Chapter 2
The Spherical Wave Functions

All acoustic radiation problems can be boiled down to solving the wave equation subject to certain initial and boundary conditions. For a constant frequency case, the problem reduces to solving the Helmholtz equation [39], $\nabla^2 \hat{p} + k^2 \hat{p} = 0$, subject to certain boundary conditions on the source surface. This sounds simple but in reality the analytic solution to the Helmholtz equation exists only for certain types of source geometry that the Helmholtz equation is separable. In most engineering applications the source geometry is arbitrary, so the analytic solution to the Helmholtz equation cannot be found. In these circumstances numerical or approximate solutions are sought.

In this chapter spherical source geometry is considered. Accordingly, the analytic solution to the Helmholtz equation is expressible as the spherical wave functions, which can be obtained by the method of separation of variables. The reasons for choosing the spherical wave functions are (1) the analytic functions are readily available; (2) they are easy to understand; (3) computer codes for the spherical wave functions exist in all software libraries; and (4) they lay the foundation for Chap. 3 of the present book.

2.1 The Helmholtz Equation Under the Spherical Coordinates

Using the spherical coordinates (r, θ, ϕ), we can rewrite the Helmholtz equation as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \hat{p}}{\partial r}\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \hat{p}}{\partial \theta}\right)+\frac{1}{r^2 \sin^2\theta}\frac{\partial \hat{p}^2}{\partial \phi^2}+k^2\hat{p}=0, \tag{2.1}$$

where $k = \omega/c$ is the acoustic wavenumber; ω and c represent the angular frequency and speed of the sound, respectively; the symbol $\hat{p}$ indicates the complex amplitude

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of the acoustic pressure. Note that a time-harmonic function of the form $e^{-i\omega t}$ is assumed in Eq. (2.1).

The solution to Eq. (2.1) can be obtained by using the separation of variable

$$\hat{p}\,(r,\theta,\phi;\omega) = R(kr)\Theta(\theta)\Phi(\phi), \tag{2.2}$$

where $R(kr)$, $\Theta(\theta)$, and $\Phi(\phi)$ are functions of the spherical coordinates r, θ, and ϕ, respectively.

Substituting Eq. (2.2) into (2.1) then leads to three separate ordinary differential equations:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{n(n+1)}{r^2}R = 0, \tag{2.3a}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0, \tag{2.3b}$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0, \tag{2.3c}$$

where n and m are integers associated with the radial function $R(kr)$ and the azimuthal function $\Phi(\phi)$, respectively. In particular, $n = 0, 1, 2, \ldots, \infty$, and $m = -n$ to $+n$, which are discussed later. Throughout this book, we use an italic n to indicate an index and a regular n to depict the unit normal.

2.2 Solution to *R*(*kr*)

The solutions to Eq. (2.3a) are expressible as the spherical Bessel functions of the first and second kinds [40], $j_n(kr)$ and $y_n(kr)$, respectively,

$$R(kr) = A_1 j_n(kr) + A_2 y_n(kr), \tag{2.4a}$$

where A_1 and A_2 are arbitrary constants to be determined by boundary conditions. Alternatively, the solutions to Eq. (2.3a) can be written as

$$R(kr) = B_1 h_n^{(1)}(kr) + B_2 h_n^{(2)}(kr), \tag{2.4b}$$

where B_1 and B_2 are arbitrary constants, and $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$ are the spherical Hankel functions of the first and second kinds [40], respectively. Since $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$ contain the exponential functions of e^{+ikr} and e^{-ikr}, respectively, and since the time-harmonic function in Eq. (2.2) is given by $e^{-i\omega t}$, these spherical Hankel functions depict the outgoing and incoming waves, respectively.

The spherical Hankel functions in Eq. (2.4b) are related to the spherical Bessel functions in Eq. (2.4a). In fact, we can write $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$ as

$$h_n^{(1)}(kr) = j_n(kr) + iy_n(kr), \tag{2.5a}$$

$$h_n^{(2)}(kr) = j_n(kr) - iy_n(kr), \tag{2.5b}$$

where $j_n(kr)$ and $y_n(kr)$ are given by

$$j_n(kr) = \sqrt{\frac{\pi}{2kr}} J_{n+1/2}(kr), \tag{2.6a}$$

$$y_n(kr) = \sqrt{\frac{\pi}{2kr}} Y_{n+1/2}(kr), \tag{2.6b}$$

where $J_{n+1/2}(kr)$ and $Y_{n+1/2}(kr)$ are the first and second kinds of Bessel functions, respectively.

Example 2.1 The first four terms of the first and second kinds of the spherical Bessel functions $j_n(kr)$ and $y_n(kr)$ and spherical Hankel functions $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$ are given, respectively, by

$$j_0(kr) = \frac{\sin(kr)}{(kr)} \quad \text{and} \quad y_0(kr) = -\frac{\cos(kr)}{(kr)};$$

$$j_1(kr) = -\frac{\cos(kr)}{(kr)} + \frac{\sin(kr)}{(kr)^2} \quad \text{and} \quad y_1(kr) = -\frac{\sin(kr)}{(kr)} - \frac{\cos(kr)}{(kr)^2};$$

$$j_2(kr) = -\frac{\sin(kr)}{(kr)} - \frac{3\cos(kr)}{(kr)^2} + \frac{3\sin(kr)}{(kr)^3} \quad \text{and}$$
$$y_2(kr) = \frac{\cos(kr)}{(kr)} - \frac{3\sin(kr)}{(kr)^2} - \frac{3\cos(kr)}{(kr)^3};$$

$$j_3(kr) = \frac{\cos(kr)}{(kr)} - \frac{6\sin(kr)}{(kr)^2} - \frac{15\cos(kr)}{(kr)^3} + \frac{15\sin(kr)}{(kr)^4} \quad \text{and}$$
$$y_3(kr) = \frac{\sin(kr)}{(kr)} + \frac{6\cos(kr)}{(kr)^2} - \frac{15\sin(kr)}{(kr)^3} - \frac{15\cos(kr)}{(kr)^4};$$

$$h_0^{(1)}(kr) = -i\frac{e^{+ikr}}{kr} \quad \text{and} \quad h_0^{(2)}(kr) = +i\frac{e^{-ikr}}{kr};$$

$$h_1^{(1)}(kr) = -\frac{(kr+i)e^{+ikr}}{(kr)^2} \quad \text{and} \quad h_2^{(2)}(kr) = -\frac{(kr-i)e^{-ikr}}{(kr)^2};$$

$$h_2^{(1)}(kr) = +i\frac{\left[(kr)^2 - 3 + i3(kr)\right]e^{+ikr}}{(kr)^3} \quad \text{and} \quad h_2^{(2)}(kr)$$

$$= -i\frac{\left[(kr)^2 - 3 - i3(kr)\right]e^{-ikr}}{(kr)^3};$$

$$h_3^{(1)}(kr) = +\frac{\left[(kr)^3 - 15(kr) + i6(kr)^2 - i15\right]e^{+ikr}}{(kr)^4} \quad \text{and}$$

$$h_3^{(1)}(kr) = +\frac{\left[(kr)^3 - 15(kr) - i6(kr)^2 + i15\right]e^{-ikr}}{(kr)^4}.$$

In general, we can write

$$j_n(kr) = (-1)^n \frac{d^n}{d(kr)^n}\left[\frac{\sin(kr)}{(kr)}\right] \quad \text{and} \quad y_n(kr) = (-1)^{n+1}\frac{d^n}{d(kr)^n}\left[\frac{\cos(kr)}{(kr)}\right];$$

$$h_n^{(1)}(kr) = (-1)^n \frac{d^n h_0^{(1)}(kr)}{d(kr)^n} \quad \text{and} \quad h_n^{(2)}(kr) = (-1)^n \frac{d^n h_0^{(2)}(kr)}{d(kr)^n}.$$

Example 2.2 The asymptotic forms of the first and second kinds of the spherical Bessel functions, $j_n(kr)$ and $y_n(kr)$, and the spherical Hankel functions, $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$, and their derivatives as $kr \rightarrow 0$ are given, respectively, by

$$j_n(kr) \approx \frac{(kr)^n}{(2n+1)!!}\left[1 - \frac{(kr)^2}{2(2n+3)} + \cdots\right] \quad \text{and}$$

$$y_n(kr) \approx -\frac{(2n-1)!!}{(kr)^{n+1}}\left[1 - \frac{(kr)^2}{2(1-2n)} + \cdots\right];$$

$$\frac{dj_n(kr)}{d(kr)} \approx \frac{(kr)^{n-1}}{(2n+1)!!}\left[n - \frac{(n+2)(kr)^2}{2(2n+3)} + \cdots\right] \quad \text{and}$$

$$\frac{dy_n(kr)}{d(kr)} \approx \frac{(2n-1)!!}{(kr)^{n+2}}\left[(n+1) - \frac{(n-1)(kr)^2}{2(1-2n)} + \cdots\right];$$

$$h_n^{(1)}(kr) \approx -i\frac{(2n-1)!!}{(kr)^{n+1}} \quad \text{and} \quad h_n^{(2)}(kr) \approx +i\frac{(2n-1)!!}{(kr)^{n+1}};$$

$$\frac{dh_n^{(1)}(kr)}{d(kr)} \approx +i\frac{(n+1)(2n-1)!!}{(kr)^{n+2}} \quad \text{and} \quad \frac{dh_n^{(2)}(kr)}{d(kr)} \approx -i\frac{(n+1)(2n-1)!!}{(kr)^{n+2}};$$

where

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!} \quad \text{and} \quad (2n-1)!! = \frac{(2n-1)!}{2^n n!}.$$

These results show that the first kind of the spherical Bessel functions $j_n(kr)$ and their derivatives are bounded, while the second kind of the spherical Bessel functions $y_n(kr)$ and their derivatives grow without a bound as $kr \to 0$. Because the spherical Hankel functions contain $y_n(kr)$, they are unbounded as well as $kr \to 0$.

Example 2.3 The asymptotic expressions of the spherical Hankel functions of the first and second kinds, $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$, and their derivatives as $kr \to \infty$ are given, respectively, by

$$h_n^{(1)}(kr) \approx (-i)^{n+1} \frac{\mathrm{e}^{ikr}}{(kr)} \quad \text{and} \quad h_n^{(2)}(kr) \approx (+i)^{n+1} \frac{\mathrm{e}^{-ikr}}{(kr)};$$
$$\frac{dh_n^{(1)}(kr)}{d(kr)} \approx (-i)^n \frac{\mathrm{e}^{ikr}}{(kr)} \quad \text{and} \quad \frac{dh_n^{(2)}(kr)}{d(kr)} \approx (+i)^n \frac{\mathrm{e}^{-ikr}}{(kr)}.$$

Note that for large real arguments, $kr \to \infty$, one cannot write the true asymptotic forms for Bessel functions of the first and second kinds because they are oscillatory and have zeros all the way to infinity, making it impossible to be matched exactly by any asymptotic expansion.

Example 2.4 The recursion relationships for the first and second kinds of the spherical Bessel functions, $j_n(kr)$ and $y_n(kr)$, and their derivatives are given, respectively, by

$$j_n(kr) = \frac{(2n-1)}{(kr)} j_{n-1}(kr) - j_{n-2}(kr) \quad \text{and}$$
$$y_n(kr) = \frac{(2n-1)}{(kr)} y_{n-1}(kr) - y_{n-2}(kr);$$

where $n \geq 2$. For $n = 0$ and 1, $j_n(kr)$ and $y_n(kr)$ are given in Example 2.1.

$$\frac{dj_n(kr)}{d(kr)} = \left(\frac{n}{kr}\right) j_n(kr) - j_{n+1}(kr) \quad \text{and} \quad \frac{dy_n(kr)}{d(kr)} = \left(\frac{n}{kr}\right) y_n(kr) - y_{n+1}(kr).$$

Example 2.5 The recursion relationships for the first and second kinds of the spherical Hankel functions, $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$, and their derivatives are given, respectively, by

$$h_n^{(1)}(kr) = \left(\frac{kr}{2n+1}\right)\left[h_{n-1}^{(1)}(kr) + h_{n+1}^{(1)}(kr)\right] \quad \text{and}$$
$$h_n^{(2)}(kr) = \left(\frac{kr}{2n+1}\right)\left[h_{n-1}^{(2)}(kr) + h_{n+1}^{(2)}(kr)\right];$$
$$\frac{dh_n^{(1)}(kr)}{d(kr)} = h_{n-1}^{(1)}(kr) - \left(\frac{n+1}{kr}\right)h_n^{(1)}(kr) \quad \text{and}$$
$$\frac{dh_n^{(2)}(kr)}{d(kr)} = h_{n-1}^{(2)}(kr) - \left(\frac{n+1}{kr}\right)h_n^{(2)}(kr);$$

where $n \geq 1$. For $n = 0$, $h_0^{(1)}(kr)$ and $h_0^{(2)}(kr)$ are given in Example 2.1, and their derivatives are

$$\frac{dh_0^{(1)}(kr)}{d(kr)} = \left(\frac{kr+i}{kr}\right)\frac{e^{ikr}}{kr} \quad \text{and} \quad \frac{dh_0^{(2)}(kr)}{d(kr)} = \left(\frac{kr-i}{kr}\right)\frac{e^{-ikr}}{kr}.$$

2.3 Solution to $\Theta(\theta)$

The solutions to Eq. (2.3b) can be written as the Legendre functions of the first and second kinds, respectively,

$$\Theta(\theta) = C_1 P_n^m(\cos\theta) + C_2 Q_n^m(\cos\theta), \tag{2.7}$$

where C_1 and C_2 are arbitrary constants. Note that the second kind of the Legendre functions is unbounded at the poles $\cos\theta = \pm 1$, and must be discarded by setting $C_2 = 0$. Also note that n is an integer and the first kind of the Legendre function $P_n^m(\cos\theta) = 0$ whenever $m > n$. Since m ranges from $-n$ to $+n$ and since $P_n^{-m}(\cos\theta)$ is related to $P_n^m(\cos\theta)$ through

$$P_n^{-m}(\cos\theta) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta), \tag{2.8}$$

where m is positive, we only need to be concerned with $P_n^m(\cos\theta)$, which is expressible as [41]

$$P_n^m(\cos\theta) = (-1)^m \sin^m\theta \frac{d^m P_n(\cos\theta)}{d(\cos\theta)^m}, \tag{2.9}$$

where $P_n(\cos\theta)$ is given by an infinite series known as the Legendre polynomials,

$$P_n(\cos\theta) = \frac{(2n-1)!!}{n!}\left[\cos^n\theta - \frac{n(n-1)}{2(2n-1)}\cos^{(n-2)}\theta + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}\cos^{(n-4)}\theta - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6(2n-1)(2n-3)(2n-5)}\cos^{(n-6)}\theta + \cdots\right], \tag{2.10}$$

where $n = 0, 1, 2, \ldots, \infty$.

The Legendre polynomials $P_n(\cos\theta)$ are orthogonal, namely,

$$\int_{-1}^{+1} P_{n'}(\cos\theta)P_n(\cos\theta)d(\cos\theta) = \left(\frac{2}{2n+1}\right)\delta_{n'n}, \tag{2.11}$$

where $\delta_{n'n}$ is the Kroneker delta function,

$$\delta_{n'n} = \begin{cases} 1, & \text{if } n = n' \\ 0, & \text{if } n \neq n' \end{cases}. \tag{2.12}$$

Since $P_n^m(\cos\theta)$ is related to the Legendre polynomials, a similar orthogonal condition exists for $P_n^m(\cos\theta)$ for any positive value of m,

$$\int_{-1}^{+1} P_{n'}^m(\cos\theta)P_n^m(\cos\theta)d(\cos\theta) = \left(\frac{2}{2n+1}\right)\frac{(n+m)!}{(n-m)!}\delta_{n'n}, \tag{2.13}$$

Example 2.6 The first five terms ($n = 0$, 1, 2, 3, and 4) of the first kind of the Legendre functions are given as follows:

For $n = 0$, $P_0^0(\cos\theta) = 1$.

For $n = 1$, $P_1^{-1}(\cos\theta) = \frac{\sin\theta}{2}$, $P_1^0(\cos\theta) = \cos\theta$, and $P_1^1(\cos\theta) = -\sin\theta$.

For $n = 2$, $P_2^{-2}(\cos\theta) = \frac{\sin^2\theta}{8}$, $P_2^{-1}(\cos\theta) = \frac{\cos\theta\sin\theta}{2}$, $P_2^0(\cos\theta) = \frac{1+3\cos 2\theta}{4}$, $P_2^1(\cos\theta) = -3\cos\theta\sin\theta$, and $P_2^2(\cos\theta) = 3\sin^2\theta$.

For $n = 3$, $P_3^{-3}(\cos\theta) = \frac{\sin^3\theta}{48}$, $P_3^{-2}(\cos\theta) = \frac{\cos\theta\sin^2\theta}{8}$, $P_3^{-1}(\cos\theta) = \frac{(3+5\cos 2\theta)\sin\theta}{16}$, $P_3^0(\cos\theta) = \frac{-3\cos\theta+5\cos^3\theta}{2}$, $P_3^1(\cos\theta) = \frac{3(1-5\cos^2\theta)\sin\theta}{2}$, $P_3^2(\cos\theta) = 15\cos\theta\sin^2\theta$, and $P_3^3(\cos\theta) = -15\sin^3\theta$.

For $n = 4$, $P_4^{-4}(\cos\theta) = \frac{7\sin^4\theta}{2688}$, $P_4^{-3}(\cos\theta) = \frac{7\cos\theta\sin^3\theta}{336}$, $P_4^{-2}(\cos\theta) = \frac{(5+7\cos 2\theta)\sin^2\theta}{96}$, $P_4^{-1}(\cos\theta) = -\frac{(3\cos\theta-7\cos^3\theta)\sin\theta}{2}$, $P_4^0(\cos\theta) = \frac{3-30\cos^2\theta+35\cos^4\theta}{8}$, $P_4^1(\cos\theta) = \frac{5(3\cos\theta-7\cos^3\theta)\sin\theta}{2}$, $P_4^2(\cos\theta) = \frac{15(5+7\cos 2\theta)\sin^2\theta}{4}$, $P_4^3(\cos\theta) = -105\cos\theta\,\sin^4\theta$, and $P_4^4(\cos\theta) = 105\sin^4\theta$.

Example 2.7 The recursion relations for the Legendre functions can be written in different ways, one of them being

$$P_n^m(\cos\theta) = \frac{(n-m+1)}{(2n+1)\cos\theta} P_{n+1}^m(\cos\theta) + \frac{(n+m)}{(2n+1)\cos\theta} P_{n-1}^m(\cos\theta),$$

where $n \geq 1$. For $n = 0$, $P_0^0(\cos\theta) = 1$ as shown in Example 2.5.

Example 2.8 Similarly, there are different recursion relations for the derivatives of the Legendre functions, one of them being

$$\frac{dP_n^m(\cos\theta)}{d(\cos\theta)} = \frac{(n+1)\cos\theta}{\sin\theta} P_n^m(\cos\theta) - \frac{(n-m+1)}{\sin\theta} P_{n+1}^m(\cos\theta).$$

2.4 Solution to $\Phi(\phi)$

The solutions to Eq. (2.3c) are simply harmonic functions of the azimuthal angle

$$\Phi(\phi) = D_1 \mathrm{e}^{im\phi} + D_2 \mathrm{e}^{-im\phi}, \tag{2.14}$$

where D_1 and D_2 are arbitrary constants.

The angular solutions $\Theta(\theta)$ and $\Phi(\phi)$ given by Eqs. (2.7) and (2.14) can be combined into a single function known as the spherical harmonics $Y_n^m(\theta, \phi)$,

$$Y_n^m(\theta, \phi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos\theta) \mathrm{e}^{im\phi}, \tag{2.15}$$

where $-n \leq m \leq n$.

Since $P_n^{-m}(\cos\theta)$ is related to $P_n^m(\cos\theta)$, so is $Y_n^{-m}(\theta, \phi)$ to $Y_n^m(\theta, \phi)$, namely,

$$Y_n^{-m}(\theta, \phi) = (-1)^m Y_n^{m*}(\theta, \phi), \tag{2.16}$$

where a superscript * implies a complex conjugation. The spherical harmonics are orthonormal,

$$\int_0^{2\pi} d\phi \int_0^{\pi} Y_{n'}^{m'}(\theta, \phi) Y_n^{m*}(\theta, \phi) \sin\theta d\theta = \delta_{n'n} \delta_{m'm}, \tag{2.17}$$

The orthonormal characteristics of the spherical harmonics enable us to use them to represent any arbitrary function $f(\theta, \phi)$ on a spherical surface. For example, we can express $f(\theta, \phi)$ as

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm} Y_n^m(\theta,\phi), \tag{2.18}$$

where A_{nm} are the expansion coefficients that can be obtained by using the orthonormal condition (2.17). Multiplying both sides of Eq. (2.17) by the spherical harmonics $Y_n^m(\theta,\varphi)$ and integrating over the solid angle of a sphere, we obtain

$$\begin{aligned}\int_0^{2\pi} d\phi \int_0^{\pi} f(\theta,\phi) Y_n^m(\theta,\phi)^* \sin\theta d\theta &= \int_0^{2\pi} d\phi \int_0^{\pi} \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm} Y_n^m(\theta,\phi) Y_{n'}^{m'}(\theta,\phi)^* \sin\theta d\theta \\ = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm} \int_0^{2\pi} d\phi \int_0^{\pi} Y_n^m(\theta,\phi) Y_{n'}^{m'}(\theta,\phi)^* \sin\theta d\theta &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\delta_{nm}\delta_{n'm'} = A_{nm}.\end{aligned} \tag{2.19}$$

Therefore, the expansion coefficients A_{nm} are given by

$$A_{nm} = \int_0^{2\pi} d\phi \int_0^{\pi} f(\theta,\phi) Y_n^m(\theta,\phi)^* \sin\theta d\theta. \tag{2.20}$$

It is important to point out that the results given by Eqs. (2.18), (2.19), (2.20) are only valid for a spherical source surface.

Example 2.9 The first few terms ($n \le 4$) of the spherical harmonics are given by:

For $n=0$, $Y_0^0(\theta,\varphi) = \frac{1}{2\sqrt{\pi}}$.

For $n=1$, $Y_1^{-1}(\theta,\phi) = \left(\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\right)e^{-i\phi}$, $Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$, and $Y_1^1(\theta,\phi) = -\left(\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\right)e^{i\phi}$.

For $n=2$, $Y_2^{-2}(\theta,\phi) = \left(\frac{3}{4}\sqrt{\frac{5}{6\pi}}\sin^2\theta\right)e^{-i2\phi}$, $Y_2^{-1}(\theta,\phi) = \left(\frac{3}{4}\sqrt{\frac{5}{6\pi}}\sin 2\theta\right)e^{-i\phi}$, $Y_1^0(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$, $Y_2^1(\theta,\phi) = -\left(\frac{3}{4}\sqrt{\frac{5}{6\pi}}\sin 2\theta\right)e^{i\phi}$, and $Y_2^2(\theta,\phi) = \left(\frac{3}{4}\sqrt{\frac{5}{6\pi}}\sin^2\theta\right)e^{i2\phi}$.

For $n=3, Y_3^{-3}(\theta,\phi) = \left(\frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta\right)e^{-i3\phi}, Y_3^{-2}(\theta,\phi) = \left(\frac{15}{4}\sqrt{\frac{7}{30\pi}}\cos\theta\sin^2\theta\right)e^{-i2\phi}$, $Y_3^{-1}(\theta,\phi) = \left[\frac{1}{16}\sqrt{\frac{21}{\pi}}(5\cos 2\theta + 3)\sin\theta\right]e^{-i\phi}$, $Y_3^0(\theta,\phi) = \frac{1}{4}\sqrt{\frac{7}{\pi}}(5\cos^3\theta - 3\cos\theta)$, $Y_3^1(\theta,\phi) = \left[\frac{1}{8}\sqrt{\frac{21}{\pi}}(-5\cos^2\theta + 1)\sin\theta\right]e^{i\phi}$, $Y_3^2(\theta,\phi) = \left(\frac{15}{4}\sqrt{\frac{7}{30\pi}}\cos\theta\sin^2\theta\right)e^{i2\phi}$, and $Y_3^3(\theta,\phi) = -\left(\frac{5}{8}\sqrt{\frac{7}{5\pi}}\sin^3\theta\right)e^{i3\phi}$.

For $n = 4$, $Y_4^{-4}(\theta,\phi) = \left(\frac{105}{16}\sqrt{\frac{1}{70\pi}}\sin^4\theta\right)e^{-i4\phi}$,

$$Y_4^{-3}(\theta,\phi) = \left(\frac{105}{8}\sqrt{\frac{1}{35\pi}}\cos\theta\sin^3\theta\right)e^{-i3\phi},$$

$$Y_4^{-2}(\theta,\phi) = \left[\frac{15}{16}\sqrt{\frac{1}{10\pi}}(7\cos 2\theta + 5)\sin^2\theta\right]e^{-i2\phi},$$

$$Y_4^{-1}(\theta,\phi) = \left[\frac{15}{8}\sqrt{\frac{1}{5\pi}}(7\cos^3\theta - 3\cos\theta)\sin\theta\right]e^{-i\phi},$$

$$Y_4^{0}(\theta,\phi) = \frac{3}{16}\sqrt{\frac{1}{\pi}}(35\cos^4\theta - 30\cos^2\theta + 3),$$

$$Y_4^{1}(\theta,\phi) = -\left[\frac{15}{8}\sqrt{\frac{1}{5\pi}}(7\cos^3\theta - 3\cos\theta)\sin\theta\right]e^{i\phi},$$

$$Y_4^{2}(\theta,\phi) = \left[\frac{15}{16}\sqrt{\frac{1}{10\pi}}(7\cos 2\theta + 5)\sin^2\theta\right]e^{i2\phi},$$

$$Y_4^{3}(\theta,\phi) = -\left(\frac{105}{8}\sqrt{\frac{1}{35\pi}}\cos\theta\sin^3\theta\right)e^{i3\phi}, \text{ and}$$

$$Y_4^{4}(\theta,\phi) = \left(\frac{105}{16}\sqrt{\frac{1}{70\pi}}\sin^4\theta\right)e^{i4\phi}.$$

2.5 Solution to $\hat{p}\,(r, \theta, \phi; \omega)$

Combining the radial functions $R(kr)$ and spherical harmonics $Y_n^m(\theta,\phi)$, we can express the solutions to the Helmholtz equation that describe standing waves in an interior region as

$$\hat{p}\,(r,\theta,\phi;\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[A_{mn}j_n(kr) + B_{mn}y_n(kr)]Y_n^m(\cos\theta), \tag{2.21a}$$

or the solutions to the Helmholtz equation that describe traveling waves in an exterior region as

$$\hat{p}\,(r,\theta,\phi;\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A_{mn}h_n^{(1)}(kr) + B_{mn}h_n^{(2)}(kr)\right]Y_n^m(\cos\theta), \tag{2.21b}$$

where the first term on the right side of Eq. (2.21b) depicts the outgoing spherical waves and second term describes the incoming spherical waves.

Now let us consider the examples of using Eq. (2.21) to predict the acoustic pressure fields generated by a vibrating sphere.

Example 2.10 Consider the case of a vibrating sphere of radius $r = a$ in a free field. Let the acoustic pressure on a spherical surface of radius $r = r^{\text{meas}}$ be specified, $\hat{p}\,(r, \theta, \phi; \omega)|_{r=r^{\text{meas}}} = \hat{p}\,(r^{\text{meas}}, \theta, \phi; \omega)$. The acoustic pressure field anywhere including the source surface is desired. This problem can be solved by using Eq. (2.21b). Since this is an exterior problem and the field is unbounded, there are only outgoing waves from the vibrating sphere to infinity. Accordingly, Eq. (2.21b) is rewritten as

$$\hat{p}\,(r, \theta, \phi; \omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn} h_n^{(1)}(kr) Y_n^m(\cos\theta).$$

The expansion coefficients A_{mn} can be specified by using the pressure boundary condition on the spherical surface of radius $r = r^{\text{meas}}$.

$$\hat{p}\,(r^{\text{meas}}, \theta, \phi; \omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn} h_n^{(1)}(kr^{\text{meas}}) Y_n^m(\cos\theta).$$

Multiplying both sides by the complex conjugate of the spherical harmonics, integrating over the solid angle of a sphere, and using the orthonormal property of the spherical harmonics, we obtain

$$\int_0^{2\pi} d\phi \int_0^{\pi} \hat{p}\,(r^{\text{meas}}, \theta, \phi; \omega) Y_n^{m*}(\theta, \phi) \sin\theta d\theta = A_{nm} h_n^{(1)}(kr^{\text{meas}}).$$

Therefore, the expansion coefficients A_{mn} are given by

$$A_{nm} = \frac{1}{h_n^{(1)}(kr^{\text{meas}})} \int_0^{2\pi} d\phi \int_0^{\pi} \hat{p}\,(r^{\text{meas}}, \theta, \phi; \omega) Y_n^{m*}(\theta, \phi) \sin\theta d\theta.$$

Once A_{mn} are specified, the acoustic pressure anywhere is expressible as

$$\hat{p}\,(r, \theta, \phi; \omega) = \sum_{n=0}^{\infty} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(kr^{\text{meas}})} \sum_{m=-n}^{n} Y_n^m(\cos\theta) \int_0^{2\pi} d\phi' \int_0^{\pi} \hat{p}\left(r^{\text{meas}}, \theta', \phi'; \omega\right) Y_n^{m*}\left(\theta', \phi'\right) \sin\theta' d\theta'.$$

Example 2.11 Consider the case of a vibrating sphere of radius $r = a$ in a free field. Assume that the normal surface of this vibrating sphere is given as $\hat{v}_{\text{n}}(a, \theta, \phi; \omega)$. With this boundary condition, we want to predict the radiated acoustic pressure anywhere, including the source surface. From the Euler's equation [42], we can write the boundary condition as

$$\hat{v}_{\mathrm{n}}(a,\theta,\phi;\omega) = \frac{1}{i\omega\rho_0} \frac{\partial \hat{p}\,(r,\theta,\phi;\omega)}{\partial \mathrm{n}}\bigg|_{r=a},$$

where ρ_0 is the ambient density of the medium surrounding the dilating sphere, and a subscript n depicts the unit normal direction.

Since there are only outgoing waves, we can rewrite Eq. (2.21b) as

$$\hat{p}\,(r,\theta,\phi;\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{mn} h_n^{(1)}(kr) Y_n^m(\cos\theta).$$

Take the normal derivative on both sides of the above expression,

$$\frac{\partial \hat{p}\,(r,\theta,\phi;\omega)}{\partial \mathrm{n}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{mn} \frac{dh_n^{(1)}(kr)}{dr} Y_n^m(\cos\theta),$$

where the symbol $\partial/\partial \mathrm{n}$ represents a normal derivative defined as

$$\frac{\partial}{\partial \mathrm{n}} = \vec{\mathrm{n}} \cdot \nabla = \frac{\partial}{\partial r}.$$

Substitute the normal derivative of the acoustic pressure evaluated at $r = a$ yields

$$\hat{v}_{\mathrm{n}}(a,\theta,\phi;\omega) = \frac{1}{i\omega\rho_0}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_{mn} \frac{dh_n^{(1)}(kr)}{dr}\bigg|_{r=a} Y_n^m(\cos\theta).$$

Next, we multiply both sides by the complex conjugate of the spherical harmonics, integrate it over the solid angle of the sphere, and use the orthonormal property of the spherical harmonics,

$$\int_0^{2\pi} d\phi \int_0^{\pi} \hat{v}_{\mathrm{n}}(a,\theta,\phi;\omega) Y_n^{m*}(\cos\theta)\sin\theta d\theta = \frac{A_{mn}}{i\omega\rho_0} \frac{dh_n^{(1)}(kr)}{dr}\bigg|_{r=a}.$$

Therefore, the expansion coefficients A_{mn} are found to be

$$A_{mn} = \frac{i\omega\rho_0}{\left.\frac{dh_n^{(1)}(kr)}{dr}\right|_{r=a}} \int_0^{2\pi} d\phi \int_0^{\pi} \hat{v}_{\mathrm{n}}(a,\theta,\phi;\omega) Y_n^{m*}(\theta,\phi)\sin\theta d\theta.$$

Accordingly, the radiated acoustic pressure is given by

$$\hat{p}\,(r,\theta,\phi;\omega) = i\omega\rho_0 \sum_{n=0}^{\infty} \frac{h_n^{(1)}(kr)}{\left.\frac{dh_n^{(1)}(kr)}{dr}\right|_{r=a}} \sum_{m=-n}^{n} Y_n^m(\cos\theta)$$

$$\int_0^{2\pi} d\phi' \int_0^{\pi} \hat{v}_{\mathrm{n}}\left(a,\theta',\phi';\omega\right) Y_n^{m*}\left(\theta',\phi'\right) \sin\theta' d\theta'.$$

Example 2.12 Let us consider a specific case of a dilating sphere for which $\hat{v}_{\mathrm{n}}(a,\theta,\phi;\omega) \equiv v_0$ is a constant. Accordingly, we set $n = m = 0$ in the above expression. Substituting the derivative of the spherical Hankel functions of the first kind with $n = 0$ (see Example 2.1) yields the expansion coefficient A_{00} as

$$A_{00} = i\rho_0 c v_0 \frac{(ka)^2}{(ka+i)} \mathrm{e}^{-ika}.$$

The corresponding acoustic pressure anywhere is given by

$$\hat{p}\,(r,\theta,\phi;\omega) = \rho_0 c v_0 \left(\frac{ka}{ka+i}\right)\left(\frac{a}{r}\right)\mathrm{e}^{ik(r-a)},$$

which agrees perfectly with the analytic solution [42].

Example 2.13 Next let us consider the acoustic pressure inside a dilating sphere of radius $r = a$. Once again let that the normal surface of this dilating sphere be constant, $v_{\mathrm{n}} = v_0$. In this case we can use Eq. (2.21a) but have to discard the second term involving the second kind of the spherical Bessel function $y_n(kr)$ because it is unbounded at the center $r = 0$. Accordingly, we have

$$\hat{p}\,(r,\theta,\phi;\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{mn} j_n(kr) Y_n^m(\cos\theta)\mathrm{e}^{im\phi},$$

where the expansion coefficients A_{nm} can be determined by the boundary condition together with the orthonormal condition. Take the normal derivative of the acoustic pressure,

$$\frac{\partial\hat{p}\,(r,\theta,\phi;\omega)}{\partial\mathrm{n}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{mn} \frac{dj_n(kr)}{dr} Y_n^m(\cos\theta)\mathrm{e}^{im\phi}.$$

Substitute the normal derivative of the acoustic pressure into the boundary condition,

$$v_0 = \frac{1}{\mathrm{i}\omega\rho_0}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{mn} \left.\frac{dj_n(kr)}{dr}\right|_{r=a} Y_n^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}.$$

Following the same procedures as those outlined in Example 2.11, we obtain

$$\int_0^{2\pi} d\phi \int_0^{\pi} v_0 Y_n^{m*}(\theta,\phi)\sin\theta d\theta = \frac{1}{i\omega\rho_0}\int_0^{2\pi} d\phi \int_0^{\pi} \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\left.\frac{dj_n(kr)}{dr}\right|_{r=a} Y_{n'}^{m'}(\theta,\phi)Y_n^{m*}(\theta,\phi)\mathrm{e}^{im\phi}\sin\theta d\theta.$$

Using the orthogonality property of the spherical harmonics and carrying out the integrations over the solid angle, we obtain

$$v_0 = \frac{1}{i\omega\rho_0}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\left.\frac{dj_n(kr)}{dr}\right|_{r=a}\mathrm{e}^{im\phi}.$$

The left side in the above equation is constant and independent of the angular variables, and the right side can match this if $n = m = 0$. Substituting the derivative of the spherical Bessel function of the first kind given in Example 2.1 with $n = 0$, we obtain

$$v_0 = \frac{A_{00}}{i\rho_0 c}\left.\frac{dj_n(kr)}{d(kr)}\right|_{r=a} = \frac{A_{00}[(ka)\cos(ka) - \sin(ka)]}{i\rho_0 c(ka)^2},$$

from which we found the expansion coefficient A_{00} to be

$$A_{00} = \frac{i\rho_0 c v_0 (ka)^2}{(ka)\cos(ka) - \sin(ka)}.$$

Substituting A_{00} into Eq. (2.21a) yields the interior acoustic pressure field,

$$\hat{p}(r,\theta,\phi;\omega) = \frac{i\rho_0 c v_0 (ka)\sin(kr)}{(ka)\cos(ka) - \sin(ka)}\left(\frac{a}{r}\right).$$

In this case resonance will occur inside the spherical surface when the frequency f is equal to one of the eigenfrequencies obtained by solving the following characteristic equation:

$$\tan\left(\frac{2\pi f_n a}{c}\right) = \frac{2\pi f_n a}{c}, \quad n = 1, 2, 3, \ldots,$$

The above equation is a transcendental equation that can only be solved numerically. The first four eigenfrequencies or the roots of this transcendental equation are:

$$f_1 = 0.715148\left(\frac{c}{a}\right) \text{ (Hz)},$$

$$f_2 = 1.229515\left(\frac{c}{a}\right) \text{ (Hz)},$$

$$f_3 = 1.735446\left(\frac{c}{a}\right) \text{ (Hz)},$$

$$f_4 = 2.238705\left(\frac{c}{a}\right) \text{ (Hz)},$$

Examples 2.10 and 2.11 demonstrate that we can use the expansions of the spherical wave functions to describe exactly the acoustic pressure field generated by a vibrating sphere. When the vibration pattern is arbitrary, the number of expansion terms may be infinite. When the vibrating surface is not spherical however, the expansion given by solution Eq. (21) is invalid. In practice, most vibrating surfaces are non-spherical. Therefore, a different methodology is needed to describe the radiated acoustic field. This is the topic of Chap. 3.

Problems

2.1. Use the recursion relations given in Example 2.4 to write the following spherical Bessel function and spherical Hankel function of the first kind: $j_n^{(1)}(kr)$ and $h_n^{(1)}(kr)$, where $n = 4$, 5, and 6.

2.2. Use the recursion relations given in Example 2.4 to write the derivatives for the following spherical Bessel function and spherical Hankel function of the first kind: $dj_n^{(1)}(kr)/d(kr)$ and $dh_n^{(1)}(kr)/d(kr)$, where $n = 4$, 5, and 6.

2.3. Use the recursion relations given in Example 2.6 and definitions of the Legendre functions to write $P_n^m(\cos\theta)$, where $n = 5$ and $m = -n$ to $+n$.

2.4. Continue Problem 2.2 and write down the spherical harmonics $Y_n^m(\theta, \phi)$, where $n = 5$ and $m = -n$ to $+n$.

2.5. Consider a vibrating sphere of radius $r = a$ in free space. Assume that the acoustic pressure on the spherical surface is given as $\hat{p} = \rho_0 c \hat{v}_z(ka)/(ka + i)$. Determine the radiated acoustic pressure anywhere in free space by using the expansion of the spherical Hankel functions and spherical harmonics.

2.6. Consider a vibrating sphere of radius $r = a$ in free space. Assume that the normal surface velocity of this sphere is a constant $\hat{v}_\text{n} = V_0$. Find the radiated acoustic pressure anywhere in free space via the expansion of the spherical Hankel functions and spherical harmonics.

2.7. Consider a vibrating sphere of radius $r = a$ in free space. Assume that the acoustic pressure on the spherical surface is given as $\hat{p} = \rho_0 c \hat{v}_z(ka)(ka + i)\cos\theta / \left(k^2a^2 - 2 + i2ka\right)$. Find the radiated acoustic pressure anywhere in free space by using the expansion of the spherical Hankel functions and spherical harmonics.

2.8. Consider a vibrating sphere of radius $r = a$ in free space. Assume that the normal surface velocity of this sphere is given as $\hat{v}_{\mathrm{n}} = V_0 \sin\theta$, where θ is the polar angle. Determine the radiated acoustic pressure in free space by using the expansion of the spherical Hankel functions and spherical harmonics. Will the acoustic pressure contain any resonance frequency? Why?

2.9. Consider the acoustic pressure field inside a sphere of radius $r = a$. Assume that the acoustic pressure on the interior surface is constant $\hat{p} = C$. Determine the acoustic pressure field in the interior region by using the expansion of the spherical Hankel functions and spherical harmonics. Will the acoustic pressure contain any resonance frequency? If so, what are these resonance frequencies?

2.10. Consider the acoustic pressure field inside a sphere of radius $r = a$. Assume that the sphere is oscillating back and forth along the x-axis direction, and the normal surface velocity is given as $\hat{v}_z = V_0 \sin\theta$, where θ is the polar angle. Solve the radiated acoustic pressure field in the interior region using the expansion of the spherical Hankel functions and spherical harmonics. Will the acoustic pressure contain any resonance frequency? If so, what are these resonance frequencies?