Functional Inequalities and Analysis of Contagion in the Financial Networks

P. Daniele, S. Giuffè, M. Lorino, A. Maugeri and C. Mirabella

Abstract In very recent papers, using delicate tools of functional analysis, a general equilibrium model of financial flows and prices is studied. In particular, without using a technical language, but using the universal language of mathematics, some significant laws, such as the Deficit formula, the Balance law and the Liability formula for the management of the world economy are provided. Further a simple but useful economical indicator $E(t)$ is considered. In this paper, considering the Lagrange dual formulation of the financial model, the Lagrange variables called "deficit" and "surplus" variables are considered. By means of these variables, we study the possible insolvencies related to the financial instruments and their propagation to the entire system, producing a "financial contagion".

Keywords Financial networks · Deficit and surplus variables · Shadow market · Balance law · Financial contagion

1 Introduction

In the papers [4–7], the authors study a general model of financial flows and prices related to individual entities called sectors. They are able to provide the equilibrium conditions and to express them in terms of a variational inequality. Then, they study

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the governing variational inequality and provide existence theorems, develop the Lagrange duality theory, and introduce an appropriate Evaluation Index $E(t)$. As a byproduct of the Lagrange duality, they get a dual formulation of the financial equilibrium in which the significance Lagrange functions $\rho_j^{*1}(t)$ and $\rho_j^{*2}(t)$ appear. These functions $\rho_j^{*1}(t), \rho_j^{*2}(t)$, $j = 1, \ldots, n$ represent the deficit and the surplus, respectively, for the financial instrument j shared by the sectors. Studying the balance of all sectors given by

$$\sum_{j=1}^{n} \rho_j^{*1}(t) - \sum_{j=1}^{n} \rho_j^{*2}(t)$$

and the single difference

$$\rho_j^{*1}(t) - \rho_j^{*2}(t) \quad j = 1, \ldots, n$$

we are able to study the possible insolvencies related to the financial instruments and to understand when they propagate to the entire system, producing a "financial contagion".

2 The Financial Network and the Equilibrium Flows and Prices

The first authors to develop a multi-sector, multi-instrument financial equilibrium model using the variational inequality theory were Nagurney et al. [34]. These results were, subsequently, extended by Nagurney in [30, 31] to include more general utility functions and by Nagurney and Siokos in [32, 33] to the international domain (see also [24, 36] for related papers). In [18], the authors apply for the first time the methodology of projected dynamical systems to develop a multi-sector, multi-instrument financial model, whose set of stationary points coincided with the set of solutions to the variational inequality model developed in [30], and then to study it qualitatively, providing stability analysis results.

Now, we describe in detail the model we are dealing with. We consider a financial economy consisting of m sectors, for example, households, domestic businesses, banks and other financial institutions, as well as state and local governments, with a typical sector denoted by i, and of n instruments, for example mortgages, mutual funds, saving deposits, money market funds, with a typical financial instrument denoted by j, in the time interval $[0, T]$. Let $s_i(t)$ denote the total financial volume held by sector i at time t as assets, and let $l_i(t)$ be the total financial volume held by sector i at time t as liabilities. Then, unlike previous papers (see [9–13] and [15]), we allow markets of assets and liabilities to have different investments $s_i(t)$ and $l_i(t)$, respectively. Since we are working in the presence of uncertainty and of risk perspectives, the volumes $s_i(t)$ and $l_i(t)$ held by each sector cannot be considered stable with respect to time and may decrease or increase. For example, depending on the crisis periods, a sector may decide not to invest on instruments and to buy goods as gold and silver. At time t, we denote the amount of instrument j held as an asset in sector i's portfolio by $x_{ij}(t)$ and the amount of instrument j held as a liability in sector i's portfolio by $y_{ij}(t)$. The assets and liabilities in all the sectors are grouped into the matrices

$$x(t) = \begin{bmatrix} x_1(t) \\ \dots \\ x_i(t) \\ \dots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) & \dots & x_{1j}(t) & \dots & x_{1n}(t) \\ \dots & \dots & \dots & \dots & \dots \\ x_{i1}(t) & \dots & x_{ij}(t) & \dots & x_{in}(t) \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1}(t) & \dots & x_{mj}(t) & \dots & x_{mn}(t) \end{bmatrix}$$

and

$$y(t) = \begin{bmatrix} y_1(t) \\ \dots \\ y_i(t) \\ \dots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} y_{11}(t) & \dots & y_{1j}(t) & \dots & y_{1n}(t) \\ \dots & \dots & \dots & \dots & \dots \\ y_{i1}(t) & \dots & y_{ij}(t) & \dots & y_{in}(t) \\ \dots & \dots & \dots & \dots & \dots \\ y_{m1}(t) & \dots & y_{mj}(t) & \dots & y_{mn}(t) \end{bmatrix}.$$

We denote the price of instrument j held as an asset at time t by $r_j(t)$ and the price of instrument j held as a liability at time t by $(1 + h_j(t))r_j(t)$, where h_j is a nonnegative function defined into $[0, T]$ and belonging to $L^\infty([0, T])$. We introduce the term $h_j(t)$ because the prices of liabilities are generally greater than or equal to the prices of assets in order to describe, in a more realistic way, the behaviour of the markets for which the liabilities are more expensive than the assets. In such a way, this paper appears as an improvement in various directions of the previous ones ([9–13] and [15]). We group the instrument prices held as assets into the vector $r(t) = [r_1(t), r_2(t), \dots, r_i(t), \dots, r_n(t)]^T$ and the instrument prices held as liabilities into the vector $(1 + h(t))r(t) = [(1 + h_1(t))r_1(t), (1 + h_2(t))r_2(t), \dots, (1 + h_i(t))r_i(t), \dots, (1 + h_n(t))r_n(t)]^T$. In our problem, the prices of each instrument appear as unknown variables. Under the assumption of perfect competition, each sector will behave as if it has no influence on the instrument prices or on the behaviour of the other sectors, whereas the instrument prices depend on the total amount of the investments and the liabilities of each sector. In order to express the time dependent equilibrium conditions by means of an evolutionary variational inequality, we choose as a functional setting the very general Lebesgue space $L^2([0, T], \mathbb{R}^p) = \{f : [0, T] \to \mathbb{R}^p : \int_0^T \|f(t)\|_p^2 dt < +\infty\}$. Then, the set of feasible assets and liabilities for each sector $i = 1, \dots, m$, becomes

$$P_i = \Big\{(x_i(t), y_i(t)) \in L^2\left([0, T], \mathbb{R}^{2n}\right) : \sum_{j=1}^n x_{ij}(t) = s_i(t), \ \sum_{j=1}^n y_{ij}(t) = l_i(t)$$

$$\text{a.e. in } [0, T], \ x_i(t) \geq 0, \ y_i(t) \geq 0, \ \text{a.e. in } [0, T]\Big\} \quad \forall i = 1, \dots, m.$$

In such a way, the set of all feasible assets and liabilities becomes

$$P = \Big\{ (x(t), y(t)) \in L^2\left([0,T], \mathbb{R}^{2mn}\right) : \sum_{j=1}^{n} x_{ij}(t) = s_i(t), \sum_{j=1}^{n} y_{ij}(t) = l_i(t),$$

$$\forall\, i = 1, \dots, m, \text{ a.e. in } [0,T],\ x_i(t) \geq 0,\ y_i(t) \geq 0,\ \forall\, i = 1, \dots, m, \text{ a.e. in } [0,T] \Big\}.$$

Now, in order to improve the model of competitive financial equilibrium described in [4], which represents a significant but still partial approach to the complex problem of financial equilibrium, we consider the possibility of policy interventions in the financial equilibrium conditions and incorporate them in the form of taxes and price controls and, mainly, we consider a more complete definition of equilibrium prices $r(t)$, based on the demand–supply law, imposing that the equilibrium prices vary between floor and ceiling prices.

To this aim, denote the ceiling price associated with instrument j by $\overline{r}_j$ and the nonnegative floor price associated with instrument j by $\underline{r}_j$, with $\overline{r}_j(t) > \underline{r}_j(t)$, a.e. in $[0,T]$. The floor price $\underline{r}_j(t)$ is determined on the basis of the official interest rate fixed by the central banks, which in turn take into account the consumer price inflation. Then, the equilibrium prices $r_j^*(t)$ cannot be less than these floor prices. The ceiling price $\overline{r}_j(t)$ derives from the financial need to control the national debt arising from the amount of public bonds and of the rise in inflation. It is a sign of the difficulty on the recovery of the economy. However, it should be not overestimated because it produced an availability of money.

In detail, the meaning of the lower and upper bounds is that to each investor a minimal price $\underline{r}_j$ for the assets held in the instrument j is guaranteed, whereas each investor is requested to pay for the liabilities in any case a minimal price $\left(1 + h_j\right)\underline{r}_j$. Analogously each investor cannot obtain for an asset a price greater than $\overline{r}_j$ and as a liability the price cannot exceed the maximum price $\left(1 + h_j\right)\overline{r}_j$.

Denote the given tax rate levied on sector i's net yield on financial instrument j, as τ_{ij}. Assume that the tax rates lie in the interval $[0, 1)$ and belong to $L^\infty([0,T])$. Therefore, the government in this model has the flexibility of levying a distinct tax rate across both sectors and instruments.

Let us group the instrument ceiling prices $\overline{r}_j$ into the column vector $\overline{r}_j(t) = [\overline{r}_1(t), \dots, \overline{r}_i(t), \dots, \overline{r}_n(t)]^T$, the instrument floor prices $\underline{r}_j$ into the column vector $\underline{r}_j(t) = [\underline{r}_1(t), \dots, \underline{r}_i(t), \dots, \underline{r}_n(t)]^T$, and the tax rates τ_{ij} into the matrix

$$\tau(t) = \begin{bmatrix} \tau_{11}(t) & \dots & \tau_{1j}(t) & \dots & \tau_{1n}(t) \\ \dots & \dots & \dots & \dots & \dots \\ \tau_{i1}(t) & \dots & \tau_{ij}(t) & \dots & \tau_{in}(t) \\ \dots & \dots & \dots & \dots & \dots \\ \tau_{m1}(t) & \dots & \tau_{mj}(t) & \dots & \tau_{mn}(t) \end{bmatrix}.$$

The set of feasible instrument prices becomes:

$$\mathcal{R} = \left\{ r \in L^2([0,T],\mathbb{R}^n) : \ \underline{r}_j(t) \leq r_j(t) \leq \overline{r}_j(t), \quad j = 1, \ldots, n, \text{ a.e. in } [0,T] \right\},$$

where $\underline{r}$ and $\overline{r}$ are assumed to belong to $L^2([0,T],\mathbb{R}^n)$.

In order to determine for each sector i, the optimal composition of instruments held as assets and as liabilities, we consider, as usual, the influence due to risk-aversion and the process of optimization of each sector in the financial economy, namely, the desire to maximize the value of the asset holdings while minimizing the value of liabilities. An example of risk aversion is given by the well-known Markowitz quadratic function based on the variance–covariance matrix denoting the sector's assessment of the standard deviation of prices for each instrument (see [25, 26]).

In our case, however, the Markowitz utility or other more general ones are considered time-dependent in order to incorporate the adjustment in time which depends on the previous equilibrium states. A way in order to obtain the adjustments is to introduce a memory term as it happens in other deterministic models (see [1–3, 8, 20–22, 29]). Then, we introduce the utility function $U_i(t, x_i(t), y_i(t), r(t))$, for each sector i, defined as follows

$$\begin{aligned} U_i(t, x_i(t), y_i(t), r(t)) &= u_i(t, x_i(t), y_i(t)) \\ &+ \sum_{j=1}^{n} r_j(t)(1 - \tau_{ij}(t)) \left[x_{ij}(t) - \left(1 + h_j(t)\right) y_{ij}(t) \right], \end{aligned}$$

where the term $-u_i(t, x_i(t), y_i(t))$ represents a measure of the risk of the financial agent and $r_j(t)\left(1 - \tau_{ij}(t)\right)\left[x_i(t) - \left(1 + h_j(t)\right) y_i(t)\right]$ represents the value of the difference between the asset holdings and the value of liabilities. We suppose that the sector's utility function $U_i(t, x_i(t), y_i(t))$ is defined on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, is measurable in t and is continuous with respect to x_i and y_i. Moreover, we assume that $\partial u_i / \partial x_{ij}$ and $\partial u_i / \partial y_{ij}$ exist and that they are measurable in t and continuous with respect to x_i and y_i. Further, we require that $\forall i = 1, \ldots, m$, $\forall j = 1, \ldots, n$, and a.e. in $[0,T]$ the following growth conditions hold true:

$$|u_i(t, x, y)| \leq \alpha_i(t) \|x\| \|y\|, \quad \forall x, y \in \mathbb{R}^n, \tag{1}$$

and

$$\left| \frac{\partial u_i(t, x, y)}{\partial x_{ij}} \right| \leq \beta_{ij}(t) \|y\|, \quad \left| \frac{\partial u_i(t, x, y)}{\partial y_{ij}} \right| \leq \gamma_{ij}(t) \|x\|, \tag{2}$$

where α_i, β_{ij}, γ_{ij} are nonnegative functions of $L^\infty([0,T])$. Finally, we suppose that the function $u_i(t, x, y)$ is concave.

We remind that the Markowitz utility function verifies conditions (1) and (2).

In order to determine the equilibrium prices, we establish the equilibrium condition which expresses the equilibration of the total assets, the total liabilities and the portion of financial transactions per unit F_j employed to cover the expenses of the financial institutions including possible dividends and manager bonus, as in [4]. Hence, the equilibrium condition for the price r_j of instrument j is the following:

$$\sum_{i=1}^{m} \left(1-\tau_{ij}(t)\right)\left[x_{ij}^*(t)-\left(1+h_j(t)\right)y_{ij}^*(t)\right]+F_j(t)\begin{cases}\geq 0 & \text{if } r_j^*(t)=\underline{r}_j(t)\\ =0 & \text{if } \underline{r}_j(t)<r_j^*(t)<\overline{r}_j(t)\\ \leq 0 & \text{if } r_j^*(t)=\overline{r}_j(t)\end{cases} \tag{3}$$

where (x^*, y^*, r^*) is the equilibrium solution for the investments as assets and as liabilities and for the prices.

In other words, the prices are determined taking into account the amount of the supply, the demand of an instrument and the charges F_j, namely, if there is an actual supply excess of an instrument as assets and of the charges F_j in the economy, then its price must be the floor price. If the price of an instrument is positive, but not at the ceiling, then the market of that instrument must clear. Finally, if there is an actual demand excess of an instrument as liabilities and of the charges F_j in the economy, then the price must be at the ceiling.

Now, we can give different but equivalent equilibrium conditions, each of which is useful to illustrate the particular features of the equilibrium.

Definition 1 A vector of sector assets, liabilities and instrument prices $(x^*(t), y^*(t), r^*(t)) \in P \times \mathcal{R}$ is an equilibrium of the dynamic financial model if and only if $\forall i = 1, \dots, m, \forall j = 1, \dots, n$, and a.e. in $[0, T]$, it satisfies the system of inequalities

$$-\frac{\partial u_i(t,x^*,y^*)}{\partial x_{ij}} - \left(1-\tau_{ij}(t)\right) r_j^*(t) - \mu_i^{(1)*}(t) \geq 0, \tag{4}$$

$$-\frac{\partial u_i(t,x^*,y^*)}{\partial y_{ij}} + \left(1-\tau_{ij}(t)\right)\left(1+h_j(t)\right) r_j^*(t) - \mu_i^{(2)*}(t) \geq 0, \tag{5}$$

and equalities

$$x_{ij}^*(t)\left[-\frac{\partial u_i(t,x^*,y^*)}{\partial x_{ij}} - \left(1-\tau_{ij}(t)\right) r_j^*(t) - \mu_i^{(1)*}(t)\right] = 0, \tag{6}$$

$$y_{ij}^*(t)\left[-\frac{\partial u_i(t,x^*,y^*)}{\partial x_{ij}} + \left(1-\tau_{ij}(t)\right)\left(1+h_j(t)\right) r_j^*(t) - \mu_i^{(2)*}(t)\right] = 0, \tag{7}$$

where $\mu_i^{(1)*}(t), \mu_i^{(2)*}(t) \in L^2([0, T])$ are Lagrange multipliers, and verifies condition (3) a.e. in $[0, T]$.

Let us explain the meaning of the above conditions. To each financial volumes s_i and l_i held by sector i, we associate the functions $\mu_i^{(1)*}(t), \mu_i^{(2)*}(t)$, related, respectively, to the assets and to the liabilities, and which represent the "equilibrium disutilities" per unit of the sector i. Then, (4) and (6) mean that the financial volume invested in instrument j as assets x_{ij}^* is greater than or equal to zero if the jth component $-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t)$ of the disutility is equal to $\mu_i^{(1)*}(t)$, whereas if $-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t) > \mu_i^{(1)*}(t)$, then $x_{ij}^*(t) = 0$. The same occurs for the liabilities and the meaning of (3) is already illustrated.

The functions $\mu_i^{(1)*}(t)$ and $\mu_i^{(2)*}(t)$ are Lagrange multipliers associated a.e. in $[0, T]$ with the constraints $\sum_{j=1}^{n} x_{ij}(t) - s_i(t) = 0$ and $\sum_{j=1}^{n} y_{ij}(t) - l_i(t) = 0$, respectively. They are unknown a priori, but this fact has no influence because we will prove in the following theorem that Definition 1 is equivalent to a variational inequality in which $\mu_i^{(1)*}(t)$ and $\mu_i^{(2)*}(t)$ do not appear.

The following Theorem is proved in [6] (see Theorem 2.1).

Theorem 1 *A vector $(x^*, y^*, r^*) \in P \times \mathcal{R}$ is a dynamic financial equilibrium if and only if it satisfies the following variational inequality:*

Find $(x^*, y^*, r^*) \in P \times \mathcal{R}$:

$$\sum_{i=1}^{m} \int_0^T \left\{ \sum_{j=1}^{n} \left[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t) \right] \times \left[x_{ij}(t) - x_{ij}^*(t) \right] \right.$$

$$\left. + \sum_{j=1}^{n} \left[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}} + (1 - \tau_{ij}(t))r_j^*(t)(1 + h_j(t)) \right] \times \left[y_{ij}(t) - y_{ij}^*(t) \right] \right\} dt$$

$$+ \sum_{j=1}^{n} \int_0^T \sum_{i=1}^{m} \left\{ \left(1 - \tau_{ij}(t)\right) \left[x_{ij}^*(t) - (1 + h_j(t))y_{ij}^*(t) \right] + F_j(t) \right\}$$

$$\times [r_j(t) - r_j^*(t)]dt \geq 0, \qquad \forall (x, y, r) \in P \times \mathcal{R}. \tag{8}$$

We are also able to provide existence theorems for the variational inequality (8).

To this end, we remind some definitions (see [27, 35]). Let X be a reflexive Banach space and let $\mathbb{K}$ be a subset of X and X^* be the dual space of X.

Definition 2 A mapping $A : \mathbb{K} \to X^*$ is pseudomonotone in the sense of Brezis (B-pseudomonotone) iff

1. For each sequence u_n weakly converging to u (in short $u_n \rightharpoonup u$) in $\mathbb{K}$ and such that $\limsup_n \langle Au_n, u_n - v \rangle \leq 0$, it results that:

$$\liminf_n \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle, \quad \forall v \in \mathbb{K}.$$

2. For each $v \in \mathbb{K}$, the function $u \mapsto \langle Au, u - v \rangle$ is lower bounded on the bounded subset of $\mathbb{K}$.

Definition 3 A mapping $A : \mathbb{K} \to X^*$ is hemicontinuous in the sense of Fan (F-hemicontinuous) iff for all $v \in \mathbb{K}$ the function $u \mapsto \langle Au, u - v\rangle$ is weakly lower semicontinuous on $\mathbb{K}$.

Now, we recall the following hemicontinuity definition, which will be used together with some kinds of monotonicity assumptions.

Definition 4 A mapping $A : \mathbb{K} \to X^*$ is lower hemicontinuous along line segments, iff the function $\xi \mapsto \langle A\xi, u - v\rangle$ is lower semicontinuous for all $u, v \in \mathbb{K}$ on the line segments $[u, v]$.

Definition 5 The map $A : \mathbb{K} \to X^*$ is said to be pseudomonotone in the sense of Karamardian (K-pseudomonotone) iff for all $u, v \in \mathbb{K}$

$$\langle Av, u - v\rangle \geq 0 \implies \langle Au, u - v\rangle \geq 0.$$

Then, the following existence theorems hold (see [27]). The first one does not require any kind of monotonicity assumptions.

Theorem 2 *Let $\mathbb{K} \subset X$ be a nonempty closed convex bounded set and let $A : \mathbb{K} \subset E \to X^*$ be B-pseudomonotone or F-hemicontinuous. Then, the variational inequality*

$$\langle Au, v - u\rangle \geq 0 \quad \forall v \in \mathbb{K} \tag{9}$$

admits a solution.

The next theorem requires the K-pseudomonotonicity assumption.

Theorem 3 *Let $\mathbb{K} \subset X$ be a closed convex bounded set and let $A : \mathbb{K} \to X^*$ be a K-pseudomonotone map which is lower hemicontinuous along line segments. Then, variational inequality (9) admits solutions.*

We can apply such theorems to our model, setting:

$$v = \left(\left(x_{ij}\right)_{i=1,\dots,m \quad j=1,\dots,n}, \left(y_{ij}\right)_{i=1,\dots,m \quad j=1,\dots,n}, \left(r_j\right)_{j=1,\dots,n} \right);$$

$$A : L^2\left([0,T], \mathbb{R}^{2mn+n}\right) \to L^2\left([0,T], \mathbb{R}^{2mn+n}\right),$$

$$\begin{aligned} A(v) = \Bigg(& \left[-\frac{\partial u_i(x,y)}{\partial x_{ij}} - (1-\tau_{ij}) r_j \right]_{i=1,\dots,m \quad j=1,\dots,n}, \\ & \left[-\frac{\partial u_i(x,y)}{\partial y_{ij}} + (1-\tau_{ij})(1+h_j) r_j \right]_{i=1,\dots,m \quad j=1,\dots,n}, \\ & \left[\sum_{i=1}^{m} (1-\tau_{ij}) \left(x_{ij} - (1+h_j) y_{ij} \right) \right]_{j=1,\dots,n} \Bigg); \end{aligned}$$

$$\mathbb{K} = P \times \mathcal{R} = \Big\{ v \in L^2\left([0,T], \mathbb{R}^{2mn+n}\right) : x_i(t) \geq 0,\ y_i(t) \geq 0, \text{a.e. in } [0,T],$$

$$\sum_{j=1}^{n} x_{ij}(t) = s_i(t), \; \sum_{j=1}^{n} y_{ij}(t) = l_i(t) \text{ a.e. in } [0, T], \; \forall i = 1, \dots, m,$$

$$\left. \underline{r}_j(t) \leq r_j(t) \leq \overline{r}_j(t), \text{a.e. in } [0, T], \forall j = 1, \dots, n \right\}.$$

Hence, evolutionary variational inequality (8) becomes (9) and we can apply Theorems 2 and 3, assuming that A is B-pseudomonotone or K-hemicontinuous, or assuming that A is K-pseudomonotone, lower hemicontinuous along line segments and noting that $\mathbb{K}$ is a nonempty closed convex and bounded set.

Moreover, we recall that condition (2) is sufficient to guarantee that the operator A is lower hemicontinuous along line segments (see [19]).

3 The Lagrange Dual Problem. The Deficit and Surplus Variables

First, let us present the infinite dimensional Lagrange duality, which represents an important and very recent achievement (see [14, 16, 17, 28]) and which we will use.

First, we recall the definition of the tangent cone. If X denote a real normed space and C is a subset of X, given an element $x \in X$, the set:

$$T_C(x) = \Big\{ h \in X :$$

$$\left. h = \lim_{n\to\infty} \lambda_n(x_n - x), \; \lambda_n \in \mathbb{R}, \; \lambda_n > 0, \; \forall n \in \mathbb{N}, \; x_n \in C \; \forall n \in \mathbb{N}, \; \lim_{n\to\infty} x_n = x \right\}$$

is called the tangent cone to C at x (see [23]).

Now, let us present the new duality principles for a convex optimization problem. Let X be a real normed space and S a nonempty convex subset of X; let $(Y, \|\cdot\|)$ be a real normed space partially ordered by a convex cone C, with $C^* = \{\lambda \in Y^* : \langle \lambda, y\rangle \geq 0 \, \forall y \in C\}$ the dual cone of C, Y^* topological dual of Y, and let $(Z, \|\cdot\|_Z)$ be a real normed space with topological dual Z^*. Let us set $-C = \{-x \in Y : \; x \in C\}$. Let $f : S \to \mathbb{R}$ and $g : S \to Y$ be two convex functions and let $h : S \to Z$ be an affine-linear function.

Let us consider the problem

$$\min_{x\in\mathbb{K}} f(x) \tag{10}$$

where $\mathbb{K} = \{x \in S : \; g(x) \in -C, \; h(x) = \theta_Z\}$ and the dual problem

$$\max_{\substack{\lambda\in C^* \\ \mu\in Z^*}} \inf_{x\in S} \{f(x) + \langle\lambda, g(x)\rangle + \langle\mu, h(x)\rangle\}. \tag{11}$$

Remember that λ and μ are the so-called Lagrange multipliers, associated to the sign constraints and to equality constraints, respectively. They play a fundamental role to better understand the behaviour of the financial equilibrium. Moreover, as it is well known, it always results:

$$\min_{x\in\mathbb{K}} f(x) \leq \max_{\substack{\lambda\in C^* \\ \mu\in Z^*}} \inf_{x\in S}\{f(x) + \langle\lambda, g(x)\rangle + \langle\mu, h(x)\rangle\}, \tag{12}$$

and, if problem (10) is solvable, and in (12), the equality holds, we say that the strong duality between primal problem (10) and dual problem (11) holds. When we have the strong duality, we may consider the so-called "shadow market", namely, the dual Lagrange problem associated to the primal problem.

In order to obtain the strong duality, we need that delicate conditions, called "constraint qualification conditions", hold. In the infinite dimensional settings, the next assumption, the so-called *Assumption S*, results to be a necessary and sufficient condition for the strong duality (see [14, 16, 17, 28]).

Definition of Assumption S We shall say that *Assumption S* is fulfilled at a point $x_0 \in \mathbb{K}$, if it results to be

$$T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \cap \big(]-\infty, 0[\, \times \{\theta_Y\} \times \{\theta_Z\}\big) = \emptyset, \tag{13}$$

where

$$\widetilde{M} = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) : \ x \in S \setminus \mathbb{K},\ \alpha \geq 0,\ y \in C\}.$$

The following theorem holds (see Theorem 1.1 in [17] for the proof).

Theorem 4 *Under the above assumptions on f, g, h and C, if problem (10) is solvable and Assumption S is fulfilled at the extremal solution $x_0 \in \mathbb{K}$, then also problem (11) is solvable, the extreme values of both problems are equal, namely, if $(x_0, \lambda^*, \mu^*) \in \mathbb{K} \times C^* \times Z^*$ is the optimal point of problem (11),*

$$\begin{aligned} f(x_0) &= \min_{x\in\mathbb{K}} f(x) = f(x_0) + \langle\lambda^*, g(x_0)\rangle + \langle\mu^*, h(x_0)\rangle \\ &= \max_{\substack{\lambda\in C^* \\ \mu\in Z^*}} \inf_{x\in S}\{f(x) + \langle\lambda, g(x)\rangle + \langle\mu, h(x)\rangle\} \end{aligned} \tag{14}$$

and, it results to be:

$$\langle\lambda^*, g(x_0)\rangle = 0.$$

Let us recall that the following one is the so-called Lagrange functional

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \langle\lambda, g(x)\rangle + \langle\mu, h(x)\rangle, \quad \forall x \in S,\ \forall\lambda \in C^*,\ \forall\mu \in Z^*. \tag{15}$$

Using the Lagrange functional, (14) may be rewritten as

$$f(x_0) = \min_{x\in\mathbb{K}} f(x) = \mathcal{L}(x_0, \lambda^*, \mu^*) = \max_{\substack{\lambda\in C^* \\ \mu\in Z^*}} \inf_{x\in S} \mathcal{L}(x, \lambda, \mu).$$

By means of Theorem 4, it is possible to show the usual relationship between a saddle point of the Lagrange functional and the solution of the constraint optimization problem (10) (see Theorem 5 in [16] for the proof).

Theorem 5 *Let us assume that the assumptions of Theorem 4 are satisfied. Then,* $x_0 \in \mathbb{K}$ *is a minimal solution to problem (10) if and only if there exist* $\overline{\lambda} \in C^*$ *and* $\overline{\mu} \in Z^*$ *such that* $(x_0, \overline{\lambda}, \overline{\mu})$ *is a saddle point of the Lagrange functional (15), namely,*

$$\mathcal{L}(x_0, \lambda, \mu) \le \mathcal{L}(x_0, \lambda^*, \mu^*) \le \mathcal{L}(x, \lambda^*, \mu^*), \quad \forall x \in S,\ \lambda \in C^*,\ \mu \in Z^*$$

and, moreover, it results that

$$\langle \lambda^*, g(x_0) \rangle = 0. \tag{16}$$

Now, we apply the infinite dimensional duality theory to our general model. To this end, as usual, let us set

$$\begin{aligned} f(x,y,r) = \int_0^T \Bigg\{ &\sum_{i=1}^m \sum_{j=1}^n \left[-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t)) r_j^*(t) \right] \\ &\times [x_{ij}(t) - x_{ij}^*(t)] \\ &+ \sum_{i=1}^m \sum_{j=1}^n \left[-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t))(1+h_j(t)) r_j^*(t) \right] \times [y_{ij}(t) - y_{ij}^*(t)] \\ &+ \sum_{j=1}^n \left[\sum_{i=1}^m (1-\tau_{ij}(t)) \left[x_{ij}^*(t) - (1+h_j(t)) y_{ij}^*(t) \right] + F_j(t) \right] \times \left[r_j(t) - r_j^*(t) \right] \Bigg\} dt. \end{aligned}$$

Then, the Lagrange functional is

$$\begin{aligned} \mathcal{L}\left(x, y, r, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)}\right) &= f(x,y,r) - \sum_{i=1}^m \sum_{j=1}^n \int_0^T \lambda_{ij}^{(1)}(t) x_{ij}(t)\, dt \\ &- \sum_{i=1}^m \sum_{j=1}^n \int_0^T \lambda_{ij}^{(2)} y_{ij}(t)\, dt - \sum_{i=1}^m \int_0^T \mu_i^{(1)}(t) \left(\sum_{j=1}^n x_{ij}(t) - s_i(t) \right) dt \\ &- \sum_{i=1}^m \int_0^T \mu_i^{(2)}(t) \left(\sum_{j=1}^n y_{ij}(t) - l_i(t) \right) dt + \sum_{j=1}^n \int_0^T \rho_j^{(1)}(t)(\underline{r}_j(t) - r_j(t))\, dt \\ &+ \sum_{j=1}^n \int_0^T \rho_j^{(2)}(t)(r_j(t) - \overline{r}_j(t))\, dt, \end{aligned} \tag{17}$$

where $(x, y, r) \in L^2([0,T], \mathbb{R}^{2mn+n})$, $\lambda^{(1)}, \lambda^{(2)} \in L^2([0,T], \mathbb{R}_+^{mn})$, $\mu^{(1)}, \mu^{(2)} \in L^2([0,T], \mathbb{R}^m)$, $\rho^{(1)}, \rho^{(2)} \in L^2([0,T], \mathbb{R}_+^n)$.

Remember that $\lambda^{(1)}$, $\lambda^{(2)}$, $\rho^{(1)}$, $\rho^{(2)}$ are the Lagrange multipliers associated, a.e. in $[0,T]$, to the sign constraints $x_i(t) \ge 0$, $y_i(t) \ge 0$, $r_j(t) - \underline{r}_j(t) \ge 0$, $\overline{r}_j(t) - r_j(t) \ge 0$, respectively. The functions $\mu^{(1)}(t)$ and $\mu^{(2)}(t)$ are the Lagrange multipliers associated, a.e. in $[0,T]$, to the equality constraints $\sum_{j=1}^{n} x_{ij}(t) - s_i(t) = 0$ and $\sum_{j=1}^{n} y_{ij}(t) - l_i(t) = 0$, respectively.

The following theorem holds (see [6] Theorem 6.1).

Theorem 6 *Let $(x^*, y^*, r^*) \in P \times \mathcal{R}$ be a solution to variational inequality (8) and let us consider the associated Lagrange functional (17). Then, Assumption S is satisfied and the strong duality holds and there exist $\lambda^{(1)*}, \lambda^{(2)*} \in L^2([0,T], \mathbb{R}^{mn}_+)$, $\mu^{(1)*}, \mu^{(2)*} \in L^2([0,T], \mathbb{R}^m)$, $\rho^{(1)*}, \rho^{(2)*} \in L^2([0,T], \mathbb{R}^n_+)$ such that $(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*})$ is a saddle point of the Lagrange functional, namely,*

$$\begin{aligned}
&\mathcal{L}\left(x^*, y^*, r^*, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)}\right)\\
&\le \mathcal{L}(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*}) \qquad (18)\\
&\le \mathcal{L}(x, y, r, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*})
\end{aligned}$$

$\forall (x,y,r) \in L^2\left([0,T], \mathbb{R}^{2mn+n}\right)$, $\forall \lambda^{(1)}, \lambda^{(2)} \in L^2([0,T], \mathbb{R}^{mn}_+)$, $\forall \mu^{(1)}, \mu^{(2)} \in L^2([0,T], \mathbb{R}^m)$, $\forall \rho^{(1)}, \rho^{(2)} \in L^2([0,T], \mathbb{R}^n_+)$ and, a.e. in $[0,T]$,

$$-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial x_{ij}} - (1 - \tau_{ij}(t)) r_j^*(t) - \lambda_{ij}^{(1)*}(t) - \mu_i^{(1)*}(t) = 0,$$

$$\forall i = 1, \ldots, m, \ \forall j = 1 \ldots, n;$$

$$-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial y_{ij}} + (1 - \tau_{ij}(t))(1 + h_j(t)) r_j^*(t) - \lambda_{ij}^{(2)*}(t) - \mu_i^{(2)*}(t) = 0,$$

$$\forall i = 1, \ldots, m, \ \forall j = 1 \ldots, n;$$

$$\sum_{i=1}^{m} (1 - \tau_{ij}(t)) \left[x_{ij}^*(t) - (1 + h_j(t)) y_{ij}^*(t)\right] + F_j(t) + \rho_j^{(2)*}(t) = \rho_j^{(1)*}(t),$$

$$\forall j = 1, \ldots, n; \qquad (19)$$

$$\lambda_{ij}^{(1)*}(t) x_{ij}^*(t) = 0, \quad \lambda_{ij}^{(2)*}(t) y_{ij}^*(t) = 0, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n \qquad (20)$$

$$\mu_i^{(1)*}(t) \left(\sum_{j=1}^{n} x_{ij}^*(t) - s_i(t)\right) = 0, \quad \mu_i^{(2)*}(t) \left(\sum_{j=1}^{n} y_{ij}^*(t) - l_i(t)\right) = 0,$$

$$\forall i = 1, \ldots, m \qquad (21)$$

$$\rho_j^{(1)*}(t)(\underline{r}_j(t) - r_j^*(t)) = 0, \quad \rho_j^{(2)*}(t)(r_j^*(t) - \overline{r}_j(t)) = 0, \quad \forall j = 1, \dots, n. \quad (22)$$

Let us now call Balance Law the following one

$$\sum_{i=1}^{m} l_i(t) = \sum_{i=1}^{m} s_i(t) - \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x_{ij}^*(t) - y_{ij}^*(t)\right] - \sum_{i=1}^{m}\sum_{j=1}^{n}(1 - \tau_{ij}(t))h_j(t)y_{ij}^*(t)$$
$$+ \sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n} \rho_j^{(1)*}(t) + \sum_{j=1}^{n} \rho_j^{(2)*}(t).$$

The following theorem holds.

Theorem 7 *Let* $(x^*, y^*, r^*) \in P \times \mathcal{R}$ *be the dynamic equilibrium solution to variational inequality* (8), *then the Balance Law*

$$\sum_{i=1}^{m} l_i(t) = \sum_{i=1}^{m} s_i(t) - \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x_{ij}^*(t) - y_{ij}^*(t)\right] - \sum_{i=1}^{m}\sum_{j=1}^{n}(1 - \tau_{ij}(t))h_j(t)y_{ij}^*(t)$$
$$+ \sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n} \rho_j^{(1)*}(t) + \sum_{j=1}^{n} \rho_j^{(2)*}(t) \quad (23)$$

is verified.

Remark 1 Let us recall that from the Liability Formula we get the following index $E(t)$, called "Evaluation Index", that is very useful for the rating procedure:

$$E(t) = \frac{\sum_{i=1}^{m} l_i(t)}{\sum_{i=1}^{m} \tilde{s}_i(t) + \sum_{j=1}^{n} \tilde{F}_j(t)},$$

where we set

$$\tilde{s}_i(t) = \frac{s_i(t)}{1 + i(t)}, \quad \tilde{F}_j(t) = \frac{F_j(t)}{1 + i(t) - \theta(t) - \theta(t)i(t)}.$$

From the Liability Formula, we obtain

$$E(t) = 1 - \frac{\sum_{j=1}^{n} \rho_j^{(1)*}(t)}{(1 - \theta(t))(1 + i(t))\left(\sum_{i=1}^{m} \tilde{s}_i(t) + \sum_{j=1}^{n} \tilde{F}_j(t)\right)}$$

$$+\frac{\sum_{j=1}^{n}\rho_j^{(2)*}(t)}{(1-\theta(t))(1+i(t))\left(\sum_{i=1}^{m}\tilde{s}_i(t)+\sum_{j=1}^{n}\tilde{F}_j(t)\right)} \quad (24)$$

4 Analysis of Financial Contagion

Let us consider (19), namely, the Deficit Formula for the generic instrument j

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^*(t)-(1+h_j(t))y_{ij}^*(t)\right]+F_j(t)+\rho_j^{(2)*}(t)=\rho_j^{(1)*}(t),$$
$$\forall j=1,\dots,n \quad \text{a.e. in } [0,T]$$

together with the complementary Eq. (22)

$$\rho_j^{(1)*}(t)(\underline{r}_j(t)-r_j^*(t))=0,\ \rho_j^{(2)*}(t)(r_j^*(t)-\overline{r}_j(t))=0,\ \rho_j^{(1)*}(t)\cdot\rho_j^{(2)*}(t)=0$$
$$\forall j=1,\dots,n \quad \text{a.e. in } [0,T].$$

Let us note that if $\rho_j^{(1)*}(t)>0$

$$r_j^*(t)=\underline{r}_j(t)$$

and hence, $\rho_j^{(2)*}(t)=0$. From (19), we get

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))x_{ij}^*(t)>\sum_{i=1}^{m}(1-\tau_{ij}(t))(1+h_j(t))y_{ij}^*(t)+F_j(t),$$

namely, the amount of the assets exceeds the one of the liabilities and of the expenses $F_j(t)$. Then, all the individual entities i, $i=1,\dots,m$, have the deficit

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))x_{ij}^*(t)\overline{\rho}_j^{(1)*}(t)-\sum_{i=1}^{m}(1-\tau_{ij}(t))(1+h_j(t))y_{ij}^*(t)\underline{r}_j(t)-F_j(t)\underline{r}_j^*(t)$$
$$=\rho_j^{(1)*}(t)\underline{r}_j(t)>0$$

because for the sectors, the quantity

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))x_{ij}^*(t)\rho_j^{(1)}(t)$$

represents the outcome, whereas

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))(1+h_j(t))y_{ij}^*(t)\underline{r}_j(t)-F_j(t)r_j^*(t)$$

represents the income.

Then, when $\rho_j^{*(1)}(t)$ is positive, formula (19) represents the deficit, whereas when $\rho_j^{*(2)}(t) > 0$, formula (19) represents the surplus. From formula (19), the Balance Law is derived as

$$\sum_{i=1}^{m} s_i(t) - \sum_{i=1}^{m} l_i(t) - \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x_{ij}^*(t) - y_{ij}^*(t)\right] - \sum_{i=1}^{m}\sum_{j=1}^{n}\left(1-\tau_{ij}(t)\right) h_j(t) y_{ij}^*(t)$$

$$+ \sum_{j=1}^{n} F_j(t) = \sum_{j=1}^{n} \rho_j^{(1)*}(t) - \sum_{j=1}^{n} \rho_j^{(2)*}(t)$$

and we see that the balance of all the financial entities depends on the difference

$$\sum_{j=1}^{n} \rho_j^{(1)*}(t) - \sum_{j=1}^{n} \rho_j^{(2)*}(t).$$

If

$$\sum_{j=1}^{n} \rho_j^{(1)*}(t) > \sum_{j=1}^{n} \rho_j^{(2)*}(t), \tag{25}$$

the balance is negative, the whole deficit exceeds the sum of all the surplus and a negative contagion appears and the insolvencies of individual entities propagate through the entire system. As we can see, it is sufficient that only one deficit $\rho_j^{(1)*}(t)$ is large to obtain, even if the other $\rho_j^{(2)*}(t)$ are lightly positive, a negative balance for all the system. Moreover, we can obtain $\rho_j^*(t) > 0$ even if for only a sector has a big insolvency.

Remark 2 When condition (25) is verified, we get $E(t) \leq 1$ and, hence, also $E(t)$ is a significant indicator that the financial contagion happens.

5 The "Shadow Financial Market"

We remark that the financial problem can be considered from two different perspectives: one from the *Point of View of the Sectors* which try to maximize the utility and a second point of view, that we can call *System Point of View*, which regards the whole equilibrium, namely, the respect of the previous laws. For example, from the point of view of the sectors, $l_i(t)$, for $i = 1, \ldots, m$, are liabilities, whereas for the economic system they are investments and, hence, the Liability Formula, from the system point of view, can be called *Investments Formula*. The system point of view coincides with the dual Lagrange problem (the so-called "shadow market") in which $\rho_j^{(1)}(t)$ and $\rho_j^{(2)}(t)$ are the dual multipliers, representing the deficit and the surplus per unit arising from instrument j. Formally, the dual problem is given as follows.

Find $\left(\rho^{(1)*}, \rho^{(2)*}\right) \in L^2([0,T], \mathbb{R}^{2n}_+)$ such that

$$\sum_{j=1}^{n} \int_0^T \left(\rho_j^{(1)}(t) - \rho_j^{(1)*}(t)\right)\left(\underline{r}_j(t) - r_j^*(t)\right) dt + \sum_{j=1}^{n} \int_0^T \left(\rho_j^{(2)}(t) - \rho_j^{(2)*}(t)\right)$$
$$\left(r_j^*(t) - \overline{r}_j(t)\right) dt \le 0, \quad \forall \left(\rho^{(1)}, \rho^{(2)}\right) \in L^2\left([0,T], \mathbb{R}^{2n}_+\right). \tag{26}$$

In fact, taking into account the inequality in the left hand side of (18), we get

$$-\sum_{i=1}^{m}\sum_{j=1}^{n} \int_0^T \left(\lambda_{ij}^{(1)}(t) - \lambda_{ij}^{(1)*}(t)\right) x_{ij}^*(t)\, dt - \sum_{i=1}^{m}\sum_{j=1}^{n} \int_0^T \left(\lambda_{ij}^{(2)} - \lambda_{ij}^{(2)*}\right) y_{ij}^*(t)\, dt$$
$$-\sum_{i=1}^{m} \int_0^T \left(\mu_i^{(1)}(t) - \mu_i^{(1)*}(t)\right)\left(\sum_{j=1}^{n} x_{ij}^*(t) - s_i(t)\right) dt$$
$$-\sum_{i=1}^{m} \int_0^T \left(\mu_i^{(2)}(t) - \mu_i^{(2)*}(t)\right)\left(\sum_{j=1}^{n} y_{ij}^*(t) - l_i(t)\right) dt$$
$$+\sum_{j=1}^{n} \int_0^T \left(\rho_j^{(1)}(t) - \rho_j^{(1)*}(t)\right)\left(\underline{r}_j(t) - r_j^*(t)\right) dt$$
$$+\sum_{j=1}^{n} \int_0^T \left(\rho_j^{(2)}(t) - \rho_j^{(2)*}(t)\right)\left(r_j^*(t) - \overline{r}_j(t)\right) dt \le 0$$

$\forall \lambda^{(1)}, \lambda^{(2)} \in L^2\left([0,T], \mathbb{R}^{mn}_+\right)$, $\mu^{(1)}, \mu^{(2)} \in L^2([0,T], \mathbb{R}^m)$, $\rho^{(1)}, \rho^{(2)} \in L^2([0,T], \mathbb{R}^n_+)$.

Choosing $\lambda^{(1)} = \lambda^{(1)*}$, $\lambda^{(2)} = \lambda^{(2)*}$, $\mu^{(1)} = \mu^{(1)*}$, $\mu^{(2)} = \mu^{(2)*}$, we obtain the dual problem (26).

Note that, from the *System Point of View*, also the expenses of the institutions $F_j(t)$ are supported by the liabilities of the sectors.

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