Some New Algorithms for Solving General Equilibrium Problems

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Abstract In this chapter, we investigate some unified iterative methods for solving the general equilibrium problems using the auxiliary principle technique. The convergence of the proposed methods is analyzed under some suitable conditions. As special cases, we obtain a number of known and new classes of equilibrium and variational inequality problems. Results obtained in this chapter continue to hold for these new and previously known problems. The ideas and techniques of this chapter may inspire the interested readers to explore applications of the general equilibrium problems in pure and applied sciences.

Keywords Variational inequalities · Algorithms · Auxiliary principle · Convergence analysis · Fixed point problems

1 Introduction

Equilibrium problems theory provides us a natural, novel, and unified framework to study a wide class of problems arising in economics, finance, transportation, network, and structural analysis, elasticity and optimization. Equilibrium problems were introduced by Blum and Oettli [1] and Noor and Oettli [20] in 1994. Since then, the ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative; see [1, 2, 3–22]. Equilibrium problems also include variational inequalities and related optimization problems as special cases. Inspired and motivated by the recent research work going in this field, Noor and Rassias [19] considered and investigated a new class of equilibrium problems, which is called *mixed quasi general equilibrium problems.* There are several methods including projection and its variant forms, Wiener–Hopf equations, and auxiliary

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principle for solving variational inequalities. It is known that projection methods and variant forms including Wiener–Hopf equations can not be extended for equilibrium. This fact has motivated to use the auxiliary principle technique. Glowinski, Lions, and Tremolieres [5] used this technique to study the existence of a solution of the mixed variational inequalities, whereas Noor–Noor–Rassias [11] used this technique to suggest and analyze an iterative method for solving mixed quasi variational inequalities. It is well known that a substantial number of numerical methods can be obtained as special cases from this technique; see [5, 13–15, 17–19]. We again use the auxiliary principle technique to suggest a class of new iterative methods for solving mixed quasi general equilibrium problems. The convergence of these methods requires only the jointly monotonicity of the trifunction in conjunction with skew symmetry of the bifunction. Since mixed quasi general equilibrium problems include equilibrium, general variational inequalities, and complementarity problems as special cases, results obtained in this chapter continue to hold for these problems. Our results can be considered an important and significant extension of the known results for solving equilibrium, variational inequalities, and complementarity problems.

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty and closed set in H. We recall the following concepts and notations, which are needed.

Definition 1 ([3, 21]). Let K be any set in H. The set K is said to be g-convex (relative convex), if there exists a function $g : K \longrightarrow K$ such that

$$g(u) + t(g(v) - g(u)) \in K, \forall u, v \in H : g(u), g(v) \in K, t \in [0,1].$$

Note that every convex set is a relative convex, but the converse is not true, see [3, 21]. In passing, we remark that the notion of the relative convex set was introduced by Noor [10] implicitly in 1988.

Definition 2 The function $f : K \longrightarrow H$ is said to be g-convex (relative convex), if there exists a function g such that

$$f(g(u) + t(g(v) - g(u))) \leq (1-t)f(g(u)) + tf(g(v)),$$
$$\forall u, v \in H : g(u), g(v) \in K, t \in [0,1].$$

Clearly every convex function is relative convex, but the converse is not true; see [3, 21]. For the properties, applications and other aspects of the relative convex functions and convex sets, see [1, 12, 16, 17] and the references therein.

For given continuous trifunction $F(.,.,.) : K \times K \times K \longrightarrow R$, continuous bifunction $\varphi(.,.) : H \times H \longrightarrow R \cup \{\infty\}$ and nonlinear operators $T, g : H \longrightarrow H$, consider the problem of finding $u \in H : g(u) \in K$ such that

$$F(g(u), T(g(u)), g(v)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \qquad \forall v \in H : g(v) \in K, \tag{1}$$

which is called the *mixed quasi general equilibrium problem with trifunction,* introduced and studied by Noor and Rassias [19].

We now discuss some special cases.

I. If $g \equiv I$, where I is the identity operator, then problem (1) is equivalent to finding $u \in K$ such that

$$F(u, T(u), v) + \varphi(v, u) - \varphi(u, u) \geq 0, \qquad \forall v \in K, \tag{2}$$

which is the mixed quasi equilibrium problem with trifunction, introduced and studied by Noor [15, 17].

II. We note that for $F(g(u), T(g(u)), g(v)) = \langle T(g(u)), g(v) - g(v)\rangle$, problem (1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$\langle T(g(u)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in H : g(v) \in K. \tag{3}$$

Inequality (3) is known as the *mixed quasi general variational inequality,* which was introduced by Noor [15].

III. If $\varphi(.,.) = \varphi(.)$ is the indicator function of a closed and relative convex-valued set $K(u)$, then problem (1) reduces to finding $u \in H : g(u) \in K(u)$ such that

$$F(g(u), T(g(u)), g(v)) \geq 0, \forall v \in H : g(v) \in K(u), \tag{4}$$

which is called the general quasi equilibrium problem and appears to be a new one.

IV. If $F(g(u), T(g(u)), g(v) = \langle T(g(u)), g(v) - g(u)\rangle$, then problem (4) is equivalent to finding $u \in H : g(u) \in K(u)$ such that

$$\langle T(g(u)), g(v) - g(u)\rangle \geq 0, \forall v \in H : g(v) \in K(u), \tag{5}$$

which is known as the general quasi variational inequality introduced by Noor [15]. For the applications and numerical methods of general quasi variational inequalities; see [3–20] and the references therein.

V. If $g = I$, the identity operator, the general quasi variational inequalities (3) are equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \ \ \forall \ \ v \in K, \tag{6}$$

which are known as the mixed quasi variational inequalities; see [3–19].

VI. We note that for $F(g(u), T(g(u)), g(v)) = B(g(u)), T(g(u)), g(v) - g(v)\rangle$, problem (1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$B(g(u), T(g(u)), g(v) - g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0,$$

$$\forall v \in H : g(v) \in K. \tag{7}$$

Inequality (7) is known as the *mixed quasi general trifunction variational inequality,* which appears to be new one.

It is clear that problems (2)–(7) are special cases of the general equilibrium problems (1). In brief, for a suitable and appropriate choice of the operators T, g, and the space H, one can obtain a wide class of equilibrium, variational inequalities, and complementarity problems. This clearly shows that problem (1) is quite general and unifying one. Furthermore, problem (1) has important applications in various branches of pure and applied sciences; see [1, 2, 3–22].

Definition 3 [19]. The trifunction $F(.,.,.) : K \times K \times K \to R$ with respect to the operators T, g, is said to be:

(i) *partially relaxed jointly strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$F(g(u), T(g(u))g(v)) + F(g(v), T(g(v)), g(z)) \leq \alpha \|g(z) - g(u)\|^2, \forall u, v, z \in K.$$

(ii) *jointly monotone*, if

$$F(g(u), T(g(u)), g(v)) + F(g(v), T(g(v)), g(u)) \leq 0, \forall u, v \in K.$$

(iii) *jointly pseudomonotone,* if

$$\begin{aligned} &F(g(u), T(g(u)), g(v)) + \varphi(g(v) - g(u)) - \varphi(g(u), g(u)) \geq 0 \\ &\qquad \Longrightarrow \\ &- F(g(v), T(g(v)), g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall u, v \in K. \end{aligned}$$

(iv) *jointly hemicontinuous,* $\forall u, v \in K, t \in [0,1]$, if the mapping $F(g(u) + t(g(v) - g(u)), T(g(u) + t(g(v) - g(u)), g(v))$ is continuous.

We remark that if $z = u$, then partially relaxed jointly strongly monotonicity is exactly jointly monotonicity of the operator $F(.,.,.)$. For $g \equiv I$, the identity operator, Definition 2.1 reduces to the standard definition of partially relaxed jointly strongly monotonicity, jointly monotonicity, and jointly pseudomonotonicity. It is known that monotonicity implies pseudomonotonicity, but not conversely. This implies that the concepts of partially relaxed strongly monotonicity and pseudomonotonicity are weaker than monotonicity.

Noor and Rassias [19] have proved that problem (1) is equivalent to its dual problem under some conditions. We include this result due to its importance. We include all the details for the sake of completeness and to convey the main idea of the technique involved.

Lemma 1 *Let $F(.,.,.)$ be jointly pseudomonotone, jointly hemicontinuous, and relative convex with respect to third argument. If the bifunction $\varphi(.,.)$ is relative convex with respect to first argument, then the general equilibrium problem* (1) *is equivalent to finding $u \in H : g(u) \in K$ such that*

$$-F(g(v), T(g(v)), g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in H : g(v) \in K. \tag{8}$$

Proof Let $u \in H : g(u) \in K$ be a solution of (1). Then

$$F(g(u), T(g(u)), g(v)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in H : g(v) \in K$$

which implies

$$-F(g(v), T(g(v)), g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in H : g(v) \in K, \tag{9}$$

since $F(.,.,.)$ is jointly monotone

Conversely, let $u \in K$ satisfy (8). Since K is a g-convex set, $\forall u, v \in H$: $g(u), g(v) \in K, t \in [0,1], g(v_t) = g(u) + t(g(v) - g(u)) \equiv (1-t)g(u) + tg(v) \in K$.

Taking $g(v) = g(v_t)$ in (9), we have

$$\begin{aligned} F(g(v_t), T(g(v_t)), g(u)) &\leq \varphi(g(v_t), g(u)) - \varphi(g(u), g(u)) \\ &\leq t\{\varphi(g(v), g(u)) - \varphi(g(u), g(u))\}. \end{aligned} \tag{10}$$

Now using (10) and relative convexity of $F(.,.)$ with respect to third argument, we have

$$\begin{aligned} 0 &\leq F(g(v_t), T(g(v_t)), g(v_t)) \\ &= F(g(v_t), T(g(v_t)), (1-t)g(u) + tg(v)) \\ &\leq tF(g(v_t), T(g(v_t)), g(v)) + (1-t)F(g(v_t), T(g(v_t)), g(u)) \\ &\leq tF(g(v_t), T(g(v_t)), g(v)) + t(1-t)\{\varphi(g(v), g(u)) - \varphi(g(u), g(u))\} \end{aligned} \tag{11}$$

Dividing (11) by t and letting $t \longrightarrow 0$, we have

$$F(g(u), T(g(u)), g(v)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in K,$$

the required (1). □

Remark 1 Problem (8) is known as the *dual mixed quasi general equilibrium problem.* One can easily show that the solution set of problem (8) is closed and relative convex set. From Lemma 2.1, it follows that the solution set of problems (1) and (8) are the same. This inter relationship has played an important role in the study of well-posedness of equilibrium problems and variational inequalities. In fact, Lemma 2.1 can be viewed as a natural generalization and extension of a well-known Minty's Lemma in variational inequalities theory; see [5, 6, 8].

Definition 4 The bifunction $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ is called *skew symmetric,* if and only if,

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) - \varphi(v, v) \geq 0, \forall u, v \in H.$$

Clearly if the skew-symmetric bifunction $\varphi(.,.)$ is bilinear, then

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) = \varphi(u - v, u - v) \geq 0, \forall u, v \in H.$$

This shows that the bifunction $\varphi(.,.)$ is positive.

3 Main Results

In this section, we suggest and analyze some new iterative methods for solving the problem (1) by using the auxiliary principle technique [5] as developed by Noor [13, 15, 17] and Noor et al. [18] in recent years.

For a given $u \in H : g(u) \in K$ satisfying (1), consider the problem of finding a unique $w \in H : g(w) \in K$ such that

$$\rho F(g(w), T(g(w)), g(v)) + \langle (1-\lambda)(g(w) - g(u)), g(v) - g(w)\rangle \\ \geq \rho\{\varphi(g(w), g(w)) - \varphi(g(v), g(w))\}, \forall v \in H : g(v) \in K, \tag{12}$$

which is called the auxiliary mixed quasi general equilibrium problem and where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is a solution of the nonconvex equilibrium problems (1). This observation enables us to suggest the following method for solving (1).

Algorithm 1 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_{n+1}), T(g(u_{n+1})), g(v)) + \langle (1-\lambda)(g(u_{n+1} - g(u_n)), g(v) - g(u_{n+1})\rangle \\ \geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall v \in H : g(v) \in K, \tag{13}$$

where $\lambda > 0$ is a constant. Algorithm 1 is called the implicit method for solving (1).

We may write Algorithm 1 in the following equivalent form, which is useful to derive other iterative methods for solving (1) and related problems.

Algorithm 2 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_n), T(g(u_n)), g(v)) + \langle g(y_n - g(u_n), g(v) - g(u_{n+1})\rangle \\ \geq \rho\{\varphi(g(y_n), g(y_n) - \varphi(g(v), g(y_n))\}, \forall g(v) \in K \\ \rho F(g(y_n), T(g(y_n)), g(v)) + \langle g(u_{n+1} - g(u_n) - \lambda(g(y_n) - g(u_n)), g(v) - g(u_{n+1})\rangle \\ \geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K$$

For $\lambda = 0$, Algorithm 2 collapses to:

Algorithm 3 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_n), T(g(u_n)), g(v)) + \langle g(y_n - g(u_n), g(v) - g(u_{n+1})\rangle \\ \geq \rho\{\varphi(g(y_n), g(y_n) - \varphi(g(v), g(y_n))\}, \forall g(v) \in K \\ \rho F(g(y_n), T(g(y_n)), g(v)) + \langle g(u_{n+1} - g(u_n), g(v) - g(u_{n+1})\rangle \\ \geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K.$$

Algorithm 3 is analogues of the extragradient method of Korpelevich, see [16] and appears to be a new one.

For $\lambda = 1$, Algorithm 3.2 reduces to the following two-step iterative method for solving (1). Such type of methods have been studied and investigated by Noor [16, 17] for general variational inequalities.

Algorithm 4 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_n), T(g(u_n)), g(v)) + \langle g(y_n - g(u_n), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(y_n), g(y_n) - \varphi(g(v), g(y_n))\}, \forall g(v) \in K$$
$$\rho F(g(y_n), T(g(y_n)), g(v)) + \langle g(u_{n+1} - g(y_n), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K$$

For $\lambda = \frac{1}{2}$, Algorithm 2 reduces to:

Algorithm 5 [17]. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_n), T(g(u_n)), g(v)) + \langle g(y_n - g(u_n), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(y_n), g(y_n) - \varphi(g(v), g(y_n))\}, \forall g(v) \in K$$
$$\rho F(g(y_n), T(g(y_n)), g(v)) + \langle g(u_{n+1} - \frac{1}{2}(g(y_n) + g(u_n)), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K$$

Note that if $g \equiv I$, the identity operator, Algorithm 1 reduces to a method for solving the equilibrium problems with trifunction (2), which are mainly due to Noor [17].

Algorithm 6 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, T(u_{n+1}, v) + (1 - \lambda)(u_{n+1} - u_n), v - u_{n+1}\rangle$$
$$\geq \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})\} \geq 0, \forall v \in K.$$

For the convergence analysis of Al; Algorithm 6, see Noor [17].

For $F(g(u), T(g(u)), (v)) = \langle T(g(u)), g(v) - g(u)\rangle$, Algorithm 1 reduces to:

Algorithm 7 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T(g(u_{n+1})) + (1 - \lambda)(g(u_{n+1} - (g(u_n)), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(v), g(u_{n+1}))\}, \forall v \in K,$$

for solving mixed quasi general variational inequalities [17].

For suitable and appropriate choice of the operators and the space H, one can obtain various new and known methods for solving general equilibrium, variational inequalities, and complementarity problems.

We now study the convergence analysis of Algorithm 1.

Theorem 1 Let the trifunction $F(.,.,.)$ be jointly pseudomonotone. If the bifunction $\varphi(.,.)$ is skew symmetric, then the approximate solution u_{n+1}obtained from Algorithm 1 satisfies the inequality

$$\|g(u)-g(u_{n+1})\|^2 \leq \|g(u)-g(u_n)\|^2 - \|g(u_n)-g(u_{n+1})\|^2, \tag{14}$$

where u is the exact solution of (1).

Proof Let $u \in H : g(u) \in K$ be a solution of (1). Then

$$F(g(u), T(g(u)), g(v)) \geq \varphi(g(u), g(u)) - \varphi(g(v), g(u))\forall v \in H : g(v) \in K,$$

which implies that

$$-F(g(v), T(g(v), g(u)) \geq \varphi(g(u), g(u)) - \varphi(g(v), g(u)), \forall v \in H : g(v) \in K, \tag{15}$$

since $F(.,.,.)$ is jointly pseudomonotone.

Taking $v = u_{n+1}$ in (15), we have

$$-F(g(u_{n+1}), T(g(u_{n+1})), g(u)) \geq \varphi(g(u), g(u)) - \varphi(g(u_{n+1}), g(u)) \tag{16}$$

Taking $v = u$ in (13), we have

$$\begin{aligned} &\rho F(g(u_{n+1}), T(g(u_{n+1})), g(u)) + \langle (1-\lambda)(g(u_{n+1}) - g(u_n)), g(u) - g(u_{n+1})\rangle \\ &\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(u), g(u_{n+1}))\}. \end{aligned} \tag{17}$$

From (16) and (17), we have

$$\begin{aligned} &(1-\lambda)\langle g(u_{n+1}) - g(u_n)\rangle \\ &\geq \rho\{\varphi(g(u_n), g(u_n)) - \varphi(g(_{n+1}), g(u)) - \varphi(g(u), g(u_{n+1}) + \varphi(g(u_{n+1}), g(u_{n+1}))\} \\ &\geq 0, \end{aligned} \tag{18}$$

where we have used the fact that the bifunction $\varphi(.,.)$ is a skew symmetric.

From (18) and using the inequality

$$2\langle v, u\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2, \forall u, v \in H,$$

we obtain

$$\|g(u)-g(u_{n+1})\|^2 \leq \|g(u)-g(u_)\|^2 - \|g(u_n)-g(u_{n+1})\|^2,$$

which is the required result. □

Theorem 2 Let H be a finite dimensional space. Let the trifunction $F(.,.,.)$ be jointly pseudomonotone and the bifunction $\varphi(.,.)$ be skew symmetric. If u_{n+1} is the approximate solution obtained from Algorithm 3.1, and g^{-1} exists, then

$$\lim_{n\longrightarrow\infty} u_n = u,$$

where $u \in H; g(u) \in K$ is a solution of (1).

Proof Let $u \in H : g(u) \in K$ be a solution of (1). From (14), we see that the sequences $\{\|g(u) - g(u_n)\|\}$ is nonincreasing under the assumptions of Theorem 2 and consequently $\{g(u_n)\}$ is bounded. Also from (14), we have

$$\sum_{n=0}^{\infty} \|g(u_{n+1} - g(u_n)\|^2 \leq \|g(u) - g(u_n)\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n\| = 0, \tag{19}$$

since g^{-1} exists.

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_i}\}$ of this sequence converges to $\hat{u} \in H : g(\hat{u}) \in K$. Replacing u_n by u_{n_i} in (13) and taking the limit as $n_i \longrightarrow \infty$ and using (19), we have

$$F(g(\hat{u}), T(g(\hat{u})), g(v)) + \varphi(g(v), g(\hat{u})) - \varphi(g(\hat{u}), g(\hat{u})) \geq 0, \forall v \in H : g(v) \in K,$$

which shows that $\hat{u}$ solves (1) and

$$\|g(u_{n+1}) - g(\hat{u})\| \leq \|g(u_n) - g(\hat{u})\|^2.$$

Thus, it follows that from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point and

$$\lim_{n\longrightarrow\infty} = \hat{u},$$

the required result. □

Algorithm 1 is an implicit method, which is its difficult to implement. In order to overcome this drawback, we again use the auxiliary principle technique to suggest an explicit iterative method for solving problem (1). This is the main motivation of next Algorithm.

For a given $u \in H : g(u) \in K$ satisfying (1), consider the problem of finding a unique $w \in H : g(w) \in K$ such that

$$\begin{aligned}&\rho F(g(u), T(g(u)), g(v)) + \langle(1 - \lambda)(g(w) - g(u)), g(v) - g(w)\rangle\\&\geq \rho\{\varphi(g(w), g(w)) - \varphi(g(v), g(w))\}, \forall v \in H : g(v) \in K,\end{aligned} \tag{20}$$

which is called the auxiliary mixed quasi general equilibrium problem. we would like to emphasize that problems (12) and (20) are quite different from each other.

We note that if $w = u$, then clearly w is a solution of the nonconvex equilibrium problems (1). This observation enables us to suggest the following method for solving (1).

Algorithm 8 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_{n+1}), T(g(u_{n+1})), g(v)) + \langle (1-\lambda)(g(u_{n+1} - g(u_n)), g(v) - g(u_{n+1}) \rangle$$
$$\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall v \in H : g(v) \in K.$$

Algorithm 1 is called the explicit method for solving (1). Using the technique of Theorem 1 and Theorem 2, one can study the convergence analysis of Algorithm 8.

Conclusion In this chapter, we have suggested some new unified iterative methods for solving a class of mixed quasi general equilibrium problems, introduced and studied by Noor and Rassias [19]. The comparison of these methods with other methods is an interesting and fascinating problem for future research. One may find the novel and innovative applications of these general equilibrium problems in various branches of pure and applied sciences.

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