Sums of squares of Littlewood–Richardson coefficients and GL_n-harmonic polynomials

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To Nolan Wallach, who has influenced generations of mathematicians

Abstract We consider the example from invariant theory concerning the conjugation action of the general linear group on several copies of the $n \times n$ matrices, and examine a symmetric function which stably describes the Hilbert series for the invariant ring with respect to the multigradation by degree. The terms of this Hilbert series may be described as a sum of squares of Littlewood–Richardson coefficients. A "principal specialization" of the gradation is then related to the Hilbert series of the K-invariant subspace in the GL_n-harmonic polynomials (in the sense of Kostant), where K denotes a block diagonal embedding of a product of general linear groups. We also consider other specializations of this Hilbert series.

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1   Introduction

We let F_n^λ denote the finite-dimensional irreducible representation of the general linear group, GL_n, (over $\mathbb{C}$) whose highest weight is indexed by the integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ (in standard coordinates). Given a finite sequence of

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integer partitions $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(m)})$, we will let $c_{\boldsymbol{\mu}}^{\lambda}$ be the multiplicity of F_n^{λ} in the m-fold tensor product $F_n^{\mu^{(1)}} \otimes \cdots \otimes F_n^{\mu^{(m)}}$ under the diagonal action of GL_n. That is, $c_{\boldsymbol{\mu}}^{\lambda}$ denotes a (generalized) Littlewood–Richardson coefficient.

We remark that the usual exposition of Littlewood–Richardson coefficients (see [4, 5, 9–11, 13]) concerns the case where $m = 2$. However, by iterating the Littlewood–Richardson rule (or its equivalents) one obtains several effective combinatorial interpretations of our $c_{\boldsymbol{\mu}}^{\lambda}$.

The subject of this exposition concerns some interpretations of the positive integer $\sum \left(c_{\boldsymbol{\mu}}^{\lambda} \right)^2$ where the sum is over certain finite subsets of nonnegative integer partitions. We believe that such sums have under-appreciated combinatorial significance. For example, one immediately observes the very simple specialization to the case where $\mu^{(j)} = (1)$ for all $j = 1, \ldots, m$, in which case the sum of squares is $m!$, which may be viewed as a consequence of Schur–Weyl duality. More generally, if $\nu = (\nu_1 \geq \cdots \geq \nu_m \geq 0)$ is a partition and $\mu^{(j)} = (\nu_j)$, then $c_{\boldsymbol{\mu}}^{\lambda}$ is equal to the Kostka number $K_{\lambda\nu}$ (i.e., the multiplicity of the weight indexed by ν in F_n^{λ}).

Our motivation for considering these numbers comes from invariant theory. On the one hand, we consider the conjugation action of the general linear group on several copies of the $n \times n$ matrices. On the other hand, we consider the K-conjugation action on one copy of the $n \times n$ matrices, where K denotes a block diagonal embedding of a product of general linear groups. These problems are related, and have been studied extensively. We make no attempt to survey the literature, but recommend [3].

Central to this work is the notion of a *Hilbert series*. Let V be a graded vector space. That is, $V = \bigoplus_{d=0}^{\infty} V_d$ where V_d is a finite-dimensional subspace. The Hilbert series, $\sum_{d=0}^{\infty} (\dim V_d) q^d$, formally records the dimensions of the graded components. Here q is an indeterminate. We also consider multivariate generalizations corresponding to situations where V is graded by a cone in a lattice.

In our setting, V will be a space of invariant polynomial functions on m copies of the $n \times n$ matrices. For fixed values of the parameters, the Hilbert series is the Taylor expansion of a rational function around zero. When these parameters are small, one can expect to write down the numerator and denominator explicitly. These polynomials encode structural information about the invariants. However, as the size of the matrix becomes large, these rational functions are difficult to compute explicitly. This motivates reorganizing the data by studying the coefficients of the Hilbert series of a fixed degree as the size of the matrix goes to infinity. The limit exists. The formal series recording this information will be referred to as the *stable Hilbert series*.

Certain sums of squares of Littlewood–Richardson coefficients describe the coefficients of the (stable) Hilbert series for the invariant algebra in each case. These Hilbert series, stably, may be expressed as a product. Furthermore, a "principal specialization" of this product is then related to the Hilbert series of the K-invariant subspace in the GL_n-harmonic polynomials. [Harmonic in the sense of Kostant (see [12]), which generalizes the usual notion of a harmonic polynomial.]

Unless otherwise stated, we will only need notation for representations with polynomial matrix coefficients, which are indexed by partitions with *nonnegative* integer components. The sum of the parts of a partition ν will be called the *size* (denoted $|\nu|$), while the number of parts will be called the *length* denoted $\ell(\nu)$. As usual, we will also write $\lambda \vdash d$ to mean $|\lambda| = d$. Furthermore, we also adapt the (non-standard) notation that if $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(m)})$ is a finite sequence of partitions, we set $|\boldsymbol{\mu}| = \sum_{j=1}^{m} |\mu^{(j)}|$, and write $\boldsymbol{\mu} \vdash d$ to mean $|\boldsymbol{\mu}| = d$. From a combinatorial point of view, the results involve a specialization of the following

Theorem 1.1 (Main Formula). *Let $t_1, t_2, t_3, \ldots$ denote a countably infinite set of indeterminates. We have*

$$\prod_{k=1}^{\infty} \frac{1}{1 - \left(t_1^k + t_2^k + t_3^k + \cdots \right)} = \sum_{\lambda} \sum_{\boldsymbol{\mu}} \left(c_{\boldsymbol{\mu}}^{\lambda} \right)^2 \boldsymbol{t}^{\boldsymbol{\mu}},$$

where the outer sum is over all partitions λ and the inner sum is over all finite sequences of partitions $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \ldots)$ with $\boldsymbol{t}^{\boldsymbol{\mu}} = t_1^{|\mu^{(1)}|} t_2^{|\mu^{(2)}|} t_3^{|\mu^{(3)}|} \cdots$.

Proof. See Section 7. □

As an application of the main formula, we turn to the space, $\mathcal{H}(\mathfrak{gl}_n)$, of GL_n-harmonic polynomial functions on the adjoint representation (with its natural gradation) by polynomial degree, $\mathcal{H}(\mathfrak{gl}_n) = \bigoplus_{d=0}^{\infty} \mathcal{H}^d(\mathfrak{gl}_n)$. The group GL_n acts on $\mathcal{H}(\mathfrak{gl}_n)$. Note that the constant functions are the only GL_n-invariant harmonic functions. However, if K is a reductive subgroup of GL_n, the space of K-invariant functions is much larger. Consider the example when the group K is the block diagonally embedded copy of $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_m}$ in GL_n, with $n_1 + \cdots + n_m = n$. We will denote this group by $K(\boldsymbol{n})$ where $\boldsymbol{n} = (n_1, \ldots, n_m)$. The purpose of this paper is to relate the dimension of the $K(\boldsymbol{n})$-invariant polynomials in $\mathcal{H}^d(\mathfrak{gl}_n)$ to a sum of squares of Littlewood–Richardson coefficients. See Theorem 5.1 for the precise statement.

We consider this question because the related algebraic combinatorics are particularly elegant, and hence have expository value in connecting harmonic analysis with algebraic combinatorics. However, this example is the tip of an iceberg. Indeed, one can replace GL_n with any algebraic group G (with $\mathfrak{g} = \mathrm{Lie}(G)$) and $K(\boldsymbol{n})$ with any subgroup of G. This area of investigation is wide open and well motivated as an examination of the special symmetries of harmonic polynomials.

In Section 2, we describe a general "answer" to this question when K is a symmetric subgroup of a reductive group G. This answer is not as explicit as would be desired, but applies to any symmetric pair (G, K). The remainder of the paper is related to the $G = \mathrm{GL}_n$ example with $K = K(\boldsymbol{n})$. Note that when $m = 2$ (i.e., $n = (n_1, n_2)$) the pair (G, K) is symmetric, but for $m > 2$ is not. We remark that in the $m = 2$ case, the results presented here were first described in [18]. Our present discussion amounts to a generalization to $m > 2$.

After setting up appropriate notation in Section 3 we provide an interpretation for a description of the Hilbert series of the $K(\boldsymbol{n})$-invariants in the GL_n-harmonic polynomials on $\mathfrak{gl}_n$ in Sections 4 and 5. Chief among these involves sums of squares of

Littlewood–Richardson coefficients. We recall other combinatorial interpretations in Section 6. These interpretations involve counting the conjugacy classes in the general linear group over a finite field.

2   The case of a symmetric pair

Let G denote a connected reductive linear algebraic group over the complex numbers and let $\mathfrak{g}$ be its complex Lie algebra. We have $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$, where $\mathfrak{z}(\mathfrak{g})$ denotes the center of $\mathfrak{g}$, while $\mathfrak{g}_{ss} = [\mathfrak{g}, \mathfrak{g}]$ denotes the semisimple part of $\mathfrak{g}$. A celebrated result of Kostant (see [12]) is that the polynomial functions on $\mathfrak{g}$, denoted $\mathbb{C}[\mathfrak{g}]$, are a free module over the invariant subalgebra, $\mathbb{C}[\mathfrak{g}]^G$, under the adjoint action. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and let Φ and W denote the corresponding set of roots and Weyl group, respectively. Choose a set of positive roots Φ^+, and let $\Phi^- = -\Phi^+$ denote the negative roots. Set $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. For $w \in W$, let $l(w)$ denote the number of positive roots sent to negative roots by w. Fix an indeterminate t. There exist positive integers $e_1 \leq e_2 \leq \cdots \leq e_r$ such that $\sum_{w \in W} t^{l(w)} = \prod_{j=1}^r \frac{1-t^{e_j}}{1-t}$ where r is the rank of $\mathfrak{g}_{ss}$. A consequence of the Chevalley restriction theorem ([2]) is that $\mathbb{C}[\mathfrak{g}]^G$ is freely generated, as a commutative ring, by $\dim \mathfrak{z}(\mathfrak{g})$ polynomials of degree 1, and r polynomials of degree $e_1, \ldots, e_r$. These polynomials are the *basic invariants*, while $e_1, \ldots, e_r$ are called the *exponents* of G.

2.1   Harmonic polynomials

We define the *harmonic polynomials* on $\mathfrak{g}$ by

$$\mathcal{H}_\mathfrak{g} = \left\{ f \in \mathbb{C}[\mathfrak{g}] \mid \Delta(f) = 0 \text{ for all } \Delta \in \mathbb{D}[\mathfrak{g}]^G \right\},$$

where $\mathbb{D}[\mathfrak{g}]^G$ is the space of constant coefficient G-invariant differential operators on $\mathfrak{g}$. In [12], it is shown that as a G-representation $\mathcal{H}_\mathfrak{g}$ is equivalent to the G-representation algebraically induced from the trivial representation of a maximal algebraic torus T in G. Thus, by Frobenius reciprocity, the irreducible rational representations of G occur with multiplicity equal to the dimension of their zero weight space. Moreover, as a representation of G, the harmonic polynomials are equivalent to the regular functions on the nilpotent cone in $\mathfrak{g}$.

The harmonic polynomials inherit a gradation by degree from $\mathbb{C}[\mathfrak{g}]$. Set $\mathcal{H}_\mathfrak{g}^d = \mathcal{H}_\mathfrak{g} \cap \mathbb{C}[\mathfrak{g}]_d$. Thus, $\mathcal{H}_\mathfrak{g} = \bigoplus_{d=0}^\infty \mathcal{H}_\mathfrak{g}^d$. We next consider the distribution of the multiplicity of an irreducible G-representation among the graded components of $\mathcal{H}_\mathfrak{g}$. A solution to this problem was originally due to Hesselink [8], which we recall next.

Let $\wp_t : \mathfrak{h}^* \to \mathbb{N}$ denote Lusztig's q-analog of Kostant's partition function and as always $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of nonnegative integers. That is $\wp_t$ is defined by the equation

$$\prod_{\alpha \in \Phi^+} \frac{1}{1 - te^\alpha} = \sum_{\xi \in Q(\mathfrak{g}, \mathfrak{h})} \wp_t(\xi) e^\xi$$

where $Q(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}^*$ denotes the lattice defined by the integer span of the roots. As usual, e^ξ denotes the corresponding character of T, with $\mathrm{Lie}(T) = \mathfrak{h}$. As usual, we set $\wp(\xi) = 0$ for $\xi \notin Q(\mathfrak{g}, \mathfrak{h})$.

Let $P(\mathfrak{g})$ denote the integral weights corresponding to the pair $(\mathfrak{g}, \mathfrak{h})$. The dominant integral weights corresponding to Φ^+ will be denoted by $P_+(\mathfrak{g})$. Let $L(\lambda)$ denote the (finite-dimensional) irreducible highest weight representation with highest weight $\lambda \in P_+(\mathfrak{g})$. The multiplicity of $L(\lambda)$ in the degree d harmonic polynomials $\mathcal{H}^d(\mathfrak{g})$ will be denoted by $m_d(\lambda)$. Set $m_\lambda(t) = \sum_{d=0}^\infty m_d(\lambda) t^d$. Hesselink's theorem asserts that

$$m_\lambda(t) = \sum_{w \in W} (-1)^{l(w)} \wp_t(w(\lambda + \rho) - \rho).$$

See [17] for a generalization of this result.

We remark that the above formula is very difficult to implement in practice. This is in part due to the fact that the order of W grows exponentially with the Lie algebra rank. Thankfully, only a small number of terms actually contribute to the overall multiplicity. See [7] for a very interesting special case where the number of contributed terms is shown to be a Fibonacci number.

2.2 The K-spherical representations of G

Let (G, K) be a symmetric pair. That is, G is a connected reductive linear algebraic group over $\mathbb{C}$ and $K = \{g \in G \mid \theta(g) = G\}$, where θ is a regular automorphism of G of order two. Since K will necessarily be reductive the quotient, G/K is an affine variety and the $\mathbb{C}$-algebra of regular function $\mathbb{C}[G/K]$ is multiplicity-free as a representation of G. This fact follows from the (complexified) Iwasawa decomposition of G. Put another way, there exists $S \subseteq P_+(\mathfrak{g})$ such that for all $\lambda \in P_+(\mathfrak{g})$ we have

$$\dim L(\lambda)^K = \begin{cases} 1, & \lambda \in S, \\ 0, & \lambda \notin S. \end{cases}$$

Note that the subset S may be read off of the data encoded in the Satake diagram associated to the pair (G, K). The above fact implies that the Hilbert series $H_t(G, K) = \sum_{d=0}^{\infty} h_d t^d$ with $h_d = \dim \mathcal{H}^d(\mathfrak{g})^K$ has the following formal expression:

$$H_t(G, K) = \sum_{w \in W} (-1)^{l(w)} \left(\sum_{\lambda \in S} \wp_t(w(\lambda + \rho) - \rho) \right).$$

This formula seems rather encouraging. Unfortunately, the inner sum is very difficult to put into a closed form for general $w \in W$. This is, in part, a reflection of the fact that the values of $\wp_t$ cannot be determined from a "closed form" expression. However, note that $w(\lambda + \rho) - \rho$ often falls outside of the support of $\wp_t$, and therefore it may be possible to obtain explicit results along these lines. Moreover, the point of this exposition is to advertise that combinatorially elegant expressions may exist. At least this is the case for the pair $(\mathrm{GL}_{n_1 + n_2}, \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2})$, as we shall see.

3 Preliminaries

We let $\mathfrak{gl}_n$ denote the complex Lie algebra of $n \times n$ matrices with the usual bracket, $[X, Y] = XY - YX$, for $X, Y \in \mathfrak{gl}_n$. Let E_{ij} denote the $n \times n$ matrix with a 1 in row i and column j, and 0 everywhere else. The Cartan subalgebra will be chosen to be the span of $\{E_{ii} \mid 1 \leq i \leq n\}$. The dual basis in $\mathfrak{h}^*$ to $(E_{11}, E_{22}, \ldots, E_{nn})$ will be denoted $(\epsilon_1, \ldots, \epsilon_n)$. Choose the simple roots as usual, $\Pi = \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n\}$. Let Φ (resp. Φ^+) denote the roots (resp. positive roots). We will identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the trace form $(H_1, H_2) = \mathrm{Tr}(H_1 H_2)$ (for $H_1, H_2 \in \mathfrak{h}$). The fundamental weights are $\omega_i = \sum_{j=1}^{i} \epsilon_j \in \mathfrak{h}^*$ for $1 \leq i \leq n - 1$. We also set $\omega_n = \sum_{j=1}^{n} \epsilon_j \in \mathfrak{h}^*$. Let $P(\mathrm{GL}_n) = \sum_{j=1}^{n} \mathbb{Z}\omega_j$, and $P_+(\mathrm{GL}_n) = \mathbb{Z}\omega_n + \sum_{j=1}^{n-1} \mathbb{N}\omega_j$.

From this point on, we will write $(a_1, \ldots, a_n)$ for $\sum a_i \epsilon_i$. Thus, we have $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_+(\mathrm{GL}_n)$ iff each λ_i in an integer and $\lambda_1 \geq \cdots \geq \lambda_n$. The finite-dimensional irreducible representation of GL_n with highest weight λ will be denoted $(\pi_\lambda, \mathrm{F}_n^\lambda)$ where

$$\pi_\lambda : \mathrm{GL}_n \to \mathrm{GL}(\mathrm{F}_n^\lambda).$$

To simply notation will write F_n^λ for $(\pi_\lambda, \mathrm{F}_n^\lambda)$.

Throughout this article, the representations of GL_n which we will consider have polynomial matrix coefficients. Thus the components of the highest weight λ will be nonnegative integers. Therefore, if λ is a (nonnegative integer) partition with at most ℓ parts ($\ell \leq n$), then the n-tuple, $(\lambda_1, \ldots, \lambda_\ell, 0, \ldots, 0)$, corresponds to the highest weight of a finite-dimensional irreducible representation of GL_n (with polynomial matrix coefficients).

Under the diagonal action, GL_n acts on the d-fold tensor product $\otimes^d \mathbb{C}^n$. Schur–Weyl duality (see [5] Chapter 9) asserts that the full commutant to the GL_n-action is generated by the symmetric group action defined by permutation of factors. Consequently, one has a multiplicity-free decomposition with respect to the joint action of $GL_n \times S_d$. Moreover, if $n \geq d$, then every irreducible representation of S_d occurs. The irreducible representation of S_d paired with F_n^λ will be denoted by V_d^λ. The full decomposition into the irreducible $GL_n \times S_d$-representation is

$$\bigotimes^d \mathbb{C}^n \cong \bigoplus_\lambda F_n^\lambda \otimes V_d^\lambda,$$

where the sum is over all nonnegative integer partitions λ of size d and length at most n. Note that when $n = d$, then the condition on $\ell(\lambda)$ is automatic. Thus, all irreducible representations of V_d^λ occur. In this manner, the highest weights of GL_n-representations provide a parametrization of the S_d-representations.

3.1 Littlewood–Richardson coefficients

Let $\boldsymbol{d} = (d_1, \ldots, d_m)$ denote a tuple of positive integers with $d = d_1 + \cdots + d_m$. Let $S_{\boldsymbol{d}}$ denote the subgroup of S_d consisting of permutations that stabilize the sets permuting the first m_1 indices, then the second m_2 indices, etc. Clearly, we have

$$S_{\boldsymbol{d}} \cong S_{d_1} \times \cdots \times S_{d_m}.$$

The irreducible representations of $S_{\boldsymbol{d}}$ are of the form

$$V(\boldsymbol{\mu}) = V_{d_1}^{\mu^{(1)}} \otimes \cdots \otimes V_{d_m}^{\mu^{(m)}},$$

where $\mu^{(j)}$ is a partition of size d_j. It is well known that if an irreducible representation V_d^λ of S_d is restricted to $S_{\boldsymbol{d}}$, then the multiplicity of $V(\boldsymbol{\mu})$ in V_d^λ is given by the Littlewood–Richardson coefficient $c_{\boldsymbol{\mu}}^\lambda$. This fact is a consequence of Schur–Weyl duality.

4 Invariant polynomials on matrices

A permutation of $\{1, 2, \ldots, m\}$ may be written as a product of disjoint cycles. This result is fundamental to combinatorial properties of the symmetric group S_m. Keeping this elementary fact in mind, let $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$ be a list of complex $n \times n$ matrices. Let Tr denote the trace of a matrix and define

$$\mathrm{Tr}_\sigma(X) = \mathrm{Tr}(X_{\sigma_1^{(1)}} X_{\sigma_2^{(1)}} \cdots X_{\sigma_{m_1}^{(1)}}) \ \mathrm{Tr}(X_{\sigma_1^{(2)}} X_{\sigma_2^{(2)}} \cdots X_{\sigma_{m_2}^{(2)}}) \cdots$$

$$\cdots \mathrm{Tr}(X_{\sigma_1^{(k)}} X_{\sigma_2^{(k)}} \cdots X_{\sigma_{m_k}^{(k)}})$$

where $\sigma = (\sigma_1^{(1)} \sigma_2^{(1)} \cdots \sigma_{m_1}^{(1)})(\sigma_1^{(2)} \sigma_2^{(2)} \cdots \sigma_{m_2}^{(2)}) \cdots (\sigma_1^{(k)} \sigma_2^{(k)} \cdots \sigma_{m_k}^{(k)})$ is a permutation of $m = \sum m_i$ written as a product of k disjoint cycles. Observe that the cycles of σ may be permuted and rotated without changing the permutation. In turn, the function Tr_σ displays these same symmetries.

Chief among the significance of Tr_σ is the fact that they are invariant under the conjugation action $g \cdot (X_1, \ldots, X_m) = (gX_1g^{-1}, \ldots, gX_mg^{-1})$, where $g \in \mathrm{GL}_n$. By setting some of the components of $(X_1, \ldots, X_m)$ equal, one defines a polynomial of equal degree but on fewer than m copies of M_n. Intuitively, this fact may be described by allowing equalities in the components of the cycles of σ. That is (formally), we consider σ up to conjugation by Levi subgroup of S_m.

In [14,15], C. Procesi described these generators for the algebra of GL_n-invariant polynomials, denoted $\mathbb{C}[V]^{\mathrm{GL}_n}$, on $V = M_n^m$, and provided a proof that these polynomials span the invariants. [Here we let M_n denote the complex vector spaces of $n \times n$ matrices (with entries from $\mathbb{C}$).] Hilbert tells us that the ring of invariants must be finitely generated. Thus, there must necessarily be algebraic relations among this (infinite) set of generators. In light of Procesi's work, these generators and relations are understood. However, recovering the Hilbert series from these data is not automatic.

In order to precisely quantify the failure of the Tr_σ being independent, we introduce the formal power series $A_n(t) = A_n(t_1, t_2, \ldots, t_m) = \sum a_n(d) t^d$, where we will use the notation $d = (d_1, \ldots, d_m)$, $t^d = t_1^{d_1} \cdots t_m^{d_m}$ and the coefficient are defined as $a_n(d) = \dim \mathbb{C}[M_n^m]_d^{\mathrm{GL}_n}$, where $\mathbb{C}[M_n^m]_d = \mathbb{C}[M_n \oplus \cdots \oplus M_n]_{(d_1, \cdots, d_m)}$ denotes the homogeneous polynomials of degree d_i on the ith copy of M_n. The multivariate series, $A_n(t)$, is called the *Hilbert Series* of the invariant ring. Except in some simple cases, a closed form for $A_n(t)$ is not known. Part of this exposition is to point out the rather simple fact that $a_n(d)$ may be expressed in terms of the squares of Littlewood–Richardson coefficients. Furthermore, we prove

Proposition 4.1. *For any natural numbers m and $d = (d_1, \ldots, d_m)$ the limit $\lim_{n \to \infty} a_n(d)$ exists. If we call the limiting value $a(d)$ and set $\widetilde{A}(t) = \sum a(d) t^d$, then*

$$\widetilde{A}(t) = \prod_{k=1}^\infty \frac{1}{1 - (t_1^k + t_2^k + \cdots + t_m^k)}.$$

Proof. The polynomial functions on M_n are multiplicity-free under the action of $\mathrm{GL}_n \times \mathrm{GL}_n$ given by $(g_1, g_2) f(X) = f(g_1^{-1} X g_2)$ for $g_1, g_2 \in \mathrm{GL}_n$, $X \in M_n$ and $f \in \mathbb{C}[M_n]$. The decomposition is a "Peter–Weyl" type, $\mathbb{C}[M_n] \cong \bigoplus (\mathrm{F}_n^\lambda)^* \otimes \mathrm{F}_n^\lambda$, where the sum is over all nonnegative integer partitions λ with $\ell(\lambda) \leq n$. We have $\mathbb{C}[\oplus_{i=1}^m M_n] \cong \otimes_{i=1}^m \mathbb{C}[M_n]$. Thus

$$\mathbb{C}[M_n^m] \cong \left(\bigoplus_{\mu^{(1)}} \left(F_n^{\mu^{(1)}} \right)^* \otimes F_n^{\mu^{(1)}} \right) \otimes \cdots \otimes \left(\bigoplus_{\mu^{(m)}} \left(F_n^{\mu^{(m)}} \right)^* \otimes F_n^{\mu^{(m)}} \right)$$

$$\cong \bigoplus_{\alpha,\beta} \left(\sum_{\boldsymbol{\mu}=(\mu^{(1)},\cdots,\mu^{(m)})} c_{\boldsymbol{\mu}}^{\alpha} c_{\boldsymbol{\mu}}^{\beta} \right) \left(F_n^{\alpha} \right)^* \otimes F_n^{\beta},$$

with respect to the $GL_n \times GL_n$ action on the diagonal of M_n^m. Note that in multidegree $(d_1, \ldots, d_m)$ the sum is over all $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}^{(j)}| = d_j$. If we restrict to the subgroup $\{(g, g) \mid g \in GL_n\}$ of $GL_n \times GL_n$, we obtain an invariant exactly when $\alpha = \beta$. The dimension of the GL_n-invariants in the degree d homogeneous polynomials on M_n^m is therefore $\sum \left(c_{\boldsymbol{\mu}}^{\lambda} \right)^2$ where the sum is over all $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(m)})$ and $\lambda \vdash m$ with length at most n. The degree d component decomposes into multi-degree components $(d_1, \ldots, d_m)$ with $d = \sum d_j$.

If $d \geq n$, then the condition that $\ell(\lambda) \leq n$ is automatic, and this fact implies that if $c_{\boldsymbol{\mu}}^{\lambda} \neq 0$, then $\ell(\mu^{(j)}) \leq n$ for all j. Thus, the dimension of the degree d invariants in $\mathbb{C}[M_n^m]$ is $\sum \left(c_{\boldsymbol{\mu}}^{\lambda} \right)^2$ where the sum is over *all* partitions of size d. If we specialize the main formula so that $t_j = 0$ for $j > m$, then the sums of squares of Littlewood–Richardson coefficients agree with $A(\boldsymbol{t})$. $\qquad\qquad\square$

For our purposes, we will specialize the multigradation on the invariants in $\mathbb{C}[M_n^m]$ to one that is more coarse. From this process, we can relate the stabilized Hilbert series of the invariants in $\mathcal{H}[M_n]$. This specialization will be the subject of the next section.

We now turn to another problem that we shall see is surprisingly related. Consider the $n \times n$ complex matrix

$$X = \begin{bmatrix} X(1,1) & X(1,2) & \cdots & X(1,m) \\ X(2,1) & X(2,2) & \cdots & X(2,m) \\ \vdots & \vdots & \ddots & \vdots \\ X(m,1) & X(m,2) & \cdots & X(m,m) \end{bmatrix}$$

where $X(i, j)$ is an $n_i \times n_j$ complex matrix with $n = \sum n_j$. Define

$$\operatorname{Tr}^{\sigma}(X) = \prod_{j=1}^{k} \operatorname{Tr}\left(X(\sigma_1^{(j)}, \sigma_2^{(j)}) X(\sigma_2^{(j)}, \sigma_3^{(j)}) X(\sigma_3^{(j)}, \sigma_4^{(j)}) \cdots X(\sigma_{m_j}^{(j)}, \sigma_1^{(j)}) \right).$$

Let $K(\boldsymbol{n})$ denote the block diagonal subgroup of GL_n of the form

$$K(\boldsymbol{n}) = \begin{bmatrix} GL_{n_1} & & \\ & \ddots & \\ & & GL_{n_m} \end{bmatrix}.$$

The group $K(\boldsymbol{n})$ acts on M_n by restricting the adjoint action of GL_n. The $K(\boldsymbol{n})$-invariant subring of $\mathbb{C}[M_n]$ (denoted $\mathbb{C}[M_n]^{K(\boldsymbol{n})}$) is spanned by $\mathrm{Tr}^\sigma(X)$. For small values of the parameter space, these expressions are far from linearly independent (as in the last example). Formally, one cannot help but notice the symbolic map $\mathrm{Tr}^\sigma \mapsto \mathrm{Tr}_\sigma$. We will try next to make a precise statement along these lines.

Define $\mathbb{C}[M_n]_d$ to be the homogeneous degree d polynomials on M_n, and let $\mathbb{C}[M_n]_d^{K(\boldsymbol{n})}$ denote the $K(\boldsymbol{n})$-invariant subspace. Set $a^{(\boldsymbol{n})}(d) = \dim \mathbb{C}[M_n]_d^{K(\boldsymbol{n})}$, and $A^{(\boldsymbol{n})}(t) = \sum_{d=0}^\infty a^{(\boldsymbol{n})}(d)\, t^d$. Analogous to Proposition 4.1, we have

Proposition 4.2. *For any nonnegative integer d, the limit*

$$\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \cdots \lim_{n_m \to \infty} a^{(n_1, \cdots, n_m)}(d)$$

exists. Denote the limiting value $a(d)$ and set $A(t) = \sum_{d=0}^\infty a(d) t^d$. We have

$$A(t) = \prod_{k=1}^\infty \frac{1}{1 - m\, t^k}.$$

Proof. We begin with the $GL_n \times GL_n$-decomposition $\mathbb{C}[M_n] = \bigoplus \left(F_n^\lambda\right)^* \otimes\ F_n^\lambda$ with respect to the action in the proof of Proposition 4.1. The irreducible GL_n-representation F_n^λ is reducible upon restriction to $K(\boldsymbol{n})$. The decomposition is given in terms of Littlewood–Richardson coefficients

$$F_n^\lambda \cong \bigoplus_{\boldsymbol{\mu} = (\mu^{(1)}, \cdots, \mu^{(m)})} c_{\boldsymbol{\mu}}^\lambda\, F_n^{\mu^{(1)}} \otimes \cdots \otimes F_n^{\mu^{(m)}}.$$

Therefore, as a $K(\boldsymbol{n}) \times K(\boldsymbol{n})$-representation we have

$$\mathbb{C}[M_n]_d = \sum_{\boldsymbol{\mu}, \boldsymbol{\nu}} \left(\sum_\lambda c_{\boldsymbol{\mu}}^\lambda c_{\boldsymbol{\nu}}^\lambda \right) \left(\otimes_{j=1}^m F_{n_j}^{\mu^{(j)}} \right)^* \otimes \left(\otimes_{j=1}^m F_{n_j}^{\nu^{(j)}} \right),$$

where the sum is over all λ with $|\lambda| = d$, $\ell(\lambda) \le n$ and $\ell(\mu^{(j)}), \ell(\nu^{(j)}) \le n_j$. If we restrict to the diagonally embedded $K(\boldsymbol{n})$-subgroup, we obtain an invariant exactly when $\boldsymbol{\mu} = \boldsymbol{\nu}$. Thus, $a^{(\boldsymbol{n})}(d) = \sum \left(c_{\boldsymbol{\mu}}^\lambda \right)^2$, with the appropriate restrictions on λ and $\boldsymbol{\mu}$.

If all $n_j \ge d$, then the condition on the lengths of partitions disappears, and we may sum over all $\lambda \vdash d$. The result follows by specializing the main formula by setting $t_j = t$ for $1 \le j \le m$ and $t_j = 0$ for $j > m$. $\qquad\qquad$ $\square$

Although we will not need it for our present purposes, it is worth pointing out that the algebra $\mathbb{C}[M_n]$ has a natural $\mathbb{N}^m$ gradation defined by the action of the center of $K(\boldsymbol{n})$. This multigradation refines the gradation by degree. The limiting multigraded

Hilbert series is the same as that of Proposition 4.1. Upon specialization to $t_1 = t_2 = \cdots = t_m = t$ we obtain the usual gradation by degree (in both situations). The advantage of considering the more refined gradation is that one can consider the direct limit as m and n go to infinity. This will be relevant in the next section.

5 Harmonic polynomials on matrices

A specific goal of this article is to understand the dimension of the space of degree d homogeneous $K(n)$-invariant harmonic polynomials on M_n. With this fact in mind, we observe the following specialization of the product in the main formula. Let $t_j = t^j$. Then we obtain

$$\prod_{k=1}^{\infty} \frac{1}{1 - (t^k + t^{2k} + t^{3k} + \cdots)} = \prod_{k=1}^{\infty} \frac{1}{1 - \frac{t^k}{1-t^k}} = \prod_{k=1}^{\infty} \frac{1 - t^k}{1 - 2t^k}.$$

For a sequence $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots)$ set $\mathrm{gr}(\boldsymbol{\mu}) = \sum_{j=1}^{\infty} j\,|\mu^{(j)}|$. The equation in the main formula becomes

$$\prod_{k=1}^{\infty} \frac{1 - t^k}{1 - 2t^k} = \sum_{\lambda} \sum_{\boldsymbol{\mu}} \left(c_{\boldsymbol{\mu}}^{\lambda}\right)^2 t^{\mathrm{gr}(\boldsymbol{\mu})}.$$

The notation gr is used for the word *grade*. We explain this choice next. Let $\mathbb{C}[M_n; d]$ denote the polynomials functions on the $n \times n$ complex matrices together with the gradation defined by d times the usual degree. That is, $\mathbb{C}[M_n; d_1]_{(d_2)}$ denotes the degree d_2 homogeneous polynomials, but regarded as the $d_1 d_2$ graded component in $\mathbb{C}[M_n; d_1]$. We consider the $\mathbb{N}$-graded complex algebra $\mathcal{A}_n$ defined as

$$\mathcal{A}_n = \mathbb{C}[M_n; 1] \otimes \mathbb{C}[M_n; 2] \otimes \mathbb{C}[M_n; 3] \otimes \cdots$$
$$= \sum_{\delta=0}^{\infty} \mathcal{A}_n[\delta],$$

where $\mathcal{A}_n[\delta]$ is the graded $\delta \in \mathbb{N}$ component (with the usual grade defined on a tensor product of algebras). The group GL_n acts on each $\mathbb{C}[M_n; d]$ by the adjoint action, and respects the grade. Under the diagonal action on the tensors, the GL_n-invariants, $\mathcal{A}_n^{\mathrm{GL}_n}$, have $A_n(\boldsymbol{t})$ as the Hilbert series when $\boldsymbol{t}$ is specialized to $(t_1, t_2, t_3, \ldots) = (t, t^2, t^3, \ldots)$. This GL_n-action respects the δ component in the gradation. That is, that the Hilbert series is

$$A_n(t, t^2, t^3, \ldots) = \sum_{\delta=0}^{\infty} \dim \left(\mathcal{A}_n[\delta]\right)^{\mathrm{GL}_n} t^{\delta}.$$

For $d = 0, \ldots, n$ the coefficient of t^d in $A_n(t, t^2, \ldots)$ is the same as the coefficient of t^d in $\widetilde{A}(t, t^2, \ldots)$. Summarizing, we can say that stably, as $n \to \infty$, the Hilbert series of $\mathcal{A}_n^{\mathrm{GL}_n}$ is $\prod_{k=1}^{\infty} \frac{1-t^k}{1-2t^k}$.

We turn now to the GL_n-harmonic polynomials on M_n together with its usual gradation by degree. Kostant's theorem [12] tells us that

$$\mathbb{C}[M_n] \cong \mathbb{C}[M_n]^{\mathrm{GL}_n} \otimes \mathcal{H}(M_n).$$

As before let $K = K(n_1, n_2)$ denote the copy of $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$ (symmetrically) embedded in $\mathrm{GL}_{n_1+n_2}$. Passing to the $K(n_1, n_2)$-invariant subspaces we obtain

$$\mathbb{C}[M_n]^K = \mathbb{C}[M_n]^{\mathrm{GL}_n} \otimes \mathcal{H}(M_n)^K.$$

The Hilbert series of $\mathbb{C}[M_n]^{\mathrm{GL}_n}$ is well known to be $\prod_{k=1}^{n} \frac{1}{1-t^k}$, while the Hilbert series $\mathbb{C}[M_n]^K$ is $A_n(t, t)$. These facts imply that the dimension of the degree d homogenous K-invariant harmonic polynomials is the coefficient of t^d in $F_n(t) = A_n(t, t) \prod_{j=1}^{n}(1 - t^j)$. We have that the coefficient of t^d in $F_n(t)$ for $d = 0, \ldots, \min(n_1, n_2)$ agrees with the coefficient of t^d in

$$\widetilde{A}(t, t^2, t^3, \cdots) \prod_{k=1}^{\infty}(1 - t^k) = \prod_{k=1}^{\infty} \frac{1 - t^k}{1 - 2t^k}.$$

Again summarizing, we say that stably, as $n_1, n_2 \to \infty$, the Hilbert series of $\mathcal{H}(M_{n_1+n_2})^{K(n_1,n_2)}$ is the same as $\mathcal{A}_n^{\mathrm{GL}_n}$ as $n, n_1, n_2 \to \infty$. That is, for *fixed* δ we have

$$\lim_{n_1, n_2 \to \infty} \dim \mathcal{H}(M_{n_1+n_2})_{\delta}^{K(n_1,n_2)} = \lim_{n \to \infty} \dim \mathcal{A}_n[\delta]^{\mathrm{GL}_n}.$$

Observe that this procedure generalizes. Let $m \geq 2$. Analogous to before, let $\mathbb{C}[M_n^m; d]$ denote the polynomial function on $M_n^m = M_n \oplus \cdots \oplus M_n$ (m-copies) together with the $\mathbb{N}$-gradation defined such that $\mathbb{C}[M_n^m; d_1]_{d_2}$ consists of the degree d_2 homogeneous polynomials but regarded as being the $d_1 d_2$-th graded component. Note that the δ-th component in the grade is (0) if δ is not a multiple of d_1.

Let $\mathcal{A}_n^m$ be the $\mathbb{N}$-graded algebra defined as

$$\mathcal{A}_n^m = \mathbb{C}[M_n^{m-1}; 1] \otimes \mathbb{C}[M_n^{m-1}; 2] \otimes \mathbb{C}[M_n^{m-1}; 3] \otimes \cdots.$$

Since $\mathcal{A}_n^m$ is a tensor product of $\mathbb{N}$-graded algebras, it has the structure of an $\mathbb{N}$-graded algebra. As before let $\mathcal{A}_n^m[\delta]$ be the δ-th graded component. The group GL_n acts on each $\mathbb{C}[M_n^m; d]$ by the adjoint action, and respects the grade.

Next, set $t_1 = t_2 = \cdots = t_{m-1} = t$, then $t_m = t_{m+1} = \cdots = t_{2m-2} = t^2$, then $t_{2m-1} = \cdots = t_{3m-3} = t^3$, and so on. The result of this procedure is

Theorem 5.1. *For all* $\boldsymbol{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$ *and* $d \in \mathbb{N}$, *let*

$$h_d(\boldsymbol{n}) = \dim \left(\mathcal{H}^d(M_n) \right)^{K(\boldsymbol{n})}.$$

Then for fixed d, *the limit*

$$\lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} h_d(n_1, \ldots, n_m)$$

exists. Let the limiting value be h_d. *Then*

$$h_d = \lim_{n \to \infty} \dim \left(\mathcal{A}_n^m[d] \right)^{\mathrm{GL}_n}.$$

Proof. After the specialization we obtain

$$\prod_{k=1}^{\infty} \frac{1}{1 - (\underbrace{t^k + \cdots + t^k}_{m-1 \text{ copies}} + \underbrace{t^{2k} + \cdots + t^{2k}}_{m-1 \text{ copies}} + \cdots)} = \prod_{k=1}^{\infty} \frac{1}{1 - (m-1)(t^k + t^{2k} + \cdots)}$$

$$= \prod_{k=1}^{\infty} \frac{1}{1 - (m-1)\frac{t^k}{1-t^k}}$$

$$= \prod_{k=1}^{\infty} \frac{1}{\frac{1-t^k-(m-1)t^k}{1-t^k}}$$

$$= \prod_{k=1}^{\infty} \frac{1 - t^k}{1 - m\, t^k}.$$

The significance of this calculation is that it allows for another interpretation of sums of Littlewood–Richardson coefficients. The rest of the proof is identical to the $m = 2$ case in the preceding discussion.   □

5.1   A bigraded algebra and a specialization of the main formula

As before, the group GL_n acts on M_n by conjugation, and then in turn acts diagonally on $M_n \oplus M_n$. That is, given $(X, Y) \in M_n \oplus M_n$, and $g \in \mathrm{GL}_n$, we have $g \cdot (X, Y) = (gXg^{-1}, gYg^{-1})$. We then obtain an action on $\mathbb{C}[M_n \oplus M_n]$.

We have already observed that the algebra $\mathbb{C}[M_n \oplus M_n]$ is bigraded. That is, let $\mathbb{C}[M_n \oplus M_n](i, j)$ denote the homogenous polynomial functions on $M_n \oplus M_n$ of degree i in the first copy of M_n and degree j in the second copy of M_n. Let a and b be positive integers. We set

$$\mathbb{C}[M_n \oplus M_n; (a,b)](ai, bj) = \mathbb{C}[M_n \oplus M_n](i,j)$$

with the other components zero. Put another way, $\mathbb{C}[M_n \oplus M_n; (a,b)]$ is the algebra of polynomial function on $M_n \oplus M_n$ together with the bi-gradation defined by (a,b) times the usual degree. As before we consider the infinite tensor product

$$\mathcal{B}_n = \bigotimes_{a=1}^{\infty} \bigotimes_{b=1}^{\infty} \mathbb{C}[M_n \oplus M_n; (a,b)].$$

Next, note the following, obvious, identity:

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{x^k y^k}{(1-x^k)(1-y^k)}} = \prod_{k=1}^{\infty} \frac{(1 - x^k)(1 - y^k)}{1 - (x^k + y^k)}.$$

We observe that this is a specialization of the product in the main formula. Specifically, let $q^i t^j = z_s$ where z_s is given as the (i, j) entry in the following table.

$q^i t^j$	q	q^2	q^3	q^5	q^6	$\cdots$
t	z_1	z_2	z_4	z_7	z_{11}	$\cdots$
t^2	z_3	z_5	z_8	z_{12}	z_{17}	$\cdots$
t^3	z_6	z_9	z_{13}	z_{18}	z_{24}	$\cdots$
t^4	z_{10}	z_{14}	z_{19}	z_{25}	z_{32}	$\cdots$
t^5	z_{15}	z_{20}	z_{26}	z_{33}	z_{41}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then,

$$\prod_{k=1}^{\infty} \frac{1}{1 - (z_1^k + z_2^k + z_3^k + \cdots)} = \prod_{k=1}^{\infty} \frac{1}{1 - \sum_{i,j \geq 1} (q^i t^j)^k}$$

$$= \prod_{k=1}^{\infty} \frac{1}{1 - \frac{q^k t^k}{(1-q^k)(1-t^k)}}$$

$$= \prod_{k=1}^{\infty} \frac{(1 - q^k)(1 - t^k)}{1 - (q^k + t^k)}.$$

In this way, the GL_n-invariants in the harmonic polynomials on $M_n \oplus M_n$ may be related to the multigraded algebra structure $\mathcal{B}$, as in the singly graded case.

This identity becomes significantly more complicated when generalized to harmonic polynomials on more than two copies of the matrices. This fact will be the subject of future work.

6 Some combinatorics related to finite fields

In this section, we collect remarks of a combinatorial nature that provide a more concrete understanding of the sum of squares that we consider in this paper.

From elementary combinatorics one knows that the infinite product

$$\prod_{k=1}^{\infty} \frac{1}{1 - q\, t^k} = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} p_{n,\ell} q^{\ell} t^n,$$

where $p_{n,\ell}$ is the number of partitions of n with exactly ℓ parts. When q is specialized to a positive integer the coefficients of this series in t has many interpretations.

6.1 The symmetric group

Fix a positive integer d, and a tuple of positive integers $\boldsymbol{d} = (d_1, d_2, \ldots, d_m)$ with $d_1 + \cdots + d_m = d$. As before, let $S_{\boldsymbol{d}}$ denote the subgroup of S_d isomorphic to $S_{d_1} \times \cdots \times S_{d_m}$ embedded by letting the ith factor permute the set J_i where $\{J_1, \ldots, J_m\}$ is the partition of $\{1, \ldots, d\}$ into the m contiguous intervals with $|J_i| = d_i$. That is, $J_1 = \{1, 2, \ldots, d_1\}$, $J_2 = \{d_1 + 1, \ldots, d_1 + d_2\}$, etc.

The group $S_{\boldsymbol{d}}$ acts on S_d by conjugation. Let the set of orbits be denoted by $\mathcal{O}(\boldsymbol{d})$. We have

Proposition 6.1. *For any* $\boldsymbol{d} = (d_1, \ldots, d_m)$,

$$|\mathcal{O}(\boldsymbol{d})| = \sum_{\lambda \vdash m} \sum_{\boldsymbol{\mu} = (\mu^{(1)}, \cdots, \mu^{(m)})} \left(c_{\boldsymbol{\mu}}^{\lambda} \right)^2$$

where the inner sum is over all tuples of partitions with $\mu^{(j)} \vdash d_j$.

Proof. We begin with the Peter–Weyl type decomposition of $\mathbb{C}[S_d]$

$$\mathbb{C}[S_d] \cong \bigoplus_{\lambda \vdash d} \left(V_d^{\lambda} \right)^* \otimes V_d^{\lambda}.$$

We then recall that Littlewood–Richardson coefficients describe the branching rule from S_d to $S_{\boldsymbol{d}}$:

$$V^{\lambda} = \bigoplus_{\boldsymbol{\mu}} c_{\boldsymbol{\mu}}^{\lambda} V^{\mu^{(1)}} \otimes \cdots \otimes V^{\mu^{(m)}}.$$

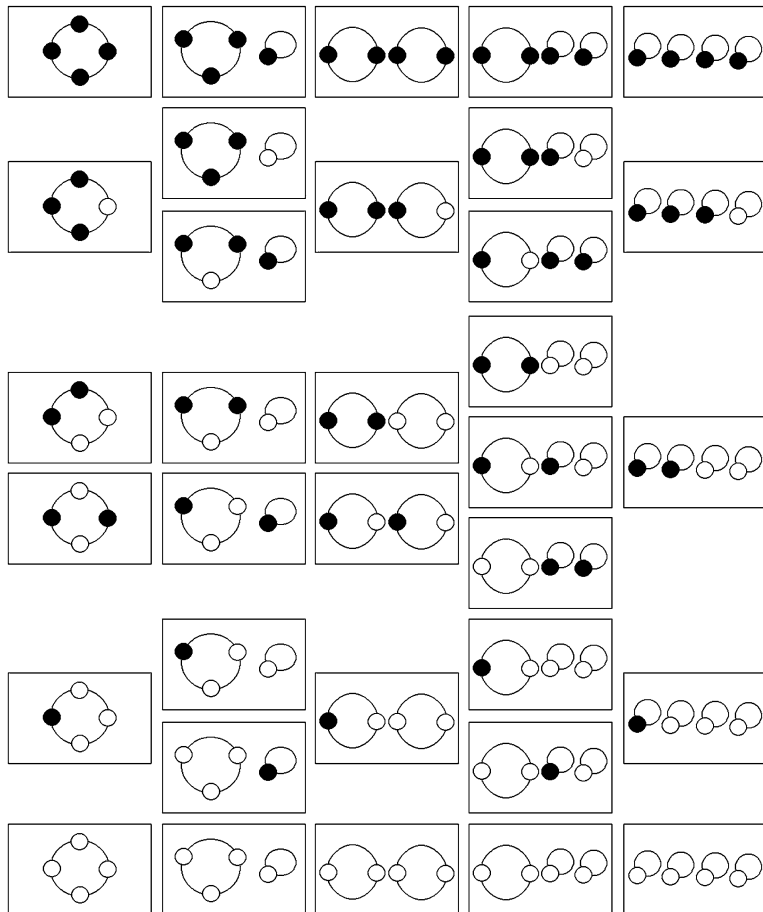Combining the above decompositions, we obtain the result from Schur's Lemma.

□

Figure 1 2-colored necklaces

It is an elementary fact that the S_d-conjugacy class of a permutation σ in S_d is determined by the lengths of the disjoint cycles of σ. A slightly more general statement is that the S_d-conjugacy class of $\sigma \in S_d$ is determined by a union of cycles in which each element of a cycle is "colored" by colors corresponding to $J_1, \ldots, J_m$. It is not difficult to write down a proof of this fact, but we omit it here for space considerations.

If one fixes d, then the sum over all d-compositions, $\sum_d |\mathcal{O}(d)|$, may be interpreted as the number of d-bead unions of necklaces with each bead colored by m colors. For example if $m = 2$, and $d = 4$, the resulting set may be depicted as noted in Figure 1.

A single k-bead necklace has $\mathbb{Z}/k\mathbb{Z}$-symmetry. If the beads of this necklace are colored with m colors, then the resulting colored necklace may have smaller group of symmetries. Using Polya enumeration (i.e., "Burnside's Lemma"), one can count such necklaces by the formula

$$N_k(m) = \frac{1}{k} \sum_{r|k} \phi(r) m^{\frac{k}{r}},$$

where ϕ denotes the Euler totient function. That is, $\phi(r)$ is the number of positive integers relatively prime to r. The theory forming the underpinnings of the above formula may be put in a larger context of *cycle index polynomials*. We refer the reader to Doron Zeilberger's survey article (IV.18 of [6]).

The generating function for disjoint unions of such necklaces can be given by the product

$$\eta_m(t) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - t^k} \right)^{N_k(m)}$$

That is, the number of d-bead necklaces, counted up to cyclic symmetry, is equal to the coefficient of t^d in $\eta(t)$. The main formula specializes to

$$\eta_m(t) = \sum_{\lambda, \boldsymbol{\mu} \vdash d} \left(c_{\boldsymbol{\mu}}^{\lambda} \right)^2 t^{|\lambda|}, \tag{1}$$

where the sum is over all partitions λ and over all m-tuples of partitions $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(m)})$ such that $\sum_{i=1}^{m} |\mu^{(i)}| = d$.

Equation 1 begs for an (explicit) bijective proof which is no doubt obtained by merging both the Littlewood–Richardson rule and the Robinson–Schensted–Knuth bijection. It is likely that more than one "natural" bijection exists.

6.2 The general linear group over a finite field

Let p denote a prime number, and $v \in \mathbb{Z}^+$. Set $q = p^v$. Let $\mathrm{GL}_m(q)$ denote the general linear group of invertible $m \times m$ matrices over the field with q elements. The set, $\mathcal{C}_m(q)$, of conjugacy classes of $\mathrm{GL}_m(q)$ has a cardinality of note in that the infinite products has the following expansion:

$$\prod_{k=1}^{\infty} \frac{1 - t^k}{1 - q\,t^k} = \sum_{m=0} |\mathcal{C}_m(q)| t^m,$$

see [1, 16], and the references within.

Note that from Equation 1 we obtain a new formula for the number of conjugacy classes of $\mathrm{GL}_m(q)$, namely,

$$\left(\prod_{k=1}^{\infty} (1 - t^k) \right) \eta_q(t) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - t^k} \right)^{N_k(q)-1}.$$

7 Proof of the main formula

Before proceeding, we require some more notation. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with v_1 ones, and v_2 twos, etc., we will call the sequence $(v_1, v_2, \ldots)$ the *type vector* of λ. Note that $\ell(\lambda) = \sum v_i$, while the size of λ is $m = |\lambda| = \sum \lambda_i = \sum i v_i$. As is standard, we set

$$z_\lambda = (v_1! 1^{v_1})(v_2! 2^{v_2})(v_3! 3^{v_3}) \cdots .$$

It is elementary that the cardinality of a conjugacy class of a permutation with cycle type λ is $\frac{m!}{z_\lambda}$. (Equivalently, the centralizer subgroup has order z_λ.)

We now prove the main formula. The product on the left side (LS) may be expanded using the sum of a geometric series, the multinomial theorem, and then the sum-product formula, as follows

$$
\begin{aligned}
LS &= \prod_{k=1}^{\infty} \frac{1}{1 - (t_1^k + t_2^k + t_3^k + \cdots)} \\
&= \prod_{k=1}^{\infty} \sum_{u=0}^{\infty} (t_1^k + t_2^k + \cdots)^u \\
&= \prod_{k=1}^{\infty} \sum_{u=0}^{\infty} \sum_{u_1+u_2+\cdots=u} \frac{u!}{u_1! u_2! \cdots} t_1^{k u_1} t_2^{k u_2} \cdots \\
&= \prod_{k=1}^{\infty} \sum_{u_1, u_2, \cdots} \frac{(u_1 + u_2 + \cdots)!}{u_1! u_2! \cdots} t_1^{k u_1} t_2^{k u_2} \cdots \\
&= \sum_{u_j^{(i)}, \cdots} \prod_{i=1}^{\infty} \frac{(u_1^{(i)} + u_2^{(i)} + \cdots)!}{u_1^{(i)}! u_2^{(i)}! \cdots} t_1^{i u_1^{(i)}} t_2^{i u_2^{(i)}} \cdots
\end{aligned}
$$

(with all sequences having finite support.) We will introduce another sequence, $\mathbf{a} = (a_1, a_2, \ldots)$ and extract the coefficient of $t^{\mathbf{a}} = \prod t_i^{a_i}$ in the above formal expression to obtain

$$
LS = \sum_{\mathbf{a}} \left(\sum_{u_j^{(i)}: \forall j, \sum_i i u_j^{(i)} = a_j} \prod_{i=1}^{\infty} \frac{(u_1^{(i)} + u_2^{(i)} + \cdots)!}{u_1^{(i)}! u_2^{(i)}! \cdots} \right) t^{\mathbf{a}}
$$

The above is not such a complicated expression, although these formal manipulations may, at first, seem daunting. Observe that the sequence $u_j^{(1)}, u_j^{(2)}, \ldots$ with $\sum_i i u_j^{(i)} = a_j$ encodes a partition of a_j with the number i occurring exactly $u_j^{(i)}$

times, while $\sum_{i\geq 1} u_j^{(i)}$ is the length of the partition. We will call this partition $\mu^{(j)}$. That is, $\mu^{(j)}$ has type vector $(u_j^{(1)}, u_j^{(2)}, \ldots)$. The coefficient of $t^{\mathbf{a}}$ may be rewritten as a sum over all double sequences $u_j^{(i)}$ such that for all j, $\sum_i i u_j^{(i)} = a_j$ of

$$\prod_{i=1}^{\infty} \frac{(u_1^{(i)} + u_2^{(i)} + \cdots)!}{u_1^{(i)}! u_2^{(i)}! \cdots} = \prod_{i=1}^{\infty} \frac{(u_1^{(i)} + u_2^{(i)} + \cdots)! i^{\sum u_j^{(i)}}}{(u_1^{(i)}! i^{u_1^{(i)}})(u_2^{(i)}! i^{u_2^{(i)}}) \cdots}. \tag{2}$$

Given a (finitely supported) sequence of partitions $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots)$, we denote the partition obtained from the (sorted) concatenation of all $\mu^{(j)}$ by $\cup \mu^{(j)}$. Thus, if $\mu^{(j)}$ has type vector $(u_1^{(i)}, u_2^{(i)}, \ldots)$, then the number of i's in $\cup \mu^{(j)}$ is $u_1^{(i)} + u_2^{(i)} + \cdots$. It therefore follows that the numerator of the right-hand side of Equation 2 is z_λ when $\lambda = \cup \mu^{(j)}$. The denominator can easily be seen to be $z_{\mu^{(j)}}$. From this observation we obtain

$$LS = \sum_{\boldsymbol{\mu}} \frac{z_{\cup_{j=1}^{\infty} \mu^{(j)}}}{\prod_{j=1}^{\infty} z_{\mu^{(j)}}} z_1^{|\mu^{(1)}|} z_2^{|\mu^{(2)}|} z_3^{|\mu^{(3)}|} \cdots.$$

7.1 An application of the Hall scalar product

Let Λ_n denote the S_n-invariant polynomials (over $\mathbb{C}$ as usual) in the indeterminates $x_1, \ldots, x_n$, let $\Lambda[\boldsymbol{x}] = \lim_{\leftarrow} \Lambda_n$ denote the inverse limit. Thus, $\Lambda[\boldsymbol{x}]$ is the algebra of symmetric functions. For a nonnegative integer partition λ, we let $s_\lambda(\boldsymbol{x})$ denote the Schur function. That is, for each n, $s_\lambda(\boldsymbol{x})$ projects to the polynomial $s_\lambda(x_1, \ldots, x_n)$ which, as a function on the diagonal matrices, coincides with the character of the GL_n-irrep F_n^λ. The set $\{s_\lambda(\boldsymbol{x}) : \lambda \vdash d\}$ is a $\mathbb{C}$-vector space basis of the homogeneous degree d symmetric functions. We will define a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ by declaring $\langle s_\alpha(\boldsymbol{x}), s_\beta(\boldsymbol{x}) \rangle = \delta_{\alpha\beta}$ for all nonnegative integer partitions α, β. This form is the *Hall scalar product*.

Given an integer m, define $p_m(\boldsymbol{x}) = x_1^m + x_2^m + \cdots$ to denote the power sum symmetric function. Given $\nu \vdash N$, set $p_\nu(\boldsymbol{x}) = \prod p_{\nu_j}(\boldsymbol{x})$. We remark that the left side of the main formula is easily seen to be $\sum_\nu p_\nu(\boldsymbol{t})$.

It is a consequence of Schur–Weyl duality that the coefficients of the Schur function expansion

$$p_\nu(\boldsymbol{x}) = \sum_\lambda \chi^\lambda(\nu) s_\lambda(\boldsymbol{x})$$

are the characters of the S_N-irrep indexed by λ evaluated at any permutation with cycle type ν. It is a standard exercise to see, from the orthogonality of the character table, that $p_\nu(\boldsymbol{x})$ are an orthogonal basis of $\Lambda[\boldsymbol{x}]$, and $\langle p_\nu(\boldsymbol{x}), p_\nu(\boldsymbol{x}) \rangle = z_\nu$.

We next consider another set of indeterminates, $\boldsymbol{y} = y_1, y_2, \ldots$. Set $\Lambda[\boldsymbol{x}, \boldsymbol{y}] = \Lambda[\boldsymbol{x}] \otimes \Lambda[\boldsymbol{y}]$. The Hall scalar product extends to $\Lambda[\boldsymbol{x}, \boldsymbol{y}]$ in the standard way as

$$\langle f(\boldsymbol{x}) \otimes g(\boldsymbol{y}), f'(\boldsymbol{x})g'(\boldsymbol{y})\rangle = \langle f(\boldsymbol{x}), f'(\boldsymbol{x})\rangle\langle g(\boldsymbol{y}), g'(\boldsymbol{y})\rangle,$$

where $f(\boldsymbol{x}), f'(\boldsymbol{x}) \in \Lambda[\boldsymbol{x}]$ and $g(\boldsymbol{y}), g'(\boldsymbol{y}) \in \Lambda[\boldsymbol{y}]$. (We then extend by linearity to all of $\Lambda[\boldsymbol{x}, \boldsymbol{y}]$.)

The character-theoretic consequence of $(\mathrm{GL}_n, \mathrm{GL}_k)$-Howe duality is the Cauchy identity:

$$\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_{\lambda}(\boldsymbol{x})s_{\lambda}(\boldsymbol{y}) \tag{3}$$

(in the infinite sets of variables). In fact, for any pair of dual bases, a_{λ}, b_{λ}, with respect to the Hall scalar product, one has $\prod_{i,j=1}^{\infty} \frac{1}{1-x_i y_j} = \sum_{\lambda} a_{\lambda}(\boldsymbol{x})b_{\lambda}(\boldsymbol{y})$. From this fact one obtains

$$\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_{\lambda} p_{\lambda}(\boldsymbol{x})p_{\lambda}(\boldsymbol{y})/z_{\lambda}. \tag{4}$$

From our point of view, we will expand the following scalar product

$$\left\langle \prod_{k=1}^{\infty}\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j t_k}, \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} \right\rangle \tag{5}$$

in two different ways, corresponding to Equations 3 and 4.

First, by homogeneity of the Schur function and Cauchy's identity

$$\prod_{k=1}^{\infty} \sum_{\mu} s_{\mu}(\boldsymbol{x})s_{\mu}(\boldsymbol{y})t_k^{|\mu|} = \sum_{\boldsymbol{\mu}=(\mu^{(1)},\mu^{(2)},\ldots)} \prod_j s_{\mu^{(j)}}(\boldsymbol{x}) \prod_j s_{\mu^{(j)}}(\boldsymbol{y})t_1^{|\mu^{(1)}|}t_2^{|\mu^{(2)}|} \cdots.$$

Since the multiplication of characters is the character of the tensor product of the corresponding representations, we have $s_{\alpha}s_{\beta} = \sum c_{\alpha\beta}^{\gamma}s_{\gamma}$ in the $\boldsymbol{x}$ (resp. $\boldsymbol{y}$) variables. Expanding the above product gives

$$\left\langle \prod_{k=1}^{\infty}\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j t_k}, \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} \right\rangle = \sum_{\boldsymbol{\mu}} \left(c_{\boldsymbol{\mu}}^{\lambda}\right)^2 t_1^{|\mu^{(1)}|}t_2^{|\mu^{(2)}|} \cdots.$$

Secondly, the scalar product in (5) may be expressed as

$$\left\langle \prod_{k=1}^{\infty} \sum_{\mu^{(k)}} p_{\mu^{(k)}}(\boldsymbol{x})p_{\mu^{(k)}}(\boldsymbol{y})/z_{\mu^{(k)}} \; t^{|\mu^{(k)}|}, \sum_{\lambda} p_{\lambda}(\boldsymbol{x})p_{\lambda}(\boldsymbol{y})/z_{\lambda} \right\rangle.$$

We observe that $\prod_{k=1}^{\infty} p_{\mu^{(k)}} = p_{\lambda}$ where $\lambda = \cup_{k=1}^{\infty}\mu^{(k)}$.

7.2 A remark from "Macdonald's book"

The results presented in this paper emphasize describing the cardinality of an orbit space by a sum of Littlewood–Richardson coefficients. It is important to note, however, that the main formula can be written simply as

$$\sum_\nu p_\nu(\boldsymbol{x}) = \sum_{\boldsymbol{\mu}} \left(\sum_\lambda \left(c_{\boldsymbol{\mu}}^\lambda \right)^2 \right) x_1^{\mu^{(1)}} x_2^{\mu^{(2)}} x_1^{\mu^{(3)}} \cdots$$

So from the point of view of [13], one realizes that the main formula is simply a way of expanding $\sum p_\nu(\boldsymbol{x})$. With this remark in mind, we recall the "standard" viewpoint.

For a nonnegative integer partition $\delta = (\delta_1 \geq \delta_2 \geq \cdots)$ let $x^\delta = x_1^{\delta_1} x_2^{\delta_2} x_3^{\delta_3} \cdots$. The monomial symmetric function, $m_\delta(\boldsymbol{x})$, is the sum over the orbit obtained by all permutations of the variables. The monomial symmetric functions are a basis for the algebra Λ.

The question becomes obtaining an expansion of $\sum_\gamma p_\gamma(\boldsymbol{x})$ into monomial symmetric functions. This question is answered immediately by observing the expansion of p_γ into monomial symmetric functions, which can be found in [13] on page 102 of Chapter I, Section 6. For partitions γ and δ define $L(\gamma, \delta)$ by the expansion

$$p_\gamma(\boldsymbol{x}) = \sum_\delta L_{\gamma\delta} m_\delta(\boldsymbol{x}).$$

We next provide a combinatorial description of $L_{\gamma\delta}$. Let γ denote a partition of length ℓ. Given an integer valued function, f, defined on $\{1, 2, 3, \ldots, \ell\}$, set

$$f(\gamma)_i = \sum_{j:f(j)=i} \nu_j$$

for each $i \geq 1$.

The sequence $(f(\gamma)_1, f(\gamma)_2, f(\gamma)_3, \ldots)$ does not have to be weakly decreasing. For example, if $\gamma = (1, 1, 1)$ and $f(1) = 1$, $f(2) = 4$ and $f(3) = 4$ then $f(\gamma)_1 = 1$, $f(\gamma)_4 = 2$ and $f(\gamma)_k = 0$ for all $k \neq 1, 4$. However, often this sequences does define a partition. We have

Proposition 7.1. $L_{\gamma,\delta}$ is equal to the number of functions f such that $f(\gamma) = \delta$.

Proof. See [13] Proposition I (6.9) □

From Proposition 7.1 and the main formula, we obtain

Corollary 1. *Given a partition δ, the cardinality of*

$$\{f \mid \text{for some partition } \gamma,\ f(\gamma) = \delta\ \}$$

is equal to

$$\sum_{\lambda} \sum_{\mu} \left(c_{\mu}^{\lambda} \right)^{2}$$

where the sum is over all μ *such that* $|\mu^{(j)}| = \delta_j$ *for all* j *and* $|\lambda| = |\mu|$.

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