Selections of Set-valued Maps Satisfying Some Inclusions and the Hyers–Ulam Stability

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Abstract We present a survey of several results on selections of some set-valued functions satisfying some inclusions and also on stability of those inclusions. Moreover, we show their consequences concerning stability of the corresponding functional equations.

Keywords Stability of functional equation · Set-valued map · Inclusion · Selection

1 Introduction

At present we know that the study of existence of selections of the set-valued maps, satisfying some inclusions, in many cases is connected to the stability problems of functional equations (see, e.g., [8, 26, 27, 29]). Let us remind the result on the stability of functional equation published in 1941 by D. H. Hyers in [6].

Let X be a linear normed space, Y a Banach space, and $\epsilon > 0$. Then, for every function $f : X \to Y$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon, \qquad x, y \in X, \tag{1}$$

there exists a unique additive function $g : X \to Y$ such that

$$\|f(x) - g(x)\| \le \epsilon, \qquad x \in X. \tag{2}$$

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For further information and references concerning that subject we refer to [1, 3, 5, 7, 10, 11, 15, 28].

W. Smajdor [29] and Z. Gajda, R. Ger [8] observed that inequality (2) can be written in the form

$$f(x+y) - f(x) - f(y) \in B(0,\epsilon), \qquad x, y \in X,$$

where $B(0,\epsilon)$ is the closed ball centered at 0 and of radius ϵ. Hence we have

$$f(x+y) + B(0,\epsilon) \subset f(x) + B(0,\epsilon) + f(y) + B(0,\epsilon), \qquad x, y \in X,$$

and the set-valued function

$$F(x) := f(x) + B(0,\epsilon), \qquad x \in X,$$

is subadditive, i.e.

$$F(x+y) \subset F(x) + F(y), \qquad x, y \in X;$$

moreover, the function g from inequality (2) satisfies

$$g(x) \in F(x), \qquad x \in X,$$

which means that F has the additive selection g.

There arises a natural question under what conditions a subadditive set-valed function admits an additive selection. An answer provides the result of Z. Gajda and R. Ger in [8] given below ($\delta(D)$ denotes the diameter of a nonempty set D).

Theorem 1 *Let $(S,+)$ be a commutative semigroup with zero, X a real Banach space and $F : S \to 2^X$ a set-valued map with nonempty, convex, and closed values such that*

$$F(x+y) \subset F(x) + F(y), \qquad x, y \in S,$$

and

$$\sup_{x\in S} \delta(F(x)) < \infty.$$

Then F admits a unique additive selection.

Some other results on the existence of the additive selections of subadditive, superadditive, or additive set-valued functions can be found in [16, 30–33].

2 Linear Inclusions

In this section X is a real vector space and Y is a real Banach space. We denote by $n(Y)$ the family of all nonempty subsets of Y and by $ccl(Y)$ the family of all nonempty closed and convex subsets of Y. The number

$$\delta(A) = \sup_{x,y\in A} \|x - y\|$$

is said to be the diameter of nonempty $A \subset Y$. For $A, B \subset Y$ and $\alpha, \beta \in \mathbb{R}$ (the set of reals) we write

$$A + B := \{a + b : a \in A, b \in B\}$$

and

$$\alpha A := \{\alpha x : x \in A\} ;$$

it is well known that

$$\alpha(A + B) = \alpha A + \alpha B$$

and

$$(\alpha + \beta)A \subset \alpha A + \beta A.$$

If $A \subset Y$ is convex and $\alpha\beta > 0$, then we have

$$(\alpha + \beta)A = \alpha A + \beta A.$$

A nonempty set $K \subset Y$ is said to be a convex cone if

$$K + K \subset K$$

and

$$tK \subset K, \qquad t > 0.$$

Any function $f : X \to Y$ such that

$$f(x) \in F(x), \qquad x \in X,$$

is said to be a selection of the multifunction $F : X \to n(Y)$.

Some generalization of Theorem 1 can be found in [20], where (α, β)-subadditive set-valued map was considered, i.e., the set valued function satisfying

$$F(\alpha x + \beta y) \subset \alpha F(x) + \beta F(y), \qquad x, y \in K.$$

It has been proved there that an (α, β)-subadditive set-valued map with closed, convex, and equibounded values in a Banach space has exactly one additive selection if α, β are positive reals and $\alpha+\beta \neq 1$. For $\alpha+\beta < 1$ a stronger result is true; namely, F is single valued and additive. The above results were extended by K. Nikodem and D. Popa [18, 22] to the case of the following general linear inclusions:

$$F(ax + by + k) \subset pF(x) + qF(y) + C, \qquad x, y \in K, \tag{3}$$

$$pF(x) + qF(y) \subset F(ax + by + k) + C, \qquad x, y \in K, \tag{4}$$

where a, b, p, q are positive reals, $K \subset X$ is a convex cone with zero, $F : K \to n(Y)$, $k \in K$, and $C \in n(Y)$. Namely, they have proved the following two theorems.

Theorem 2 *Suppose that $a+b \neq 1$, $p+q \neq 1$, and $F : K \to ccl(Y)$ satisfies the general linear inclusion*

$$F(ax+by+k) \subset pF(x)+qF(y), \qquad x, y \in K,$$

and

$$\sup_{x\in K} \delta(F(x)) < \infty. \tag{5}$$

Then,

(i) *in the case $p+q > 1$, there exists a unique selection $f : K \to Y$ of F that satisfies the general linear equation*

$$f(ax+by+k) = pf(x)+qf(y), \qquad x, y \in K; \tag{6}$$

(ii) *in the case $p+q < 1$, F is single valued.*

Making a suitable substitutions, we easily deduce from the above theorem the following corollary.

Corollary 1 *Suppose that $a+b \neq 1$, $p+q > 1$, $C \subset Y$ is nonempty, compact, and convex and $F : K \to ccl(Y)$ satisfies (5) and the general linear inclusion (3).*

Then there exists a unique single valued mapping $f : K \to Y$ satisfying Eq. (6) *and such that*

$$f(x) \in F(x) + \frac{1}{p+q-1}C, \qquad x \in K.$$

The next theorem is complementary to the above one.

Theorem 3 *Suppose that $p+q \neq 1$ and $F : K \to ccl(Y)$ satisfies the general linear inclusion*

$$pF(x)+qF(y) \subset F(ax+by), \qquad x, y \in K, \tag{7}$$

and

$$\sup_{x\in L_z} \delta(F(x)) < \infty, \qquad z \in K,$$

where

$$L_z = \{tz : t \geq 0\}.$$

Then,

(i) *in the case $p+q < 1$, there exists a unique selection $f : K \to Y$ of F satisfying the general linear equation*

$$pf(x)+qf(y) = f(ax+by), \qquad x, y \in K;$$

(ii) *in the case $p+q > 1$, F is single-valued.*

It can be easily shown that Theorem 3 yields the following.

Corollary 2 *Let $a+b \neq 1$, $p+q < 1$, $C \subset Y$ be nonempty, compact, and convex, and*

$$x_0 := \frac{k}{1-a-b}.$$

Suppose that $F : K + x_0 \to ccl(Y)$ satisfies the general linear inclusion (4) for $x, y \in K + x_0$ and

$$\sup_{x \in L_z + x_0} \delta(F(x)) < \infty, \qquad z \in K.$$

Then there exists a unique single valued mapping $f : K + x_0 \to Y$ satisfying Eq. (6) *for $x, y \in K + x_0$ and such that*

$$f(x) \in F(x) + \frac{1}{1-p-q} C, \qquad x \in K + x_0.$$

Now, we recall some results concerning the linear inclusions when $p + q = 1$. The special cases are the following two Jensen inclusions

$$F\left(\frac{x+y}{2}\right) \subset \frac{F(x)+F(y)}{2}$$

and

$$\frac{F(x)+F(y)}{2} \subset F\left(\frac{x+y}{2}\right).$$

First we show some examples. Namely, the multifunction $F : \mathbb{R} \to ccl(\mathbb{R})$ given by

$$F(x) = [x-1, x+1], \qquad x \in \mathbb{R},$$

satisfies the Jensen equation

$$F\left(\frac{x+y}{2}\right) = \frac{F(x)+F(y)}{2}, \qquad x, y \in \mathbb{R},$$

and each function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = x + b, \qquad x \in \mathbb{R},$$

where $b \in [-1, 1]$ is fixed, is a selection of F and satisfies the Jensen functional equation.

Observe also that, in the case $p+q=1$, a constant function $F: K \to ccl(Y)$, $F(x) = M$ for $x \in K$, where $K \subset X$ is a cone and $M \in ccl(Y)$ is fixed, satisfies the equation

$$F(ax+by) = pF(x) + qF(y), \qquad x, y \in K,$$

and each constant function $f: K \to Y$, $f(x) = m$ for $x \in K$, where $m \in M$ is fixed, satisfies

$$f(ax+by) = pf(x) + qf(y), \qquad x, y \in K.$$

The subsequent results, concerning this case, have been obtained by K. Nikodem [17] and by A. Smajdor and W. Smajdor in [34] (as before, $K \subset X$ is a convex cone containing zero).

Theorem 4 *Let $\alpha \in (0,1)$, $a, b > 0$, C be a nonempty, compact, and convex subset of Y containing zero. Suppose that $F: K \to ccl(Y)$ satisfies*

$$(1-\alpha)F(x) + \alpha F(y) \subset F(px+qy) + C, \qquad x, y \in K,$$

and

$$\sup_{x\in K} \delta(F(x)) < \infty.$$

Then there exists a function $f: K \to Y$ satisfying

$$(1-\alpha)f(x) + \alpha f(y) = f(px+qy), \qquad x, y \in K,$$

and such that

$$f(x) \in F(x) + \frac{1}{\alpha}C, \qquad x \in K.$$

Recently D. Inoan and D. Popa in [9] generalized the above theorem onto the case of inclusion

$$(1-\alpha)F(x) + \alpha F(y) \subset F(x \star y) + C, \qquad x, y \in G, \tag{8}$$

where $(G, \star)$ is a groupoid with an operation that is bisymmetric, i.e.,

$$(x_1 \star y_1) \star (x_2 \star y_2) = (x_1 \star x_2) \star (y_1 \star y_2), \qquad x_1, x_2, y_1, y_2 \in G,$$

and fulfills the property:
there exists an idempotent element $a \in G$ (i.e. $a \star a = a$) such that for every $x \in G$ there exists a unique $t_a(x) \in G$ with $t_a(x) \star a = x$.

They have proved the following (we write $t_a^{n+1}(x) := t_a(t_a^n(x))$ for $x \in G$ and each positive integer n).

Theorem 5 *Let* $p \in (0, 1)$ *and* $F : G \to n(Y)$ *satisfy inclusion* (8) *and*

$$\sup_{n \in \mathbb{N}} \delta(F(t_a^n(x))) < \infty, \qquad x \in G.$$

Then there exists a function $f : G \to Y$ *with the following properties:*

$$f(x) \in \mathrm{cl} F(x) + \frac{1}{p} C, \qquad x \in G,$$

$$(1 - p) f(x) + p f(y) = f(x \star y), \qquad x, y \in G.$$

To present the further generalizations of those results, we need to remind the notion of the square symmetric operation. Let $(G, \star)$ be a groupoid (i.e., G is a nonempty set endowed with a binary operation $\star : G^2 \to G$). We say that $\star$ is square symmetric provided

$$(x \star y) \star (x \star y) = (x \star x) \star (y \star y), \qquad x, y \in G.$$

D. Popa in [21, 23] have proved that a set-valued map $F : X \to n(Y)$ satisfying one of the following two functional inclusions

$$F(x \star y) \subset F(x) \diamond F(y), \qquad x, y \in X,$$

$$F(x) \diamond F(y) \subset F(x \star y), \qquad x, y \in X,$$

in appropriate conditions admits a unique selection $f : X \to Y$ satisfying the functional equation

$$f(x) \diamond f(y) = f(x \star y),$$

where $(X, \star)$, $(Y, \diamond)$ are square-symmetric groupoids.

Those results extend the previous ones, because it is easy to check that if $K \subset X$ is a convex cone, $k \in T$ and a, b are fixed positive reals, then $\star : T^2 \to T$ defined by

$$x \star y := ax + by + k, \qquad x, y \in T,$$

is square symmetric. Actually, even more general property is valid: the operation $* : T^2 \to T$, given by

$$x * y := \alpha(x) + \beta(y) + \gamma_0, \qquad x, y \in T,$$

is square symmetric, where $\alpha, \beta : T \to T$ are fixed additive mappings with

$$\alpha \circ \beta = \beta \circ \alpha$$

and γ_0 is a fixed element of T.

3 Inclusions in a Single Variable

Now, we present some results corresponding to inclusions in a single variable and applications to the inclusions in several variables.

In this section, K stands for a nonempty set and (Y, d) denotes a metric space, unless explicitly stated otherwise. For $F : K \to n(Y)$ we denote by $\mathrm{cl} F$ the multifunction defined by

$$(\mathrm{cl} F)(x) = \mathrm{cl} F(x), \qquad x \in K.$$

Given $\alpha : K \to K$ we write $\alpha^0(x) = x$ for $x \in K$ and

$$\alpha^{n+1} = \alpha^n \circ \alpha, \qquad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

($\mathbb{N}$ is the set of positive integers). The following result has been obtained in [24].

Theorem 6 *Let $F : K \to n(Y)$, $\Psi : Y \to Y$, $\alpha : K \to K$, $\lambda \in (0, +\infty)$,*

$$d(\Psi(x), \Psi(y)) \le \lambda d(x, y), \qquad x, y \in Y,$$

and

$$\lim_{n\to\infty} \lambda^n \delta(F(\alpha^n(x))) = 0, \qquad x \in K.$$

1) *If Y is complete and*

$$\Psi(F(\alpha(x))) \subset F(x), \qquad x \in K,$$

then, for each $x \in K$, the limit

$$\lim_{n\to\infty} \mathrm{cl}\Psi^n \circ F \circ \alpha^n(x) =: f(x)$$

exists and f is a unique selection of the multifunction $\mathrm{cl} F$ *such that*

$$\Psi \circ f \circ \alpha = f.$$

2) *If*

$$F(x) \subset \Psi(F(\alpha(x))), \qquad x \in K,$$

then F is a single-valued function and

$$\Psi \circ F \circ \alpha = F.$$

Obviously, if Ψ is a contraction (i.e., $\lambda < 1$) and

$$\sup_{x\in K} \delta(F(x)) < \infty,$$

then it is easily seen that

$$\lim_{n\to\infty} \lambda^n \delta(F(\alpha^n(x))) = 0$$

and consequently the assertions of Theorem 6 are satisfied.

It has been shown in [24] that from Theorem 6 we can derive results on the selections of the set-valued functions satisfying inclusions in several variables, especially the general linear inclusions. Indeed, it is enough to take

$$\Psi(x) = \frac{1}{p+q}x, \qquad \alpha(x) = (a+b)x, \qquad x \in K,$$

or

$$\Psi(x) = (p+q)x, \qquad \alpha(x) = \frac{1}{a+b}x, \qquad x \in K,$$

to obtain the results on selections for the inclusions

$$F(ax+by) \subset pF(x) + qF(y), \qquad x, y \in K,$$

and

$$pF(x) + qF(y) \subset F(ax+by), \qquad x, y \in K,$$

respectively. Analogously, we can also obtain results for the quadratic inclusions:

$$F(x+y) + F(x-y) \subset 2F(x) + 2F(y)$$

and

$$2F(x) + 2F(y) \subset F(x+y) + F(x-y),$$

the cubic inclusions:

$$F(2x+y) + F(2x-y) \subset 2F(x+y) + 2F(x-y) + 12F(x)$$

and

$$2F(x+y) + 2F(x-y) + 12F(x) \subset F(2x+y) + F(2x-y),$$

and the quartic inclusions:

$$F(2x+y) + F(2x-y) + 6F(y) \subset 4F(x+y) + 4F(x-y) + 24F(x), \tag{9}$$

$$4F(x+y) + 4F(x-y) + 24F(x) \subset F(2x+y) + F(2x-y) + 6F(y) \tag{10}$$

(some of them have been investigated in [19]), or the following one in three variables

$$F(x+y+z) \subset 2F\left(\frac{x+y}{2}\right) + F(z),$$

considered in [14].

From Theorem 6 we can deduce the same conclusions as in [14, 19] (cf. also, e.g., [13]), but under weaker assumptions. As an example we present below such a result for the quartic inclusions, with a proof.

Corollary 3 *Let Y be a real Banach space, $(K, +)$ be a commutative group, $F : K \to ccl(Y)$ and*

$$\sup_{x \in K} \delta(F(x)) < \infty.$$

(i) *If (9) holds for all $x, y \in K$, then there exists a unique selection $f : K \to Y$ of the multifunction F such that*

$$f(2x+y) + f(2x-y) + 6f(y) = 4f(x+y) + 4f(x-y) + 24f(x), \quad x, y \in K.$$

(ii) *If (10) holds for all $x, y \in K$, then F is single-valued.*

Proof (i) Setting $x = y = 0$ in (9) we have

$$8F(0) \subset 32F(0).$$

and, by the Rådström cancellation lemma, we get $0 \in F(0)$. Next setting $y = 0$ in (9) and using the last condition we obtain

$$2F(2x) \subset 2F(2x) + 6F(0) \subset 32F(x), \qquad x \in K,$$

whence we derive the inclusion

$$\frac{F(2x)}{16} \subset F(x), \qquad x \in K.$$

Next, by Theorem 6, with

$$\Psi(x) = \frac{1}{16}x, \qquad \alpha(x) = 2x, \qquad x \in K,$$

for each $x \in K$ there exists the limit

$$\lim_{n\to\infty} \Psi^n(F(\alpha^n(x))) = \lim_{n\to\infty} \frac{F(2^n x)}{16^n} = f(x);$$

moreover,

$$f(x) \in F(x), \qquad x \in K.$$

Since, for every $x, y \in K, n \in \mathbb{N}$,

$$\frac{F(2^n(2x+y))}{16^n} + \frac{F(2^n(2x-y))}{16^n} + 6\frac{F(2^n y)}{16^n} \subset 4\frac{F(2^n(x+y))}{16^n} + 4\frac{F(2^n(x-y))}{16^n} + 24\frac{F(2^n x)}{16^n},$$

letting $n \to \infty$ we also get

$$f(2x+y)+f(2x-y)+6f(y)=4f(x+y)+4f(x-y)+24f(x), \qquad x,y\in K.$$

Also the uniqueness of f can be easily deduced from Theorem 6.

(ii) Setting $x=y=0$ in (10) and using the Rådström cancellation lemma we get

$$F(0)=\{0\}.$$

Thus and by (10) (with $y=0$) we have

$$32F(x)\subset 2F(2x)+6F(0)=2F(2x), \qquad x\in K,$$

and consequently

$$F(x)\subset \frac{F(2x)}{16}, \qquad x\in K.$$

So, using Theorem 6 with Ψ and α defined as in the previous case, we deduce that F must be single-valued. □

Some generalization of Theorem 6 can be found in [25]; they are given below.

Theorem 7 *Let $F: K\to n(Y)$, $k\in\mathbb{N}$, $\alpha_1,\dots,\alpha_k: K\to K$, $\lambda_1,\dots,\lambda_k: K\to [0,\infty)$, $\Psi: K\times Y^k\to Y$,*

$$d(\Psi(x,w_1,\dots,w_k),\Psi(x,z_1,\dots,z_k))\le\sum_{i=1}^{k}\lambda_i(x)d(w_i,z_i)$$

for $x\in K$, $w_1,\dots,w_k,z_1,\dots,z_k\in Y$ and

$$\liminf_{n\to\infty}\sum_{i_1=1}^{k}\lambda_{i_1}(x)\sum_{i_2=1}^{k}(\lambda_{i_2}\circ\alpha_{i_1})(x)\dots\sum_{i_n=1}^{k}(\lambda_{i_n}\circ\alpha_{i_{n-1}}\circ\dots\circ\alpha_{i_1})(x)$$
$$\times\,\delta(F((\alpha_{i_n}\circ\dots\circ\alpha_{i_1})(x)))=0, \qquad x\in K.$$

(a) *If Y is complete and*

$$\Psi(x,F(\alpha_1(x)),\dots,F(\alpha_k(x)))\subset F(x), \qquad x\in K,$$

then there exists a unique selection $f: K\to Y$ of the multifunction cl$F$ *such that*

$$\Psi(x,f(\alpha_1(x)),\dots,f(\alpha_k(x)))=f(x), \qquad x\in K.$$

(b) *If*

$$F(x)\subset\Psi(x,F(\alpha_1(x)),\dots,F(\alpha_k(x))), \qquad x\in K,$$

then F is a single-valued function and

$$\Psi(x,F(\alpha_1(x)),\dots,F(\alpha_k(x)))=F(x), \qquad x\in K.$$

From this theorem we can easily deduce similar results for the following two gamma-type inclusions in single variable

$$\phi(x)F(a(x)) \subset F(x), \qquad x \in K,$$

and

$$F(x) \subset \phi(x)F(a(x)), \qquad x \in K,$$

where $F : K \to n(Y)$, $a : K \to K$, $\phi : K \to \mathbb{R}$ (for some recent stability results connected with those inclusions see [12]); or for the subsequent two inclusions

$$\lambda_1 F(\alpha_1(x)) + \cdots + \lambda_k F(\alpha_k(x)) \subset F(x), \qquad x \in K,$$

and

$$F(x) \subset \lambda_1 F(\alpha_1(x)) + \cdots + \lambda_k F(\alpha_k(x)), \qquad x \in K,$$

where $\Psi : K \times Y^k \to Y, \alpha_1, \ldots, \alpha_k : K \to K, \lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$ (nonegative reals), and $\lambda_1 + \cdots + \lambda_k \in (0, 1)$.

A different generalization of Theorem 6 have been suggested in [25], with the right side of inclusions as a sum of two set-valued functions. But in this situation we do not obtain existence of the selection but of a suitable single valued function close to F. Namely, we have the following two theorems.

Theorem 8 *Assume that* Y *is complete,* $F, G : K \to n(Y)$*,* $0 \in G(x)$ *for all* $x \in K$*,* $\Psi : Y \to Y$*,* $\alpha : K \to K$*,* $\lambda \in (0, 1)$*,*

$$d(\Psi(x), \Psi(y)) \le \lambda d(x, y), \qquad x, y \in Y,$$

$$M := \sup_{x \in K} \delta(F(x) + G(x)) < \infty$$

and

$$\Psi(F(\alpha(x))) \subset F(x) + G(x), \qquad x \in K. \tag{11}$$

Then there exists a unique function $f : K \to Y$ *such that*

$$\Psi \circ f \circ \alpha = f$$

and

$$\sup_{y \in F(x)} d(f(x), y) \le \frac{1}{1 - \lambda} M, \qquad x \in K.$$

Theorem 9 *Assume that* Y *is complete,* $F, G : K \to n(Y)$*,* $0 \in G(x)$ *for all* $x \in K$*,* $k \in \mathbb{N}$*,* $\Psi : K \times Y^k \to Y$*,* $\alpha_1, \ldots, \alpha_k : K \to K$*,* $\lambda_1, \ldots, \lambda_k : K \to [0, \infty)$*,*

$$d(\Psi(x, w_1, \ldots, w_k), \Psi(x, z_1, \ldots, z_k)) \le \sum_{i=1}^{k} \lambda_i(x) d(w_i, z_i)$$

for $x \in K$*,* $w_1, \ldots, w_k, z_1, \ldots, z_k \in Y$*,*

$$
\begin{aligned}
k(x) &:= \delta(F(x)+G(x)) \\
&\quad + \sum_{l=1}^{\infty}\sum_{i_1=1}^{k}\lambda_{i_1}(x)\sum_{i_2=1}^{k}(\lambda_{i_2}\circ\alpha_{i_1})(x)\dots\sum_{i_l=1}^{k}(\lambda_{i_l}\circ\alpha_{i_{l-1}}\circ\dots\circ\alpha_{i_1})(x) \\
&\quad \times \delta(F((\alpha_{i_l}\circ\dots\circ\alpha_{i_1})(x))+G((\alpha_{i_l}\circ\dots\circ\alpha_{i_1})(x))) < \infty
\end{aligned}
$$

for $x \in K$ and

$$\Psi(x, F(\alpha_1(x)), \dots, F(\alpha_k(x))) \subset F(x)+G(x), \qquad x \in K.$$

Then there exists a unique function $f : K \to Y$ such that

$$\Psi(x, f(\alpha_1(x)), \dots, f(\alpha_k(x))) = f(x), \qquad x \in K,$$

and

$$\sup_{y \in F(x)} d(f(x), y) \le k(x), \qquad x \in K.$$

A special case of inclusion (11), without the assumption $0 \in G(x)$, has been investigated in [4]. In what follows X is a Banach space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $a : K \to \mathbb{K}$, $b : K \to [0, \infty)$, $\phi : K \to K$, $\psi : K \to X$ are given functions and $B \in n(X)$ is a fixed balanced and convex set with $\delta(B) < \infty$. Moreover, we write

$$a_{-1}(x) := 1, \qquad a_n(x) := \prod_{j=0}^{n} a(\phi^j(x)),$$

$$c_n(x) := b(\phi^n(x))a_{n-1}(x),$$

and

$$s_{-1}(x) := 0, \qquad s_n(x) := -\sum_{k=0}^{n} a_{k-1}(x)\psi(\phi^k(x))$$

for every $n \in \mathbb{N}_0$, $x \in K$.

Theorem 10 *Assume that $F : K \to n(X)$ is a set-valued map and the following three conditions hold:*

$$a(x)F(\phi(x)) \subset F(x) + \psi(x) + b(x)B, \qquad x \in K,$$

$$\liminf_{n\to\infty} \delta(F(\phi^{n+1}(x)))|a_n(x)| = 0, \qquad x \in K,$$

$$\omega(x) := \sum_{n=0}^{\infty} |c_n(x)| < \infty, \qquad x \in K. \tag{12}$$

Let

$$\Phi_n(x) := \mathrm{cl}\left(a_{n-1}(x)F(\phi^n(x)) + s_{n-1}(x) + \left(\sum_{k=n}^{\infty}|c_k(x)|\right)B\right)$$

for $x \in K$, $n \in \mathbb{N}_0$. *Then, for each* $x \in K$, *the sequence* $(\Phi_n(x))_{n\in\mathbb{N}_0}$ *is decreasing (i.e.,* $\Phi_{n+1}(x) \subset \Phi_n(x)$*), the set*

$$\widehat{\Phi}(x) := \bigcap_{n=0}^{\infty}\Phi_n(x)$$

has exactly one point and the function $f : K \to X$ *given by* $f(x) \in \widehat{\Phi}(x)$ *is the unique solution of the equation*

$$a(x)f(\phi(x)) = f(x) + \psi(x), \qquad x \in K, \tag{13}$$

with

$$f(x) \in \Phi_0(x) = cl(F(x) + \omega(x)B), \qquad x \in K.$$

4 Applications

In this section we present a few applications of the results, presented in the previous sections, to the stability of some functional equations.

Let V be nonempty, compact, and convex subset of a real Banach space Y, $0 \in V$, and $a, b, p, q \in \mathbb{R}$.

Corollary 4 *Let* K *be a convex cone in a real vector space and* $c \in K$. *Suppose that* $a + b \neq 1$, $p + q > 1$, *and* $f : K \to Y$ *satisfies*

$$f(ax + by + c) - pf(x) - qf(y) \in V, \qquad x, y \in K.$$

Then there exists a unique function $h : K \to Y$ *such that*

$$h(ax + by + c) = ph(x) + qh(y), \qquad x, y \in K,$$

and

$$h(x) - f(x) \in \frac{1}{p+q-1}V, \qquad x \in K.$$

Proof Let

$$F(x) := f(x) + \frac{1}{p+q-1}V, \qquad x \in K.$$

Then

$$F(ax + by + c) = f(ax + by + c) + \frac{1}{p+q-1}V$$

$$\subset pf(x) + qf(y) + \frac{p+q}{p+q-1}V$$
$$= p\left(f(x) + \frac{1}{p+q-1}V\right) + q\left(f(y) + \frac{1}{p+q-1}V\right)$$
$$= pF(x) + qF(y), \qquad x, y \in K.$$

By Theorem 2 there exists a unique function $h : K \to Y$ with

$$h(x) \in f(x) + \frac{1}{p+q-1}V, \qquad x \in K,$$

and such that

$$h(ax + by + c) = ph(x) + qh(y), \qquad x, y \in K.$$

□

Corollary 5 *Let $(K, +)$ be a commutative group and $f : K \to Y$ satisfies*

$$f(2x + y) + f(2x - y) + 6f(y) - 4f(x + y) - 4f(x - y) - 24f(x) \in V$$

for every $x, y \in K$. Then there exists a unique function $h : K \to Y$ such that

$$h(2x+y) + h(2x-y) + 6h(y) = 4h(x + y) + 4h(x - y) + 24h(x), \qquad x, y \in K,$$

$$h(x) - f(x) \in \frac{1}{24}V, \qquad x \in K.$$

Proof Let $F(x) := f(x) + \frac{1}{24}V$ for $x \in K$. Then

$$F(2x + y) + F(2x - y) + 6F(y)$$
$$= f(2x + y) + f(2x - y) + 6f(y) + \frac{8}{24}V$$
$$\subset 4f(x + y) + 4f(x - y) + 24f(x) + \frac{8}{24}V + V$$
$$= 4\left(f(x + y) + \frac{1}{24}V\right) + 4\left(f(x - y) + \frac{1}{24}V\right) + 24\left(f(x) + \frac{1}{24}V\right)$$
$$= 4F(x + y) + 4F(x - y) + 24F(x), \qquad x, y \in K.$$

Now, according to Corollary 3 there exists a unique function $h : K \to X$ such that $h(2x + y) + h(2x - y) + 6h(y) = 4h(x + y) + 4h(x - y) + 24h(x)$ for $x, y \in K$ and

$$h(x) \in f(x) + \frac{1}{24}V, \qquad x \in K.$$

□

In similar way we can obtain the stability results for some other equations. In particular, from Theorem 7 with

$$F(x) = f(x) + \frac{1}{1-(\lambda_1 + \cdots + \lambda_k)} V, \qquad x \in K,$$

and $\lambda_1 + \cdots + \lambda_k \in (0, 1)$, we can derive analogous as in Corollary 5 results for functions f satisfying the condition

$$\lambda_1 f(\alpha_1(x)) + \cdots + \lambda_k f(\alpha_k(x)) - f(x) \in V, \qquad x \in K.$$

The following corollary follows from Theorem 10 (see [4]).

Corollary 6 *Let* (12) *be valid and* $g : K \to X$ *satisfy*

$$a(x)g(\phi(x)) - g(x) - \psi(x) \in b(x)B, \qquad x \in K.$$

Then there exists a unique solution $f : K \to X$ *of* Eq. (13) *with*

$$f(x) - g(x) \in \omega(x)\mathrm{cl}B, \qquad x \in K.$$

Moreover, for each $x \in K$,

$$f(x) = \lim_{n\to\infty} [a_{n-1}(x)g(\phi^n(x)) + s_{n-1}(x)].$$

Finally, let us recall the result in [2].

Theorem 11 *Let* $(S, +)$ *be a left amenable semigroup and let* X *be a Hausdorff locally convex linear space. Let* $F : S \to n(X)$ *be set-valued function such that* $F(s)$ *is convex and weakly compact for all* $s \in S$. *Then* F *admits an additive selection* $a : S \to X$ *if and only if there exists* $f : S \to X$ *such that*

$$f(s + t) - f(t) \in F(s), \qquad s, t \in S.$$

As a consequence of it we obtain the following corollaries.

Corollary 7 *Let* $(S, +)$ *be a left amenable semigroup and let* X *be a reflexive Banach space. In addition, let* $\rho : S \to [0, \infty)$ *and* $g : S \to X$ *be arbitrary functions. Then there exists an additive function* $a : S \to X$ *such that*

$$\|a(s) - g(s)\| \le \rho(s), \qquad s \in S,$$

if and only if there exists a function $f : S \to X$ *such that*

$$\|f(s + t) - f(t) - g(s)\| \le \rho(s), \qquad s, t \in S.$$

Corollary 8 *Let* $(S, +)$ *be a left amenable semigroup,* X *be a reflexive Banach space, and let* $\rho : S \to [0, \infty)$ *be an arbitrary function. Assume that a function* $f : S \to X$ *satisfies*

$$\|f(s + t) - f(t) - f(s)\| \le \rho(s), \qquad s, t \in S.$$

Then there exists an additive function $a : S \to X$ *such that*

$$\|a(s) - f(s)\| \le \rho(s), \qquad s, t \in S.$$

References

1. Baak, C., Boo, D.-H., Rassias, Th. M.: Generalized additive mappings in Banach modules and isomorphisms between C^*–algebras. J. Math. Anal. Appl. **314**, 150–161 (2006)
2. Badora, R., Ger, R., Páles, Z.: Additive selections and the stability of the Cauchy functional equation. ANZIAM J. **44**, 323–337 (2003)
3. Brillouët-Belluot, N., Brzdęk, J., Ciepliński, K.: On some recent developments in Ulam's type stability. Abstr. Appl. Anal. **2012**, Article ID 716936 (2012)
4. Brzdęk, J., Popa, D., Xu, B.: Selections of set-valued maps satisfying a linear inclusions in single variable. Nonlinear Anal. **74**, 324–330 (2011)
5. Czerwik, S.: Functional Equations and Inequalities in Several Variables. World Scientific, London (2002)
6. Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. U S A **27**, 222–224 (1941)
7. Hyers, D.H., Isac, G., Rassias, Th.M.: Stability of Functional Equations in Several Variables. Birkhäuser, Boston (1998)
8. Gajda, Z., Ger, R.: Subadditive multifunctions and Hyers-Ulam stability. In: General Inequalities, 5 (Oberwolfach, 1986), pp. 281–291, Internat. Schriftenreihe Numer. Math. **80**. Birkhäuser, Basel (1987)
9. Inoan, D., Popa, D.: On selections of general convex set-valued maps. Aequ. Math. (2013). doi:10.1007/s00010-013-0219-5
10. Jung, S.-M.: Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis. Hadronic Press, Palm Harbor (2001)
11. Jung, S.-M.: Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis. Springer, New York (2011)
12. Jung, S.-M., Popa, D., Rassias, M.Th.: On the stability of the linear functional equation in a single variable on complete metric groups. J. Glob. Optim. (2014). doi:10.1007/s10898-013-0083-9
13. Lee, Y.-H., Jung, S.-M., Rassias, M.Th.: On an n–dimensional mixed type additive and quadratic functional equation. Appl. Math. Comput. (to appear)
14. Lu, G., Park, C.: Hyers-Ulastability of additive set-valued functional equations. Appl. Math. Lett. **24**, 1312–1316 (2011)
15. Moszner, Z.: On stability of some functional equations and topology of their target spaces. Ann. Univ. Paedagog. Crac. Stud. Math. **11**, 69–94 (2012)
16. Nikodem, K.: Additive selections of additive set-valued functions. Univ. u Novum Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **18**, 143–148 (1988)
17. Nikodem, K.: Characterization of midconvex set-valued functions. Acta Univ. Carolin. Math. Phys. **30**, 125–129 (1989)
18. Nikodem, K., Popa, D.: On selections of general linear inclusions. Publ. Math. Debr. **75**, 239–249 (2009)
19. Park, C., O'Regan, D., Saadati, R.: Stability of some set-valued functional equations. Appl. Math. Lett. **24**, 1910–1914 (2011)
20. Popa, D.: Additive selections of (α, β)-subadditive set valued maps. Glas. Math. **36**, 11–16 (2001)
21. Popa, D.: Functional inclusions on square-symetric grupoid and Hyers-Ulam stability. Math. Inequal. Appl. **7**, 419–428 (2004)
22. Popa, D.: A stability result for a general linear inclusion. Nonlinear Funct. Anal. Appl. **3**, 405–414 (2004)
23. Popa, D.: A property of a functional incusion connected with Hyers-Ulam stability. J. Math. Inequal. **4**, 591–598 (2009)
24. Piszczek, M.: On selections of set-valued inclusions in a single variable with applications to several variables. Results Math. **64**, 1–12 (2013)
25. Piszczek, M.: On selections of set-valued maps satisfying some inclusions in a single variable. Math. Slov. (to appear)

26. Rassias, Th.M.: Stability and set-valued functions. In: Cazacu C.A., Lehto O.E., Rassias Th.M. (eds.) Analysis and Topology, pp. 585–614. World Scientific, River Edge (1998)
27. Rassias, Th.M.: Stability and set-valued functions. In: Andreian Cazacu, C., Lehto, O.E., Rassias, Th.M. (eds.) Analysis and Topology, pp. 585–614. World Scientific, River Edge (1998)
28. Rassias, Th.M., Tabor, J. (eds.): Stability of Mappings of Hyers - Ulam Type. Hadronic Press, Palm Harbor (1994)
29. Smajdor, W.: Superadditive set-valued functions. Glas. Mat. **21**, 343–348 (1986)
30. Smajdor, W.: Subadditive and subquadratic set-valued functions. Prace Naukowe Uniwersytetu Śląskiego w Katowicach [Scientific Publications of the University of Silesia], **889**, 75 pp., Uniwersytet Śląski, Katowice (1987)
31. Smajdor, W.: Superadditive set-valued functions and Banach-Steinhauss theorem. Radovi Mat. **3**, 203–214 (1987)
32. Smajdor, A.: Additive selections of superadditive set-valued functions. Aequ. Math. **39**, 121–128 (1990)
33. Smajdor, A.: Additive selections of a composition of additive set-valued functions. In: Iteration theory (Batschuns, 1992), pp. 251–254. Word Scientific, River Edge (1996)
34. Smajdor, A., Smajdor, W.: Affine selections of convex set-valued functions. Aequ. Math. **51**, 12–20 (1996)