On a Weak Version of Hyers–Ulam Stability Theorem in Restricted Domains

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Abstract In this chapter we consider a weak version of the Hyers–Ulam stability problem for the Pexider equation, Cauchy equation satisfied in restricted domains in a group when the target space of the functions is a 2-divisible commutative group. As the main result we find an approximate sequence for the unknown function satisfying the Pexider functional inequality, the limit of which is the approximate function in the Hyers–Ulam stability theorem.

Keywords Hyers–Ulam stability · Functional equations · Restricted domains · Pexider equation · 2-divisible commutative group

1 Introduction

The Hyers–Ulam stability problems of functional equations were originated by S. M. Ulam in 1940 when he proposed the following question [36]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot, \cdot)$ such that

$$d(f(xy),\ f(x)f(y)) \leq \varepsilon.$$

Then does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that

$$d(f(x),\ h(x)) \leq \delta_\epsilon$$

for all $x \in G_1$?

One of the first assertions to be obtained is the following result, essentially due to D. H. Hyers [20], that gives an answer for the question of Ulam.

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Theorem 1 *Suppose that S is a commutative semigroup, B is a Banach space, $\epsilon \geq 0$, and $f : S \to B$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \tag{1}$$

for all $x, y \in S$. Then there exists a unique function $A : S \to B$ satisfying

$$A(x+y) = A(x) + A(y) \tag{2}$$

and

$$\|f(x) - A(x)\| \leq \epsilon \tag{3}$$

for all $x \in S$.

In 1950, this result was generalized by T. Aoki [4] and D.G. Bourgin [9, 8]. In 1978 T.M. Rassias generalized the Hyers' result to new approximately linear mappings [?]. Since then the stability problems have been investigated in various directions for many other functional equations. Among the results, the stability problem in a restricted domain was investigated by F. Skof, who proved the stability problem of the inequality (1) in a restricted domain [35]. Several papers have been published on the Hyers–Ulam stability in restricted domains for a large variety of functional equations including the Jensen functional equation [24], quadratic type functional equations [23], mixed type functional equations [30], and Jensen type functional equations [31]. The results can be summarized as follows: Let X and B be a real normed space and a real Banach space, respectively. For fixed $d \geq 0$, if $f : X \to B$ satisfies the functional inequalities (such as that of Cauchy, quadratic, Jensen, and Jensen type, etc.) for all $x, y \in X$ with $\|x\| + \|y\| \geq d$, then the inequalities hold for all $x, y \in X$.

In [14, 15], generalizing the restricted domains such as $\|x\|+\|y\| \geq d$ in a normed space to some abstract domains in a group, we consider the stability problem of Pexider equation and Jensen-type equations in the restricted domains. In the present paper, we consider a weak version of Hyers–Ulam stability of the Pexider equation when the target space of the functions in given functional inequalities are not a normed space but a 2-divisible commutative group. Note that the existence of the approximate additive function A in Theorem 1 is due to the completeness of the target space B. For example, if Y is a noncomplete normed space and $f : S \to Y$ satisfies (1), then we can only find a Cauchy sequence $a_n : S \to Y$ such that

$$|a_n(x+y) - a_n(x) - a_n(y)| \leq 2^{-n}\epsilon \tag{4}$$

for all $x, y \in S$, $n = 1, 2, 3, \ldots$, and

$$|f(x) - a_n(x)| \leq \epsilon \tag{5}$$

for all $x \in S$ and $n = 1, 2, 3, \ldots$. Throughout this paper, we denote a commutative group by G and a 2-divisible commutative group by H respectively, $0 \in V \subset H$ and $W \subset G \times G$. Also, we denote a Banach space and a real normed space by B

and Y, respectively, and $f, g, h : G \to H$(or Y, B). In Sect. 2 of this chapter, we consider the behavior of $f : G \to H$ satisfying

$$f(x+y) - f(x) - f(y) \in V \tag{6}$$

for all $x, y \in G$. As a result we prove that there exists a Cauchy-type sequence $a_n : G \to H$ (which is a Cauchy sequence when $H = Y$) such that

$$f(x) - a_n(x) \in 2^{-n}(V + 2V + \ldots + 2^{n-1}V) \tag{7}$$

for all $x \in G$. In Sect. 3, we consider

$$f(x+y) - g(x) - h(y) \in V \tag{8}$$

for all $(x, y) \in W \subset G \times G$. As the main result we prove that under some assumptions on W, if f, g, h satisfy (8) then there exist approximate Cauchy-type sequences a_n, b_n, and c_n for f, g, and h respectively. From our result we obtain the Hyers–Ulam stability theorem for Pexider equation when $f, g, h : G \to B$.

2 A Weak Stability of Pexider Equation

For subsets V, V_1, V_2 of H, $v \in V$, and $n \in \mathbb{N}$, we define

$$nv = \underbrace{v + \cdots + v}_{n-\text{times}}, \quad nV = \{nv : v \in V\}, \quad 2^{-n}V = \{h \in H : 2^n h \in V\},$$

and

$$V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}.$$

We call $a_n : G \to H$ a V-Cauchy sequence if

$$a_{m+n}(x) - a_m(x) \in 2^{-m-n}(V + 2V + \ldots + 2^{n-1}V)$$

for all $m, n = 1, 2, 3, \ldots$, and $x \in G$.

First we consider the weak version of the Hyers–Ulam stability theorem for the Cauchy equation.

Theorem 2 *Suppose that $f : G \to H$ satisfies*

$$f(x+y) - f(x) - f(y) \in V \tag{9}$$

for all $x, y \in G$. Then there exists a V-Cauchy sequence $a_n : G \to H$ satisfying

$$a_n(x+y) - a_n(x) - a_n(y) \in 2^{-n}V, \tag{10}$$

and

$$a_n(x) - f(x) \in 2^{-n}(V + 2V + \ldots + 2^{n-1}V) \tag{11}$$

for all $x, y \in G$ *and* $n \in \mathbb{N}$.

Proof Note that since H is 2-divisible, for each $n \in \mathbb{N}$ and $x \in G$ we can choose an $a_n(x)$ such that

$$2^n a_n(x) = f(2^n x). \tag{12}$$

Replacing y by x in (9) and using induction argument we have

$$\begin{aligned} 2^{n-1} f(2x) - 2^n f(x) &\in 2^{n-1} V \\ 2^{n-2} f(4x) - 2^{n-1} f(2x) &\in 2^{n-2} V \\ &\cdots\cdots\cdots \\ 2 f(2^{n-1}x) - 4 f(2^{n-2}x) &\in 2V \\ f(2^n x) - 2 f(2^{n-1}x) &\in V \end{aligned}$$

for all $x \in G$. Thus it follows that

$$f(2^n x) - 2^n f(x) \in V + 2V + \ldots + 2^{n-1} V \tag{13}$$

for all $x \in G$. Now it follows from (12) and (13) that

$$a_n(x) - f(x) \in 2^{-n}(V + 2V + \ldots + 2^{n-1} V) \tag{14}$$

for all $x \in G$. Replacing x by $2^m x$ in (13) and using (12) we have

$$a_{m+n}(x) - a_m(x) \in 2^{-m-n}(V + 2V + \ldots + 2^{n-1} V) \tag{15}$$

for all $x \in G$, which implies that a_n is V-Cauchy. Replacing x by $2^n x$ and y by $2^n y$ in (9) and using (12) we have

$$a_n(x + y) - a_n(x) - a_n(y) \in 2^{-n} V \tag{16}$$

for all $n \in \mathbb{N}$ and $x \in G$. This completes the proof.

Let $\langle Y, \|\cdot\|\rangle$ be a normed space and $V = \{x \in Y : \|x\| \le \epsilon\}$. Then we have

$$2^{-n}(V + 2V + \ldots + 2^{n-1} V) \subset \{x \in Y : \|x\| \le \epsilon\}$$

for all $n \in \mathbb{N}$, and

$$2^{-m-n}(V + 2V + \ldots + 2^{n-1} V) \subset \{x \in Y : \|x\| \le 2^{-m}\epsilon\}$$

for all $m, n \in \mathbb{N}$. Thus in this case, every V-Cauchy sequence is a Cauchy sequence. Now as a direct consequence of Theorem 2 we have the following.

Corollary 1 *Let* $\epsilon > 0$. *Suppose that* $f : G \to Y$ *satisfies*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon \tag{17}$$

for all $x, y \in G$. Then there exists a Cauchy sequence $a_n : G \to Y$ satisfying

$$\|a_n(x+y) - a_n(x) - a_n(y)\| \le 2^{-n}\epsilon \tag{18}$$

for all $n \in \mathbb{N}$ and $x, y \in G$, and

$$\|a_n(x) - f(x)\| \le \epsilon \tag{19}$$

for all $x \in G$.

In particular, if $f : G \to B$, then there exists $A : G \to B$ such that

$$\lim_{n\to\infty} a_n(x) = A(x).$$

Letting $n \to \infty$ in (18) we have

$$A(x+y) - A(x) - A(y) = 0 \tag{20}$$

for all $x, y \in G$. We call a function $A : G \to B$ satisfying (20) an *additive function*. Thus as a direct consequence of Corollary 1 we have the well known Hyers–Ulam stability theorem.

Corollary 2 *Let $\epsilon > 0$. Suppose that $f : G \to B$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon \tag{21}$$

for all $x, y \in G$. Then there exists an additive function $A : G \to B$ such that

$$\|f(x) - A(x)\| \le \epsilon \tag{22}$$

for all $x \in G$.

Throughout this chapter we denote

$$V^* = \{v_1 + v_2 - v_3 - v_4 : v_j \in V,\ j = 1,2,3,4\}.$$

Theorem 3 *Suppose that $f, g, h : G \to H$ satisfy*

$$f(x+y) - g(x) - h(y) \in V \tag{23}$$

for all $x, y \in G$. Then there exist V^*-Cauchy sequences $a_n, b_n, c_n : G \to H$ satisfying

$$a_n(x+y) - a_n(x) - a_n(y) \in 2^{-n}V^* \tag{24}$$

$$b_n(x+y) - b_n(x) - b_n(y) \in 2^{-n}V^* \tag{25}$$

$$c_n(x+y) - c_n(x) - c_n(y) \in 2^{-n}V^* \tag{26}$$

for all $n \in \mathbb{N}$ and $x, y \in G$, and

$$a_n(x) - f(x) + f(0) \in V_n^*, \tag{27}$$

$$b_n(x) - g(x) + g(0) \in V_n^*, \tag{28}$$

$$c_n(x) - h(x) + h(0) \in V_n^*, \tag{29}$$

and

$$a_n(x + y) - b_n(x) - c_n(y) \in V_n^{**} \tag{30}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$, *where*

$$V_n^* = 2^{-n}(V^* + 2V^* + \ldots + 2^{n-1}V^*),$$

$$V_n^{**} = V - V + V_n^* - V_n^* - V_n^*.$$

Proof Let $D(x, y) = f(x + y) - g(x) - h(y)$. Then we have

$$f(x + y) - f(x) - f(y) + f(0) = D(x, y) + D(0, 0) - D(x, 0) - D(y, 0) \in V^* \tag{31}$$

$$g(x + y) - g(x) - g(y) + g(0) = D(x, y) + D(y, 0) - D(x + y, 0) - D(0, y) \in V^* \tag{32}$$

$$h(x + y) - h(x) - h(y) + h(0) = D(x, y) + D(0, x) - D(0, x + y) - D(x, 0) \in V^* \tag{33}$$

for all $x, y \in G$. Thus, in view of (31), (32), and (33), using Theorem 2 for $f(x) - f(0)$, $g(x) - g(0)$, $h(x) - h(0)$, we obtain (24)–(29). Now, putting $x = y = 0$ in (23), we have

$$f(0) - g(0) - h(0) \in V. \tag{34}$$

Then, by (23), (27), (28), (29), and (34) we get (30).

This completes the proof.

In particular, let $V = \{x \in Y : \|x\| \le \epsilon\}$. Then we have

$$V_n^* \subset \{x \in Y : \|x\| \le 4\epsilon\}, \quad V_n^{**} \subset \{x \in Y : \|x\| \le 14\epsilon\}$$

for all $n \in \mathbb{N}$. Thus as a direct consequence of Theorem 3 we have the following.

Corollary 3 *Let* $\epsilon > 0$. *Suppose that* $f, g, h : G \to Y$ *satisfy*

$$\|f(x + y) - g(x) - h(y)\| \le \epsilon \tag{35}$$

for all $x, y \in G$. *Then there exist Cauchy sequences* $a_n, b_n, c_n : G \to Y$ *satisfying*

$$\|a_n(x + y) - a_n(x) - a_n(y)\| \le 2^{-n+2}\epsilon \tag{36}$$

$$\|b_n(x + y) - b_n(x) - b_n(y)\| \le 2^{-n+2}\epsilon \tag{37}$$

$$\|c_n(x+y)-c_n(x)-c_n(y)\| \le 2^{-n+2}\epsilon \tag{38}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$, *and*

$$\|f(x)-a_n(x)-f(0)\| \le 4\epsilon, \tag{39}$$

$$\|g(x)-b_n(x)-g(0)\| \le 4\epsilon, \tag{40}$$

$$\|h(x)-c_n(x)-h(0)\| \le 4\epsilon \tag{41}$$

and

$$\|a_n(x+y)-b_n(x)-c_n(y)\| \le 14\epsilon \tag{42}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$.

Corollary 4 *Let* $\epsilon > 0$. *Suppose that* $f, g, h : G \to B$ *satisfy*

$$\|f(x+y)-g(x)-h(y)\| \le \epsilon \tag{43}$$

for all $x, y \in G$. *Then there exists an additive function* $A : G \to B$ *such that*

$$\|f(x)-A(x)-f(0)\| \le 4\epsilon,$$
$$\|g(x)-A(x)-g(0)\| \le 4\epsilon,$$
$$\|h(x)-A(x)-h(0)\| \le 4\epsilon$$

for all $x \in G$.

Proof Let $A_1(x) = \lim_{n\to\infty} a_n(x)$, $A_2(x) = \lim_{n\to\infty} b_n(x)$, $A_3(x) = \lim_{n\to\infty} c_n(x)$. Then it follows from (36)–(38) that for each $j = 1, 2, 3$, A_j is an additive function. Letting $n \to \infty$ in (39)–(41) we have

$$\|f(x)-A_1(x)-f(0)\| \le 4\epsilon,$$
$$\|g(x)-A_2(x)-g(0)\| \le 4\epsilon,$$
$$\|h(x)-A_3(x)-h(0)\| \le 4\epsilon$$

for all $x \in G$. Finally, letting $n \to \infty$ in (42) we have

$$\|A_1(x+y)-A_2(x)-A_3(y)\| \le 14\epsilon \tag{44}$$

for all $x, y \in G$. Putting $y = 0$ and $x = 0$ in (44) separately, we have

$$\|A_1(x)-A_2(x)\| \le 14\epsilon$$
$$\|A_1(y)-A_3(y)\| \le 14\epsilon$$

for all $x, y \in G$, which implies that $A_1 = A_2$ and $A_1 = A_3$. This completes the proof.

3 Weak Stability of Pexider Equation in Restricted Domains

It is a frequent situation to consider a functional equation satisfied in a restricted domain or satisfied under a restricted condition [3, 5–7, 10–12, 15, 18, 28, 32–35]. In this section we consider the weak version of the Hyers–Ulam stability theorem in some restricted domains in G. We use the following usual notations. Let $G \times G = \{(a_1, a_2) : a_1, a_2 \in G\}$ be the product group. For a subset K of $G \times G$ and $a \in G \times G$, we define $a + K = \{a + k : k \in K\}$. For given $x, y \in G$ we denote the sets of points of the forms (not necessarily distinct) in $G \times G$ by $P_{x,y}$, $Q_{x,y}$, and$R_{x,y}$, respectively as,

$$P_{x,y} = \{(0, 0), (x, 0), (0, y), (x, y)\},$$
$$Q_{x,y} = \{(y, 0), (0, y), (x, y), (x + y, 0)\},$$
$$R_{x,y} = \{(x, 0), (0, x), (x, y), (0, x + y)\},$$

where 0 is the identity element of G. The set $P_{x,y}$ can be viewed as the vertices of a rectangle in $G \times G$, and $Q_{x,y}$ and $R_{x,y}$ can be viewed as the vertices of parallelograms in $G \times G$.

Definition 1 Let $W \subset G \times G$. We introduce the following conditions $(C1)$, $(C2)$, and $(C3)$ on W: For any $x, y \in G$, there exist $z_1, z_2, z_3 \in G$ such that

$$(C1) \quad (-z_1, z_1) + P_{x,y} \subset W,$$
$$(C2) \quad (0, z_2) + Q_{x,y} \subset W,$$
$$(C3) \quad (z_3, 0) + R_{x,y} \subset W,$$

respectively.

Example 1 Let G be a real normed space. For $\alpha, \beta, d \in \mathbb{R}$, let

$$U = \{(x, y) \in G \times G \ : \ \alpha\|x\| + \beta\|y\| \ge d\}, \tag{45}$$
$$V = \{(x, y) \in G \times G \ : \ \|\alpha x + \beta y\| \ge d\}. \tag{46}$$

Then U satisfies $(C1)$ if $\alpha + \beta > 0$, $(C2)$ if $\beta > 0$ and $(C3)$ if $\alpha > 0$, and V satisfies $(C1)$ if $\alpha \ne \beta$, $(C2)$ if $\beta \ne 0$ and $(C3)$ if $\alpha \ne 0$.

Example 2 Let G be a real inner product space. For $d \ge 0$, $x_0, y_0 \in G$

$$U = \{(x, y) \in G \times G \ : \ \langle x_0, x\rangle + \langle y_0, y\rangle \ge d\}. \tag{47}$$

Then U satisfies $(C1)$, if $x_0 \ne y_0$, $(C2)$ if $y_0 \ne 0$ and $(C3)$ if $x_0 \ne 0$.

Example 3 Let G be the group of nonsingular square matrices with the operation of matrix multiplication. For $\alpha, \beta \in \mathbb{R}$, $\delta, d \ge 0$, let

$$U = \{(P_1, P_2) \in G \times G \; : |\det P_1|^{\alpha} |\det P_2|^{\beta} \le \delta\}, \tag{48}$$

$$U = \{(P_1, P_2) \in G \times G \; : |\det P_1|^{\alpha} |\det P_2|^{\beta} \ge d\}. \tag{49}$$

Then U satisfies $(C1)$ if $\alpha \neq \beta$, $(C2)$ if $\beta \neq 0$, and $(C3)$ if $\alpha \neq 0$.

In the following one can see that if $P_{x,y}$, $Q_{x,y}$, and $R_{x,y}$ are replaced by arbitrary subsets of four points (not necessarily distinct) in $G \times G$, respectively, the conditions become stronger, that is, there are subsets U_j, $j = 1,2,3$, which satisfy the conditions $(C1)$, $(C2)$, and $(C3)$, respectively, but U_j, $j = 1,2,3$, fail to fulfill the following conditions (2.6), (2.7), and (2.8), respectively: For any subset $\{p_1, p_2, p_3, p_4\}$ of points (not necessarily distinct) in $G \times G$, there exists a $z \in G$ such that

$$(e,z)\{p_1, p_2, p_3, p_4\}(z^{-1}, e) \subset U_1, \tag{50}$$

$$\{p_1, p_2, p_3, p_4\}(e,z) \subset U_2, \tag{51}$$

$$(z,e)\{p_1, p_2, p_3, p_4\} \subset U_3, \tag{52}$$

respectively.

Now we give examples of U_1, U_2, U_3 which satisfy $(C1), (C2)$, and $(C3)$, respectively, but not (50), (51), and (52), respectively.

Example 4 Let $G = \mathbb{Z}$ be the group of integers. Enumerating

$$\mathbb{Z} \times \mathbb{Z} = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n), \dots\}$$

such that

$$|a_1| + |b_1| \le |a_2| + |b_2| \le \cdots \le |a_n| + |b_n| \le \cdots,$$

and let $P_n = \{(0,0), (a_n, 0), (0, b_n), (a_n, b_n)\}$, $n = 1, 2, \dots$. Then it is easy to see that $U_1 = \bigcup_{n=1}^{\infty} (P_n + (-2^n, 2^n))$ satisfies the condition $(C1)$. Now let $P = \{(x_1, y_1), (x_2, y_2)\} \subset \mathbb{Z} \times \mathbb{Z}$ with $x_2 > x_1$, $y_2 > y_1$, $(x_1 + y_1)(x_2 + y_2) > 0$. Then $P + (-z, z)$ is not contained in U_1 for all $z \in \mathbb{Z}$. Indeed, let $(a,b) \in P_n + (-2^n, 2^n), (c,d) \in P_{n+1} + (-2^{n+1}, 2^{n+1})$. Then we have $a > c$, $b < d$ for all $n = 1, 2, \dots$. Thus it follows from $x_2 > x_1$, $y_2 > y_1$ that if $P + (-z, z) \subset U_1$, then $P + (-z, z) \subset P_n + (-2^n, 2^n)$ for some $n \in \mathbb{N}$, which implies that the line segment joining the points of $P + (-z, z)$ intersects the line $y = -x$ in $\mathbb{R}^2$, contradicting to the condition $(x_1 + y_1)(x_2 + y_2) > 0$. Similarly, let $Q_n = \{(b_n, 0), (0, b_n), (a_n, b_n), (a_n + b_n, 0)\}$ and $R_n = \{(a_n, 0), (0, a_n), (a_n, b_n), (0, a_n + b_n)\}$, $n = 1, 2, \dots$. Then it is easy to see that $U_2 = \bigcup_{n=1}^{\infty} (Q_n + (0, 2^n))$ satisfies the condition $(C2)$ but not (2.7) and $U_3 = \bigcup_{n=1}^{\infty} (R_n + (2^n, 0))$ satisfies the condition $(C3)$ but not (52).

As in Sect. 2, we denote

$$V^* = \{v_1 + v_2 - v_3 - v_4 : v_j \in V, \; j = 1, 2, 3, 4\}.$$

Theorem 4 *Let W satisfy the condition $(C1)$. Suppose that $f, g, h : G \to H$ satisfy*

$$f(x + y) - g(x) - h(y) \in V \tag{53}$$

for all $(x, y) \in W$. *Then there exists a* V^**-Cauchy sequence* $a_n : G \to H$ *satisfying*

$$a_n(x + y) - a_n(x) - a_n(y) \in 2^{-n} V^* \tag{54}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$ *and*

$$a_n(x) - f(x) + f(0) \in 2^{-n}(V^* + 2V^* + \ldots + 2^{n-1} V^*) \tag{55}$$

for all $x \in G$.

Proof For given $x, y \in G$, choose $z \in G$ such that $(-z, z) + P_{x,y} \subset W$. Then we have

$$\begin{aligned} f(x + y) - g(x - z) - h(z + y) &\in V, \\ - f(x) + g(x - z) + h(z) &\in -V, \\ - f(y) + g(-z) + h(z + y) &\in -V, \\ + f(0) - g(-z) - h(z) &\in V. \end{aligned}$$

Thus it follows that

$$f(x + y) - f(x) - f(y) + f(0) \in V + (-V) + (-V) + V = V^* \tag{56}$$

for all $x, y \in G$.

Now by Theorem 2, there exists a V^*-Cauchy sequence $a_n : G \to H$ satisfying (54) and (55). This completes the proof.

In particular, let $V = \{x \in Y : \|x\| \le \epsilon\}$. Then we have

$$V^* \subset \{x \in Y : \|x\| \le 4\epsilon\}, \quad 2^{-n}(V^* + 2V^* + \ldots + 2^{n-1} V^*) \subset \{x \in Y : \|x\| \le 4\epsilon\}$$

for all $n \in \mathbb{N}$, and

$$2^{-m-n}(V^* + 2V^* + \ldots + 2^{n-1} V^*) \subset \{x \in Y : \|x\| \le 2^{-m+2}\epsilon\}$$

for all $m, n \in \mathbb{N}$. Thus in this case, every V^*-Cauchy sequence is a Cauchy sequence. Now as a direct consequence of Theorem 4 we have the following.

Corollary 5 *Let* W *satisfy the condition* $(C1)$ *and* $\epsilon \ge 0$. *Suppose that* $f, g, h : G \to Y$ *satisfy*

$$\|f(x + y) - g(x) - h(y)\| \le \epsilon \tag{57}$$

for all $(x, y) \in W$. *Then there exists a Cauchy sequence* $a_n : G \to Y$ *satisfying*

$$\|a_n(x + y) - a_n(x) - a_n(y)\| \le 2^{-n+2}\epsilon \tag{58}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$, *and*

$$\|a_n(x) - f(x) + f(0)\| \le 4\epsilon \tag{59}$$

for all $x \in G$.

As a direct consequence of Corollary 5 we have the following.

Corollary 6 *Let* W *satisfy the condition* $(C1)$ *and* $\epsilon \geq 0$. *Suppose that* $f, g, h : G \to B$ *satisfy*

$$\|f(x+y) - g(x) - h(y)\| \leq \epsilon \tag{60}$$

for all $(x, y) \in W$. *Then there exists an additive function* $A_1 : G \to B$ *and*

$$\|f(x) - A_1(x) - f(0)\| \leq 4\epsilon \tag{61}$$

for all $x \in G$.

Theorem 5 *Let* W *satisfy the condition* $(C2)$. *Suppose that* $f, g, h : G \to H$ *satisfy*

$$f(x+y) - g(x) - h(y) \in V \tag{62}$$

for all $(x, y) \in W$. *Then there exists a* V^**-Cauchy sequence* $b_n : G \to H$ *satisfying*

$$b_n(x+y) - b_n(x) - b_n(y) \in 2^{-n} V^* \tag{63}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$, *and*

$$b_n(x) - g(x) + g(0) \in 2^{-n}(V^* + 2V^* + \ldots + 2^{n-1}V^*) \tag{64}$$

for all $x \in G$.

Proof For given $x, y \in G$, choose $z \in G$ such that $(0, z) + Q_{x,y} \subset W$. Then we have

$$\begin{aligned} -f(x+y+z) + g(x+y) + h(z) &\in -V, \\ f(x+y+z) - g(x) - h(y+z) &\in V, \\ f(y+z) - g(y) - h(z) &\in V, \\ -f(y+z) + g(0) + h(y+z) &\in -V. \end{aligned}$$

Thus it follows that

$$g(x+y) - g(x) - g(y) + g(0) \in -V + V + V - V = V^* \tag{65}$$

for all $x, y \in G$. Now by Theorem 2, there exists a sequence $b_n : G \to H$ satisfying (63) and (64). This completes the proof.

In particular, if $f, g, h : G \to Y$ we have the following.

Corollary 7 *Let* W *satisfy the condition* $(C2)$ *and* $\epsilon \geq 0$. *Suppose that* $f, g, h : G \to Y$ *satisfy*

$$\|f(x+y) - g(x) - h(y)\| \leq \epsilon \tag{66}$$

for all $(x, y) \in W$. *Then there exists a Cauchy sequence* $b_n : G \to Y$ *satisfying*

$$\|b_n(x+y) - b_n(x) - b_n(y)\| \le 2^{-n+2}\epsilon \tag{67}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$, *and*

$$\|b_n(x) - g(x) + g(0)\| \le 4\epsilon \tag{68}$$

for all $x \in G$.

In particular, if $f, g, h : G \to B$ we have the following.

Corollary 8 *Let* W *satisfy the condition* $(C2)$ *and* $\epsilon \ge 0$. *Suppose that* $f, g, h : G \to B$ *satisfy*

$$\|f(x+y) - g(x) - h(y)\| \le \epsilon \tag{69}$$

for all $(x, y) \in W$. *Then there exists a unique additive function* $A_2 : G \to B$ *such that*

$$\|g(x) - A_2(x) - g(0)\| \le 4\epsilon \tag{70}$$

for all $x \in G$.

Theorem 6 *Let* W *satisfy the condition* $(C3)$. *Suppose that* $f, g, h : G \to H$ *satisfy*

$$f(x+y) - g(x) - h(y) \in V \tag{71}$$

for all $(x, y) \in W$. *Then there exists a* V^*-*Cauchy sequence* $c_n : G \to H$ *satisfying*

$$c_n(x+y) - c_n(x) - c_n(y) \in 2^{-n}V^* \tag{72}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$ *and*

$$c_n(x) - h(x) + h(0) \in 2^{-n}(V^* + 2V^* + \ldots + 2^{n-1}V^*) \tag{73}$$

for all $x \in G$.

Proof For given $x, y \in G$, choose $z \in G$ such that $(0, z) + Q_{x,y} \subset W$. Then we have

$$\begin{aligned} -f(z+x+y) + g(z) + h(x+y) &\in -V, \\ f(z+x+y) - g(z+x) - h(y) &\in V, \\ f(z+x) - g(z) - h(x) &\in V, \\ -f(z+x) + g(z+x) + h(0) &\in -V. \end{aligned}$$

Thus it follows that

$$h(x+y) - h(x) - h(y) + h(0) \in -V + V + V - V = V^* \tag{74}$$

for all $x, y \in G$. Now by Theorem 2, there exists a sequence $c_n : G \to H$ satisfying (72) and (73). This completes the proof.

In particular, if $f, g, h : G \to Y$ we have the following.

Corollary 9 *Let W satisfy the condition $(C3)$ and $\epsilon \geq 0$. Suppose that $f, g, h : G \to Y$ satisfy*

$$\|f(x+y) - g(x) - h(y)\| \leq \epsilon \tag{75}$$

for all $(x, y) \in W$. Then there exists a Cauchy sequence $c_n : G \to Y$ satisfying

$$\|c_n(x+y) - c_n(x) - c_n(y)\| \leq 2^{-n+2}\epsilon \tag{76}$$

for all $n \in \mathbb{N}$ and $x, y \in G$, and

$$\|c_n(x) - h(x) + h(0)\| \leq 4\epsilon \tag{77}$$

for all $x \in G$.

In particular, if $f, g, h : G \to B$ we have the following.

Corollary 10 *Let W satisfy the condition $(C3)$ and $\epsilon \geq 0$. Suppose that $f, g, h : G \to B$ satisfy*

$$\|f(x+y) - g(x) - h(y)\| \leq \epsilon \tag{78}$$

for all $(x, y) \in W$. Then there exists a unique additive function $A_3 : G \to B$ such that

$$\|h(x) - A_3(x) - h(0)\| \leq 4\epsilon \tag{79}$$

for all $x \in G$.

Theorem 7 *Let W satisfy all the conditions $(C1)$, $(C2)$, and $(C3)$. Suppose that $f, g, h : G \to H$ satisfy*

$$f(x+y) - g(x) - h(y) \in V \tag{80}$$

for all $(x, y) \in W$. Then there exist V^*-Cauchy sequences $a_n, b_n, c_n : G \to H$ satisfying

$$a_n(x+y) - a_n(x) - a_n(y) \in 2^{-n}V^* \tag{81}$$

$$b_n(x+y) - b_n(x) - b_n(y) \in 2^{-n}V^* \tag{82}$$

$$c_n(x+y) - c_n(x) - c_n(y) \in 2^{-n}V^* \tag{83}$$

for all $n \in \mathbb{N}$ and $x, y \in G$, and

$$a_n(x) - f(x) + f(0) \in V_n^*, \tag{84}$$

$$b_n(x) - g(x) + g(0) \in V_n^*, \tag{85}$$

$$c_n(x) - h(x) + h(0) \in V_n^* \tag{86}$$

for all $n \in \mathbb{N}$ *and* $x \in G$, *and*

$$a_n(x+y) - b_n(x) - c_n(y) \in V_n^{**}. \tag{87}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$, *where*

$$V_n^* = 2^{-n}(V^* + 2V^* + \ldots + 2^{n-1}V^*),$$

$$V_n^{**} = V + V + V + V + V - V - V - V - V - V + V_n^* - V_n^* - V_n^*.$$

Proof From Theorems 4, 5, and 6, it remains to show (87). By the condition $(C1)$, for given $x, y \in G$, choose $z \in G$ such that $(-z, z), (x - z, z + y) \in W$. Then from (80) we have

$$f(x+y) - g(x-z) - h(z+y) \in V, \tag{88}$$

$$-f(0) + g(-z) + h(z) \in -V. \tag{89}$$

Also, by (65) and (74) we have

$$g(x-z) - g(x) - g(-z) + g(0) \in V + V - V - V, \tag{90}$$

$$h(z+y) - h(z) - h(y) + h(0) \in V + V - V - V. \tag{91}$$

for all $x, y, z \in G$. From (88)–(91), we have

$$f(x+y) - g(x) - h(y) - f(0) + g(0) + h(0) \in V + V + V + V + V - V - V - V - V - V \tag{92}$$

for all $x, y \in G$. Using (84), (85), (86), and (92) we have

$$a_n(x+y) - b_n(x) - c_n(y) \in V + V + V + V + V - V - V - V - V - V + V_n^* - V_n^* - V_n^*. \tag{93}$$

This completes the proof.

In particular, let $V = \{x \in Y : \|x\| \le \epsilon\}$. Then we have

$$V_n^* \subset \{x \in Y : \|x\| \le 4\epsilon\}, \quad V_n^{**} \subset \{x \in Y : \|x\| \le 22\epsilon\}$$

for all $n \in \mathbb{N}$. Thus as a direct consequence of Theorem 7 we have the following.

Corollary 11 *Let* W *satisfy the conditions* $(C1)$, $(C2)$, *and* $(C3)$ *and* $\epsilon \ge 0$. *Suppose that* $f, g, h : G \to Y$ *satisfy*

$$\|f(x+y) - g(x) - h(y)\| \le \epsilon \tag{94}$$

for all $(x, y) \in W$. *Then there exist Cauchy sequences* $a_n, b_n, c_n : G \to Y$ *satisfying*

$$\|a_n(x+y) - a_n(x) - a_n(y)\| \le 2^{-n+2}\epsilon \tag{95}$$

$$\|b_n(x+y) - b_n(x) - b_n(y)\| \le 2^{-n+2}\epsilon \tag{96}$$

$$\|c_n(x+y) - c_n(x) - c_n(y)\| \le 2^{-n+2}\epsilon \tag{97}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$,

$$\|f(x) - a_n(x) - f(0)\| \le 4\epsilon, \tag{98}$$

$$\|g(x) - b_n(x) - g(0)\| \le 4\epsilon, \tag{99}$$

$$\|h(x) - c_n(x) - h(0)\| \le 4\epsilon \tag{100}$$

for all $n \in \mathbb{N}$ *and* $x \in G$, *and*

$$\|a_n(x+y) - b_n(x) - c_n(y)\| \le 22\epsilon \tag{101}$$

for all $n \in \mathbb{N}$ *and* $x, y \in G$.

Corollary 12 *Let* W *satisfy the conditions* $(C1)$, $(C2)$, *and* $(C3)$ *and* $\epsilon \ge 0$. *Suppose that* $f, g, h : G \to B$ *satisfy*

$$\|f(x+y) - g(x) - h(y)\| \le \epsilon \tag{102}$$

for all $(x, y) \in W$. *Then there exists an additive function* $A : G \to B$ *such that*

$$\|f(x) - A(x) - f(0)\| \le 4\epsilon,$$

$$\|g(x) - A(x) - g(0)\| \le 4\epsilon,$$

$$\|h(x) - A(x) - h(0)\| \le 4\epsilon$$

for all $x \in G$.

Proof Let $A_1(x) = \lim_{n\to\infty} a_n(x)$, $A_2(x) = \lim_{n\to\infty} b_n(x)$, $A_3(x) = \lim_{n\to\infty} c_n(x)$. Then it follows from (95)–(97) that for each $j = 1, 2, 3$, A_j is additive. Letting $n \to \infty$ in (98)–(100) we have

$$\|f(x) - A_1(x) - f(0)\| \le 4\epsilon,$$

$$\|g(x) - A_2(x) - g(0)\| \le 4\epsilon,$$

$$\|h(x) - A_3(x) - h(0)\| \le 4\epsilon$$

for all $x \in G$. Finally letting $n \to \infty$ in (101) we have

$$\|A_1(x+y) - A_2(x) - A_3(y)\| \le 22\epsilon \tag{103}$$

for all $x, y \in G$. Putting $y = 0$ and $x = 0$ in (103) separately, we have

$$\|A_1(x) - A_2(x)\| \le 22\epsilon$$

$$\|A_1(y) - A_3(y)\| \leq 22\epsilon$$

for all $x, y \in G$, which implies that $A_1 = A_2$ and $A_1 = A_3$. This completes the proof.

In particular, if G is a normed vector space we have the following.

Corollary 13 *Let $d > 0$. Suppose that $f, g, h : G \to B$ satisfy*

$$\|f(x+y) - g(x) - h(y)\| \leq \epsilon \tag{104}$$

for all $\|x\| + \|y\| \geq d$. Then there exists an additive function $A : G \to B$ such that

$$\|f(x) - A(x) - f(0)\| \leq 4\epsilon,$$
$$\|g(x) - A(x) - g(0)\| \leq 4\epsilon,$$
$$\|h(x) - A(x) - h(0)\| \leq 4\epsilon$$

for all $x \in G$.

Finally we give another interesting example of the set $W \subset \mathbb{R}^n \times \mathbb{R}^n$ with finite Lebesgue measure satisfying all the conditions $(C1)$.

Lemma 1 *Let $D := \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots\}$ be a countable dense subset of $\mathbb{R}^2$. For each $j = 1, 2, 3, \dots$, we denote by*

$$R_j = \{(x, y) \in \mathbb{R}^2 : |x - x_j| < 1, \ |y - y_j| < 2^{-j}\epsilon\}$$

the rectangle in $\mathbb{R}^2$ with center (x_j, y_j) and let $W = \bigcup_{j=1}^{\infty} R_j$. It is easy to see that the Lebesgue measure $m(W)$ of U satisfies $m(W) \leq \epsilon$. Now for $d > 0$, let

$$W_d = W \cap \{(x, y) \in \mathbb{R}^2 : |x| + |y| > d\}.$$

Then W_d satisfies $(C1)$.

Proof For given $x, y \in \mathbb{R}$ we choose a $p \in \mathbb{R}$ such that

$$|p| \geq d + |x| + |y| + 1. \tag{105}$$

We first choose $(x_{i_1}, y_{i_1}) \in K$ such that

$$|-p - x_{i_1}| + |p - y_{i_1}| < \frac{1}{4}, \tag{106}$$

and then we choose $(x_{i_2}, y_{i_2}) \in K$, $(x_{i_3}, y_{i_3}) \in K$ and $(x_{i_4}, y_{i_4}) \in K$ with $1 < i_1 < i_2 < i_3 < i_4$, step by step, satisfying

$$|x - y_{i_1} - x_{i_2}| + |y_{i_1} - y_{i_2}| < 2^{-i_1 - 1}, \tag{107}$$

$$|x - y_{i_2} - x_{i_3}| + |y + y_{i_2} - y_{i_3}| < 2^{-i_2 - 1}, \tag{108}$$

$$|y - y_{i_3} - x_{i_4}| + |y_{i_3} - y_{i_4}| < 2^{-i_3 - 1}. \tag{109}$$

Let

$$z_1 = y_{i_1} - p,$$
$$z_2 = y_{i_2} - y_{i_1},$$
$$z_3 = y_{i_3} - y_{i_2} - y,$$
$$z_4 = y_{i_4} - y_{i_3},$$

and

$$z = z_1 + z_2 + z_3 + z_4.$$

Then from (106)–(109) we have

$$|z_1| < \frac{1}{4},\ |z_2| < 2^{-i_1-1},\ |z_3| < 2^{-i_2-1},\ |z_4| < 2^{-i_3-1},\ |z| < \frac{1}{2}. \tag{110}$$

Thus from (105), (106), and (110) we have

$$|-p-z| + |p+z| \geq 2(|p| - |z|) \geq 2(|p| - \frac{1}{2}) \tag{111}$$
$$> 2d \geq d,$$
$$|-p-z-x_{i_1}| \leq |-p-x_{i_1}| + |z| \tag{112}$$
$$< \frac{1}{4} + \frac{1}{2} < 1,$$

and

$$|p+z-y_{i_1}| = |z_2+z_3+z_4| < 2^{-i_1-1} + 2^{-i_2-1} + 2^{-i_3-1} < 2^{-i_1}. \tag{113}$$

The inequalities (111), (112), and (113) imply

$$(-p-z, p+z) \in W_d. \tag{114}$$

Also from the inequalities

$$|x-p-z| + |p+z| \geq 2(|p| - |x| - |z|) > 2(|p| - |x| - \frac{1}{2}) > d,$$
$$|x-p-z-x_{i_2}| \leq |x-y_{i_1}-x_{i_2}| + |z_2| + |z_3| + |z_4|$$
$$< \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} < 1,$$

and

$$|p+z-y_{i_2}| = |z_3+z_4| < 2^{-i_2-1} + 2^{-i_3-1} < 2^{-i_2},$$

we have

$$(x-p-z, p+z) \in W_d. \tag{115}$$

Similarly, using the inequalities

$$\begin{aligned}
&|x - p - z - x_{i_3}| \le |x - y_{i_2} - x_{i_3}| + |z_3| + |z_4| < 1,\\
&|y + p + z - y_{i_3}| = |z_4| < 2^{-i_3},\\
&|-p - z - x_{i_4}| \le |y - y_{i_3} - x_{i_4}| + |z_4| < 1,\\
&|y + p + z - y_{i_4}| = 0,
\end{aligned}$$

we have

$$(x - p - z, y + p + z), (-p - z, y + p + z) \in W_d. \tag{116}$$

Let $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots\}$ be defined as above. For each $j = 1, 2, 3, \dots,$ let

$$S_j = \{(x, y) : x, y \in \mathbb{R} : |x + y - x_j - y_j| < 1, |x - y - x_j + y_j| < 2^{-j}\epsilon\}$$

and let $V = \bigcup_{j=1}^{\infty} S_j$. Then V satisfies $m(V) \le \epsilon$. For fixed $d > 0$, let

$$V_d = V \cap \{(x, y) \in \mathbb{R}^2 : |x| + |y| > d\}.$$

Using the similar method as in the proof of Lemma 1 we can show that V_d satisfies the conditions $(C1), (C2)$, and $(C3)$.

As a direct consequence of Lemma 1 we have the following.

Theorem 8 *Let $d > 0$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$|f(x + y) - f(x) - f(y)| \le \epsilon \tag{117}$$

for all $(x, y) \in W_d$. Then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x)| \le 3\epsilon \tag{118}$$

for all $x \in \mathbb{R}$.

Proof It follows from (115) and (116) that for given $x, y \in \mathbb{R}$ there exist $p, z \in \mathbb{R}$ satisfying

$$\begin{aligned}
|f(x + y) - f(x) - f(y)| &\le |-f(x) + f(x - p - z) + f(p + z)|\\
&\quad + |f(x + y) - f(x - p - z) - f(y + p + z)|\\
&\quad + |-f(y) + f(-p - z) + f(y + p + z)|\\
&\le 3\epsilon.
\end{aligned}$$

Using Theorem A we get the result.

As a consequence of Theorem 8 we obtain an asymptotic behavior of

$$C_d(f) := \sup_{(x,y) \in W_d} |f(x + y) - f(x) - f(y)| \to 0 \tag{119}$$

as $d \to \infty$.

Theorem 9 *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the condition*

$$C_d(f) \to 0 \tag{120}$$

as $d \to \infty$. *Then* f *is an additive function.*

Proof By the condition (120), for each $j \in \mathbb{N}$, there exists $d_j > 0$ such that

$$|f(x+y) - f(x) - f(y)| \le \frac{1}{j}$$

for all $(x, y) \in W_{d_j}$. By Theorem 8, there exists a unique additive function $A_j : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A_j(x)| \le \frac{3}{j} \tag{121}$$

for all $x \in \mathbb{R}$. From (121), using the triangle inequality we have

$$|A_j(x) - A_k(x)| \le \frac{3}{j} + \frac{3}{k} \le 6 \tag{122}$$

for all $x \in \mathbb{R}$ and all positive integers j, k. Now, the inequality (122) implies $A_j = A_k$. Indeed, for all $x \in \mathbb{R}$ and all rational numbers $r > 0$ we have

$$|A_j(x) - A_k(x)| = \frac{1}{r}|A_j(rx) - A_k(rx)| \le \frac{6}{r}. \tag{123}$$

Letting $r \to \infty$ in (123) we have $A_j = A_k$. Thus, letting $j \to \infty$ in (121) we get the result.

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