Chapter 7
Weighted Inequalities

Weighted inequalities arise naturally in Fourier analysis, but their use is best justified by the variety of applications in which they appear. For example, the theory of weights plays an important role in the study of boundary value problems for Laplace's equation on Lipschitz domains. Other applications of weighted inequalities include extrapolation theory, vector-valued inequalities, and estimates for certain classes of nonlinear partial differential equations.

The theory of weighted inequalities is a natural development of the principles and methods we have acquainted ourselves with in earlier chapters. Although a variety of ideas related to weighted inequalities appeared almost simultaneously with the birth of singular integrals, it was only in the 1970s that a better understanding of the subject was obtained. This was spurred by Muckenhoupt's characterization of positive functions w for which the Hardy–Littlewood maximal operator M maps $L^p(\mathbf{R}^n, w(x)\,dx)$ to itself. This characterization led to the introduction of the class A_p and the development of weighted inequalities. We pursue exactly this approach in the next section to motivate the introduction of the A_p classes.

7.1 The A_p Condition

A *weight* is a nonnegative locally integrable function on $\mathbf{R}^n$ that takes values in $(0,\infty)$ almost everywhere. Therefore, weights are allowed to be zero or infinite only on a set of Lebesgue measure zero. Hence, if w is a weight and $1/w$ is locally integrable, then $1/w$ is also a weight.

Given a weight w and a measurable set E, we use the notation

$$w(E) = \int_E w(x)\,dx$$

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to denote the w-measure of the set E. Since weights are locally integrable functions, $w(E) < \infty$ for all sets E contained in some ball. The weighted L^p spaces are denoted by $L^p(\mathbf{R}^n, w)$ or simply $L^p(w)$. Recall the uncentered Hardy–Littlewood maximal operators on $\mathbf{R}^n$ over balls

$$M(f)(x) = \sup_{B \ni x} \operatorname*{Avg}_{B} |f| = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

and over cubes

$$M_c(f)(x) = \sup_{Q \ni x} \operatorname*{Avg}_{Q} |f| = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the suprema are taken over all balls B and cubes Q (with sides parallel to the axes) that contain the given point x. A classical result (Theorem 2.1.6) states that for all $1 < p < \infty$ there is a constant $C_p(n) > 0$ such that

$$\int_{\mathbf{R}^n} M(f)(x)^p\, dx \le C_p(n)^p \int_{\mathbf{R}^n} |f(x)|^p\, dx \tag{7.1.1}$$

for all functions $f \in L^p(\mathbf{R}^n)$. We are concerned with the situation in which the measure dx in (7.1.1) is replaced by $w(x)\, dx$ for some weight $w(x)$.

7.1.1 Motivation for the A_p Condition

The question we raise is whether there is a characterization of all weights $w(x)$ such that the strong type (p,p) inequality

$$\int_{\mathbf{R}^n} M(f)(x)^p\, w(x)\, dx \le C_p^p \int_{\mathbf{R}^n} |f(x)|^p\, w(x)\, dx \tag{7.1.2}$$

is valid for all $f \in L^p(w)$.

Suppose that (7.1.2) is valid for some weight w and all $f \in L^p(w)$ for some $1 < p < \infty$. Apply (7.1.2) to the function $f\chi_B$ supported in a ball B and use that $\operatorname{Avg}_B |f| \le M(f\chi_B)(x)$ for all $x \in B$ to obtain

$$w(B)\big(\operatorname*{Avg}_{B} |f|\big)^p \le \int_B M(f\chi_B)^p\, w\, dx \le C_p^p \int_B |f|^p\, w\, dx. \tag{7.1.3}$$

It follows that

$$\left(\frac{1}{|B|} \int_B |f(t)|\, dt\right)^p \le \frac{C_p^p}{w(B)} \int_B |f(x)|^p\, w(x)\, dx \tag{7.1.4}$$

for all balls B and all functions f. At this point, it is tempting to choose a function such that the two integrands are equal. We do so by setting $f = w^{-p'/p}$, which gives

$f^p w = w^{-p'/p}$. Under the assumption that $\inf_B w > 0$ for all balls B, it would follow from (7.1.4) that

$$\sup_{B \text{ balls}} \left(\frac{1}{|B|} \int_B w(x)\,dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}}\,dx \right)^{p-1} \le C_p^p. \tag{7.1.5}$$

If $\inf_B w = 0$ for some balls B, we take $f = (w+\varepsilon)^{-p'/p}$ to obtain

$$\left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B (w(x)+\varepsilon)^{-\frac{p'}{p}}\,dx \right)^p \left(\frac{1}{|B|} \int_B \frac{w(x)\,dx}{(w(x)+\varepsilon)^{p'}} \right)^{-1} \le C_p^p \tag{7.1.6}$$

for all $\varepsilon > 0$. Replacing $w(x)\,dx$ by $(w(x)+\varepsilon)\,dx$ in the last integral in (7.1.6) we obtain a smaller expression, which is also bounded by C_p^p. Since $-p'/p = -p'+1$, (7.1.6) implies that

$$\left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B (w(x)+\varepsilon)^{-\frac{p'}{p}}\,dx \right)^{p-1} \le C_p^p, \tag{7.1.7}$$

from which we can still deduce (7.1.5) via the Lebesgue monotone convergence theorem by letting $\varepsilon \to 0$. We have now obtained that every weight w that satisfies (7.1.2) must also satisfy the rather strange-looking condition (7.1.5), which we refer to in the sequel as the A_p condition. It is a remarkable fact, to be proved in this chapter, that the implication obtained can be reversed, that is, (7.1.2) is a consequence of (7.1.5). This is the first significant achievement of the theory of weights [i.e., a characterization of all functions w for which (7.1.2) holds]. This characterization is based on some deep principles discussed in the next section and provides a solid motivation for the introduction and careful examination of condition (7.1.5).

Before we study the converse statements, we consider the case $p = 1$. Assume that for some weight w the weak type $(1,1)$ inequality

$$w\big(\{x \in \mathbf{R}^n : M(f)(x) > \alpha\}\big) \le \frac{C_1}{\alpha} \int_{\mathbf{R}^n} |f(x)| w(x)\,dx \tag{7.1.8}$$

holds for all functions $f \in L^1(\mathbf{R}^n)$. Since $M(f)(x) \ge \operatorname{Avg}_B |f|$ for all $x \in B$, it follows from (7.1.8) that for all $\alpha < \operatorname{Avg}_B |f|$ we have

$$w(B) \le w\big(\{x \in \mathbf{R}^n : M(f)(x) > \alpha\}\big) \le \frac{C_1}{\alpha} \int_{\mathbf{R}^n} |f(x)| w(x)\,dx. \tag{7.1.9}$$

Taking $f\chi_B$ instead of f in (7.1.9), we deduce that

$$\operatorname*{Avg}_B |f| = \frac{1}{|B|} \int_B |f(t)|\,dt \le \frac{C_1}{w(B)} \int_B |f(x)|\,w(x)\,dx \tag{7.1.10}$$

for all functions f and balls B. Taking $f = \chi_S$, we obtain

$$\frac{|S|}{|B|} \le C_1 \frac{w(S)}{w(B)}, \tag{7.1.11}$$

where S is any measurable subset of the ball B.

Recall that the *essential infimum* of a function w over a set E is defined as

$$\operatorname*{ess.inf}_E(w) = \inf\big\{b > 0 : \ |\{x \in E : \ w(x) < b\}| > 0\big\}.$$

Then for every $a > \text{ess.inf}_B(w)$ there exists a subset S_a of B with positive measure such that $w(x) < a$ for all $x \in S_a$. Applying (7.1.11) to the set S_a, we obtain

$$\frac{1}{|B|}\int_B w(t)\,dt \le \frac{C_1}{|S_a|}\int_{S_a} w(t)\,dt \le C_1 a, \tag{7.1.12}$$

which implies

$$\frac{1}{|B|}\int_B w(t)\,dt \le C_1 w(x) \qquad \text{for all balls } B \text{ and almost all } x \in B. \tag{7.1.13}$$

It remains to understand what condition (7.1.13) really means. For every ball B, there exists a null set $N(B)$ such that (7.1.13) holds for all x in $B \setminus N(B)$. Let N be the union of all the null sets $N(B)$ for all balls B with centers in $\mathbf{Q}^n$ and rational radii. Then N is a null set and for every x in $B \setminus N$, (7.1.13) holds for all balls B with centers in $\mathbf{Q}^n$ and rational radii. By density, (7.1.13) must also hold for all balls B that contain a fixed x in $\mathbf{R}^n \setminus N$. It follows that for $x \in \mathbf{R}^n \setminus N$ we have

$$M(w)(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B w(t)\,dt \le C_1 w(x). \tag{7.1.14}$$

Therefore, assuming (7.1.8), we have arrived at the condition

$$M(w)(x) \le C_1 w(x) \qquad \text{for almost all } x \in \mathbf{R}^n, \tag{7.1.15}$$

where C_1 is the same constant as in (7.1.13).

We later see that this deduction can be reversed and we can obtain (7.1.8) as a consequence of (7.1.15). This motivates a careful study of condition (7.1.15), which we refer to as the A_1 condition. Since in all the previous arguments we could have replaced balls with cubes, we give the following definitions in terms of cubes.

Definition 7.1.1. A function $w(x) \ge 0$ is called an A_1 *weight* if

$$M(w)(x) \le C_1 w(x) \qquad \text{for almost all } x \in \mathbf{R}^n \tag{7.1.16}$$

for some constant C_1. If w is an A_1 weight, then the (finite) quantity

$$[w]_{A_1} = \sup_{Q \text{ cubes in } \mathbf{R}^n} \left(\frac{1}{|Q|}\int_Q w(t)\,dt\right)\|w^{-1}\|_{L^\infty(Q)} \tag{7.1.17}$$

is called the A_1 *Muckenhoupt characteristic constant* of w, or simply the A_1 *characteristic constant* of w. Note that A_1 weights w satisfy

$$\frac{1}{|Q|}\int_Q w(t)\,dt \le [w]_{A_1} \operatorname*{ess.inf}_{y\in Q} w(y) \tag{7.1.18}$$

for all cubes Q in $\mathbf{R}^n$.

Remark 7.1.2. We also define

$$[w]_{A_1}^{\text{balls}} = \sup_{B \text{ balls in } \mathbf{R}^n} \left(\frac{1}{|B|}\int_B w(t)\,dt\right) \left\|w^{-1}\right\|_{L^\infty(B)}. \tag{7.1.19}$$

Using (7.1.13), we see that the smallest constant C_1 that appears in (7.1.16) is equal to the A_1 characteristic constant of w as defined in (7.1.19). This is also equal to the smallest constant that appears in (7.1.13). All these constants are bounded above and below by dimensional multiples of $[w]_{A_1}$.

We now recall condition (7.1.5), which motivates the following definition of A_p weights for $1 < p < \infty$.

Definition 7.1.3. Let $1 < p < \infty$. A weight w is said to be *of class* A_p if

$$\sup_{Q \text{ cubes in } \mathbf{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty. \tag{7.1.20}$$

The expression in (7.1.20) is called the A_p *Muckenhoupt characteristic constant of* w (or simply the A_p characteristic constant of w) and is denoted by $[w]_{A_p}$.

Remark 7.1.4. Note that Definitions 7.1.1 and 7.1.3 could have been given with the set of all cubes in $\mathbf{R}^n$ replaced by the set of all balls in $\mathbf{R}^n$. Defining $[w]_{A_p}^{\text{balls}}$ as in (7.1.20) except that cubes are replaced by balls, we see that

$$\left(v_n 2^{-n}\right)^p \le \frac{[w]_{A_p}}{[w]_{A_p}^{\text{balls}}} \le \left(n^{n/2} v_n 2^{-n}\right)^p. \tag{7.1.21}$$

7.1.2 Properties of A_p Weights

It is straightforward that translations, isotropic dilations, and scalar multiples of A_p weights are also A_p weights with the same A_p characteristic. We summarize some basic properties of A_p weights in the following proposition.

Proposition 7.1.5. *Let $w \in A_p$ for some $1 \le p < \infty$. Then*

(1) $[\delta^\lambda(w)]_{A_p} = [w]_{A_p}$, *where* $\delta^\lambda(w)(x) = w(\lambda x_1, \dots, \lambda x_n)$.

(2) $[\tau^z(w)]_{A_p} = [w]_{A_p}$, *where* $\tau^z(w)(x) = w(x - z)$, $z \in \mathbf{R}^n$.

(3) $[\lambda w]_{A_p} = [w]_{A_p}$ *for all* $\lambda > 0$.

(4) When $1 < p < \infty$, *the function* $w^{-\frac{1}{p-1}}$ *is in* $A_{p'}$ *with characteristic constant*

$$\big[w^{-\frac{1}{p-1}}\big]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}.$$

Therefore, $w \in A_2$ *if and only if* $w^{-1} \in A_2$ *and both weights have the same* A_2 *characteristic constant.*

(5) $[w]_{A_p} \geq 1$ *for all* $w \in A_p$. *Equality holds if and only if* w *is a constant.*

(6) The classes A_p *are increasing as* p *increases; precisely, for* $1 \leq p < q < \infty$ *we have*

$$[w]_{A_q} \leq [w]_{A_p}.$$

(7) $\lim_{q \to 1+} [w]_{A_q} = [w]_{A_1}$ *if* $w \in A_1$.

(8) The following is an equivalent characterization of the A_p *characteristic constant of* w*:*

$$[w]_{A_p} = \sup_{\substack{Q \text{ cubes} \\ \text{in } \mathbf{R}^n}} \; \sup_{\substack{f \in L^p(Q, w\,dt) \\ \int_Q |f|^p w\,dt > 0}} \left\{ \frac{\left(\frac{1}{|Q|}\int_Q |f(t)|\,dt\right)^p}{\frac{1}{w(Q)}\int_Q |f(t)|^p w(t)\,dt} \right\}.$$

(9) The measure $w(x)\,dx$ *is doubling: precisely, for all* $\lambda > 1$ *and all cubes* Q *we have*

$$w(\lambda Q) \leq \lambda^{np}[w]_{A_p} w(Q).$$

(λQ *denotes the cube with the same center as* Q *and side length* λ *times the side length of* Q*.)*

Proof. The simple proofs of (1), (2), and (3) are left as an exercise. Property (4) is also easy to check and plays the role of duality in this context. To prove (5) we use Hölder's inequality with exponents p and p' to obtain

$$1 = \frac{1}{|Q|}\int_Q dx = \frac{1}{|Q|}\int_Q w(x)^{\frac{1}{p}} w(x)^{-\frac{1}{p}}\,dx \leq [w]_{A_p}^{\frac{1}{p}},$$

with equality holding only when $w(x)^{\frac{1}{p}} = c\,w(x)^{-\frac{1}{p}}$ for some $c > 0$ (i.e., when w is a constant). To prove (6), observe that $0 < q' - 1 < p' - 1 \leq \infty$ and that the statement

$$[w]_{A_q} \leq [w]_{A_p}$$

is equivalent to the fact

$$\big\|w^{-1}\big\|_{L^{q'-1}(Q, \frac{dx}{|Q|})} \leq \big\|w^{-1}\big\|_{L^{p'-1}(Q, \frac{dx}{|Q|})}.$$

Property (7) is a consequence of part (a) of Exercise 1.1.3.

To prove (8), apply Hölder's inequality with exponents p and p' to get

$$\begin{aligned}
(\operatorname*{Avg}_Q |f|)^p &= \left(\frac{1}{|Q|}\int_Q |f(x)|\,dx\right)^p \\
&= \left(\frac{1}{|Q|}\int_Q |f(x)| w(x)^{\frac{1}{p}} w(x)^{-\frac{1}{p}}\,dx\right)^p \\
&\le \frac{1}{|Q|^p}\left(\int_Q |f(x)|^p w(x)\,dx\right)\left(\int_Q w(x)^{-\frac{p'}{p}}\,dx\right)^{\frac{p}{p'}} \\
&= \left(\frac{1}{\omega(Q)}\int_Q |f(x)|^p w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \\
&\le [w]_{A_p}\left(\frac{1}{\omega(Q)}\int_Q |f(x)|^p w(x)\,dx\right).
\end{aligned}$$

This argument proves the inequality $\ge$ in (8) when $p > 1$. In the case $p = 1$ the obvious modification yields the same inequality. The reverse inequality follows by taking $f = (w+\varepsilon)^{-p'/p}$ as in (7.1.6) and letting $\varepsilon \to 0$.

Applying (8) to the function $f = \chi_Q$ and putting λQ in the place of Q in (8), we obtain

$$w(\lambda Q) \le \lambda^{np}[w]_{A_p} w(Q),$$

which says that $w(x)\,dx$ is a doubling measure. This proves (9). □

Example 7.1.6. A positive measure $d\mu$ is called doubling if for some $C < \infty$,

$$\mu(2B) \le C\mu(B) \tag{7.1.22}$$

for all balls B. We show that the measures $|x|^a\,dx$ are doubling when $a > -n$. We divide all balls $B(x_0,R)$ in $\mathbf{R}^n$ into two categories: balls of type I that satisfy $|x_0| \ge 3R$ and type II that satisfy $|x_0| < 3R$. For balls of type I we observe that

$$\int_{B(x_0,2R)} |x|^a\,dx \le v_n(2R)^n \begin{cases} (|x_0|+2R)^a & \text{when } a \ge 0, \\ (|x_0|-2R)^a & \text{when } a < 0, \end{cases}$$

$$\int_{B(x_0,R)} |x|^a\,dx \ge v_n R^n \begin{cases} (|x_0|-R)^a & \text{when } a \ge 0, \\ (|x_0|+R)^a & \text{when } a < 0. \end{cases}$$

Since $|x_0| \ge 3R$, we have $|x_0| + 2R \le 4(|x_0| - R)$ and $|x_0| - 2R \ge \frac{1}{4}(|x_0| + R)$, from which (7.1.22) follows with $C = 2^{3n}4^{|a|}$.

For balls of type II, we have $|x_0| \le 3R$ and we note two things: first

$$\int_{B(x_0,2R)} |x|^a\,dx \le \int_{|x|\le 5R} |x|^a\,dx = c_n R^{n+a},$$

and second, since $|x|^a$ is radially decreasing for $a < 0$ and radially increasing for $a \geq 0$, we have

$$\int_{B(x_0,R)} |x|^a\,dx \geq \begin{cases} \displaystyle\int_{B(0,R)} |x|^a\,dx & \text{when } a \geq 0, \\ \displaystyle\int_{B(3R\frac{x_0}{|x_0|},R)} |x|^a\,dx & \text{when } a < 0. \end{cases}$$

For $x \in B(3R\frac{x_0}{|x_0|},R)$ we must have $|x| \geq 2R$, and hence both integrals on the right are at least a multiple of R^{n+a}. This establishes (7.1.22) for balls of type II.

Example 7.1.7. We investigate for which real numbers a the power function $|x|^a$ is an A_p weight on $\mathbf{R}^n$. For $1 < p < \infty$, we examine for which a the following expression is finite:

$$\sup_{B \text{ balls}} \left(\frac{1}{|B|}\int_B |x|^a\,dx\right)\left(\frac{1}{|B|}\int_B |x|^{-a\frac{p'}{p}}\,dx\right)^{\frac{p}{p'}}. \tag{7.1.23}$$

As in the previous example we split the balls in $\mathbf{R}^n$ into those of type I and those of type II. If $B = B(x_0,R)$ is of type I, then for x satisfying $|x - x_0| \leq R$ we must have

$$\frac{2}{3}|x_0| \leq |x_0| - R \leq |x| \leq |x_0| + R \leq \frac{4}{3}|x_0|,$$

thus the expression inside the supremum in (7.1.23) is comparable to

$$|x_0|^a\big(|x_0|^{-a\frac{p'}{p}}\big)^{\frac{p}{p'}} = 1.$$

If $B(x_0,R)$ is a ball of type II, then $B(0,5R)$ has size comparable to $B(x_0,R)$ and contains it. Since the measure $|x|^a\,dx$ is doubling, the integrals of the function $|x|^a$ over $B(x_0,R)$ and over $B(0,5R)$ are comparable. It suffices therefore to estimate the expression inside the supremum in (7.1.23), in which we have replaced $B(x_0,R)$ by $B(0,5R)$. But this is

$$\left(\frac{1}{v_n(5R)^n}\int_{B(0,5R)} |x|^a\,dx\right)\left(\frac{1}{v_n(5R)^n}\int_{B(0,5R)} |x|^{-a\frac{p'}{p}}\,dx\right)^{\frac{p}{p'}}$$
$$= \left(\frac{n}{(5R)^n}\int_0^{5R} r^{a+n-1}dr\right)\left(\frac{n}{(5R)^n}\int_0^{5R} r^{-a\frac{p'}{p}+n-1}dr\right)^{\frac{p}{p'}},$$

which is seen easily to be finite and independent of R exactly when $-n < a < n\frac{p}{p'}$. We conclude that $|x|^a$ is an A_p weight, $1 < p < \infty$, if and only if $-n < a < n(p-1)$.

The previous proof can be suitably modified to include the case $p = 1$. In this case we obtain that $|x|^a$ is an A_1 weight if and only if $-n < a \leq 0$. As we have seen, the measure $|x|^a\,dx$ is doubling on the larger range $-n < a < \infty$. Thus for $a > n(p-1)$, the function $|x|^a$ provides an example of a doubling measure that is not in A_p.

Example 7.1.8. On $\mathbf{R}^n$ the function

$$u(x) = \begin{cases} \log \frac{1}{|x|} & \text{when } |x| < \frac{1}{e}, \\ 1 & \text{otherwise,} \end{cases}$$

is an A_1 weight. Indeed, to check condition (7.1.19) it suffices to consider balls of type I and type II as defined in Example 7.1.6. In either case the required estimate follows easily.

We now return to a point alluded to earlier, that the A_p condition implies the boundedness of the Hardy–Littlewood maximal function M on the space $L^p(w)$. To this end we introduce four maximal functions acting on functions f that are locally integrable with respect to w:

$$M^w(f)(x) = \sup_{B \ni x} \frac{1}{w(B)} \int_B |f|\, w\, dy,$$

where the supremum is taken over open balls B that contain the point x and

$$\begin{aligned} \mathcal{M}^w(f)(x) &= \sup_{\delta>0} \frac{1}{w(B(x,\delta))} \int_{B(x,\delta)} |f|\, w\, dy, \\ M_c^w(f)(x) &= \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f|\, w\, dy, \end{aligned}$$

where Q is an open cube containing the point x, and

$$\mathcal{M}_c^w(f)(x) = \sup_{\delta>0} \frac{1}{w(Q(x,\delta))} \int_{Q(x,\delta)} |f|\, w\, dy,$$

where $Q(x,\delta) = \prod_{j=1}^n (x_j - \delta, x_j + \delta)$ is a cube of side length 2δ centered at $x = (x_1, \dots, x_n)$. When $w = 1$, these maximal functions reduce to the standard ones $M(f)$, $\mathcal{M}(f)$, $M_c(f)$, and $\mathcal{M}_c(f)$, the uncentered and centered Hardy–Littlewood maximal functions with respect to balls and cubes, respectively.

Theorem 7.1.9. *(a) Let $w \in A_1$. Then we have*

$$\big\|\mathcal{M}_c\big\|_{L^1(w) \to L^{1,\infty}(w)} \le 3^n [w]_{A_1}. \tag{7.1.24}$$

(b) Let $w \in A_p(\mathbf{R}^n)$ for some $1 < p < \infty$. Then there is a constant $C_{n,p}$ such that

$$\big\|\mathcal{M}_c\big\|_{L^p(w) \to L^p(w)} \le C_{n,p} [w]_{A_p}^{\frac{1}{p-1}}. \tag{7.1.25}$$

Since the operators $\mathcal{M}_c$, M_c, $\mathcal{M}$, and M are pointwise comparable, a similar conclusions hold for the other three as well.

Proof. (a) Since $d\mu = w\,dx$ is a doubling measure and $d\mu(3Q) \le 3^n [w]_{A_1} \mu(Q)$, using Proposition 7.1.5 (9) and Exercise 2.1.1 we obtain that M_c^w maps $L^1(w)$ to $L^{1,\infty}(w)$ with norm at most $3^n [w]_{A_1}$. This proves (7.1.24).

(b) Fix a weight w in A_p and let $\sigma = w^{-\frac{1}{p-1}}$ be its dual weight. Fix an open cube $Q = Q(x_0, r)$ in $\mathbf{R}^n$ with center x_0 and side length $2r$ and write

$$\frac{1}{|Q|}\int_Q |f|\,dy = \frac{w(Q)^{\frac{1}{p-1}}\sigma(3Q)}{|Q|^{\frac{p}{p-1}}}\left\{\frac{|Q|}{w(Q)}\left(\frac{1}{\sigma(3Q)}\int_Q |f|\,dy\right)^{p-1}\right\}^{\frac{1}{p-1}}. \tag{7.1.26}$$

For any $x \in Q$, consider the cube $Q(x, 2r)$. Then $Q \subsetneqq Q(x,2r) \subsetneqq 3Q = Q(x_0, 3r)$ and thus

$$\frac{1}{\sigma(3Q)}\int_Q |f|\,dy \le \frac{1}{\sigma(Q(x,2r))}\int_{Q(x,2r)} |f|\,dy \le \mathcal{M}_c^\sigma(|f|\sigma^{-1})(x)$$

for any $x \in Q$. Inserting this expression in (7.1.26), we obtain

$$\frac{1}{|Q|}\int_Q |f|\,dy \le \frac{w(Q)^{\frac{1}{p-1}}\sigma(3Q)}{|Q|^{\frac{p}{p-1}}}\left\{\frac{1}{w(Q)}\int_Q \mathcal{M}_c^\sigma(|f|\sigma^{-1})^{p-1}\,dy\right\}^{\frac{1}{p-1}}. \tag{7.1.27}$$

Since one may easily verify that

$$\frac{w(Q)\sigma(3Q)^{p-1}}{|Q|^p} \le 3^{np}[w]_{A_p},$$

it follows that

$$\frac{1}{|Q|}\int_Q |f|\,dy \le 3^{\frac{np}{p-1}}[w]_{A_p}^{\frac{1}{p-1}}\left(\mathcal{M}_c^w\left[\left(\mathcal{M}_c^\sigma(|f|\sigma^{-1})\right)^{p-1}w^{-1}\right](x_0)\right)^{\frac{1}{p-1}},$$

since x_0 is the center of Q. Hence, we have

$$\mathcal{M}_c(f) \le 3^{\frac{np}{p-1}}[w]_{A_p}^{\frac{1}{p-1}}\left(\mathcal{M}_c^w\left[\left(\mathcal{M}_c^\sigma(|f|\sigma^{-1})\right)^{p-1}w^{-1}\right]\right)^{\frac{1}{p-1}}.$$

Applying $L^p(w)$ norms, we deduce

$$\begin{aligned}
\big\|\mathcal{M}_c(f)\big\|_{L^p(w)} &\le 3^{\frac{np}{p-1}}[w]_{A_p}^{\frac{1}{p-1}}\big\|\mathcal{M}_c^w\left[\left(\mathcal{M}_c^\sigma(|f|\sigma^{-1})\right)^{p-1}w^{-1}\right]\big\|_{L^{p'}(w)}^{\frac{1}{p-1}}\\
&\le 3^{\frac{np}{p-1}}[w]_{A_p}^{\frac{1}{p-1}}\big\|\mathcal{M}_c^w\big\|_{L^{p'}(w)\to L^{p'}(w)}^{\frac{1}{p-1}}\big\|\left(\mathcal{M}_c^\sigma(|f|\sigma^{-1})\right)^{p-1}w^{-1}\big\|_{L^{p'}(w)}^{\frac{1}{p-1}}\\
&= 3^{\frac{np}{p-1}}[w]_{A_p}^{\frac{1}{p-1}}\big\|\mathcal{M}_c^w\big\|_{L^{p'}(w)\to L^{p'}(w)}^{\frac{1}{p-1}}\big\|\mathcal{M}_c^\sigma(|f|\sigma^{-1})\big\|_{L^p(\sigma)}\\
&\le 3^{\frac{np}{p-1}}[w]_{A_p}^{\frac{1}{p-1}}\big\|\mathcal{M}_c^w\big\|_{L^{p'}(w)\to L^{p'}(w)}^{\frac{1}{p-1}}\big\|\mathcal{M}_c^\sigma\big\|_{L^p(\sigma)\to L^p(\sigma)}\big\|f\big\|_{L^p(w)},
\end{aligned}$$

and conclusion (7.1.25) follows, provided we show that

$$\big\|\mathcal{M}_c^w\big\|_{L^q(w)\to L^q(w)} \le C(q,n) < \infty \tag{7.1.28}$$

for any $1 < q < \infty$ and any weight w.

We obtain this estimate by interpolation. Obviously (7.1.28) is valid when $q = \infty$ with $C(\infty, n) = 1$. If we prove that

$$\big\|\mathcal{M}_c^w\big\|_{L^1(w)\to L^{1,\infty}(w)} \le C(1,n) < \infty, \tag{7.1.29}$$

then (7.1.28) will follow from Theorem 1.3.2.

To prove (7.1.29) we fix $f \in L^1(\mathbf{R}^n, w\,dx)$. We first show that the set

$$E_\lambda = \{\mathcal{M}_c^w(f) > \lambda\}$$

is open. For any $r > 0$, let $Q(x,r)$ denote an open cube of side length $2r$ with center $x \in \mathbf{R}^n$. If we show that for any $r > 0$ and $x \in \mathbf{R}^n$ the function

$$x \mapsto \frac{1}{w(Q(x,r))}\int_{Q(x,r)} |f|\,w\,dy \tag{7.1.30}$$

is continuous, then $\mathcal{M}_c^w(f)$ is the supremum of continuous functions; hence it is lower semicontinuous and thus the set E_λ is open. But this is straightforward. If $x_n \to x_0$, then $w(Q(x_n,r)) \to w(Q(x_0,r))$ and also $\int_{Q(x_n,r)} |f|\,w\,dy \to \int_{Q(x_0,r)} |f|\,w\,dy$ by the Lebesgue dominated convergence theorem. Since $w(Q(x_0,r)) \neq 0$, it follows that the function in (7.1.30) is continuous.

Given K a compact subset of E_λ, for any $x \in K$ select an open cube Q_x centered at x such that

$$\frac{1}{w(Q_x)}\int_{Q_x} |f|\,w\,dy > \lambda\,.$$

Applying Lemma 7.1.10 (proved immediately afterward) we find a subfamily $\{Q_{x_j}\}_{j=1}^m$ of the family of the balls $\{Q_x : x \in K\}$ such that (7.1.31) and (7.1.32) hold. Then

$$w(K) \le \sum_{j=1}^m w(Q_{x_j}) \le \sum_{j=1}^m \frac{1}{\lambda}\int_{Q_{x_j}} |f|\,w\,dy \le \frac{24^n}{\lambda}\int_{\mathbf{R}^n} |f|\,w\,dy,$$

where the last inequality follows by multiplying (7.1.32) by $|f|w$ and integrating over $\mathbf{R}^n$. Taking the supremum over all compact subsets K of E_λ and using the inner regularity of $w\,dx$, which is a consequence of the Lebesgue monotone convergence theorem, we deduce that $\mathcal{M}_c^w$ maps $L^1(w)$ to $L^{1,\infty}(w)$ with constant at most 24^n. Thus (7.1.29) holds with $C(1,n) = 24^n$. □

Lemma 7.1.10. *Let K be a bounded set in $\mathbf{R}^n$ and for every $x \in K$, let Q_x be an open cube with center x and sides parallel to the axes. Then there are an $m \in \mathbf{Z}^+ \cup \{\infty\}$ and a sequence of points $\{x_j\}_{j=1}^m$ in K such that*

$$K \subseteq \bigcup_{j=1}^m Q_{x_j} \tag{7.1.31}$$

and for almost all $y \in \mathbf{R}^n$ one has

$$\sum_{j=1}^{m} \chi_{Q_{x_j}}(y) \le 24^n \,. \tag{7.1.32}$$

Proof. Let $s_0 = \sup\{\ell(Q_x) : x \in K\}$. If $s_0 = \infty$, then there exists $x_1 \in K$ such that $\ell(Q_{x_1}) > 4L$, where $[-L,L]^n$ contains K. Then K is contained in Q_{x_1} and the statement of the lemma is valid with $m = 1$.

Suppose now that $s_0 < \infty$. Select $x_1 \in K$ such that $\ell(Q_{x_1}) > s_0/2$. Then define

$$K_1 = K \setminus Q_{x_1}\,, \qquad s_1 = \sup\{\ell(Q_x) : x \in K_1\}\,,$$

and select $x_2 \in K_1$ such that $\ell(Q_{x_2}) > s_1/2$. Next define

$$K_2 = K \setminus (Q_{x_1} \cup Q_{x_2})\,, \qquad s_2 = \sup\{\ell(Q_x) : x \in K_2\}\,,$$

and select $x_3 \in K_2$ such that $\ell(Q_{x_3}) > s_2/2$. Continue until the first integer m is found such that K_m is an empty set. If no such integer exists, continue this process indefinitely and set $m = \infty$.

We claim that for all $i \neq j$ we have $\frac{1}{3}Q_{x_i} \cap \frac{1}{3}Q_{x_j} = \emptyset$. Indeed, suppose that $i > j$. Then $x_i \in K_{i-1} = K \setminus (Q_{x_1} \cup \cdots \cup Q_{x_{i-1}})$; thus $x_i \notin Q_j$. Also $x_i \in K_{i-1} \subsetneqq K_{j-1}$, which implies that $\ell(Q_{x_i}) \le s_{j-1} < 2\ell(Q_{x_j})$. If $x_i \notin Q_j$ and $\ell(Q_{x_j}) > \frac{1}{2}\ell(Q_{x_i})$, it easily follows that $\frac{1}{3}Q_{x_i} \cap \frac{1}{3}Q_{x_j} = \emptyset$.

We now prove (7.1.31). If $m < \infty$, then $K_m = \emptyset$ and therefore $K \subsetneqq \bigcup_{j=1}^{m} Q_{x_j}$. If $m = \infty$, then there is an infinite number of selected cubes Q_{x_j}. Since the cubes $\frac{1}{3}Q_{x_j}$ are pairwise disjoint and have centers in a bounded set, it must be the case that some subsequence of the sequence of their lengths converges to zero. If there exists a $y \in K \setminus \bigcup_{j=1}^{\infty} Q_{x_j}$, this y would belong to all K_j, $j = 1,2,\dots$, and then $s_j \ge \ell(Q_y)$ for all j. Since some subsequence of the s_j's tends to zero, it would follow that $\ell(Q_y) = 0$, which would force the open cube Q_y to be the empty set, a contradiction. Thus (7.1.31) holds.

Finally, we show that $\sum_{j=1}^{m} \chi_{Q_{x_j}}(y) \le 24^n$ for almost every point $y \in \mathbf{R}^n$. To prove this we consider the n hyperplanes H_i that are parallel to the coordinate hyperplanes and pass through the point y. Then we write $\mathbf{R}^n$ as a union of n hyperplanes H_i of n-dimensional Lebesgue measure zero and 2^n higher-dimensional open "octants" O_r, henceforth called orthants. We fix a $y \in \mathbf{R}^n$ and we show that there are only 12^n points x_j such that y lies in $O_r \cap Q_{x_j}$ for a given open orthant O_r. To prove this assertion, setting $|z|_{\ell^\infty} = \sup_{1 \le i \le n} |z_i|$ for points $z = (z_1,\dots,z_n)$ in $\mathbf{R}^n$, we pick an $x_{k_0} \in K \cap O_r$ such that $Q_{x_{k_0}}$ contains y and $|x_{k_0} - y|_{\ell^\infty}$ is the largest possible among all $|x_j - y|_{\ell^\infty}$. If x_j is another point in $K \cap O_r$ such that Q_{x_j} contains y, then we claim that $x_j \in Q_{x_{k_0}}$. Indeed, to show this we notice that for each $i \in \{1,\dots,n\}$ we have

$$\begin{aligned} |x_{j,i} - x_{k_0,i}| &= \big|x_{j,i} - y_i - (x_{k_0,i} - y_i)\big| \\ &= \big||x_{j,i} - y_i| - |x_{k_0,i} - y_i|\big| \end{aligned}$$

$$
\begin{aligned}
&\leq \max\left(|x_{k_0,i}-y_i|,|x_{j,i}-y_i|\right)\\
&\leq \max\left(|x_{k_0}-y|_{\ell^\infty},|x_j-y|_{\ell^\infty}\right)\\
&= |x_{k_0}-y|_{\ell^\infty}\\
&< \tfrac{1}{2}\ell(Q_{x_{k_0}}),
\end{aligned}
$$

where the second equality is due to the fact that x_j, x_{k_0} lie in the same orthant and the last inequality in the fact that $y \in Q_{x_{k_0}}$; it follows that x_j lies in $Q_{x_{k_0}}$.

We observed previously that $i > j$ implies $x_i \notin Q_j$. Since x_j lies in $Q_{x_{k_0}}$, one must then have $j \leq k_0$, which implies that $\frac{1}{2}\ell(Q_{x_{k_0}}) < \ell(Q_{x_j})$. Thus all cubes Q_{x_j} with centers in $K \cap O_r$ that contain the fixed point y have side lengths comparable to that of $Q_{x_{k_0}}$. A simple geometric argument now gives that there are at most finitely many cubes Q_{x_j} of side length between α and 2α that contain the given point y such that $\frac{1}{3}Q_{x_j}$ are pairwise disjoint. Indeed, let $\alpha = \frac{1}{2}\ell(Q_{x_{k_0}})$ and let $\{Q_{x_r}\}_{r\in I}$ be the cubes with these properties. Then we have

$$
\frac{\alpha^n|I|}{3^n} \leq \sum_{r\in I}\left|\tfrac{1}{3}Q_{x_r}\right| = \Big|\bigcup_{r\in I}\tfrac{1}{3}Q_{x_r}\Big| \leq \Big|\bigcup_{r\in I}Q_{x_r}\Big| \leq (4\alpha)^n,
$$

since all the cubes Q_{x_r} contain the point y and have length at most 2α and they must therefore be contained in a cube of side length 4α centered at y. This observation shows that $|I| \leq 12^n$, and since there are 2^n sets O_r, we conclude the proof of (7.1.32). □

Remark 7.1.11. Without use of the covering Lemma 7.1.10, (7.1.29) can be proved via the doubling property of w (cf. Exercise 2.1.1(a)), but then the resulting constant $C(q,n)$ would depend on the doubling constant of the measure $w\,dx$ and thus on $[w]_{A_p}$; this would yield a worse dependence on $[w]_{A_p}$ in the constant in (7.1.25).

Exercises

7.1.1. Let k be a nonnegative measurable function such that k, k^{-1} are in $L^\infty(\mathbf{R}^n)$. Prove that if w is an A_p weight for some $1 \leq p < \infty$, then so is kw.

7.1.2. Let w_1, w_2 be two A_1 weights and let $1 < p < \infty$. Prove that $w_1 w_2^{1-p}$ is an A_p weight by showing that

$$
[w_1 w_2^{1-p}]_{A_p} \leq [w_1]_{A_1}[w_2]_{A_1}^{p-1}.
$$

7.1.3. Suppose that $w \in A_p$ for some $p \in [1,\infty)$ and $0 < \delta < 1$. Prove that $w^\delta \in A_q$, where $q = \delta p + 1 - \delta$, by showing that

$$
[w^\delta]_{A_q} \leq [w]_{A_p}^\delta.
$$

7.1.4. Show that if the A_p characteristic constants of a weight w are uniformly bounded for all $p>1$, then $w\in A_1$.

7.1.5. Let $w_0\in A_{p_0}$ and $w_1\in A_{p_1}$ for some $1\le p_0,p_1<\infty$. Let $0\le\theta\le 1$ and define

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\qquad\text{and}\qquad w^{\frac{1}{p}}=w_0^{\frac{1-\theta}{p_0}}\,w_1^{\frac{\theta}{p_1}}\,.$$

Prove that

$$[w]_{A_p}\le[w_0]_{A_{p_0}}^{(1-\theta)\frac{p}{p_0}}[w_1]_{A_{p_1}}^{\theta\frac{p}{p_1}}\,;$$

thus w is in A_p.

7.1.6. ([122]) Fix $1<p<\infty$. A pair of weights (u,w) that satisfies

$$[u,w]_{(A_p,A_p)}=\sup_{\substack{Q\text{ cubes}\\ \text{in }\mathbf{R}^n}}\left(\frac{1}{|Q|}\int_Q u\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\,dx\right)^{p-1}<\infty$$

is said to be of class (A_p,A_p). The quantity $[u,w]_{(A_p,A_p)}$ is called the (A_p,A_p) *characteristic constant of the pair* (u,w).
(a) Suppose that pair of weights (u,w) is of class (A_p,A_p). Show that for all nonnegative measurable functions f and all cubes Q' we have

$$\left(\frac{1}{|Q'|}\int_{Q'}|f|\,dx\right)^p u(Q')\le C_0\int_{Q'}|f|^p w\,dx,$$

where $C_0=[u,w]_{(A_p,A_p)}$.
(b) Suppose that a pair of weights (u,w) satisfies the inequality in part (a) for some constant C_0. Prove that M maps $L^p(w)$ to $L^{p,\infty}(u)$ with norm at most $C(n,p)C_0^{1/p}$, where $C(n,p)$ is a fixed constant.
(c) Suppose that for a pair of weights (u,w), M maps $L^p(w)$ to $L^{p,\infty}(u)$. Show that the pair (u,w) is of class (A_p,A_p).
[*Hint:* Part (b): Replacing f by $f\chi_Q$ in part (a), where $Q\subseteq Q'$, obtain that

$$u(Q')\le C_0|Q'|^p\frac{\int_Q|f|^p w\,dx}{\big(\int_Q|f|\,dx\big)^p}\,.$$

Then use Exercise 5.3.9 to find disjoint cubes Q_j such that the set $E_\alpha=\{x\in\mathbf{R}^n: M_c(f)(x)>\alpha\}$ is contained in the union of $3Q_j$ and $\frac{\alpha}{4^n}<\frac{1}{|Q_j|}\int_{Q_j}|f(t)|\,dt\le\frac{\alpha}{2^n}$. Then $u(E_\alpha)\le\sum_j u(3Q_j)$, and bound each $u(3Q_j)$ by taking $Q'=3Q_j$ and $Q=Q_j$ in the preceding estimate. Part (c): First prove the assertion in part (b) and then derive the inequality in part (a) by adapting the idea in the discussion in the beginning of Subsection 7.1.1.]

7.1.7. ([122]) Let $1 < p < \infty$ and let (u,w) be a pair of weights of class (A_p,A_p). Show that for any q with $p < q < \infty$ there is a constant $C_{p,q,n} < \infty$ such that for all $f \in L^q(w)$ we have

$$\left(\int_{\mathbf{R}^n} M(f)(x)^q u(x)\,dx\right)^{1/q} \le C_{p,q,n}\left(\int_{\mathbf{R}^n} f(x)^q w(x)\,dx\right)^{1/q}.$$

[*Hint:* Use Exercise 7.1.6 and interpolate between L^p and L^∞.]

7.1.8. Let $k > 0$. For an A_1 weight w show that $[\min(w,k)]_{A_1} \le [w]_{A_1}$. If $1 < p < \infty$ and $w \in A_p$, show that

$$[\min(w,k)]_{A_p} \le c_p [w]_{A_p},$$

where $c_p = 1$ if $1 < p \le 2$ and $c_p = 2^{p-1}$ if $2 < p < \infty$.
[*Hint:* Use the inequality $\frac{1}{|Q|}\int_Q \min(w,k)^{-\frac{1}{p-1}}dx \le \frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}dx + k^{-\frac{1}{p-1}}$ and also $\frac{1}{|Q|}\int_Q \min(w,k)dx \le \min\left\{k, \frac{1}{|Q|}\int_Q w\,dx\right\}$.]

7.1.9. Suppose that $w_j \in A_{p_j}$ with $1 \le j \le m$ for some $1 \le p_1,\dots,p_m < \infty$ and let $0 < \theta_1,\dots,\theta_m < 1$ be such that $\theta_1 + \cdots + \theta_m = 1$. Show that

$$w_1^{\theta_1}\cdots w_m^{\theta_m} \in A_{\max\{p_1,\dots,p_m\}}.$$

[*Hint:* First note that each weight w_j lies in $A_{\max\{p_1,\dots,p_m\}}$ and then apply Hölder's inequality.]

7.1.10. Let $w_1 \in A_{p_1}$ and $w_2 \in A_{p_2}$ for some $1 \le p_1, p_2 < \infty$. Prove that

$$[w_1 + w_2]_{A_p} \le [w_1]_{A_{p_1}} + [w_2]_{A_{p_2}},$$

where $p = \max(p_1,p_2)$.

7.1.11. Show that the function

$$u(x) = \begin{cases} \log\frac{1}{|x|} & \text{when } |x| < \frac{1}{e}, \\ 1 & \text{otherwise,} \end{cases}$$

in Example 7.1.8 is an A_1 weight on $\mathbf{R}^n$.
[*Hint:* Use $[u]_{A_1}^{\text{balls}}$ instead of $[u]_{A_1}$ and consider balls of type I and II as in Example 7.1.7.]

7.1.12. Let $1 < p < \infty$ and $w \in A_1$. Show that the uncentered Hardy-Littlewood maximal function M maps $L^{p,\infty}(w)$ to itself.
[*Hint:* Prove first the inequality

$$w(\{M(g) > \lambda\}) \le \frac{3^n([w]_{A_1}^{balls})^2}{\lambda}\int_{\{M(g)>\lambda\}} |g|\,w\,dx$$

and then use the characterization of $L^{p,\infty}$ given in Exercise 1.1.12.]

7.2 Reverse Hölder Inequality for A_p Weights and Consequences

An essential property of A_p weights is that they assign to subsets of balls mass proportional to the percentage of the Lebesgue measure of the subset within the ball. The following lemma provides a way to quantify this statement.

Lemma 7.2.1. *Let $w \in A_p$ for some $1 \le p < \infty$ and let $0 < \alpha < 1$. Then there exists $\beta < 1$ such that whenever S is a measurable subset of a cube Q that satisfies $|S| \le \alpha|Q|$, we have $w(S) \le \beta\, w(Q)$.*

Proof. Taking $f = \chi_A$ in property (8) of Proposition 7.1.5, we obtain

$$\left(\frac{|A|}{|Q|}\right)^p \le [w]_{A_p} \frac{w(A)}{w(Q)}\,. \tag{7.2.1}$$

We write $S = Q \setminus A$ to get

$$\left(1 - \frac{|S|}{|Q|}\right)^p \le [w]_{A_p} \left(1 - \frac{w(S)}{w(Q)}\right). \tag{7.2.2}$$

Given $0 < \alpha < 1$, set

$$\beta = 1 - \frac{(1-\alpha)^p}{[w]_{A_p}} \tag{7.2.3}$$

and use (7.2.2) to obtain the required conclusion. □

7.2.1 *The Reverse Hölder Property of A_p Weights*

We are now ready to state and prove one of the main results of the theory of weights, the reverse Hölder inequality for A_p weights.

Theorem 7.2.2. *Let $w \in A_p$ for some $1 \le p < \infty$. Then there exist constants C and $\gamma > 0$ that depend only on the dimension n, on p, and on $[w]_{A_p}$ such that for every cube Q we have*

$$\left(\frac{1}{|Q|}\int_Q w(t)^{1+\gamma}\,dt\right)^{\frac{1}{1+\gamma}} \le \frac{C}{|Q|}\int_Q w(t)\,dt\,. \tag{7.2.4}$$

Proof. Let us fix a cube Q and set

$$\alpha_0 = \frac{1}{|Q|}\int_Q w(x)\,dx\,.$$

We also fix $0 < \alpha < 1$. We define an increasing sequence of scalars

$$\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < \cdots$$

for $k \ge 0$ by setting

$$\alpha_{k+1} = 2^n \alpha^{-1} \alpha_k \qquad \text{or} \qquad \alpha_k = (2^n \alpha^{-1})^k \alpha_0,$$

and for each $k \ge 1$ we apply a Calderón–Zygmund decomposition to w at height α_k. Precisely, for dyadic subcubes R of Q, we let

$$\frac{1}{|R|} \int_R w(x)\, dx > \alpha_k \tag{7.2.5}$$

be the selection criterion. Since Q does not satisfy the selection criterion, it is not selected. We divide the cube Q into a mesh of 2^n subcubes of equal side length, and among these cubes we select those that satisfy (7.2.5). We subdivide each unselected subcube into 2^n cubes of equal side length and we continue in this way indefinitely. We denote by $\{Q_{k,j}\}_j$ the collection of all selected subcubes of Q. We observe that the following properties are satisfied:

(1) $\alpha_k < \dfrac{1}{|Q_{k,j}|} \displaystyle\int_{Q_{k,j}} w(t)\, dt \le 2^n \alpha_k$.

(2) For almost all $x \notin U_k$ we have $w(x) \le \alpha_k$, where $U_k = \bigcup_j Q_{k,j}$.

(3) Each $Q_{k+1,j}$ is contained in some $Q_{k,l}$.

Property (1) is satisfied since the unique dyadic parent of $Q_{k,j}$ was not chosen in the selection procedure. Property (2) follows from the Lebesgue differentiation theorem using the fact that for almost all $x \notin U_k$ there exists a sequence of unselected cubes of decreasing lengths whose closures' intersection is the singleton $\{x\}$. Property (3) is satisfied since each $Q_{k,j}$ is the maximal subcube of Q satisfying (7.2.5). And since the average of w over $Q_{k+1,j}$ is also bigger than α_k, it follows that $Q_{k+1,j}$ must be contained in some maximal cube that possesses this property.

We now compute the portion of $Q_{k,l}$ that is covered by cubes of the form $Q_{k+1,j}$ for some j. We have

$$\begin{aligned}
2^n \alpha_k &\ge \frac{1}{|Q_{k,l}|} \int_{Q_{k,l} \cap U_{k+1}} w(t)\, dt \\
&= \frac{1}{|Q_{k,l}|} \sum_{j:\, Q_{k+1,j} \subseteq Q_{k,l}} |Q_{k+1,j}| \frac{1}{|Q_{k+1,j}|} \int_{Q_{k+1,j}} w(t)\, dt \\
&> \frac{\big|Q_{k,l} \cap U_{k+1}\big|}{|Q_{k,l}|} \alpha_{k+1} \\
&= \frac{\big|Q_{k,l} \cap U_{k+1}\big|}{|Q_{k,l}|} 2^n \alpha^{-1} \alpha_k .
\end{aligned}$$

It follows that $\big|Q_{k,l} \cap U_{k+1}\big| \le \alpha |Q_{k,l}|$; thus, applying Lemma 7.2.1, we obtain

$$\frac{w(Q_{k,l} \cap U_{k+1})}{w(Q_{k,l})} < \beta = 1 - \frac{(1-\alpha)^p}{[w]_{A_p}},$$

from which, summing over all l, we obtain

$$w(U_{k+1}) \le \beta w(U_k)\,.$$

The latter gives $w(U_k) \le \beta^k w(U_0)$. We also have $|U_{k+1}| \le \alpha |U_k|$; hence $|U_k| \to 0$ as $k \to \infty$. Therefore, the intersection of the U_k's is a set of Lebesgue measure zero. We can therefore write

$$Q = (Q\setminus U_0) \bigcup \Big(\bigcup_{k=0}^{\infty} U_k \setminus U_{k+1} \Big)$$

modulo a set of Lebesgue measure zero. Let us now find a $\gamma > 0$ such that the reverse Hölder inequality (7.2.4) holds. We have $w(x) \le \alpha_k$ for almost all x in $Q \setminus U_k$ and therefore

$$\begin{aligned}
\int_Q w(t)^{1+\gamma}\,dt &= \int_{Q\setminus U_0} w(t)^{\gamma} w(t)\,dt + \sum_{k=0}^{\infty} \int_{U_k \setminus U_{k+1}} w(t)^{\gamma} w(t)\,dt \\
&\le \alpha_0^{\gamma} w(Q\setminus U_0) + \sum_{k=0}^{\infty} \alpha_{k+1}^{\gamma} w(U_k) \\
&\le \alpha_0^{\gamma} w(Q\setminus U_0) + \sum_{k=0}^{\infty} ((2^n\alpha^{-1})^{k+1}\alpha_0)^{\gamma} \beta^k w(U_0) \\
&\le \alpha_0^{\gamma} \Big(1 + (2^n\alpha^{-1})^{\gamma} \sum_{k=0}^{\infty} (2^n\alpha^{-1})^{\gamma k} \beta^k \Big) w(Q) \\
&= \Big(\frac{1}{|Q|} \int_Q w(t)\,dt \Big)^{\gamma} \Big(1 + \frac{(2^n\alpha^{-1})^{\gamma}}{1-(2^n\alpha^{-1})^{\gamma}\beta} \Big) \int_Q w(t)\,dt\,,
\end{aligned}$$

provided $\gamma > 0$ is chosen small enough that $(2^n\alpha^{-1})^{\gamma}\beta < 1$. Keeping track of the constants, we conclude the proof of the theorem with

$$\gamma = \frac{1}{2} \frac{-\log\beta}{\log 2^n - \log\alpha} = \frac{\log\big([w]_{A_p}\big) - \log\big([w]_{A_p} - (1-\alpha)^p\big)}{2\log\frac{2^n}{\alpha}} \tag{7.2.6}$$

and

$$\begin{aligned}
C^{\gamma+1} &= 1 + \frac{(2^n\alpha^{-1})^{\gamma}}{1-(2^n\alpha^{-1})^{\gamma}\beta} \\
&= 1 + \frac{(2^n\alpha^{-1})^{\gamma}}{1-(2^n\alpha^{-1})^{\gamma}\big(1-\frac{(1-\alpha)^p}{[w]_{A_p}}\big)} \\
&= 1 + \frac{1}{(2^n\alpha^{-1})^{-\gamma} - \big(1-\frac{(1-\alpha)^p}{[w]_{A_p}}\big)}\,,
\end{aligned}$$

which yields

$$C = \left[1 + \frac{1}{\left(1 - \frac{(1-\alpha)^p}{[w]_{A_p}}\right)^{\frac{1}{2}} - \left(1 - \frac{(1-\alpha)^p}{[w]_{A_p}}\right)} \right]^{\frac{2\log\frac{2^n}{\alpha}}{2\log\frac{2^n}{\alpha} - \log\left(1 - \frac{(1-\alpha)^p}{[w]_{A_p}}\right)}}. \tag{7.2.7}$$

Note that up to this point, α was an arbitrary number in $(0,1)$. □

Remark 7.2.3. It is worth observing that for α such that $(1-\alpha)^p = \frac{3}{4}$, the constant γ in (7.2.6) decreases as $[w]_{A_p}$ increases, while the constant C in (7.2.7) increases as $[w]_{A_p}$ increases. This is because $1 - \frac{3}{4}[w]_{A_p}^{-1} \geq \frac{1}{4}$ and for $t \in (\frac{1}{4}, 1)$ the function $\sqrt{t} - t$ is decreasing. This allows us to obtain the following stronger version of Theorem 7.2.2: For any $1 \leq p < \infty$ and $B > 1$, there exist positive constants $C = C(n,p,B)$ and $\gamma = \gamma(n,p,B)$ such that for all $w \in A_p$ satisfying $[w]_{A_p} \leq B$ the reverse Hölder condition (7.2.4) holds for every cube Q. See Exercise 7.2.4(a) for details.

Observe that in the proof of Theorem 7.2.2 it was crucial to know that for some $0 < \alpha, \beta < 1$ we have

$$|S| \leq \alpha\,|Q| \implies w(S) \leq \beta\, w(Q) \tag{7.2.8}$$

whenever S is a subset of the cube Q. No special property of Lebesgue measure was used in the proof of Theorem 7.2.2 other than its doubling property. Therefore, it is reasonable to ask whether Lebesgue measure in (7.2.8) can be replaced by a general measure μ satisfying the doubling property

$$\mu(3Q) \leq C_n\, \mu(Q) < \infty \tag{7.2.9}$$

for all cubes Q in $\mathbf{R}^n$. A straightforward adjustment of the proof of the previous theorem indicates that this is indeed the case.

Corollary 7.2.4. *Let w be a weight and let μ be a measure on $\mathbf{R}^n$ satisfying (7.2.9). Suppose that there exist $0 < \alpha, \beta < 1$, such that*

$$\mu(S) \leq \alpha\, \mu(Q) \implies \int_S w(t)\, d\mu(t) \leq \beta \int_Q w(t)\, d\mu(t)$$

whenever S is a μ-measurable subset of a cube Q. Then there exist $0 < C, \gamma < \infty$ [which depend only on the dimension n, the constant C_n in (7.2.9), α, and β] such that for every cube Q in $\mathbf{R}^n$ we have

$$\left(\frac{1}{\mu(Q)} \int_Q w(t)^{1+\gamma}\, d\mu(t) \right)^{\frac{1}{1+\gamma}} \leq \frac{C}{\mu(Q)} \int_Q w(t)\, d\mu(t). \tag{7.2.10}$$

Proof. The proof of the corollary can be obtained almost verbatim from that of Theorem 7.2.2 by replacing Lebesgue measure with the doubling measure $d\mu$ and the constant 2^n by C_n.

Precisely, we define $\alpha_k = (C_n\alpha^{-1})^k\alpha_0$, where α_0 is the μ-average of w over Q; then properties (1), (2), (3) concerning the selected cubes $\{Q_{k,j}\}_j$ are replaced by

(1_μ) $\alpha_k < \frac{1}{\mu(Q_{k,j})}\int_{Q_{k,j}} w(t)\,d\mu(t) \le C_n\,\alpha_k$.

(2_μ) On $Q\setminus U_k$ we have $w \le \alpha_k$ μ-almost everywhere, where $U_k = \bigcup_j Q_{k,j}$.

(3_μ) Each $Q_{k+1,j}$ is contained in some $Q_{k,l}$.

To prove the upper inequality in (1_μ) we use that the dyadic parent of each selected cube $Q_{k,j}$ was not selected and is contained in $3Q_{k,j}$. To prove (2_μ) we need a differentiation theorem for doubling measures, analogous to that in Corollary 2.1.16. This can be found in Exercise 2.1.1. The remaining details of the proof are trivially adapted to the new setting. The conclusion is that for

$$0 < \gamma < \frac{-\log\beta}{\log C_n - \log\alpha} \tag{7.2.11}$$

and

$$C = \left[1 + \frac{(C_n\alpha^{-1})^\gamma}{1-(C_n\alpha^{-1})^\gamma\beta}\right]^{\frac{1}{\gamma+1}}, \tag{7.2.12}$$

(7.2.10) is satisfied. Notice that the choice of the constants (7.2.6) and (7.2.7) is valid in this case with C_n in place of 2^n. □

7.2.2 Consequences of the Reverse Hölder Property

Having established the crucial reverse Hölder inequality for A_p weights, we now pass to some very important applications. Among them, the first result of this section yields that an A_p weight that lies a priori in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ must actually lie in the better space $L^{1+\sigma}_{\mathrm{loc}}(\mathbf{R}^n)$ for some $\sigma > 0$ depending on the weight.

Theorem 7.2.5. *If $w \in A_p$ for some $1 \le p < \infty$, then there exists a number $\gamma > 0$ (that depends on n, p, and $[w]_{A_p}$) such that $w^{1+\gamma} \in A_p$.*

Proof. Let C be the constant in the proof of Theorem 7.2.2. When $p = 1$, we apply the reverse Hölder inequality of Theorem 7.2.2 to the weight w to obtain

$$\frac{1}{|Q|}\int_Q w(t)^{1+\gamma}\,dt \le \left(\frac{C}{|Q|}\int_Q w(t)\,dt\right)^{1+\gamma} \le C^{1+\gamma}[w]_{A_1}^{1+\gamma}w(x)^{1+\gamma}$$

for almost all x in the cube Q. Therefore, $w^{1+\gamma}$ is an A_1 weight with characteristic constant at most $C^{1+\gamma}[w]_{A_1}^{1+\gamma}$. When $p > 1$, there exist $\gamma_1, \gamma_2 > 0$ and $C_1, C_2 > 0$ such that the reverse Hölder inequality of Theorem 7.2.2 holds for the weights $w \in A_p$ and $w^{-\frac{1}{p-1}} \in A_{p'}$, that is,

$$\left(\frac{1}{|Q|}\int_Q w(t)^{1+\gamma_1}\,dt\right)^{\frac{1}{1+\gamma_1}} \le \frac{C_1}{|Q|}\int_Q w(t)\,dt,$$
$$\left(\frac{1}{|Q|}\int_Q w(t)^{-\frac{1}{p-1}(1+\gamma_2)}\,dt\right)^{\frac{1}{1+\gamma_2}} \le \frac{C_2}{|Q|}\int_Q w(t)^{-\frac{1}{p-1}}\,dt\,.$$

Taking $\gamma=\min(\gamma_1,\gamma_2)$, both inequalities are satisfied with γ in the place of γ_1, γ_2. It follows that $w^{1+\gamma}$ is in A_p and satisfies

$$[w^{1+\gamma}]_{A_p} \le (C_1C_2^{p-1})^{1+\gamma}[w]_{A_p}^{1+\gamma}. \tag{7.2.13}$$

This concludes the proof of the theorem. □

Corollary 7.2.6. *For any $1<p<\infty$ and for every $w\in A_p$ there is a $q=q(n,p,[w]_{A_p})$ with $q<p$ such that $w\in A_q$. In other words, we have*

$$A_p = \bigcup_{q\in(1,p)} A_q\,.$$

Proof. Given $w\in A_p$, let γ, C_1, C_2 be as in the proof of Theorem 7.2.5. In view of the result in Exercise 7.1.3 with $\delta=1/(1+\gamma)$, if $w^{1+\gamma}\in A_p$ and

$$q = p\,\frac{1}{1+\gamma}+1-\frac{1}{1+\gamma} = \frac{p+\gamma}{1+\gamma},$$

then $w\in A_q$ and

$$[w]_{A_q} = [(w^{1+\gamma})^{\frac{1}{1+\gamma}}]_{A_q} \le \big[w^{1+\gamma}\big]_{A_p}^{\frac{1}{1+\gamma}} \le C_1C_2^{p-1}[w]_{A_p}\,,$$

where the last estimate comes from (7.2.13). Since $1<q=\frac{p+\gamma}{1+\gamma}<p$, the required conclusion follows. Observe that the constants $C_1C_2^{p-1}$, q, and $\frac{1}{\gamma}$ increase as $[w]_{A_p}$ increases. □

Another powerful consequence of the reverse Hölder property of A_p weights is the following characterization of all A_1 weights.

Theorem 7.2.7. *Let w be an A_1 weight. Then there exist $0<\varepsilon<1$, a nonnegative function k such that $k,k^{-1}\in L^\infty$, and a nonnegative locally integrable function f that satisfies $M(f)<\infty$ a.e. such that*

$$w(x) = k(x)\,M(f)(x)^{\varepsilon}\,. \tag{7.2.14}$$

Conversely, given a nonnegative function k such that $k,k^{-1}\in L^\infty$ and given a nonnegative locally integrable function f that satisfies $M(f)<\infty$ a.e., define w via (7.2.14). *Then w is an A_1 weight that satisfies*

$$[w]_{A_1} \le \frac{C_n}{1-\varepsilon}\,\|k\|_{L^\infty}\|k^{-1}\|_{L^\infty}\,, \tag{7.2.15}$$

where C_n is a universal dimensional constant.

Proof. In view of Theorem 7.2.2, there exist $0 < \gamma, C < \infty$ such that the reverse Hölder condition

$$\left(\frac{1}{|Q|}\int_Q w(t)^{1+\gamma}\,dt\right)^{\frac{1}{1+\gamma}} \le \frac{C}{|Q|}\int_Q w(t)\,dt \le C\,[w]_{A_1}w(x) \tag{7.2.16}$$

holds for all cubes Q and for all x in $Q\setminus E_Q$, where E_Q is a null subset of Q. We set

$$\varepsilon = \frac{1}{1+\gamma} \qquad \text{and} \qquad f(x) = w(x)^{1+\gamma} = w(x)^{\frac{1}{\varepsilon}}.$$

Letting N be the union of E_Q over all Q with rational radii and centers in $\mathbf{Q}^n$, it follows from (7.2.16) that the uncentered Hardy–Littlewood maximal function $M_c(f)$ with respect to cubes satisfies

$$M_c(f)(x) \le C^{1+\gamma}[w]_{A_1}^{1+\gamma} f(x) \qquad \text{for } x \in \mathbf{R}^n \setminus N.$$

This implies that $M(f) \le C_n C^{1+\gamma}[w]_{A_1}^{1+\gamma} f$ a.e. for some constant C_n that depends only on the dimension. We now set

$$k(x) = \frac{f(x)^{\varepsilon}}{M(f)(x)^{\varepsilon}},$$

and we observe that $C^{-1}C_n^{-\varepsilon}[w]_{A_1}^{-1} \le k \le 1$ a.e.

It remains to prove the converse. Given a weight $w = kM(f)^{\varepsilon}$ in the form (7.2.14) and a cube Q, it suffices to show that

$$\frac{1}{|Q|}\int_Q M(f)(t)^{\varepsilon}\,dt \le \frac{C_n}{1-\varepsilon}M(f)^{\varepsilon}(x) \qquad \text{for almost all } x \in Q, \tag{7.2.17}$$

since then (7.2.15) follows trivially from (7.2.17) with $w = kM(f)^{\varepsilon}$ using that $k, k^{-1} \in L^{\infty}$. To prove (7.2.17), we write

$$f = f\chi_{3Q} + f\chi_{(3Q)^c}.$$

Then

$$\frac{1}{|Q|}\int_Q M(f\chi_{3Q})(t)^{\varepsilon}\,dt \le \frac{C_n'}{1-\varepsilon}\left(\frac{1}{|Q|}\int_{\mathbf{R}^n}(f\chi_{3Q})(t)\,dt\right)^{\varepsilon} \tag{7.2.18}$$

in view of Kolmogorov's inequality (Exercise 2.1.5). But the last expression in (7.2.18) is at most a dimensional multiple of $M(f)(x)^{\varepsilon}$ for almost all $x \in Q$, which proves (7.2.17) when f is replaced by $f\chi_{3Q}$ on the left-hand side of the inequality. And for $f\chi_{(3Q)^c}$ we only need to notice that

$$M(f\chi_{(3Q)^c})(t) \le 2^n \mathcal{M}(f\chi_{(3Q)^c})(t) \le 2^n n^{\frac{n}{2}} M(f)(x)$$

for all x,t in Q, since any ball B centered at t that gives a nonzero average for $f\chi_{(3Q)^c}$ must have radius at least the side length of Q, and thus $\sqrt{n}B$ must also contain x. (Here $\mathcal{M}$ is the centered Hardy–Littlewood maximal operator introduced in Definition 2.1.1.) Hence (7.2.17) also holds when f is replaced by $f\chi_{(3Q)^c}$ on the left-hand side. Combining these two estimates and using the subadditivity property $M(f_1+f_2)^\varepsilon \le M(f_1)^\varepsilon + M(f_2)^\varepsilon$, we obtain (7.2.17). □

We end this section with the following consequence of the reverse Hölder property of A_p weights which can be viewed as a reverse property to (7.2.1).

Proposition 7.2.8. *Let $1 \le p < \infty$ and $w \in A_p$. Then there exist $\delta \in (0,1)$ and $C > 0$ depending only on n, p, and $[w]_{A_p}$ such that for any cube Q and any measurable subset S of Q we have*

$$\frac{w(S)}{w(Q)} \le C\left(\frac{|S|}{|Q|}\right)^{\delta}.$$

Proof. Let C and γ be as in Theorem 7.2.2. We use Hölder's inequality to write

$$\begin{aligned}
\frac{w(S)}{w(Q)} &= \frac{1}{w(Q)}\int_Q w(x)\chi_S(x)\,dx \\
&\le \frac{1}{w(Q)}\left(\int_Q w(x)^{1+\gamma}\,dx\right)^{\frac{1}{1+\gamma}}|S|^{\frac{\gamma}{1+\gamma}} \\
&= \frac{1}{w(Q)}\left(\frac{1}{|Q|}\int_Q w(x)^{1+\gamma}\,dx\right)^{\frac{1}{1+\gamma}}|Q|^{\frac{1}{1+\gamma}}|S|^{\frac{\gamma}{1+\gamma}} \\
&= \frac{C}{w(Q)}\left(\int_Q w(x)\,dx\right)|Q|^{-\frac{\gamma}{1+\gamma}}|S|^{\frac{\gamma}{1+\gamma}} \\
&= C\left(\frac{|S|}{|Q|}\right)^{\delta},
\end{aligned}$$

where $\delta = \frac{\gamma}{1+\gamma}$. This proves the assertion. □

Exercises

7.2.1. Let $w \in A_p$ for some $1 < p < \infty$ and let $1 \le q < \infty$. Prove that the sublinear operator

$$S(f) = \left(M(|f|^q w)w^{-1}\right)^{\frac{1}{q}}$$

is bounded on $L^{p'q}(w)$.

7.2.2. Let v be a real-valued locally integrable function on $\mathbf{R}^n$ and let $1 < p < \infty$. For a cube Q, let v_Q be the average of v over Q.

(a) If e^v is an A_p weight, show that

$$\sup_{Q \text{ cubes}} \frac{1}{|Q|}\int_Q e^{v(t)-v_Q}\,dt \;\le\; [e^v]_{A_p}\,,$$
$$\sup_{Q \text{ cubes}} \frac{1}{|Q|}\int_Q e^{-(v(t)-v_Q)\frac{1}{p-1}}\,dt \;\le\; [e^v]_{A_p}\,.$$

(b) Conversely, if the preceding inequalities hold with some constant C in place of $[v]_{A_p}$, then v lies in A_p with $[v]_{A_p} \le C$.
$\big[$*Hint:* Part (a): If $e^v \in A_p$, use that

$$\frac{1}{|Q|}\int_Q e^{v(t)-v_Q}\,dt \le \Big(\operatorname*{Avg}_Q e^{-\frac{v}{p-1}}\Big)^{p-1}\Big(\operatorname*{Avg}_Q e^v\Big)$$

and obtain a similar estimate for the second quantity.$\big]$

7.2.3. This exercise assumes familiarity with the space BMO.
(a) Show that if $\varphi \in A_2$, then $\log\varphi \in BMO$ and $\|\log\varphi\|_{BMO} \le [\varphi]_{A_2}$.
(b) Prove that every BMO function is equal to a constant multiple of the logarithm of an A_2 weight. Precisely, given $f \in BMO$ show that

$$\big[e^{cf}\big]_{A_2} \le 1+2e\,,$$

where $c = 1/(2^{n+1}\|f\|_{BMO})$.
(c) Prove that if φ is in A_p for some $1<p<\infty$, then $\log\varphi$ is in BMO by showing that

$$\big\|\log\varphi\big\|_{BMO} \le \begin{cases} [\varphi]_{A_p} & \text{when } 1<p\le 2,\\ (p-1)[\varphi]_{A_p}^{\frac{1}{p-1}} & \text{when } 2<p<\infty\,. \end{cases}$$

$\big[$*Hint:* Part (a): Use Exercise 7.2.2 with $p=2$. Part (b): Use Exercise 7.2.2 and Corollary 3.1.7 in [131]. Use Part (c): Use that $\varphi^{-\frac{1}{p-1}} \in A_{p'}$ when $p>2$.$\big]$

7.2.4. Prove the following quantitative versions of Theorem 7.2.2 and Corollary 7.2.6.
(a) For any $1\le p<\infty$ and $B>1$, there exists a positive constant $C_3(n,p,B)$ and $\gamma = \gamma(n,p,B)$ such that for all $w\in A_p$ satisfying $[w]_{A_p}\le B$, (7.2.4) holds for every cube Q with $C_3(n,p,B)$ in place of C.
(b) Given any $1<p<\infty$ and $B>1$ there exists a constant $C_4(n,p,B)$ and $\delta = \delta(n,p,B)$ such that for all $w\in A_p$ we have

$$[w]_{A_p}\le B \implies [w]_{A_{p-\delta}} \le C_4(n,p,B)\,.$$

7.2.5. Given a positive doubling measure μ on $\mathbf{R}^n$, define the characteristic constant $[w]_{A_p(\mu)}$ and the class $A_p(\mu)$ for $1<p<\infty$.
(a) Show that statement (8) of Proposition 7.1.5 remains valid if Lebesgue measure is replaced by μ.

(b) Obtain as a consequence that if $w \in A_p(\mu)$, then for all cubes Q and all μ-measurable subsets A of Q we have

$$\left(\frac{\mu(A)}{\mu(Q)}\right)^p \leq [w]_{A_p(\mu)} \frac{w(A)}{w(Q)}.$$

Conclude that if Lebesgue measure is replaced by μ in Lemma 7.2.1, then the lemma is valid for $w \in A_p(\mu)$.
(c) Use Corollary 7.2.4 to obtain that weights in $A_p(\mu)$ satisfy a reverse Hölder condition.
(d) Prove that given a weight $w \in A_p(\mu)$, there exists $1 < q < p$, which depends on $[w]_{A_p(\mu)}$, such that $w \in A_q(\mu)$.

7.2.6. Let $1 < q < \infty$ and μ a positive measure on $\mathbf{R}^n$. We say that a positive function K on $\mathbf{R}^n$ satisfies a *reverse Hölder condition* of order q with respect to μ, symbolically $K \in RH_q(\mu)$, if

$$[K]_{RH_q(\mu)} = \sup_{Q \text{cubes in } \mathbf{R}^n} \frac{\left(\frac{1}{\mu(Q)} \int_Q K^q \, d\mu\right)^{\frac{1}{q}}}{\frac{1}{\mu(Q)} \int_Q K \, d\mu} < \infty.$$

For positive functions u, v on $\mathbf{R}^n$ and $1 < p < \infty$, show that

$$[vu^{-1}]_{RH_{p'}(u\,dx)} = [uv^{-1}]_{A_p(v\,dx)}^{\frac{1}{p}},$$

that is, vu^{-1} satisfies a reverse Hölder condition of order p' with respect to $u\,dx$ if and only if uv^{-1} is in $A_p(v\,dx)$. Conclude that

$$w \in RH_{p'}(dx) \iff w^{-1} \in A_p(w\,dx),$$
$$w \in A_p(dx) \iff w^{-1} \in RH_{p'}(w\,dx).$$

7.2.7. ([125]) Suppose that a positive function K on $\mathbf{R}^n$ lies in $RH_p(dx)$ for some $1 < p < \infty$. Show that there exists a $\delta > 0$ such that K lies in $RH_{p+\delta}(dx)$.
[*Hint:* By Exercise 7.2.6, $K \in RH_p(dx)$ is equivalent to the fact that $K^{-1} \in A_{p'}(K\,dx)$, and the index p' can be improved by Exercise 7.2.5 (d).]

7.2.8. (a) Show that for any $w \in A_1$ and any cube Q in $\mathbf{R}^n$ and $a > 1$ we have

$$\operatorname*{ess.inf}_Q w \leq a^n [w]_{A_1} \operatorname*{ess.inf}_{aQ} w.$$

(b) Prove that there is a constant C_n such that for all locally integrable functions f on $\mathbf{R}^n$ and all cubes Q in $\mathbf{R}^n$ we have

$$\operatorname*{ess.inf}_Q M(f) \leq C_n \operatorname*{ess.inf}_{3Q} M(f),$$

and an analogous statement is valid for M_c.

[*Hint:* Part (a): Use (7.1.18). Part (b): Apply part (a) to $M(f)^{\frac{1}{2}}$, which is an A_1 weight in view of Theorem 7.2.7.]

7.2.9. ([223]) For a weight $w \in A_1(\mathbf{R}^n)$ define a quantity $r = 1 + \frac{1}{2^{n+1}[w]_{A_1}}$. Show that

$$M_c(w^r)^{\frac{1}{r}} \leq 2\,[w]_{A_1}\, w \qquad \text{a.e.}$$

[*Hint:* Fix a cube Q and consider the family $\mathscr{F}_Q$ of all cubes obtained by subdividing Q into a mesh of $(2^n)^m$ subcubes of side length $2^{-m}\ell(Q)$ for all $m = 1, 2, \dots$. Define $M_Q^d(f)(x) = \sup_{R \in \mathscr{F}_Q, R \ni x} |R|^{-1} \int_R |f|\,dy$. Using Corollary 2.1.21 obtain

$$\int_{Q \cap \{M_Q^d(w) > \lambda\}} w(x)\,dx \leq 2^n \lambda \,|\{x \in Q:\ M_Q^d(w)(x) > \lambda\}|$$

for $\lambda > w_Q = \frac{1}{|Q|}\int_Q w\,dt$. Multiply by $\lambda^{\delta-1}$ and integrate to obtain

$$\int_Q M_Q^d(w)^\delta w\,dx \leq (w_Q)^\delta \int_Q w\,dx + \frac{2^n \delta}{\delta+1}\int_Q M_Q^d(w)^{\delta+1}dx\,.$$

Replace w by $w_k = \min(k, w)$ and select $\delta = \frac{1}{2^{n+1}[w]_{A_1}}$ to deduce

$$\frac{1}{|Q|}\int_Q w_k^{\delta+1}dx \leq \frac{1}{|Q|}\int_Q M_Q^d(w_k)^\delta w_k\,dx \leq 2(w_Q)^{\delta+1},$$

using $[w_k]_{A_1} \leq [w]_{A_1}$. Then let $k \to \infty$.]

7.2.10. Let $1 < p < \infty$. Recall that a pair of weights (u, w) that satisfies

$$[u,w]_{(A_p,A_p)} = \sup_{\substack{Q \text{ cubes} \\ \text{in } \mathbf{R}^n}} \left(\frac{1}{|Q|}\int_Q u\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty$$

is said to be of *class* (A_p, A_p). The quantity $[u,w]_{(A_p,A_p)}$ is called the (A_p, A_p) *characteristic constant* of the pair (u, w).
(a) Show that for any $g \in L^1_{\text{loc}}(\mathbf{R}^n)$ with $0 < g < \infty$ a.e., the pair $(g, M(g))$ is of class (A_p, A_p) with characteristic constant independent of f.
(b) If (u, w) is of class (A_p, A_p), then the Hardy–Littlewood maximal operator M may not map $L^p(w)$ to $L^p(u)$.
(c) Given $g \in L^1_{\text{loc}}(\mathbf{R}^n)$ with $0 < g < \infty$ a.e., conclude that Hardy–Littlewood maximal operator M maps $L^p(M(g)dx)$ to $L^{p,\infty}(g dx)$ and also $L^q(M(g)dx)$ to $L^q(g dx)$ for any q with $p < q < \infty$.
[*Hint:* Part (a): Use Hölder's inequality and Theorem 7.2.7. Part (b): Try the pair $(M(g)^{1-p}, |g|^{1-p})$ for a suitable g. Part (c): Use Exercises 7.1.6 and 7.1.7.]

7.3 The A_∞ Condition

In this section we examine more closely the class of all A_p weights. It turns out that A_p weights possess properties that are p-independent but delicate enough to characterize them without reference to a specific value of p. The A_p classes increase as p increases, and it is only natural to consider their limit as $p \to \infty$. Not surprisingly, a condition obtained as a limit of the A_p conditions as $p \to \infty$ provides some unexpected but insightful characterizations of the class of all A_p weights.

7.3.1 The Class of A_∞ Weights

Let us start by recalling a simple consequence of Jensen's inequality:

$$\left(\int_X |h(t)|^q \, d\mu(t)\right)^{\frac{1}{q}} \geq \exp\left(\int_X \log|h(t)| \, d\mu(t)\right), \tag{7.3.1}$$

which holds for all measurable functions h on a probability space (X,μ) and all $0 < q < \infty$. See Exercise 1.1.3(b). Moreover, part (c) of the same exercise says that the limit of the expressions on the left in (7.3.1) as $q \to 0$ is equal to the expression on the right in (7.3.1).

We apply (7.3.1) to the function $h = w^{-1}$ for some weight w in A_p with $q = 1/(p-1)$. We obtain

$$\frac{w(Q)}{|Q|}\left(\frac{1}{|Q|}\int_Q w(t)^{-\frac{1}{p-1}} \, dt\right)^{p-1} \geq \frac{w(Q)}{|Q|} \exp\left(\frac{1}{|Q|}\int_Q \log w(t)^{-1} \, dt\right), \tag{7.3.2}$$

and the limit of the expressions on the left in (7.3.2) as $p \to \infty$ is equal to the expression on the right in (7.3.2). This observation provides the motivation for the following definition.

Definition 7.3.1. A weight w is called an A_∞ weight if

$$[w]_{A_\infty} = \sup_{Q \text{ cubes in } \mathbf{R}^n} \left\{\left(\frac{1}{|Q|}\int_Q w(t)\, dt\right) \exp\left(\frac{1}{|Q|}\int_Q \log w(t)^{-1} \, dt\right)\right\} < \infty.$$

The quantity $[w]_{A_\infty}$ is called the A_∞ *characteristic constant* of w.

It follows from the previous definition and (7.3.2) that for all $1 \leq p < \infty$ we have

$$[w]_{A_\infty} \leq [w]_{A_p}.$$

This means that

$$\bigcup_{1 \leq p < \infty} A_p \subseteq A_\infty, \tag{7.3.3}$$

but the remarkable thing is that equality actually holds in (7.3.3), a deep property that requires some work.

Before we examine this and other characterizations of A_∞ weights, we discuss some of their elementary properties.

Proposition 7.3.2. *Let $w \in A_\infty$. Then*

(1) $[\delta^\lambda(w)]_{A_\infty} = [w]_{A_\infty}$, *where* $\delta^\lambda(w)(x) = w(\lambda x_1, \dots, \lambda x_n)$ *and* $\lambda > 0$.

(2) $[\tau^z(w)]_{A_\infty} = [w]_{A_\infty}$, *where* $\tau^z(w)(x) = w(x-z)$, $z \in \mathbf{R}^n$.

(3) $[\lambda w]_{A_\infty} = [w]_{A_\infty}$ *for all* $\lambda > 0$.

(4) $[w]_{A_\infty} \geq 1$.

(5) The following is an equivalent characterization of the A_∞ characteristic constant of w:

$$[w]_{A_\infty} = \sup_{\substack{Q \text{ cubes} \\ \text{in } \mathbf{R}^n}} \sup_{\substack{\log|f| \in L^1(Q) \\ \int_Q |f| w\, dt > 0}} \left\{ \frac{w(Q)}{\int_Q |f(t)| w(t)\, dt} \exp\left(\frac{1}{|Q|} \int_Q \log|f(t)|\, dt \right) \right\}.$$

(6) The measure $w(x)\,dx$ is doubling; precisely, for all $\lambda > 1$ and all cubes Q we have

$$w(\lambda Q) \leq 2^{\lambda^n} [w]_{A_\infty}^{\lambda^n} w(Q).$$

As usual, λQ here denotes the cube with the same center as Q and side length λ times that of Q.

We note that estimate (6) is not as good as $\lambda \to \infty$ but it can be substantially improved using the case $\lambda = 2$. We refer to Exercise 7.3.1 for an improvement.

Proof. Properties (1)–(3) are elementary, while property (4) is a consequence of Exercise 1.1.3(b). To show (5), first observe that by taking $f = w^{-1}$, the expression on the right in (5) is at least as big as $[w]_{A_\infty}$. Conversely, (7.3.1) gives

$$\exp\left(\frac{1}{|Q|} \int_Q \log\big(|f(t)| w(t)\big)\, dt \right) \leq \frac{1}{|Q|} \int_Q |f(t)| w(t)\, dt,$$

which, after a simple algebraic manipulation, can be written as

$$\frac{w(Q)}{\int_Q |f| w\, dt} \exp\left(\frac{1}{|Q|} \int_Q \log|f|\, dt \right) \leq \frac{w(Q)}{|Q|} \exp\left(-\frac{1}{|Q|} \int_Q \log|w|\, dt \right),$$

whenever f does not vanish almost everywhere on Q. Taking the supremum over all such f and all cubes Q in $\mathbf{R}^n$, we obtain that the expression on the right in (5) is at most $[w]_{A_\infty}$.

To prove the doubling property for A_∞ weights, we fix $\lambda > 1$ and we apply property (5) to the cube λQ in place of Q and to the function

$$f = \begin{cases} c & \text{on } Q, \\ 1 & \text{on } \mathbf{R}^n \setminus Q, \end{cases} \tag{7.3.4}$$

where c is chosen so that $c^{1/\lambda^n} = 2[w]_{A_\infty}$. We obtain

$$\frac{w(\lambda Q)}{w(\lambda Q \setminus Q) + c\, w(Q)} \exp\left(\frac{\log c}{\lambda^n}\right) \le [w]_{A_\infty},$$

which implies (6) if we take into account the chosen value of c. □

7.3.2 Characterizations of A_∞ Weights

Having established some elementary properties of A_∞ weights, we now turn to some of their deeper properties, one of which is that every A_∞ weight lies in some A_p for $p < \infty$. It also turns out that A_∞ weights are characterized by the reverse Hölder property, which as we saw is a fundamental property of A_p weights. The following is the main theorem of this section.

Theorem 7.3.3. *Suppose that w is a weight. Then w is in A_∞ if and only if any one of the following conditions holds:*
(a) There exist $0 < \gamma, \delta < 1$ such that for all cubes Q in $\mathbf{R}^n$ we have

$$\left|\left\{x \in Q : \; w(x) \le \gamma \operatorname{Avg}_Q w\right\}\right| \le \delta\, |Q|.$$

(b) There exist $0 < \alpha, \beta < 1$ such that for all cubes Q and all measurable subsets A of Q we have

$$|A| \le \alpha\, |Q| \implies w(A) \le \beta\, w(Q).$$

(c) The reverse Hölder condition holds for w, that is, there exist $0 < C_1, \varepsilon < \infty$ such that for all cubes Q we have

$$\left(\frac{1}{|Q|}\int_Q w(t)^{1+\varepsilon}\, dt\right)^{\frac{1}{1+\varepsilon}} \le \frac{C_1}{|Q|}\int_Q w(t)\, dt.$$

(d) There exist $0 < C_2, \varepsilon_0 < \infty$ such that for all cubes Q and all measurable subsets A of Q we have

$$\frac{w(A)}{w(Q)} \le C_2 \left(\frac{|A|}{|Q|}\right)^{\varepsilon_0}.$$

(e) There exist $0 < \alpha', \beta' < 1$ such that for all cubes Q and all measurable subsets A of Q we have

$$w(A) < \alpha'\, w(Q) \implies |A| < \beta'\, |Q|.$$

(f) There exist $p, C_3 < \infty$ *such that* $[w]_{A_p} \le C_3$. *In other words,* w *lies in* A_p *for some* $p \in [1,\infty)$.

All the constants $C_1, C_2, C_3, \alpha, \beta, \gamma, \delta, \alpha', \beta', \varepsilon, \varepsilon_0$, *and* p *in (a)–(f) depend only on the dimension* n *and on* $[w]_{A_\infty}$. *Moreover, if any of the statements in (a)–(f) is valid, then so is any other statement in (a)–(f) with constants that depend only on the dimension n and the constants that appear in the assumed statement.*

Proof. The proof follows from the sequence of implications

$$w \in A_\infty \implies (a) \implies (b) \implies (c) \implies (d) \implies (e) \implies (f) \implies w \in A_\infty .$$

At each step we keep track of the way the constants depend on the constants of the previous step. This is needed to validate the last assertion of the theorem.

$w \in A_\infty \implies (a)$

Fix a cube Q. Since multiplication of an A_∞ weight with a positive scalar does not alter its A_∞ characteristic, we may assume that $\int_Q \log w(t)\,dt = 0$. This implies that $\operatorname{Avg}_Q w \le [w]_{A_\infty}$. Then we have

$$\begin{aligned}
\Big|\{x \in Q:\ w(x) \le \gamma \operatorname*{Avg}_Q w\}\Big| &\le \Big|\{x \in Q:\ w(x) \le \gamma [w]_{A_\infty}\}\Big| \\
&= \Big|\{x \in Q:\ \log(1+w(x)^{-1}) \ge \log(1+(\gamma[w]_{A_\infty})^{-1})\}\Big| \\
&\le \frac{1}{\log(1+(\gamma[w]_{A_\infty})^{-1})} \int_Q \log \frac{1+w(t)}{w(t)}\,dt \\
&= \frac{1}{\log(1+(\gamma[w]_{A_\infty})^{-1})} \int_Q \log(1+w(t))\,dt \\
&\le \frac{1}{\log(1+(\gamma[w]_{A_\infty})^{-1})} \int_Q w(t)\,dt \\
&\le \frac{[w]_{A_\infty}|Q|}{\log(1+(\gamma[w]_{A_\infty})^{-1})} \\
&= \frac{1}{2}|Q|,
\end{aligned}$$

which proves (a) with $\gamma = [w]_{A_\infty}^{-1}(e^{2[w]_{A_\infty}} - 1)^{-1}$ and $\delta = \frac{1}{2}$.

$(a) \implies (b)$

Let Q be fixed and let A be a subset of Q with $w(A) > \beta w(Q)$ for some β to be chosen later. Setting $S = Q \setminus A$, we have $w(S) < (1-\beta)w(Q)$. We write $S = S_1 \cup S_2$, where

$$S_1 = \{x \in S:\ w(x) > \gamma \operatorname{Avg}_Q w\} \quad \text{and} \quad S_2 = \{x \in S:\ w(x) \le \gamma \operatorname{Avg}_Q w\}.$$

For S_2 we have $|S_2| \le \delta |Q|$ by assumption (a). For S_1 we use Chebyshev's inequality to obtain

$$|S_1| \le \frac{1}{\gamma \operatorname*{Avg}_Q w} \int_S w(t)\,dt = \frac{|Q|}{\gamma} \frac{w(S)}{w(Q)} \le \frac{1-\beta}{\gamma}\,|Q|.$$

Adding the estimates for $|S_1|$ and $|S_2|$, we obtain

$$|S| \le |S_1| + |S_2| \le \frac{1-\beta}{\gamma}|Q| + \delta|Q| = \Big(\delta + \frac{1-\beta}{\gamma}\Big)|Q|.$$

Choosing numbers α, β in $(0,1)$ such that $\delta + \frac{1-\beta}{\gamma} = 1-\alpha$, for example $\alpha = \frac{1-\delta}{2}$ and $\beta = 1 - \frac{(1-\delta)\gamma}{2}$, we obtain $|S| \le (1-\alpha)|Q|$, that is, $|A| > \alpha|Q|$.
$(b) \implies (c)$

This was proved in Corollary 7.2.4. To keep track of the constants, we note that the choices

$$\varepsilon = \frac{-\frac{1}{2}\log\beta}{\log 2^n - \log\alpha} \quad \text{and} \quad C_1 = 1 + \frac{(2^n\alpha^{-1})^\varepsilon}{1-(2^n\alpha^{-1})^\varepsilon\beta}$$

as given in (7.2.6) and (7.2.7) serve our purposes.
$(c) \implies (d)$

We apply first Hölder's inequality with exponents $1+\varepsilon$ and $(1+\varepsilon)/\varepsilon$ and then the reverse Hölder estimate to obtain

$$\begin{aligned}\int_A w(x)\,dx &\le \Big(\int_A w(x)^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}}|A|^{\frac{\varepsilon}{1+\varepsilon}}\\ &\le \Big(\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}}|Q|^{\frac{1}{1+\varepsilon}}\,|A|^{\frac{\varepsilon}{1+\varepsilon}}\\ &\le \frac{C_1}{|Q|}\int_Q w(x)\,dx\,|Q|^{\frac{1}{1+\varepsilon}}\,|A|^{\frac{\varepsilon}{1+\varepsilon}},\end{aligned}$$

which gives

$$\frac{w(A)}{w(Q)} \le C_1\Big(\frac{|A|}{|Q|}\Big)^{\frac{\varepsilon}{1+\varepsilon}}.$$

This proves (d) with $\varepsilon_0 = \frac{\varepsilon}{1+\varepsilon}$ and $C_2 = C_1$.
$(d) \implies (e)$

Pick an $0 < \alpha'' < 1$ small enough that $\beta'' = C_2(\alpha'')^{\varepsilon_0} < 1$. It follows from (d) that

$$|A| < \alpha''|Q| \implies w(A) < \beta'' w(Q) \tag{7.3.5}$$

for all cubes Q and all A measurable subsets of Q. Replacing A by $Q \setminus A$, the implication in (7.3.5) can be equivalently written as

$$|A| \ge (1-\alpha'')|Q| \implies w(A) \ge (1-\beta'')w(Q).$$

In other words, for measurable subsets A of Q we have

$$w(A) < (1-\beta'')w(Q) \implies |A| < (1-\alpha'')|Q|, \tag{7.3.6}$$

which is the statement in (e) if we set $\alpha' = (1-\beta'')$ and $\beta' = 1-\alpha''$. Note that (7.3.5) and (7.3.6) are indeed equivalent.

$(e) \implies (f)$

We begin by examining condition (e), which can be written as

$$\int_A w(t)\,dt \le \alpha' \int_Q w(t)\,dt \implies \int_A w(t)^{-1} w(t)\,dt \le \beta' \int_Q w(t)^{-1} w(t)\,dt\,,$$

or, equivalently, as

$$\mu(A) \le \alpha' \mu(Q) \implies \int_A w(t)^{-1}\,d\mu(t) \le \beta' \int_Q w(t)^{-1}\,d\mu(t)$$

after defining the measure $d\mu(t) = w(t)\,dt$. As we have already seen, the assertions in (7.3.5) and (7.3.6) are equivalent. Therefore, we may use Exercise 7.3.2 to deduce that the measure μ is doubling, i.e., it satisfies property (7.2.9) for some constant $C_n = C_n(\alpha', \beta')$, and hence the hypotheses of Corollary 7.2.4 are satisfied. We conclude that the weight w^{-1} satisfies a reverse Hölder estimate with respect to the measure μ, that is, if γ, C are defined as in (7.2.11) and (7.2.12) [in which α is replaced by α', β by β', and C_n is the doubling constant of $w(x)\,dx$], then we have

$$\left(\frac{1}{\mu(Q)} \int_Q w(t)^{-1-\gamma}\,d\mu(t)\right)^{\frac{1}{1+\gamma}} \le \frac{C}{\mu(Q)} \int_Q w(t)^{-1}\,d\mu(t) \tag{7.3.7}$$

for all cubes Q in $\mathbf{R}^n$. Setting $p = 1 + \frac{1}{\gamma}$ and raising to the pth power, we can rewrite (7.3.7) as the A_p condition for w. We can therefore take $C_3 = C^p$ to conclude the proof of (f).

$(f) \implies w \in A_\infty$

This is trivial, since $[w]_{A_\infty} \le [w]_{A_p}$. □

An immediate consequence of the preceding theorem is the following result relating A_∞ to A_p.

Corollary 7.3.4. *The following equality is valid:*

$$A_\infty = \bigcup_{1 \le p < \infty} A_p.$$

Exercises

7.3.1. Let $\lambda > 0$, Q be a cube in $\mathbf{R}^n$, and $w \in A_\infty(\mathbf{R}^n)$.

(a) Show that property (6) in Proposition 7.3.2 can be improved to

$$w(\lambda Q) \le \min_{\varepsilon > 0} \frac{(1+\varepsilon)^{\lambda^n} [w]_{A_\infty}^{\lambda^n} - 1}{\varepsilon}\, w(Q)\,.$$

(b) Prove that

$$w(\lambda Q) \le (2\lambda)^{2^n(1+\log_2[w]_{A_\infty})} w(Q).$$

$\big[$*Hint:* Part (a): Take c in (7.3.4) such that $c^{1/\lambda^n} = (1+\varepsilon)[w]_{A_\infty}$. Part (b): Use the estimate in property (6) of Proposition 7.3.2 with $\lambda = 2$.$\big]$

7.3.2. Suppose that μ is a positive Borel measure on $\mathbf{R}^n$ with the property that for all cubes Q and all measurable subsets A of Q we have

$$|A| < \alpha|Q| \implies \mu(A) < \beta\mu(Q)$$

for some fixed $0 < \alpha, \beta < 1$. Show that μ is doubling [i.e., it satisfies (7.2.9)].
$\big[$*Hint:* Use that $|S| > (1-\alpha)|Q| \Rightarrow \mu(S) > (1-\beta)\mu(Q)$ when $S \subsetneqq Q$.$\big]$

7.3.3. Prove that a weight w is in A_p if and only if both w and $w^{-\frac{1}{p-1}}$ are in A_∞.
$\big[$*Hint:* You may want to use the result of Exercise 7.2.2.$\big]$

7.3.4. ([33], [343]) Prove that if $P(x)$ is a polynomial of degree k in $\mathbf{R}^n$, then $\log|P(x)|$ is in BMO with norm depending only on k and n and not on the coefficients of the polynomial.
$\big[$*Hint:* Use that all norms on the finite-dimensional space of polynomials of degree at most k are equivalent to show that $|P(x)|$ satisfies a reverse Hölder inequality. Therefore, $|P(x)|$ is an A_∞ weight and thus Exercise 7.2.3 (c) is applicable.$\big]$

7.3.5. Show that the product of two A_1 weights may not be an A_∞ weight.

7.3.6. Let g be in $L^p(w)$ for some $1 \le p \le \infty$ and $w \in A_p$. Prove that $g \in L^1_{\text{loc}}(\mathbf{R}^n)$.
$\big[$*Hint:* Let B be a ball. In the case $p < \infty$, write $\int_B |g|\,dx = \int_B (|g|w^{-\frac{1}{p}})w^{\frac{1}{p}}\,dx$ and apply Hölder's inequality. In the case $p = \infty$, use that $w \in A_{p_0}$ for some $p_0 < \infty$.$\big]$

7.3.7. ([278]) Show that a weight w lies in A_∞ if and only if there exist $\gamma, C > 0$ such that for all cubes Q we have

$$w\big(\{x \in Q:\ w(x) > \lambda\}\big) \le C\lambda\big|\{x \in Q:\ w(x) > \gamma\lambda\}\big|$$

for all $\lambda > \operatorname{Avg}_Q w$.
$\big[$*Hint:* The displayed condition easily implies that

$$\frac{1}{|Q|}\int_Q w_k^{1+\varepsilon}\,dx \le \left(\frac{w(Q)}{|Q|}\right)^{\varepsilon+1} + \frac{C'\delta}{\gamma^{1+\varepsilon}}\frac{1}{|Q|}\int_Q w_k^{1+\varepsilon}\,dx,$$

where $k > 0$, $w_k = \min(w,k)$ and $\delta = \varepsilon/(1+\varepsilon)$. Take $\varepsilon > 0$ small enough to obtain the reverse Hölder condition (c) in Theorem 7.3.3 for w_k. Let $k \to \infty$ to obtain the same conclusion for w. Conversely, find constants $\gamma, \delta \in (0,1)$ as in condition (a) of Theorem 7.3.3 and for $\lambda > \operatorname{Avg}_Q w$ write the set $\{w > \lambda\} \cap Q$ as a union of maximal dyadic cubes Q_j such that $\lambda < \operatorname{Avg}_{Q_j} w \le 2^n\lambda$ for all j. Then $w(Q_j) \le 2^n\lambda|Q_j| \le \frac{2^n\lambda}{1-\delta}|Q_j \cap \{w > \gamma\lambda\}|$ and the required conclusion follows by summing on j.$\big]$

7.4 Weighted Norm Inequalities for Singular Integrals

We now address a topic of great interest in the theory of singular integrals, their boundedness properties on weighted L^p spaces. It turns out that a certain amount of regularity must be imposed on the kernels of these operators to obtain the aforementioned weighted estimates.

7.4.1 Singular Integrals of Non Convolution type

We introduce some definitions.

Definition 7.4.1. Let $0 < \delta, A < \infty$. A function $K(x,y)$ defined for $x,y \in \mathbf{R}^n$ with $x \neq y$ is called a *standard kernel* (with constants δ and A) if

$$|K(x,y)| \le \frac{A}{|x-y|^n}, \qquad x \neq y, \tag{7.4.1}$$

and whenever $|x-x'| \le \frac{1}{2}\max\big(|x-y|,|x'-y|\big)$ we have

$$|K(x,y)-K(x',y)| \le \frac{A|x-x'|^\delta}{(|x-y|+|x'-y|)^{n+\delta}} \tag{7.4.2}$$

and also when $|y-y'| \le \frac{1}{2}\max\big(|x-y|,|x-y'|\big)$ we have

$$|K(x,y)-K(x,y')| \le \frac{A|y-y'|^\delta}{(|x-y|+|x-y'|)^{n+\delta}}\,. \tag{7.4.3}$$

The class of all kernels that satisfy (7.4.1), (7.4.2), and (7.4.3) is denoted by $SK(\delta,A)$.

Definition 7.4.2. Let $0 < \delta, A < \infty$ and K in $SK(\delta,A)$. A *Calderón–Zygmund operator* associated with K is a linear operator T defined on $\mathscr{S}(\mathbf{R}^n)$ that admits a bounded extension on $L^2(\mathbf{R}^n)$,

$$\big\|T(f)\big\|_{L^2} \le B\big\|f\big\|_{L^2}, \tag{7.4.4}$$

and that satisfies

$$T(f)(x) = \int_{\mathbf{R}^n} K(x,y)f(y)\,dy \tag{7.4.5}$$

for all $f \in \mathscr{C}_0^\infty$ and x not in the support of f. The class of all Calderón–Zygmund operators associated with kernels in $SK(\delta,A)$ that are bounded on L^2 with norm at most B is denoted by $CZO(\delta,A,B)$. Note that there is no unique T associated with a given K. Given a Calderón–Zygmund operator T in $CZO(\delta,A,B)$, we define the truncated operator $T^{(\varepsilon)}$ as

$$T^{(\varepsilon)}(f)(x) = \int_{|x-y|>\varepsilon} K(x,y)\,f(y)\,dy$$

and the *maximal operator* associated with T as follows:

$$T^{(*)}(f)(x) = \sup_{\varepsilon>0} \left|T^{(\varepsilon)}(f)(x)\right| .$$

We note that if T is in $CZO(\delta,A,B)$, then $T^{(\varepsilon)}(f)$ and $T^{(*)}(f)$ are well defined for all f in $\bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$. It is also well defined whenever f is locally integrable and satisfies $\int_{|x-y|\ge\varepsilon} |f(y)|\,|x-y|^{-n}dy < \infty$ for all $x \in \mathbf{R}^n$ and $\varepsilon > 0$.

The class of kernels in $SK(\delta,A)$ extends the family of convolution kernels that satisfy conditions (5.3.10), (5.3.11), and (5.3.12). Obviously, the associated operators in $CZO(\delta,A,B)$ generalize the associated convolution operators.

A fundamental property of operators in $CZO(\delta,A,B)$ is that they have bounded extensions on all the $L^p(\mathbf{R}^n)$ spaces and also from $L^1(\mathbf{R}^n)$ to weak $L^1(\mathbf{R}^n)$. This is proved via an adaptation of Theorem 5.3.3; see Theorem 4.2.2 in [131]. There are analogous results for the maximal counterparts $T^{(*)}$ of elements of $CZO(\delta,A,B)$. In fact, an analogue of Theorem 5.3.5 yields that $T^{(*)}$ is L^p bounded for $1 < p < \infty$ and weak type $(1,1)$; this result is contained in Theorem 4.2.4 in [131].

We discuss weighted inequalities for singular integrals for general operators in $CZO(\delta,A,B)$. In Subsections 7.4.2 and 7.4.3, the reader may wish to replace kernels in $SK(\delta,A)$ by the more familiar functions $K(x)$ defined on $\mathbf{R}^n \setminus \{0\}$ that satisfy (5.3.10), (5.3.11), and (5.3.12).

7.4.2 A Good Lambda Estimate for Singular Integrals

The following theorem is the main result of this section.

Theorem 7.4.3. *Let* $1 \le p \le \infty$, $w \in A_p$, *and* T *in* $CZO(\delta,A,B)$. *Then there exist positive constants*[1] $C_0 = C_0(n,p,[w]_{A_p})$, $\varepsilon_0 = \varepsilon_0(n,p,[w]_{A_p})$, *and* $c_0(n,\delta)$, *such that if* $\gamma_0 = c_0(n,\delta)/A$, *then for all* $0 < \gamma < \gamma_0$ *we have*

$$w\big(\{T^{(*)}(f) > 3\lambda\} \cap \{M(f) \le \gamma\lambda\}\big) \le C_0\gamma^{\varepsilon_0}(A+B)^{\varepsilon_0} w\big(\{T^{(*)}(f) > \lambda\}\big), \quad (7.4.6)$$

for all locally integrable functions f *for which*

$$\int_{|x-y|\ge\varepsilon} |f(y)|\,|x-y|^{-n}dy < \infty$$

for all $x \in \mathbf{R}^n$ *and* $\varepsilon > 0$. *Here* M *denotes the Hardy–Littlewood maximal operator.*

Proof. We write the open set

$$\Omega = \{T^{(*)}(f) > \lambda\} = \bigcup_j Q_j\,,$$

[1] the dependence on p is relevant only when $p < \infty$

where Q_j are the Whitney cubes (see Appendix J). We set

$$Q_j^* = 10\sqrt{n}\,Q_j\,,$$
$$Q_j^{**} = 10\sqrt{n}\,Q_j^*\,,$$

where aQ denotes the cube with the same center as Q whose side length is $a\ell(Q)$, where $\ell(Q)$ is the side length of Q. We note that in view of the properties of the Whitney cubes, the distance from Q_j to Ω^c is at most $4\sqrt{n}\,\ell(Q_j)$. But the distance from Q_j to the boundary of Q_j^* is $(5\sqrt{n}-\frac{1}{2})\,\ell(Q_j)$, which is bigger than $4\sqrt{n}\,\ell(Q_j)$. Therefore, Q_j^* must meet Ω^c and for every cube Q_j we fix a point y_j in $\Omega^c \cap Q_j^*$. See Figure 7.1.

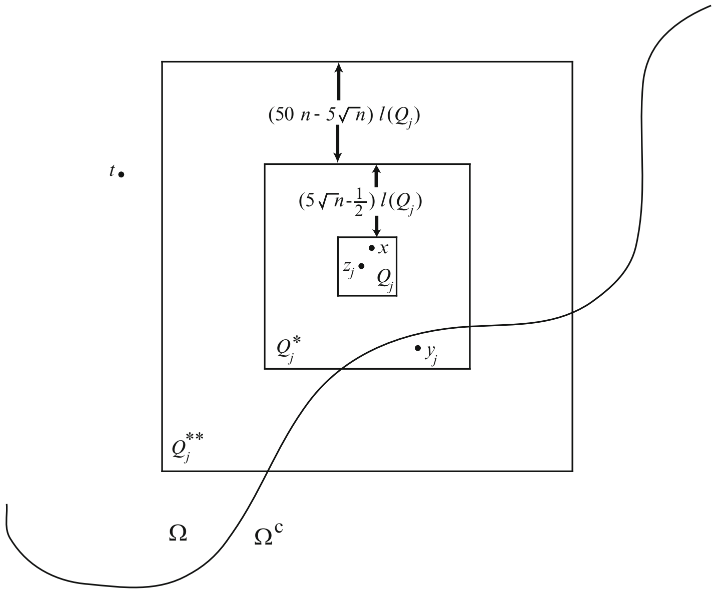

Fig. 7.1 A picture of the proof.

We also fix f in $\bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$, and for each j we write $f = f_0^j + f_\infty^j$, where $f_0^j = f\chi_{Q_j^{**}}$ is the part of f near Q_j and $f_\infty^j = f\chi_{(Q_j^{**})^c}$ is the part of f away from Q_j.

We now claim that the following estimate is true:

$$\big|Q_j \cap \{T^{(*)}(f) > 3\lambda\} \cap \{M(f) \le \gamma\lambda\}\big| \le C_n\,\gamma\,(A+B)\,\big|Q_j\big|\,. \tag{7.4.7}$$

Once the validity of (7.4.7) is established, we apply Theorem 7.3.3 (d) when $p=\infty$ or Proposition 7.2.8 when $p<\infty$ to obtain constants $\varepsilon_0, C_2 > 0$, which depend on $[w]_{A_p}$, p, n when $p<\infty$ and on $[w]_{A_\infty}$ and n when $p=\infty$, such that

$$w\big(Q_j \cap \{T^{(*)}(f) > 3\lambda\} \cap \{M(f) \le \gamma\lambda\}\big) \le C_2\,(C_n)^{\varepsilon_0}\,\gamma^{\varepsilon_0}\,(A+B)^{\varepsilon_0}\,w(Q_j)\,.$$

Then a simple summation in j gives (7.4.6) with $C_0 = C_2(C_n)^{\varepsilon_0}$, and recall that C_2 and ε_0 depend on n and $[w]_{A_p}$ and on p if $p < \infty$.

In proving estimate (7.4.7), we may assume that for each cube Q_j there exists a $z_j \in Q_j$ such that $M(f)(z_j) \le \gamma\lambda$; otherwise, the set on the left in (7.4.7) is empty.

We now invoke Theorem 4.2.4 in [131], which states that $T^{(*)}$ maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ with norm at most $C(n)(A+B)$. We have the estimate

$$\big|Q_j \cap \{T^{(*)}(f) > 3\lambda\} \cap \{M(f) \le \gamma\lambda\}\big| \le I_0^\lambda + I_\infty^\lambda\,, \tag{7.4.8}$$

where

$$\begin{aligned} I_0^\lambda &= \big|Q_j \cap \{T^{(*)}(f_0^j) > \lambda\} \cap \{M(f) \le \gamma\lambda\}\big|\,,\\ I_\infty^\lambda &= \big|Q_j \cap \{T^{(*)}(f_\infty^j) > 2\lambda\} \cap \{M(f) \le \gamma\lambda\}\big|\,. \end{aligned}$$

To control I_0^λ we note that f_0^j is in $L^1(\mathbf{R}^n)$ and we argue as follows:

$$\begin{aligned} I_0^\lambda &\le \big|\{T^{(*)}(f_0^j) > \lambda\}\big|\\ &\le \frac{\|T^{(*)}\|_{L^1\to L^{1,\infty}}}{\lambda}\int_{\mathbf{R}^n} |f_0^j(x)|\,dx\\ &\le C(n)\,(A+B)\,\frac{|Q_j^{**}|}{\lambda}\,\frac{1}{|Q_j^{**}|}\int_{Q_j^{**}} |f(x)|\,dx\\ &\le C(n)\,(A+B)\,\frac{|Q_j^{**}|}{\lambda} M_c(f)(z_j)\\ &\le \widetilde{C}(n)\,(A+B)\,\frac{|Q_j^{**}|}{\lambda} M(f)(z_j)\\ &\le \widetilde{C}(n)\,(A+B)\,\frac{|Q_j^{**}|}{\lambda}\,\lambda\,\gamma\\ &= C_n\,(A+B)\,\gamma|Q_j|\,. \end{aligned} \tag{7.4.9}$$

Next we claim that $I_\infty^\lambda = 0$ if we take γ sufficiently small. We first show that for all $x \in Q_j$ we have

$$\sup_{\varepsilon>0}\big|T^{(\varepsilon)}(f_\infty^j)(x) - T^{(\varepsilon)}(f_\infty^j)(y_j)\big| \le C_{n,\delta}^{(1)} A\, M(f)(z_j)\,. \tag{7.4.10}$$

Indeed, let us fix an $\varepsilon > 0$. We have

$$\begin{aligned} \big|T^{(\varepsilon)}(f_\infty^j)(x) - T^{(\varepsilon)}(f_\infty^j)(y_j)\big| &= \Bigg|\int\limits_{|t-x|>\varepsilon} K(x,t)f_\infty^j(t)\,dt - \int\limits_{|t-y_j|>\varepsilon} K(y_j,t)f_\infty^j(t)\,dt\Bigg|\\ &\le L_1 + L_2 + L_3\,, \end{aligned}$$

where

$$L_1 = \Bigg| \int_{|t-y_j|>\varepsilon} \big[K(x,t) - K(y_j,t)\big] f_\infty^j(t)\,dt \Bigg|,$$

$$L_2 = \Bigg| \int_{\substack{|t-x|>\varepsilon \\ |t-y_j|\le\varepsilon}} K(x,t) f_\infty^j(t)\,dt \Bigg|,$$

$$L_3 = \Bigg| \int_{\substack{|t-x|\le\varepsilon \\ |t-y_j|>\varepsilon}} K(x,t) f_\infty^j(t)\,dt \Bigg|,$$

in view of identity (5.4.7).

We now make a couple of observations. For $t \notin Q_j^{**}$, $x, z_j \in Q_j$, and $y_j \in Q_j^*$ we have

$$\frac{3}{4} \le \frac{|t-x|}{|t-y_j|} \le \frac{5}{4}, \qquad \frac{48}{49} \le \frac{|t-x|}{|t-z_j|} \le \frac{50}{49}. \tag{7.4.11}$$

Indeed,

$$|t-y_j| \ge (50n - 5\sqrt{n})\,\ell(Q_j) \ge 44\,n\,\ell(Q_j)$$

and

$$|x-y_j| \le \frac{1}{2}\sqrt{n}\,\ell(Q_j) + \sqrt{n}\,10\sqrt{n}\,\ell(Q_j) \le 11\,n\,\ell(Q_j) \le \frac{1}{4}\,|t-y_j|.$$

Using this estimate and the inequalities

$$\frac{3}{4}\,|t-y_j| \le |t-y_j| - |x-y_j| \le |t-x| \le |t-y_j| + |x-y_j| \le \frac{5}{4}\,|t-y_j|,$$

we obtain the first estimate in (7.4.11). Likewise, we have

$$|x-z_j| \le \sqrt{n}\,\ell(Q_j) \le n\,\ell(Q_j)$$

and

$$|t-z_j| \ge (50n - \tfrac{1}{2})\ell(Q_j) \ge 49\,n\,\ell(Q_j),$$

and these give

$$\frac{48}{49}\,|t-z_j| \le |t-z_j| - |x-z_j| \le |t-x| \le |t-z_j| + |x-z_j| \le \frac{50}{49}\,|t-z_j|,$$

yielding the second estimate in (7.4.11).

Since $|x-y_j| \le \frac{1}{2}|t-y_j| \le \frac{1}{2}\max\big(|t-x|, |t-y_j|\big)$, we have

$$|K(x,t) - K(y_j,t)| \le \frac{A|x-y_j|^\delta}{(|t-x| + |t-y_j|)^{n+\delta}} \le C'_{n,\delta} A\,\frac{\ell(Q_j)^\delta}{|t-z_j|^{n+\delta}};$$

hence, we obtain

$$L_1 \le \int_{|t-z_j|\ge 49n\ell(Q_j)} C'_{n,\delta} A \frac{\ell(Q_j)^\delta}{|t-z_j|^{n+\delta}} |f(t)|\,dt \le C''_{n,\delta} A M(f)(z_j)$$

using Theorem 2.1.10. Using (7.4.11) we deduce

$$L_2 \le \int_{|t-z_j|\le \frac{5}{4}\cdot\frac{49}{48}\varepsilon} \frac{A}{|x-t|^n} \chi_{|t-x|\ge\varepsilon} |f_\infty^j(t)|\,dt \le C'_n A M(f)(z_j).$$

Again using (7.4.11), we obtain

$$L_3 \le \int_{|t-z_j|\le \frac{49}{48}\varepsilon} \frac{A}{|x-t|^n} \chi_{|t-x|\ge\frac{3}{4}\varepsilon} |f_\infty^j(t)|\,dt \le C''_n A M(f)(z_j).$$

This proves (7.4.10) with constant $C^{(1)}_{n,\delta} = C''_{n,\delta} + C'_n + C''_n$.

Having established (7.4.10), we next claim that

$$\sup_{\varepsilon>0} \left|T^{(\varepsilon)}(f_\infty^j)(y_j)\right| \le T^{(*)}(f)(y_j) + C_n^{(2)} A M(f)(z_j). \tag{7.4.12}$$

To prove (7.4.12) we fix a cube Q_j and $\varepsilon > 0$. We let R_j be the smallest number such that

$$Q_j^{**} \subseteq B(y_j, R_j).$$

See Figure 7.2. We consider the following two cases.

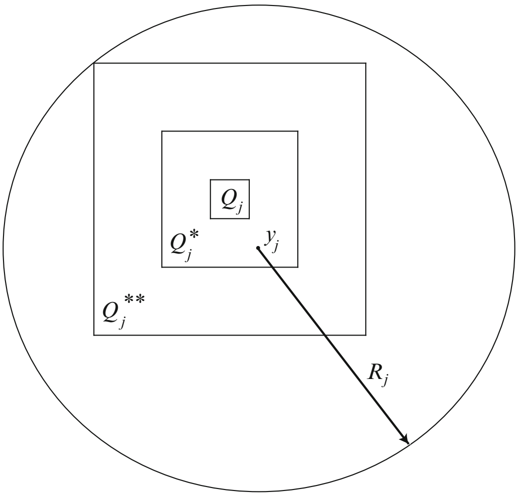

Fig. 7.2 The ball $B(y_j, R_j)$.

Case (1): $\varepsilon \geq R_j$. Since $Q_j^{**} \subseteq B(y_j,\varepsilon)$, we have $B(y_j,\varepsilon)^c \subseteq (Q_j^{**})^c$ and therefore

$$T^{(\varepsilon)}(f_\infty^j)(y_j) = T^{(\varepsilon)}(f)(y_j),$$

so (7.4.12) holds easily in this case.
Case (2): $0 < \varepsilon < R_j$. Note that if $t \in (Q_j^{**})^c$, then $|t-y_j| \geq 40n\,\ell(Q_j)$. On the other hand, $R_j \leq \operatorname{diam}(Q_j^{**}) = 100n^{\frac{3}{2}}\,\ell(Q_j)$. This implies that

$$R_j \leq \tfrac{5\sqrt{n}}{2}|t-y_j|, \qquad \text{when} \quad t \in (Q_j^{**})^c.$$

Notice also that in this case we have $B(y_j,R_j)^c \subseteq (Q_j^{**})^c$, hence

$$T^{(R_j)}(f_\infty^j)(y_j) = T^{(R_j)}(f)(y_j).$$

Therefore, we have

$$\begin{aligned}
\big|T^{(\varepsilon)}(f_\infty^j)(y_j)\big| &\leq \big|T^{(\varepsilon)}(f_\infty^j)(y_j) - T^{(R_j)}(f_\infty^j)(y_j)\big| + \big|T^{(R_j)}(f)(y_j)\big| \\
&\leq \int\limits_{\varepsilon \leq |y_j-t| \leq R_j} |K(y_j,t)|\,|f_\infty^j(t)|\,dt + T^{(*)}(f)(y_j) \\
&\leq \int\limits_{\frac{2}{5\sqrt{n}}R_j \leq |y_j-t| \leq R_j} |K(y_j,t)||f_\infty^j(t)|\,dt + T^{(*)}(f)(y_j) \\
&\leq \frac{A(\frac{2}{5\sqrt{n}})^{-n}}{R_j^n} \int\limits_{|z_j-t| \leq \frac{5}{4}\cdot\frac{49}{48}R_j} |f(t)|\,dt + T^{(*)}(f)(y_j) \\
&\leq C_n^{(2)} A M(f)(z_j) + T^{(*)}(f)(y_j),
\end{aligned}$$

where in the penultimate estimate we used (7.4.11). The proof of (7.4.12) follows with the required bound $C_n^{(2)}A$.

Combining (7.4.10) and (7.4.12), we obtain

$$T^{(*)}(f_\infty^j)(x) \leq T^{(*)}(f)(y_j) + \big(C_{n,\delta}^{(1)} + C_n^{(2)}\big) A M(f)(z_j).$$

Recalling that $y_j \notin \Omega$ and that $M(f)(z_j) \leq \gamma\lambda$, we deduce

$$T^{(*)}(f_\infty^j)(x) \leq \lambda + \big(C_{n,\delta}^{(1)} + C_n^{(2)}\big) A\,\gamma\lambda.$$

Setting $\gamma_0 = \big(C_{n,\delta}^{(1)} + C_n^{(2)}\big)^{-1} A^{-1} = c_0(n,\delta)A^{-1}$, for $0 < \gamma < \gamma_0$, we have that the set

$$Q_j \cap \{T^{(*)}(f_\infty^j) > 2\lambda\} \cap \{M(f) \leq \gamma\lambda\}$$

is empty. This shows that the quantity I_∞^γ vanishes if γ is smaller than γ_0. Returning to (7.4.8) and using the estimate (7.4.9) proved earlier, we conclude the proof of (7.4.7), which, as indicated earlier, implies the theorem. □

Remark 7.4.4. We observe that for any $\delta > 0$, estimate (7.4.6) also holds for the operator

$$T_\delta^{(*)}(f)(x) = \sup_{\varepsilon \geq \delta} |T^{(\varepsilon)}(f)(x)| \tag{7.4.13}$$

with the same constant (which is independent of δ).

To see the validity of (7.4.6) for $T_\delta^{(*)}$, it suffices to prove

$$\left|T_\delta^{(*)}(f_\infty^j)(y_j)\right| \leq T_\delta^{(*)}(f)(y_j) + C_n^{(2)} A\, M(f)(z_j)\,, \tag{7.4.14}$$

which is a version of (7.4.12) with $T^{(*)}$ replaced by $T_\delta^{(*)}$. The following cases arise:
Case (1′): $R_j \leq \delta \leq \varepsilon$ or $\delta \leq R_j \leq \varepsilon$. Here, as in Case (1) we have

$$|T^{(\varepsilon)}(f_\infty^j)(y_j)| = |T^{(\varepsilon)}(f)(y_j)| \leq T_\delta^{(*)}(f)(y_j)\,.$$

Case (2′): $\delta \leq \varepsilon < R_j$. As in Case (2) we have

$$T^{(R_j)}(f_\infty^j)(y_j) = T^{(R_j)}(f)(y_j),$$

thus

$$\left|T^{(\varepsilon)}(f_\infty^j)(y_j)\right| \;\leq\; \left|T^{(\varepsilon)}(f_\infty^j)(y_j) - T^{(R_j)}(f_\infty^j)(y_j)\right| + \left|T^{(R_j)}(f)(y_j)\right|.$$

As in the proof of Case (2), we bound the first term on the right of the last displayed expression by $C_n^{(2)} A\, M(f)(z_j)$ while the second term is at most $T_\delta^{(*)}(f)(y_j)$.

7.4.3 Consequences of the Good Lambda Estimate

Having obtained the important good lambda weighted estimate for singular integrals, we now pass to some of its consequences. We begin with the following lemma:

Lemma 7.4.5. *Let $1 \leq p < \infty$, $\varepsilon > 0$, $w \in A_p$, $x \in \mathbf{R}^n$, and $f \in L^p(w)$. Then we have*

$$\int_{|x-y|\geq\varepsilon} \frac{|f(y)|}{|x-y|^n}\, dy \leq C_{00}(w,n,p,x,\varepsilon)\, \big\|f\big\|_{L^p(w)}$$

for some constant C_{00} depending on the stated parameters. In particular, $T^{(\varepsilon)}(f)$ and $T^{(*)}(f)$ are defined for $f \in L^p(w)$.

Proof. For each $\varepsilon > 0$ and x pick a cube $Q_0 = Q_0(x,\varepsilon)$ of side length $c_n\varepsilon$ (for some constant c_n) such that $Q_0 \subsetneqq B(x,\varepsilon)$. Set $Q_j = 2^j Q_0$ for $j \geq 0$. We have

$$
\begin{aligned}
\int_{|y-x|\geq\varepsilon}\frac{|f(y)|}{|x-y|^n}\,dy &\leq C_n\sum_{j=0}^{\infty}(2^j\varepsilon)^{-n}\int_{Q_{j+1}\setminus Q_j}|f(y)|\,dy\\
&\leq C_n\sum_{j=1}^{\infty}\left(\frac{1}{|Q_j|}\int_{Q_j}|f(y)|^p w\,dy\right)^{\frac{1}{p}}\left(\frac{1}{|Q_j|}\int_{Q_j}w^{-\frac{p'}{p}}\,dy\right)^{\frac{1}{p'}}\\
&\leq C_n\,[w]_{A_p}^{\frac{1}{p}}\sum_{j=1}^{\infty}\left(\int_{Q_j}|f(y)|^p w\,dy\right)^{\frac{1}{p}}\left(\frac{1}{w(Q_j)}\right)^{\frac{1}{p}}\\
&\leq C_n\,[w]_{A_p}^{\frac{1}{p}}\,\big\|f\big\|_{L^p(w)}\sum_{j=1}^{\infty}\big(w(Q_j)\big)^{-\frac{1}{p}}\,.
\end{aligned}
$$

But Proposition 7.2.8 gives for some $\delta=\delta(n,p,[w]_{A_p})$ that

$$\frac{w(Q_0)}{w(Q_j)}\leq C(n,p,[w]_{A_p})\frac{|Q_0|^\delta}{|Q_j|^\delta}\,,$$

from which it follows that

$$w(Q_j)^{-\frac{1}{p}}\leq C'(n,p,[w]_{A_p})\,2^{-j\frac{n\delta}{p}}\,w(Q_0)^{-\frac{1}{p}}\,.$$

In view of this estimate, the previous series converges. Note that C' and thus C_{00} depend on $[w]_{A_p},n,p,x,\varepsilon$, and $w(Q_0)$.

This argument is also valid in the case $p=1$ by an obvious modification. □

Theorem 7.4.6. *Let $A,B,\beta>0$ and let T be a $CZO(\beta,A,B)$. Then given $1<p<\infty$, there is a constant $C_p=C_p(n,\beta,[w]_{A_p})$ such that*

$$\big\|T^{(*)}(f)\big\|_{L^p(w)}\leq C_p\,(A+B)\big\|f\big\|_{L^p(w)} \tag{7.4.15}$$

for all $w\in A_p$ and $f\in L^p(w)$. There is also a constant $C_1=C_1(n,\beta,[w]_{A_1})$ such that

$$\big\|T^{(*)}(f)\big\|_{L^{1,\infty}(w)}\leq C_1\,(A+B)\,\big\|f\big\|_{L^1(w)} \tag{7.4.16}$$

for all $w\in A_1$ and $f\in L^1(w)$.

Proof. This theorem is a consequence of the estimate proved in the previous theorem. For technical reasons, it is useful to fix a $\delta>0$ and work with the auxiliary maximal operator $T_\delta^{(*)}$ defined in (7.4.13) instead of $T^{(*)}$. We begin by taking $1<p<\infty$ and $f\in L^p(w)$ for some $w\in A_p$. We write

$$
\begin{aligned}
\big\|T_\delta^{(*)}(f)\big\|_{L^p(w)}^p &= \int_0^\infty p\lambda^{p-1}w\big(\{T_\delta^{(*)}(f)>\lambda\}\big)\,d\lambda\\
&= 3^p\int_0^\infty p\lambda^{p-1}w\big(\{T_\delta^{(*)}(f)>3\lambda\}\big)\,d\lambda\,,
\end{aligned}
$$

which we control by

$$3^p \int_0^\infty p\lambda^{p-1} w\big(\{T_\delta^{(*)}(f) > 3\lambda\} \cap \{M(f) \le \gamma\lambda\}\big)\, d\lambda$$
$$+ \; 3^p \int_0^\infty p\lambda^{p-1} w\big(\{M(f) > \gamma\lambda\}\big)\, d\lambda\,.$$

Using Theorem 7.4.3 (or rather Remark 7.4.4), there are $C_0 = C_0(n, [w]_{A_p})$, $\varepsilon_0 = \varepsilon_0(n, [w]_{A_p})$, and $\gamma_0 = c_0(n,\beta)A^{-1}$, such that the preceding displayed expression is bounded by

$$3^p C_0 \gamma^{\varepsilon_0} (A+B)^{\varepsilon_0} \int_0^\infty p\lambda^{p-1} w\big(\{T_\delta^{(*)}(f) > \lambda\}\big)\, d\lambda$$
$$+ \; \frac{3^p}{\gamma^p} \int_0^\infty p\lambda^{p-1} w\big(\{M(f) > \lambda\}\big)\, d\lambda\,,$$

which is equal to

$$3^p C_0 \gamma^{\varepsilon_0} (A+B)^{\varepsilon_0} \big\|T_\delta^{(*)}(f)\big\|_{L^p(w)}^p + \frac{3^p}{\gamma^p} \big\|M(f)\big\|_{L^p(w)}^p\,.$$

Taking $\gamma = \min\big(\frac{1}{2}c_0(n,\beta)A^{-1}, \frac{1}{2}(2C_0 3^p)^{-\frac{1}{\varepsilon_0}}(A+B)^{-1}\big) < \gamma_0$, we conclude that

$$\begin{aligned} &\big\|T_\delta^{(*)}(f)\big\|_{L^p(w)}^p \\ &\qquad \le \frac{1}{2}\big\|T_\delta^{(*)}(f)\big\|_{L^p(w)}^p + \widetilde{C}_p(n,\beta,[w]_{A_p})(A+B)^p \big\|M(f)\big\|_{L^p(w)}^p\,. \end{aligned} \tag{7.4.17}$$

We now prove a similar estimate when $p = 1$. For $f \in L^1(w)$ and $w \in A_1$ we have

$$\begin{aligned} &3\lambda w\big(\{T_\delta^{(*)}(f) > 3\lambda\}\big) \\ &\qquad \le 3\lambda w\big(\{T_\delta^{(*)}(f) > 3\lambda\} \cap \{M(f) \le \gamma\lambda\}\big) + 3\lambda w\big(\{M(f) > \gamma\lambda\}\big)\,, \end{aligned}$$

and this expression is controlled by

$$3\lambda C_0 \gamma^{\varepsilon_0} (A+B)^{\varepsilon_0} w\big(\{T_\delta^{(*)}(f) > \lambda\}\big) + \frac{3}{\gamma}\big\|M(f)\big\|_{L^{1,\infty}(w)}\,.$$

Recalling that $\gamma_0 = c_0(n,\beta)A^{-1}$ and choosing $\gamma = \min\big(\frac{1}{2}\gamma_0, \frac{1}{2}(6C_0)^{-\frac{1}{\varepsilon_0}}(A+B)^{-1}\big)$, it follows that

$$\begin{aligned} &\big\|T_\delta^{(*)}(f)\big\|_{L^{1,\infty}(w)} \\ &\qquad \le \frac{1}{2}\big\|T_\delta^{(*)}(f)\big\|_{L^{1,\infty}(w)} + \widetilde{C}_1(n,\beta,[w]_{A_1})(A+B)\big\|M(f)\big\|_{L^{1,\infty}(w)}. \end{aligned} \tag{7.4.18}$$

Estimate (7.4.15) would follow from (7.4.17) if we knew that $\|T_\delta^{(*)}(f)\|_{L^p(w)} < \infty$ whenever $1 < p < \infty$, $w \in A_p$ and $f \in L^p(w)$, while (7.4.16) would follow from (7.4.18) if we had $\|T_\delta^{(*)}(f)\|_{L^{1,\infty}(w)} < \infty$ whenever $w \in A_1$ and $f \in L^1(w)$. Since we do not know that these quantities are finite, a certain amount of work is needed.

To deal with this problem we momentarily restrict attention to a special class of functions on $\mathbf{R}^n$, the class of bounded functions with compact support. Such functions are dense in $L^p(w)$ when $w \in A_p$ and $1 \le p < \infty$; see Exercise 7.4.1. Let h be a bounded function with compact support on $\mathbf{R}^n$. Then $T_\delta^{(*)}(h) \le C_1 \delta^{-n} \|h\|_{L^1}$ and $T_\delta^{(*)}(h)(x) \le C_2(h)|x|^{-n}$ for x away from the support of h. It follows that

$$T_\delta^{(*)}(h)(x) \le C_3(h,\delta)(1+|x|)^{-n}$$

for all $x \in \mathbf{R}^n$. Furthermore, if h is nonzero, then

$$M(h)(x) \ge \frac{C_4(h)}{(1+|x|)^n},$$

and therefore for $w \in A_1$,

$$\big\|T_\delta^{(*)}(h)\big\|_{L^{1,\infty}(w\,dx)} \le C_5(h,\delta)\big\|M(h)\big\|_{L^{1,\infty}(w\,dx)} < \infty,$$

while for $1 < p < \infty$ and $w \in A_p$,

$$\int_{\mathbf{R}^n} (T_\delta^{(*)}(h)(x))^p w(x)\,dx \le C_5(h,p,\delta) \int_{\mathbf{R}^n} M(h)(x)^p w(x)\,dx < \infty$$

in view of Theorem 7.1.9. Using these facts, (7.4.17), (7.4.18), and Theorem 7.1.9 once more, we conclude that for all $\delta > 0$ and $1 < p < \infty$ we have

$$\begin{aligned} \big\|T_\delta^{(*)}(h)\big\|_{L^p(w)}^p &\le 2\widetilde{C}_p\big\|M(h)\big\|_{L^p(w)}^p \le \widetilde{C}'_p[w]_{A_p}^{\frac{p}{p-1}}\big\|h\big\|_{L^p(w)}^p = C_p^p\big\|h\big\|_{L^p(w)}^p, \\ \big\|T_\delta^{(*)}(h)\big\|_{L^{1,\infty}(w)} &\le 2\widetilde{C}_1\big\|M(h)\big\|_{L^{1,\infty}(w)} \le \widetilde{C}_1[w]_{A_1}\big\|h\big\|_{L^1(w)} = C_1\big\|h\big\|_{L^1(w)}, \end{aligned} \tag{7.4.19}$$

whenever h a bounded function with compact support. The constants $\widetilde{C}_p$, $\widetilde{C}'_p$, and C_p depend only on the parameters n, β, p, and $[w]_{A_p}$.

We now extend estimates (7.4.16) and (7.4.15) to functions in $L^p(\mathbf{R}^n, w\,dx)$. Given $1 \le p < \infty$, $w \in A_p$, and $f \in L^p(w)$, let

$$f_N(x) = f(x)\chi_{|f|\le N}\chi_{|x|\le N}.$$

Then f_N is a bounded function with compact support that converges to f in $L^p(w)$ (i.e., $\|f_N - f\|_{L^p(w)} \to 0$ as $N \to \infty$) by the Lebesgue dominated convergence theorem. Also $|f_N| \le |f|$ for all N. Sublinearity and Lemma 7.4.5 give for all $x \in \mathbf{R}^n$,

$$\begin{aligned} |T_\delta^{(*)}(f_N)(x) - T_\delta^{(*)}(f)(x)| &\le T_\delta^{(*)}(f - f_N)(x) \\ &\le A\,C_{00}(w,n,p,x,\delta)\,\big\|f_N - f\big\|_{L^p(w)}, \end{aligned}$$

and this converges to zero as $N \to \infty$ since $C_{00}(w,n,p,x,\delta) < \infty$. Therefore

$$T_\delta^{(*)}(f) = \lim_{N\to\infty} T_\delta^{(*)}(f_N)$$

pointwise, and Fatou's lemma for weak type spaces [see Exercise 1.1.12 (d)] gives for $w \in A_1$ and $f \in L^1(w)$,

$$\begin{aligned} \big\|T_\delta^{(*)}(f)\big\|_{L^{1,\infty}(w)} &= \big\|\liminf_{N\to\infty} T_\delta^{(*)}(f_N)\big\|_{L^{1,\infty}(w)} \\ &\le \liminf_{N\to\infty} \big\|T_\delta^{(*)}(f_N)\big\|_{L^{1,\infty}(w)} \\ &\le C_1 \liminf_{N\to\infty} \big\|M(f_N)\big\|_{L^{1,\infty}(w)} \\ &\le C_1 \big\|M(f)\big\|_{L^{1,\infty}(w)}, \end{aligned}$$

since $|f_N| \le |f|$ for all N. An analogous argument gives the estimate

$$\big\|T_\delta^{(*)}(f)\big\|_{L^p(w)} \le C_p \big\|f\big\|_{L^p(w)}$$

for $w \in A_p$ and $f \in L^p(w)$ when $1 < p < \infty$.

It remains to prove (7.4.15) and (7.4.16) for $T^{(*)}$. But this is also an easy consequence of Fatou's lemma, since the constants C_p and C_1 are independent of δ and

$$\lim_{\delta\to 0} T_\delta^{(*)}(f) = T^{(*)}(f)$$

for all $f \in L^p(w)$. □

We end this subsection by making the comment that if a given T in $CZO(\delta,A,B)$ is pointwise controlled by $T^{(*)}$, then the estimates of Theorem 7.4.6 also hold for it. This is the case for the Hilbert transform, the Riesz transforms, and other classical singular integral operators.

7.4.4 Necessity of the A_p Condition

We have established the main theorems relating Calderón–Zygmund operators and A_p weights, namely that such operators are bounded on $L^p(w)$ whenever w lies in A_p. It is natural to ask whether the A_p condition is necessary for the boundedness of singular integrals on L^p. We end this section by indicating the necessity of the A_p condition for the boundedness of the Riesz transforms on weighted L^p spaces.

Theorem 7.4.7. *Let w be a weight in $\mathbf{R}^n$ and let $1 \le p < \infty$. Suppose that each of the Riesz transforms R_j is of weak type (p,p) with respect to w. Then w must be an A_p weight. Similarly, let w be a weight in $\mathbf{R}$. If the Hilbert transform H is of weak type (p,p) with respect to w, then w must be an A_p weight.*

Proof. We prove the n-dimensional case, $n \geq 2$. The one-dimensional case is essentially contained in following argument, suitably adjusted.

Let Q be a cube and let f be a nonnegative function on $\mathbf{R}^n$ supported in Q that satisfies $\operatorname{Avg}_Q f > 0$. Let Q' be the cube that shares a corner with Q, has the same length as Q, and satisfies $x_j \geq y_j$ for all $1 \leq j \leq n$ whenever $x \in Q'$ and $y \in Q$. Then for $x \in Q'$ we have

$$\left|\sum_{j=1}^{n} R_j(f)(x)\right| = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{j=1}^{n} \int_Q \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,dy \geq \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_Q \frac{f(y)}{|x-y|^n}\,dy.$$

But if $x \in Q'$ and $y \in Q$ we must have that $|x-y| \leq 2\sqrt{n}\,\ell(Q)$, which implies that $|x-y|^{-n} \geq (2\sqrt{n})^{-n}|Q|^{-1}$. Let $C_n = \Gamma(\frac{n+1}{2})(2\sqrt{n})^{-n}\pi^{-\frac{n+1}{2}}$. It follows that for all $0 < \alpha < C_n \operatorname{Avg}_Q f$ we have

$$Q' \subseteq \left\{x \in \mathbf{R}^n : \Big|\sum_{j=1}^{n} R_j(f)(x)\Big| > \alpha\right\}.$$

Since the operator $\sum_{j=1}^n R_j$ is of weak type (p,p) with respect to w (with constant C), we must have

$$w(Q') \leq \frac{C^p}{\alpha^p} \int_Q f(x)^p w(x)\,dx$$

for all $\alpha < C_n \operatorname{Avg}_Q f$, which implies that

$$\big(\operatorname*{Avg}_Q f\big)^p \leq \frac{C_n^{-p} C^p}{w(Q')} \int_Q f(x)^p w(x)\,dx. \tag{7.4.20}$$

We observe that we can reverse the roles of Q and Q' and obtain

$$\big(\operatorname*{Avg}_{Q'} g\big)^p \leq \frac{C_n^{-p} C^p}{w(Q)} \int_{Q'} g(x)^p w(x)\,dx \tag{7.4.21}$$

for all g supported in Q'. In particular, taking $g = \chi_{Q'}$ in (7.4.21) gives that

$$w(Q) \leq C_n^{-p} C^p w(Q').$$

Using this estimate and (7.4.20), we obtain

$$\big(\operatorname*{Avg}_Q f\big)^p \leq \frac{(C_n^{-p} C^p)^2}{w(Q)} \int_Q f(x)^p w(x)\,dx. \tag{7.4.22}$$

Using the characterization of the A_p characteristic constant in Proposition 7.1.5 (8), it follows that

$$[w]_{A_p} \leq (C_n^{-p} C^p)^2 < \infty;$$

hence $w \in A_p$. $\square$

Exercises

7.4.1. Let $1 \le p < \infty$ and let $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ satisfy $w > 0$ a.e. Show that $\mathscr{C}_0^\infty(\mathbf{R}^n)$ is dense in $L^p(w)$. In particular this assertion holds for any $w \in A_\infty$.

7.4.2. ([74]) Let T be in $CZO(\delta,A,B)$. Show that for all $\varepsilon > 0$ and all $1 < p < \infty$ there exists a constant $C_{n,p,\varepsilon,\delta}$ such that for all $f \in L^p(\mathbf{R}^n)$ and for all measurable nonnegative functions u with $u^{1+\varepsilon} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $M(u^{1+\varepsilon}) < \infty$ a.e. we have

$$\int_{\mathbf{R}^n} |T^{(*)}(f)|^p \, u \, dx \le C_{n,p,\varepsilon,\delta}(A+B)^p \int_{\mathbf{R}^n} |f|^p M(u^{1+\varepsilon})^{\frac{1}{1+\varepsilon}} \, dx .$$

[*Hint:* Obtain this result as a consequence of Theorems 7.4.6 and 7.2.7.]

7.4.3. Use the idea of the proof of Theorem 7.4.6 to prove the following result. Suppose that for some fixed $A,B > 0$ the nonnegative μ-measurable functions F and G on a σ-finite measure space (X,μ) satisfy the distributional inequality

$$\mu\big(\{G > \alpha\} \cap \{F \le c\alpha\}\big) \le A\,\mu\big(\{G > B\alpha\}\big)$$

for all $\alpha > 0$. Given $0 < p < \infty$, if $A < B^p$ and $\|G\|_{L^p(\mu)} < \infty$, show that

$$\|G\|_{L^p(\mu)} \le \frac{B}{(B^p - A)^{1/p}} \frac{1}{c} \|F\|_{L^p(\mu)} .$$

7.4.4. Let $\alpha > 0$, $w \in A_1$, and $f \in L^1(\mathbf{R}^n,w) \cap L^1(\mathbf{R}^n)$. Let $f = g + b$ be the Calderón–Zygmund decomposition of f at height $\alpha > 0$ given in Theorem 5.3.1, such that $b = \sum_j b_j$, where each b_j is supported in a dyadic cube Q_j, $\int_{Q_j} b_j(x)\,dx = 0$, and Q_j and Q_k have disjoint interiors when $j \ne k$. Prove that

(a) $\|g\|_{L^1(w)} \le [w]_{A_1} \|f\|_{L^1(w)}$ and $\|g\|_{L^\infty(w)} = \|g\|_{L^\infty} \le 2^n \alpha$,

(b) $\|b_j\|_{L^1(w)} \le (1 + [w]_{A_1}) \|f\|_{L^1(Q_j,w)}$ and $\|b\|_{L^1(w)} \le (1 + [w]_{A_1}) \|f\|_{L^1(w)}$,

(c) $\sum_j w(Q_j) \le \frac{[w]_{A_1}}{\alpha} \|f\|_{L^1(w)}$.

7.4.5. Assume that T is an operator associated with a kernel in $SK(\delta,A)$. Suppose that T maps $L^2(w)$ to $L^2(w)$ for all $w \in A_1$ with bound B_w. Prove that there is a constant $C_{n,\delta}$ such that

$$\|T\|_{L^1(w) \to L^{1,\infty}(w)} \le C_{n,\delta}(A + B_w)\,[w]_{A_1}^2$$

for all $w \in A_1$.
[*Hint:* Apply the idea of the proof of Theorem 5.3.3 using the Calderón-Zygmund decomposition $f = g + b$ of Exercise 7.4.4 at height $\gamma\alpha$ for a suitable γ. To estimate

$T(g)$ use an $L^2(w)$ estimate and Exercise 7.4.4. To estimate $T(b)$ use the mean value property, the fact that

$$\int_{\mathbf{R}^n\setminus Q_j^*} \frac{|y-c_j|^\delta}{|x-c_j|^{n+\delta}}\, w(x)\, dx \le C_{\delta,n} M(w)(y) \le C'_{\delta,n}[w]_{A_1}\, w(y)\,,$$

and Exercise 7.4.4 to obtain the required estimate.]

7.4.6. Recall that the transpose T^t of a linear operator T is defined by

$$\big\langle T(f),g\big\rangle = \big\langle f,T^t(g)\big\rangle$$

for all suitable f and g. Suppose that T is a linear operator that maps $L^p(\mathbf{R}^n, v dx)$ to itself for some $1 < p < \infty$ and some $v \in A_p$. Show that the transpose operator T^t maps $L^{p'}(\mathbf{R}^n, w dx)$ to itself with the same norm, where $w = v^{1-p'} \in A_{p'}$.

7.4.7. Suppose that T is a linear operator that maps $L^2(\mathbf{R}^n, v dx)$ to itself for all v such that $v^{-1} \in A_1$. Show that the transpose operator T^t of T maps $L^2(\mathbf{R}^n, w dx)$ to itself for all $w \in A_1$.

7.4.8. Let $1 < p < \infty$. Suppose that T is a linear operator that maps $L^p(v)$ to itself for all v satisfying $v^{-1} \in A_p$. Show that the transpose operator T^t of T maps $L^{p'}(w)$ to itself for all w satisfying $w^{-1} \in A_{p'}$.

7.5 Further Properties of A_p Weights

In this section we discuss other properties of A_p weights. Many of these properties indicate deep connections with other branches of analysis. We focus attention on three such properties: factorization, extrapolation, and relations of weighted inequalities to vector-valued inequalities.

7.5.1 Factorization of Weights

Recall the simple fact that if w_1, w_2 are A_1 weights, then $w = w_1 w_2^{1-p}$ is an A_p weight (Exercise 7.1.2). The factorization theorem for weights says that the converse of this statement is true. This provides a surprising and striking representation of A_p weights.

Theorem 7.5.1. *Suppose that w is an A_p weight for some $1 < p < \infty$. Then there exist A_1 weights w_1 and w_2 such that*

$$w = w_1 w_2^{1-p}\,.$$

Proof. Let us fix a $p \geq 2$ and $w \in A_p$. We define an operator T as follows:

$$T(f) = \big(w^{-\frac{1}{p}} M(f^{p-1} w^{\frac{1}{p}})\big)^{\frac{1}{p-1}} + w^{\frac{1}{p}} M(f w^{-\frac{1}{p}}),$$

where M is the Hardy–Littlewood maximal operator. We observe that T is well defined and bounded on $L^p(\mathbf{R}^n)$. This is a consequence of the facts that $w^{-\frac{1}{p-1}}$ is an $A_{p'}$ weight and that M maps $L^{p'}(w^{-\frac{1}{p-1}})$ to itself and also $L^p(w)$ to itself. Thus the norm of T on L^p depends only on the A_p characteristic constant of w. Let $B(w) = \|T\|_{L^p \to L^p}$, the norm of T on L^p. Next, we observe that for $f, g \geq 0$ in $L^p(\mathbf{R}^n)$ and $\lambda \geq 0$ we have

$$T(f+g) \leq T(f) + T(g), \quad T(\lambda f) = \lambda T(f). \tag{7.5.1}$$

To see the first assertion, we need only note that for every ball B, the operator

$$f \to \left(\frac{1}{|B|}\int_B |f|^{p-1} w^{\frac{1}{p}}\, dx\right)^{\frac{1}{p-1}}$$

is sublinear as a consequence of Minkowski's integral inequality, since $p-1 \geq 1$.

We now fix an L^p function f_0 with $\|f_0\|_{L^p} = 1$ and we define a function φ in $L^p(\mathbf{R}^n)$ as the sum of the L^p convergent series

$$\varphi = \sum_{j=1}^{\infty} (2B(w))^{-j} T^j(f_0). \tag{7.5.2}$$

We define

$$w_1 = w^{\frac{1}{p}} \varphi^{p-1}, \qquad w_2 = w^{-\frac{1}{p}} \varphi,$$

so that $w = w_1 w_2^{1-p}$. It remains to show that w_1, w_2 are A_1 weights. Applying T and using (7.5.1), we obtain

$$\begin{aligned} T(\varphi) &\leq 2B(w) \sum_{j=1}^{\infty} (2B(w))^{-j-1} T^{j+1}(f_0) \\ &= 2B(w)\left(\varphi - \frac{T(f_0)}{2B(w)}\right) \\ &\leq 2B(w)\, \varphi, \end{aligned}$$

that is,

$$\big(w^{-\frac{1}{p}} M(\varphi^{p-1} w^{\frac{1}{p}})\big)^{\frac{1}{p-1}} + w^{\frac{1}{p}} M(\varphi w^{-\frac{1}{p}}) \leq 2B(w)\, \varphi.$$

Using that $\varphi = (w^{-\frac{1}{p}} w_1)^{\frac{1}{p-1}} = w^{\frac{1}{p}} w_2$, we obtain

$$M(w_1) \leq (2B(w))^{p-1} w_1 \qquad \text{and} \qquad M(w_2) \leq 2B(w) w_2.$$

These show that w_1 and w_2 are A_1 weights whose characteristic constants depend on $[w]_{A_p}$ (and also the dimension n and p). This concludes the case $p \geq 2$.

We now turn to the case $p < 2$. Given a weight $w \in A_p$ for $1 < p < 2$, we consider the weight $w^{-1/(p-1)}$, which is in $A_{p'}$. Since $p' > 2$, using the result we obtained, we write $w^{-1/(p-1)} = v_1 v_2^{1-p'}$, where v_1, v_2 are A_1 weights. It follows that $w = v_1^{1-p} v_2$, and this completes the asserted factorization of A_p weights. □

Combining the result just obtained with Theorem 7.2.7, we obtain the following description of A_p weights.

Corollary 7.5.2. *Let w be an A_p weight for some $1 < p < \infty$. Then there exist locally integrable functions f_1 and f_2 with*

$$M(f_1) + M(f_2) < \infty \qquad a.e.,$$

constants $0 < \varepsilon_1, \varepsilon_2 < 1$, and a nonnegative function k satisfying $k, k^{-1} \in L^\infty$ such that

$$w = k M(f_1)^{\varepsilon_1} M(f_2)^{\varepsilon_2(1-p)}. \tag{7.5.3}$$

7.5.2 Extrapolation from Weighted Estimates on a Single L^{p_0}

Our next topic concerns a striking application of the class of A_p weights. It says that an estimate on $L^{p_0}(v)$ for a single p_0 and all A_{p_0} weights v implies a similar L^p estimate for all p in $(1,\infty)$. This property is referred to as extrapolation.

Surprisingly the operator T is not needed to be linear or sublinear in the following extrapolation theorem. The only condition required is that T be well defined on $\bigcup_{1\leq q<\infty} \bigcup_{w\in A_q} L^q(w)$. If T happens to be a linear operator, this condition can be relaxed to T being well defined on $\mathscr{C}_0^\infty(\mathbf{R}^n)$.

Theorem 7.5.3. *Suppose that T is defined on $\bigcup_{1\leq q<\infty} \bigcup_{w\in A_q} L^q(w)$ and takes values in the space of measurable complex-valued functions. Let $1 \leq p_0 < \infty$ and suppose that there exists a positive increasing function N on $[1,\infty)$ such that for all weights v in A_{p_0} we have*

$$\big\|T\big\|_{L^{p_0}(v)\to L^{p_0}(v)} \leq N\big([v]_{A_{p_0}}\big). \tag{7.5.4}$$

Then for any $1 < p < \infty$ and for all weights w in A_p we have

$$\big\|T\big\|_{L^p(w)\to L^p(w)} \leq K\big(n, p, p_0, [w]_{A_p}\big), \tag{7.5.5}$$

where

$$K\big(n,p,p_0,[w]_{A_p}\big) = \begin{cases} 2N\Big(\kappa_1(n,p,p_0)\,[w]_{A_p}^{\frac{p_0-1}{p-1}}\Big) & \text{when } p < p_0, \\ 2^{\frac{p-p_0}{p_0(p-1)}} N\big(\kappa_2(n,p,p_0)\;[w]_{A_p}\big) & \text{when } p > p_0, \end{cases}$$

and $\kappa_1(n,p,p_0)$ and $\kappa_2(n,p,p_0)$ are constants that depend on n, p, and p_0.

Proof. Let $1 < p < \infty$ and $w \in A_p$. We define an operator

$$M'(f) = \frac{M(fw)}{w},$$

where M is the Hardy–Littlewood maximal operator. We observe that since $w^{1-p'}$ is in $A_{p'}$, the operator M' maps $L^{p'}(w)$ to itself; indeed, we have

$$\begin{aligned} \big\|M'\big\|_{L^{p'}(w)\to L^{p'}(w)} &= \big\|M\big\|_{L^{p'}(w^{1-p'})\to L^{p'}(w^{1-p'})} \\ &\le C_{n,p}[w^{1-p'}]_{A_{p'}}^{\frac{1}{p'-1}} \\ &= C_{n,p}[w]_{A_p} \end{aligned} \tag{7.5.6}$$

in view of Theorem 7.1.9 and property (4) of Proposition 7.1.5.

We introduce operators $M^0(f) = |f|$ and $M^k = M \circ M \circ \cdots \circ M$, where M is the Hardy–Littlewood maximal function and the composition is taken k times. Likewise, we introduce powers $(M')^k$ of M' for $k \in \mathbf{Z}^+ \cup \{0\}$. The following lemma provides the main tool in the proof of Theorem 7.5.3. Its simple proof uses Theorem 7.1.9 and (7.5.6) and is omitted.

Lemma 7.5.4. *Let $1 < p < \infty$ and $w \in A_p$. Define operators R and R'*

$$R(f) = \sum_{k=0}^{\infty} \frac{M^k(f)}{\big(2\|M\|_{L^p(w)\to L^p(w)}\big)^k}$$

for functions f in $L^p(w)$ and also

$$R'(f) = \sum_{k=0}^{\infty} \frac{(M')^k(f)}{\big(2\big\|M'\big\|_{L^{p'}(w)\to L^{p'}(w)}\big)^k}$$

for functions f in $L^{p'}(w)$. Then there exist constants $C_1(n,p)$ and $C_2(n,p)$ that depend on n and p such that

$$|f| \le R(f), \tag{7.5.7}$$

$$\big\|R(f)\big\|_{L^p(w)} \le 2\big\|f\big\|_{L^p(w)}, \tag{7.5.8}$$

$$M(R(f)) \le C_1(n,p)\,[w]_{A_p}^{\frac{1}{p-1}}\,R(f), \tag{7.5.9}$$

for all functions f in $L^p(w)$ and such that

$$|h| \le R'(h), \tag{7.5.10}$$

$$\big\|R'(h)\big\|_{L^{p'}(w)} \le 2\big\|h\big\|_{L^{p'}(w)}, \tag{7.5.11}$$

$$M'(R'(h)) \le C_2(n,p)\,[w]_{A_p}\,R'(h), \tag{7.5.12}$$

for all functions h in $L^{p'}(w)$.

We now proceed with the proof of the theorem. It is natural to split the proof into the cases $p < p_0$ and $p > p_0$.

Case (1): $p < p_0$. Assume momentarily that $R(f)^{-\frac{p_0}{(p_0/p)'}}$ is an A_{p_0} weight. Then we have

$$\begin{aligned}
&\big\|T(f)\big\|_{L^p(w)}^p \\
&= \int_{\mathbf{R}^n} |T(f)|^p R(f)^{-\frac{p}{(p_0/p)'}} R(f)^{\frac{p}{(p_0/p)'}} w\,dx \\
&\le \bigg(\int_{\mathbf{R}^n} |T(f)|^{p_0} R(f)^{-\frac{p_0}{(p_0/p)'}} w\,dx\bigg)^{\frac{p}{p_0}} \bigg(\int_{\mathbf{R}^n} R(f)^p\, w\,dx\bigg)^{\frac{1}{(p_0/p)'}} \\
&\le N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}}\big]_{A_{p_0}}\Big)^p \bigg(\int_{\mathbf{R}^n} |f|^{p_0} R(f)^{-\frac{p_0}{(p_0/p)'}} w\,dx\bigg)^{\frac{p}{p_0}} \bigg(\int_{\mathbf{R}^n} R(f)^p\, w\,dx\bigg)^{\frac{1}{(p_0/p)'}} \\
&\le N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}}\big]_{A_{p_0}}\Big)^p \bigg(\int_{\mathbf{R}^n} R(f)^{p_0} R(f)^{-\frac{p_0}{(p_0/p)'}} w\,dx\bigg)^{\frac{p}{p_0}} \bigg(\int_{\mathbf{R}^n} R(f)^p\, w\,dx\bigg)^{\frac{1}{(p_0/p)'}} \\
&= N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}}\big]_{A_{p_0}}\Big)^p \bigg(\int_{\mathbf{R}^n} R(f)^p w\,dx\bigg)^{\frac{p}{p_0}} \bigg(\int_{\mathbf{R}^n} R(f)^p\, w\,dx\bigg)^{\frac{1}{(p_0/p)'}} \\
&\le N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}}\big]_{A_{p_0}}\Big)^p \big(2\big\|f\big\|_{L^p(w)}\big)^p,
\end{aligned}$$

where we used Hölder's inequality with exponents p_0/p and $(p_0/p)'$, the hypothesis of the theorem, (7.5.7), and (7.5.8). Thus, we have the estimate

$$\big\|T(f)\big\|_{L^p(w)} \le 2N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}}\big]_{A_{p_0}}\Big)\big\|f\big\|_{L^p(w)} \tag{7.5.13}$$

and it remains to obtain a bound for the A_{p_0} characteristic constant of $R(f)^{-\frac{p_0}{(p_0/p)'}}$. In view of (7.5.9), the function $R(f)$ is an A_1 weight with characteristic constant at most a constant multiple of $[w]_{A_p}^{\frac{1}{p-1}}$. Consequently, there is a constant C_1' such that

$$R(f)^{-1} \le C_1'\,[w]_{A_p}^{\frac{1}{p-1}} \bigg(\frac{1}{|Q|}\int_Q R(f)\,dx\bigg)^{-1}$$

for any cube Q in $\mathbf{R}^n$. Thus we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q R(f)^{-\frac{p_0}{(p_0/p)'}} w\,dx \\
&\qquad \le \big(C_1'\,[w]_{A_p}^{\frac{1}{p-1}}\big)^{\frac{p_0}{(p_0/p)'}} \bigg(\frac{1}{|Q|}\int_Q R(f)\,dx\bigg)^{-\frac{p_0}{(p_0/p)'}} \bigg(\frac{1}{|Q|}\int_Q w\,dx\bigg).
\end{aligned} \tag{7.5.14}$$

Next we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \Big(R(f)^{-\frac{p_0}{(p_0/p)'}}w\Big)^{1-p_0'}dx\right)^{p_0-1} \\
&\qquad = \left(\frac{1}{|Q|}\int_Q R(f)^{\frac{p_0(p_0'-1)}{(p_0/p)'}}w^{1-p_0'}\,dx\right)^{p_0-1} \qquad (7.5.15)\\
&\qquad \le \left(\frac{1}{|Q|}\int_Q R(f)\,dx\right)^{\frac{p_0}{(p_0/p)'}}\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1},
\end{aligned}$$

where we applied Hölder's inequality with exponents

$$\left(\frac{p'-1}{p_0'-1}\right)' \qquad \text{and} \qquad \frac{p'-1}{p_0'-1},$$

and we used that

$$\frac{p_0(p_0'-1)}{(p_0/p)'}\left(\frac{p'-1}{p_0'-1}\right)'=1 \qquad \text{and} \qquad \frac{p_0-1}{\left(\frac{p'-1}{p_0'-1}\right)'}=\frac{p_0}{(p_0/p)'}.$$

Multiplying (7.5.14) by (7.5.15) and taking the supremum over all cubes Q in $\mathbf{R}^n$ we deduce that

$$\Big[R(f)^{-\frac{p_0}{(p_0/p)'}}\Big]_{A_{p_0}} \le \big(C_1'\,[w]_{A_p}^{\frac{1}{p-1}}\big)^{\frac{p_0}{(p_0/p)'}}[w]_{A_p} = \kappa_1(n,p,p_0)\,[w]_{A_p}^{\frac{p_0-1}{p-1}}.$$

Combining this estimate with (7.5.13) and using the fact that N is an increasing function, we obtain the validity of (7.5.5) in the case $p<p_0$.

Case (2): $p>p_0$. In this case we set $r=p/p_0>1$. Then we have

$$\big\|T(f)\big\|_{L^p(w)}^p=\big\|\,|T(f)|^{p_0}\big\|_{L^r(w)}^r=\left(\int_{\mathbf{R}^n}|T(f)|^{p_0}h\,w\,dx\right)^r \qquad (7.5.16)$$

for some nonnegative function h with $L^{r'}(w)$ norm equal to 1. We define a function

$$H=\big[R'\big(h^{\frac{r'}{p'}}\big)\big]^{\frac{p'}{r'}}.$$

Obviously, we have $0\le h\le H$ and thus

$$\begin{aligned}
\int_{\mathbf{R}^n}|T(f)|^{p_0}h\,w\,dx &\le \int_{\mathbf{R}^n}|T(f)|^{p_0}H\,w\,dx \\
&\le N\big([H\,w]_{A_{p_0}}\big)^{p_0}\big\|f\big\|_{L^{p_0}(H\,w)}^{p_0} \\
&\le N\big([H\,w]_{A_{p_0}}\big)^{p_0}\big\|\,|f|^{p_0}\big\|_{L^r(w)}\big\|H\big\|_{L^{r'}(w)} \qquad (7.5.17)\\
&\le 2^{\frac{p'}{r'}}N\big([H\,w]_{A_{p_0}}\big)^{p_0}\big\|f\big\|_{L^p(w)}^{p_0},
\end{aligned}$$

noting that

$$\|H\|_{L^{r'}(w)}^{r'} = \int_{\mathbf{R}^n} R'(h^{r'/p'})^{p'}\, w\, dx \le 2^{p'} \int_{\mathbf{R}^n} h^{r'}\, w\, dx = 2^{p'} ,$$

which is valid in view of (7.5.11). Moreover, this argument is based on the hypothesis of the theorem and requires that $H\,w$ be an A_{p_0} weight. To see this, we observe that condition (7.5.12) implies that $H^{r'/p'}w$ is an A_1 weight with characteristic constant at most a multiple of $[w]_{A_1}$. Thus, there is a constant C_2' that depends only on n and p such that

$$\frac{1}{|Q|}\int_Q H^{\frac{r'}{p'}}\, w\, dx \le C_2'\,[w]_{A_p} H^{\frac{r'}{p'}}\, w$$

for all cubes Q in $\mathbf{R}^n$. From this it follows that

$$(H\,w)^{-1} \le \kappa_2(n,p,p_0)\,[w]_{A_p}^{\frac{p'}{r'}}\left(\frac{1}{|Q|}\int_Q H^{\frac{r'}{p'}}\, w\, dx\right)^{-\frac{p'}{r'}} w^{\frac{p'}{r'}-1},$$

where we set $\kappa_2(n,p,p_0) = (C_2')^{p'/r'}$. We raise the preceding displayed expression to the power $p_0'-1$, we average over the cube Q, and then we raise to the power p_0-1. We deduce the estimate

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q (H\,w)^{1-p_0'}\, dx\right)^{p_0-1}\\ &\qquad \le \kappa_2(n,p,p_0)\,[w]_{A_p}^{\frac{p'}{r'}}\left(\frac{1}{|Q|}\int_Q H^{\frac{r'}{p'}}\, w\, dx\right)^{-\frac{p'}{r'}}\left(\frac{1}{|Q|}\int_Q w^{1-p'}\, dx\right)^{p_0-1},\end{aligned} \tag{7.5.18}$$

where we use the fact that

$$\left(\frac{p'}{r'}-1\right)(p_0'-1) = 1-p' .$$

Note that $r'/p' \ge 1$, since $p_0 \ge 1$. Using Hölder's inequality with exponents r'/p' and $(r'/p')^{-1}$ we obtain that

$$\frac{1}{|Q|}\int_Q H\,w\, dx \le \left(\frac{1}{|Q|}\int_Q H^{\frac{r'}{p'}}\, w\, dx\right)^{\frac{p'}{r'}}\left(\frac{1}{|Q|}\int_Q w\, dx\right)^{\frac{p_0-1}{p-1}}, \tag{7.5.19}$$

where we used that

$$\frac{1}{(\frac{r'}{p'})'} = \frac{p_0-1}{p-1} .$$

Multiplying (7.5.18) by (7.5.19), we deduce the estimate

$$\big[H\,w\big]_{A_{p_0}} \le \kappa_2(n,p,p_0)\,[w]_{A_p}^{\frac{p'}{r'}}\,[w]_{A_p}^{\frac{p_0-1}{p-1}} = \kappa_2(n,p,p_0)\,[w]_{A_p} .$$

Inserting this estimate in (7.5.17) we obtain

$$\int_{\mathbf{R}^n} |T(f)|^{p_0} h\,w\,dx \le 2^{\frac{p'}{r'}} N\big(\kappa_2(n,p,p_0)\,[w]_{A_p}\big)^{p_0} \|f\|_{L^p(w)}^{p_0},$$

and combining this with (7.5.16) we conclude that

$$\big\|T(f)\big\|_{L^p(w)}^p \le 2^{\frac{p'r}{r'}} N\big(\kappa_2(n,p,p_0)\,[w]_{A_p}\big)^{p_0 r} \|f\|_{L^p(w)}^{p_0 r}.$$

This proves the required estimate (7.5.5) in the case $p > p_0$. □

There is a version of Theorem 7.5.3 in which the initial strong type assumption is replaced by a weak type estimate.

Theorem 7.5.5. *Suppose that T is a well defined operator on $\bigcup_{1<q<\infty}\bigcup_{w\in A_q} L^q(w)$ that takes values in the space of measurable complex-valued functions. Fix $1 \le p_0 < \infty$ and suppose that there is an increasing function N on $[1,\infty)$ such that for all weights v in A_{p_0} we have*

$$\big\|T\big\|_{L^{p_0}(v)\to L^{p_0,\infty}(v)} \le N([v]_{A_{p_0}}). \tag{7.5.20}$$

Then for any $1 < p < \infty$ and for all weights w in A_p we have

$$\big\|T\big\|_{L^p(w)\to L^{p,\infty}(w)} \le K\big(n,p,p_0,[w]_{A_p}\big), \tag{7.5.21}$$

where $K\big(n,p,p_0,[w]_{A_p}\big)$ is as in Theorem 7.5.3.

Proof. For every fixed $\lambda > 0$ we define

$$T_\lambda(f) = \lambda\chi_{|T(f)|>\lambda}.$$

The operator T_λ is not linear but is well defined on $\bigcup_{1<q<\infty}\bigcup_{w\in A_q} L^q(w)$, since T is well defined on this union. We show that T_λ maps $L^{p_0}(v)$ to $L^{p_0}(v)$ for every $v \in A_{p_0}$. Indeed, we have

$$\begin{aligned}\left(\int_{\mathbf{R}^n} |T_\lambda(f)|^{p_0}\,v\,dx\right)^{\frac{1}{p_0}} &= \left(\int_{\mathbf{R}^n} \lambda^{p_0}\chi_{|T(f)|>\lambda}\,v\,dx\right)^{\frac{1}{p_0}}\\ &= \Big(\lambda^{p_0} v\big(\{|T(f)|>\lambda\}\big)\Big)^{\frac{1}{p_0}}\\ &\le N([v]_{A_{p_0}})\big\|f\big\|_{L^{p_0}(v)}\end{aligned}$$

using the hypothesis on T. Applying Theorem 7.5.3, we obtain that T_λ maps $L^p(w)$ to itself for all $1 < p < \infty$ and all $w \in A_p$ with a constant independent of λ. Precisely, for any $w \in A_p$ and any $f \in L^p(w)$ we have

$$\big\|T_\lambda(f)\big\|_{L^p(w)} \le K\big(n,p,p_0,[w]_{A_p}\big)\big\|f\big\|_{L^p(w)}.$$

Since

$$\big\|T(f)\big\|_{L^{p,\infty}(w)} = \sup_{\lambda>0}\big\|T_\lambda(f)\big\|_{L^p(w)},$$

it follows that T maps $L^p(w)$ to $L^{p,\infty}(w)$ with the asserted norm. □

Assuming that the operator T in the preceding theorem is sublinear (or quasi-sublinear), we obtain the following result that contains a stronger conclusion.

Corollary 7.5.6. *Suppose that T is a sublinear operator on $\bigcup_{1<q<\infty}\bigcup_{w\in A_q} L^q(w)$ that takes values in the space of measurable complex-valued functions. Fix $1 \le p_0 < \infty$ and suppose that there is an increasing function N on $[1,\infty)$ such that for all weights v in A_{p_0} we have*

$$\big\|T\big\|_{L^{p_0}(v)\to L^{p_0,\infty}(v)} \le N([v]_{A_{p_0}}). \tag{7.5.22}$$

Then for any $1 < p < \infty$ and any weight w in A_p there is a constant $K'(n,p,p_0,[w]_{A_p})$ such that

$$\big\|T(f)\big\|_{L^p(w)} \le K'(n,p,p_0,[w]_{A_p})\big\|f\big\|_{L^p(w)}.$$

Proof. The proof follows from Theorem 7.5.5 and the Marcinkiewicz interpolation theorem. □

We end this subsection by observing that the conclusion of the extrapolation Theorem 7.5.3 can be strengthened to yield vector-valued estimates. This strengthening may be achieved by a simple adaptation of the proof discussed.

Corollary 7.5.7. *Suppose that T is defined on $\bigcup_{1\le q<\infty}\bigcup_{w\in A_q} L^q(w)$ and takes values in the space of all measurable complex-valued functions. Fix $1 \le p_0 < \infty$ and suppose that there is an increasing function N on $[1,\infty)$ such that for all weights v in A_{p_0} we have*

$$\big\|T\big\|_{L^{p_0}(v)\to L^{p_0}(v)} \le N\big([v]_{A_{p_0}}\big).$$

Then for every $1 < p < \infty$ and every weight $w \in A_p$ we have

$$\Big\|\Big(\sum_j |T(f_j)|^{p_0}\Big)^{\frac{1}{p_0}}\Big\|_{L^p(w)} \le K(n,p,p_0,[w]_{A_p})\Big\|\Big(\sum_j |f_j|^{p_0}\Big)^{\frac{1}{p_0}}\Big\|_{L^p(w)}$$

for all sequences of functions f_j in $L^p(w)$, where $K\big(n,p,p_0,[w]_{A_p}\big)$ is as in Theorem 7.5.3.

Proof. To derive the claimed vector-valued inequality follow the proof of Theorem 7.5.3 replacing the function f by $(\sum_j |f_j|^{p_0})^{\frac{1}{p_0}}$ and $T(f)$ by $(\sum_j |T(f_j)|^{p_0})^{\frac{1}{p_0}}$. □

7.5.3 Weighted Inequalities Versus Vector-Valued Inequalities

We now discuss connections between weighted inequalities and vector-valued inequalities. The next result provides strong evidence that there is a nontrivial connection of this sort. The following is a general theorem saying that any vector-valued inequality is equivalent to some weighted inequality. The proof of the theorem is based on a minimax lemma whose precise formulation and proof can be found in Appendix H.

Theorem 7.5.8. *(a) Let* $0 < p < q, r < \infty$*. Let* $\{T_j\}_j$ *be a sequence of sublinear operators that map* $L^q(\mu)$ *to* $L^r(\nu)$*, where* μ *and* ν *are arbitrary measures. Then the vector-valued inequality*

$$\Big\|\big(\sum_j |T_j(f_j)|^p\big)^{\frac{1}{p}}\Big\|_{L^r} \le C\Big\|\big(\sum_j |f_j|^p\big)^{\frac{1}{p}}\Big\|_{L^q} \tag{7.5.23}$$

holds for all $f_j \in L^q(\mu)$ *if and only if for every* $u \ge 0$ *in* $L^{\frac{r}{r-p}}(\nu)$ *there exists* $U \ge 0$ *in* $L^{\frac{q}{q-p}}(\mu)$ *with*

$$\begin{aligned} \|U\|_{L^{\frac{q}{q-p}}} &\le \|u\|_{L^{\frac{r}{r-p}}}, \\ \sup_j \int |T_j(f)|^p\, u\, d\nu &\le C^p \int |f|^p\, U\, d\mu. \end{aligned} \tag{7.5.24}$$

(b) Let $0 < q, r < p < \infty$*. Let* $\{T_j\}_j$ *be as before. Then the vector-valued inequality (7.5.23) holds for all* $f_j \in L^q(\mu)$ *if and only if for every* $u \ge 0$ *in* $L^{\frac{q}{p-q}}(\mu)$ *there exists* $U \ge 0$ *in* $L^{\frac{r}{p-r}}(\nu)$ *with*

$$\begin{aligned} \|U\|_{L^{\frac{r}{p-r}}} &\le \|u\|_{L^{\frac{q}{p-q}}}, \\ \sup_j \int |T_j(f)|^p\, U^{-1}\, d\nu &\le C^p \int |f|^p\, u^{-1}\, d\mu. \end{aligned} \tag{7.5.25}$$

Proof. We begin with part (a). Given $f_j \in L^q(\mathbf{R}^n, \mu)$, we use (7.5.24) to obtain

$$\begin{aligned} \Big\|\Big(\sum_j |T_j(f_j)|^p\Big)^{\frac{1}{p}}\Big\|_{L^r(\nu)} &= \Big\|\sum_j |T_j(f_j)|^p\Big\|_{L^{\frac{r}{p}}(\nu)}^{\frac{1}{p}} \\ &= \sup_{\|u\|_{L^{\frac{r}{r-p}}} \le 1} \left(\int_{\mathbf{R}^n} \sum_j |T_j(f_j)|^p\, u\, d\nu\right)^{\frac{1}{p}} \\ &\le \sup_{\|u\|_{L^{\frac{r}{r-p}}} \le 1} C\left(\int_{\mathbf{R}^n} \sum_j |f_j|^p\, U\, d\mu\right)^{\frac{1}{p}} \\ &\le \sup_{\|u\|_{L^{\frac{r}{r-p}}} \le 1} C\Big\|\sum_j |f_j|^p\Big\|_{L^{\frac{q}{p}}(\mu)}^{\frac{1}{p}} \|U\|_{L^{\frac{q}{q-p}}}^{\frac{1}{p}} \\ &\le C\Big\|\Big(\sum_j |f_j|^p\Big)^{\frac{1}{p}}\Big\|_{L^q(\mu)}, \end{aligned}$$

which proves (7.5.23) with the same constant C as in (7.5.24). To prove the converse, given a nonnegative $u \in L^{\frac{r}{r-p}}(\nu)$ with $\|u\|_{L^{\frac{r}{r-p}}} = 1$, we define

$$A = \Big\{a = (a_0, a_1):\ a_0 = \sum_j |f_j|^p, \quad a_1 = \sum_j |T_j(f_j)|^p, \quad f_j \in L^q(\mu)\Big\}$$

and

$$B = \Big\{b \in L^{\frac{q}{q-p}}(\mu):\ b \geq 0, \quad \|b\|_{L^{\frac{q}{q-p}}} \leq 1 = \|u\|_{L^{\frac{r}{r-p}}}\Big\}.$$

Notice that A and B are convex sets and B is weakly compact. (The sublinearity of each T_j is used here.) We define the function Φ on $A \times B$ by setting

$$\Phi(a,b) = \int a_1 u\,d\nu - C^p \int a_0 b\,d\mu = \sum_j \left(\int |T_j(f_j)|^p u\,d\nu - C^p \int |f_j|^p b\,d\mu \right).$$

Then Φ is concave on A and weakly continuous and convex on B. Thus the *minimax lemma* in Appendix H is applicable. This gives

$$\min_{b\in B} \sup_{a\in A} \Phi(a,b) = \sup_{a\in A} \min_{b\in B} \Phi(a,b). \tag{7.5.26}$$

At this point observe that for a fixed $a = \big(\sum_j |f_j|^p, \sum_j |T_j(f_j)|^p\big)$ in A we have

$$\begin{aligned} \min_{b\in B} \Phi(a,b) &\leq \Big\| \sum_j |T_j(f_j)|^p \Big\|_{L^{\frac{r}{p}}(\nu)} \|u\|_{L^{\frac{r}{r-p}}} - C^p \max_{b\in B} \int \sum_j |f_j|^p\, b\,d\mu \\ &\leq \Big\| \sum_j |T_j(f_j)|^p \Big\|_{L^{\frac{r}{p}}(\nu)} - C^p \Big\| \sum_j |f_j|^p \Big\|_{L^{\frac{q}{p}}(\mu)} \leq 0 \end{aligned}$$

using the hypothesis (7.5.23). It follows that $\sup_{a\in A} \min_{b\in B} \Phi(a,b) \leq 0$ and hence (7.5.26) yields $\min_{b\in B} \sup_{a\in A} \Phi(a,b) \leq 0$. Thus there exists a $U \in B$ such that $\Phi(a,U) \leq 0$ for every $a \in A$. This completes the proof of part (a).

The proof of part (b) is similar. Using the result of Exercise 7.5.1 and (7.5.25), given $f_j \in L^q(\mathbf{R}^n, \mu)$ we have

$$\begin{aligned} \Big\| \Big(\sum_j |f_j|^p\Big)^{\frac{1}{p}} \Big\|_{L^q(\mu)} &= \Big\| \sum_j |f_j|^p \Big\|_{L^{\frac{q}{p}}(\mu)}^{\frac{1}{p}} \\ &= \inf_{\|u\|_{L^{\frac{q}{p-q}}} \leq 1} \left(\int_{\mathbf{R}^n} \sum_j |f_j|^p\, u^{-1}\,d\mu \right)^{\frac{1}{p}} \\ &\geq \frac{1}{C} \inf_{\|U\|_{L^{\frac{r}{p-r}}} \leq 1} \left(\int_{\mathbf{R}^n} \sum_j |T_j(f_j)|^p\, U^{-1}\,d\nu \right)^{\frac{1}{p}} \end{aligned}$$

$$= \frac{1}{C}\Big\|\sum_j |T_j(f_j)|^p\Big\|_{L^{\frac{r}{p}}(\nu)}^{\frac{1}{p}}$$

$$= \frac{1}{C}\Big\|\Big(\sum_j |T_j(f_j)|^p\Big)^{\frac{1}{p}}\Big\|_{L^r(\nu)}.$$

To prove the converse direction in part (b), given a fixed $u \ge 0$ in $L^{\frac{q}{p-q}}(\mu)$ with $\|u\|_{L^{\frac{q}{p-q}}} = 1$, we define A as in part (a) and

$$B = \Big\{ b \in L^{\frac{p}{p-r}}(\nu) : \; b \ge 0, \quad \|b\|_{L^{\frac{p}{p-r}}} \le 1 = \|u\|_{L^{\frac{q}{p-q}}} \Big\}.$$

We also define the function Φ on $A \times B$ by setting

$$\begin{aligned} \Phi(a,b) &= \int a_1 b^{-1} d\nu - C^p \int a_0 u^{-1}\, d\mu \\ &= \sum_j \Big(\int |T_j(f_j)|^p b^{-1}\, d\nu - C^p \int |f_j|^p u^{-1}\, d\mu \Big). \end{aligned}$$

Then Φ is concave on A and weakly continuous and convex on B. Also, using Exercise 7.5.1, for any $a = \big(\sum_j |f_j|^p, \sum_j |T_j(f_j)|^p\big)$ in A, we have

$$\min_{b\in B} \Phi(a,b) \le \Big\|\sum_j |T_j(f_j)|^p\Big\|_{L^{\frac{r}{p}}(\nu)} - C^p \Big\|\sum_j |f_j|^p\Big\|_{L^{\frac{q}{p}}(\mu)} \le 0.$$

Thus $\sup_{a\in A} \min_{b\in B} \Phi(a,b) \le 0$. Using (7.5.26), yields $\min_{b\in B} \sup_{a\in A} \Phi(a,b) \le 0$, and the latter implies the existence of a U in B such that $\Phi(a,U) \le 0$ for all $a \in A$. This proves (7.5.25). □

Example 7.5.9. We use the previous theorem to obtain another proof of the vector-valued Hardy–Littlewood maximal inequality in Corollary 5.6.5. We take $T_j = M$ for all j. For given $1 < p < q < \infty$ and u in $L^{\frac{q}{q-p}}$ we set $s = \frac{q}{q-p}$ and $U = \|M\|_{L^s \to L^s}^{-1} M(u)$. In view of Exercise 7.1.7 we have

$$\|U\|_{L^s} \le \|u\|_{L^s} \qquad \text{and} \qquad \int_{\mathbf{R}^n} M(f)^p\, u\, dx \le C^p \int_{\mathbf{R}^n} |f|^p\, U\, dx.$$

Using Theorem 7.5.8, we obtain

$$\Big\|\Big(\sum_j |M(f_j)|^p\Big)^{\frac{1}{p}}\Big\|_{L^q} \le C_{n,p,q} \Big\|\Big(\sum_j |f_j|^p\Big)^{\frac{1}{p}}\Big\|_{L^q} \tag{7.5.27}$$

whenever $1 < p < q < \infty$, an inequality obtained earlier in (5.6.25).

It turns out that no specific properties of the Hardy–Littlewood maximal function were used in the preceding inequality, and one could obtain a general result along these lines.

Exercises

7.5.1. Let (X,μ) be a measure space, $0<s<1$, and $f\in L^s(X,\mu)$. Show that

$$\big\|f\big\|_{L^s}=\inf\bigg\{\int_X |f|u^{-1}\,d\mu:\ \|u\|_{L^{\frac{s}{1-s}}}\le 1\bigg\}$$

and that the infimum is attained.
$\big[$*Hint:* Try $u=c\,|f|^{1-s}$ for a suitable constant c.$\big]$

7.5.2. (*K. Yabuta*) Let $w\in A_p$ for some $1<p<\infty$ and let f be in $L^p_{\mathrm{loc}}(\mathbf{R}^n,w\,dx)$. Show that f lies in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$.
$\big[$*Hint:* Write $w=w_1/w_2^{p-1}$ via Theorem 7.5.1.$\big]$

7.5.3. Use the same idea of the proof of Theorem 7.5.1 to prove the following general result: Let μ be a positive measure on a measure space X and let T be a bounded sublinear operator on $L^p(X,\mu)$ for some $1\le p<\infty$. Suppose that $T(f)\ge 0$ for all f in $L^p(X,\mu)$. Prove that for all $f_0\in L^p(X,\mu)$, there exists an $f\in L^p(X,\mu)$ such that

(a) $f_0(x)\le f(x)$ for μ-almost all $x\in X$.

(b) $\|f\|_{L^p(X)}\le 2\,\|f_0\|_{L^p(X)}$.

(c) $T(f)(x)\le 2\,\|T\|_{L^p\to L^p}\,f(x)$ for μ-almost all $x\in X$.

$\big[$*Hint:* Try the expression in (7.5.2) starting the sum at $j=0$.$\big]$

7.5.4. ([100]) Suppose that T is an operator defined on $\bigcup_{1<q<\infty}\bigcup_{w\in A_q}L^q(w)$ that satisfies $\|T\|_{L^r(v)\to L^r(v)}\le N([v]_{A_r})$ for some increasing function $N:[1,\infty)\to\mathbf{R}^+$. Without using Theorem 7.5.3 prove that for $1<q<r$ and all $v\in A_1$, T maps $L^q(v)$ to $L^q(v)$ with constant depending on q,r, n, and $[v]_{A_1}$.
$\big[$*Hint:* Hölder's inequality gives that

$$\big\|T(f)\big\|_{L^q(v)}\le\bigg(\int_{\mathbf{R}^n}|T(f)(x)|^rM(f)(x)^{q-r}\,v(x)\,dx\bigg)^{\frac{1}{r}}\bigg(\int_{\mathbf{R}^n}M(f)(x)^q v(x)\,dx\bigg)^{\frac{r-q}{rq}}.$$

Then use the fact that the weight $M(f)^{\frac{r-q}{r-1}}$ is in A_1 and Exercise 7.1.2.$\big]$

7.5.5. Let T be a sublinear operator defined on $\bigcup_{2\le q<\infty}L^q$. Suppose that for all functions f and u we have

$$\int_{\mathbf{R}^n}|T(f)|^2u\,dx\le\int_{\mathbf{R}^n}|f|^2M(u)\,dx.$$

Prove that T maps $L^p(\mathbf{R}^n)$ to itself for all $2<p<\infty$.
$\big[$*Hint:* Use that

$$\big\|T(f)\big\|_{L^p}=\sup_{\|u\|_{L^{(p/2)'}}\le 1}\bigg(\int_{\mathbf{R}^n}|T(f)|^2u\,dx\bigg)^{\frac{1}{2}}$$

and Hölder's inequality.$\big]$

7.5.6. (*X. C. Li*) Let T be a sublinear operator defined on $\bigcup_{1<q\le 2}\bigcup_{w\in A_q} L^q(w)$. Suppose that T maps $L^2(w)$ to $L^2(w)$ for all weights w that satisfy $w^{-1}\in A_1$. Prove that T maps L^p to itself for all $1<p<2$.
$\big[$*Hint:* We have

$$\big\|T(f)\big\|_{L^p}\le\bigg(\int_{\mathbf{R}^n}|T(f)|^2M(f)^{-(2-p)}\,dx\bigg)^{\frac12}\bigg(\int_{\mathbf{R}^n}M(f)^p\,dx\bigg)^{\frac{2-p}{2p}}$$

by Hölder's inequality. Apply the hypothesis to the first term of the product.$\big]$

HISTORICAL NOTES

Weighted inequalities can probably be traced back to the beginning of integration, but the A_p condition first appeared in a paper of Rosenblum [298] in a somewhat different form. The characterization of A_p when $n=1$ in terms of the boundedness of the Hardy–Littlewood maximal operator was obtained by Muckenhoupt [260]. The estimate on the norm in (7.1.25) can also be reversed, as shown by Buckley [38]. The simple proof of Theorem 7.1.9 is contained in Lerner's article [218] and yields both the Muckenhoupt theorem and Buckley's optimal growth of the norm of the Hardy–Littlewood maximal operator in terms of the A_p characteristic constant of the weight. Another proof of this result is given by Christ and Fefferman [61]. Versions of Lemma 7.1.10 for balls were first obtained by Besicovitch [27] and independently by Morse [258]. The particular version of Lemma 7.1.10 that appears in the text is adapted from that in de Guzmán [93]. Another version of this lemma is contained in the book of Mattila [246]. The fact that A_∞ is the union of the A_p spaces was independently obtained by Muckenhoupt [261] and Coifman and Fefferman [66]. The latter paper also contains a proof that A_p weights satisfy the crucial reverse Hölder condition. This condition first appeared in the work of Gehring [125] in the following context: If F is a quasiconformal homeomorphism from $\mathbf{R}^n$ into itself, then $|\det(\nabla F)|$ satisfies a reverse Hölder inequality. The characterization of A_1 weights is due to Coifman and Rochberg [68]. The fact that $M(f)^\delta$ is in A_∞ when $\delta<1$ was previously obtained by Córdoba and Fefferman [74]. The different characterizations of A_∞ (Theorem 7.3.3) are implicit in [260] and [66]. Another characterization of A_∞ in terms of the Gurov-Reshetnyak condition $\sup_Q\frac{1}{|Q|}\int_Q|f-\operatorname{Avg}_Q f|\,dx\le\varepsilon\operatorname{Avg}_Q f$ for $f\ge 0$ and $0<\varepsilon<2$ was obtained by Korenovskyy, Lerner, and Stokolos [201]. The definition of A_∞ using the reverse Jensen inequality herein was obtained as an equivalent characterization of that space by García-Cuerva and Rubio de Francia [122] (p. 405) and independently by Hruščev [161]. The reverse Hölder condition was extensively studied by Cruz-Uribe and Neugebauer [82].

Weighted inequalities with weights of the form $|x|^a$ for the Hilbert transform were first obtained by Hardy and Littlewood [147] and later by Stein [332] for other singular integrals. The necessity and sufficiency of the A_p condition for the boundedness of the Hilbert transform on weighted L^p spaces was obtained by Hunt, Muckenhoupt, and Wheeden [167]. Historically, the first result relating A_p weights and the Hilbert transform is the Helson-Szegő theorem [149], which says that the Hilbert transform is bounded on $L^2(w)$ if and only if $\log w=u+Hv$, where $u,v\in L^\infty(\mathbf{R})$ and $\|v\|_{L^\infty}<\frac{\pi}{2}$. The Helson-Szegő condition easily implies the A_2 condition, but the only known direct proof for the converse gives $\|v\|_{L^\infty}<\pi$; see Coifman, Jones, and Rubio de Francia [67]. A related result in higher dimensions was obtained by Garnett and Jones [123]. Weighted L^p estimates controlling Calderón–Zygmund operators by the Hardy–Littlewood maximal operator were obtained by Coifman [65]. Coifman and Fefferman [66] extended one-dimensional weighted norm inequalities to higher dimensions and also obtained good lambda inequalities for A_∞ weights for more general singular integrals and maximal singular integrals (Theorem 7.4.3). Bagby and Kurtz [19], and later Alvarez and Pérez [4], gave a sharper version of Theorem 7.4.3, by replacing the good lambda inequality by a rearrangement inequality. See also the related work of Lerner [217]. The following relation $\|M_d(f)\|_{L^p(w)}\le C(p,n,[w]_{A_\infty})\|M^\#(f)\|_{L^p(w)}$ between the dyadic maximal function and the sharp maximal function is valid for any $w\in A_\infty$ under the condition $M(f)\in L^{p_0}$ but also under the weaker assumption that $w(\{|f|>t\})<\infty$ for every $t>0$; see Kurtz [208]. Using that $\min(M,w)$ is an A_∞ weight with constant independent of M and Fatou's lemma, this condition can be relaxed to $|\{|f|>t\}|<\infty$ for every $t>0$. A rearrangement inequality relating f and $M^\#(f)$ is given in Bagby and Kurtz [18].

The factorization of A_p weights was conjectured by Muckenhoupt and proved by Jones [179]. The simple proof given in the text can be found in [67]. Extrapolation of operators (Theorem 7.5.3) is due to Rubio de Francia [300]. An alternative proof of this theorem was given later by García-Cuerva [121]. The value of the constant $K(n,p,p_0,[w]_{A_p})$ first appeared in Dragičević, Grafakos, Pereyra, and Petermichl [98]. Another proof with sharp bounds (in terms of the characteristic constant of the weights) was given by Duoandikoetxea [101]. The present treatment of Theorem 7.5.3, based on crucial Lemma 7.5.4, was communicated to the author by J. M. Martell. One may also consult the related work of Cruz-Uribe, Martell, and Pérez [80]. The simple proof of Theorem 7.5.5 was conceived by J. M. Martell and first appeared in the treatment of extrapolation of operators of many variables; see Grafakos and Martell [135]. The idea of extrapolation can be carried to general pairs of functions, see Cruz-Uribe, Martell, and Pérez [78]. Estimates for the distribution function in extrapolation theory were obtained by Carro, Torres, and Soria [58]. The equivalence between vector-valued inequalities and weighted norm inequalities of Theorem 7.5.8 is also due to Rubio de Francia [299]. The difficult direction in this equivalence is obtained using a minimax principle (see Fan [111]). Alternatively, one can use the factorization theory of Maurey [247], which brings an interesting connection with Banach space theory. The book of García-Cuerva and Rubio de Francia [122] provides an excellent reference on this and other topics related to weighted norm inequalities.

A primordial double-weighted norm inequality is the observation of Fefferman and Stein [115] that the maximal function maps $L^p(M(w))$ to $L^p(w)$ for nonnegative measurable functions w (Exercise 7.1.7). Sawyer [312] obtained that the condition $\sup_Q\big(\int_Q v^{1-p'}dx\big)^{-1}\int_Q M(v^{1-p'}\chi_Q)^p w\,dx<\infty$ provides a characterization of all pairs of weights (v,w) for which the Hardy–Littlewood maximal operator M maps $L^p(v)$ to $L^p(w)$. Simpler proofs of this result were obtained by Cruz-Uribe [77] and Verbitsky [367]. The fact that Sawyer's condition reduces to the usual A_p condition when $v=w$ was shown by Hunt, Kurtz, and Neugebauer [166]. The two-weight problem for singular integrals is more delicate, since they are not necessarily bounded from $L^p(M(w))$ to $L^p(w)$. Known results in this direction are that singular integrals map $L^p(M^{[p]+1}(w))$ to $L^p(w)$, where M^r denotes the rth iterate of the maximal operator. See Wilson [377] (for $1<p<2$) and Pérez [277] for the remaining p's. A necessary condition for the boundedness of the Hilbert transform from $L^p(v)$ to $L^p(w)$ was obtained by Muckenhoupt and Wheeden [262].

For an approach to two-weighted inequalities using Bellman functions, we refer to the article of Nazarov, Treil, and Volberg [266]. The notion of Bellman functions originated in control theory; the article [267] of the previous authors analyzes the connections between optimal control and harmonic analysis. Bellman functions have been used to derive estimates for the norms of classical operators on weighted Lebesgue spaces; for instance, Petermichl [279] showed that for $w\in A_2(\mathbf{R})$, the norm of the Hilbert transform from $L^2(\mathbf{R},w)$ to $L^2(\mathbf{R},w)$ is bounded by a constant times the characteristic constant $[w]_{A_2}$.

The theory of A_p weights in this chapter carries through to the situation in which Lebesgue measure is replaced by a general doubling measure. This theory also has a substantial analogue when the underlying measure is nondoubling but satisfies $\mu(\partial Q)=0$ for all cubes Q in $\mathbf{R}^n$ with sides parallel to the axes; see Orobitg and Pérez [272]. A thorough account of weighted Littlewood–Paley theory and exponential-square function integrability is contained in the book of Wilson [378].

The conjecture whether $\|T\|_{L^1(M(w))\to L^{1,\infty}(w)}<\infty$ holds for a weight w was disproved by Reguera [287] when T is a Haar multiplier and then by Reguera and Thiele [288] for the Hilbert transform. However, the slightly weaker version of this inequality, in which $M(w)$ is replaced by the Orlicz maximal operator $M_{L(\log L)^\varepsilon}(w)$, holds for any $\varepsilon>0$ and any Calderón-Zygmund operator T, as shown by Pérez [277]. For A^1 weights w the aforementioned conjecture would imply $\|T\|_{L^1(w)\to L^{1,\infty}(w)}\le C[w]_{A_1}$. However, Nazarov, Reznikov, Vasyunin, and Volberg [265] disproved the weaker inequality $\|T\|_{L^1(w)\to L^{1,\infty}(w)}\le C[w]_{A_1}\big(\log(e+[w]_{A_1})\big)^\alpha$ for $\alpha<\frac{1}{5}$. Lerner, Ombrosi, and Pérez [223] had previously shown that the preceding inequality holds with $\alpha=1$ for any Calderón-Zygmund operator T.

Concerning the sharp weighted bound $\|T\|_{L^2(w)\to L^2(w)} \le c_T\,[w]_{A_2}$ for a Calderón-Zygmund operator T we have the work of Petermichl and Volberg [281] which answered a question by Astala, Iwaniecz and Saksman [14] on the regularity of solutions to the Beltrami equation. The proofs of this inequality for the Hilbert and Riesz transforms via the Bellman function technique were obtained soon afterwards by Petermichl [279], [280]. The use of Bellman functions was first avoided in the work of Lacey, Petermichl, and Reguera [210], whose proof recovered the already known cases and used Haar shift operators, the two-weight theory for them of Nazarov, Treil and Volberg [268], and corona decompositions. The simplest proof for these classical operators was obtained by Cruz-Uribe, Martell, and Pérez [79], [81] using a very powerful inequality due to Lerner [219]. The complete proof for a general Calderón-Zygmund operator was given by Hytönen [168]. A simplified proof was provided by Lerner [220], [221]. For other improvements and estimates involving A_p and A_∞ constants see the work of Lerner [222], Hytönen and Pérez [170], and Lacey, Hytönen, and Pérez [169].