Chapter 9
Transient HELS

Most vibrating structures are subject to impulsive or transient force excitations in practice. Oftentimes transient excitations are unknown and therefore the resultant acoustic field cannot be predicted. Even if the excitations are given, prediction of a transient acoustic field produced by an arbitrarily shaped source is very difficult. The scarcity in literature on predicting, not to mention reconstructing a transient acoustic field, is the testimony of how challenging this problem is.

One possibility of determining the transient acoustic field generated by an arbitrary object is to reconstruct the acoustic quantities in the frequency domain first, and take an inverse Fourier transform to retrieve the time-domain signals. Wang is the first to reconstruct a transient acoustic field in this manner [137]. Needless to say, numerical computations involved in this process are very intensive, if possible at all.

Another possibility is to utilize the so-called non-stationary spatial transformation of sound field (NS-STSF) [138]. NS-STSF is based on the time-domain holography (TDH) that processes the acoustic pressures measured by a planar array of microphones with the neighboring microphones separated by one-half the wavelength of a target acoustic wave. Basically, TDH produces "a sequence of snapshots of instantaneous pressure over the array area, the time separation between subsequent snapshots being equal to the sampling interval in A/D conversion. Similarly, the output of TDH is a time sequence of snapshots of a selected acoustic quantity in a calculation plane parallel to the measurement plane" [138]. Therefore, what one sees is a series of the acoustic pressure images in the frequency domain at fixed time instances over the recorded measurement time period. As pointed out in Sect. 5.3, NS-STSF is actually non-stationary acoustical holography because it gauges with respect to the acoustic frequency or the acoustic wavelength, not the spatial frequency or the spatial wavelength.

In this chapter we develop the transient NAH formulations by using the HELS method to visualize acoustic waves traveling in both space and time. Note that Hansen [101] has used a spherical wave expansion to predict time-domain acoustic radiation by scanning the acoustic pressure over a minimal spherical surface

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enclosing target sources. The major difference between Hansen's work and the present one is that the former is based on infinite series of the spherical Hankel functions and spherical harmonics, and expansion coefficients are determined using the orthogonal property of the spherical harmonics; while the latter utilizes a finite expansion and expansion coefficients are determined by matching the expansion solution to the measured data and the errors involved in this process are minimized through regularization. This infinite series is called Rayleigh series and Sect. 4.2 has discussed in detail the differences between the Rayleigh series and the HELS formulations. We have learned that the Rayleigh series is in general invalid for reconstructing the acoustic field on a corrugated or arbitrarily shaped surface based on the acoustic pressure specified on a measurement surface above the source surface.

Theoretically, the transient acoustic field generated by an arbitrary object can be calculated by using the Kirchhoff–Helmholtz integral formulation, provided that the normal component of the surface velocity is specified. For an arbitrarily shaped object, there is no analytic solution to this integral formulation. Hence numerical solutions are sought. A direct approach is to discretize the Kirchhoff–Helmholtz integral formulation in both spatial and temporal domains simultaneously. Such an approach is unrealistic in practice because the corresponding numerical computations are prohibitively expensive and time consuming. One alternative is to find numerical solutions to the radiated acoustic quantities in the frequency domain first, and take an inverse Fourier transform to obtain the time-domain signals [137]. Needless to say, numerical computations involved are intensive, if possible at all. The reality is that in most cases the normal surface velocity is not specified. Thus these numerical solutions strategies, no matter how plausible they are, cannot be utilized.

In Chap. 9 explicit formulations for reconstructing the transient acoustic field generated by an arbitrarily shaped 3D object in free space subject to an arbitrarily time-dependent excitation are derived using the Kirchhoff–Helmholtz integral theory. The reconstructed acoustic quantities are expressed in the frequency domain, and the corresponding time-domain quantities are obtained by taking an inverse Fourier transform, which is facilitated by using the residue theorem. The final formulation for reconstructing a transient acoustic quantity is expressed in a convolution integral of the acoustic pressure signal measured in the time-domain and a unit impulse response function.

It is emphasized that these explicit formulations are applicable to an arbitrary object with a uniformly distributed surface velocity. Input data to these explicit formulations are the acoustic pressure signals measured on a hologram surface in the near field of the target object.

For simplicity yet without loss of generality, background noise and interfering signals are assumed negligible as compared to the measured acoustic pressure signals. Reconstruction of the transient acoustic field is carried out by using BEM- [25–27, 91] and HELS [36, 37, 91, 102]-based NAH.

9.1 Transient Acoustic Radiation

To tackle transient acoustic radiation problems, let us first define the Fourier transform as

$$F\left(\vec{x};\omega\right) = \int_{-\infty}^{\infty} f\left(\vec{x};t\right)\mathrm{e}^{i\omega t}dt \text{ and } f\left(\vec{x};t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\left(\vec{x};\omega\right)\mathrm{e}^{-i\omega t}d\omega, \tag{9.1}$$

where $f\left(\vec{x};t\right)$ is a continuous and bounded function as $t \to \infty$, namely, $\int_{-\infty}^{\infty} \left|f\left(\vec{x};t\right)\right|dt < \infty$.

Assume that the transient acoustic field is generated by an arbitrary object subject to an arbitrarily time-dependent force excitation. Also, assume that the velocity is uniformly distributed on the surface of the object, which has a closed, smooth, and impermeable surface immersed in an inviscid, isotropic, and unbounded fluid medium. This object is initially stationary and excited by an unknown forcing function at $t = t_s$, causing the amplitude of the velocity to rise from 0 to V_s instantly in a specific direction $\vec{e}_{\mathrm{c}}$, where $\vec{e}_{\mathrm{c}}$ is a unit vector at the center of the object,

$$\vec{v}\left(\vec{x}_s;t\right) = V_s\vec{e}_cH(t - t_s), \tag{9.2}$$

where V_s is a constant and $H(t - t_s)$ represents the Heaviside step function defined as

$$H(t - t_s) = \begin{cases} 0, & t < t_s \\ 1/2, & t = t_s \\ 1, & t > t_s \end{cases}. \tag{9.3}$$

The derivative of the Heaviside step function is the Dirac delta function [139],

$$H'(t - t_s) = \delta(t - t_s). \tag{9.4}$$

The acoustic pressure $p\left(\vec{x};t\right)$ generated by this accelerated body in free space satisfies the homogeneous wave equation,

$$\nabla^2 p\left(\vec{x};t\right) - \frac{1}{c^2}\frac{\partial^2 p\left(\vec{x};t\right)}{\partial t^2} = 0, \tag{9.5}$$

subject to the Sommerfeld radiation condition,

$$\lim_{\left|\vec{x}\right|\to\infty}\left|\vec{x}\right|\left(\frac{\partial p}{\partial\left|\vec{x}\right|}+\frac{1}{c}\frac{\partial\hat{p}}{\partial t}\right)=0,\ \ \text{as}\ \ \left|\vec{x}\right|\to\infty. \tag{9.6}$$

In addition, $p\left(\vec{x};t\right)$ satisfies the causality condition,

$$p\left(\vec{x};t\right)\equiv 0,\ \ \text{for}\ \ t<t_s. \tag{9.7}$$

In other words, the field is perfectly silent before the body is suddenly excited at $t=t_s$.

To find an integral representation of the wave equation (9.5), we make use of the temporal free-space Green's function

$$g\left(\vec{x};t\middle|\vec{x}_s;t_s\right)=\frac{\delta(t-t_s-R/c)}{R}, \tag{9.8}$$

where $\delta(t-t_s-R/c)$ is the Dirac delta function, $(t-R/c)$ is known as the retarded time because it takes additional time R/c for the acoustic signal to travel from the source at $\vec{x}_s$ to a receiver at $\vec{x}$, here $R=\left|\vec{x}-\vec{x}_s\right|$ is the distance between the source and receiver in field space.

The temporal free-space Green's function satisfies the homogeneous wave equation,

$$\nabla^2 g\left(\vec{x};t\middle|\vec{x}_s;t_s\right)-\frac{1}{c^2}\frac{\partial^2 g\left(\vec{x};t\middle|\vec{x}_s;t_s\right)}{\partial t^2}=-4\pi\delta\left(\vec{x}-\vec{x}_s\right)\delta(t-t_s), \tag{9.9}$$

subject to the initial condition,

$$g\left(\vec{x};t\middle|\vec{x}_s;t_s\right)=\frac{\partial g\left(\vec{x};t\middle|\vec{x}_s;t_s\right)}{\partial t}\equiv 0,\ \ \text{for}\ \ t<t_s, \tag{9.10}$$

and the reciprocal relation,

$$g\left(\vec{x};t\middle|\vec{x}_s;t_s\right)=g\left(\vec{x}_s;-t_s\middle|\vec{x};-t\right). \tag{9.11}$$

Physically, Eq. (9.10) states that if the source is excited at t_s, no sound is detected before time t_s. Equation (9.11) is the reciprocity principle, which states that when the source location and emission time are interchanged with the receiver location and time, the effect remains unchanged.

Multiply Eq. (9.5) by $g\left(\vec{x};t\middle|\vec{x}_s;t_s\right)$ and Eq. (9.9) by $p\left(\vec{x};t\right)$ and utilize the chain rule to replace $\nabla\cdot(A\nabla B)$ by $A\nabla^2 B-\nabla A\cdot\nabla B$. Doing so yields

$$\nabla\cdot(g\nabla p)-\nabla p\cdot\nabla g-\frac{1}{c^2}\frac{\partial}{\partial t}\left(g\frac{\partial p}{\partial t}\right)+\frac{1}{c^2}\frac{\partial p}{\partial t}\frac{\partial g}{\partial t}=0, \tag{9.12}$$

$$\begin{aligned}\nabla\cdot(p\nabla g)-\nabla p\cdot\nabla g-\frac{1}{c^2}\frac{\partial}{\partial t}\left(p\frac{\partial g}{\partial t}\right)+\frac{1}{c^2}\frac{\partial p}{\partial t}\frac{\partial g}{\partial t}\\ =-4\pi p\delta\left(\vec{x}-\vec{x}_s\right)\delta(t-t_s),\end{aligned} \tag{9.13}$$

where the arguments of $p\left(\vec{x};t\right)$ and $g\left(\vec{x};t\middle|\vec{x}_s;t_s\right)$ in Eqs. (9.12) and (9.13) are suppressed for brevity. Subtracting Eq. (9.13) from (9.12), we obtain

$$\nabla\cdot(g\nabla p-p\nabla g)-\frac{1}{c^2}\frac{\partial}{\partial t}\left(g\frac{\partial p}{\partial t}-p\frac{\partial g}{\partial t}\right)=4\pi p\delta\left(\vec{x}-\vec{x}_s\right)\delta(t-t_s). \tag{9.14}$$

Integrating Eq. (9.15) over the entire time history and three-dimensional space leads to

$$\begin{aligned}\iiint\limits_{\Omega s}\int\limits_{-\infty}^{\infty}\nabla\cdot(g\nabla p-p\nabla g)-\frac{1}{c^2}\frac{\partial}{\partial t_s}\left(g\frac{\partial p}{\partial t_s}-p\frac{\partial g}{\partial t_s}\right)dt_s d\Omega_s\\ =4\pi\iiint\limits_{\Omega s}\int\limits_{-\infty}^{\infty}p\delta\left(\vec{x}-\vec{x}_s\right)\delta(t-t_s)dt_s d\Omega_s.\end{aligned} \tag{9.15}$$

The integrations on the right side of Eq. (9.15) are readily obtained by the sifting property of the Dirac delta function (6.4). Changing the order of volume and temporal integrations of the first term on the left side of Eq. (9.15) and using the Gauss theorem, we can replace the volume integral by a surface integral. As for the second term on the left side of Eq. (9.15), the temporal integration and time derivative cancel each other. Therefore, we obtain

$$\begin{aligned}4\pi p\left(\vec{x};t\right)=\int\limits_{-\infty}^{\infty}\iint\limits_{S}\left(p_s\frac{\partial g}{\partial \mathrm{n}_s}-g\frac{\partial p_s}{\partial \mathrm{n}_s}\right)dS dt_s\\ -\frac{1}{c^2}\iiint\limits_{\Omega s}\left(g\frac{\partial p_s}{\partial t_s}-p\frac{\partial g}{\partial t_s}\right)\Bigg|_{-\infty}^{\infty}d\Omega_s,\end{aligned} \tag{9.16}$$

where a subscript s in Eq. (9.16) indicates that the quantities are evaluated at a surface point. The second term on the right side of Eq. (9.16) is identically zero because of the property of the Dirac delta function (6.4). Changing the order of temporal and surface integrations in the first term on the left side of Eq. (9.16) once again then leads to

$$4\pi p\left(\vec{x};t\right)=\iint\limits_{S}\int\limits_{-\infty}^{\infty}\left[p\frac{\partial}{\partial \mathrm{n}_s}\frac{\delta(t-t_s-R/c)}{R}-\frac{\partial p_s}{\partial \mathrm{n}_s}\frac{\delta(t-t_s-R/c)}{R}\right]dt_s dS. \quad (9.17)$$

Using the chain rule and property of the Dirac delta function, we can rewrite the first term on the right side of Eq. (9.17) as

$$\begin{aligned}
&\iint\limits_{S}\int\limits_{-\infty}^{\infty}p\frac{\partial}{\partial \mathrm{n}_s}\frac{\delta(t-t_s-R/c)}{R}dt_s dS\\
&=-\iint\limits_{S}\int\limits_{-\infty}^{\infty}\frac{p}{R^2}\frac{\partial R}{\partial \mathrm{n}_s}\delta(t-t_s-R/c)dt_s dS-\iint\limits_{S}\int\limits_{-\infty}^{\infty}\frac{p}{cR}\frac{\partial R}{\partial \mathrm{n}_s}\delta'(t-t_s-R/c)dt_s dS\\
&=-\iint\limits_{S}\frac{p}{R^2}\frac{\partial R}{\partial \mathrm{n}_s}\bigg|_{t_s=t-R/c}dS+\iint\limits_{S}\frac{1}{cR}\frac{\partial R}{\partial \mathrm{n}_s}\int\limits_{-\infty}^{\infty}\frac{\partial}{\partial t_s}[p\delta(t-t_s-R/c)]dt_s dS\\
&\quad-\iint\limits_{S}\frac{1}{cR}\frac{\partial R}{\partial \mathrm{n}_s}\int\limits_{-\infty}^{\infty}\frac{\partial p}{\partial t_s}\delta(t-t_s-R/c)dt_s dS=-\iint\limits_{S}\frac{p}{R^2}\frac{\partial R}{\partial \mathrm{n}_s}\bigg|_{t_s=t-R/c}dS\\
&\quad+\iint\limits_{S}\frac{1}{cR}\frac{\partial R}{\partial \mathrm{n}_s}[p\delta(t-t_s-R/c)]|_{-\infty}^{\infty}dS-\iint\limits_{S}\frac{1}{cR}\frac{\partial R}{\partial \mathrm{n}_s}\frac{\partial p}{\partial t_s}\bigg|_{t_s=t-R/c}dS\\
&=-\iint\limits_{S}\left[\frac{1}{R}\frac{\partial R}{\partial \mathrm{n}_s}\left(\frac{1}{R}+\frac{1}{c}\frac{\partial}{\partial t_s}\right)p\right]\bigg|_{t_s=t-R/c}dS.
\end{aligned} \quad (9.18)$$

Substituting Eqs. (9.18) into (9.17), we obtain

$$\begin{aligned}
p\left(\vec{x};t\right)=&-\frac{1}{4\pi}\iint\limits_{S}\left[\frac{1}{R}\frac{\partial R}{\partial \mathrm{n}_s}\left(\frac{1}{R}+\frac{1}{c}\frac{\partial}{\partial t_s}\right)p\left(\vec{x}_s;t_s\right)\right]\bigg|_{t_s=t-R/c}dS\\
&-\frac{1}{4\pi}\iint\limits_{S}\left[\frac{1}{R}\frac{\partial p\left(\vec{x}_s;t_s\right)}{\partial \mathrm{n}_s}\right]\bigg|_{t_s=t-R/c}dS.
\end{aligned} \quad (9.19)$$

Equation (9.19) is known as the Kirchhoff–Helmholtz integral formulation for predicting the transient acoustic pressure in free space. The surface acoustic pressure $p\left(\vec{x}_s;t_s\right)$ on the right side of Eq. (9.19) is related to its normal derivative

$\partial p\left(\vec{x}_s; t_s\right)/\partial \mathrm{n}\left(\vec{x}_s\right)$ via the surface Kirchhoff–Helmholtz integral equation obtained by taking the limit as the field point approaches the surface $\vec{x} \to \vec{x}_s$. The processes of taking this limit are the same as those described in Sect. 6.3 and the result is

$$p\left(\vec{x}_s; t_s\right) = -\frac{1}{2\pi}\iint_{S'} \left[\frac{1}{R_s}\frac{\partial R_s}{\partial \mathrm{n}_{s'}}\left(\frac{1}{R_s} + \frac{1}{c}\frac{\partial}{\partial t_{s'}}\right)p\left(\vec{x}_{s'}; t_{s'}\right)\right]\Bigg|_{t_{s'}=t_s-R_s/c} dS' - \frac{1}{2\pi}\iint_{S'}\frac{1}{R_s}\frac{\partial p\left(\vec{x}_{s'}; t_{s'}\right)}{\partial \mathrm{n}_{s'}}\Bigg|_{t_{s'}=t_s-R_s/c} dS'. \tag{9.20}$$

The normal derivative of the surface acoustic pressure in the second term on the right side of Eq. (9.20) can be rewritten by using the Euler's equation, the initial condition (9.2) and the derivative of the Heaviside step function (9.4) as

$$\frac{\partial p\left(\vec{x}_{s'}; t_{s'}\right)}{\partial \mathrm{n}_{s'}} = -\rho_0 \frac{\partial v_\mathrm{n}\left(\vec{x}_{s'}; t_{s'}\right)}{\partial t_{s'}} = -\rho_0 V_s\left(\vec{\mathrm{n}}_{s'} \cdot \vec{e}_c\right)\delta(t_s - t_{s'}). \tag{9.21}$$

Substituting Eq. (9.21) into the second term on the right side of Eq. (9.20) and taking the Fourier transform, we obtain

$$P\left(\vec{x}_s; \omega\right) = -\frac{1}{2\pi}\iint_{S'} \frac{\partial R_s}{\partial \mathrm{n}_{s'}}\left(\frac{1 - ikR_s}{R_s^2}\right)P\left(\vec{x}_{s'}; \omega\right)dS' + \frac{\rho_0 V_s}{2\pi}\iint_{S'}\left(\frac{\vec{\mathrm{n}}_{s'} \cdot \vec{e}_z}{R_s}\right)e^{ikR_s}dS'. \tag{9.22}$$

Equation (9.22) is the surface Helmholtz integral equation for solving the surface acoustic pressure, given the initial condition (9.2). Note that the surface velocity V_s is independent of the spatial variable. This often happens in practice when an object is hit by a force and starts to move impulsively. This sudden motion may result in an impulsive-like sound. The Fourier transform of the resultant acoustic pressure is expressible as

$$P\left(\vec{x}_s; \omega\right) = \xi\left(\vec{x}_s; \omega\right)V_s, \tag{9.23}$$

where $\xi\left(\vec{x}_s; \omega\right)$ may be obtained by substituting Eq. (9.23) into (9.22),

$$\xi\left(\vec{x}_s; \omega\right) = \rho_0\iint_{S'}\left(\frac{\vec{\mathrm{n}}_{s'} \cdot \vec{e}_c}{R_s}\right)\mathrm{e}^{ikR_s}dS'/\varsigma(\omega), \tag{9.24a}$$

where

$$\varsigma(\omega) = 2\pi + \iint_{S'} \left(\frac{\vec{n}_{s'} \cdot \vec{e}_R}{R_s^3} \right) (1 - ikR_s) e^{ikR_s} dS'. \tag{9.24b}$$

Once the surface acoustic pressure is specified, the acoustic pressure anywhere in free space is completely determined by the Fourier transformed version of Eq. (9.19),

$$P\left(\vec{x};\omega\right) = \left[\frac{\eta\left(\vec{x};\omega\right)}{\varsigma(\omega)} \right] V_s, \tag{9.25}$$

where

$$\eta\left(\vec{x};\omega\right) = \frac{\rho_0}{4\pi} \iint_S \left[\left(\vec{n}_s \cdot \vec{e}_c\right) R^2 \varsigma(\omega) - \left(\vec{n}_s \cdot \vec{e}_R\right)(1 - ikR) \iint_{S'} \left(\frac{\vec{n}_{s'} \cdot \vec{e}_z}{R_s} \right) e^{ikR_s} dS' \right] \frac{e^{ikR}}{R^3} dS. \tag{9.26}$$

Equation (9.25) offers the closed-form solution for the acoustic pressure in the frequency domain generated by an arbitrary object subject to the initial condition (9.2) in free space. The temporal acoustic pressure can be obtained by taking an inverse Fourier transform of Eq. (9.25),

$$p\left(\vec{x};t\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P\left(\vec{x};\omega\right) e^{-i\omega t} d\omega\,. \tag{9.27}$$

For an early portion of the transient event, Eq. (9.27) can be evaluated asymptotically by taking the limit as $\omega \to \infty$ [140],

$$\lim_{t\to 0} p\left(\vec{x};t\right) = -i \lim_{\omega\to\infty} P\left(\vec{x};\omega\right). \tag{9.28a}$$

On the other hand, for a latter portion of the transient event, the inverse Fourier transform (9.27) can be evaluated asymptotically by taking the limit as $\omega \to 0$,

$$\lim_{t\to\infty} p\left(\vec{x};t\right) = -i \lim_{\omega\to 0} P\left(\vec{x};\omega\right). \tag{9.28b}$$

These two extreme cases indicate that the early portion of the transient event is governed by the high-frequency content, whereas the late portion of the transient event is controlled by the low-frequency contents of the spectrum. However, these asymptotic solutions are undesirable as far as the tractability of any transient event is concerned. Alternatively, one can utilize the residue theorem to evaluate the inverse Fourier transform (9.27) as discussed below.

9.2 Residue Theorem

Mathematically, the evaluation of an infinite integral such as the one given by Eq. (9.27) can be facilitated by a contour integral. Namely, one can replace the infinite line integral along the real axis by a finite one from $-\mathbf{R}$ to $+\mathbf{R}$ in the complex frequency domain, and close the integration path by a semicircle in the lower half plane in the clockwise direction. The reason for choosing the lower half plane in the complex frequency domain is to ensure that the integration remains finite. The radius $\mathbf{R}$ is then extended to infinity. The integration along the semicircle is finite because by definition the infinite integral satisfies the boundedness condition [see Eq. (9.1)] [141],

$$p\left(\vec{x};t\right) = \left[\frac{1}{2\pi}\oint_{C} \frac{\eta\left(\vec{x};\omega\right)}{\varsigma(\omega)} \mathrm{e}^{-i\omega t} d\omega\right] V_s. \tag{9.29}$$

Equation (9.29) shows that the temporal acoustic pressure is expressible as V_s multiplied by a contour integral of $\eta\left(\vec{x};\omega\right)/\varsigma(\omega)$ with $\eta\left(\vec{x};\omega\right)$ and $\varsigma(\omega)$ being given by Eqs. (9.26) and (9.24b), respectively. The contour integral in Eq. (9.29) can be evaluated by the residue theory.

Figure 9.1 shows this contour integration path $\boldsymbol{C}$. If there are singularities on the real axis, they must be excluded by drawing a small semicircle of radius $r = \varepsilon$. For example, suppose that the integrand has singularities at $\pm x_0$ on the real axis. Then the contour integration path can be broken up into segments from $-\mathbf{R}$ to $(-x_0 - \varepsilon)$, a semicircle from $(-x_0 - \varepsilon)$ to $(-x_0 + \varepsilon)$, a straight line from $(-x_0 + \varepsilon)$ to $(x_0 - \varepsilon)$, another semicircle from $(x_0 - \varepsilon)$ to $(x_0 + \varepsilon)$, another straight line from $(x_0 + \varepsilon)$ to $+\mathbf{R}$, and a semicircle from $+\mathbf{R}$ to $-\mathbf{R}$. The integration along the small semicircle is with respect to $\varepsilon d\theta$, where θ varies from π to 0, which vanishes in the limit as $\varepsilon \to 0$. The integration along the large semicircle is with respect to $\mathbf{R}d\theta$, where θ varies from π to 0, which is identically zero because the boundedness condition is satisfied as $\mathbf{R} \to \infty$.

Therefore, the infinite line integral in Eq. (9.26) is equivalent to the contour integral in Eq. (9.29), which reduces to the line integral from $-\mathbf{R}$ to $+\mathbf{R}$ with $\mathbf{R} \to \infty$. Meanwhile, this contour integral is equal to the sum of residues enclosed by the contour $\boldsymbol{C}$. Consequently, we obtain

$$p\left(\vec{x};t\right) = \hbar\left(\vec{x}\,\middle|\,\vec{x}_s;t\right) V_s, \tag{9.30}$$

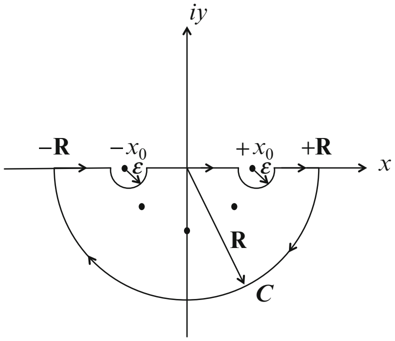

Fig. 9.1 Schematic of a contour integral in the complex frequency domain

where $\hbar\left(\vec{x}\middle|\vec{x}_s;t\right)$ is the sum of residues enclosed by the contour $\boldsymbol{C}$,

$$\hbar\left(\vec{x}\middle|\vec{x}_s;t\right) = -i\sum_q \frac{\eta\left(\vec{x};\omega_q\right)}{\varsigma'\left(\omega_q\right)} \mathrm{e}^{-i\omega_q(t-t_s-R/c)} H\left(t-t_s-\frac{R}{c}\right), \tag{9.31}$$

where the Heaviside step function appears as the field acoustic pressure is felt only after the source is suddenly excited at $t=t_s$ plus the retarded time r/c, which is needed for the impulsive acoustic signal to travel from the source to any receiver. Also, we have adopted a minus sign because the contour is completed by a semicircle in the lower half plane. The symbol ς ' (ω_q) represents the derivative of $\varsigma(\omega_q)$ with respect to ω, and ω_q is the qth singularity of the ratio $\eta\left(\vec{x};\omega_q\right)/\varsigma(\omega)$, which can be obtained by setting $\varsigma(\omega_q)=0$.

Example 9.1 Consider the case of a sudden-expansion sphere of radius $r=a$ subject to the initial condition (9.2) with $\vec{n}\left(\vec{x}_s\right)\cdot\vec{e}_c=1$. Suppose that this sudden expansion occurs at $t=t_s$ $(=a/c)$. The surface acoustic pressure in the frequency domain can be obtained by using Eq. (9.23) with $\xi\left(\vec{x}_s;\omega\right)$ given by Eq. (9.24a), which for a spherical surface is given by [142]

$$\xi\left(\vec{x}_s;\omega\right) = \frac{a}{ika-1}.$$

Therefore, Eq. (9.23) gives the surface acoustic pressure in the frequency domain,

$$P\left(\vec{x}_s;\omega\right) = \frac{\rho_0 V_s a \mathrm{e}^{ika}}{1-ika}.$$

Similarly, substituting $\xi\left(\vec{x}_s;\omega\right)$ into Eq. (9.25) yields the field acoustic pressure as

$$P\left(\vec{x};\omega\right) = \frac{\rho_0 V_s}{(1-ika)}\left(\frac{a^2}{r}\right)\mathrm{e}^{ikr}.$$

The temporal acoustic pressure anywhere in free space is given by Eq. (9.27), which can be replaced by the residue theorem through Eq. (9.30), where

$$\eta\left(\vec{x};\omega_q\right) = \rho_0\left(\frac{a^2}{r}\right)\mathrm{e}^{i\omega_q(r-a)/c},$$
$$\varsigma\left(\omega_q\right) = 1 - ik_q a, \quad \text{and} \quad \varsigma'\left(\omega_q\right) = -i\frac{a}{c},$$

where ω_q is the qth root of the characteristic equation, $\varsigma\left(\omega_q\right) = 0$. In this case there is only one root, $\omega_1 = -ic/a$. Accordingly, the residue theorem leads to

$$p\left(\vec{x};t\right) = \rho_0 c V_s\left(\frac{a}{r}\right)\mathrm{e}^{-(ct-r)/a}H\left(t-\frac{r}{c}\right),$$

which agrees perfectly with the analytic result. This transient sound field is typically seen in an explosion, where the amplitude of the acoustic pressure decays exponentially in all direction.

Example 9.2 Next, consider the case of a sphere of radius $r = a$ that is impulsively accelerated in the z-axis direction such that the normal surface velocity is given by

$$v_\mathrm{n}\left(\vec{x}_s;t\right) = V_s\left(\vec{\mathrm{n}}_s\cdot\vec{e}_z\right)H(t-t_s),$$

where $t_s = a/c$. Following the same procedures as those in Example 9.1, we obtain

$$\xi\left(\vec{x}_s;\omega\right) = a(ika-1)/\left[2-(ka)^2-i2ka\right].$$

The surface and field acoustic pressures in the frequency domain are given, respectively, by

$$P\left(\vec{x}_s;\omega\right) = \frac{\rho_0 V_s a(1-ika)\cos\theta}{2-(ka)^2-i2ka}\mathrm{e}^{ika} \quad \text{and}$$
$$P\left(\vec{x};\omega\right) = \frac{\rho_0 V_s a(1-ikr)\cos\theta}{2-(ka)^2-i2ka}\left(\frac{a}{r}\right)^2\mathrm{e}^{ikr}.$$

Setting the denominator in the above to zero gives two roots in the lower half complex frequency domain, $\omega_1 = (1 - i)c/a$ and $\omega_2 = (-1 - i)c/a$. Accordingly, the temporal acoustic pressure at a field point is found to be

$$p\left(\vec{x};t\right) = \rho_0 c V_s \cos\theta(a/r)\mathrm{e}^{-(ct-r)/a}H(t - r/c)$$
$$\{\cos[(ct-r)/a] - (1 - a/r)\sin[(ct-r)/a]\},$$

which once again agrees perfectly with the analytic result. This transient sound field is typically seen during an impact where the acoustic pressure is highly directional yet decays exponentially.

9.3 Extension to Arbitrary Time-Dependent Excitations

The formulations derived in Sect. 9.2 for predicting the transient acoustic pressure field can be extended to arbitrary time-dependent excitations acting on rigid bodies in free space. To the end, we consider a rigid body subject to a temporal rectangle function, which consists of two unit step functions in the opposite signs.

$$\Delta\vec{v}\left(\vec{x}_s;t\right) = V_s\vec{e}_c[H(t - t_s) - H(t - t_s - \Delta t)], \tag{9.32}$$

where Δt is the gap between two unit step functions.

Following the same procedures as those described in Sect. 9.2, we derive the resultant surface acoustic pressure in the frequency domain as

$$\Delta P\left(\vec{x};\omega\right) = \left[\frac{(1 - \mathrm{e}^{i\omega\Delta t})\eta\left(\vec{x};\omega\right)}{\varsigma(\omega)}\right]V_s. \tag{9.33}$$

The corresponding temporal acoustic pressure anywhere in free space may be obtained by taking the inverse Fourier transform of Eq. (9.33), which can be evaluated via the residue theorem and be expressible as

$$\Delta p\left(\vec{x};t\right) = \left[\hbar\left(\vec{x}\middle|\vec{x}_s;t\right) - \hbar\left(\vec{x}\middle|\vec{x}_s;t - \Delta t\right)\right]V_s, \tag{9.34}$$

where $\hbar\left(\vec{x}\middle|\vec{x}_s;t\right)$ is the same as that given by Eq. (9.31) and $\hbar\left(\vec{x}\middle|\vec{x}_s;t - \Delta t\right)$ is defined as

$$\hbar\left(\vec{x}\middle|\vec{x}_s;t-\Delta t\right) = -i\sum_q \frac{\eta\left(\vec{x};\omega_q\right)}{\varsigma'\left(\omega_q\right)} \mathrm{e}^{-i\omega_q(t-\Delta t-t_s-R/c)} H\left(t-\Delta t-t_s-\frac{R}{c}\right). \tag{9.35}$$

Consequently, the transient acoustic pressure radiated from an object subject to a velocity rectangle impulse of constant amplitude is the superposition of two step response functions of the same amplitudes but opposite signs with a separation of Δt in time. Meanwhile, any continuous and arbitrarily time-dependent excitation may be approximated as a sum of rectangle impulses of constant amplitudes with a small duration Δt. Therefore, for an object subject to a continuous and arbitrarily time-dependent velocity excitation, we can write the field acoustic pressure as a sum of individual acoustic pressure pulses,

$$p\left(\vec{x};t\right) = \sum_\ell \left[\hbar\left(\vec{x}\middle|\vec{x}_s;t_\ell\right) - \hbar\left(\vec{x}\middle|\vec{x}_s;t_\ell-\Delta t\right)\right] V_s. \tag{9.36}$$

Equation (9.36) is now ready to be extended to a general, continuous, and time-dependent excitation. For this purpose, we rewrite Eq. (9.34) in the following manner,

$$\Delta p\left(\vec{x};t\right) = \left\{\left[\frac{\hbar\left(\vec{x}\middle|\vec{x}_s;t\right) - \hbar\left(\vec{x}\middle|\vec{x}_s;t-\Delta t\right)}{\Delta t}\right]\Delta t\right\} V_s. \tag{9.37}$$

Equation (9.37) represents an acoustic pressure pulse at a field point $\vec{x}$ and time t due to a velocity rectangle pulse at time t_s. As $\Delta t \to 0$, the square bracket term of Eq. (9.37) becomes an impulse response function. The transient field acoustic pressure at $\vec{x}$ due to all the velocity impulses prior to time t can be expressed as the Duhamel integral [143],

$$p\left(\vec{x};t\right) = \int_0^t h\left(\vec{x}\middle|\vec{x}_s;t-\tau\right) V_s d\tau, \tag{9.38}$$

where $h\left(\vec{x}\middle|\vec{x}_s;t-\tau\right)$ is known as the impulse response function since it is the response to a velocity impulse at time τ, and can be obtained by using the residue theorem as

$$h\left(\vec{x}\middle|\vec{x}_s;t-\tau\right) = -i\sum_q \frac{\eta\left(\vec{x};\omega_q\right)}{\varsigma'\left(\omega_q\right)} \mathrm{e}^{-i\omega_q\tau}. \tag{9.39}$$

Equation (9.38) states that the transient acoustic pressure a field point $\vec{x}$ and time t can be expressed as the convolution integral of the impulse response function and the time history of the surface velocity of an object. Sometimes this convolution integral is abbreviated as

$$p\left(\vec{x};t\right) = h\left(\vec{x};t\right) \times v_{\mathrm{n}}\left(\vec{x}_s;t\right), \tag{9.40}$$

where $v_{\mathrm{n}}\left(\vec{x}_s;t\right) = V_s\left(\vec{\mathrm{n}}_s \cdot \vec{e}_c\right)H(t - t_s)$ represents the normal surface velocity of the source and the symbol * indicates the convolution integral given in Eq. (9.38).

9.4 Transient NAH Formulations

The transient formulations developed in Sects. 9.1–9.3 have laid a solid foundation for performing transient NAH. Two types of implementation schemes, namely, the Helmholtz integral formulation and HELS method-based NAH are considered in this section.

9.4.1 Reconstruction Through BEM-Based NAH

Suppose that the input data consist of the acoustic pressure signals measured at $\vec{x}_m^{\Gamma}$ on the hologram surface, $m = 1, 2, \ldots, M$, which is positioned around the source surface in the near field. Taking the Fourier transform of the measured acoustic pressure signals and using Eq. (9.25) lead to the following general, discretized BEM-based formulations:

$$\left\{P\left(\vec{x}_m^{\Gamma};\omega\right)\right\}_{M\times 1} = \left\{T_{pv}\left(\vec{x}_m^{\Gamma}\middle|\vec{x}_s;\omega\right)\right\}_{M\times 1} V_s(\omega), \tag{9.41}$$

where $\left\{P\left(\vec{x}_m^{\Gamma};\omega\right)\right\}_{M\times 1}$ is the acoustic pressure measured on the hologram surface in the frequency domain and $\left\{T_{pv}\left(\vec{x}_m^{\Gamma}\middle|\vec{x}_s;\omega\right)\right\}_{M\times 1}$ is the transfer function correlating the measured acoustic pressure at $\vec{x}_m^{\Gamma}$ to the velocity magnitude on the source surface $\vec{x}_s$, whose elements are defined as

$$T_{pv,m}\left(\vec{x}_m^{\Gamma}\middle|\vec{x}_s;\omega\right) = \frac{\eta\left(\vec{x}_m^{\Gamma};\omega\right)}{\varsigma(\omega)}. \tag{9.42}$$

The symbol $V_s(\omega)$ on the right side of Eq. (9.41) indicates the magnitude of the surface velocity, which is frequency dependent but spatially invariant on the source surface, and $R_m = \left| \vec{x}_m^{\Gamma} - \vec{x}_s \right|$. The value of $V_s(\omega)$ may be obtained by taking a pseudo inversion of Eq. (9.41),

$$V_s(\omega) = \left\{ T_{pv}\left(\vec{x}_s \middle| \vec{x}_m^{\Gamma}; \omega \right) \right\}_{1\times M}^{\dagger} \left\{ P\left(\vec{x}_m^{\Gamma}; \omega \right) \right\}_{M\times 1}, \tag{9.43}$$

where

$$\left\{ T_{pv}\left(\vec{x}_s \middle| \vec{x}_m^{\Gamma}; \omega \right) \right\}_{1\times M}^{\dagger} = \left[\left\{ T_{pv}\left(\vec{x}_s \middle| \vec{x}_m^{\Gamma}; \omega \right) \right\}_{1\times M}^{H} \left\{ T_{pv}\left(\vec{x}_m^{\Gamma} \middle| \vec{x}_s; \omega \right) \right\}_{M\times 1} \right]^{-1} \left\{ T_{pv}\left(\vec{x}_s \middle| \vec{x}_m^{\Gamma}; \omega \right) \right\}_{1\times M}^{H}. \tag{9.44}$$

In practice Eq. (9.43) must be regularized because the errors involved in the input data may make the pseudo-inversion matrix singular and cause solutions to diverge without a bound. There are many choices for conduct regularization, ranging from the simplest TSVD, L-Curve, to MTR [46, 49, 50], which have been discussed extensively in the past and are omitted here for brevity.

Once the surface velocity is reconstructed, the surface acoustic pressure can be obtained by substituting Eq. (9.43) into Eq. (9.23), and the result is

$$P\left(\vec{x}_s; \omega \right) = \xi\left(\vec{x}_s; \omega \right) \left\{ T_{pv}\left(\vec{x}_s \middle| \vec{x}_m^{\Gamma}; \omega \right) \right\}_{1\times M}^{\dagger} \left\{ P\left(\vec{x}_m^{\Gamma}; \omega \right) \right\}_{M\times 1}. \tag{9.45}$$

Meanwhile, the reconstructed acoustic pressure at any field point $\vec{x}$ can be determined by substituting Eq. (9.43) into Eq. (9.25), which is expressible as

$$P\left(\vec{x}; \omega \right) = \frac{\eta\left(\vec{x}; \omega \right)}{\varsigma(\omega)} \left\{ T_{pv}\left(\vec{x}_s \middle| \vec{x}_m^{\Gamma}; \omega \right) \right\}_{1\times M}^{\dagger} \left\{ P\left(\vec{x}_m^{\Gamma}; \omega \right) \right\}_{M\times 1}. \tag{9.46}$$

The normal component of the particle velocity at $\vec{x}$ can be obtained by taking the normal derivative of Eq. (9.46),

$$V_{\mathrm{n}}\left(\vec{x}; \omega \right) = -i \frac{1}{\rho_0 \omega \varsigma(\omega)} \frac{\partial \eta\left(\vec{x}; \omega \right)}{\partial \mathrm{n}} \left\{ T_{pv}\left(\vec{x}_s \middle| \vec{x}_m^{\Gamma}; \omega \right) \right\}_{1\times M}^{\dagger} \left\{ P\left(\vec{x}_m^{\Gamma}; \omega \right) \right\}_{M\times 1}. \tag{9.47}$$

9.4.2 Reconstruction Through HELS-Based NAH

Alternatively, the expansion theory can be used to reconstruct the acoustic field. One such approach is the so-called HELS method that employs the spherical Hankel functions and spherical harmonics as the basis functions to describe the acoustic quantities [37, 38].

Suppose that the acoustic pressure is specified on a hologram surface Γ in the same way as that depicted in the preceding section. The acoustic pressure and normal component of the particle velocity anywhere in the field, including the source surface, can be reconstructed by using the HELS formulations and the results are

$$P\left(\vec{x};\omega\right) = \left\{G_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};\omega\right)\right\}_{1\times M}\left\{P\left(\vec{x}_m^{\Gamma};\omega\right)\right\}_{M\times 1}, \tag{9.48}$$

$$V_{\mathrm{n}}\left(\vec{x};\omega\right) = \left\{G_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};\omega\right)\right\}_{1\times M}\left\{P\left(\vec{x}_m^{\Gamma};\omega\right)\right\}_{M\times 1}, \tag{9.49}$$

where $\left\{G_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};\omega\right)\right\}_{1\times M}^{\mathrm{T}}$ and $\left\{G_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};\omega\right)\right\}_{1\times M}^{\mathrm{T}}$ are the transfer functions that correlate $P\left(\vec{x};\omega\right)$ and $V_{\mathrm{n}}\left(\vec{x};\omega\right)$ anywhere in the field to $P\left(\vec{x}_m^{\Gamma};\omega\right)$ on the hologram surface Γ, respectively,

$$\left\{G_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};\omega\right)\right\}_{1\times M}^{\mathrm{T}} = \left\{\Psi\left(\vec{x};\omega\right)\right\}_{1\times J}^{\mathrm{T}}\left(\left[\Psi\left(\vec{x}_m^{\Gamma};\omega\right)\right]_{J\times M}^{H}\left[\Psi\left(\vec{x}_m^{\Gamma};\omega\right)\right]_{M\times J}\right)^{-1} \left[\Psi\left(\vec{x}_m^{\Gamma};\omega\right)\right]_{J\times M}^{H}, \tag{9.50}$$

$$\left\{G_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};\omega\right)\right\}_{1\times M}^{\mathrm{T}} = \frac{1}{i\omega\rho_0}\left\{\frac{\partial\Psi\left(\vec{x};\omega\right)}{\partial \mathrm{n}}\right\}_{1\times J}^{\mathrm{T}}\left(\left[\Psi\left(\vec{x}_m^{\Gamma};\omega\right)\right]_{J\times M}^{H}\left[\Psi\left(\vec{x}_m^{\Gamma};\omega\right)\right]_{M\times J}\right)^{-1}\left[\Psi\left(\vec{x}_m^{\Gamma};\omega\right)\right]_{J\times M}^{H}, \tag{9.51}$$

where the elements of the matrix $\left[\Psi\left(\vec{x}_m^{\Gamma};\omega\right)\right]_{J\times M}$ consist of the particular solution to the Helmholtz equation, which are expressible in the spherical coordinates as

$$\Psi_j(r,\theta,\phi;\omega) \equiv \Psi_{nl}(r,\theta,\phi;\omega) = h_n^{(1)}(kr)Y_n^l(\theta,\phi), \tag{9.52}$$

where $h_n^{(1)}(kr)$ and $Y_n^l(\theta,\phi)$ are the spherical Hankel functions of the first kind and the spherical harmonics, respectively, and the indices j, n, and l in Eq. (9.52) are related through $j = n^2 + n + l + 1$, where the order of expansion in the radial function n starts from 0 to N and l ranges from $-n$ to $+n$.

9.4.3 Transient NAH Formulations

Once the acoustic quantities in the frequency domain are determined by utilizing either the BEM- or HELS-based NAH formulations, the corresponding time-domain signals are obtained by taking an inverse Fourier transform of either Eqs. (9.46) and (9.47) or Eqs. (9.48) and (9.49). These equations may be evaluated by using the residue theorem and expressed as a convolution integral (9.40), except that input data consist of the measured acoustic pressure signal $p\left(\vec{x}_m^{\Gamma}; t\right)$ rather than velocity signal on the source surface,

$$p\left(\vec{x}; t\right) = g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right) \times p\left(\vec{x}_m^{\Gamma}; t\right), \tag{9.53}$$

$$v_{\mathrm{n}}\left(\vec{x}; t\right) = g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right) \times p\left(\vec{x}_m^{\Gamma}; t\right), \tag{9.54}$$

where the temporal kernels $g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right)$ and $g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right)$ are expressible, respectively, as

$$g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t-\tau\right) = -i\sum_q \frac{\eta_{pp}\left(\vec{x}; \omega_q^{pp}\right)}{\varsigma'_{pp}(\omega_q^{pp})} \mathrm{e}^{-i\omega_q^{pp}\tau}, \tag{9.55}$$

$$g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t-\tau\right) = -i\sum_q \frac{\eta_{vp}\left(\vec{x}; \omega_q^{vp}\right)}{\varsigma'_{vp}(\omega_q^{vp})} \mathrm{e}^{-i\omega_q^{vp}\tau}, \tag{9.56}$$

where ω_q^{pp} and ω_q^{vp} are, respectively, the roots of the characteristic equations of

$$\varsigma_{pp}\left(\omega_q^{pp}\right) = 0, \tag{9.57}$$

$$\varsigma_{vp}\left(\omega_q^{vp}\right) = 0. \tag{9.58}$$

It is emphasized that there are no closed-form solutions for $g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right)$ and $g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right)$ in general because the source surfaces, measurements, and reconstruction locations are arbitrary. Mathematically, $g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right)$ implies the impulse response function correlating the reconstructed acoustic pressure $p\left(\vec{x}; t\right)$ at $\vec{x}$ to the measured acoustic pressure signal $p\left(\vec{x}_m^{\Gamma}; t\right)$ at $\vec{x}_m^{\Gamma}$. Similarly, $g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right)$ is the impulse response function that correlates the reconstructed normal component of the particle velocity $v_{\mathrm{n}}\left(\vec{x}; t\right)$ at $\vec{x}$ to the measured acoustic pressure signal $p\left(\vec{x}_m^{\Gamma}; t\right)$ at $\vec{x}_m^{\Gamma}$. Note that because the residue theorem is used in Eqs. (9.53) and (9.54) to reconstruct the transient acoustic field, rather than a direct inverse Fourier transform, the conventional discretization and the minimal sampling rate requirement in the time domain are avoided.

9.4.4 Applications of the Transient NAH Formulations

In this section both the integral theory and HELS-based NAH formulations are utilized to reconstruct the transient acoustic pressure fields, and results are compared with the analytic ones.

Example 9.3 (A Sudden-Expansion Sphere) Consider a sudden-expansion sphere of radius $r = a$ subject to the initial condition (9.2) with $\vec{\mathrm{n}}\left(\vec{x}_s\right) \cdot \vec{e}_c = 1$. Suppose that sudden expansion occurs at $t = a/c$. The analytic acoustic pressure signal on a hologram surface is taken as the input. For simplicity, we assume that the time history of the acoustic pressure signal measured at any field point is (see Example 9.1)

$$p\left(\vec{x}_m^{\Gamma}; t\right) = \rho_0 c V_s \left(\frac{a}{r_m^{\Gamma}}\right) \mathrm{e}^{-\left(ct - r_m^{\Gamma}\right)/a} H\left(t - \frac{r_m^{\Gamma}}{c}\right). \tag{9.59}$$

The reconstructed acoustic pressure signal at any field point $\vec{x}$ can be determined by using Eq. (9.53). Since the normal surface velocity is constant, it suffices to take one measurement on a hologram surface, i.e., $M = 1$. First, we use the BEM-based NAH formulation to reconstruct the acoustic pressure field. Accordingly, the pseudo inversion defined in Eq. (9.43) reduces to

$$T_{pp}^{\dagger}\left(\vec{x}_m^{\Gamma}\middle|\vec{x}_s; \omega_q^{pp}\right) = \frac{\varsigma(\omega)}{\eta\left(\vec{x}_m^{\Gamma}; \omega\right)}. \tag{9.60}$$

Substituting Eq. (9.60) into Eq. (9.46) yields the reconstructed acoustic pressure at any field point $\vec{x}$ in the frequency domain,

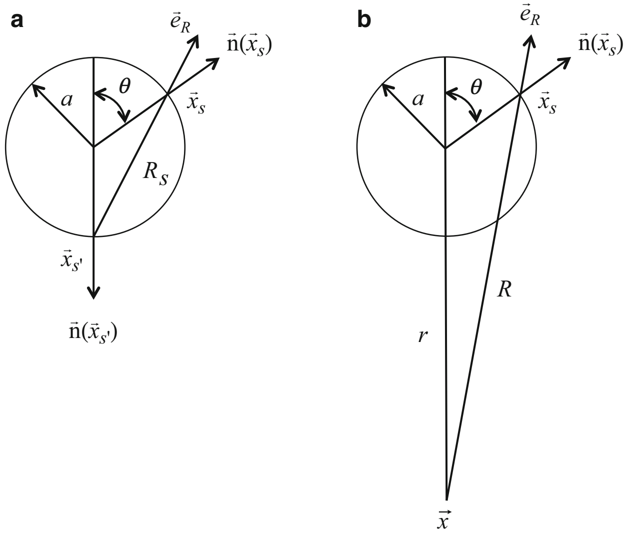

Fig. 9.2 Schematic of points on the surface of the sphere of radius $r = a$ and in the field, and corresponding distances R_s and R, respectively. (**a**): Both points on the source surface; (**b**): One point on the source surface and another in the field

$$P\left(\vec{x};\omega\right) = \frac{\eta\left(\vec{x};\omega\right)}{\eta\left(\vec{x}^{\Gamma}_m;\omega\right)} P\left(\vec{x}^{\Gamma}_m;\omega\right), \tag{9.61}$$

The temporal acoustic pressure at any field point $p\left(\vec{x};\omega\right)$ may be obtained by Eq. (9.27), which can be evaluated using the residue Eq. (9.53), where the temporal kernel $g_{pp}\left(\vec{x}\middle|\vec{x}^{\Gamma}_m;t-\tau\right)$ is

$$g_{pp}\left(\vec{x}\middle|\vec{x}^{\Gamma}_m;t-\tau\right) = -i\sum_q \frac{\eta\left(\vec{x};\omega_q^{pp}\right)}{\eta\left(\vec{x}^{\Gamma}_m;\omega_q^{pp}\right)}, \tag{9.62}$$

where ω_q^{pp} is the qth root of the characteristic equation $\varsigma_{pp}(\omega_q^{pp}) = 0$. In this case, there is only one root, $\omega_1^{pp} = -ic/a$, so $q = 1$.

Figure 9.2 displays the schematic of relative positions of the locations of surface and field points with respect to a sudden-expansion sphere. The quantity $\xi\left(\vec{x}_s;\omega_1^{pp}\right)$ involved in $\eta\left(\vec{x};\omega_1^{pp}\right)$ and $\eta\left(\vec{x}^{\Gamma}_m;\omega_1^{pp}\right)$ is given by Eq. (9.26). The distance between two points on the source surface is $R_s = 2a\cos(\theta/2)$, $\partial R_s/\partial \mathrm{n}_s = \cos(\theta/2)$ and $dS' = a^2\sin\theta d\theta d\phi$, with θ varying from 0 to π and ϕ from 0 to 2π. Since integrands are independent of the azimuthal angle ϕ, integration over ϕ can be done separately, yielding 2π. The distance between a surface and field point is $R = \sqrt{r^2 + a^2 + 2ar\cos\theta}$. For simplicity, the radial distance r is assumed much larger than radius a, so $R \approx r$, $e^{ikR} \approx e^{ikR + ika\cos\theta}$ and $\partial R/\partial n_s = \cos\theta$. Detailed integrations for $\xi(a; \omega_1^{pp})$ and $\eta\left(\vec{x}_m^{\Gamma}; \omega_1^{pp}\right)$ are shown in reference [142] and omitted here for brevity.

Substituting $\xi\left(\vec{x}_s; \omega_1^{pp}\right)$ and $\eta\left(\vec{x}_m^{\Gamma}; \omega_1^{pp}\right)$ into Eq. (9.62) yields

$$g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t-\tau\right) = \frac{(a/r)e^{ik_1^{pp}r}}{\left(a/r_m^{\Gamma}\right)e^{ik_1^{pp}r_m^{\Gamma}}} = \left(\frac{r_m^{\Gamma}}{r}\right)e^{\left(r-r_m^{\Gamma}\right)/a}. \tag{9.63}$$

Substituting Eqs. (9.59) and (9.63) into Eq. (9.53) then leads to

$$p\left(\vec{x}; t\right) = \rho_0 c V_s\left(\frac{a}{r}\right)e^{-(ct-r)/a}H\left(t-\frac{r}{c}\right), \tag{9.64}$$

which matches the analytic solution for a sudden-expansion sphere [142].

Next the HELS-based NAH formulation is used to reconstruct the transient acoustic field. The basis function in the HELS expansion is given by Eq. (9.52). For a sudden-expansion sphere, it suffices to use a one-term expansion. Accordingly, we have $\Psi_1(r, \theta, \phi; \omega) = e^{ikr}/r$. The temporal kernels $g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t-\tau\right)$ as defined by Eq. (9.62) reduces to

$$\begin{aligned} g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t-\tau\right) &= -i\frac{\eta_{pp}\left(\vec{x}; \omega_1^{pp}\right)}{\varsigma'_{pp}\left(\omega_1^{pp}\right)}e^{-i\omega_1^{pp}\tau} = \frac{\left(ar_m^{\Gamma}/cr\right)e^{-i[c/(-ia)]\left(r-r_m^{\Gamma}\right)/c}}{a/c} \\ &= \left(\frac{r_m^{\Gamma}}{r}\right)e^{\left(r-r_m^{\Gamma}\right)/a}. \end{aligned} \tag{9.65}$$

Substituting (9.59) and (9.65) into the convolution integral (9.53) yields

$$p\left(\vec{x}; t\right) = g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right) \times p\left(\vec{x}_m^{\Gamma}; t\right) = \rho_0 c V_s e^{-(ct-r)/a}\left(\frac{a}{r}\right)H\left(t-\frac{r}{c}\right), \tag{9.66}$$

which agrees with the analytic solution [142].

Figure 9.3 demonstrates three-dimensional images of acoustic pressure fields at arbitrarily selected time instances $t = 3.24$ (ms), 4.41 (ms), 5.88 (ms), and 7.35 (ms).

Similarly, the normal component of the particle velocity at any field point is reconstructed by using Eq. (9.54) by using $p\left(\vec{x}_m^{\Gamma}; t\right)$ and $g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t\right)$ given by

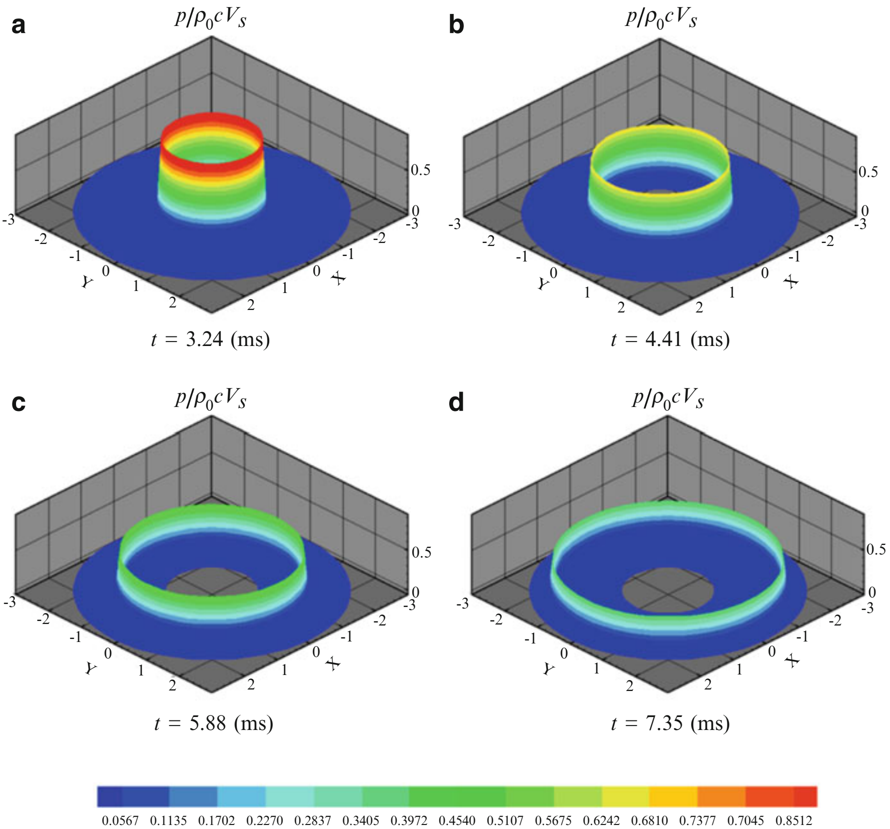

Fig. 9.3 Reconstructed temporal acoustic pressure fields resulting from a sudden-expansion sphere of radius a at different time instances. (**a**): $t = 3.24$ ms; (**b**): $t = 4.41$ ms; (**c**): $t = 5.88$ ms; and (**d**): $t = 7.35$ ms

$$g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t-\tau\right) = -i\frac{\eta_{vp}\left(\vec{x};\omega_1^{vp}\right)}{\varsigma_{vp}'\left(\omega_1^{vp}\right)}\mathrm{e}^{-i\omega_1^{vp}\tau}$$

$$= \frac{\left(ar_m^{\Gamma}/\rho_0c^2r\right)\mathrm{e}^{-i[c/(-ia)]\left(r-r_m^{\Gamma}\right)/c}}{a/c} = \frac{1}{\rho_0c}\left(\frac{r_m^{\Gamma}}{r}\right)\mathrm{e}^{\left(r-r_m^{\Gamma}\right)/a}. \quad (9.67)$$

Substituting Eqs. (9.59) and (9.67) into Eq. (9.54) leads to

$$v_{\mathrm{n}}\left(\vec{x};t\right) = g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};t\right)\times p\left(\vec{x}_m^{\Gamma};t\right) = V_s\mathrm{e}^{-(ct-r)/a}\left(\frac{a}{r}\right)H\left(t-\frac{r}{c}\right), \quad (9.68)$$

which reduces to the initial condition (9.2) when r is set on the source surface and $a/c = t_s$.

Example 9.4 (An Impulsively Accelerated Sphere) Consider the case of a sphere of radius $r = a$ that is impulsively accelerated in the z-axis direction (see Example 9.2). Accordingly, the normal surface velocity is given by

$$v_{\mathrm{n}}\left(\vec{x}_s;t\right) = V_s\left(\vec{\mathrm{n}}_s \cdot \vec{e}_z\right)H(t-t_s), \tag{9.69}$$

where $\vec{\mathrm{n}}\left(\vec{x}_s\right) \cdot \vec{e}_z = \cos\theta$.

Again, the integral theory-based NAH is utilized to reconstruct the transient acoustic field first. The analytic acoustic pressure signal at the hologram surface is taken as the input,

$$\begin{aligned} p\left(\vec{x}_m^{\Gamma};t\right) = \rho_0 c V_s \cos\theta_m^{\Gamma}\left(\frac{a}{r_m^{\Gamma}}\right)\mathrm{e}^{-\left(ct-r_m^{\Gamma}\right)/a}H\left(t-\frac{r_m^{\Gamma}}{c}\right) \\ \left[\cos\left(\frac{ct-r_m^{\Gamma}}{a}\right)-\left(1-\frac{a}{r_m^{\Gamma}}\right)\sin\left(\frac{ct-r_m^{\Gamma}}{a}\right)\right]. \end{aligned} \tag{9.70}$$

As in the previous sudden-expansion sphere, the reconstructed temporal acoustic pressure $p\left(\vec{x};\omega\right)$ can be obtained using Eq. (9.53) with its temporal kernel $g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};t-\tau\right)$ given by Eq. (9.62). The quantity $\xi\left(\vec{x}_s;\omega_q^{pp}\right)$ involved in $\eta_{pp}\left(\vec{x};\omega_q^{pp}\right)$ and $\eta_{pp}\left(\vec{x}_m^{\Gamma};\omega_q^{pp}\right)$ can be shown as

$$\xi\left(\vec{x}_s;\omega_q^{pp}\right) = \frac{a\left(ik_q^{pp}a-1\right)}{2-\left(k_q^{pp}a\right)^2-i2k_q^{pp}a}. \tag{9.71}$$

In this case $q = 2$, $\omega_1^{pp} = (1-i)c/a$ and $\omega_1^{pp} = (-1-i)c/a$. Thus the temporal kernel becomes

$$g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};t-\tau\right) = -i\sum_{q=1}^{2}\left(\frac{r_m^{\Gamma}}{r}\right)^2\frac{\left(1-ik_q^{pp}r\right)}{\left(1-ik_q^{pp}r_m^{\Gamma}\right)}\mathrm{e}^{ik_q^{pp}\left(r-r_m^{\Gamma}\right)}. \tag{9.72}$$

Detailed derivations of Eq. (9.72) are omitted here for brevity. Substituting Eqs. (9.70) and (9.72) into Eq. (9.53) and summing the residues yield

$$p\left(\vec{x};t\right) = \rho_0 c V_s \cos\theta\left(\frac{a}{r}\right)\mathrm{e}^{-(ct-r)/a}H\left(t-\frac{r}{c}\right)\left[\cos\left(\frac{ct-r}{a}\right)-\left(1-\frac{a}{r}\right)\sin\left(\frac{ct-r}{a}\right)\right], \tag{9.73}$$

which is the analytic solution to the acoustic pressure due to an impulsively accelerated sphere [142].

Alternatively, the HELS-based NAH can be used to reconstruct the transient acoustic field. Suppose that a two-term HELS expansion is used,

$$\Psi_1(r,\theta,\phi;\omega) = \frac{e^{ikr}}{r} \quad \text{and} \quad \Psi_2(r,\theta,\phi;\omega) = \frac{(kr+i)\cos\theta}{(kr)^2} e^{ikr}. \tag{9.74}$$

Accordingly, the temporal kernel involved in Eq. (9.55) can be written as

$$g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t-\tau\right) = -i\sum_{q=1}^{2} \frac{\eta_{pp}\left(\vec{x};\omega_q^{pp}\right)}{\varsigma'_{pp}(\omega_q^{pp})} e^{-i\omega_q^{pp}\tau}, \tag{9.75}$$

where $q=1$ and 2, $\omega_1^{pp}=(1-i)c/a$, $\omega_1^{pp}=(-1-i)c/a$, $\eta_{pp}\left(\vec{x};\omega_q^{pp}\right)$, and $\varsigma'_{pp}\left(\omega_q^{pp}\right)$ are given, respectively, by

$$\eta_{pp}\left(\vec{x};\omega_1^{pp}\right) = \frac{\left(\omega_1^{pp} r/c + i\right)a^3\cos\theta}{r^2} e^{i\omega_1^{pp} r/c}, \tag{9.76a}$$

$$\eta_{pp}\left(\vec{x};\omega_2^{pp}\right) = \frac{\left(\omega_2^{pp} r/c + i\right)a^3\cos\theta}{r^2} e^{i\omega_2^{pp} r/c}, \tag{9.76b}$$

$$\varsigma'_{pp}\left(\omega_1^{pp}\right) = -2(a/c)\left(a\omega_1^{pp} + i\right), \tag{9.76c}$$

$$\varsigma'_{pp}\left(\omega_2^{pp}\right) = -2(a/c)\left(a\omega_2^{pp} + i\right). \tag{9.76d}$$

Substituting Eqs. (9.73) and (9.75) into Eq. (9.53) and summing the residues yield

$$\begin{aligned} p\left(\vec{x};t\right) &= g_{pp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};t\right) \times p\left(\vec{x}_m^{\Gamma};t\right) \\ &= \rho_0 c V_s \cos\theta\left(\frac{a}{r}\right) e^{-(ct-r)/a} H\left(t-\frac{r}{c}\right)\left[\cos\left(\frac{ct-r}{a}\right) - \left(1-\frac{a}{r}\right)\sin\left(\frac{ct-r}{a}\right)\right], \end{aligned} \tag{9.77}$$

which matches the analytic solution for the temporal acoustic pressure emitted by an impulsively accelerated sphere of radius a in free space [142].

Figure 9.4 shows three-dimensional images of the acoustic pressure fields at arbitrarily selected time instances [149].

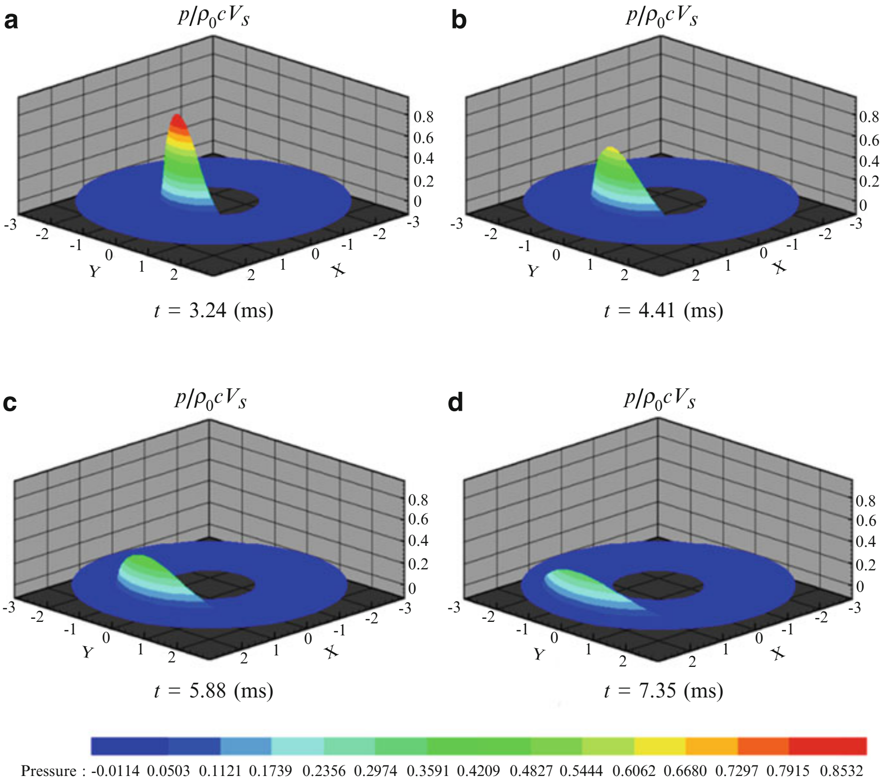

Fig. 9.4 Reconstructed temporal acoustic pressure fields generated by an impulsively accelerated sphere of radius a at different time instances. (**a**): $t = 3.24$ ms; (**b**): $t = 4.41$ ms; (**c**): $t = 5.88$ ms; and (**d**): $t = 7.35$ ms

The normal component of the particle velocity in the time domain can be reconstructed by Eq. (9.54), where the impulse response function $g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t-\tau\right)$ is given by

$$g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma}; t-\tau\right) = -i\sum_{q=1}^{2} \frac{\eta_{vp}\left(\vec{x}; \omega_q^{vp}\right)}{\varsigma_{vp}'\left(\omega_q^{vp}\right)} \mathrm{e}^{-i\omega_q^{vp}\tau}, \tag{9.78}$$

where

$$\eta_{vp}\left(\vec{x}; \omega_1^{vp}\right) = \frac{\left(\omega_1^{vp} r/c + i\right) a^3 \cos\theta}{\rho_0 c r^2} \mathrm{e}^{i\omega_1^{vp} r/c}, \tag{9.79a}$$

$$\eta_{vp}\left(\vec{x};\omega_2^{vp}\right)=\frac{(\omega_2^{vp}r/c+i)a^3\cos\theta}{\rho_0 cr^2}\mathrm{e}^{i\omega_2^{vp}r/c}, \tag{9.79b}$$

$$\varsigma'_{vp}\left(\omega_1^{vp}\right)=-2(a/c)\left(a\omega_1^{vp}+i\right), \tag{9.79c}$$

$$\varsigma'_{pv}\left(\omega_2^{vp}\right)=-2(a/c)\left(a\omega_2^{vp}+i\right). \tag{9.79d}$$

where $\omega_1^{pp}=(1-i)c/a$ and $\omega_1^{pp}=(-1-i)c/a$.

Substituting Eqs. (9.73) and (9.78) into Eq. (9.54) yields the normal component of the particle velocity anywhere in the field,

$$\begin{aligned} v_{\mathrm{n}}\left(\vec{x};t\right) &= g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};t\right)\times p\left(\vec{x}_m^{\Gamma};t\right) \\ &= V_s\cos\theta\left(\frac{a}{r}\right)\mathrm{e}^{-(ct-r)/a}H\left(t-\frac{r}{c}\right)\left[\cos\left(\frac{ct-r}{a}\right)-\left(1-\frac{a}{r}\right)\sin\left(\frac{ct-r}{a}\right)\right], \end{aligned} \tag{9.80}$$

which reduces the initial condition (9.2) when the distance is set to $r=a$ and $t=a/c$.

Example 9.5 (An Impulsively Accelerated Baffled Piston on a Sphere) Consider acoustic radiation from a piston mounted on a sphere of radius a. In general, the acoustic pressure generated by a spherical source in the frequency domain can be described by an infinite series of the spherical Hankel functions of the first kind and the spherical harmonics [99],

$$P(r,\theta,\phi;\omega)=\sum_{n=0}^{\infty}\sum_{l=-n}^{n}A_{nl}h_n^{(1)}(kr)Y_n^l(\theta,\phi), \tag{9.81}$$

where the expansion coefficients A_{nl} can be obtained by the orthonormal property of the spherical harmonics. Suppose that the normal surface velocity is given in the boundary condition. Then the coefficients A_{nl} can be obtained by the Euler's equation (9.21) and the orthonormal property of the spherical harmonics,

$$A_{nl}=i\frac{\rho_0 c}{h_n^{(1)'}(ka)}\int_0^{2\pi}\int_0^{\pi}V_s(a,\theta,\phi;\omega)Y_n^{l*}(\theta,\phi)\sin\theta d\theta d\phi, \tag{9.82}$$

where $V_s(a,\theta,\phi;\omega)$ is specified on the surface of the sphere; $h_n^{(1)'}(ka)=(c/\omega)[dh_n^{(1)}(kr)/dr]|_{r=a}$ is the normal derivative evaluated on the surface of the sphere.

Once the expansion coefficients A_{nl} are specified, the acoustic pressure at any field point in the frequency domain can be determined by Eq. (9.81). The temporal acoustic pressure can be obtained by taking the inverse Fourier transform and evaluated by using Eq. (9.53).

For simplicity, the piston is assumed axisymmetric with respect to the polar axis at $\theta = 0$, and is impulsively accelerated at $t = t_s$. Moreover, the normal surface velocity is non-zero over a vertex angle, $\pm\theta_0$, and zero elsewhere,

$$v_n(a,\theta;t) = V_s[H(\theta+\theta_0) - H(\theta-\theta_0)]H(t-t_s), \tag{9.83}$$

where $t_s = a/c$.

Accordingly, Eq. (9.81) is reduced to [99]

$$P(r,\theta;\omega) = \sum_{n=0}^{\infty} A_n h_n^{(1)}(kr) Q_n^{(1)}(\cos\theta), \tag{9.84}$$

where $Q_n^{(1)}(\cos\theta)$ are the Legendre functions of the first kind.

Note that there is no closed-form solution for the radiated acoustic pressure signal $P(r,\theta;\omega)$ in this case. Hence numerical solutions are sought. As an example, a circular piston with a vertex angle of $\pm\theta_0 = \angle 15°$ is considered in this section. Theoretically, the normal surface velocity given by Eq. (9.83) requires an infinite series to depict the sharp edges at $\theta_0 = \pm\angle 15°$. For the purpose of demonstrating the application of the transient NAH formulations, a finite expansion is utilized to approximate the velocity profile as specified in Eq. (9.83),

$$v_n(a,\theta;t) = V_s \sum_{n=1}^{N} B_n Q_n^{(1)}(\cos\theta) H(t-a/c), \tag{9.85}$$

where N is finite. The larger the value of N is, the better the approximation to the velocity profile is, but the more intensive numerical computations are. For simplicity yet without loss of generality, $N = 11$ is selected in this numerical example. The expansion coefficients B_n can be determined by using the orthonormal property of the Legendre functions [99],

$$B_n = \left(\frac{2n+1}{2}\right) \int_{-\theta_0}^{\theta_0} Q_n^{(1)*}(\cos\theta) \sin\theta d\theta. \tag{9.86}$$

Accordingly, the expansion coefficients A_n for the acoustic pressure, Eq. (9.84), can be obtained by using the orthonormal property of the Legendre functions and boundary condition

$$A_n = i\frac{(2n+1)\rho_0 c V_s}{2h_n^{(1)'}(ka)} \sum_{n'=1}^{N} \left[\int_{-\theta_0}^{\theta_0} Q_{n'}^{(1)*}(\cos\theta) \sin\theta d\theta\right]. \tag{9.87}$$

The temporal acoustic pressure at any field point can be determined by taking the inverse Fourier transform of Eq. (9.84) and facilitated by Eq. (9.53). The resultant

Table 9.1 Singularities of the impulse response functions for reconstructing the transient acoustic field generated by a partially impulsively accelerating piston mounted on a sphere of radius a with a vertex angle of $\pm\theta^\circ = \angle 15^\circ$

No.	Singularities ω_q^{vp}
1	$-5.53363E+02 - i3.28058E+03$
2	$-1.41101E+03 - i2.46182E+03$
3	$-1.87779E+03 - i1.80707E+03$
4	$-2.16515E+03 - i1.19157E+03$
5	$-2.32467E+03 - i5.92509E+02$
6	$-2.37603E+03 + i0.00000E+00$
7	$-2.32467E+03 + i5.29509E+02$
8	$-2.16515E+03 + i1.19157E+03$
9	$-1.87779E+03 + i1.80707E+03$
10	$-1.41101E+03 + i2.46182E+03$
11	$-5.53363E+02 + i3.28058E+03$

Table 9.2 Comparison of the expansion coefficients C_j reconstructed by the HELS method and the analytic ones for an impulsively accelerating piston mounted on a sphere of radius a with a vertex angle of $\pm\theta^\circ = \angle 15^\circ$

C_j	Reconstructed values	Benchmark values
C_1	$+6.69999E-02$	$+6.70000E-02$
C_2	$+1.87500E-01$	$+1.87500E-01$
C_3	$+2.70633E-01$	$+2.70633E-01$
C_4	$+3.00783E-01$	$+3.00781E-01$
C_5	$+2.74016E-01$	$+2.74016E-01$
C_6	$+1.98730E-01$	$+1.98730E-01$
C_7	$+9.34531E-02$	$+9.34529E-02$
C_8	$-1.76239E-02$	$-1.76239E-02$
C_9	$-1.10301E-01$	$-1.10301E-01$
C_{10}	$-1.65869E-01$	$-1.65869E-01$
C_{11}	$-1.75139E-01$	$-1.75140E-01$

acoustic pressure signals on the hologram surface can be taken as input to Eqs. (9.53) and (9.54) to reconstruct the acoustic pressure and particle velocity.

In this example the reconstructed acoustic pressure and particle velocity are obtained using the HELS-based NAH. Numerical computations involved in the BEM-based NAH are excessively intensive as compared to those of the HELS-based NAH and are omitted here for brevity.

Specifically, Eq. (9.54) is used to reconstruct the normal surface velocity with its temporal kernel $g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};t\right)$ determined by Eq. (9.56). Table 9.1 lists the singularities of $g_{vp}\left(\vec{x}\middle|\vec{x}_m^{\Gamma};t\right)$ that are obtained by using Eq. (9.58), namely, ω_q^{vp}, $q=1$ to 11, in this case.

Substituting ω_q^{vp} into Eq. (9.54) and evaluating the residues give the reconstructed normal surface velocity. Table 9.2 shows the comparison of the reconstructed expansion coefficients with benchmark values. Results indicate that the accuracy in the reconstructed expansion coefficients is guaranteed up to the 5th decimal point.

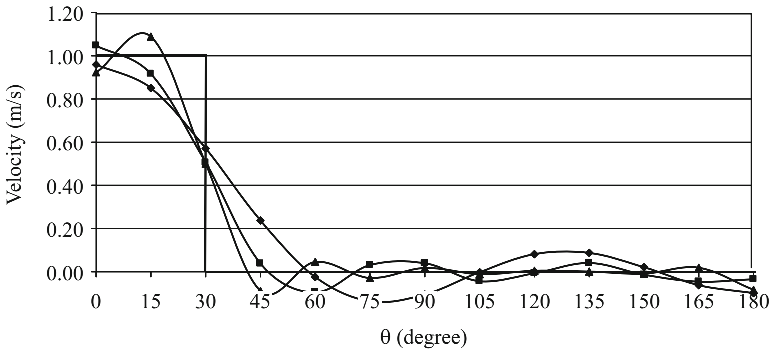

Fig. 9.5 Comparison of the reconstructed normal surface velocity distributions along the generator of the surface of a partially impulsively accelerated sphere of radius a. *dashed line*: Exact velocity profile; *filled diamond*: Reconstructed with $J = 4$; *filled square*: Reconstructed with $J = 8$; *filled triangle*: Reconstructed with $J = 11$

Figure 9.5 displays the comparison of the normal surface velocity reconstructed by using Eq. (9.54) under various numbers of the expansion terms with Eq. (9.83). Results indicate that the reconstructed normal surface velocity converges to the correct velocity profile as the number of expansion terms increases from $J = 4$, 8, and 11.

Figure 9.6 demonstrates the acoustic pressure fields reconstructed by using Eq. (9.53) at four different time instances: $t = 3.24$ ms; $t = 4.41$ ms; $t = 5.88$ ms; and $t = 7.35$ ms [145].

Example 9.6 (An Impulsively Accelerated Baffled Circular Disk) Finally, consider reconstruction of transient acoustic radiation from a non-spherical object. Specifically, Eq. (9.53) is utilized to reconstruct the acoustic pressure generated by an impulsive accelerated circular disk of radius a mounted on an infinite baffled. The normal surface velocity of this baffled disk is given by

$$v_{\mathrm{n}}\left(\vec{x}_s; t\right) = V_s H(a - r) H(t). \tag{9.88}$$

The procedures for reconstruction are exactly the same as those described in Example 9.5 and not repeated here. The reconstructed temporal acoustic pressures are obtained by Eq. (9.53). Figure 9.7 shows the locations at which the input acoustic pressure signals are collected. Assume that the baffled circular disk is axisymmetric with respect to the z-axis. The temporal acoustic pressure signals on the hologram surface are given by

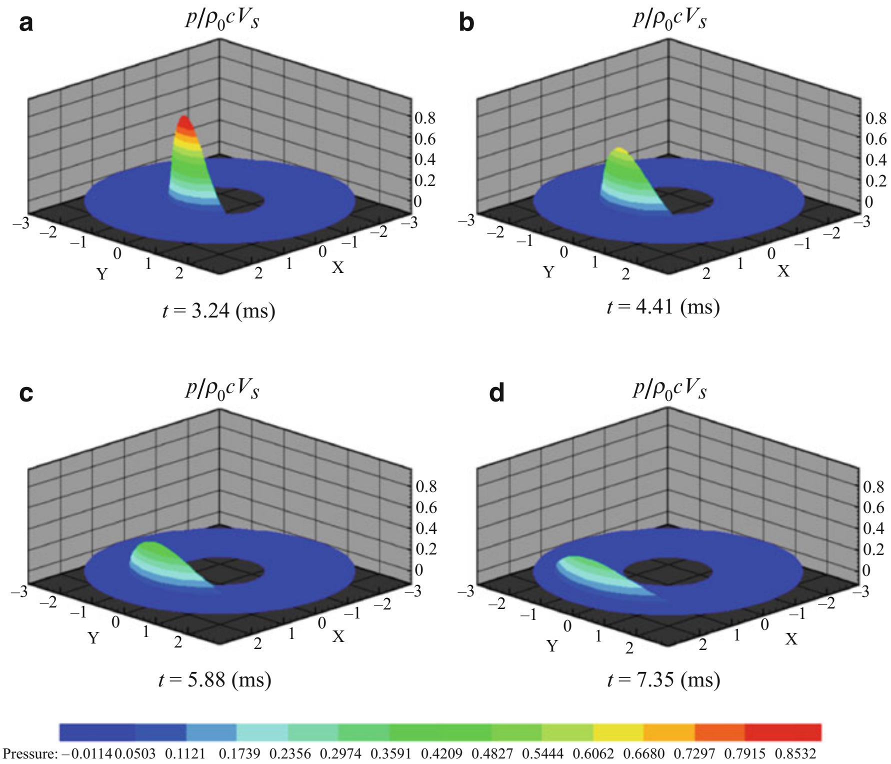

Fig. 9.6 Reconstructed acoustic pressure fields generated by a partially impulsively accelerated sphere of radius a at different time instances. (**a**): $t = 3.24$ ms; (**b**): $t = 4.41$ ms; (**c**): $t = 5.88$ ms; and (**d**): $t = 7.35$ ms

$$p\left(\vec{x}_m; t\right) = \begin{cases} 0, & ct < z_m \\ 0, & w > a, z_m \le ct < \sqrt{(a-w)^2 + z_m^2} \\ \rho_0 c V_s, & w < a, z_m \le ct < \sqrt{(a-w)^2 + z_m^2} \\ \left(\dfrac{\rho_0 c V_s}{\pi}\right) \cos^{-1}\left[\dfrac{\left(c^2 t^2 - z_m^2 + w^2 - a^2\right)}{2w\sqrt{c^2 t^2 - z_m^2}}\right]; & \sqrt{(a-w)^2 + z_m^2} \le ct < \sqrt{(a+w)^2 + z_m^2} \\ 0 & ct \ge \sqrt{(a+w)^2 + z_m^2} \end{cases}, \tag{9.89}$$

Note that in this case the standoff distance z_m is of no concern because the input data given by Eq. (9.89) are analytic. The array of microphones extends to twice the diameter of the baffled disk with respect to its geometric center. The total number of microphones is 40 and microphone spacing is $\Delta = a/10$. Figure 9.8 illustrates the

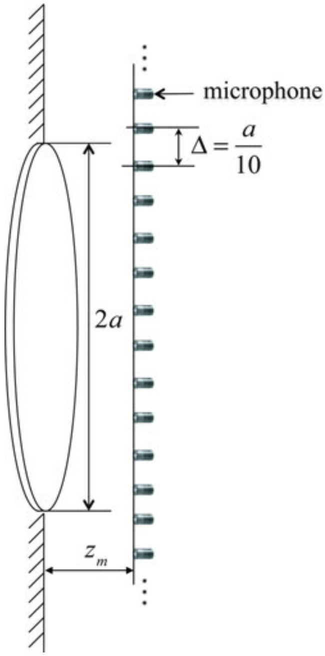

Fig. 9.7 Schematic of an impulsively accelerated baffled disk of radius a and an array of microphones that covers twice the diameter of the disk. The total number of microphones is 40 and the microphone spacing is $a/10$

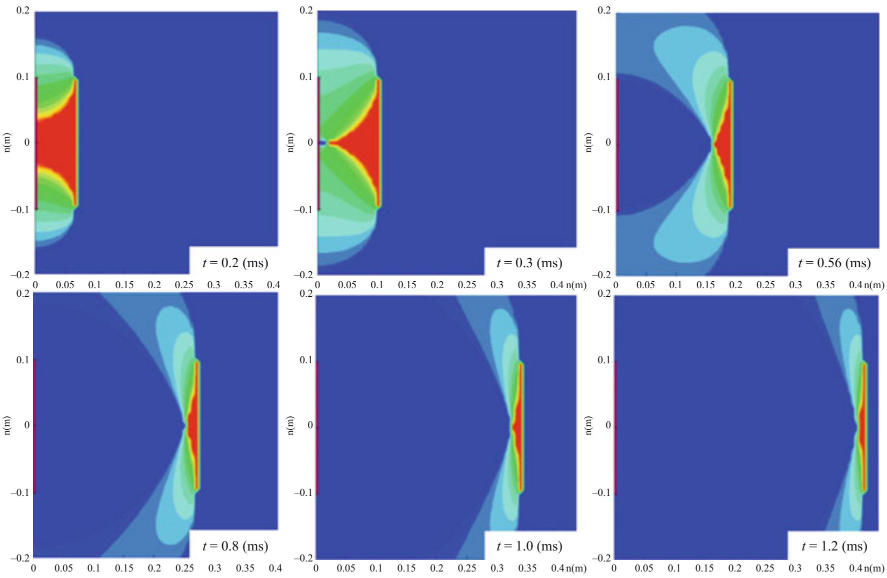

Fig. 9.8 Reconstructed acoustic pressure fields generated by an impulsively accelerated baffled disk of radius a at different time instances. Clockwise: $t = 0.2$ ms, $t = 0.3$ ms, $t = 0.56$ ms, $t = 0.8$ ms, $t = 1.0$ ms, and $t = 1.2$ ms

transient acoustic pressure fields captured at six time instances $t = 0.2$ ms, 0.3 ms, 0.56 ms, 0.8 ms, 1.0 ms, and 1.2 ms, respectively.

Problems

9.1. Show that the Kirchhoff–Helmholtz integral formulation for predicting transient acoustic radiation is given by Eq. (9.19).

9.2. Show that the Kirchhoff–Helmholtz integral equation for determining the transient surface acoustic pressure is given by Eq. (9.20).

9.3. Show that the Fourier transformed version of the Kirchhoff–Helmholtz integral formulation for predicting the acoustic pressure produced by an impulsively accelerating body is given by Eq. (9.25).

9.4. Show that the transient acoustic pressure radiated from an impulsively accelerating object can be written as Eq. (9.30) through the residue theorem.

9.5. Consider an explosion that occurs in free space at $t = t_0$. Assume that the particle velocity rises from near zero to a very high constant value (like a step function) omnidirectionally. Determine the resultant transient acoustic pressure anywhere in the field.

9.6. Consider the case in which an object is impacted by an external force and accelerates in a particular direction in free space. Assume that this impact occurs at $t = t_0$ and the velocity of the entire body rises from zero to a constant value (like a step function) in this direction. Determine the resultant transient acoustic pressure anywhere in the field.

9.7. Show that the transient acoustic pressure field generated by an arbitrarily shaped rigid body subject to an arbitrarily time-dependent excitation is expressible as a convolution integral given by Eq. (9.45).

9.8. Similarly, show that the normal component of the particle velocity in free space generated by an arbitrarily shaped rigid body subject to an arbitrarily time-dependent excitation can be written as a convolution integral given by Eq. (9.46).

9.9. Continue Problem 9.5. Assume that the time histories of the acoustic pressure signals that are measured at two arbitrary points in free space are $p\left(\vec{x}_m; t\right) = (Q/r_m)\mathrm{e}^{-(ct-r_m)/a}$, where Q is a constant and a is the characteristic dimension of the initial explosion region, $m = 1$ and 2. Determine the transient acoustic pressure field anywhere.

9.10. Continue Problem 9.6. Assume that the time histories of the acoustic pressure signals that are measured at two arbitrary points in free space are given

$$p\left(\vec{x}_m^{\Gamma}; t\right) = \left(\frac{Q}{r_m}\right)\mathrm{e}^{-(ct-r_m)/a}\cos\theta_m H\left(t - \frac{r_m}{c}\right)$$
$$\left[\cos\left(\frac{ct - r_m}{a}\right) - \left(1 - \frac{a}{r_m}\right)\sin\left(\frac{ct - r_m}{a}\right)\right],$$

where Q is a constant, θ indicates the polar angle, a is the characteristic dimension of the rigid body, and $m = 1$ and 2. Determine the transient acoustic pressure field anywhere.