Chapter 4
Validity of the HELS Method

The validity challenges came at the joint meetings of the 136th Meeting of the Acoustical Society of America (ASA), the 2nd Convention of the European Acoustics Association (EAA), and the 25th German Annual Conference on Acoustics (DAGA) held in Berlin, Germany, 1999 [56]. The major questions were as follows: "How can the acoustic field on the surface of any non-spherical structure be described by the spherical wave functions?" "Is this a Rayleigh hypothesis in NAH that pushes a solution formulation beyond its region of validity?"

4.1 Rayleigh Hypothesis

At the turn of the last century, Rayleigh used a series expansion of plane waves to depict the acoustic pressure field resulting from an incident time-harmonic acoustic plane wave scattered on a one-dimensional periodic, impenetrable corrugated surface S (see Fig. 4.1). The corrugations can be expressed mathematically as [57, 58]

$$\varsigma = b\cos\left(2\pi x/\lambda_x\right), \tag{4.1}$$

where b and λ_x are, respectively, the amplitude and wavelength of corrugation of the surface S and θ is the angle of incidence with respect to the unit normal of the surface S.

Above the highest point of the corrugations of surface S ($z > b$), the complex amplitude of the total acoustic pressure (with the time-harmonic function $e^{-i\omega t}$ omitted for brevity) as given by Rayleigh was

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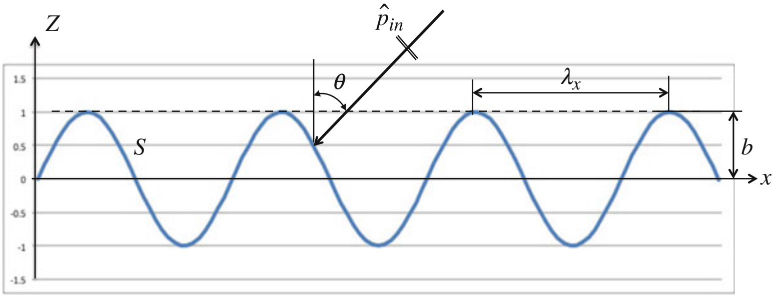

Fig. 4.1 Schematic of an incident time-harmonic acoustic pressure plane wave on a periodic corrugated surface S

$$\hat{p}\,(x,z;\omega) = \hat{p}_{\,\text{in}}\left[\mathrm{e}^{ikz\cos\theta} + R(\omega)\mathrm{e}^{-ikz\cos\theta}\right] + \sum_{n=-\infty}^{\infty} A_n \mathrm{e}^{i(nk_x x + kx\sin\theta - kz\cos\theta_n)}, \quad (4.2)$$

where $\hat{p}_{\,in}$ is the complex amplitude of the incident acoustic pressure, $R(\omega)$ is the acoustic pressure reflection coefficient, A_n represent the expansion coefficients that are determined by the boundary conditions on the corrugated surface S, $k_x = 2\pi/\lambda_x$ is the spatial wavenumber of the corrugations, and $\cos\theta_n = \sqrt{1 - \sin^2\theta_n}$, here $\sin_n = \sin\theta + n(k_x/k)$. The first and second terms on the right side of Eq. (4.2) represent, respectively, the incident and reflected acoustic pressure waves acting on a smooth surface and the infinite series imply the acoustic pressure scattered from the corrugated surface as shown in Fig. 4.1.

In an attempt to use the boundary conditions on S, Rayleigh assumed that the infinite series (4.42) was valid everywhere, including the corrugated surface S. This is known as the Rayleigh hypothesis. This hypothesis was tested on various acoustic scattering problems and had aroused many controversies over the next 60 years. Sometimes the results given by the Rayleigh series (4.2) were correct, but most of times they were completely wrong.

The validity of Rayleigh hypothesis may be examined through analyticity of the solution. If the solution to the acoustic pressure can be analytically continued from the field to the surface, then the expansion coefficients may be determined by the boundary conditions, and the Rayleigh hypothesis is correct [59]. Therefore to answer the question of the validity of Rayleigh hypothesis, it is necessary to find the distribution of singularities using the analytic continuation of the acoustic field across the surface of any scatterer.

These controversies were eventually settled by Millar [59–61] who proved that the Rayleigh hypothesis was neither completely right nor completely wrong. In fact, the validity of a Rayleigh series was governed by the locations of the singularities of the analytic continuation of the exterior scattered field across a scattering surface. For example, in the case of scattering from the gratings and periodic corrugated structures, the Rayleigh series solution would be valid if singularities lay below the lowest point of a corrugated surface. If the singularities lay above the lowest point of a corrugated surface, the series solution is valid only in the region above the highest singularities.

For the case of a periodic corrugated surface as depicted in Fig. 4.1, Millar showed that the Rayleigh hypothesis would be wrong and Eq. (4.2) be invalid when $\lambda_x b > 0.0448$, and neither wrong nor right when $\lambda_x b \leq 0.0448$ [62]. For example, consider a corrugated surface of a wavelength $\lambda_x = 1$ m and corrugation height $b = 0.05$ m. Because $\lambda_x b = 0.05 > 0.0448$, it would be wrong to use Eq. (4.2) to depict the scattered acoustic pressure on the corrugated surface S. On the other hand, if $\lambda_x = 1$ m and $b = 0.045$ m, then $\lambda_x b = 0.045 < 0.0448$ and it might be acceptable to use Eq. (4.2) to describe the scattered acoustic pressure on and above the corrugated surface S.

Millar gave the formal proof of the method for determining the singularities of the acoustic field for a periodically corrugated surface [60]. Hill and Celli offered a heuristic method to estimate the singularities of a periodic corrugated surface [63]. van der Berg and Fokkema studied the acoustic scattering from a nonperiodic corrugated surface [64].

Similarly, in a two-dimensional acoustic scattering scenario, we can use a Rayleigh series in terms of the outgoing cylindrical waves to describe the scattered acoustic pressure field,

$$\hat{p}_{\text{scattered}}(r, \phi; \omega) = \sum_{n=-\infty}^{\infty} A_n H_n^{(1)}(kr) \mathrm{e}^{in\phi}, \tag{4.3}$$

where $H_n^{(1)}(kr)$ represents the nth-order cylindrical Hankel functions of the first kind.

Once again, the validity of Eq. (4.3) will be correlated to the distribution of singularities in the analytic continuation of the acoustic pressure field across the surface of a two-dimensional scatterer. Figure 4.2 depicts an arbitrary circle S, which is the cross section of an infinite cylinder. When the singularities all lie inside the maximum circle $S_{\max}$ that inscribes the circle S, the series solution (4.3) converges absolutely and uniformly in the compact subsets of the exterior of $S_{\min}$ that circumscribes the scatterer. When the singularities lie on or outside the maximum circle $S_{\max}$, the series solution (4.3) will be valid to depict the scattered acoustic pressure above the highest singularities, but invalid below these singularities, because Eq. (4.3) only converges absolutely and uniformly outside the circle defined by the locations of the singularities.

A number of people have looked into the problem of locating possible singularities of the analytic continuation of solutions to the Helmholtz equation for a two-dimensional scatterer with analytic data across analytic boundaries [65–67]. In particular, Maystre and Cadilhac developed a method for determining possible singularities [68], and Keller gave the proof of its validity [69].

Note that in general there is no way of determining the locations of the singularities in the analyticity of solution to the Helmholtz equation because analytic solutions for arbitrary geometry do not exist. In an attempt to determine possible

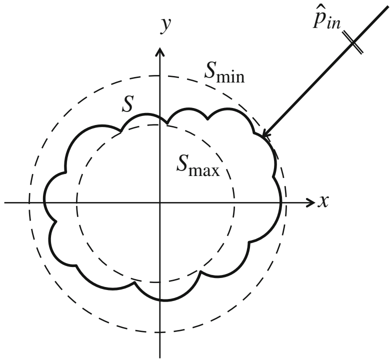

Fig. 4.2 Schematic of an acoustic scattering problem in two-dimensional space. The Rayleigh series solution is valid outside the minimum circle $S_{\min}$ circumscribing the scatterer S when singularities all lie inside the maximum circle $S_{\max}$ inscribing the scatterer S

singularities of the analytic continuation without the explicit knowledge of the solution, Millar made use of the Schwarz function [70], which utilized of the geometric properties of the boundary. By locating the singularities of the Schwarz function, possible singularities in the analytic continuation of the solution might be determined. However, there is no way of knowing if these possible singularities are indeed the actual singularities. Thus in practice the Rayleigh series solution (4.3) is utilized for domains that are free of singularities. Examples of these include separable geometry such as a sphere and an infinite cylinder.

In three-dimensional acoustic scattering problems, the Rayleigh series can be expanded in terms of the spherical Hankel functions and spherical harmonics, with their expansion coefficients determined by the orthogonality properties of the spherical harmonics.

$$\hat{p}\,(r,\theta,\phi;\omega)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{mn}h_{n}^{(1)}(kr)Y_{n}^{m}(\cos\theta). \tag{4.4}$$

In Examples 2.10 and 2.11 it has been shown that the infinite series expansion (4.4) can be used to predict acoustic radiation from a vibrating sphere, given the normal surface velocity on a spherical source surface as the boundary condition; or reconstruct the acoustic pressure anywhere including the spherical source surface, given the acoustic pressure on a spherical surface at some distance away from the source surface.

Note that there is a major difference between prediction and reconstruction problems. The former represents a forward problem, whereas the latter stands for an inverse problem. A forward problem is mathematically well defined and errors in input data are bounded in prediction. On the other hand, an inverse problem is mathematically ill posed and errors in input data may increase without a bound in reconstruction. To get a bounded reconstruction, regularization must be used.

Another complication for the infinite series solution may arise in practice when the source is non-spherical. Figure 4.3 demonstrates the schematic of acoustic scattering from an arbitrarily shaped source in three-dimensional space. Analyses have shown the infinite series solution (4.4) is only valid outside the minimum

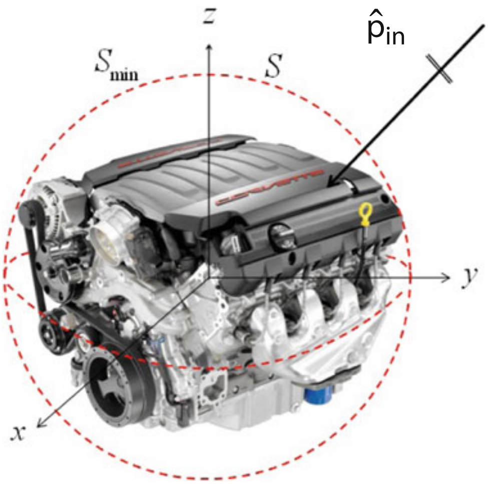

Fig. 4.3 Schematic of an acoustic scattering problem in the three-dimensional space. The Rayleigh series solution (4.4) is valid outside the minimum sphere S_{min} circumscribing the scatterer but invalid inside S_{min}

sphere S_{min} that circumscribes an arbitrary source surface S, but invalid inside the minimum sphere S_{min} in general [71].

On the surface, it looks as though the infinite series solution (4.4) is quite similar to the HELS formulation (3.1), which is expressible as

$$\hat{p}\,(r,\theta,\phi;\omega)=\sum_{n=0}^{N}\sum_{l=-n}^{n}C_{ln}h_n^{(1)}(kr)Y_n^l(\cos\theta). \tag{4.5}$$

Equation (4.5) shows that the acoustic pressure can be described by a superposition of the spherical Hankel functions and spherical harmonics, which is the same as the Rayleigh series (4.4) in three dimensions. Therefore a natural question is as follows: "Will HELS expansions be subject to the same restriction as the Rayleigh series does? Specifically, will Eq. (4.5) be valid only inside the region bounded by spheres?" Moreover, "How will the HELS formulations be related to the Rayleigh series?" These questions are answered in the next section.

4.2 The Rayleigh Series Versus HELS Formulations

Section 4.1 has discussed in detail that the Rayleigh hypothesis is valid and the Rayleigh series converges absolutely and uniformly when the singularities of the analytic continuation of the solution lie inside the maximum sphere that inscribes the source surface of interest. Since in general the analytic solution to the Helmholtz equation for an arbitrarily shaped surface does not exist, there is no way of knowing if the Rayleigh series (4.4) is a valid solution, and where the region of validity is. One thing for sure is that the infinite series will diverge when Eq. (4.4) is evaluated on an arbitrarily shaped source surface. Even if this series is truncated, the solution can still diverge when input data are noisy. So a safe tactic is to use the Rayleigh series to predict the scattered acoustic pressure outside the minimum that circumscribes an arbitrarily shaped surface. The trouble with this approach is that the expansion coefficients cannot be determined because the boundary conditions are given on the source surface.

On the other hand, Chap. 3 has demonstrated examples of using the HELS formulations (4.5) to reconstruct very accurately all the acoustic quantities on a flat vibrating panel, which is way inside the minimum sphere circumscribing this highly non-spherical surface. Thus the HELS formulations (4.5) must be different from the Rayleigh series (4.4), even though they both use the expansion of the spherical wave functions.

The differences between the Rayleigh series (4.4) and the HELS formulations (4.5) are as follows. First of all, the Rayleigh series is infinity, while the HELS expansion is finite. Second, the expansion coefficients in the Rayleigh series are specified by using the orthonormal property of the spherical harmonics and integrating over the solid angle of a sphere, while those in the HELS formulation are specified by matching the expansion (4.5) to the measured data, and the errors involved in this process are minimized by using the least-squares method. Last but not the least, the orthonormal property of the spherical harmonics holds true for a spherical surface, but not an arbitrary surface. So the Rayleigh series is bound to fail when applying it to an arbitrarily shaped surface. In contrast, the HELS formulations always utilize an optimal number of expansion terms to best approximate the reconstructed acoustic quantities. In other words, the HELS formulations always attempt to produce the best approximation for the acoustic quantities radiated from a non-spherical source surface under any given set of input data.

The interrelationships between HELS and Rayleigh series are revealed by Semenova and Wu [72] in reconstructing the acoustic field generated by an arbitrary surface in the exterior region. For simplicity, Semenova and Wu consider infinite cylinders with arbitrary cross sections. They discover that outside the minimum circle that circumscribes the singularities of the cylinder, the Rayleigh series yields an identical result as HELS does when the input data are error free. This is because the high-order terms are negligibly small, so the differences between the Rayleigh series and HELS solutions (a truncated expansion) are minuscule.

When the input data are noisy, the results are different. The normalized errors are the same for all expansion terms in the Rayleigh series because in calculating the coefficients of the series solution by integration, the noise affects all the coefficients equally. In order to obtain a bounded solution, the Rayleigh series must be truncated. Meanwhile, the normalized errors change with the number of expansion terms for the HELS formulations, and are minimal at the optimal number of expansion terms. This is because errors embedded in measurements affect the higher-order terms more than the lower-order ones, as demonstrated in Eq. (3.49) in Sect. 3.4.

Of particular concern is the difference between Rayleigh series and HELS solution inside the minimum circle circumscribing a source. Semenova and Wu illustrates that the Rayleigh series diverges once it is extended inside the minimum circle bounded by the singularities. This confirms Millar's theory on the validity of the Rayleigh hypothesis. In contrast, HELS formulations are not subject to this restriction and may provide satisfactory reconstruction everywhere. Of course, the further the reconstruction point is extended into the minimum circle, the larger the reconstruction errors may become. Note that even if the Rayleigh series is truncated at the same order as that of the HELS expansion, its reconstruction errors inside the minimum circle are still much larger than those of HELS.

These results suggest that the HELS formulations are different from the Rayleigh series as far as back propagating an acoustic field to an arbitrary surface is concerned. However, knowing this difference is not enough to justify the validity of HELS inside a minimum sphere. Moreover, previous results have demonstrated that the accuracy of reconstruction on a non-spherical surface using HELS decreases continuously as the aspect ratio of a source surface and dimensionless frequency ka increase, where a is the characteristic dimension [73].

Therefore, a rigorous mathematical justification of the validity of the HELS formulations to reconstruct the acoustic quantities on an arbitrary surface is needed, which are given rigorously in the next section.

4.3 Justification of the HELS Formulations

Since its first publication in 1997, the HELS-based NAH method has been successfully used to reconstruct the acoustic pressure fields generated by arbitrarily shaped vibrating structures in both exterior and interior regions. Of course, in these cases the structures are not highly elongated, but nonetheless arbitrary. From the acoustics point of view, one can claim that the HELS method may yield satisfactory reconstruction of the acoustic field by using superposition of the spherical wave functions, which explains many phenomena observed in the previous studies. However, this is contradictory to the belief that the expansion solutions using the spherical wave functions and spherical harmonics are valid only inside the regions bounded by spheres and invalid outside these regions.

In this section we present rigorous mathematical justifications for the HELS formulations [74]. Basically, we show that for reconstructing acoustic radiation from an arbitrary source surface, the solutions given by the HELS formulations are approximate; but nonetheless, reconstruction errors are bounded.

Consider reconstruction of acoustic radiation from a finite, arbitrarily shaped object, which includes the acoustic pressure and the normal component of the velocity on the source surface and those in the field. Mathematically, this is equivalent to solving the Helmholtz equation in a three-dimensional domain Ω bounded by the source surface Γ and a surface at infinity Γ_∞.

The acoustic field u with the acoustic wavenumber k satisfies the Helmholtz equation in Ω,

$$\nabla^2 u(r,\theta,\phi;\omega) + k^2 u(r,\theta,\phi;\omega) = 0 \quad \text{in} \;\; \Omega\big(\text{or in} \;\; \Omega_e = \mathrm{R}^3 \backslash \overline{\Omega}\big). \tag{4.6}$$

In practice, the domain Ω can be either the exterior or interior region of a passenger vehicle or an aircraft cabinet. For the exterior problems, solutions to Eq. (4.6) satisfy the Sommerfeld radiation condition,

$$\lim_{r\to\infty} r\left[\frac{\partial u(r,\theta,\phi;\omega)}{\partial r} - iku(r,\theta,\phi;\omega)\right] = 0, \quad as \;\; r \to \infty. \tag{4.7}$$

Such a solution u is called a radiating solution. In what follows the arguments of u are omitted for brevity.

To reconstruct the acoustic field, we need to measure the acoustic pressures u around the source. Suppose that the acoustical sensors are placed on a surface Γ_0 either inside or outside the source surface. These measured data are utilized to reconstruct u on the source surface Γ and in Ω and, in particular, the normal surface velocity v_n defined by

$$v_n = \frac{1}{i\omega\rho_0}\partial_r u \quad \text{on} \;\; \Gamma, \tag{4.8}$$

where the subscript n indicates the unit outward normal on Γ and the symbol ∂_r indicates a partial derivative with respect to r. Note that here we assume that there are no sources other than the one under consideration.

The steps involved in our mathematical justifications are outlined as follows. First, we show that any radiating solution to the Helmholtz equation outside a bounded Lipschitz domain Ω with a connected complement can be approximated by a family of certain known special solutions, for example, the spherical wave functions. Next, by using this approximation together with conditional stability estimates in the Cauchy problem for an elliptic equation, namely, the Helmholtz equation, we demonstrate that these special solutions are bounded on Ω_e and their convergence on Γ_0 implies convergence in Ω_e. Finally, we derive estimates of the convergence of Hölder type at a distance from Ω and that of logarithmic type in Ω_e. These results justify mathematically the validity of the HELS formulations, in which the measured acoustic pressures on Γ_0 are approximated by a linear combination of the special solutions. For an exterior problem this linear combination is well defined everywhere except at the origin, and gives an approximate solution in the exterior region.

Note that we use Ω_e to denote the complement $\mathrm{R}^3\backslash\overline{\Omega}$ and fix a (large) ball B that contains $\overline{\Omega}$. Also we use $H_{(\ell)}(\Omega)$ to imply the Sobolev space of functions in Ω, whose partial derivatives up to the order n are square integrable, and use

$$||u||_{(\ell)}(\Omega) = \sqrt{\sum_{\alpha \leq \ell} \int_{\Omega} |\partial^{\alpha} u|^2}, \tag{4.9}$$

to denote the standard norm in this space. Note that in Eq. (4.9), we let $\|\|_2 = \|\|_{(0)}$ be the norm in the space $L^2(\Omega)$ and use the symbol ∂^{α} to indicate the αth-order partial derivative. Accordingly, $\|\|_{\ell+\lambda}$, where $0 < \lambda < 1$ means the norm in the Hoelder space $\Upsilon^{\ell+\lambda}$ of the functions whose partial derivatives up to order ℓ are Hoelder continuous of an exponent λ, where Υ is a generic constant depending only on Ω, Γ_0, and k.

Now we focus on the approximation of u through the simplest solutions. Our purpose is to interpolate the measured data on Γ_0 for solutions to some integral equations, which can be crucial for higher acoustic wavenumbers k.

Theorem 4.1 *Let Ω be a bounded Lipschitz domain in $\boldsymbol{R}^3$ with connected Ω_e. Let $u \in H_{(1)}\left(B \backslash \overline{\Omega}\right)$ be a radiating solution to the Helmholtz equation (4.6) in Ω_e. Let $B_0 \subset \overline{B}_0 \subset \Omega$ for a ball B_0. Then for any positive ε there is a radiating solution u_{ε} to the Helmholtz equation outside B_0 such that*

$$||u - u_{\varepsilon}||_{(\ell)}(\Omega_{\mathrm{e}}) < \varepsilon. \tag{4.10}$$

In the proofs we will use the following Green's formula:

$$\int_{\Omega_{\mathrm{e}}} \left[\left(\nabla^2 u + k^2 u\right)u^* - u\left(\nabla^2 u^* + k^2 u^*\right)\right] = \int_{\partial\Omega_{\mathrm{e}}} \left(u\partial_{\gamma}u^* - u^*\partial_{\gamma}u\right), \tag{4.11}$$

for u *and* $\mathrm{u}^* \in \mathrm{H}_{(2)}(\mathrm{B}\backslash\Omega)$*, which are radiating solutions to the Helmholtz equation in* B_{1e} *for some ball $B_1 \subset \overline{B}_1 \subset \Omega$. Also, we need the following Runge property of radiating solutions.*

Lemma 4.1 *Let Ω_1 and Ω_2 be two bounded domains that contain B_0 with connected Ω_{1e} and Ω_{2e}, $\overline{\Omega}_1 \subset \Omega_2$. Let u_1 be a radiating solution to the Helmholtz equation $(\nabla^2 + \mathrm{k}^2)\mathrm{u}_1 = 0$ in Ω_{1e}. Then for any $\varepsilon > 0$ there is a radiating solution u outside B_0 such that $//\mathrm{u}_1 - \mathrm{u}//_{(\ell)}(\Omega_{2\mathrm{e}} \cap \mathrm{B}) < \varepsilon$.*

Proof Due to interior Schauder-type estimates for elliptic equations, it suffices to consider $\ell = 0$. By shrinking Ω_2 we can achieve that $\partial\Omega_2 \in \Upsilon^{\infty}$.

Let us assume the opposite. Let $\Omega^* = \Omega_{2\mathrm{e}} \cap B$. Then by the Hahn-Banach theorem there is a function $f^* \in L^2(\Omega^*)$ such that

$$\int_{\Omega^*} uf^* = 0, \tag{4.12}$$

for all functions u, but

$$\int_{\Omega^*} u_1 f^* \neq 0, \tag{4.13}$$

for some functions u_1. We will extend f^* outside Ω^* as zero.

To obtain a contradiction, we introduce a ball B^* centered at the origin and contained in Ω_1. Since there is a unique radiating solution $u^* \in H_2$ to the equation $(\nabla^2 + k^2)u^* = f^*$ in B_e^* with zero Dirichlet data $u^* = 0$ on ∂B^*, we can find radiating solutions from the Green's formula (4.11),

$$-\int_{\Omega^*} u f^* = \int_{\partial B^*} u \partial_\nu u^*. \tag{4.14}$$

Using Eq. (4.11) and completeness of u in $L^2(\partial B^*)$, we conclude that $\partial_\nu u^* = 0$ on ∂B^*.

Since u^* solves the Helmholtz equation (4.6) in the connected open set $\Omega_2 \backslash \overline{B}^*$, we can conclude from the uniqueness in the Cauchy problem for elliptic equations [75] that $u^* = 0$ on $\Omega_2 \backslash \overline{B}^*$. Now applying again the Green's formula (4.11) to the radiating solutions u_1 and u^*, we obtain

$$\int_{\Omega^*} u_1 f^* = \int_{\partial \Omega_{2e}} (u^* \partial_\nu u_1 - u_1 \partial_\nu u^*) = 0 \tag{4.15}$$

which contradicts Eq. (4.13).

Proof of Theorem 4.1 By extension theorems for Sobolev space in Lipschitz domains, there is an extension $u^* \in H_{(1)}(B)$ of u onto $\mathbf{R}^3$. Let $f^* = \nabla^2 u^* + k^2 u^*$. Then $f^* \in H_{(-1)}(\mathbf{R}^3)$ and supp$f^* \subset \overline{\Omega}$. It is known that $f^* = f_0 + \sum_j \partial_j f_j$ for some $f_0, \ldots, f_3 \in L^2(\mathbf{R}^3)$ that are supported in $\overline{\Omega}$. Let χ_n be a sequence of measurable functions with values 0 or 1 supported in Ω and pointwise convergent to 1 on Ω. Then f_n^* defined as $f_0 \chi_n + \sum_j \partial_j \left(f_j \chi_n \right)$ will converge to f in $H_{(-1)}(\mathbf{R}^3)$ with $\sup \mathrm{p} f_n^* \subset \Omega$. From the theories of elliptic equations and scattering [76], it follows that radiating solutions to the Helmholtz equation $(\Delta + k^2)u_{\mathrm{n}}^* = f_n^*$ in $\mathbf{R}^3$ converge to u in $H_{(1)}(B \backslash \Omega)$ for any ball B. So one can write u_{n} such that

$$\left|\left| u - u_{\mathrm{n}} \right|\right|_{(\ell)}(B \backslash \Omega) < \frac{\varepsilon}{2}. \tag{4.16}$$

By the Runge property for scattering solutions in $\mathbf{R}^3 \backslash \overline{B}_0$ (Lemma 1), there is a radiating solution u_ε to the Helmholtz equation outside $\overline{B}_0$ such that

$$\left|\left|u_{\mathrm{n}} - u_\varepsilon\right|\right|_{(\ell)}(B\backslash\Omega) < \frac{\varepsilon}{2}. \tag{4.17}$$

From Eqs. (4.16) and (4.17) we obtain Eq. (4.10).

The proof is complete.

In practice it is very helpful to use a special family of radiating solutions to the Helmholtz equation ε_{mn}, which are expressible as

$$e_{mn}(r, \theta, \phi; \omega) = h_n^{(1)}(kr)Y_n^m(\cos\theta), \tag{4.18}$$

where $h_n^{(1)}$ represent the spherical Hankel functions of the first kind, and $Y_n^m(\cos\theta)$ stand for the spherical harmonics orthonormal in $L^2(S^2)$ on a unit sphere S^2. It is convenient to approximate the solution u to the Helmholtz equation by a linear combination of

$$u_{\mathrm{e}}(r, \theta, \phi; \omega) = \sum_{n=0}^{N}\sum_{m=-n}^{n} C_{mn}e_{mn}(r, \theta, \phi; \omega), \tag{4.19}$$

where C_{mn} are the expansion coefficients to be determined.

Corollary 4.1 *Let $0 \in \Omega$.*

For any positive ε there is u_ε such that

$$\left|\left|u - u_{\mathrm{e}}\right|\right|_{(1)}(B\backslash\Omega) < \varepsilon. \tag{4.20}$$

Proof By Theorem 4.1 there is a radiating solution u_ε to the Helmholtz equation in $\mathbf{R}^3\backslash\overline{B}_0$ so that $\|u - u_\varepsilon\|_{(1)}(B\backslash\Omega) < \varepsilon/2$. Let B_1 be a ball of radius r_1 ($r_1 > r_0$) centered at 0 such that $\overline{B}_1 \subset \Omega$. The spherical harmonics $Y_n^m(\cos\theta)$ form an orthonormal basis in $L^2(S^2)$. Expanding the function u_ε at r_1 with respect to this basis, we can conclude that the partial sums of the corresponding series are convergent in $L^2(\partial B_1)$ and therefore, due to the known results of these series (Theorem 2.14 in Ref. [77]), these partial sums are convergent to u_ε on $B\backslash\Omega$ in $H_1(B\backslash\Omega)$. Consequently, we can find a partial sum u_ε such that $\|u_\varepsilon - u_e\|_{(1)}(B\backslash\Omega) < \varepsilon/2$, and the claim follows from the triangle inequality.

A similar result is valid for interior problems.

Theorem 4.2 *Let $u \in H_1(\Omega)$ be a solution to the Helmholtz equation (4.6) in Ω. Then for a positive ε, there is a solution u_ε to the Helmholtz equation in $\boldsymbol{R}^3$ such that*

$$\left|\left|u - u_{\mathrm{e}}\right|\right|_{(1)}(\Omega) < \varepsilon. \tag{4.21}$$

For interior problems a partial family of useful solutions can be spanned by the functions

$$E_{mn}(r,\theta,\phi;\omega) = j_n(kr)Y_n^m(\cos\theta). \tag{4.22}$$

Now we discuss how to use these results to approximate u *via* u_ε.

Let $\varepsilon = 1$ *in Eq. (4.21). Since for* $\varepsilon < 1$*, there are approximate functions* u_ε *such that*

$$||u_e||_{(1)}(\Omega_0\backslash\Omega) \leq M_1 = ||u||_{(1)}(B\backslash\Omega) + 1. \tag{4.23}$$

Replacing u_ε *by its definition, we have*

$$\int\limits_{\Omega_0\backslash\Omega}\left(\left|\sum_{n=0}^{N}\sum_{m=-n}^{n} C_{mn}e_{mn}(r,\theta,\phi;\omega)\right|^2 + \left|\sum_{n=0}^{N}\sum_{m=-n}^{n} C_{mn}\nabla e_{mn}(r,\theta,\phi;\omega)\right|^2\right)d(r,\theta,\phi) \leq M_1^2, \tag{4.24}$$

where $d(r, \theta, \phi)$ represents integrations over the source region in Ω_0. Since input data are given on Γ_0, we can approximate u via u_ε by solving a minimization problem,

$$\min_{u_e}||u - u_e||_{(0)}(\Gamma_0) \tag{4.25}$$

subject to the constraint (4.23). By solving this problem for sufficiently large $N = N(\delta)$, we find $u_\varepsilon(;\delta)$ such that

$$||u - u_e(;\delta)||_{(0)}(\Gamma_0) < \delta, \tag{4.26}$$

so that the constraint (4.23) holds.

Lemma 4.2 *Let* Ω_0 *be a bounded domain,* $\overline{\Omega} \subset \Omega_0$*. Let either* $\Gamma_0 = \partial\Omega_0$ *or* $\partial\Omega_0$ *be analytic and* Γ_0 *be a non-void open part of* $\partial\Omega_0$*. Let* $\Omega_1 \subset \Omega_0$*. Then there is a function* $\omega(\delta) \to 0$ *as* $\delta \to 0$ *such that*

$$||u - u_e(;\delta)||_{(0)}(B\backslash\Omega_1) < \omega(\delta). \tag{4.27}$$

In addition, if $\overline{\Omega} \subset \Omega_1$*, then one can choose* $\omega(\delta) = (C/d^2)M_1^{1-\theta}\delta^\theta$ *[*$\theta \in (0, 1), \theta > d/\Upsilon$*, where d is the distance from* $\partial\Omega_1$ *to* Ω*]; and if* $\Omega_1 = \Omega_0$*, then one can let* $\omega(\delta) = M(-\Upsilon/\log\delta)^{1/4}$.

Proof We use the Carleman-type estimates (Sect. 3.3 in Ref. [75]) for the Helmholtz operator,

$$\sum_{\alpha\leq 1}\int_B \tau|\partial^{\alpha}u_0|^2 \mathrm{e}^{2\tau\phi} \leq C\int_B \left|\nabla^2 u_0 + k^2 u_0\right|^2 \mathrm{e}^{2\tau\phi} \tag{4.28}$$

for any $H_{(2)}(B)$-function u_0 compactly supported in B and $0 < \tau$. Here $\varphi \in \Upsilon^2(\mathbf{R}^3)$ is the so-called strongly pseudo-convex function for the Laplace operator in $\mathbf{R}^3$ (see Sects. 3.2 and 3.3 in Ref. [75]). In particular $\nabla\varphi \neq 0$ on $\overline{\Omega}$. We can show that there is such a function satisfying the conditions $\varphi = 0$ on $\partial\Omega$, $\varphi > 0$ on $B\backslash\Omega$ (see Sect. 3.3 in Ref. [75]).

Let $\delta = \|u - u_{\mathrm{e}}\|_{(0)}(\Gamma_0)$. First, we consider the case in which $\Gamma_0 = \partial\Omega_0$. Let Ω_2 be a domain containing $\overline{\Omega}_0$ with $\overline{\Omega}_2 \subset B$. By using the Green's function for the exterior Dirichlet problem for the Helmholtz equation (4.6) in Ω_0 we conclude that

$$\left|\left|u - u_e(;\delta)\right|\right|_{(1)}(B\backslash\Omega_2) < C\delta. \tag{4.29}$$

To obtain an interior bound, we introduce the cutoff function $\chi \in \Upsilon^{\infty}$, which is 1 on $\Omega_2\backslash\Omega(d)$ and which is supported in $B\backslash\overline{\Omega}$. We utilize χ with $0 \leq \chi \leq 1$, $|\nabla^j\chi| \leq d^{-j}$, when $j = 1$ and 2, where $\Omega(d) = B\cap\{\varphi < d\}$. Observe that due to our choice of φ, $d(r, \theta, \phi)/\Upsilon < \varphi(r, \theta, \phi) < \Upsilon d(r, \theta, \phi)$, thus we can obtain χ with the above bounds. Let $u_0 = \chi(u - u_{\mathrm{e}})$. Then u_0 is compactly supported in B. Using Eq. (4.28) and the equality $(\nabla^2 + k^2)u_0 = 2\nabla(u - u_e)\cdot\nabla\chi + (u - u_e)\nabla^2\chi$, we have

$$\begin{aligned}
&\sum_{\alpha\leq 1}\tau \int_{\Omega_0\backslash\Omega(2d)} \left(\left|\nabla(u - u_e)\right|^2 + \left|u - u_e\right|^2\right)\mathrm{e}^{2\tau\phi} \\
&\leq C \int_{\left(\Omega(d)\backslash\Omega\right)\cap\left(B\backslash\Omega_2\right)} \left|2\nabla(u - u_e)\cdot\nabla\chi + (u - u_e)\nabla^2\chi\right|^2 \mathrm{e}^{2\tau\phi} \\
&\leq Ce^{2\tau d}\int_{\Omega_0\backslash\Omega}\left(\frac{|\nabla u|^2}{d^2} + \frac{|u|^2}{d^4}\right) + Ce^{2\tau\Phi}\int_{B\backslash\Omega_2}\left(\left|\nabla(u - u_e)\right|^2 + \left|u - u_e\right|^2\right),
\end{aligned} \tag{4.30}$$

where $\Phi = \max\varphi$ over $\overline{B}$ and the inequality $\varphi < d$ on $\Omega(d)\backslash\Omega$ is used. In addition, using $2d < \varphi$ on $\Omega(2d)$ and replacing φ on the left side of Eq. (4.30) by $2d$, we obtain

$$\mathrm{e}^{4\tau d}\left|\left|u - u_{\mathrm{e}}\right|\right|^2_{(1)}(\Omega_0\backslash\Omega(2d)) \leq C\left(\frac{e^{2\tau d}}{d^4}\left|\left|u - u_e\right|\right|^2_{(1)}(\Omega_0\backslash\Omega) + e^{2\tau\Phi}\left|\left|u - u_e\right|\right|^2_{(1)}(B\backslash\Omega_2)\right). \tag{4.31}$$

Dividing the both parts by $\mathrm{e}^{4\tau d}$ and using that due to Eq. (4.23) $\|u - u_e\|_{(1)}(B\backslash\Omega)$, we find

$$\left|\left|u-u_e\right|\right|^2_{(1)}(\Omega_0\backslash\Omega(2d))\leq C\left(\frac{e^{-2\tau d}}{d^4}M_1^2+e^{2\tau(\Phi-2d)}\varepsilon^2\right). \tag{4.32}$$

Minimizing the right side of Eq. (4.32) with respect to $\tau > 0$ yields the minimum point,

$$2\tau=\frac{1}{\Phi-d}\ln\left[\frac{M_1^2}{d^3(\Phi-2d)\varepsilon^2}\right], \tag{4.33}$$

where the value of the minimized function is less than $Cd^{-4}M_1^{2(1-\theta)}\varepsilon^{2\theta}$, here $\theta=\frac{d}{\Phi-d}$.

This proves the interior bound.

To obtain the logarithmic bound, we will split the norm

$$\left|\left|u-u_e\right|\right|^2_{(0)}(\Omega_0\backslash\Omega)=\left|\left|u-u_e\right|\right|^2_{(0)}(\Omega\backslash\Omega(d))+\left|\left|u-u_{\mathrm{e}}\right|\right|^2_{(0)}(\Omega(d)\backslash\Omega). \tag{4.34}$$

By the Hoelder inequality the second term on the right side of Eq. (4.34) is

$$\begin{aligned}\int\limits_{\Omega(d)\backslash\Omega}1\left|u-u_e\right|^2&\leq\left(\int\limits_{\Omega(d)\backslash\Omega}1\right)^{\frac{2}{3}}\left(\int\limits_{\Omega(d)\backslash\Omega}\left|u-u_{\mathrm{e}}\right|^6\right)^{\frac{1}{3}}\\&\leq Cd^{\frac{2}{3}}\left|\left|u-u_{\mathrm{e}}\right|\right|^2_{(1)}(\Omega_0\backslash\Omega),\end{aligned} \tag{4.35}$$

by Sobolev embedding theorems (see the Appendix of Ref. [75]). Now using the interior bound, we obtain

$$\left|\left|u-u_{\mathrm{e}}\right|\right|^2_{(0)}(\Omega_0\backslash\Omega)\leq C\left[\frac{1}{d^4}M_1^2\left(\frac{\delta}{M_1}\right)^{\frac{d}{C}}+M_1^2d^{\frac{2}{3}}\right]. \tag{4.36}$$

Letting $d=[-\ln(\delta/M_1)]^{-3/4}$, we conclude that Eq. (4.36) can be rewritten as

$$\left|\left|u-u_e\right|\right|^2_{(0)}(\Omega_0\backslash\Omega)\leq CM_1^2\left(L^3e^{-\frac{L}{C}}+L^{-2}\right), \tag{4.37}$$

where $L=[-\ln(\delta/M_1)]^{1/4}$. Using $L^3e^{-L/C}\leq CL^{-2}$, we complete the proof of a logarithmic bound.

The case of analytic $\partial\Omega_0$ is similar. We only have to observe that due to known conditional stability estimates of the analytic continuation for the analytic (in some two-dimensional complex neighborhood of the analytic surface $\partial\Omega_0$) function $u-u_e$ from Γ_0 to $\partial\Omega_0$ (see Corollary 1.2.2 of Ref. [78]), we have

$$\left|\left|u-u_{\mathrm{e}}\right|\right|_{(0)}(\partial\Omega_0)\leq C\left|\left|u-u_{\mathrm{e}}\right|\right|_{(0)}^{\theta_1}(\Gamma_0). \tag{4.38}$$

Here, as in Sect. 6.3 of Ref. [75], the neighborhood of $\partial\Omega_0$ and a bound of the complex-analytic continuation onto this neighborhood depend only on Ω, $\partial\Omega_0$, and M_1. The bound in Ref. [75] is given for a plane domain and for a function of a single complex variable, but by using the local analytic coordinates and continuation in each of two coordinate variables, we obtain the same bound on $\partial\Omega_0$. After that we proceed as above.

The proof is complete.

By some standard but more complicated argument, we can replace the exponent 1/4 in the logarithmic bound by any value smaller than 1. Also we can demonstrate that the interior bound of Lemma 4.2 holds when the bound (4.23) in $H_{(1)}(\Omega\backslash\Omega_0)$ is weakened to the following,

$$\|u_{\mathrm{e}}\|_{(0)}(\Omega_0\backslash\Omega)\leq M_0, \tag{4.39}$$

and correspondingly the constraint (4.24) is weakened to the bound,

$$\int\limits_{\Omega_0\backslash\Omega}\left|\sum_n\sum_m u_{n,m}e_{n,m}(x)\right|^2 dx\leq M_0^2,\ \ m=0,\ldots,2n+1\ \text{ and }\ n$$
$$=0,\ldots,N. \tag{4.40}$$

The approximation and stability results in this section suggest the following strategies for finding the approximate solution u_{e}. First, guess N, which is the number of the expansion term for radial functions. Next, find the convex constraint minimization, namely, Eqs. (4.25) and (4.24). Note that sometimes it might be easier to solve Eqs. (4.25) and (4.39) instead.

4.4 Significance of the Justification

The rigorous mathematical justification of the HELS formulations provided by Isakov and Wu is significant in that:

1. It demonstrates that any radiating solution to the Helmholtz equation outside a bounded Lipschitz domain with connected complement can be approximated by using a family of special solutions.
2. Using these approximations and conditional stability estimates in the Cauchy problem for the Helmholtz equation, these special solutions are proven to be bounded outside a vibrating surface and converge to the exact solution, provided that they converge to the exact solution on the measurement surface.

3. Moreover, the estimates of convergence of Hölder and logarithmic types in different regions are derived.

Isakov and Wu's work has provided definitive answers to the question of the validity of the HELS formulations: one can indeed use the spherical wave functions to approximate an acoustic field on a non-spherical surface. This conclusion also holds for the interior region.

The most significant impact of Isakov and Wu's work on HELS method is the suggestion of an effective regularization to overcome ill-posedness difficulty inherent in all inverse acoustic problems. Specifically, Isakov and Wu propose a regularization technique using quasi-solutions [79] by imposing a limit on the growth of reconstructed acoustic quantities in the entire exterior region, including the source surface S. Using the same symbols as those defined in Chap. 3, we obtain

$$\left|\left|\hat{p}_j\left(\vec{x}_S;\omega\right)\right|\right|_2^2 \equiv \iint_S \left|\hat{p}_J\left(\vec{x}_S;\omega\right)\right|^2 \leq K^2, (r_s,\theta_s,\phi_s;\omega) \in S, \tag{4.41}$$

where $j = 1$ to J_{op}. The constant on the right side of Eq. (4.41) has been shown to be correlated to the time-averaged acoustic power [71], which is a constant for any given source and is independent of measurement locations, or be correlated to the propagating component of the acoustic pressure,

$$K = \max_{(r_m,\theta_m,\phi_m)\in\Gamma;\ m=1,2,\ldots,M} \left|\hat{p}\left(r_m,\theta_m,\phi_m;\omega\right)\right| \left(\frac{r_m}{a}\right), \tag{4.42}$$

where a is the characteristic radius of the source surface.

Using this constraint on the source surface together with an iteration scheme to obtain J_{op}, Semenova and Wu [72] illustrate unambiguously that reconstruction errors remain finite everywhere including the source surface, whereas errors in reconstruction using HELS with the least-squares minimization alone can grow to an unsatisfactory level as the reconstruction point approaches the source surface. This explains why sometimes the accuracy of reconstruction on the surface of an arbitrarily shaped structure may be unsatisfactory.

By the way, the validity of using the spherical wave functions and spherical harmonics to reconstruct the acoustic quantities on a non-spherical surface was investigated by Prager [80] as well. In particular, Prager proposed a method to approximate the sound field not fulfilling the Rayleigh hypothesis by transforming a non-converging spherical wave function expansion to a converging one.

Isakov and Wu's theory has laid a solid foundation for the HELS method, answered any questions surrounding its validity in reconstructing acoustic radiation from an arbitrary object and provided the stability estimates for regularizing an ill-posed inverse acoustic problem. The work described in [71, 72] further reveals the interrelationship between a Rayleigh series and HELS solution and most significantly demonstrates that HELS solutions are convergent with bounded errors whenever a surface constraint condition is imposed.

Problems

4.1. What is the Rayleigh hypothesis? What does it attempt to do?

4.2. Consider the solution to the Helmholtz equation that describes the standing waves inside a spherical surface as given by Eq. (2.21a)

$$\hat{p}\,(r,\theta,\phi;\omega)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A_{mn}j_n(kr)+B_{mn}y_n(kr)\right]Y_n^m\left(\cos\theta\right)$$

Does this solution subject to the Rayleigh hypothesis? In other words, will this formulation work if the interior surface is corrugated, namely, not exactly spherical?

4.3. Consider the solution to the Helmholtz equation that describes the traveling waves outside a spherical surface as given by Eq. (2.21b). Suppose that this infinite series is truncated to a finite one as follows,

$$\hat{p}\,(r,\theta,\phi;\omega)=\sum_{n=0}^{N}\sum_{m=-n}^{n}\left[A_{mn}h_n^{(1)}(kr)+B_{mn}h_n^{(2)}(kr)\right]Y_n^m\left(\cos\theta\right),$$

and regularization is applied to the expansion. Will this modified solution subject to the Rayleigh hypothesis? Will it be applicable to a corrugated, namely, not exactly spherical surface?

4.4. What are the differences between the Rayleigh hypothesis and HELS formulations?

4.5. Will the HELS formulations be subject to the same restrictions as the Rayleigh hypothesis does?

4.6. How are the HELS formulations related to the Rayleigh hypothesis?

4.7. What does the mathematical justification prove for the HELS formulations?

4.8. What is the significance of this mathematical justification?