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On the Equivalence of Polygons (1924)

This chapter contains an English translation of Alfred Tarski's paper *O równoważności wielokątów*, [1924] 2014b, written as he was completing his doctoral studies. It appeared in volume 24 of the journal *Przegląd matematyczno-fizyczny*. This is its first translation. A description of the journal, background for the paper, and a summary are provided in sections 4.1–4.3.

The translation is meant to be as faithful as possible to the original. Its only intentional modernizations are punctuation and some changes in symbols, where Tarski's conflict with others used throughout this book. Bibliographic references and some personal names have been adjusted to conform with the conventions used here. The original paper sometimes employed barely discernible a u g m e n t e d l e t t e r s p a c i n g to emphasize a phrase.[1] In some cases the translation uses italics instead; in others, this emphasis has been suppressed. As an aspect of adjusting punctuation, the editors greatly increased use of white space to enhance visual organization of the paper. All [square] brackets in the translation enclose editorial comments. Those are inserted, usually as footnotes, to indicate changes in notation and explain passages that seem obscure.

[1] In the letter translated in section 15.12, Tarski conveyed his dislike of that style.

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On the Equivalence of Polygons

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On the Equivalence of Polygons

In elementary geometry,[2] we call two polygons *equivalent* if it is possible to divide them into the same finite number of respectively congruent polygons not having common interior points. In the theory of the equivalence of polygons, the following statement, usually accepted without proof in elementary geometry and sometimes called *De Zolt's axiom*, plays a fundamental role:

> *If polygon V is a part of polygon W, then these polygons are not equivalent.*

As is well known, David Hilbert showed[3] that the preceding statement can be proved with the help of axioms usually cited in elementary geometry textbooks. Because of the difficulty of that proof, however, one does not make use of it in a secondary-school class.

Relying on De Zolt's axiom, among others, it is possible in the theory of mensuration to prove the following theorem, which provides a necessary and sufficient condition for the equivalence of two polygons:

> *In order for polygons V and W to be equivalent, it is necessary and sufficient that they have equal areas.*

The question arises, do the [italicized] formulations of both statements above remain true sentences if equivalence is understood in a broader sense than it usually is in elementary geometry: that is to say, if two geometric figures (thus in particular, two polygons) are called equivalent when it is possible to divide them into the same finite number of respectively congruent *arbitrary* geometric figures not having *any* common points.

In the present article I show that this question ought to be given an *affirmative answer*. It is also interesting that the proofs of both of these very straightforward statements, the first of which may seem almost obvious, rely on results obtained by Prof. Stefan Banach with the aid of the entire apparatus of contemporary mathematical knowledge: in particular, with the help of the so-called *axiom of choice*.[4]

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[2] The definitions and theorems of elementary geometry to which I refer in the present article can be found, for instance, in the textbook Enriques and Amaldi [1903] 1916.

[3] See Hilbert [1899] 1922.

[4] Banach 1923 [discussed in section 4.2 of the present book]. All those notions and principles of set theory to which I refer in the present article—just a few—are contained in the book Sierpiński 1923.

Notation

By means of letters $p,q,s,\ldots$ I denote *points*, while the letters $A,B,K,P,V,\ldots$ [denote] *geometric figures*—that is, *point sets*.

The symbol $A \cup B$ denotes the *union of sets* A and B: that is, the set consisting of all those points that belong either to set A or to B. The notion of the union of sets can be extended with ease to an arbitrary finite number of components; it is even possible to consider the union of all sets that are terms of a certain infinite sequence. We use the symbols

$$\bigcup_{k=1}^{n} A_k \quad \text{and} \quad \bigcup_{k=1}^{\infty} A_k,$$

respectively, as well.[5]

The symbol $A - B$ denotes the *difference of sets* A and B: that is, the set consisting of all those points of the set A that do not belong to B.

As an expression that sets A and B *are identical*—that is, that they have all points in common—I write $A = B$. To express that the sets A and B *are disjoint*—that is, that they do not have any points in common—I will write A][B. Finally, to express that the set A *is a proper part of* set B —that is, that every point belonging to A also belongs to B, but not conversely —I will write $A \subsetneq B$.[6]

In this article I do not distinguish a point p from the set consisting solely of that same point p. In this way, for example, the symbol

$$\bigcup_{k=1}^{n} p_k$$

denotes the set consisting of points $p_1,p_2,\ldots,p_n$.

§1

I begin by recalling the familiar definition of the congruence of two arbitrary geometric figures, based on the notion of *equal distances between two pairs of points*. (This notion should, of course, either be defined earlier or assumed as a primitive notion.)

Definition 1. Point sets A and B are *congruent*— $A \cong B$ —if between their points a perfect (one-to-one) correspondence can be established that satisfies the following condition: if p and q are arbitrary points of set A, while p' and q' are their corresponding points in set B, then the distances between the pairs of points p and q, and p' and q', are equal.

In my following discussions, I will assume familiarity with elementary properties of the congruence relation.[7]

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[5] [For *union* Tarski used the Polish equivalent of the English term *sum*, and for $\cup$ and $\bigcup$ he used $+$ and Σ, respectively.]

[6] [In the original, Tarski used the symbol $\subset$ for *is a proper part of*. He employed it only once.]

[7] [The correspondence between $p,q,\ldots$ and $p',q',\ldots$ need not be a *direct* isometry; it may reverse orientation. Moreover, Tarski did *not* require that it be a restriction of an isometry of the entire plane.]

Definition 2. Point sets A and B *are equivalent*— $A \equiv B$ —if there exist sets $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_n$, where n [is] a natural number, that satisfy the conditions

(a) $A = \bigcup_{k=1}^{n} A_k$ and $B = \bigcup_{k=1}^{n} B_k$,

(b) $A_k \cong B_k$ whenever $1 \le k \le n$,

(c) $A_k \,][\, A_l$ and $B_k \,][\, B_l$ whenever $1 \le k < l \le n$.

To express that the figures A and B *are not equivalent* I shall write $A \not\equiv B$. In the following five theorems, I shall present several elementary properties of the relation of equivalence.

Theorem 1. If $A \cong B$, then $A \equiv B$. In particular, an arbitrary point set A satisfies the condition $A \equiv A$.

Theorem 2. If $A \equiv B$, then $B \equiv A$.

Both of those theorems follow directly from definition 2.

Theorem 3. If $A \equiv B$ and $B \equiv C$, then $A \equiv C$.

Proof. For the proof I shall apply a method similar to the one that we use in the proof of an analogous theorem in elementary geometry, the so-called "method of double networks."

In view of the equivalence of sets A and B as well as of B and C, there exist point sets $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_n$, as well as $B_1', B_2', \dots, B_m'$ and $C_1, C_2, \dots, C_m$ that satisfy all the conditions of definition 2. Let us denote by $B_{k,l}$ the set of all those points that belong simultaneously[8] to B_k and B_l'. Since every point in set B_k belongs to one of the sets B_l', [where] $1 \le l \le m$, and conversely, it is thus easy to check that

(1) $B_k = \bigcup_{l=1}^{m} B_{k,l}$ when $1 \le k \le n$,

(2) $B_l' = \bigcup_{k=1}^{n} B_{k,l}$ when $1 \le l \le m$.

In addition, according to condition (c) of definition 2, we have

(3) $B_{k,l} \,][\, B_{k_1,l_1}$ whenever $k \ne k_1$, or $k = k_1$ but $l \ne l_1$.

In accordance with condition (b) of definition 2, figures A_k and B_k are congruent [when] $1 \le k \le n$. From (1), (3), and the general properties of congruence, we thus infer

[8] Of course, the possibility is not excluded that some of the sets $B_{k,l}$ may be empty: that is, that they should not contain any points. A small modification to the proof would permit removing such sets from our consideration.

with ease the possibility of dividing each of the sets A_k into parts $A_{k,1}, A_{k,2}, \dots, A_{k,n}$ that satisfy the conditions

(4) $A_k = \bigcup_{l=1}^{m} A_{k,l}$ when $1 \leq k \leq n$,

(5) $A_{k,l} \cong B_{k,l}$ when $1 \leq k \leq n$ and $1 \leq l \leq m$,

(6) $A_{k,l}][A_{k_1,l_1}$ whenever $k \neq k_1$, or $k = k_1$ but $l \neq l_1$.

Similarly, from the congruence of figures B_l' and C_l [when] $1 \leq l \leq m$, [and] from (2) and (3), follows the possibility of analogous division of each of the sets C_l into parts $C_{1,l}, C_{2,l}, \dots, C_{n,l}$:

(7) $C_l = \bigcup_{k=1}^{n} C_{k,l}$ when $1 \leq l \leq m$,

(8) $C_{k,l} \cong B_{k,l}$ when $1 \leq k \leq n$ and $1 \leq l \leq m$,

(9) $C_{k,l}][C_{k_1,l_1}$ whenever $k \neq k_1$, or $k = k_1$ but $l \neq l_1$.

Since according to condition (a) of definition 2 the set A is the union of sets $A_1, A_2, \dots, A_n$, and set C [is] the union of sets $C_1, C_2, \dots, C_m$, we may thus conclude from (4) and (7),

(10) $A = \bigcup_{k=1}^{n} \bigcup_{l=1}^{m} A_{k,l}$,

(11) $C = \bigcup_{l=1}^{m} \bigcup_{k=1}^{n} C_{k,l} = \bigcup_{k=1}^{n} \bigcup_{l=1}^{m} C_{k,l}$.

Moreover, from (5) and (8) we immediately obtain

(12) $A_{k,l} \cong C_{k,l}$ when $1 \leq k \leq n$ and $1 \leq l \leq m$.

Equations (10) and (11) show that each of the sets A and C can be divided into $n \cdot m$ parts, which in view of (6) and (9) have no points in common, and in accordance with (12) are respectively congruent to each other. Therefore, according to definition 2, $A \equiv C$, Q.E.D.

Theorems 1–3 express that the relation of equivalence is reflexive, symmetric, and transitive.

Theorem 4. If

(1) $A_k \equiv B_k$ (possibly $A_k \cong B_k$ or $A_k = B_k$) whenever $1 \leq k \leq n$,

(2) $A_k][A_l$ and $B_k][B_l$ whenever $1 \leq k < l \leq n$,

then $\bigcup_{k=1}^{n} A_k \equiv \bigcup_{k=1}^{n} B_k$.

Proof. [To facilitate typesetting, those two unions will be denoted by A and B, respectively.] Taking theorem 1 into account, we can restrict ourselves in the proof to considering the hypothesis that $A_k \equiv B_k$ for each value of k [such that] $1 \le k \le n$. Now according to definition 2, for each pair of sets A_k and B_k there exist sets $A_{k,1}, A_{k,2}, \ldots, A_{k,m_k}$ and $B_{k,1}, B_{k,2}, \ldots, B_{k,m_k}$ that satisfy the conditions

(1) $A_k = \bigcup_{l=1}^{m_k} A_{k,l}$ and $B_k = \bigcup_{l=1}^{m_k} B_{k,l}$ when $1 \le k \le n$,

(2) $A_{k,l} \cong B_{k,l}$ when $1 \le k \le n$ and $1 \le l \le m_k$,

(3) $A_{k,l} \mathbin{][} A_{k_1,l_1}$ and $B_{k,l} \mathbin{][} B_{k_1,l_1}$ whenever $k \ne k_1$, or $k = k_1$ but $l \ne l_1$.

From (1) we immediately obtain

(4) $A = \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} \bigcup_{l=1}^{m_k} A_{k,l}$ and $B = \bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} \bigcup_{l=1}^{m_k} B_{k,l}$.

From statements 2–4 it follows that the point sets A and B can be divided into the same finite number of respectively congruent parts without common points. From this, in accordance with definition 2, $A \equiv B$, Q.E.D.

The preceding theorem can be expressed in words in the following way:

If two given point sets can be divided into the same finite number of respectively equivalent parts having no common points, then these sets are equivalent.

Theorem 5. If sets A and B consist of the same finite number of points, then $A \equiv B$.

For the proof, it suffices to note that according to definition 1, two arbitrary points are congruent figures, hence definition 2 can be applied directly.

§2

I turn now to the proof of theorem 6, which can be regarded as a *generalization of De Zolt's axiom*. The proof will rely on the theorem of Banach mentioned already in the introduction, which for our purposes can be adequately formulated in the following way.

Banach's Theorem. Each point set A that is part of any polygon can be assigned some nonnegative real number $m(A)$, called the *measure* of that set. Moreover, the following conditions are satisfied:

(1) if $A \cong B$ then $m(A) = m(B)$,

(2) if $A \mathbin{][} B$ then $m(A \cup B) = m(A) + m(B)$,

(3) if W is a polygon then $m(W)$ is its area.[9]

[9] To the word *area* one ought to append throughout the words, *in relation to some square, chosen as the unit of area*. [Banach's theorem and its proof are discussed in section 4.2.]

Theorem 6. If V and W are polygons and $V \subsetneq W$, then $V \not\equiv W$.

Proof. Suppose, contrary to the conclusion of the theorem, that

(1) $V \equiv W$;

then there would exist point sets $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ that satisfy all the conditions of definition 2 [with point sets V, W in place of A, B].

Further, in accordance with Banach's theorem, let us assign to every bounded planar set—that is, part of any polygon—its measure. Condition 2 of the theorem mentioned can be extended with ease, by applying the principle of mathematical induction, to a sum of an arbitrary finite number of sets having no common points. In view of conditions (a) and (c) of definition 2, we infer from this that

(2) $$m(V) = \sum_{k=1}^{n} m(A_k), \quad m(W) = \sum_{k=1}^{n} m(B_k).$$

From condition 1 of Banach's theorem and condition (b) of definition 2 we obtain

(3) $m(A_k) = m(B_k)$ whenever $1 \leq k \leq n$.

Equations (2) and (3) entail immediately

(4) $m(V) = m(W)$.

Therefore, according to condition 3 of the cited theorem, polygons V and W must have equal areas, which contradicts the assumption of our theorem, since V is a proper part of W.

Assumption (1) thus leads to a contradiction, and we must accept that $V \not\equiv W$, Q.E.D.

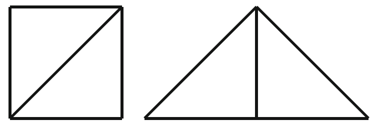
Reasoning in an analogous way, [we can] prove the more general

Theorem 7. If V and W are polygons with different areas, then $V \not\equiv W$.

§3

We now take up the proof of the theorem converse to the one just presented. First of all, we note that despite what might at first glance be supposed, this theorem does not follow directly from an analogous theorem of elementary geometry. I will illustrate this circumstance with a straightforward example.

Let V be an arbitrary square and W, an isosceles right triangle with base twice as long as the edge of the square. Having equal areas, V and W are thus equivalent in the sense of elementary geometry. In fact, each of these polygons can be divided into two



triangles without common interior points, respectively congruent. (See the figure.) From this subdivision, however, the subdivision that would satisfy definition 2 cannot be obtained via a direct route. Although the interiors of the [smaller] triangles are in fact congruent, nevertheless the [parts] that stand out in the figure are broken lines, [and] since these unions of the boundaries of the triangles have different length, it is not hard to demonstrate that they are not equivalent in the sense that we established in the present article.[10]

The proof of the theorem that interests us will rely on several lemmas.

Lemma I. If A is a plane set having interior points,[11] whereas set B consists of a finite number of points, and $A\ ][\ B$, then $A \equiv A \cup B$.

Proof. Certainly there exists some disk K that is part of the set A; let us denote its center by s. Let us choose some positive irrational number α and some point p_0 lying on the circumference of the disk K. For each natural number k let us denote by p_k the point resulting from the rotation of point p_0 about point s through an angle whose degree measure is the number $k \cdot \alpha$ (or a number differing from $k \cdot \alpha$ by a multiple of 360) —and we always carry out the rotation in some specified direction. From this, since the angle of α degrees is incommensurate with a full angle, we infer with ease that no two points p_k and p_l with different indices are identical.

Let n be the number of points in the set B. Set

$$(1)\quad B' = \bigcup_{k=0}^{n-1} p_k\,,$$

$$(2)\quad C = \bigcup_{k=0}^{\infty} p_k\,,$$

$$(3)\quad C' = \bigcup_{k=n}^{\infty} p_k\,,$$

$$(4)\quad D = A - C.$$

From (1) to (4) and the definition of the points p_k we immediately obtain

$$(5)\quad A = C \cup D = B' \cup C' \cup D,$$

$$(6)\quad A \cup B = B \cup C \cup D.$$

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[10] [If the correspondence of the interiors of the triangles were extended by somehow subdividing the segments shown on the left and rearranging the parts to form those on the right, the total length of the left-hand segments would equal that on the right. But they differ: $4 + \sqrt{2} \neq 3 + 2\sqrt{2}$.]

[11] We call point p an *interior* point of a plane set A if there exists a disk with center p that is a part of the set A.

According to theorem 5, since each of the sets B and B' consists of n points, this statement follows:

(7) $B \equiv B'$.

With ease, we also convince ourselves that

(8) $C \cong C'$.

In fact, if the set C is rotated about an angle of $n \cdot a$ degrees, then it covers the set C'. In other words, if we assign to an arbitrary point p_k of the set C the point p_{k+n} of the set C', then we define a perfect correspondence between the points of these sets, that satisfies the conditions of definition 1.

Statements 5 to 8 show that the sets A and $A \cup B$ can be divided into the same finite number of parts, respectively equivalent, or even congruent or identical. It is also easy to check, [by] relying on statements 1 to 4, by the way of specifying the points p_k, and [by] the hypothesis of the theorem, that no two of the three parts into which we divide each of these sets have common points. From this, in accordance with theorem 4, we infer that $A \equiv A \cup B$, Q.E.D.

Lemma II. If A is a plane set having interior points, while the set B consists of all points of some segment except at most the end points, and A][B, then $A \equiv A \cup B$.

Proof. The ideas behind the proofs of lemmas I and II are similar to each other. Let us denote by δ the length of the segment from which B differs by at most the absence of the end points. Certainly there exists a natural number n large enough that some disk K that has a radius of length equal to δ/n is part of the set A.

Clearly, the set B can be divided into n segments of length δ/n without common interior points, and two of these might have only one endpoint each. Let us denote the interiors of these segments by $C_0, C_1, \ldots, C_{n-1}$ and set

(1) $C = \bigcup_{k=0}^{n-1} C_k$,

(2) $D = B - C$.

It is easy to see that D is a set consisting of a finite number of points.[12]

Let us choose some positive irrational number α [and] denote by C_0' the interior of some radial segment of the disk K; when k is an arbitrary natural number, [denote] by C_k' the set formed by rotating the set C_0' through an angle of $k \cdot \alpha$ degrees about the center of the disk K in a certain specified direction. As in the proof of lemma I, we convince ourselves that no two of the sets C_k' and C_l' with different indices have common points. Set

[12] [This number is] equal to $n + 1$, n, or $n - 1$, depending on whether the set B has both endpoints, or just one, or, lastly, does not have any.

(3) $C' = \bigcup_{k=0}^{n-1} C'_k,$

(4) $E = \bigcup_{k=0}^{\infty} C'_k,$

(5) $E' = \bigcup_{k=n}^{\infty} C'_k,$

(6) $F = A - E.$

From (1) to (6), we immediately obtain

$$B = C \cup D, \quad E = C' \cup E', \quad A = E \cup F = C' \cup E' \cup F,$$

from which [follow]

(7) $A \cup D = C' \cup D \cup E' \cup F,$

(8) $A \cup B = C \cup D \cup E \cup F.$

As the interiors of segments of the same length δ/n, the sets C_k and C'_k are congruent. Therefore, from (1) and (3) we infer

(9) $C \equiv C'.$

Furthermore, reasoning as in the proof of lemma I, we reach the conclusion that

(10) $E \cong E'.$

Finally, as it is not difficult to be convinced, no two of the sets C', D, E', F nor of C, D, E, F have common points. In view of this, we can apply theorem 4; by virtue of statements 7 to 10 we have

(11) $A \cup D \equiv A \cup B.$

On the other hand, the set D, as we already noticed, consists of a finite number of points. Therefore, according to lemma I,

(12) $A \equiv A \cup D.$

From (11) and (12) it follows, in accordance with theorem 3, that $A \equiv A \cup B$, Q.E.D.

Lemma III. If A is a plane set having interior points, while B [is] the union of a finite number of segments, and $A\ ][\ B$, then $A \equiv A \cup B$.

Proof. It is nearly obvious that the set B can be regarded as a union of a finite number of segments $B_1, B_2, \ldots, B_n$ without common interior points. Let us set $B'_1 = B_1$, and when $2 \le k \le n$ denote by B'_k the set differing from the segment B_k in at most the absence of one or two endpoints and that of the points belonging to any of the segments $B_1, B_2, \ldots, B_{k-1}$ [—that is,]

$$B'_k = B_k - \bigcup_{l=1}^{k-1} B_l\,.$$[13]

We obtain

$$(1)\quad B = \bigcup_{k=1}^{n} B'_k,$$

$$(2)\quad B'_k \,][\, B'_l \ \text{ whenever } \ 1 \le k < l \le n.$$

On the other hand, there certainly exist sets $A_1, A_2, \dots, A_n$ having interior points and also satisfying the conditions

$$(3)\quad A = \bigcup_{k=1}^{n} A_k,$$

$$(4)\quad A_k \,][\, A_l \ \text{ whenever } \ 1 \le k < l \le n.$$

In fact, if we divide into n circular sectors some disk K that is part of the set A, denote by $A_1, A_2, \dots, A_{n-1}$ the interiors of all but one of these sectors, and set

$$A_n = A - \bigcup_{k=1}^{n-1} A_k,$$

then we will at that time obtain sets with the desired properties.

In view of (1), (3), and the condition $A \,][\, B$ given in the hypothesis of the theorem, we have

$$(5)\quad A_k \,][\, B_l \ \text{ whenever } 1 \le k \le n \ \text{ and } \ 1 \le l \le n.$$

Thus, we can assert with ease that every pair of sets A_k and B_k, where $1 \le k \le n$, satisfies the conditions of lemma II. Therefore,

$$(6)\quad A_k \equiv A_k \cup B'_k \ \text{ when } \ 1 \le k \le n.$$

From statements (2), (4), and (5) we conclude further that

$$(7)\quad A_k \cup B'_k \,][\, A_l \cup B'_l \ \text{ whenever } 1 \le k < l \le n.$$

Moreover, from (1) and (3) also follows

$$(8)\quad A = \bigcup_{k=1}^{n} A_k, \qquad A \cup B = \bigcup_{k=1}^{n} \left(A_k \cup B'_k\right).$$

In accordance with (6) to (8) the sets $A_1, A_2, \dots, A_n$ and $A_1 \cup B'_1, A_2 \cup B'_2, \dots, A_n \cup B'_n$, [whose unions[14] are] sets A and $A \cup B$, satisfy all the conditions of theorem 4. Thus, we finally obtain $A \equiv A \cup B$, Q.E.D.

Lemma III now enables us [to give] a direct proof of the theorem converse to theorem 7.

[13] [This sentence and the next might not fully explain the first sentence of the proof. One can apply mathematical induction as follows: if B is the union of a finite number of segments without common interior points, and C is a segment, then $C - B$ is also such a union, and $B \cup C = B \cup (C - B)$.]

[14] [In the original, the phrase here in brackets was vague: *w stosunku do*.]

Theorem 8. If V and W are polygons with the equal areas, then $V \equiv W$.

Proof. As is well known, polygons V and W are equivalent in the sense of elementary geometry. Thus, they can be divided into the same number of polygons having no common interior points. Let $V_1, V_2, \ldots, V_n$ and $W_1, W_2, \ldots, W_n$ be the interiors of the polygons obtained as a result of such a division. Certainly we have

(1) $V_k \cong W_k$ when $1 \le k \le n$,

(2) $V_k \,][\, V_l$, and also $W_k \,][\, W_l$, whenever $1 \le k < l \le n$.

According to definition 2 we infer from (1) and (2) that

$$(3) \quad \bigcup_{k=1}^{n} V_k \equiv \bigcup_{k=1}^{n} W_k.$$

Set

$$(4) \quad A = V - \bigcup_{k=1}^{n} V_k, \qquad B = W - \bigcup_{k=1}^{n} W_k.$$

From this we immediately obtain

$$(5) \quad A \,][\, \bigcup_{k=1}^{n} V_k, \qquad B \,][\, \bigcup_{k=1}^{n} W_k,$$

and

$$(6) \quad V = A \cup \bigcup_{k=1}^{n} V_k, \qquad W = B \cup \bigcup_{k=1}^{n} W_k.$$

In view of (4) it is easy to see that A and B are broken lines, the unions of the boundaries of the polygons that we obtained by the subdivision of V and W; each is thus the union of a finite number of segments. Moreover, since the sets

$$\bigcup_{k=1}^{n} V_k \quad \text{and} \quad \bigcup_{k=1}^{n} W_k$$

certainly have interior points, after applying lemma III [and] in accordance with (5) and (6), we thus obtain

$$(7) \quad \bigcup_{k=1}^{n} V_k \equiv A \cup \bigcup_{k=1}^{n} V_k = V,$$

$$(8) \quad \bigcup_{k=1}^{n} W_k \equiv B \cup \bigcup_{k=1}^{n} W_k = W.$$

From statements (3), (7), and (8), according to theorem 3, it follows that $V \equiv W$, Q.E.D.

Theorems 7 and 8 immediately entail

Conclusion 9. In order for polygons V and W to be equivalent, it is necessary and sufficient that they have equal areas.

Theorem 6 and conclusion 9 settle the question posed at the beginning of the present article.

The question arises here whether the statements are true [that are] analogous to the theorems proved in this article but relate to polyhedra instead of polygons. As it happens, such statements are false. Specifically, the following theorem can be proved:

Two arbitrary polyhedra are equivalent.

The proof of this statement is complicated enough not to include it here: it is contained in a joint article by Banach and me, entitled *On Decomposition of Point Sets into Respectively Congruent Parts*.[15]

We are easily made aware of how greatly the above theorem contradicts our intuitions, if we consider so much as the following conclusion that flows from it:

An arbitrary cube can be divided into a finite number of parts without common points, which then can be rearranged to form a cube with an edge twice as long.

The theorem becomes even more striking when we recall that, as Max Dehn showed,[16] even two polyhedra with equal volumes may not be equivalent in the sense of elementary geometry.

In conclusion, I pose here the following problem, which as far as is known, is to this day not settled:

Can theorem 8 be extended to arbitrary plane regions bounded by closed curves? Specifically, can a disk and a polygon with equal areas be equivalent in the sense of definition 2?[17]

Summary[18]

On the Equivalence of Polygons

In elementary geometry two polygons (or polyhedra) are called *equivalent by decomposition* if they can be decomposed into the same finite number of respectively congruent polygons (or polyhedra) that have no common interior points. In the theory of equivalence of polygons the following theorem, sometimes called *De Zolt's axiom*, plays an important role:

1. *Two arbitrary polygons, one of which is [properly] contained in the other, are never equivalent by decomposition.*

[15] [Banach and Tarski [1924] 2014, translated in chapter 6, with background and summary in section 4.4.]

[16] Compare Amaldi [1900] 1914, §11, 161–172. [Tarski failed to mention the author, Ugo Amaldi. See also Dehn 1901–1902.]

[17] A disk and a polygon with equal areas are not equivalent in the sense of elementary geometry: compare Amaldi [1900] 1914, §§6–7, 151–157. [See the previous footnote. This problem, known as *Tarski's circle-squaring problem*, was published separately as Tarski 1925b. It has since been solved, affirmatively: see the discussion in section 4.3 of the present book. The wrong text was printed for Tarski 1925b in the *Collected Papers* volume Tarski 1986a; the original text is reproduced and translated in section 4.3.]

[18] [In the original, the summary was in French.]

Starting from this principle, one establishes the following theorem, which presents a necessary and sufficient condition for the equivalence of polygons:

2. *In order that two polygons should be equivalent by decomposition, it is necessary and sufficient that they should have equal areas.*

In my note I envisage the notion of equivalence in a sense more general than that of elementary geometry: two point sets (thus, in particular, two polygons or polyhedra) are termed *equivalent by decomposition* if they can be decomposed into the same finite number of respectively congruent *arbitrary* point sets that have no common points.

I prove that, *even admitting this definition of equivalence, theorems 1 and 2 remain valid*.

In demonstrating the cited theorems I rely on results obtained by Banach in measure theory (Banach 1923). Establishing theorem 2, I also make use of the following lemma:

P being the interior of a polygon and Q the point set obtained from P by adding a finite number of segments, the sets P and Q are equivalent by decomposition.

It is interesting to remark that in attributing to equivalence the sense established in this note, theorems 1 and 2 may not be extended to polyhedra. This results from the following theorem, which perhaps seems paradoxical:

Two arbitrary polyhedra (with equal volumes or not) are equivalent by decomposition.

This theorem is demonstrated in the note Banach and Tarski [1924] 2014.[19]

[19] [Summarized in section 4.4 of the present book and translated in full in chapter 6.]