A Note on the Functions that Are Approximately p-Wright Affine

Janusz Brzdęk

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Abstract Let W be a Banach space, $(V, +)$ be a commutative group, p be an endomorphism of V, and $\overline{p} : V \to V$ be defined by $\overline{p}(x) := x - p(x)$ for $x \in V$. We present some results on the Hyers–Ulam type stability for the following functional equation

$$f(p(x) + \overline{p}(x)) + f(\overline{p}(x) + p(y)) = f(x) + f(y),$$

in the class of functions $f : V \to W$.

Keywords Hyers–Ulam stability · p-Wright affine function · Polynomial function

1 Introduction

Let $0 < p < 1$ be a fixed real number and P be a nonempty subset of a real linear space X. Assume that P is p-convex, i.e., $px + (1-p)y \in P$ for $x, y \in P$. We say that a function f mapping P into the set of reals $\mathbb{R}$ is p-Wright convex (see, e.g., [7, 8, 14, 17, 26]) if it satisfies the inequality

$$f(px + (1-p)y) + f((1-p)x + py) \leq f(x) + f(y) \qquad x, y \in P. \tag{1}$$

Note that we obtain (1) by adding the following usual p-convexity inequality

$$f(px + (1-p)y) \leq pf(x) + (1-p)f(y) \qquad x, y \in P \tag{2}$$

to its corresponding version (with x and y interchanged)

$$f(py + (1-p)x) \leq pf(y) + (1-p)f(x) \qquad x, y \in P. \tag{3}$$

J. Brzdęk (✉)
Department of Mathematics, Pedagogical University,
Podchorążych 2, 30-084 Kraków, Poland
e-mail: jbrzdek@up.krakow.pl

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Analogously, we say that $g : P \to \mathbb{R}$ is p-Wright concave provided the subsequent inequality holds:

$$f(px + (1-p)y) + f((1-p)x + py) \geq f(x) + f(y) \qquad x, y \in P.$$

The functions that are simultaneously p-Wright convex and p-Wright concave, i.e., satisfy the functional equation

$$f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y), \tag{4}$$

are called p-Wright affine (see [7]).

Note that for $p = 1/2$, Eq. (4) is just the well-known Jensen functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

If $p = 1/3$, then Eq. (4) can be written in the form

$$f(x + 2y) + f(2x + y) = f(3x) + f(3y). \tag{5}$$

Solutions and stability of the latter equation have been investigated in [16] (cf. [5]) in connection with a generalized notion of the Jordan derivations on Banach algebras. Solutions and stability of Eq. (4), for more arbitrary p, have been studied in [4, 6, 7] (see also [13, 23]). (For further information and references on stability of functional equations, we refer to, e.g., [3, 10, 11, 15, 18–22, 25]). In particular, the following results have been obtained in [4] ($\mathbb{C}$ denotes the set of complex numbers).

Theorem 1 *Let X be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, Y be a Banach space, $p \in \mathbb{F}$, $A, k \in (0, \infty)$, $|p|^k + |1-p|^k < 1$, and $g : X \to Y$ satisfy*

$$\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \leq A(\|x\|^k + \|y\|^k) \tag{6}$$

for all $x, y \in X$. Then there is a unique solution $G : X \to Y$ of Eq. (4) *with*

$$\|g(x) - G(x)\| \leq \frac{A\|x\|^k}{1 - |p|^k - |1-p|^k} \qquad x \in X. \tag{7}$$

Theorem 2 *Let X be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, Y be a Banach space, $p \in \mathbb{F}$, $A, k \in (0, \infty)$, $|p|^{2k} + |1-p|^{2k} < 1$, and $g : X \to Y$ satisfy*

$$\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \leq A\|x\|^k\|y\|^k$$

for all $x, y \in X$. Then g is a solution to (4).

In this chapter, we complement these two theorems by considering the inequality

$$\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \leq \delta \qquad x, y \in X \tag{8}$$

with a fixed positive real δ. In particular, we also obtain a description of solutions to (4).

Note that if we write $\overline{p} := 1 - p$, then Eq. (4) can be rewritten as follows:

$$f(px + \overline{p}y) + f(\overline{p}x + py) = f(x) + f(y). \tag{9}$$

We use this form of (4) in the sequel. Moreover, we consider a generalization of it with p and $\overline{p}$ being suitable functions, using the notions $px := p(x)$ and $\overline{p}x := px - x$ $(x \in X)$ for simplicity.

Actually, some results in such situation can be derived from [23]. Namely, from [23, Theorem 2] we can deduce the following.

Theorem 3 *Let $\delta \in (0, \infty)$, $(X, +)$ be a commutative group, $p : X \to \mathbb{X}$ be additive (i.e., $p(x + y) = p(x) + p(y)$ for $x, y \in X$), $\overline{p}(X) = p(X)$, and $g : X \to \mathbb{C}$ satisfy*

$$|g(px + (1 - p)y) + g((1 - p)x + py) - g(x) - g(y)| \le \delta \qquad x, y \in X$$

for all $x, y \in X$. Then there is a solution $G : X \to \mathbb{C}$ of Eq. (4) *with*

$$\sup_{x \in X} |g(x) - G(x)| < \infty. \tag{10}$$

In this chapter, we provide a bit more precise estimations than (10), though we apply reasonings similar to those in [23].

2 Auxiliary Information and Lemmas

Let us start with a result that follows easily from [2, 24] (cf. [9]). We need for it the notion of the Fréchet difference operator. Let us recall that for a function f, mapping a semigroup $(S, +)$ into a group $(G, +)$,

$$\Delta_y f(x) = \Delta_y^1 f(x) := f(x + y) - f(x) \qquad x, y \in S,$$

$$\Delta_{t,z}^2 := \Delta_t \circ \Delta_z, \qquad \Delta_t^2 := \Delta_{t,t}^2 \qquad t, z \in S,$$

$$\Delta_{t,u,z}^3 := \Delta_t \circ \Delta_u \circ \Delta_z, \qquad \Delta_t^3 := \Delta_{t,t,t}^3 \qquad t, u, z \in S.$$

It is easy to check that

$$\Delta_{t,z}^2 f(x) = f(x + t + z) - f(x + t) - f(x + z) + f(x) \qquad x, t, z \in S,$$

$$\Delta_{t,z,u}^3 f(x) = f(x + t + z + u) - f(x + t + z) - f(x + t + u) - f(x + z + u) + f(x + t) + f(x + z) + f(x + u) - f(x) \qquad x, t, z, u \in S.$$

We refer to [12] for more information and further references concerning this subject. From [2, Theorem 4] (cf. [10, Theorem 7.6]) and [24, Theorem 9.1] we can easily derive the following proposition.

Proposition 1 *Let W be a normed space, $(V,+)$ be a commutative group, $\varepsilon \geq 0$, and $G : V \to W$ satisfy the inequality*

$$\|(\Delta_y^3 G)(x)\| \leq \varepsilon \qquad x, y \in V. \tag{11}$$

Assume that one of the following two hypotheses is valid.

(a) $\varepsilon = 0$.
(b) W is complete and V is divisible by 6 (i.e., for each $x \in V$, there is $y \in V$ with $x = 6y$).

Then there exist a constant $c \in W$, an additive mapping $a : V \to W$, and a symmetric biadditive mapping $b : V^2 \to W$ such that

$$\|G(x) - b(x,x) - a(x) - c\| \leq \frac{2\varepsilon}{3} \qquad x \in V.$$

Let us now recall two more stability results (see, e.g., [10, p. 13 and Theorem 3.1]).

Lemma 1 *Let $(V,+)$ be a commutative group, W be a Banach space, $\varepsilon \geq 0$, and $g : V \to W$ satisfy the inequality*

$$\|g(x+y) - g(x) - g(y)\| \leq \varepsilon \qquad x, y \in V.$$

Then there exists the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} g(2^n x) \qquad x \in V \tag{12}$$

and the function $A : V \to W$, defined in this way, is additive and

$$\|g(x) - A(x)\| \leq \varepsilon \qquad x \in V.$$

Lemma 2 *Let $(V,+)$ be a commutative group, W be a Banach space, $\varepsilon \geq 0$, and $g : V \to W$ satisfy the inequality*

$$\|g(x+y) + g(x-y) - 2\,g(x) - 2\,g(y)\| \leq \varepsilon \qquad x, y \in V.$$

Then there exists the limit

$$b(x) = \lim_{n\to\infty} 4^{-n} g(2^n x) \qquad x \in V \tag{13}$$

and the function $b : V \to W$, defined in this way, is quadratic and fulfills the inequality

$$\|g(x) - b(x)\| \leq \frac{\varepsilon}{2} \qquad x \in V.$$

In what follows, given a function p mapping a group $(V,+)$ into itself, for the sake of simplicity we write,

$$px := p(x), \qquad \overline{p}x := x - px \qquad x \in V.$$

The next proposition will be very useful in the proofs of our main results.

Lemma 3 *Let $(V,+)$ be a commutative group, $\varepsilon \geq 0$, $p : V \to V$ be a homomorphism with $p(V) = \overline{p}(V)$, and W be a normed space. Assume that $g : V \to W$ satisfies the inequality*

$$\|g(px + \overline{p}y) + g(\overline{p}x + py) - g(x) - g(y)\| \leq \epsilon \qquad x, y \in V. \tag{14}$$

Then the following two statements are valid.

(i) *If g is odd, then $\|\Delta^2_{z,u} g(x)\| \leq 4\varepsilon$ for $x, z, u \in V$.*

(ii) $\|\Delta^3_{t,u,z}\, g(x)\| \leq 8\varepsilon$ *for* $x, z, u, t \in V$.

Proof This proof is patterned on some reasonings from [23].

Take $z \in V$. There exists $w \in V$ with $pw = -\overline{p}z$, because $p(V) = \overline{p}(V)$ is a subgroup of V. Note that

$$\overline{p}(x + z) + p(y + w) = \overline{p}x + py \qquad x, y \in V,$$

whence replacing x by $x + z$ and y by $y + w$ in (14), we get

$$\begin{aligned}\|g(px &+ \overline{p}y + pz + \overline{p}w) + g(\overline{p}x + py) \\ &- g(x + z) - g(y + w)\| \leq \varepsilon \qquad x, y \in V.\end{aligned} \tag{15}$$

Now, (14) and (15) yield

$$\begin{aligned}\|g(x + z) &- g(x) - g(px + \overline{p}y + pz + \overline{p}w) \\ &+ g(px + \overline{p}y) + g(y + w) - g(y)\| \\ &\leq \|g(px + \overline{p}y + pz + \overline{p}w) + g(\overline{p}x + py) - g(x + z) - g(y + w)\| \\ &\quad + \|g(px + \overline{p}y) + g(\overline{p}x + py) - g(x) - g(y)\| \leq 2\varepsilon \qquad x, y \in V.\end{aligned} \tag{16}$$

Take $u \in V$. Analogously as before, we deduce that there is $v \in V$ with $\overline{p}v = -pu$. Clearly

$$p(x + u) + \overline{p}(y + v) = px + \overline{p}y \qquad x, y \in V.$$

Hence, replacing x by $x + u$ and y by $y + v$ in (16), we have

$$\begin{aligned}\|g(x + u + z) &- g(x + u) - g(px + \overline{p}y + pz + \overline{p}w) + g(px + \overline{p}y) \\ &+ g(y + w + v) - g(y + v)\| \leq 2\varepsilon \qquad x, y \in V.\end{aligned} \tag{17}$$

It is easily seen that (16) and (17) imply

$$\begin{aligned}\|g(x + u + z) &- g(x + u) - g(x + z) + g(x) \\ &+ g(y + w + v) - g(y + w) - g(y + v) + g(y)\| \\ &\leq \|g(px + \overline{p}y + pz + \overline{p}w) - g(px + \overline{p}y)\end{aligned} \tag{18}$$

$$\begin{aligned}&- g(x+z) - g(y+w) + g(x) + g(y)\| \\ &+ \|g(px + \overline{p}y + pz + \overline{p}w) - g(px + \overline{p}y) \\ &- g(x+u+z) - g(y+w+v) \\ &+ g(x+u) + g(y+v)\| \le 4\varepsilon \qquad x, y \in V,\end{aligned}$$

which with x replaced by $x + t$ yields

$$\begin{aligned}\|g(x+t+u+z) &- g(x+t+u) - g(x+t+z) + g(x+t) + g(y+w+v) \\ &- g(y+w) - g(y+v) + g(y)\| \le 4\varepsilon \qquad t, x, y \in V.\end{aligned}$$

Combining (18) and the latter inequality, we get statement (ii).

For the proof of (i), observe that (18) with x replaced by $-x - z - u$, under the assumption of the oddness of g, brings

$$\begin{aligned}\| - g(x) &+ g(x+z) + g(x+u) - g(x+z+u) \\ &+ g(y+w+v) - g(y+w) - g(y+v) + g(y)\| \le 4\varepsilon \qquad x, y \in V,\end{aligned} \tag{19}$$

whence and from (18) we have

$$\|2\, g(x) - 2\, g(x+z) - 2\, g(x+u) + 2\, g(x+z+u)\| \le 8\varepsilon \qquad x, y \in V. \tag{20}$$

This yields statement (i). □

The next corollary provides a description of solutions to (9), which will be useful in the sequel.

Corollary 1 *Let V and W be as in Proposition 1 and $p : V \to V$ be a homomorphism with $p(V) = \overline{p}(V)$. Then $f : V \to W$ satisfies* Eq. (9) *if and only if there exist $c \in W$, an additive $a : V \to W$ and a biadditive and symmetric $L : V^2 \to W$ such that*

$$f(x) = L(x, x) + a(x) + c \qquad x \in V, \tag{21}$$

$$L(px, \overline{p}x) = 0 \qquad x \in V. \tag{22}$$

Proof Let $f : V \to W$ be a solution of Eq. (9). Then (14) holds with $\varepsilon = 0$. Consequently, according to Lemma 3 (ii),

$$(\Delta_y^3 f)(x) = 0 \qquad x, y \in V.$$

Hence, on account of Proposition 1, there exist $c \in W$, an additive $a : V \to W$, and a quadratic $b : V \to W$ such that $f(x) = b(x) + a(x) + c$ for $x \in V$. Further, it is well known (see, e.g., [1]) that there exists a symmetric biadditive $L : V^2 \to W$ such that $b(x) = L(x, x)$ for $x \in V$, whence (21) holds. Now, it is easily seen that (9) (with $y = 0$) and (21) yield

$$L(px, px) + L(\overline{p}x, \overline{p}x) = L(x, x) \qquad x \in V$$

and consequently

$$-2\,L(px,\overline{p}x) = L(px,px) + L(\overline{p}x,\overline{p}x) - L(x,x) = 0 \qquad x \in V, \tag{23}$$

which gives (22).

The converse is a routine task. □

We need yet the following very simple lemma.

Lemma 4 *Let $(V,+)$ be a commutative group, W be a normed space, $a, a_0 : V \to W$ be additive, $L, L_0 : V^2 \to W$ be biadditive, $c \in W$ and*

$$M := \sup_{x\in V} \|a_0(x) - a(x) + L_0(x,x) - L(x,x) + c\| < \infty. \tag{24}$$

Then $a = a_0$ and $L = L_0$.

Proof That proof is actually a routine by now, but we present it here for the convenience of readers.

Note that

$$\|L_0(x,x) - L(x,x)\| \le \|a(x) - a_0(x)\| + \|c\| + M \qquad x \in V,$$

whence

$$\begin{aligned}\|L(x,x) - L_0(x,x)\| &= n^{-2}\|L(nx,nx) - L_0(nx,nx)\| \\ &\le n^{-2}(\|a(nx) - a_0(nx)\| + \|c\| + M) \\ &= n^{-1}\|a(x) - a_0(x)\| + n^{-2}(\|c\| + M) \qquad x \in V, n \in \mathbb{N},\end{aligned}$$

which yields $L = L_0$. Hence, by (24),

$$\begin{aligned}\|a(x) - a_0(x)\| &= n^{-1}\|a(nx) - a_0(nx)\| \\ &\le n^{-1}(\|c\| + M) \qquad x \in V, n \in \mathbb{N},\end{aligned}$$

and consequently $a = a_0$. □

3 The Main Stability Results

We start with two theorems describing odd and even solutions of functional inequality (14). They will help us to obtain the main result of the chapter (but they seem to be interesting, as well).

Theorem 4 *Let $(V,+)$ be a commutative group, $\epsilon \ge 0$, $p : V \to V$ be a homomorphism, $p(V) = \overline{p}(V)$, and W be a Banach space. Assume that $g : V \to W$ is odd and satisfies the inequality*

$$\|g(px + \overline{p}y) + g(\overline{p}x + py) - g(x) - g(y)\| \le \varepsilon \qquad x, y \in V. \tag{25}$$

Then there exists a unique additive function, $A : V \to W$, such that

$$\|g(x) - A(x)\| \leq 4\varepsilon \qquad x \in V. \tag{26}$$

Moreover, (12) *holds and for every solution $h : V \to W$ of* (9) *such that*

$$\sup_{x \in V} \|g(x) - h(x)\| < \infty,$$

the function $A - h$ is constant.

Proof According to Lemma 3 (i),

$$\|g(x + z + u) - g(x + z) - g(x + u) + g(x)\| \leq 4\varepsilon \qquad x, z, u \in V,$$

which with $x = 0$ yields

$$\|g(z + u) - g(z) - g(u)\| \leq 4\varepsilon \qquad z, u \in V.$$

Hence Lemma 1 implies the existence and the form of A. It remains to show the statements on the uniqueness of A.

So, suppose that $A_0 : V \to W$ is additive and

$$\sup_{x \in V} \|g(x) - A_0(x)\| \leq 4\varepsilon.$$

Then

$$\sup_{x \in V} \|A(x) - A_0(x)\| \leq 8\varepsilon,$$

which implies that $A = A_0$.

Now, let $h : V \to W$ be a solution of (9) such that

$$\sup_{x \in V} \|g(x) - h(x)\| < \infty.$$

Then

$$M := \sup_{x \in V} \|A(x) - h(x)\| < \infty.$$

Further, by Corollary 1, $h(x) = a(x) + L(x, x) + c$ with some $c \in W$, an additive $a : V \to W$, and a biadditive and symmetric $L : V^2 \to W$. So, Lemma 4 implies that

$$L(x, x) = 0 \qquad x \in V$$

and $A = a$. □

Theorem 5 *Let $(V, +)$ be a commutative group, $\varepsilon \geq 0$, $p : V \to V$ be a homomorphism, $p(V) = \overline{p}(V)$, and W be a Banach space. Assume that $g : V \to W$ is even and satisfies inequality* (25)*. Then there exists a unique biadditive and symmetric mapping $L : V^2 \to W$ such that*

$$\|L(x, x) - g(x) + g(0)\| \leq 4\varepsilon \qquad x \in V. \tag{27}$$

Moreover, (22) *holds,*

$$L(x,x) = \lim_{n\to\infty} 4^{-n} g(2^n x) \qquad x \in V \tag{28}$$

and, for every solution $h : V \to W$ *of* (9) *with*

$$\sup_{x\in V} \|g(x) - h(x)\| < \infty,$$

there is $c \in W$ *such that* $h(x) = L(x,x) + c$ *for* $x \in V$.

Proof Let $g_0 := g - g(0)$. Then g_0 fulfills (25) as well. According to Lemma 3 (ii),

$$\begin{aligned}\|g_0(x+t+z+u) - g_0(x+t+z) - g_0(x+t+u) - g_0(x+z+u)\\ + g_0(x+t) + g_0(x+z) + g_0(x+u) - g_0(x)\| \le 8\varepsilon \qquad x,t,z,u \in S,\end{aligned}$$

whence (with $x = 0$ and $u = -t$) we obtain

$$\begin{aligned}\|g_0(z) - g_0(t+z) - g_0(0) - g_0(z-t) + g_0(t) + g_0(z) + g_0(-t) - g_0(0)\|\\ \le 8\varepsilon \qquad t,u,z \in V\end{aligned}$$

and consequently

$$\|2g_0(z) - g_0(t+z) - g_0(z-t) + 2g_0(t)\| \le 8\varepsilon \qquad t,z \in V.$$

Hence Lemma 2 implies the existence of L and (28).

Now we show that (22) holds. Clearly, (25) (with $y = 0$) yields

$$\|g(px) + g(\overline{p}x) - g(x) - g(0)\| \le \varepsilon \qquad x \in V.$$

So, (27) implies that

$$\begin{aligned}&\|L(px,px) + L(\overline{p}x,\overline{p}x) - L(x,x)\|\\ &\quad\le \|L(px,px) + g(0) - g(px)\|\\ &\qquad + \|L(\overline{p}x,\overline{p}x) + g(0) - g(\overline{p}x)\|\\ &\qquad + \|g(x) - L(x,x) - g(0)\|\\ &\qquad + \|g(px) + g(\overline{p}x) - g(x) - g(0)\| \le 13\varepsilon \qquad x \in V.\end{aligned} \tag{29}$$

Since b is biadditive and it is very easy to check that

$$-2\,L(px,\overline{p}x) = L(px,px) + L(\overline{p}x,\overline{p}x) - L(x,x) \qquad x \in V,$$

from (29), we get

$$\begin{aligned}2k^2\|L(px,\overline{p}x)\| &= \|L(pkx,pkx) + L(\overline{p}kx,\overline{p}kx) - L(kx,kx)\|\\ &\le 13\varepsilon \qquad x \in V, k \in \mathbb{N},\end{aligned} \tag{30}$$

which means that (22) holds.

It remains to show the statements on the uniqueness of L. So, first suppose that $L_0 : V^2 \to W$ is symmetric, biadditive, and

$$\sup_{x\in V} \|L_0(x,x) - g(x) + g(0)\| \leq 4\varepsilon.$$

Then

$$\sup_{x\in V} \|L_0(x,x) - L(x,x)\| \leq 8\varepsilon,$$

whence from Lemma 4 we deduce that $L_0 = L$.

Now, assume that $h : V \to W$ is a solution of (9) with

$$\sup_{x\in V} \|g(x) - h(x)\| < \infty.$$

This implies that

$$M := \sup_{x\in V} \|L(x,x) - h(x)\| < \infty.$$

Further, according to Corollary 1,

$$h(x) = a(x) + S(x,x) + c \qquad x \in V$$

with some $c \in W$, an additive $a : V \to W$, and a biadditive and symmetric $S : V^2 \to W$. Clearly, by Lemma 4, $L = S$ and $a(x) = 0$ for every $x \in V$. Hence

$$h(x) = L(x,x) + c \qquad x \in V. \qquad \square$$

In what follows, given a function g mapping a group $(V,+)$ into a real linear space W, by g_o and g_e, we denote the odd and even parts of g, i.e.,

$$g_o(x) := \frac{g(x) - g(-x)}{2} \qquad x \in V,$$

$$g_e(x) := \frac{g(x) + g(-x)}{2} \qquad x \in V.$$

The next theorem is the main result in this chapter.

Theorem 6 *Let $(V,+)$ be a commutative group, $p : V \to V$ be a homomorphism such that $p(V) = \overline{p}(V)$, W be a Banach space, $\varepsilon \geq 0$ and $g : V \to W$ satisfy inequality (25). Then there exist a unique additive function $a : V \to W$ and a unique biadditive function $L : V^2 \to W$ such that*

$$\|g(x) - a(x) - L(x,x) - g(0)\| \leq 8\varepsilon \qquad x \in V. \tag{31}$$

Moreover, (22) holds,

$$a(x) = \lim_{n\to\infty} 2^{-n} g_o(2^n x), \qquad L(x,x) = \lim_{n\to\infty} 4^{-n} g_e(2^n x) \qquad x \in V \tag{32}$$

and, for every solution $h : V \to W$ of (9) *with*

$$\sup_{x\in V} \|g(x) - h(x)\| < \infty, \tag{33}$$

there is $c \in W$ such that $h(x) = a(x) + L(x,x) + c$ for $x \in V$.

If V is divisible by 6*, then there exists $c_0 \in W$ with*

$$\|g(x) - a(x) - L(x,x) - c_0\| \le \frac{16\varepsilon}{3} \qquad x \in V. \tag{34}$$

Proof It is easily seen that g_o and g_e satisfy inequalities analogous to (25). So, by Theorems 4 and 5, there exist an additive function $a : V \to W$ and a symmetric biadditive function $L : V^2 \to W$ such that

$$\|g_o(x) - a(x)\| \le 4\varepsilon, \qquad \|g_e(x) - L(x,x) - g(0)\| \le 4\varepsilon \qquad x \in V. \tag{35}$$

Moreover, (32) holds and, clearly,

$$\begin{aligned}\|g(x) - a(x) - L(x,x) - g(0)\| &\le \|g_o(x) - a(x)\| \\ &+ \|g_e(x) - L(x,x) - g(0)\| \le 8\varepsilon \qquad x \in V.\end{aligned} \tag{36}$$

Further, (25) (with $y = 0$) yields

$$\|g_e(px) + g_e(\overline{p}x) - g_e(x) - g(0)\| \le \varepsilon \qquad x \in V.$$

Hence analogous to (29), from (35) we derive that

$$\|L(px,px) + L(\overline{p}x,\overline{p}x) - L(x,x)\| \le 13\varepsilon \qquad x \in V, \tag{37}$$

whence (30) holds, which implies (22).

For the proof of uniqueness of a and L, suppose that $a_0 : V \to W$ is additive, $L_0 : V^2 \to W$ is biadditive, and

$$\|g(x) - a_0(x) - L_0(x,x) - g(0)\| \le 8\varepsilon \qquad x \in V. \tag{38}$$

Then

$$\|a_0(x) - a(x) - L_0(x,x) - L(x,x)\| \le 16\varepsilon \qquad x \in V \tag{39}$$

and consequently, by Lemma 4, $L = L_0$ and $a = a_0$.

Now, let $h : V \to W$ be a solution of (9) fulfilling condition (33). Then, in view of (31),

$$M := \sup_{x\in V} \|a(x) + L(x,x) + g(0) - h(x)\| < \infty \tag{40}$$

and, according to Corollary 1, $h(x) = a_0(x) + L_0(x,x) + c$ with some $c \in W$, an additive $a_0 : V \to W$ and a biadditive and symmetric $L_0 : V^2 \to W$. Hence, again

Lemma 4 implies that $L = L_0$ and $a = a_0$. Consequently $h(x) = L(x, x) + a(x) + c$ for $x \in V$.

Finally assume that V is divisible by 6. Then, in view of Lemma 3 (ii), we have

$$\|(\Delta_y^3 g)(x)\| \leq 8\varepsilon \qquad x, y \in V.$$

Further, by Proposition 1, there are $c_0 \in W$, an additive $a_0 : V \to W$ and a biadditive and symmetric $b_0 : V^2 \to W$ such that

$$\|g(x) - b_0(x, x) - a_0(x) - c\| \leq \frac{16}{3}\varepsilon \qquad x \in V. \tag{41}$$

In view of (31) and Lemma 4, we must have $a_0 = a$ and $L_0 = L$. □

For some discussions on a special case of condition (22), we refer to [7] (see also [6, 8, 13]).

Remark 1 There arises natural questions whether (under reasonable suitable assumptions) we can get some better estimations than in (31) and (34) and whether the assumption of divisibility of V by 6 is necessary to get (34). Also, it would be interesting to know if we can have $c_0 = g(0)$ in (34).

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