in terms of an integral of the product of the step response solution, which is time dependent, and the time derivative of the unsteady effect.

Transient Semi-infinite (Thermally Thick) One-Dimensional Examples

For thermally thick problems in which the thermal penetration wave never reaches the back side, analytical solutions are available for a range of boundary conditions. These solutions are generally specified in terms of tabulated functions [13]. Useful approximate solutions can also be developed for these problems using integral approximations and the scaling properties of diffusive transport (Fig. 2.17).

For the thermally thick conduction problem, the solution is often described as being for a semi-infinite domain (Fo<<1). Using either a Laplace transform solution approach or any number of other approaches, one can arrive at the temperature variation for a constant surface temperature boundary condition.

$$\theta = \frac{T - T_0}{T_s - T_0} = 1 - \frac{2}{\pi^{1/2}}\int_0^\eta e^{-u^2}du = \operatorname{erfc}\left(\frac{x}{(4\alpha t)^{1/2}}\right)$$

We see that the solution is defined using the complementary error function (erfc). The surface heat flux variation with time is:

$$q_s = -k(T_s - T_0)\left[-\frac{2}{\pi^{1/2}}e^{-\eta^2}\frac{1}{(4\alpha t)^{1/2}}\right]_{\eta=0} = \frac{k(T_s - T_0)}{(\pi\alpha t)^{1/2}}$$

Interestingly, an approximation for the heat flux that depends on the thermal penetration depth does a reasonable job of predicting the overall trends.

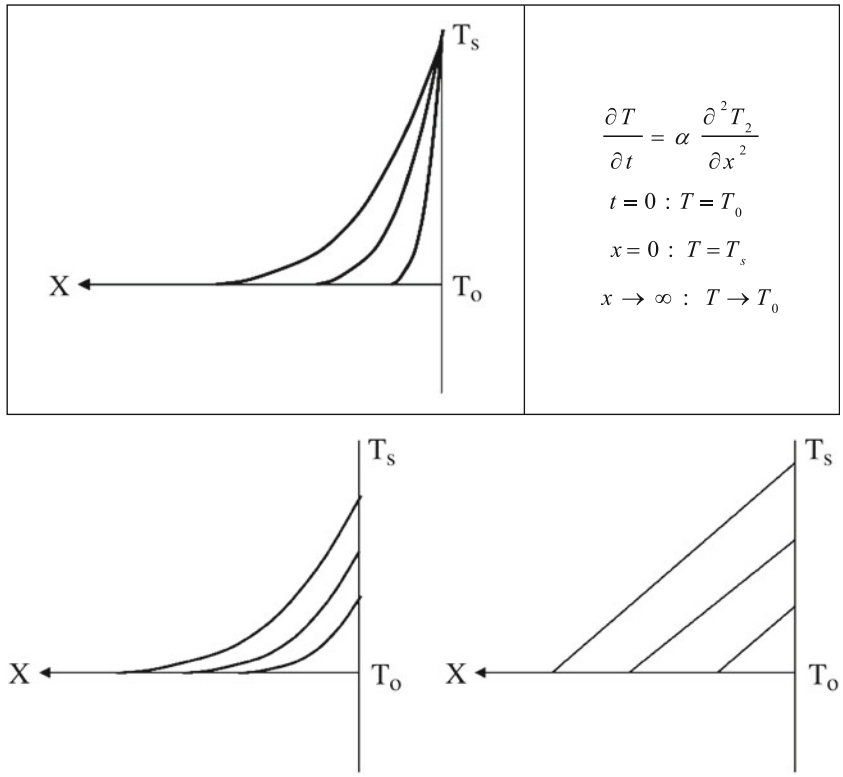


Fig. 2.17 Governing equation for semi-infinite model of conduction in a slab and associated temperature profiles

$$q_s \cong \frac{k(T_s - T_0)}{\delta(t)} = \frac{k(T_s - T_0)}{\sqrt{\alpha t}}$$

In the approximation, we have used the sing the simplified scaling law $\delta \approx \sqrt{\alpha t}$

A similar approximation can be used for the constant surface heat flux boundary condition. We recognize that the approximate variation is as follows:

$$T_s = T_0 + \frac{q_s\sqrt{\alpha t}}{k} = T_0 + \frac{q_s\sqrt{t}}{\sqrt{k\rho c}}$$

The surface temperature has $t^{1/2}$ dependence which we can compare to exact analytical result

$$T(x,t) = T_0 + \frac{q_s}{k}\left[\left(\frac{4\alpha t}{\pi}\right)^{1/2} e^{-x^2/4\alpha t} - x\,\mathrm{erfc}\frac{x}{(4\alpha t)^{1/2}}\right]$$

which at

$$\mathrm{x} = 0 \text{ is } T_s = T_0 + \frac{q_s}{k}\left[\left(\frac{4\alpha t}{\pi}\right)^{1/2}\right]$$

The Newton's law of cooling convective boundary condition case is analytically quite challenging, but can be reduced to a very simple form using the type of scaling previously described (Fig. 2.18).

We note that the internal heat flux can be approximated using a diffusion model and must be balanced by the Newton's law of cooling convection term.

$$q_s \cong \frac{k(T_s(t) - T_0)}{\delta} = h(T_e - T_s)$$

In solving for the wall temperature we arrive at:

$$\frac{T_s(t) - T_0}{(T_e - T_0)} = \frac{\frac{h\sqrt{\alpha t}}{k}}{\left(1 + \frac{h\sqrt{\alpha t}}{k}\right)}$$

At long time, we get constant surface temperature. We can compare this to the exact solution (Fig. 2.19):

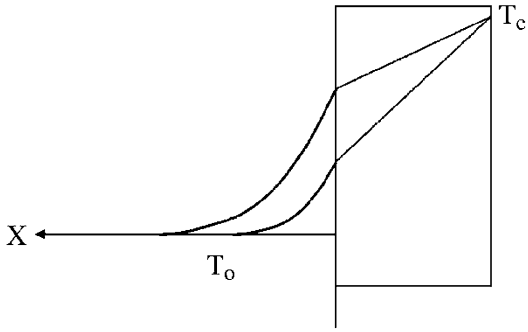


Fig. 2.18 Schematic of semi-infinite model of conduction in a slab with convective boundary conditions

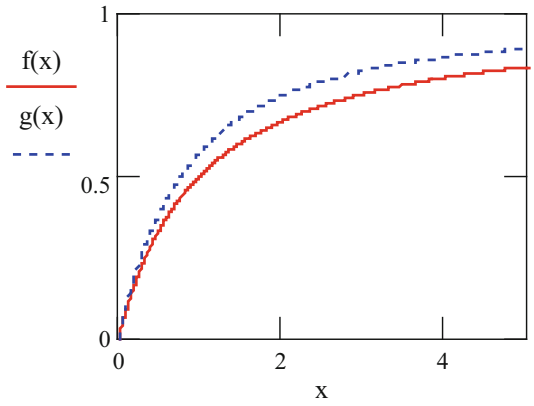


Fig. 2.19 Comparison of approximate and exact solution of semi-infinite model of conduction in a slab with convective boundary conditions

$$\frac{T_s - T_0}{T_e - T_0} = 1 - e^{(h_c/k)^2 \alpha t} \text{erfc}\left[\frac{h_c}{k}(\alpha t)^{1/2}\right]$$

Transient Multidimensional Examples

Multidimensional transient problems represent a class of problems that pose sufficient analytical complexity that we often revert to computational solutions for these systems. For some simple geometries, however, there are tabulated results that are useful for determining the overall

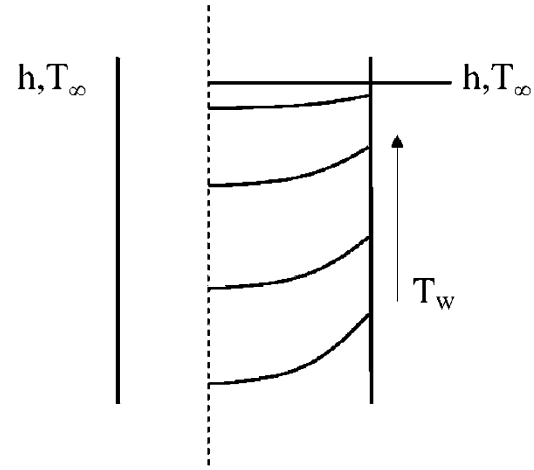


Fig. 2.20 Schematic of one dimensional conduction in a slab with convective boundary conditions

thermal behavior of such systems. The starting point for constructing such solutions is a one dimensional transient problem. As an example, for the slab/wall geometry for which there is convective heating on two sides, there is a separation of variables series solution for the problem. For relatively long times as is defined by the Fourier number (Fo>1), the series solution converges with a small number of terms. Physically, the temperature profile (i.e., the solution) becomes smoother with increasing time. Because so few terms are required to represent the temperature field evolution, it is possible to define the solution in terms of a simple product of a time dependent function and a spatially varying function (Fig. 2.20). The governing equation, initial condition and boundary conditions for the one dimensional problem are:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$t = 0: \quad T = T_0$$

$$x = 0: \quad \frac{\partial T}{\partial x} = 0$$

$$x = L: \quad -k\frac{\partial T}{\partial x} = h(T - T_\infty)$$

The solution of the one dimensional problem is:

$$\frac{T-T_\infty}{T_0-T_\infty}=\frac{\theta}{\theta_0}=\sum_{n=0}^{\infty} \times \frac{\sin(\lambda_n L)}{\lambda_n L+\sin(\lambda_n L)\cos(\lambda_n L)} \times e^{-\lambda_n{}^2 \alpha t}\cos(\lambda_n x)$$

The eigenvalues of the problem can be written as: $\lambda_n L = \mathrm{Bi}\cot(\lambda_n L)$. Also, the argument of the exponential function can be written as:$\lambda_n^2 \alpha t = (\lambda_n L)^2\left(\frac{\alpha t}{L^2}\right) = (\lambda_n L)^2\,\mathrm{Fo}$. This suggests that the solution is defined by the following parameters:

$$\mathrm{Bi},\ \mathrm{Fo},\ \frac{x}{L}$$

For long enough times, the one term expansion generates [2]:

$$\frac{T-T_\infty}{T_0-T_\infty}=\frac{\theta}{\theta_0}=\frac{\theta}{\theta_{CL}}\times\frac{\theta_{CL}}{\theta_0} = C(\mathrm{Bi})\ \ f(t)g(x)$$

The centerline temperature varies only with time, while the temperature at any location normalized by the centerline temperature is essentially a function of the spatial location.

It can be easily shown that the temperature within a block can be found as the product of the temperatures found from three separate one dimensional solutions (Fig. 2.21):

$$\theta(t,x,y,x)=P(x,t)P(y,t)P(z,t)$$

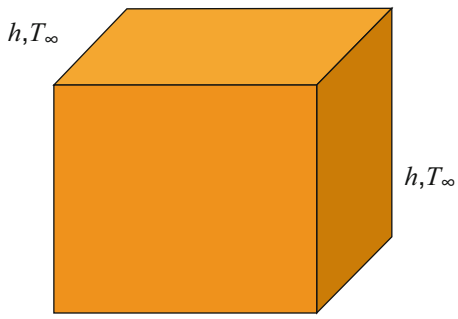


Fig. 2.21 Schematic diagram of cube with external convective heating

Overview of Computational Issues in Conduction

For complex geometries or for problems with nonlinear effects and multiple physical processes, analytical solutions are usually not easily found. For the majority of practical heat transfer processes in fire, a fire engineer will resort to the use of computational tools to characterize the system. Olenick and Carpenter [14] summarize the key features of a number of computational tools available for simulating thermal response associated with fire endurance testing. These tools use a variety of numerical discretization techniques. In the following section, we discuss the types of computational techniques most often used for fire problems and present simple examples for some of these cases to help clarify what underlying steps are taken in some of the tools and codes described in [14]. Useful reference texts for numerical solution of the differential equations associated with conduction are Shih [15] and Ferziger [16].

To start, we define a very simple conduction/diffusion problem with a source (S).

$$\Gamma\frac{d^2\phi}{dx^2}+S=0$$

The analytical solution can be easily found by integrating this equation twice. For the case of a constant source we get:

$$\phi = C_2 + C_1 x - \frac{S}{2\Gamma}x^2$$

Consider the case with boundary conditions:

$$\left.\frac{d\phi}{dx}\right|_{x=0} = 0 \text{ and } \phi(L)=0$$

C_1 is zero by the adiabatic condition, and $C_2 = \frac{S}{2\Gamma}$ such that;

$$\phi=\frac{S}{2\Gamma}(L-x^2)$$

We will compare this solution to those generated by the numerical solutions.

Finite Difference Approximations

The finite difference technique is used to solve differential equations by expanding the dependent variable, at specified positions in the computational space into a Taylor series, appropriately adding and subtracting terms from other series terms together to generate approximations for the derivative terms in the underlying differential equation. As an example, the dependent variable is expanded below in a Taylor expansion at locations 1 and 3 (Fig. 2.22).

$$\phi_1 = \phi_2 - \left.\frac{\partial \phi}{\partial x}\right|_2 \Delta x + \frac{1}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_2 \Delta x^2 + O\left(\Delta x^3\right)$$

$$\phi_3 = \phi_2 + \left.\frac{\partial \phi}{\partial x}\right|_2 \Delta x + \frac{1}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_2 \Delta x^2 + O\left(\Delta x^3\right)$$

By subtracting the two series from each other, we arrive at one representation for the derivatives at location 2.

$$\left.\frac{\partial \phi}{\partial x}\right|_2 = \frac{\phi_3 - \phi_1}{2\Delta x} + O\left(\Delta x^2\right)$$

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_2 = \frac{\phi_3 - 2\phi_2 + \phi_1}{\Delta x^2} + O\left(\Delta x^2\right)$$

Substituting the approximations into the original differential equation then provides an algebraic rule by which to evaluate the values of the dependent variable at the prespecified (grid) points.

$$\Gamma \left.\frac{\partial^2 \phi}{\partial x^2}\right|_2 = \Gamma\left(\frac{\phi_3 - 2\phi_2 + \phi_1}{\Delta x^2}\right)$$

And the source term can be approximated as $S_2 = S(\phi_2)$

We see that the final form of the algebraic system of equations is

$$\Gamma\left(\frac{\phi_3 - 2\phi_2 + \phi_1}{\Delta x^2}\right) = -S(\phi_2)$$

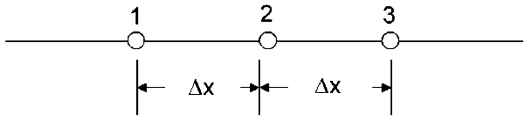


Fig. 2.22 Schematic of finite difference stencil

Symbolically, we can arrange the system of algebraic equations into a form shown below.

$$a_{i-1}\phi_{i-1} - a_i\phi_i + a_{i+1}\phi_{i+1} = b$$

$$a_{i-1} = \frac{\Gamma}{\Delta x}, a_{i+1} = \frac{\Gamma}{\Delta x}, a_i = \frac{-2\Gamma}{\Delta x}, b = -\overline{S}\ \Delta x$$

For the case in which $\Gamma = 0.2$, $0 < x < L$, and $S = 4$

We will use $N = 5$ grid points, $\Delta x = 0.25L$ for $L = 1$

$$a_{i-1} = 1, a_{i+1} = 1, a_i = -2, \quad b = 1$$

$$\begin{bmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \text{ with solution } \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 8 \\ 7.5 \\ 6 \\ 3.5 \\ 0 \end{bmatrix}$$

Which is exactly the same as the answer provided by the exact solution.

Finite Volume Approximation

An alternative way of generating algebraic equations representing the differential equation is the finite volume technique. The finite volume technique is closely related to Galerkin finite element techniques. Both are members of a family of solution techniques called method of weighted residuals. For both types of solution techniques, an integrated form of the underlying differential equation is used to develop the algebraic equations. In both types of techniques, one weights the equation before integrating. Because the weight in the control volume method is just equal to unity, the process to developing the algebraic equations has a simple physical interpretation. The finite volume technique is sometimes called the control volume technique or the subdomain method [17] (Fig. 2.23).

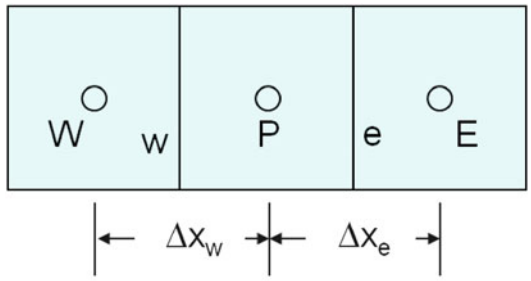


Fig. 2.23 Schematic of finite volume cells

Consider the diffusion equation in conservative form.

$$\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S=0$$

The equation is integrated over the control volume

$\int_w^e \frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S\ dx=0$ to become

$$\left(\Gamma\frac{d\phi}{dx}\right)_e-\left(\Gamma\frac{d\phi}{dx}\right)_w+\int_w^e S\ dx=0$$

A linear profile assumption is made between cell centroids for ϕ. Assume S varies linearly over the control volume

$$\frac{\Gamma_e(\phi_E-\phi_P)}{\Delta x_e}-\frac{\Gamma_w(\phi_P-\phi_W)}{\Delta x_w}+\overline{S}\ \Delta x=0$$

Collecting terms and casting into an algebraic equation yields:

$$a_E\phi_E+a_P\phi_P+a_W\phi_W=b$$

$$a_E=\frac{\Gamma_e}{\Delta x_e}, a_W=\frac{\Gamma_w}{\Delta x_w}, a_P=\frac{-2\Gamma_w}{\Delta x_w},\quad b=-\overline{S}\ \Delta x$$

Again, for the case in which $\Gamma=0.25,\quad 0<x<L,$ and $\quad S=4$

We will use $N=5$ grid points, $\Delta x=0.25L$ for $L=1$

$$a_{i-1}=1, a_{i+1}=1, a_i=-2,\quad b=1$$

$$\begin{bmatrix}-2&2&&&\\1&-2&1&&\\&1&-2&1&\\&&1&-2&1\\&&&&1\end{bmatrix}\begin{bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\\\phi_5\end{bmatrix}=\begin{bmatrix}-1\\-1\\-1\\-1\\0\end{bmatrix}$$

This system of linear equations, written in matrix form has a banded structure and can be solved using a number of linear algebraic solution strategies. For tridiagonal matrices, an efficient algorithm exists (Thomas algorithm) for solving the system [18]. The solution is:

$$\begin{bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\\\phi_5\end{bmatrix}=\begin{bmatrix}8\\7.5\\6\\3.5\\0\end{bmatrix}$$

The solution is exactly the same as the answer provided by the exact solution.

Finite Element Approximations

This discussion is on Galerkin finite element techniques which are a subclass of the method of weighted residuals (MWR) [15, 17]. The method explicitly identifies an approximate solution to the governing equation, for which the approximate solution is generated from a broader class of functions than the true solution resides in. There is an approximation, $\widetilde{\phi}$, to ϕ that produces a residual in the solution of the diffusion equation when the approximation is used.

$$\Gamma\frac{d^2\widetilde{\phi}}{dx^2}+S=R$$

In MWR, the weighted average of the residual is zero, rather than the residual. The weight function is called W(x).

$$\int_{domain} R(x)W(x)dx=0$$

The approximating function is assumed to be expandable in a set of linearly independent "basis functions" $N_j(x)$.

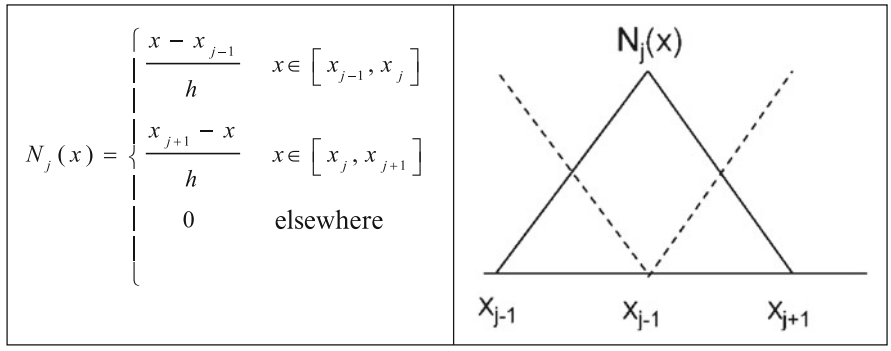


Fig. 2.24 Triangular basis functions and schematic of overlapping functions

$$\widetilde{\phi} = \sum \phi_j N_j(x)$$

In Galerkin FEM the weight function is specified to be the same as the basis function. $W_j(x) = N_j(x)$.

$$\int_{domain} R(x)W(x)dx = 0$$

Effectively, we are forcing the residual to be orthogonal to the basis functions. Properties of the basis function in terms of ease of evaluation, accuracy etc. suggest the use of polynomials.

An example of basis functions is shown below (Fig. 2.24).

The weighted residual becomes:

$$\int_a^b \left(\Gamma \frac{d^2\widetilde{\phi}}{dx^2} + S \right) N_j(x)dx = 0$$

While ϕ should be twice differentiable, the "weak form" allows a wider class of functions to be used. We integrate by parts. We get an equation in terms of integrals of basis functions and coefficients of the unknown function at nodal points.

$$A\phi_{j+1} + B\phi_j + C\phi_{j-1} + D_j = 0$$

Inverse Conduction Heat Transfer Approximations and Examples

Inverse analysis refers to a variety of approaches used to determine model parameters, boundary conditions, initial conditions, etc. The use of the term inverse relates to the idea that the analysis inverts the normal rules of causality in order to find an answer that best agrees with observations. In conduction problems, typical examples of use of inverse analysis are determining the heat flux on a slab given a limited number of interior temperature measurement or identifying material properties that allow a model to best match measured temperature data. There are many approaches to generating an inverse solution, and some of these techniques are essentially optimization methods. One might summarize optimization based analysis as the search for parameters that when used in a forward formulation of the problem of interest generate solutions whose deviation from experimental data is quantified., There are, however, specialized techniques that have been developed for inverse analysis that have not been specifically developed from an optimization perspective. These techniques generally require that the problem be formulated in an inverse sense. For problems formulated in an inverse sense, there ultimately becomes an ill posed mathematical problem. Ill posed problems [19] are those for which the solution is not unique or does not smoothly change relative to small changes in an input parameter. For such problems, there is a need to regularize the underlying system of equations.

If one considers the simple diffusion equation that has been presented in earlier sections and considers the need to specify the source in the medium given some number of temperature

measurements in the slab, the process to determine the source is an inverse analysis. An important fundamental consideration in inverse analysis is the notion that experimental noise exists in the measured data and that such noise corrupts the inversion process.

Given the ODE, we can formulate an integral equation for the temperatures using a delta function source [19, 20]. This is simply a Greens' function solution for the problem. In a forward analysis, we would simply integrate the source distribution in such a way as to generate a temperature at every point given the source distribution. The inversion requires an estimate of the source, given some measurements of temperature. Further, it is assumed that the temperature data are noisy. We see that for each temperature measurement, there is a corresponding quadrature rule that maps the integral equation to an algebraic equation. We see that this process results in a matrix equation in which the source description explicitly shows up as an unknown. Using typical linear algebra techniques, we find that the results for the source are quite noisy and not representative of the input source that was used to drive the forward solution. Note that for this exercise, synthetic measured data are sampled from the forward solution.

The Greens' function solution is constructed using a delta function source:

$$\Gamma\frac{d^2\phi}{dx^2} = -S'''(x) \text{ which for the Green's function is } \frac{\Gamma}{S''_o}\frac{d^2G}{dx^2} = -\delta(x-x_0)$$

Integrating the delta function yields:

$$\begin{aligned} G &= ax+b && x<x_0 \\ \Gamma\frac{dG}{dx}\Big|_{x_0-\varepsilon}^{x_0+\varepsilon} &= S''_o H_s(x-x_0) \\ G &= Ax+B && x>x_0 \end{aligned}$$

For a case in which ϕ is zero at −L and L, we get the Green's function solutions to be:

$$G(x,x_0) = \frac{-S''_o L}{2\Gamma}(1-x_0/L)(1+x/L) \qquad x<x_0$$

$$G(x,x_0) = \frac{-S''_o L}{2\Gamma}(1+x_0/L)(1-x/L) \qquad x>x_0$$

The final solution for the dependent variable is

$$\begin{aligned} \phi(x_0) = &\frac{L}{2\Gamma}(1-x_0/L)\int_{-L}^{x_0}(1+x/L)S''_o S'(x)dx \\ &+\frac{L}{2\Gamma}(1+x_0/L)\int_{x_0}^{L}(1-x/L)S''_o S'(x)dx \end{aligned}$$

For the case of a constant source we can show that we retrieve the exact solution. For an inverse formulation, we can imagine that several experimental values of ϕ are assumed to be available through measurements. The linear algebraic system is then:

$$\vec{\phi}(x_i) = \frac{L}{2\Gamma}(1-x_i/L)\sum_{j=1}^{1+(x_i+L)/\Delta x}(1+x_j/L)S'''(x_j)\Delta x + \frac{L}{2\Gamma}(1+x_i/L)\sum_{j=1+(x_i+L)/\Delta x}^{N}(1-x_j/L)S'''(x_j)\Delta x$$

$$x_j = -L + \frac{2L}{N-1}(j-1)$$

$$\vec{\phi}_i = A_{i,j}\vec{S}_j$$

We see that the m×n matrix operator $A_{i,j}$ is not square when there are a limited number (m) of measurements $\vec{\phi}_i$ that require many more evaluations (n) of the source term. To overcome the sensitivity to the data noise and the limited number of available measurements, regularization methods are used to generate smooth approximations to inversely formulated problems. Examples of regularization methods include truncated singular value decomposition (TSVD), Tikhonov regularization, and conjugate gradient based approaches. More details can be found in Hansen [19] and Press et al. [18], on details of these methods. When applied to the example problem, TSVD regularization allows us to easily invert for the source profile.

For the case in which $\Gamma = 0.25$, $-L < x < L$, and $S = 4$, we assume that measurements are made at x = (−.67 L, −0.33 L, 0.33 L, 0.67 L). We will use N = 7 grid points to evaluate the integral in the Greens function solution. This results in a grid spacing of $\Delta x = 0.33L$. For this problem we add random noise to the measured $\vec{\phi}_i$ values to simulate measurement error.

The matrix operations are then

$$\begin{bmatrix} 0 & 0.37 & 0.296 & 0.222 & 0.148 & 0.074 & 0 \\ 0 & 0.296 & 0.593 & 0.444 & 296 & 0.148 & 0 \\ 0 & 0.148 & 0.296 & 0.444 & 0.593 & 0.296 & 0 \\ 0 & 0.074 & 0.148 & 0.222 & 0.296 & 0.37 & 0 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_6 \\ S_7 \end{bmatrix} = \begin{bmatrix} 4.94 \\ 7.13 \\ 7.61 \\ 4.47 \end{bmatrix}$$

The singular value decomposition of the matrix A results in $A = U\Sigma V^T$. The matrix Σ is a diagonal matrix with the singular values ordered from largest to smallest as the diagonal elements. The number of singular values provides an indication of the number of independent equations in the system of equations. Also the ratio of the largest singular value to the smallest singular value, called the condition number of the matrix, reflects how singular the matrix is and how much amplification of error might be associated with small perturbations propagated through the inversion. For this problem, the singular value decomposition is:

$$\begin{pmatrix} -0.357 & 0.5 & -0.61 & -0.5 \\ -0.61 & 0.5 & 0.357 & 0.5 \\ -0.61 & -0.5 & 0.357 & -0.5 \\ -0.357 & -0.5 & -0.61 & 0.5 \end{pmatrix} \begin{pmatrix} 1.358 & 0 & 0 & 0 \\ 0 & 0.444 & 0 & 0 \\ 0 & 0 & 0.178 & 0 \\ 0 & 0 & 0 & 0.148 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.317 & 0.5 & -0.632 & -0.5 \\ -0.516 & 0.5 & 0.258 & 0.5 \\ -0.516 & 0 & 0.258 & 0 \\ -0.516 & -0.5 & 0.258 & -0.5 \\ -0.317 & -0.5 & -0.632 & 0.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The system of equations is inverted using the numerically accurate inversion (SVD pseudo-inverse) and also using a so-called truncated SVD (TSVD) pseudo-inverse. In the TSVD inversion, the smallest singular value (0.148) is not used in the inversion. The inversion is given by

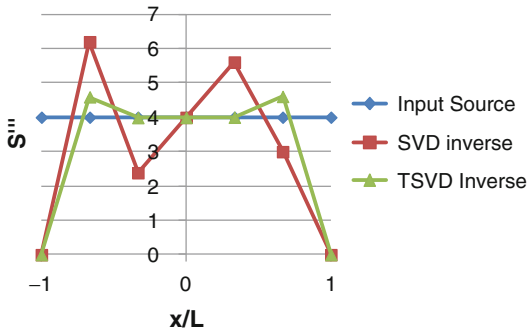


Fig. 2.25 Comparison of SVD inverse and TSVD inverse of noisy input data

$$\vec{S} = \sum_{i=1}^{k} \frac{\vec{u}_i \cdot \vec{\phi}}{\sigma_i} \vec{v}_i$$

In this expression the $\vec{u}_i$ and $\vec{v}_i$ are the column vectors of the U and V matrices in the SVD decomposition. The summation is over the total number of retained singular values. Figure 2.25 shows a comparison of the SVD and TSVD inversions of the noisy measurements. We see that there is considerably more variation and deviation from the exact source values for the SVD inversion relative to the TSVD inversion. The process of truncating the representation of the A matrix acts to smooth or regularize the solution.

This problem is only one example of the many types of techniques that can be applied in inverse heat transfer analysis. Many other types of inversion processes might be used for fire problems. There are many examples of time dependent inversion, as might be done to determine a surface heat flux when measuring internal temperatures that present many different solution approaches for inversion [21, 22]. An interesting analogy exists between the steady example using TSVD based inversion and some types of temporally evolving inverse problems. For linear time dependent problems, analytical solution forms exist for time dependent heating using the Duhamel integral formulation. A simple example from Beck et al. [21] shows the process for determining the heat flux when a single temperature measurement is made using a modification of the Duhamel integral. The Duhamel integral representation looks very much like the Greens function formulation that was detailed. In both cases, one is solving a Fredholm integral equation for some underlying system forcing.

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