Chapter 6
Littlewood–Paley Theory and Multipliers

In this chapter we are concerned with orthogonality properties of the Fourier transform. This orthogonality is easily understood on L^2, but at this point it is not clear how it manifests itself on other spaces. Square functions introduce a way to express and quantify orthogonality of the Fourier transform on L^p and other function spaces. The introduction of square functions in this setting was pioneered by Littlewood and Paley, and the theory that subsequently developed is named after them. The extent to which Littlewood–Paley theory characterizes function spaces is remarkable.

Historically, Littlewood–Paley theory first appeared in the context of one-dimensional Fourier series and depended on complex function theory. With the development of real-variable methods, the whole theory became independent of complex methods and was extended to $\mathbf{R}^n$. This is the approach that we follow in this chapter. It turns out that the Littlewood–Paley theory is intimately related to the Calderón–Zygmund theory introduced in the previous chapter. This connection is deep and far-reaching, and its central feature is that one is able to derive the main results of one theory from the other.

The thrust and power of the Littlewood–Paley theory become apparent in some of the applications we discuss in this chapter. Such applications include the derivation of certain multiplier theorems, that is, theorems that yield sufficient conditions for bounded functions to be L^p multipliers. As a consequence of Littlewood–Paley theory we also prove that the lacunary partial Fourier integrals $\int_{|\xi|\le 2^N} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$ converge almost everywhere to an L^p function f on $\mathbf{R}^n$.

6.1 Littlewood–Paley Theory

We begin by examining more closely what we mean by orthogonality of the Fourier transform. If the functions f_j defined on $\mathbf{R}^n$ have Fourier transforms $\widehat{f_j}$ supported in disjoint sets, then they are *orthogonal* in the sense that

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$$\|\sum_j f_j\|_{L^2}^2 = \sum_j \|f_j\|_{L^2}^2. \tag{6.1.1}$$

Unfortunately, when 2 is replaced by some $p \neq 2$ in (6.1.1), the previous quantities may not even be comparable, as we show in Examples 6.1.8 and 6.1.9. The Littlewood–Paley theorem provides a substitute inequality to (6.1.1) expressing the fact that certain orthogonality considerations are also valid in $L^p(\mathbf{R}^n)$.

6.1.1 The Littlewood–Paley Theorem

The orthogonality we are searching for is best seen in the context of one-dimensional Fourier series (which was the setting in which Littlewood and Paley formulated their result). The primary observation is that the exponential $e^{2\pi i 2^k x}$ oscillates half as much as $e^{2\pi i 2^{k+1} x}$ and is therefore nearly constant in each period of the latter. This observation was instrumental in the proof of Theorem 3.6.4, which implied in particular that for all $1 < p < \infty$ we have

$$\Big\| \sum_{k=1}^N a_k e^{2\pi i 2^k x} \Big\|_{L^p[0,1]} \approx \Big(\sum_{k=1}^N |a_k|^2 \Big)^{\frac{1}{2}}. \tag{6.1.2}$$

In other words, we can calculate the L^p norm of $\sum_{k=1}^N a_k e^{2\pi i 2^k x}$ in almost a precise fashion to obtain (modulo multiplicative constants) the same answer as in the L^2 case. Similar calculations are valid for more general blocks of exponentials in the dyadic range $\{2^k+1, \dots, 2^{k+1}-1\}$, since the exponentials in each such block behave independently from those in each previous block. In particular, the L^p integrability of a function on $\mathbf{T}^1$ is not affected by the randomization of the sign of its Fourier coefficients in the previous dyadic blocks. This is the intuition behind the Littlewood–Paley theorem.

Motivated by this discussion, we introduce the Littlewood–Paley operators in the continuous setting.

Definition 6.1.1. Let Ψ be an integrable function on $\mathbf{R}^n$ and $j \in \mathbf{Z}$. We define the *Littlewood–Paley operator* Δ_j associated with Ψ by

$$\Delta_j(f) = f * \Psi_{2^{-j}},$$

where $\Psi_{2^{-j}}(x) = 2^{jn}\Psi(2^j x)$ for all x in $\mathbf{R}^n$. Thus we have $\widehat{\Psi_{2^{-j}}}(\xi) = \widehat{\Psi}(2^{-j}\xi)$ for all ξ in $\mathbf{R}^n$. We note that whenever Ψ is a Schwartz function and f is a tempered distribution, the quantity $\Delta_j(f)$ is a well defined function.

These operators depend on the choice of the function Ψ; in most applications we choose Ψ to be a smooth function with compactly supported Fourier transform. Observe that if $\widehat{\Psi}$ is supported in some annulus $0 < c_1 < |\xi| < c_2 < \infty$, then the Fourier transform of Δ_j is supported in the annulus $c_1 2^j < |\xi| < c_2 2^j$; in other words, it is localized near the frequency $|\xi| \approx 2^j$. Thus the purpose of Δ_j is to isolate the part of frequency of a function concentrated near $|\xi| \approx 2^j$.

The *square function* associated with the Littlewood–Paley operators Δ_j is defined by

$$f \mapsto \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}}.$$

This quadratic expression captures the intrinsic orthogonality of the function f.

Theorem 6.1.2. ***(Littlewood–Paley theorem)*** *Suppose that Ψ is an integrable $\mathscr{C}^1$ function on $\mathbf{R}^n$ with mean value zero that satisfies*

$$|\Psi(x)| + |\nabla\Psi(x)| \le B(1+|x|)^{-n-1}. \tag{6.1.3}$$

Then there exists a constant $C_n < \infty$ such that for all $1 < p < \infty$ and all f in $L^p(\mathbf{R}^n)$ we have

$$\Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R}^n)} \le C_n B \max\big(p,(p-1)^{-1}\big) \big\|f\big\|_{L^p(\mathbf{R}^n)}. \tag{6.1.4}$$

There also exists a $C_n' < \infty$ such that for all f in $L^1(\mathbf{R}^n)$ we have

$$\Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n' B \big\|f\big\|_{L^1(\mathbf{R}^n)}. \tag{6.1.5}$$

Conversely, let Ψ be a Schwartz function such that either $\widehat{\Psi}(0) = 0$ and

$$\sum_{j\in\mathbf{Z}} |\widehat{\Psi}(2^{-j}\xi)|^2 = 1, \qquad \text{for all } \xi \in \mathbf{R}^n \setminus \{0\}, \tag{6.1.6}$$

or $\widehat{\Psi}$ is compactly supported away from the origin and

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \qquad \text{for all } \xi \in \mathbf{R}^n \setminus \{0\}. \tag{6.1.7}$$

Then there is a constant $C_{n,\Psi}$, such that for any $f \in \mathscr{S}'(\mathbf{R}^n)$ with $\big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2\big)^{\frac{1}{2}}$ in $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$, there exists a unique polynomial Q such that the tempered distribution $f - Q$ coincides with an L^p function, and we have

$$\big\|f - Q\big\|_{L^p(\mathbf{R}^n)} \le C_{n,\Psi} B \max\big(p,(p-1)^{-1}\big) \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R}^n)}. \tag{6.1.8}$$

Consequently, if g lies in $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$, then

$$\big\|g\big\|_{L^p(\mathbf{R}^n)} \approx \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(g)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R}^n)}.$$

Proof. We first prove (6.1.4) when $p=2$. Using Plancherel's theorem, we see that (6.1.4) is a consequence of the inequality

$$\sum_j |\widehat{\Psi}(2^{-j}\xi)|^2 \le C_n B^2 \tag{6.1.9}$$

for some $C_n < \infty$. Because of (6.1.3), Fourier inversion holds for Ψ. Furthermore, Ψ has mean value zero and we may write

$$\widehat{\Psi}(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i x\cdot\xi}\Psi(x)\,dx = \int_{\mathbf{R}^n} (e^{-2\pi i x\cdot\xi}-1)\Psi(x)\,dx\,, \tag{6.1.10}$$

from which we obtain the estimate

$$|\widehat{\Psi}(\xi)| \le \sqrt{4\pi|\xi|}\int_{\mathbf{R}^n} |x|^{\frac{1}{2}}|\Psi(x)|\,dx \le C_n B|\xi|^{\frac{1}{2}}\,. \tag{6.1.11}$$

For $\xi=(\xi_1,\dots,\xi_n)\neq 0$, let j be such that $|\xi_j|\ge|\xi_k|$ for all $k\in\{1,\dots,n\}$. Integrate by parts with respect to ∂_j in (6.1.10) to obtain

$$\widehat{\Psi}(\xi) = -\int_{\mathbf{R}^n} (-2\pi i\xi_j)^{-1} e^{-2\pi i x\cdot\xi}(\partial_j\Psi)(x)\,dx,$$

from which we deduce the estimate

$$|\widehat{\Psi}(\xi)| \le \sqrt{n}\,|\xi|^{-1}\int_{\mathbf{R}^n} |\nabla\Psi(x)|\,dx \le C_n B|\xi|^{-1}. \tag{6.1.12}$$

We now break the sum in (6.1.9) into the parts where $2^{-j}|\xi|\le 1$ and $2^{-j}|\xi|\ge 1$ and use (6.1.11) and (6.1.12), respectively, to obtain (6.1.9). (See also Exercise 6.1.2.) This proves (6.1.4) when $p=2$.

We now turn our attention to the case $p\neq 2$ in (6.1.4). We view (6.1.4) and (6.1.5) as vector-valued inequalities in the spirit of Section 5.5. Define an operator $\vec{T}$ acting on functions on $\mathbf{R}^n$ as follows:

$$\vec{T}(f)(x) = \{\Delta_j(f)(x)\}_j\,.$$

The inequalities (6.1.4) and (6.1.5) we wish to prove say simply that $\vec{T}$ is a bounded operator from $L^p(\mathbf{R}^n,\mathbf{C})$ to $L^p(\mathbf{R}^n,\ell^2)$ and from $L^1(\mathbf{R}^n,\mathbf{C})$ to $L^{1,\infty}(\mathbf{R}^n,\ell^2)$. We just proved that this statement is true when $p=2$, and therefore the first hypothesis of Theorem 5.6.1 is satisfied. We observe that the operator $\vec{T}$ can be written in the form

$$\vec{T}(f)(x) = \left\{\int_{\mathbf{R}^n} \Psi_{2^{-j}}(x-y)f(y)\,dy\right\}_j = \int_{\mathbf{R}^n} \vec{K}(x-y)(f(y))\,dy,$$

where for each $x\in\mathbf{R}^n$, $\vec{K}(x)$ is a bounded linear operator from $\mathbf{C}$ to ℓ^2 given by

$$\vec{K}(x)(a) = \{\Psi_{2^{-j}}(x)a\}_j. \tag{6.1.13}$$

We clearly have that $\|\vec{K}(x)\|_{\mathbf{C}\to\ell^2} = \big(\sum_j |\Psi_{2^{-j}}(x)|^2\big)^{\frac{1}{2}}$, and to be able to apply Theorem 5.6.1 we need to know that for some constant C_n we have

$$\big\|\vec{K}(x)\big\|_{\mathbf{C}\to\ell^2} \le C_n B\,|x|^{-n}, \tag{6.1.14}$$

$$\lim_{\varepsilon\downarrow 0}\int_{\varepsilon\le|y|\le 1}\vec{K}(y)\,dy = \left\{\int_0^1 \Psi_{2^j}(y)\,dy\right\}_{j\in\mathbf{Z}}, \tag{6.1.15}$$

$$\sup_{y\neq 0}\int_{|x|\ge 2|y|}\big\|\vec{K}(x-y)-\vec{K}(x)\big\|_{\mathbf{C}\to\ell^2}\,dx \le C_n B. \tag{6.1.16}$$

Of these, (6.1.14) is easily obtained using (6.1.3), (6.1.15) i.e. trivial, and so we focus on (6.1.16). Since Ψ is a $\mathscr{C}^1$ function, for $|x|\ge 2|y|$ we have

$$\begin{aligned}
&|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)| \\
&\quad\le 2^{(n+1)j}|\nabla\Psi(2^j(x-\theta y))|\,|y| && \text{for some } \theta\in[0,1],\\
&\quad\le B\,2^{(n+1)j}\big(1+2^j|x-\theta y|\big)^{-(n+1)}|y| \\
&\quad\le B\,2^{nj}\big(1+2^{j-1}|x|\big)^{-(n+1)}2^j|y| && \text{since } |x-\theta y|\ge \tfrac{1}{2}|x|.
\end{aligned} \tag{6.1.17}$$

We also have that

$$\begin{aligned}
&|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)| \\
&\quad\le 2^{nj}|\Psi(2^j(x-y))|+2^{jn}|\Psi(2^jx)| \\
&\quad\le B\,2^{nj}\big(1+2^j|x|\big)^{-(n+1)}+B2^{jn}\big(1+2^{j-1}|x|\big)^{-(n+1)} \\
&\quad\le 2B\,2^{nj}\big(1+2^{j-1}|x|\big)^{-(n+1)}.
\end{aligned} \tag{6.1.18}$$

Taking the geometric mean of (6.1.17) and (6.1.18), we obtain for any $\gamma\in[0,1]$

$$|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)| \le 2^{1-\gamma}B\,2^{nj}(2^j|y|)^{\gamma}\big(1+2^{j-1}|x|\big)^{-(n+1)}. \tag{6.1.19}$$

Using this estimate, when $|x|\ge 2|y|$, we obtain

$$\begin{aligned}
\big\|\vec{K}(x-y)-\vec{K}(x)\big\|_{\mathbf{C}\to\ell^2} &= \bigg(\sum_{j\in\mathbf{Z}}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big|^2\bigg)^{1/2} \\
&\le \sum_{j\in\mathbf{Z}}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big| \\
&\le 2B\bigg(|y|\sum_{2^j<\frac{2}{|x|}}2^{(n+1)j}+|y|^{\frac{1}{2}}\sum_{2^j\ge\frac{2}{|x|}}2^{(n+\frac{1}{2})j}(2^{j-1}|x|)^{-(n+1)}\bigg) \\
&\le C_nB\big(|y||x|^{-n-1}+|y|^{\frac{1}{2}}|x|^{-n-\frac{1}{2}}\big),
\end{aligned}$$

where we used (6.1.19) with $\gamma=1$ in the first sum and (6.1.19) with $\gamma=1/2$ in the second sum. Using this bound, we easily deduce (6.1.16) by integrating over the region $|x| \geq 2|y|$. Finally, using Theorem 5.6.1 we conclude the proofs of (6.1.4) and (6.1.5), which establishes one direction of the theorem.

We now turn to the converse direction. Let Δ_j^* be the adjoint operator of Δ_j given by $\widehat{\Delta_j^* f} = \widehat{f}\,\widehat{\Psi_{2^{-j}}}$. Let f be in $\mathscr{S}'(\mathbf{R}^n)$. Then the series $\sum_{j\in\mathbf{Z}} \Delta_j^*\Delta_j(f)$ converges in $\mathscr{S}'(\mathbf{R}^n)$. To see this, it suffices to show that the sequence of partial sums $u_N = \sum_{|j|<N} \Delta_j^*\Delta_j(f)$ converges in $\mathscr{S}'$. This means that if we test this sequence against a Schwartz function g, then it is a Cauchy sequence and hence it converges as $N\to\infty$. But an easy argument using duality and the Cauchy–Schwarz and Hölder's inequalities shows that for $M>N$ we have

$$|\langle u_N, g\rangle - \langle u_M, g\rangle| \leq \Big\|\Big(\sum_j |\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\Big\|\Big(\sum_{N\leq|j|\leq M} |\Delta_j(g)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p'}},$$

and this can be made small by picking $M>N\geq N_0(g)$. Since the sequence $\langle u_N, g\rangle$ is Cauchy, it converges to some $\Lambda(g)$. Now it remains to show that the map $g\mapsto\Lambda(g)$ is a tempered distribution. Obviously $\Lambda(g)$ is a linear functional. Also,

$$\begin{aligned}|\Lambda(g)| &\leq \Big\|\Big(\sum_j |\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\Big\|\Big(\sum_j |\Delta_j(g)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p'}}\\ &\leq C_{p'}\Big\|\Big(\sum_j |\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\big\|g\big\|_{L^{p'}},\end{aligned}$$

and since $\|g\|_{L^{p'}}$ is controlled by a finite number of Schwartz seminorms of g, it follows that Λ is in $\mathscr{S}'$. The distribution Λ is the limit of the series $\sum_j \Delta_j^*\Delta_j$.

Under hypothesis (6.1.6), the Fourier transform of the tempered distribution $f - \sum_{j\in\mathbf{Z}}\Delta_j^*\Delta_j(f)$ is supported at the origin. This implies that there exists a polynomial Q such that $f - Q = \sum_{j\in\mathbf{Z}}\Delta_j^*\Delta_j(f)$. Now let g be a Schwartz function. We have

$$\begin{aligned}\big|\langle f-Q, \overline{g}\rangle\big| &= \Big|\Big\langle \sum_{j\in\mathbf{Z}} \Delta_j^*\Delta_j(f), \overline{g}\Big\rangle\Big|\\ &= \Big|\sum_{j\in\mathbf{Z}} \big\langle \Delta_j^*\Delta_j(f), \overline{g}\big\rangle\Big|\\ &= \Big|\sum_{j\in\mathbf{Z}} \big\langle \Delta_j(f), \overline{\Delta_j(g)}\big\rangle\Big|\\ &= \Big|\int_{\mathbf{R}^n}\sum_{j\in\mathbf{Z}} \Delta_j(f)\,\overline{\Delta_j(g)}\,dx\Big|\\ &\leq \int_{\mathbf{R}^n}\Big(\sum_{j\in\mathbf{Z}}|\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big(\sum_{j\in\mathbf{Z}}|\Delta_j(g)|^2\Big)^{\frac{1}{2}}dx\\ &\leq \Big\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\Big\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j(g)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p'}}\end{aligned}$$

$$\leq \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} C_n B \max\big(p', (p'-1)^{-1}\big) \big\|g\big\|_{L^{p'}}, \tag{6.1.20}$$

having used the definition of the adjoint (Section 2.5.2), the Cauchy–Schwarz inequality, Hölder's inequality, and (6.1.4). Taking the supremum over all g in $L^{p'}$ with norm at most one, we obtain that the tempered distribution $f-Q$ is a bounded linear functional on $L^{p'}$. By the Riesz representation theorem, $f-Q$ coincides with an L^p function whose norm satisfies the estimate

$$\big\|f-Q\big\|_{L^p} \leq C_n B \max\big(p, (p-1)^{-1}\big) \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p}.$$

We now show uniqueness. If Q_1 is another polynomial, with $f-Q_1 \in L^p$, then $Q-Q_1$ must be an L^p function; but the only polynomial that lies in L^p is the zero polynomial. This completes the proof of the converse of the theorem under hypothesis (6.1.6).

To obtain the same conclusion under the hypothesis (6.1.7) we argue in a similar way but we leave the details as an exercise. (One may adapt the argument in the proof of Corollary 6.1.7 to this setting.) □

Remark 6.1.3. We make some observations. If $\widehat{\Psi}$ is real-valued, then the operators Δ_j are self-adjoint. Indeed,

$$\int_{\mathbf{R}^n} \Delta_j(f)\,\overline{g}\,dx = \int_{\mathbf{R}^n} \widehat{f}\,\widehat{\Psi_{2^{-j}}}\,\overline{\widehat{g}}\,d\xi = \int_{\mathbf{R}^n} \widehat{f}\,\overline{\widehat{\Psi_{2^{-j}}}\,\widehat{g}}\,d\xi = \int_{\mathbf{R}^n} f\,\overline{\Delta_j(g)}\,dx.$$

Moreover, if Ψ is a radial function, we see that the operators Δ_j are self-transpose, that is, they satisfy

$$\int_{\mathbf{R}^n} \Delta_j(f)\,g\,dx = \int_{\mathbf{R}^n} f\,\Delta_j(g)\,dx.$$

Assume now that Ψ is both radial and has a real-valued Fourier transform. Suppose also that Ψ satisfies (6.1.3) and that it has mean value zero. Then the inequality

$$\Big\| \sum_{j\in\mathbf{Z}} \Delta_j(f_j) \Big\|_{L^p} \leq C_n B \max\big(p, (p-1)^{-1}\big) \Big\| \Big(\sum_{j\in\mathbf{Z}} |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \tag{6.1.21}$$

is true for sequences of functions $\{f_j\}_j$. To see this we use duality. Let

$$\vec{T}(f) = \{\Delta_j(f)\}_j.$$

Then

$$\vec{T}^*(\{g_j\}_j) = \sum_j \Delta_j(g_j).$$

Inequality (6.1.4) says that the operator $\vec{T}$ maps $L^p(\mathbf{R}^n,\mathbf{C})$ to $L^p(\mathbf{R}^n,\ell^2)$, and its dual statement is that $\vec{T}^*$ maps $L^{p'}(\mathbf{R}^n,\ell^2)$ to $L^{p'}(\mathbf{R}^n,\mathbf{C})$. This is exactly the statement in (6.1.21) if p is replaced by p'. Since p is any number in $(1,\infty)$, (6.1.21) is proved.

6.1.2 Vector-Valued Analogues

We now obtain a vector-valued extension of Theorem 6.1.2. We have the following.

Proposition 6.1.4. *Let Ψ be an integrable $\mathscr{C}^1$ function on $\mathbf{R}^n$ with mean value zero that satisfies (6.1.3) and let Δ_j be the Littlewood–Paley operator associated with Ψ. Then there exists a constant $C_n < \infty$ such that for all $1 < p, r < \infty$ and all sequences of L^p functions f_j we have*

$$\Big\| \Big(\sum_{j\in\mathbf{Z}} \Big(\sum_{k\in\mathbf{Z}} |\Delta_k(f_j)|^2 \Big)^{\frac{r}{2}} \Big)^{\frac{1}{r}} \Big\|_{L^p(\mathbf{R}^n)} \le C_n B \widetilde{C}_{p,r} \Big\| \Big(\sum_{j\in\mathbf{Z}} |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{L^p(\mathbf{R}^n)},$$

where $\widetilde{C}_{p,r} = \max(p,(p-1)^{-1})\max(r,(r-1)^{-1})$. *Moreover, for some $C_n' > 0$ and all sequences of L^1 functions f_j we have*

$$\Big\| \Big(\sum_{j\in\mathbf{Z}} \Big(\sum_{k\in\mathbf{Z}} |\Delta_k(f_j)|^2 \Big)^{\frac{r}{2}} \Big)^{\frac{1}{r}} \Big\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n' B \max(r,(r-1)^{-1}) \Big\| \Big(\sum_{j\in\mathbf{Z}} |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{L^1(\mathbf{R}^n)}.$$

In particular,

$$\Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f_j)|^r \Big)^{\frac{1}{r}} \Big\|_{L^p(\mathbf{R}^n)} \le C_n B \widetilde{C}_{p,r} \Big\| \Big(\sum_{j\in\mathbf{Z}} |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{L^p(\mathbf{R}^n)}. \tag{6.1.22}$$

Proof. We introduce Banach spaces $\mathscr{B}_1 = \mathbf{C}$ and $\mathscr{B}_2 = \ell^2$ and for $f \in L^p(\mathbf{R}^n)$ define an operator

$$\vec{T}(f) = \{\Delta_k(f)\}_{k\in\mathbf{Z}}.$$

In the proof of Theorem 6.1.2 we showed that $\vec{T}$ has a kernel $\vec{K}$ that satisfies condition (6.1.16). Furthermore, $\vec{T}$ obviously maps $L^r(\mathbf{R}^n,\mathbf{C})$ to $L^r(\mathbf{R}^n,\ell^r)$. Applying Proposition 5.6.4, we obtain the first two statements of the proposition. Restricting to $k = j$ yields (6.1.22). □

6.1.3 L^p Estimates for Square Functions Associated with Dyadic Sums

Let us pick a Schwartz function Ψ whose Fourier transform is compactly supported in the annulus $2^{-1} \le |\xi| \le 2^2$ such that (6.1.6) is satisfied. (Clearly (6.1.6) has no chance of being satisfied if $\widehat{\Psi}$ is supported only in the annulus $1 \le |\xi| \le 2$.) The Littlewood–Paley operation $f \mapsto \Delta_j(f)$ represents the smoothly truncated frequency localization of a function f near the dyadic annulus $|\xi| \approx 2^j$. Theorem 6.1.2 says that the square function formed by these localizations has L^p norm comparable to that of the original function. In other words, this square function characterizes the L^p norm of a function. This is the main feature of Littlewood–Paley theory.

One may ask whether Theorem 6.1.2 still holds if the Littlewood–Paley operators Δ_j are replaced by their nonsmooth versions

$$f \mapsto \big(\chi_{2^j \le |\xi| < 2^{j+1}} \widehat{f}(\xi)\big)^\vee(x). \tag{6.1.23}$$

This question has a surprising answer that already signals that there may be some fundamental differences between one-dimensional and higher-dimensional Fourier analysis. The square function formed by the operators in (6.1.23) can be used to characterize $L^p(\mathbf{R})$ in the same way Δ_j did, but not $L^p(\mathbf{R}^n)$ when $n > 1$ and $p \neq 2$. The problem lies in the fact that the characteristic function of the unit disk is not an L^p multiplier on $\mathbf{R}^n$ when $n \ge 2$ unless $p = 2$; see Section 5.1 in [131]. The one-dimensional result we alluded to earlier is the following.

For $j \in \mathbf{Z}$ we introduce the one-dimensional operator

$$\Delta_j^\#(f)(x) = (\widehat{f}\chi_{I_j})^\vee(x), \tag{6.1.24}$$

where

$$I_j = [2^j, 2^{j+1}) \cup (-2^{j+1}, -2^j],$$

and $\Delta_j^\#$ is a version of the operator Δ_j in which the characteristic function of the set $2^j \le |\xi| < 2^{j+1}$ replaces the function $\widehat{\Psi}(2^{-j}\xi)$.

Theorem 6.1.5. *There exists a constant C_1 such that for all $1 < p < \infty$ and all f in $L^p(\mathbf{R})$ we have*

$$\frac{\|f\|_{L^p(\mathbf{R}^n)}}{C_1(p+\frac{1}{p-1})^2} \le \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j^\#(f)|^2 \Big)^{\frac12} \Big\|_{L^p(\mathbf{R}^n)} \le C_1(p+\tfrac{1}{p-1})^2 \|f\|_{L^p(\mathbf{R}^n)}. \tag{6.1.25}$$

Proof. Pick a Schwartz function ψ on the line whose Fourier transform is supported in the set $2^{-1} \le |\xi| \le 2^2$ and is equal to 1 on the set $1 \le |\xi| \le 2$. Let Δ_j be the Littlewood–Paley operator associated with ψ. Observe that $\Delta_j\Delta_j^\# = \Delta_j^\#\Delta_j = \Delta_j^\#$, since $\widehat{\psi}$ is equal to one on the support of $\Delta_j^\#(f)\widehat{\ }$. We now use Exercise 5.6.1(a) to obtain

$$\begin{aligned}\Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j^\#(f)|^2 \Big)^{\frac12} \Big\|_{L^p} &= \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j^\#\Delta_j(f)|^2 \Big)^{\frac12} \Big\|_{L^p} \\ &\le C\max(p,(p-1)^{-1}) \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac12} \Big\|_{L^p} \\ &\le CB\max(p,(p-1)^{-1})^2 \|f\|_{L^p},\end{aligned}$$

where the last inequality follows from Theorem 6.1.2. The reverse inequality for $1 < p < \infty$ follows just like the reverse inequality (6.1.8) of Theorem 6.1.2 by simply replacing the Δ_j's by the $\Delta_j^\#$'s and setting the polynomial Q equal to zero. (There is no need to use the Riesz representation theorem here, just the fact that the L^p norm of f can be realized as the supremum of expressions $|\langle f,g\rangle|$ where g has $L^{p'}$ norm at most 1.) $\square$

There is a higher-dimensional version of Theorem 6.1.5 with dyadic rectangles replacing the dyadic intervals. As has already been pointed out, the higher-dimensional version with dyadic annuli replacing the dyadic intervals is false.

Let us introduce some notation. For $j \in \mathbf{Z}$, we denote by I_j the dyadic set $[2^j,2^{j+1})\bigcup(-2^{j+1},-2^j]$ as in the statement of Theorem 6.1.5. For $j_1,\dots,j_n \in \mathbf{Z}$ define a dyadic rectangle

$$R_{j_1,\dots,j_n} = I_{j_1} \times \cdots \times I_{j_n}$$

in $\mathbf{R}^n$. Actually $R_{j_1,\dots,j_n}$ is not a rectangle but a union of 2^n rectangles; with some abuse of language we still call it a rectangle. For notational convenience we write

$$R_{\mathbf{j}} = R_{j_1,\dots,j_n}\,, \qquad \text{where } \mathbf{j} = (j_1,\dots,j_n) \in \mathbf{Z}^n.$$

Observe that for different $\mathbf{j},\mathbf{j}' \in \mathbf{Z}^n$ the rectangles $R_{\mathbf{j}}$ and $R_{\mathbf{j}'}$ have disjoint interiors and that the union of all the $R_{\mathbf{j}}$'s is equal to $\mathbf{R}^n \setminus \{0\}$. In other words, the family of $R_{\mathbf{j}}$'s, where $\mathbf{j} \in \mathbf{Z}^n$, forms a *tiling* of $\mathbf{R}^n$, which we call the *dyadic decomposition* of $\mathbf{R}^n$. We now introduce operators

$$\Delta_{\mathbf{j}}^{\#}(f)(x) = (\widehat{f}\chi_{R_{\mathbf{j}}})^{\vee}(x)\,, \tag{6.1.26}$$

and we have the following n-dimensional extension of Theorem 6.1.5.

Theorem 6.1.6. *For a Schwartz function ψ on the line with integral zero we define the operator*

$$\Delta_{\mathbf{j}}(f)(x) = \big(\widehat{\psi}(2^{-j_1}\xi_1)\cdots\widehat{\psi}(2^{-j_n}\xi_n)\widehat{f}(\xi)\big)^{\vee}(x)\,, \tag{6.1.27}$$

where $\mathbf{j} = (j_1,\dots,j_n) \in \mathbf{Z}^n$. Then there is a dimensional constant C_n such that

$$\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n(p+(p-1)^{-1})^n\big\|f\big\|_{L^p(\mathbf{R}^n)}\,. \tag{6.1.28}$$

Let $\Delta_{\mathbf{j}}^{\#}$ be the operators defined in (6.1.26). Then there exists a positive constant C_n such that for all $1<p<\infty$ and all $f \in L^p(\mathbf{R}^n)$ we have

$$\frac{\big\|f\big\|_{L^p(\mathbf{R}^n)}}{C_n(p+\frac{1}{p-1})^{2n}} \le \Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}^{\#}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n(p+\tfrac{1}{p-1})^{2n}\big\|f\big\|_{L^p(\mathbf{R}^n)}. \tag{6.1.29}$$

Proof. We first prove (6.1.28). Note that if $\mathbf{j} = (j_1,\dots,j_n) \in \mathbf{Z}^n$, then the operator $\Delta_{\mathbf{j}}$ is equal to

$$\Delta_{\mathbf{j}}(f) = \Delta_{j_1}^{(j_1)}\cdots\Delta_{j_n}^{(j_n)}(f)\,,$$

where the $\Delta_{j_r}^{(j_r)}$ are one-dimensional operators given on the Fourier transform by multiplication by $\widehat{\psi}(2^{-j_r}\xi_r)$, with the remaining variables fixed. Inequality in (6.1.28) is a consequence of the one-dimensional case. For instance, we discuss the case $n = 2$. Using Proposition 6.1.4, we obtain

$$\begin{aligned}
&\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^2}|\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^2)}^p \\
&\qquad = \int_{\mathbf{R}}\Big[\int_{\mathbf{R}}\Big(\sum_{j_1\in\mathbf{Z}}\sum_{j_2\in\mathbf{Z}}|\Delta_{j_1}^{(1)}\Delta_{j_2}^{(2)}(f)(x_1,x_2)|^2\Big)^{\frac{p}{2}}dx_1\Big]dx_2 \\
&\qquad \le C^p\max(p,(p-1)^{-1})^p\int_{\mathbf{R}}\Big[\int_{\mathbf{R}}\Big(\sum_{j_2\in\mathbf{Z}}|\Delta_{j_2}^{(2)}(f)(x_1,x_2)|^2\Big)^{\frac{p}{2}}dx_1\Big]dx_2 \\
&\qquad = C^p\max(p,(p-1)^{-1})^p\int_{\mathbf{R}}\Big[\int_{\mathbf{R}}\Big(\sum_{j_2\in\mathbf{Z}}|\Delta_{j_2}^{(2)}(f)(x_1,x_2)|^2\Big)^{\frac{p}{2}}dx_2\Big]dx_1 \\
&\qquad \le C^{2p}\max(p,(p-1)^{-1})^{2p}\int_{\mathbf{R}}\Big[\int_{\mathbf{R}}|f(x_1,x_2)|^p\,dx_2\Big]dx_1 \\
&\qquad = C^{2p}\max(p,(p-1)^{-1})^{2p}\big\|f\big\|_{L^p(\mathbf{R}^2)}^p\,,
\end{aligned}$$

where we also used Theorem 6.1.2 in the calculation. Higher-dimensional versions of this estimate may easily be obtained by induction.

We now turn to the upper inequality in (6.1.29). We pick a Schwartz function ψ whose Fourier transform is supported in the union $[-4,-1/2]\bigcup[1/2,4]$ and is equal to 1 on $[-2,-1]\bigcup[1,2]$. Then we clearly have

$$\Delta_{\mathbf{j}}^{\#} = \Delta_{\mathbf{j}}^{\#}\Delta_{\mathbf{j}}\,,$$

since $\widehat{\psi}(2^{-j_1}\xi_1)\cdots\widehat{\psi}(2^{-j_n}\xi_n)$ is equal to 1 on the rectangle $R_{\mathbf{j}}$. We now use Exercise 5.6.1(b) and estimate (6.1.28) to obtain

$$\begin{aligned}
\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}^{\#}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} &= \Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}^{\#}\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \\
&\le C\max(p,(p-1)^{-1})^n\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \\
&\le CB\max(p,(p-1)^{-1})^{2n}\big\|f\big\|_{L^p}\,.
\end{aligned}$$

The lower inequality in (6.1.29) for $1 < p < \infty$ is proved like inequality (6.1.8) in Theorem 6.1.2. The fundamental ingredient in the proof is that $f = \sum_{\mathbf{j}\in\mathbf{Z}^n}\Delta_{\mathbf{j}}^{\#}\Delta_{\mathbf{j}}^{\#}(f)$ for all Schwartz functions f, where the sum is interpreted as the L^2-limit of the sequence of partial sums. Thus the series converges in $\mathscr{S}'$, and pairing with a Schwartz function $\overline{g}$, we obtain the lower inequality in (6.1.29) for Schwartz functions, by applying the steps that prove (6.1.20) (with $Q = 0$). To prove the lower inequality in (6.1.29) for a general function $f \in L^p(\mathbf{R}^n)$ we approximate an L^p function by a sequence of Schwartz functions in the L^p norm. Then both sides of the lower inequality in (6.1.29) for the approximating sequence converge to the corresponding sides of the lower inequality in (6.1.29) for f; the convergence of the sequence of L^p norms of the square functions requires the upper inequality in (6.1.29) that was previously established. This concludes the proof of the theorem. $\square$

Next we observe that if the Schwartz function ψ is suitably chosen, then the reverse inequality in estimate (6.1.28) also holds. More precisely, suppose $\widehat{\psi}(\xi)$ is an even smooth real-valued function supported in the set $\frac{9}{10} \le |\xi| \le \frac{21}{10}$ in $\mathbf{R}$ that satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\psi}(2^{-j}\xi) = 1, \qquad \xi \in \mathbf{R}\setminus\{0\}; \tag{6.1.30}$$

then we have the following.

Corollary 6.1.7. *Suppose that ψ satisfies (6.1.30) and let $\Delta_{\mathbf{j}}$ be as in (6.1.27). Let f be an L^p function on $\mathbf{R}^n$ such that the function $\big(\sum_{\mathbf{j}\in\mathbf{Z}^n} |\Delta_{\mathbf{j}}(f)|^2\big)^{\frac{1}{2}}$ is in $L^p(\mathbf{R}^n)$. Then there is a constant C_n that depends only on the dimension and ψ such that the lower estimate*

$$\frac{\big\|f\big\|_{L^p}}{C_n(p+\frac{1}{p-1})^n} \le \Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n} |\Delta_{\mathbf{j}}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \tag{6.1.31}$$

holds.

Proof. If we had $\sum_{j\in\mathbf{Z}} |\widehat{\psi}(2^{-j}\xi)|^2 = 1$ instead of (6.1.30), then we could apply the method used in the lower estimate of Theorem 6.1.2 to obtain the required conclusion. In this case we provide another argument that is very similar in spirit.

We first prove (6.1.31) for Schwartz functions f. Then the series $\sum_{j\in\mathbf{Z}^n} \Delta_{\mathbf{j}}(f)$ converges in L^2 (and hence in $\mathscr{S}'$) to f. Now let g be another Schwartz function. We express the inner product $\langle f, \overline{g}\rangle$ as the action of the distribution $\sum_{\mathbf{j}\in\mathbf{Z}^n} \Delta_{\mathbf{j}}(f)$ on the test function $\overline{g}$:

$$\begin{aligned}
\big|\big\langle f,\overline{g}\big\rangle\big| &= \Big|\Big\langle \sum_{\mathbf{j}\in\mathbf{Z}^n} \Delta_{\mathbf{j}}(f), \overline{g}\Big\rangle\Big| \\
&= \Big|\sum_{\mathbf{j}\in\mathbf{Z}^n} \big\langle \Delta_{\mathbf{j}}(f), \overline{g}\big\rangle\Big| \\
&= \Big|\sum_{\mathbf{j}\in\mathbf{Z}^n} \sum_{\substack{\mathbf{k}=(k_1,\dots,k_n)\in\mathbf{Z}^n \\ \exists r\ |k_r-j_r|\le 1\}}} \big\langle \Delta_{\mathbf{j}}(f), \overline{\Delta_{\mathbf{k}}(g)}\big\rangle\Big| \\
&\le \int_{\mathbf{R}^n} \sum_{\mathbf{j}\in\mathbf{Z}^n} \sum_{\substack{\mathbf{k}=(k_1,\dots,k_n)\in\mathbf{Z}^n \\ \exists r\ |k_r-j_r|\le 1\}}} \big|\Delta_{\mathbf{j}}(f)\big|\,\big|\Delta_{\mathbf{k}}(g)\big|\,dx \\
&\le 3^n \int_{\mathbf{R}^n} \Big(\sum_{\mathbf{j}\in\mathbf{Z}^n} \big|\Delta_{\mathbf{j}}(f)\big|^2\Big)^{\frac{1}{2}} \Big(\sum_{\mathbf{k}\in\mathbf{Z}^n} \big|\Delta_{\mathbf{k}}(g)\big|^2\Big)^{\frac{1}{2}}\,dx
\end{aligned}$$

$$\leq 3^n \Big\| \Big(\sum_{\mathbf{j}\in\mathbf{Z}^n} |\Delta_{\mathbf{j}}(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \Big\| \Big(\sum_{\mathbf{k}\in\mathbf{Z}^n} |\Delta_{\mathbf{k}}(g)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p'}}$$

$$\leq C_n^{-1} \max\big(p', (p'-1)^{-1}\big)^n \|g\|_{L^{p'}} \Big\| \Big(\sum_{\mathbf{j}\in\mathbf{Z}^n} |\Delta_{\mathbf{j}}(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p},$$

where we used the fact that $\Delta_{\mathbf{j}}(f)$ and $\Delta_{\mathbf{k}}(g)$ are orthogonal operators unless every coordinate of $\mathbf{k}$ is within 1 unit of the corresponding coordinate of $\mathbf{j}$; this is an easy consequence of the support properties of $\widehat{\psi}$. We now take the supremum over all g in $L^{p'}$ with norm at most 1, to obtain (6.1.31) for Schwartz functions f.

To extend this estimate to general L^p functions f, we use the density argument described in the last paragraph in the proof of Theorem 6.1.6. □

6.1.4 Lack of Orthogonality on L^p

We discuss two examples indicating why (6.1.1) cannot hold if the exponent 2 is replaced by some other exponent $q \neq 2$. More precisely, we show that if the functions f_j have Fourier transforms supported in disjoint sets, then the inequality

$$\Big\| \sum_j f_j \Big\|_{L^p}^p \leq C_p \sum_j \|f_j\|_{L^p}^p \tag{6.1.32}$$

cannot hold if $p > 2$, and similarly, the inequality

$$\sum_j \|f_j\|_{L^p}^p \leq C_p \Big\| \sum_j f_j \Big\|_{L^p}^p \tag{6.1.33}$$

cannot hold if $p < 2$. In both (6.1.32) and (6.1.33) the constants C_p are supposed to be independent of the functions f_j.

Example 6.1.8. Pick a Schwartz function ζ whose Fourier transform is positive and supported in the interval $|\xi| \leq 1/4$. Let N be a large integer and let

$$f_j(x) = e^{2\pi i j x}\zeta(x).$$

Then

$$\widehat{f_j}(\xi) = \widehat{\zeta}(\xi - j)$$

and the $\widehat{f_j}$'s have disjoint Fourier transforms. We obviously have

$$\sum_{j=0}^{N} \|f_j\|_{L^p}^p = (N+1)\|\zeta\|_{L^p}^p.$$

On the other hand, we have the estimate

$$\begin{aligned}\Big\|\sum_{j=0}^{N} f_j\Big\|_{L^p}^p &= \int_{\mathbf{R}} \Big|\tfrac{e^{2\pi i(N+1)x}-1}{e^{2\pi ix}-1}\Big|^p |\zeta(x)|^p\,dx\\ &\ge c\int_{|x|<\frac{1}{10}(N+1)^{-1}} \tfrac{(N+1)^p|x|^p}{|x|^p}|\zeta(x)|^p\,dx\\ &= C_\zeta (N+1)^{p-1},\end{aligned}$$

since ζ does not vanish in a neighborhood of zero. We conclude that (6.1.32) cannot hold for this choice of f_j's for $p>2$.

Example 6.1.9. We now indicate why (6.1.33) cannot hold for $p<2$. We pick a smooth function Ψ on the line whose Fourier transform $\widehat{\Psi}$ is supported in $\big[\frac{7}{8},\frac{17}{8}\big]$, is nonnegative, is equal to 1 on $\big[\frac{9}{8},\frac{15}{8}\big]$, and satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi)^2 = 1, \qquad\qquad \xi>0.$$

Extend $\widehat{\Psi}$ to be an even function on the whole line and let Δ_j be the Littlewood–Paley operator associated with Ψ. Also pick a nonzero Schwartz function φ on the real line whose Fourier transform is nonnegative and supported in the set $\big[\frac{11}{8},\frac{13}{8}\big]$. Fix N a large positive integer and let

$$f_j(x) = e^{2\pi i \frac{12}{8}2^j x}\varphi(x), \tag{6.1.34}$$

for $j=1,2,\dots,N$. Then the function $\widehat{f_j}(\xi)=\widehat{\varphi}(\xi-\frac{12}{8}2^j)$ is supported in the set $\big[\frac{11}{8}+\frac{12}{8}2^j,\frac{13}{8}+\frac{12}{8}2^j\big]$, which is contained in $\big[\frac{9}{8}2^j,\frac{15}{8}2^j\big]$ for $j\ge 3$. In other words, $\widehat{\Psi}(2^{-j}\xi)$ is equal to 1 on the support of $\widehat{f_j}$. This implies that

$$\Delta_j(f_j)=f_j \qquad \text{for} \qquad j\ge 3.$$

This observation combined with (6.1.21) gives for $N\ge 3$,

$$\Big\|\sum_{j=3}^{N} f_j\Big\|_{L^p} = \Big\|\sum_{j=3}^{N}\Delta_j(f_j)\Big\|_{L^p} \le C_p\Big\|\Big(\sum_{j=3}^{N}|f_j|^2\Big)^{\frac12}\Big\|_{L^p} = C_p\|\varphi\|_{L^p}(N-2)^{\frac12},$$

where $1<p<\infty$. On the other hand, (6.1.34) trivially yields that

$$\Big(\sum_{j=3}^{N}\|f_j\|_{L^p}^p\Big)^{\frac1p} = \|\varphi\|_{L^p}(N-2)^{\frac1p}.$$

Letting $N\to\infty$ we see that (6.1.33) cannot hold for $p<2$ even when the f_j's have Fourier transforms supported in disjoint sets.

Example 6.1.10. A similar idea illustrates the necessity of the ℓ^2 norm in (6.1.4). To see this, let Ψ and Δ_j be as in Example 6.1.9. Let us fix $1 < p < \infty$ and $q < 2$. We show that the inequality

$$\Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p} \le C_{p,q} \big\| f \big\|_{L^p} \tag{6.1.35}$$

cannot hold. Take $f = \sum_{j=3}^N f_j$, where the f_j are as in (6.1.34) and $N \ge 3$. Then the left-hand side of (6.1.35) is bounded from below by $\|\varphi\|_{L^p}(N-2)^{1/q}$, while the right-hand side is bounded above by $\|\varphi\|_{L^p}(N-2)^{1/2}$. Letting $N \to \infty$, we deduce that (6.1.35) is impossible when $q < 2$.

Example 6.1.11. For $1 < p < \infty$ and $2 < q < \infty$, the inequality

$$\big\| g \big\|_{L^p} \le C_{p,q} \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(g)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p} \tag{6.1.36}$$

cannot hold even under assumption (6.1.6) on Ψ. Let Δ_j be as in Example 6.1.9. Let us suppose that (6.1.36) did hold for some $q > 2$ for these Δ_j's. Then the self-adjointness of the Δ_j's and duality would give

$$\begin{aligned}
&\Big\| \Big(\sum_{k\in\mathbf{Z}} |\Delta_k(g)|^{q'} \Big)^{\frac{1}{q'}} \Big\|_{L^{p'}} \\
&\quad = \sup_{\big\| \|\{h_k\}_k\|_{\ell^q} \big\|_{L^p} \le 1} \Big| \int_{\mathbf{R}} \sum_{k\in\mathbf{Z}} \Delta_k(g)\, \overline{h_k}\, dx \Big| \\
&\quad \le \big\| g \big\|_{L^{p'}} \sup_{\big\| \|\{h_k\}_k\|_{\ell^q} \big\|_{L^p} \le 1} \Big\| \sum_{k\in\mathbf{Z}} \overline{\Delta_k(h_k)} \Big\|_{L^p} \\
&\quad \le C \big\| g \big\|_{L^{p'}} \sup_{\big\| \|\{h_k\}_k\|_{\ell^q} \big\|_{L^p} \le 1} \Big\| \Big(\sum_{j\in\mathbf{Z}} \Big| \Delta_j \Big(\sum_{k\in\mathbf{Z}} \Delta_k(h_k) \Big) \Big|^q \Big)^{\frac{1}{q}} \Big\|_{L^p} \qquad \text{by (6.1.36)} \\
&\quad \le C' \big\| g \big\|_{L^{p'}} \sup_{\big\| \|\{h_k\}_k\|_{\ell^q} \big\|_{L^p} \le 1} \Big\{ \sum_{l=-1}^{1} \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j \Delta_{j+l}(h_j)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p} \Big\} \\
&\quad \le C'' \big\| g \big\|_{L^{p'}} \sup_{\big\| \|\{h_k\}_k\|_{\ell^q} \big\|_{L^p} \le 1} \Big\| \Big(\sum_{j\in\mathbf{Z}} |h_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p} = C'' \big\| g \big\|_{L^{p'}},
\end{aligned}$$

where the next-to-last inequality follows from (6.1.22) applied twice, while the one before that follows from support considerations. But since $q' < 2$, this exactly proves (6.1.35), previously shown to be false, a contradiction.

We conclude that if both assertions (6.1.4) and (6.1.8) of Theorem 6.1.2 were to hold, then the ℓ^2 norm inside the L^p norm could not be replaced by an ℓ^q norm for some $q \ne 2$. Exercise 6.1.6 indicates the crucial use of the fact that ℓ^2 is a Hilbert space in the converse inequality (6.1.8) of Theorem 6.1.2.

Exercises

6.1.1. Construct a Schwartz function Ψ that satisfies $\sum_{j\in\mathbf{Z}}|\widehat{\Psi}(2^{-j}\xi)|^2 = 1$ for all $\xi \in \mathbf{R}^n\setminus\{0\}$ and whose Fourier transform is supported in the annulus $\frac{6}{7}\le|\xi|\le 2$ and is equal to 1 on the annulus $1\le|\xi|\le\frac{14}{7}$.
[*Hint:* Set $\widehat{\Psi}(\xi)=\eta(\xi)\big(\sum_{k\in\mathbf{Z}}|\eta(2^{-k}\xi)|^2\big)^{-1/2}$ for a suitable $\eta\in\mathscr{C}_0^\infty(\mathbf{R}^n)$.]

6.1.2. Suppose that Ψ is an integrable function on $\mathbf{R}^n$ that satisfies $|\widehat{\Psi}(\xi)|\le B\min(|\xi|^{\varepsilon},|\xi|^{-\varepsilon'})$ for some $\varepsilon',\varepsilon>0$. Show that for some constant $C_{\varepsilon,\varepsilon'}<\infty$ we have

$$\sup_{\xi\in\mathbf{R}^n}\left(\int_0^\infty|\widehat{\Psi}(t\xi)|^2\frac{dt}{t}\right)^{\frac12}+\sup_{\xi\in\mathbf{R}^n}\Big(\sum_{j\in\mathbf{Z}}|\widehat{\Psi}(2^{-j}\xi)|^2\Big)^{\frac12}\le C_{\varepsilon,\varepsilon'}B.$$

6.1.3. Let Ψ be an integrable function on $\mathbf{R}^n$ with mean value zero that satisfies

$$|\Psi(x)|\le B(1+|x|)^{-n-\varepsilon},\qquad\int_{\mathbf{R}^n}|\Psi(x-y)-\Psi(x)|\,dx\le B|y|^{\varepsilon'},$$

for some $B,\varepsilon',\varepsilon>0$ and for all $y\neq0$.
(a) Prove that $|\widehat{\Psi}(\xi)|\le c_{n,\varepsilon,\varepsilon'}B\min(|\xi|^{\min(\frac{\varepsilon}{2},1)},|\xi|^{-\varepsilon})$ for some constant $c_{n,\varepsilon,\varepsilon'}$ and conclude that (6.1.4) holds for $p=2$.
(b) Deduce the validity of (6.1.4) and (6.1.5).
(c) If $\varepsilon<1$ and the assumption $|\Psi(x)|\le B(1+|x|)^{-n-\varepsilon}$ is weakened to $|\Psi(x)|\le B|x|^{-n-\varepsilon}$ for all $x\in\mathbf{R}^n$, then show that $|\widehat{\Psi}(\xi)|\le c_{n,\varepsilon,\varepsilon'}B\min(|\xi|^{\frac{\varepsilon}{2}},|\xi|^{-\varepsilon})$ and thus (6.1.4) and (6.1.5) are valid.
[*Hint:* Part (a): Make use of the identity

$$\widehat{\Psi}(\xi)=\int_{\mathbf{R}^n}e^{-2\pi ix\cdot\xi}\Psi(x)\,dx=-\int_{\mathbf{R}^n}e^{-2\pi ix\cdot\xi}\Psi(x-y)\,dx,$$

where $y=\frac12\frac{\xi}{|\xi|^2}$ when $|\xi|\ge1$. For $|\xi|\le1$ use the mean value property of Ψ to write $\widehat{\Psi}(\xi)=\int_{|x|\le1}\Psi(x)(e^{-2\pi ix\cdot\xi}-1)\,dx$ and split the integral in the regions $|x|\le1$ and $|x|\ge1$. Part (b): If $\vec{K}$ is defined by (6.1.13), then control the $\ell^2(\mathbf{Z})$ norm by the $\ell^1(\mathbf{Z})$ norm to prove (6.1.16). Then split the sum $\sum_{j\in\mathbf{Z}}\int_{|x|\ge2|y|}\big|\Psi_{2^{-j}}(x-y)-\Psi_{2^{-j}}(x)\big|\,dx$ into the parts $\sum_{2^j\le|y|^{-1}}$ and $\sum_{2^j>|y|^{-1}}$. Part (c): Notice that when $\varepsilon<1$, we have $|\int_{|x|\le1}\Psi(x)(e^{-2\pi ix\cdot\xi}-1)\,dx|\le C_nB|\xi|^{\frac{\varepsilon}{2}}$.]

6.1.4. Let Ψ be an integrable function on $\mathbf{R}^n$ with mean value zero that satisfies

$$|\Psi(x)|\le B(1+|x|)^{-n-\varepsilon},\qquad\int_{\mathbf{R}^n}|\Psi(x-y)-\Psi(x)|\,dx\le B|y|^{\varepsilon'},$$

for some $B, \varepsilon', \varepsilon > 0$ and for all $y \neq 0$. Let $\Psi_t(x) = t^{-n}\Psi(x/t)$. (a) Prove that there are constants c_n, c_n' such that

$$\left(\int_0^\infty |\Psi_t(x)|^2 \frac{dt}{t}\, dx\right)^{\frac{1}{2}} \le c_n B\, |x|^{-n},$$

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}} \int_{|x|\ge 2|y|} \left(\int_0^\infty |\Psi_t(x-y) - \Psi_t(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}} dx \le c_n' B.$$

(b) Show that there exist constants C_n, C_n' such that for all $1 < p < \infty$ and for all $f \in L^p(\mathbf{R}^n)$ we have

$$\left\| \left(\int_0^\infty |f * \Psi_t|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)} \le C_n B \max(p, (p-1)^{-1}) \big\|f\big\|_{L^p(\mathbf{R}^n)}$$

and also for all $f \in L^1(\mathbf{R}^n)$ we have

$$\left\| \left(\int_0^\infty |f * \Psi_t|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \right\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n' B \big\|f\big\|_{L^1(\mathbf{R}^n)}.$$

(c) Under the additional hypothesis that $0 < \int_0^\infty |\widehat{\Psi}(t\xi)|^2 \frac{dt}{t} = c_0$ for all $\xi \in \mathbf{R}^n \setminus \{0\}$, prove that for all $f \in L^p(\mathbf{R}^n)$ we have

$$\big\|f\big\|_{L^p(\mathbf{R}^n)} \le C_n'' B \max(p, (p-1)^{-1}) \left\| \left(\int_0^\infty |f * \Psi_t|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)}$$

[*Hint:* Part (a): Use the Cauchy-Schwarz inequality to obtain

$$\int_{|x|\ge 2|y|} \left(\int_0^\infty |\Psi_t(x-y) - \Psi_t(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}} dx$$
$$\le c_n |y|^{-\frac{\varepsilon}{2}} \left(\int_{|x|\ge 2|y|} |x|^{n+\varepsilon} \int_0^\infty |\Psi_t(x-y) - \Psi_t(x)|^2 \frac{dt}{t}\, dx\right)^{\frac{1}{2}},$$

and split the integral on the right into the regions $t \le |y|$ and $t > |y|$. In the second region use that Ψ is bounded to replace the square by the first power. Part (b): Use Exercise 6.1.2 and part (a) of Exercise 6.1.3 and to deduce the inequality when $p = 2$. Then apply Theorem 5.6.1. Part (c): Prove the inequality first for $f \in \mathscr{S}(\mathbf{R}^n)$ using duality.]

6.1.5. Prove the following generalization of Theorem 6.1.2. Let $A > 0$. Suppose that $\{K_j\}_{j\in\mathbf{Z}}$ is a sequence of locally integrable functions on $\mathbf{R}^n \setminus \{0\}$ that satisfies

$$\sup_{x\neq 0} |x|^n \Big(\sum_{j\in\mathbf{Z}} |K_j(x)|^2\Big)^{\frac{1}{2}} \le A,$$

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}} \int_{|x|\ge 2|y|} \Big(\sum_{j\in\mathbf{Z}} |K_j(x-y) - K_j(x)|^2\Big)^{\frac{1}{2}} dx \le A < \infty,$$

and for each $j \in \mathbf{Z}$ there is a number L_j such that

$$\lim_{\varepsilon_k \downarrow 0} \int_{\varepsilon_k \le |y| \le 1} K_j(y)\, dy = L_j\,.$$

If the K_j coincide with tempered distributions W_j that satisfy

$$\sum_{j \in \mathbf{Z}} |\widehat{W_j}(\xi)|^2 \le B^2\,,$$

then the operator

$$f \to \Big(\sum_{j \in \mathbf{Z}} |K_j * f|^2 \Big)^{\frac{1}{2}}$$

maps $L^p(\mathbf{R}^n)$ to itself and is weak type $(1,1)$ norms at most multiples of $A+B$.

6.1.6. Suppose that $\mathscr{H}$ is a Hilbert space with inner product $\langle \cdot\,,\,\cdot \rangle_{\mathscr{H}}$. Let $A > 0$ and $1 < p < \infty$. Suppose that an operator T from $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n, \mathscr{H})$ is a multiple of an isometry, that is,

$$\big\|T(g)\big\|_{L^2(\mathbf{R}^n,\mathscr{H})} = A\big\|g\big\|_{L^2(\mathbf{R}^n)}$$

for all $g \in L^2(\mathbf{R}^n, \mathscr{H})$. Then the inequality $\|T(f)\|_{L^p(\mathbf{R}^n,\mathscr{H})} \le C_p \|f\|_{L^p(\mathbf{R}^n)}$ for all $f \in \mathscr{S}(\mathbf{R}^n)$ implies

$$\big\|f\big\|_{L^{p'}(\mathbf{R}^n)} \le C_{p'} A^{-2} \big\|T(f)\big\|_{L^{p'}(\mathbf{R}^n,\mathscr{H})}$$

for all in $f \in \mathscr{S}(\mathbf{R}^n)$.
$\big[$*Hint:* Use the inner product structure and polarization to obtain

$$A^2 \bigg| \int_{\mathbf{R}^n} f(x)\overline{g(x)}\, dx \bigg| = \bigg| \int_{\mathbf{R}^n} \Big\langle T(f)(x), T(g)(x) \Big\rangle_{\mathscr{H}} dx \bigg|$$

and then argue as in the proof of inequality (6.1.8).$\big]$

6.1.7. Suppose that $\{m_j\}_{j \in \mathbf{Z}}$ is a sequence of bounded functions supported in the intervals $[2^j, 2^{j+1}]$. Let $T_j(f) = (\widehat{f} m_j)^{\vee}$ be the corresponding multiplier operators. Assume that for all sequences of functions $\{f_j\}_j$ the vector-valued inequality

$$\bigg\| \Big(\sum_j |T_j(f_j)|^2 \Big)^{\frac{1}{2}} \bigg\|_{L^p} \le A_p \bigg\| \Big(\sum_j |f_j|^2 \Big)^{\frac{1}{2}} \bigg\|_{L^p}$$

is valid for some $1 < p < \infty$. Prove there is a $C_p > 0$ such that for all finite subsets S of $\mathbf{Z}$ we have

$$\bigg\| \sum_{j \in S} m_j \bigg\|_{\mathscr{M}_p} \le C_p A_p.$$

$\big[$*Hint:* Use that $\big\langle \sum_{j \in S} T_j(f), g \big\rangle = \sum_{j \in S} \big\langle \Delta_j^{\#} T_j(f), \Delta_j^{\#}(g) \big\rangle$.$\big]$

6.1.8. Let m be a bounded function on $\mathbf{R}^n$ that is supported in the annulus $1 \le |\xi| \le 2$ and define $T_j(f) = \big(\widehat{f}(\xi)m(2^{-j}\xi)\big)^{\vee}$. Suppose that the square function $f \mapsto \big(\sum_{j\in\mathbf{Z}} |T_j(f)|^2\big)^{1/2}$ is bounded on $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$. Show that for every finite subset S of the integers we have

$$\Big\| \sum_{j\in S} T_j(f) \Big\|_{L^p(\mathbf{R}^n)} \le C_{p,n} \big\| f \big\|_{L^p(\mathbf{R}^n)}$$

for some constant $C_{p,n}$ independent of S.

6.1.9. Fix a nonzero Schwartz function h on the line whose Fourier transform is supported in the interval $\big[-\frac{1}{8}, \frac{1}{8}\big]$. For $\{a_j\}$ a sequence of numbers, set

$$f(x) = \sum_{j=1}^{\infty} a_j e^{2\pi i 2^j x} h(x)\,.$$

Prove that for all $1 < p < \infty$ there exists a constant C_p such that

$$\|f\|_{L^p(\mathbf{R})} \le C_p \Big(\sum_j |a_j|^2\Big)^{\frac{1}{2}} \|h\|_{L^p}\,.$$

$\big[$*Hint:* Write $f = \sum_{j=1}^{\infty} \Delta_j(a_j e^{2\pi i 2^j(\cdot)} h)$, where Δ_j is given by convolution with $\varphi_{2^{-j}}$ for some φ whose Fourier transform is supported in the interval $\big[\frac{6}{8}, \frac{10}{8}\big]$ and is equal to 1 on $\big[\frac{7}{8}, \frac{9}{8}\big]$. Then use (6.1.21).$\big]$

6.1.10. Let Ψ be a Schwartz function whose Fourier transform is supported in the annulus $\frac{1}{2} \le |\xi| \le 2$ and that satisfies (6.1.7). Define a Schwartz function Φ by setting

$$\widehat{\Phi}(\xi) = \begin{cases} \sum_{j\le 0} \widehat{\Psi}(2^{-j}\xi) & \text{when } \xi \neq 0, \\ 1 & \text{when } \xi = 0. \end{cases}$$

Let S_0 be the operator given by convolution with Φ. Let $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n)$. Show that

$$\big\|f\big\|_{L^p} \approx \big\|S_0(f)\big\|_{L^p} + \Big\| \Big(\sum_{j=1}^{\infty} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p}.$$

$\big[$*Hint:* Use Theorem 6.1.2 together with the identity $S_0 + \sum_{j=1}^{\infty} \Delta_j = I$.$\big]$

6.2 Two Multiplier Theorems

We now return to the spaces $\mathscr{M}_p$ introduced in Section 2.5. We seek sufficient conditions on L^∞ functions defined on $\mathbf{R}^n$ to be elements of $\mathscr{M}_p$. In this section we are concerned with two fundamental theorems that provide such sufficient conditions.

These are the Marcinkiewicz and the Hörmander–Mihlin multiplier theorems. Both multiplier theorems are consequences of the Littlewood–Paley theory discussed in the previous section.

Using the dyadic decomposition of $\mathbf{R}^n$, we can write any L^∞ function m as the sum

$$m = \sum_{\mathbf{j}\in\mathbf{Z}^n} m\chi_{R_{\mathbf{j}}} \qquad \text{a.e.,}$$

where $\mathbf{j} = (j_1,\dots,j_n)$, $R_{\mathbf{j}} = I_{j_1}\times\cdots\times I_{j_n}$, and $I_k = [2^k,2^{k+1})\bigcup(-2^{k+1},-2^k]$. For $\mathbf{j}$ in $\mathbf{Z}^n$ we set $m_{\mathbf{j}} = m\chi_{R_{\mathbf{j}}}$. A consequence of the ideas developed so far is the following characterization of $\mathscr{M}_p(\mathbf{R}^n)$ in terms of a vector-valued inequality.

Proposition 6.2.1. *Let $m\in L^\infty(\mathbf{R}^n)$ and let $m_{\mathbf{j}} = m\chi_{R_{\mathbf{j}}}$. Then m lies in $\mathscr{M}_p(\mathbf{R}^n)$, that is, for some c_p we have*

$$\big\|(\widehat{f}m)^\vee\big\|_{L^p} \le c_p\big\|f\big\|_{L^p}, \qquad f\in L^p(\mathbf{R}^n),$$

if and only if for some $C_p>0$ we have

$$\bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|(\widehat{f_{\mathbf{j}}}m_{\mathbf{j}})^\vee|^2\Big)^{\frac12}\bigg\|_{L^p} \le C_p\bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|f_{\mathbf{j}}|^2\Big)^{\frac12}\bigg\|_{L^p} \tag{6.2.1}$$

for all sequences of functions $f_{\mathbf{j}}$ in $L^p(\mathbf{R}^n)$.

Proof. Suppose that $m\in\mathscr{M}_p(\mathbf{R}^n)$. Exercise 5.6.1 gives the first inequality below

$$\bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|(\chi_{R_{\mathbf{j}}}m\widehat{f_{\mathbf{j}}})^\vee|^2\Big)^{\frac12}\bigg\|_{L^p} \le C_p\bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|(m\widehat{f_{\mathbf{j}}})^\vee|^2\Big)^{\frac12}\bigg\|_{L^p} \le C_p\bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|f_{\mathbf{j}}|^2\Big)^{\frac12}\bigg\|_{L^p},$$

while the second inequality follows from Theorem 5.5.1. (Observe that when $p=q$ in Theorem 5.5.1, then $C_{p,q}=1$.) Conversely, suppose that (6.2.1) holds for all sequences of functions $f_{\mathbf{j}}$. Fix a function f and apply (6.2.1) to the sequence $(\widehat{f}\chi_{R_{\mathbf{j}}})^\vee$, where $R_{\mathbf{j}}$ is the dyadic rectangle indexed by $\mathbf{j}=(j_1,\dots,j_n)\in\mathbf{Z}^n$. We obtain

$$\bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|(\widehat{f}m\chi_{R_{\mathbf{j}}})^\vee|^2\Big)^{\frac12}\bigg\|_{L^p} \le C_p\bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}|(\widehat{f}\chi_{R_{\mathbf{j}}})^\vee|^2\Big)^{\frac12}\bigg\|_{L^p}.$$

Using Theorem 6.1.6, we obtain that the previous inequality is equivalent to the inequality

$$\big\|(\widehat{f}m)^\vee\big\|_{L^p} \le c_p\big\|f\big\|_{L^p},$$

which implies that $m\in\mathscr{M}_p(\mathbf{R}^n)$. □

6.2.1 The Marcinkiewicz Multiplier Theorem on $\mathbf{R}$

Proposition 6.2.1 suggests that the behavior of m on each dyadic rectangle R_j should play a crucial role in determining whether m is an L^p multiplier. The Marcinkiewicz multiplier theorem provides such sufficient conditions on m restricted to any dyadic rectangle R_j. Before stating this theorem, we illustrate its main idea via the following example. Suppose that m is a bounded function that vanishes near $-\infty$, that is differentiable at every point, and whose derivative is integrable. Then we may write

$$m(\xi) = \int_{-\infty}^{\xi} m'(t)\,dt = \int_{-\infty}^{+\infty} \chi_{[t,\infty)}(\xi) m'(t)\,dt\,,$$

from which it follows that for a Schwartz function f we have

$$(\widehat{f}m)^\vee = \int_{\mathbf{R}} (\widehat{f}\chi_{[t,\infty)})^\vee m'(t)\,dt.$$

Since the operators $f \mapsto (\widehat{f}\chi_{[t,\infty)})^\vee$ map $L^p(\mathbf{R})$ to itself independently of t, it follows that

$$\big\|(\widehat{f}m)^\vee\big\|_{L^p} \le C_p \big\|m'\big\|_{L^1} \big\|f\big\|_{L^p}\,,$$

thus yielding that m is in $\mathscr{M}_p(\mathbf{R})$. The next multiplier theorem is an improvement of this result and is based on the Littlewood–Paley theorem. We begin with the one-dimensional case, which already captures the main ideas.

Theorem 6.2.2. ***(Marcinkiewicz multiplier theorem)*** *Let* $m: \mathbf{R} \to \mathbf{R}$ *be a bounded function that is* $\mathscr{C}^1$ *in every dyadic set* $(2^j, 2^{j+1}) \bigcup (-2^{j+1}, -2^j)$ *for* $j \in \mathbf{Z}$*. Assume that the derivative* m' *of* m *satisfies*

$$\sup_j \left[\int_{-2^{j+1}}^{-2^j} |m'(\xi)|\,d\xi + \int_{2^j}^{2^{j+1}} |m'(\xi)|\,d\xi \right] \le A < \infty. \tag{6.2.2}$$

Then for all $1 < p < \infty$ *we have that* $m \in \mathscr{M}_p(\mathbf{R})$ *and for some* $C > 0$ *we have*

$$\big\|m\big\|_{\mathscr{M}_p(\mathbf{R})} \le C \max\big(p, (p-1)^{-1}\big)^6 \big(\big\|m\big\|_{L^\infty} + A\big). \tag{6.2.3}$$

Proof. Since the function m has an integrable derivative on $(2^j, 2^{j+1})$, it has bounded variation in this interval and hence it is a difference of two increasing functions. Therefore, m has left and right limits at the points 2^j and 2^{j+1}, and by redefining m at these points we may assume that m is right continuous at the points 2^j and left continuous at the points -2^j.

Set $I_j = [2^j, 2^{j+1}) \cup (-2^{j+1}, -2^j]$ and $I_j^+ = [2^j, 2^{j+1})$ whenever $j \in \mathbf{Z}$. Given an interval I in $\mathbf{R}$, we introduce an operator Δ_I defined by $\Delta_I(f) = (\widehat{f}\chi_I)^\vee$. With this notation $\Delta_{I_j^+}(f)$ is "half" of the operator $\Delta_j^\#$ introduced in the previous section. Given m as in the statement of the theorem, we write $m(\xi) = m_+(\xi) + m_-(\xi)$, where $m_+(\xi) = m(\xi)\chi_{\xi \ge 0}$ and $m_-(\xi) = m(\xi)\chi_{\xi<0}$. We show that both m_+ and m_-

are L^p multipliers. Since m' is integrable over all intervals of the form $[2^j,\xi]$ when $2^j \le \xi < 2^{j+1}$, the fundamental theorem of calculus gives

$$m(\xi) = m(2^j) + \int_{2^j}^{\xi} m'(t)\,dt, \qquad \text{for } 2^j \le \xi < 2^{j+1},$$

from which it follows that for a Schwartz function f on the real line we have

$$m(\xi)\widehat{f}(\xi)\chi_{I_j^+}(\xi) = m(2^j)\widehat{f}(\xi)\chi_{I_j^+}(\xi) + \int_{2^j}^{2^{j+1}} \widehat{f}(\xi)\chi_{[t,\infty)}(\xi)\chi_{I_j^+}(\xi)\,m'(t)\,dt\,.$$

We therefore obtain the identity

$$(\widehat{f}\chi_{I_j}m_+)^\vee = (\widehat{f}m\chi_{I_j^+})^\vee = m(2^j)\Delta_{I_j^+}(f) + \int_{2^j}^{2^{j+1}} \Delta_{[t,\infty)}\Delta_{I_j^+}(f)\,m'(t)\,dt\,,$$

which implies that

$$|(\widehat{f}\chi_{I_j}m_+)^\vee| \le \|m\|_{L^\infty}|\Delta_{I_j^+}(f)| + A^{\frac{1}{2}}\bigg(\int_{2^j}^{2^{j+1}} \big|\Delta_{[t,\infty)}\Delta_{I_j^+}(f)\big|^2\,|m'(t)|\,dt\bigg)^{\frac{1}{2}},$$

using the hypothesis (6.2.2). Taking $\ell^2(\mathbf{Z})$ norms we obtain

$$\begin{aligned}\Big(\sum_{j\in\mathbf{Z}}|(\widehat{f}\chi_{I_j}m_+)^\vee|^2\Big)^{\frac{1}{2}} &\le \|m\|_{L^\infty}\Big(\sum_{j\in\mathbf{Z}}|\Delta_{I_j^+}(f)|^2\Big)^{\frac{1}{2}} \\ &\quad + A^{\frac{1}{2}}\bigg(\int_0^\infty \big|\Delta_{[t,\infty)}\Delta^{\#}_{[\log_2 t]}(f)\big|^2\,|m'(t)|\,dt\bigg)^{\frac{1}{2}}.\end{aligned}$$

Exercise 5.6.2 gives

$$\begin{aligned}&A^{\frac{1}{2}}\bigg\|\bigg(\int_0^\infty \big|\Delta_{[t,\infty)}\Delta^{\#}_{[\log_2 t]}(f)\big|^2|m'(t)|\,dt\bigg)^{\frac{1}{2}}\bigg\|_{L^p} \\ &\qquad \le C\max(p,(p-1)^{-1})A^{\frac{1}{2}}\bigg\|\bigg(\int_0^\infty \big|\Delta^{\#}_{[\log_2 t]}(f)\big|^2|m'(t)|\,dt\bigg)^{\frac{1}{2}}\bigg\|_{L^p},\end{aligned}$$

while the hypothesis on m' implies the inequality

$$\bigg\|\bigg(\sum_{j\in\mathbf{Z}}\big|\Delta_{I_j^+}(f)\big|^2\int_{I_j^+}|m'(t)|\,dt\bigg)^{\frac{1}{2}}\bigg\|_{L^p} \le A^{\frac{1}{2}}\bigg\|\Big(\sum_j|\Delta_{I_j^+}(f)|^2\Big)^{\frac{1}{2}}\bigg\|_{L^p}.$$

Using Theorem 6.1.5 we obtain that

$$\bigg\|\Big(\sum_j|\Delta_{I_j^+}(f)|^2\Big)^{\frac{1}{2}}\bigg\|_{L^p} \le C'\max(p,(p-1)^{-1})^2\big\|\big(\widehat{f}\chi_{(0,\infty)}\big)^\vee\big\|_{L^p},$$

and the latter is at most a constant multiple of $\max(p,(p-1)^{-1})^3\big\|f\big\|_{L^p}$. Putting things together we deduce that

$$\Big\|\Big(\sum_j |(\widehat{f}\chi_{I_j}m_+)^\vee|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C''\max(p,(p-1)^{-1})^4\big(A+\|m\|_{L^\infty}\big)\big\|f\big\|_{L^p}\,, \tag{6.2.4}$$

from which we obtain the estimate

$$\big\|(\widehat{f}m_+)^\vee\big\|_{L^p} \le C\max(p,(p-1)^{-1})^6\big(A+\|m\|_{L^\infty}\big)\big\|f\big\|_{L^p}\,,$$

using the lower estimate of Theorem 6.1.5. This proves (6.2.3) for m_+. A similar argument also works for m_-, and this concludes the proof by summing the corresponding estimates for m_+ and m_-. □

We remark that the same proof applies under the more general assumption that m is a function of bounded variation on every interval $[2^j,2^{j+1}]$ and $[-2^{j+1},-2^j]$. In this case the measure $|m'(t)|\,dt$ should be replaced by the total variation $|dm(t)|$ of the Lebesgue–Stieltjes measure $dm(t)$.

Example 6.2.3. Any bounded function that is constant on dyadic intervals is an L^p multiplier. Also, the function

$$m(\xi) = |\xi|2^{-[\log_2|\xi|]}$$

is an L^p multiplier on $\mathbf{R}$ for $1<p<\infty$.

6.2.2 The Marcinkiewicz Multiplier Theorem on $\mathbf{R}^n$

We now extend this theorem on $\mathbf{R}^n$. As usual we denote the coordinates of a point $\xi\in\mathbf{R}^n$ by $(\xi_1,\dots,\xi_n)$. We recall the notation $I_j=(-2^{j+1},-2^j]\bigcup[2^j,2^{j+1})$ and $R_{\mathbf{j}}=I_{j_1}\times\cdots\times I_{j_n}$ whenever $\mathbf{j}=(j_1,\dots,j_n)\in\mathbf{Z}^n$.

Theorem 6.2.4. *Let m be a bounded function on $\mathbf{R}^n$ such that for all $\alpha=(\alpha_1,\dots,\alpha_n)$ with $|\alpha_1|,\dots,|\alpha_n|\le 1$ the derivatives $\partial^\alpha m$ are continuous up to the boundary of $R_{\mathbf{j}}$ for all $\mathbf{j}\in\mathbf{Z}^n$. Assume that there is a constant $A<\infty$ such that for all partitions $\{s_1,\dots,s_k\}\cup\{r_1,\dots,r_\ell\}=\{1,2,\dots,n\}$ with $n=k+\ell$ and all $\xi\in R_{\mathbf{j}}$ we have*

$$\sup_{\xi_{r_1}\in I_{j_{r_1}}}\cdots\sup_{\xi_{r_\ell}\in I_{j_{r_\ell}}}\int_{I_{j_{s_1}}}\cdots\int_{I_{j_{s_k}}}\big|(\partial_{s_1}\cdots\partial_{s_k}m)(\xi_1,\dots,\xi_n)\big|\,d\xi_{s_k}\cdots d\xi_{s_1}\le A \tag{6.2.5}$$

for all $\mathbf{j}=(j_1,\dots,j_n)\in\mathbf{Z}^n$. Then m is in $\mathscr{M}_p(\mathbf{R}^n)$ whenever $1<p<\infty$ and there is a constant $C_n<\infty$ such that

$$\|m\|_{\mathscr{M}_p(\mathbf{R}^n)}\le C_n\big(A+\|m\|_{L^\infty}\big)\max\big(p,(p-1)^{-1}\big)^{6n}. \tag{6.2.6}$$

Proof. We prove this theorem only in dimension $n = 2$, since the general case presents no substantial differences but only some notational inconvenience. We decompose the given function m as

$$m(\xi) = m_{++}(\xi) + m_{-+}(\xi) + m_{+-}(\xi) + m_{--}(\xi),$$

where each of the last four terms is supported in one of the four quadrants. For instance, the function $m_{+-}(\xi_1, \xi_2)$ is supported in the quadrant $\xi_1 \geq 0$ and $\xi_2 < 0$. As in the one-dimensional case, we work with each of these pieces separately. By symmetry we choose to work with m_{++} in the following argument.

Using the fundamental theorem of calculus, we obtain the following simple identity, valid for $2^{j_1} \leq \xi_1 < 2^{j_1+1}$ and $2^{j_2} \leq \xi_2 < 2^{j_2+1}$:

$$\begin{aligned} m(\xi_1, \xi_2) \;=\; & m(2^{j_1}, 2^{j_2}) + \int_{2^{j_1}}^{\xi_1} (\partial_1 m)(t_1, 2^{j_2})\, dt_1 \\ & + \int_{2^{j_2}}^{\xi_2} (\partial_2 m)(2^{j_1}, t_2)\, dt_2 \\ & + \int_{2^{j_1}}^{\xi_1} \int_{2^{j_2}}^{\xi_2} (\partial_1 \partial_2 m)(t_1, t_2)\, dt_2\, dt_1 \,. \end{aligned} \tag{6.2.7}$$

We introduce operators $\Delta_I^{(r)}$, $r \in \{1, 2\}$, acting in the rth variable (with the other variable remaining fixed) given by multiplication on the Fourier transform side by the characteristic function of the interval I. Likewise, we introduce operators $\Delta_j^{\#(r)}$, $r \in \{1, 2\}$ (also acting in the rth variable), given by multiplication on the Fourier transform side by the characteristic function of the set $(-2^{j+1}, -2^j] \bigcup [2^j, 2^{j+1})$. For notational convenience, for a given Schwartz function f we write

$$f_{++} = \big(\widehat{f}\chi_{(0,\infty)^2}\big)^\vee,$$

and likewise we define f_{+-}, f_{-+}, and f_{--}.

Multiplying both sides of (6.2.7) by the function $\widehat{f}\chi_{R_\mathbf{j}}\chi_{(0,\infty)^2}$ and taking inverse Fourier transforms yields

$$\begin{aligned} (\widehat{f}\chi_{R_\mathbf{j}} m_{++})^\vee = \; & m(2^{j_1}, 2^{j_2}) \Delta_{j_1}^{\#(1)} \Delta_{j_2}^{\#(2)}(f_{++}) \\ & + \int_{2^{j_1}}^{2^{j_1+1}} \Delta_{j_2}^{\#(2)} \Delta_{[t_1,\infty)}^{(1)} \Delta_{j_1}^{\#(1)}(f_{++})\, (\partial_1 m)(t_1, 2^{j_2})\, dt_1 \\ & + \int_{2^{j_2}}^{2^{j_2+1}} \Delta_{j_1}^{\#(1)} \Delta_{[t_2,\infty)}^{(2)} \Delta_{j_2}^{\#(2)}(f_{++})\, (\partial_2 m)(2^{j_1}, t_2)\, dt_2 \\ & + \int_{2^{j_1}}^{2^{j_1+1}} \int_{2^{j_2}}^{2^{j_2+1}} \Delta_{[t_1,\infty)}^{(1)} \Delta_{j_1}^{\#(1)} \Delta_{[t_2,\infty)}^{(2)} \Delta_{j_2}^{\#(2)}(f_{++})\, (\partial_1 \partial_2 m)(t_1, t_2)\, dt_2\, dt_1 \,. \end{aligned} \tag{6.2.8}$$

We apply the Cauchy–Schwarz inequality in the last three terms of (6.2.8) with respect to the measures $|(\partial_1 m)(t_1, 2^{j_2})|dt_1$, $|(\partial_2 m)(2^{j_1}, t_2)|dt_2$, $|(\partial_1 \partial_2 m)(t_1, t_2)|\, dt_2 dt_1$ and we use hypothesis (6.2.5) to deduce

$$
\begin{aligned}
&\big|(\widehat{f}\chi_{R_{\mathbf{j}}} m_{++})^{\vee}\big| \le \|m\|_{L^\infty}\big|\Delta_{j_1}^{\#(1)}\Delta_{j_2}^{\#(2)}(f_{++})\big| \\
&+ A^{\frac{1}{2}}\bigg(\int_{2^{j_1}}^{2^{j_1+1}} \big|\Delta_{j_2}^{\#(2)}\Delta_{[t_1,\infty)}^{(1)}\Delta_{j_1}^{\#(1)}(f_{++})\big|^2 \,|(\partial_1 m)(t_1,2^{j_2})|\,dt_1\bigg)^{\frac{1}{2}} \\
&+ A^{\frac{1}{2}}\bigg(\int_{2^{j_2}}^{2^{j_2+1}} \big|\Delta_{j_1}^{\#(1)}\Delta_{[t_2,\infty)}^{(2)}\Delta_{j_2}^{\#(2)}(f_{++})\big|^2 \,|(\partial_2 m)(2^{j_1},t_2)|\,dt_2\bigg)^{\frac{1}{2}} \\
&+ A^{\frac{1}{2}}\bigg(\int_{2^{j_1}}^{2^{j_1+1}}\int_{2^{j_2}}^{2^{j_2+1}} \big|\Delta_{[t_1,\infty)}^{(1)}\Delta_{j_1}^{\#(1)}\Delta_{[t_2,\infty)}^{(2)}\Delta_{j_2}^{\#(2)}(f_{++})\big|^2 \,|(\partial_1\partial_2 m)(t_1,t_2)|\,dt_2\,dt_1\bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Both sides of the preceding inequality are sequences indexed by $\mathbf{j} \in \mathbf{Z}^2$. We apply $\ell^2(\mathbf{Z}^2)$ norms and use Minkowski's inequality to deduce the pointwise estimate

$$
\begin{aligned}
&\Big(\sum_{\mathbf{j}\in\mathbf{Z}^2}\big|(\widehat{f}\chi_{R_{\mathbf{j}}} m_{++})^{\vee}\big|^2\Big)^{\frac{1}{2}} \le \|m\|_{L^\infty}\Big(\sum_{\mathbf{j}\in\mathbf{Z}^2}\big|\Delta_{\mathbf{j}}^{\#}(f_{++})\big|^2\Big)^{\frac{1}{2}} \\
&\quad + A^{\frac{1}{2}}\bigg(\int_0^\infty\int_0^\infty \big|\Delta_{[t_1,\infty)}^{(1)}\Delta_{[\log_2 t_2]}^{\#(2)}\Delta_{[\log_2 t_1]}^{\#(1)}(f_{++})\big|^2 \big|(\partial_1 m)(t_1,2^{[\log_2 t_2]})\big|\,dt_1 d\nu(t_2)\bigg)^{\frac{1}{2}} \\
&\quad + A^{\frac{1}{2}}\bigg(\int_0^\infty\int_0^\infty \big|\Delta_{[t_2,\infty)}^{(2)}\Delta_{[\log_2 t_1]}^{\#(1)}\Delta_{[\log_2 t_2]}^{\#(2)}(f_{++})\big|^2 \big|(\partial_2 m)(2^{[\log_2 t_1]},t_2)\big|\,d\nu(t_1) dt_2\bigg)^{\frac{1}{2}} \\
&\quad + A^{\frac{1}{2}}\bigg(\int_0^\infty\int_0^\infty \big|\Delta_{[t_1,\infty)}^{(1)}\Delta_{[t_2,\infty)}^{(2)}\Delta_{[\log_2 t_1]}^{\#(1)}\Delta_{[\log_2 t_2]}^{\#(2)}(f_{++})\big|^2 \big|(\partial_1\partial_2 m)(t_1,t_2)\big|\,dt_1 dt_2\bigg)^{\frac{1}{2}},
\end{aligned}
$$

where ν is the counting measure $\sum_{j\in\mathbf{Z}}\delta_{2^j}$ defined by $\nu(A) = \#\{j \in \mathbf{Z} :\ 2^j \in A\}$ for subsets A of $(0,\infty)$. We now take $L^p(\mathbf{R}^2)$ norms and we estimate separately the contribution of each of the four terms on the right side. Using Exercise 5.6.2 we obtain

$$
\begin{aligned}
&\bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^2}\big|(\widehat{f}\chi_{R_{\mathbf{j}}} m_{++})^{\vee}\big|^2\Big)^{\frac{1}{2}}\bigg\|_{L^p} \le \|m\|_{L^\infty}\bigg\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^2}\big|\Delta_{\mathbf{j}}^{\#}(f_{++})\big|^2\Big)^{\frac{1}{2}}\bigg\|_{L^p} \\
&\quad + C_2 A^{\frac{1}{2}}\max\big(p,(p-1)^{-1}\big)^2 \\
&\quad\times\bigg\{\bigg\|\bigg(\int_0^\infty\int_0^\infty \big|\Delta_{[\log_2 t_2]}^{\#(2)}\Delta_{[\log_2 t_1]}^{\#(1)}(f_{++})\big|^2 \big|(\partial_1 m)(t_1,2^{[\log_2 t_2]})\big|\,dt_1\,d\nu(t_2)\bigg)^{\frac{1}{2}}\bigg\|_{L^p} \\
&\qquad + \bigg\|\bigg(\int_0^\infty\int_0^\infty \big|\Delta_{[\log_2 t_1]}^{\#(1)}\Delta_{[\log_2 t_2]}^{\#(2)}(f_{++})\big|^2 \big|(\partial_2 m)(2^{[\log_2 t_1]},t_2)\big|\,d\nu(t_1)\,dt_2\bigg)^{\frac{1}{2}}\bigg\|_{L^p} \\
&\qquad + \bigg\|\bigg(\int_0^\infty\int_0^\infty \big|\Delta_{[\log_2 t_1]}^{\#(1)}\Delta_{[\log_2 t_2]}^{\#(2)}(f_{++})\big|^2 \big|(\partial_1\partial_2 m)(t_1,t_2)\big|\,dt_1 dt_2\bigg)^{\frac{1}{2}}\bigg\|_{L^p}\bigg\}.
\end{aligned}
$$

But the functions $(t_1,t_2) \mapsto \Delta^{\#(1)}_{[\log_2 t_1]}\Delta^{\#(2)}_{[\log_2 t_2]}(f_{++})$ are constant on products of intervals of the form $[2^{j_1},2^{j_1+1})\times[2^{j_2},2^{j_2+1})$; hence using hypothesis (6.2.5) again we deduce the estimate

$$\begin{aligned}
&\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^2}\big|(\widehat{f}\chi_{R_\mathbf{j}}m_{++})^\vee\big|^2\Big)^{\frac12}\Big\|_{L^p(\mathbf{R}^2)}\\
&\qquad\le C_2\big(\|m\|_{L^\infty}+A\big)\max\big(p,(p-1)^{-1}\big)^2\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^2}\big|\Delta^\#_\mathbf{j}(f_{++})\big|^2\Big)^{\frac12}\Big\|_{L^p(\mathbf{R}^2)}\\
&\qquad\le C_2\big(\|m\|_{L^\infty}+A\big)\max\big(p,(p-1)^{-1}\big)^6\big\|(\widehat{f}\chi_{(0,\infty)^2})^\vee\big\|_{L^p(\mathbf{R}^2)}\\
&\qquad\le C_2\big(\|m\|_{L^\infty}+A\big)\max\big(p,(p-1)^{-1}\big)^8\big\|f\big\|_{L^p(\mathbf{R}^2)},
\end{aligned}$$

where the penultimate estimate follows from Theorem 6.1.6 and the last estimate by the boundedness of the Hilbert transform (Theorem 5.1.7). We now appeal to inequality (6.1.29) which yields the required estimate for the $L^p(\mathbf{R}^2)$ norm of $(\widehat{f}m_{++})^\vee$. A similar argument also works for the remaining parts of m_{+-}, m_{-+}, m_{--}, and summing concludes the proof of (6.2.6).

The analogous estimate on $\mathbf{R}^n$ is

$$\Big\|\Big(\sum_{\mathbf{j}\in\mathbf{Z}^n}\big|(\widehat{f}\chi_{R_\mathbf{j}}m_{+\cdots+})^\vee\big|^2\Big)^{\frac12}\Big\|_{L^p(\mathbf{R}^n)}\le C_n\big(\|m\|_{L^\infty}+A\big)\max\big(p,(p-1)^{-1}\big)^{4n}\big\|f\big\|_{L^p(\mathbf{R}^n)}$$

which is obtained in a similar fashion. Using (6.1.29), this implies that

$$\big\|(\widehat{f}m_{+\cdots+})^\vee\big\|_{L^p(\mathbf{R}^n)}\le C_n\big(\|m\|_{L^\infty}+A\big)\max\big(p,(p-1)^{-1}\big)^{6n}\big\|f\big\|_{L^p(\mathbf{R}^n)}.$$

A similar inequality holds when some (or all) $+$'s are replaced by $-$'s. □

We now give a condition that implies (6.2.5) and is well suited for a variety of applications.

Corollary 6.2.5. *Let m be a bounded $\mathscr{C}^n$ function defined away from the coordinate axes on $\mathbf{R}^n$. Assume that for all $k\in\{1,\dots,n\}$, all distinct $j_1,\dots,j_k\in\{1,2,\dots,n\}$, and all $\xi_r\in\mathbf{R}\setminus\{0\}$ for $r\notin\{j_1,\dots,j_k\}$ we have*

$$\big|(\partial_{j_1}\cdots\partial_{j_k}m)(\xi_1,\dots,\xi_n)\big|\le A\,|\xi_{j_1}|^{-1}\cdots|\xi_{j_k}|^{-1}. \tag{6.2.9}$$

Then m satisfies (6.2.6).

Proof. Simply observe that condition (6.2.9) implies (6.2.5). □

Example 6.2.6. The following are examples of functions that satisfy the hypotheses of Corollary 6.2.5:

$$m_1(\xi) = \frac{\xi_1}{\xi_1 + i(\xi_2^2 + \cdots + \xi_n^2)},$$

$$m_2(\xi) = \frac{|\xi_1|^{\alpha_1} \cdots |\xi_n|^{\alpha_n}}{(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^{\alpha/2}},$$

where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \alpha$, $\alpha_j > 0$,

$$m_3(\xi) = \frac{\xi_2 \xi_3^2}{i\xi_1 + \xi_2^2 + \xi_3^4}.$$

The functions m_1 and m_2 are defined on $\mathbf{R}^n \setminus \{0\}$ and m_3 on $\mathbf{R}^3 \setminus \{0\}$.

The previous examples and many other examples that satisfy the hypothesis (6.2.9) of Corollary 6.2.5 are invariant under a set of dilations in the following sense: suppose that there exist $k_1, \dots, k_n \in \mathbf{R}^+$ and $s \in \mathbf{R}$ such that the smooth function m on $\mathbf{R}^n \setminus \{0\}$ satisfies

$$m(\lambda^{k_1}\xi_1, \dots, \lambda^{k_n}\xi_n) = \lambda^{is} m(\xi_1, \dots, \xi_n)$$

for all $\xi_1, \dots, \xi_n \in \mathbf{R}$ and $\lambda > 0$. Then m satisfies condition (6.2.9). Indeed, differentiation gives

$$\lambda^{\alpha_1 k_1 + \cdots + \alpha_n k_n} \partial^\alpha m(\lambda^{k_1}\xi_1, \dots, \lambda^{k_n}\xi_n) = \lambda^{is} \partial^\alpha m(\xi_1, \dots, \xi_n)$$

for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. Now for every $\xi \in \mathbf{R}^n \setminus \{0\}$ pick the unique $\lambda_\xi > 0$ such that $(\lambda_\xi^{k_1}\xi_1, \dots, \lambda_\xi^{k_n}\xi_n) \in \mathbf{S}^{n-1}$. Then $\lambda_\xi^{k_j \alpha_j} \le |\xi_j|^{-\alpha_j}$, and it follows that

$$|\partial^\alpha m(\xi_1, \dots, \xi_n)| \le \Big[\sup_{\mathbf{S}^{n-1}} |\partial^\alpha m| \Big] \lambda_\xi^{\alpha_1 k_1 + \cdots + \alpha_n k_n} \le C_\alpha |\xi_1|^{-\alpha_1} \cdots |\xi_n|^{-\alpha_n}.$$

6.2.3 The Mihlin–Hörmander Multiplier Theorem on $\mathbf{R}^n$

We now discuss another multiplier theorem that also requires decay of derivatives. We will consider the situation where each differentiation produces uniform decay in all variables, quantitatively expressed via the condition

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{6.2.10}$$

for each multi-index α. The decay can also be expressed in terms of a square integrable estimate that has the form

$$\left(\int_{R<|\xi|<2R} |\partial_\xi^\alpha m(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \le C'_\alpha R^{\frac{n}{2}-|\alpha|} < \infty \tag{6.2.11}$$

for all multi-indices α and all $R>0$. Obviously (6.2.10) implies (6.2.11)

Theorem 6.2.7. *Let $m(\xi)$ be a complex-valued bounded function on $\mathbf{R}^n \setminus \{0\}$ that satisfies for some $A<\infty$*

$$\left(\int_{R<|\xi|<2R} |\partial_\xi^\alpha m(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \le A R^{\frac{n}{2}-|\alpha|} < \infty \tag{6.2.12}$$

for all multi-indices $|\alpha| \le [n/2]+1$ and all $R>0$.
Then for all $1<p<\infty$, m lies in $\mathscr{M}_p(\mathbf{R}^n)$ and the following estimate is valid:

$$\|m\|_{\mathscr{M}_p} \le C_n \max(p,(p-1)^{-1})\big(A+\|m\|_{L^\infty}\big)\,. \tag{6.2.13}$$

Moreover, the operator $f \mapsto (\widehat{f}m)^\vee$ maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ with norm at most a dimensional constant multiple of $A+\|m\|_{L^\infty}$.

We remark that in most applications, condition (6.2.12) appears in the form

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|}, \tag{6.2.14}$$

which should be, in principle, easier to verify.

Proof. Since m is a bounded function, the operator given by convolution with $W = m^\vee$ is bounded on $L^2(\mathbf{R}^n)$. To prove that this operator maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$, it suffices to prove that the distribution W coincides with a function K on $\mathbf{R}^n \setminus \{0\}$ that satisfies Hörmander's condition.

Let $\widehat{\zeta}$ be a smooth function supported in the annulus $\frac{1}{2} \le |\xi| \le 2$ such that

$$\sum_{j\in\mathbf{Z}} \widehat{\zeta}(2^{-j}\xi) = 1, \qquad \text{when } \xi \neq 0.$$

Set $m_j(\xi) = m(\xi)\widehat{\zeta}(2^{-j}\xi)$ for $j \in \mathbf{Z}$ and $K_j = m_j^\vee$. We begin by observing that $\sum_{-N}^N K_j$ converges to W in $\mathscr{S}'(\mathbf{R}^n)$. Indeed, for all $\varphi \in \mathscr{S}(\mathbf{R}^n)$ we have

$$\left\langle \sum_{j=-N}^N K_j, \varphi \right\rangle = \left\langle \sum_{j=-N}^N m_j, \varphi^\vee \right\rangle \to \langle m, \varphi^\vee \rangle = \langle W, \varphi \rangle .$$

We set $n_0 = [\frac{n}{2}]+1$. We claim that there is a constant $\widetilde{C}_n$ such that

$$\sup_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} |K_j(x)|\,(1+2^j|x|)^{\frac{1}{4}}\,dx \le \widetilde{C}_n A\,, \tag{6.2.15}$$

$$\sup_{j\in\mathbf{Z}} 2^{-j} \int_{\mathbf{R}^n} |\nabla K_j(x)|\,(1+2^j|x|)^{\frac{1}{4}}\,dx \le \widetilde{C}_n A\,. \tag{6.2.16}$$

To prove (6.2.15) we multiply and divide the integrand in (6.2.15) by the expression $(1+2^j|x|)^{n_0}$. Applying the Cauchy–Schwarz inequality to $|K_j(x)|\,(1+2^j|x|)^{n_0}$ and $(1+2^j|x|)^{-n_0+\frac{1}{4}}$, we control the integral in (6.2.15) by the product

$$\left(\int_{\mathbf{R}^n}|K_j(x)|^2(1+2^j|x|)^{2n_0}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbf{R}^n}(1+2^j|x|)^{-2n_0+\frac{1}{2}}\,dx\right)^{\frac{1}{2}}. \tag{6.2.17}$$

We now note that $-2n_0+\frac{1}{2}<-n$, and hence the second factor in (6.2.17) is equal to a constant multiple of $2^{-jn/2}$. To estimate the first integral in (6.2.17) we use the simple fact that

$$(1+2^j|x|)^{n_0}\le C(n)\sum_{|\gamma|\le n_0}|(2^jx)^\gamma|.$$

We now have that the expression inside the supremum in (6.2.15) is controlled by

$$C'(n)2^{-jn/2}\sum_{|\gamma|\le n_0}\left(\int_{\mathbf{R}^n}|K_j(x)|^2 2^{2j|\gamma|}|x^\gamma|^2\,dx\right)^{\frac{1}{2}}, \tag{6.2.18}$$

which, by Plancherel's theorem, is equal to

$$2^{-jn/2}\sum_{|\gamma|\le n_0}C_\gamma 2^{j|\gamma|}\left(\int_{\mathbf{R}^n}|(\partial^\gamma m_j)(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \tag{6.2.19}$$

for some constants C_γ.

For multi-indices $\delta=(\delta_1,\dots,\delta_n)$ and $\gamma=(\gamma_1,\dots,\gamma_n)$ we introduce the notation $\delta\le\gamma$ to mean $\delta_j\le\gamma_j$ for all $j=1,\dots,n$. For any $|\gamma|\le n_0$ we use Leibniz's rule to obtain for some constants $C_{\delta,\gamma}$

$$\begin{aligned}\left(\int_{\mathbf{R}^n}|(\partial^\gamma m_j)(\xi)|^2\,d\xi\right)^{\frac{1}{2}} &\le \sum_{\delta\le\gamma}C_{\delta,\gamma}\left(\int_{\mathbf{R}^n}\left|2^{-j|\gamma-\delta|}(\partial_\xi^{\gamma-\delta}\widehat{\zeta})(2^{-j}\xi)(\partial_\xi^\delta m)(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}\\ &\le \sum_{\delta\le\gamma}C_{\delta,\gamma}2^{-j|\gamma|}2^{j|\delta|}\left(\int_{2^{j-1}\le|\xi|\le 2^{j+1}}\left|(\partial_\xi^\delta m)(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}\\ &\le \sum_{\delta\le\gamma}C_{\delta,\gamma}2^{-j|\gamma|}2^{j|\delta|}\,2A\,2^{jn/2}2^{-j|\delta|}\\ &=\widetilde{C}_n A\,2^{jn/2}2^{-j|\gamma|},\end{aligned}$$

which inserted in (6.2.19) and combined with (6.2.18) yields (6.2.15). To obtain (6.2.16) we repeat the same argument for every derivative $\partial_r K_j$. Since the Fourier transform of $(\partial_r K_j)(x)x^\gamma$ is equal to a constant multiple of $\partial^\gamma\big(\xi_r m(\xi)\widehat{\zeta}(2^{-j}\xi)\big)$, we observe that the extra factor 2^{-j} in (6.2.16) can be combined with ξ_r to write $2^{-j}\partial^\gamma\big(\xi_r m(\xi)\widehat{\zeta}(2^{-j}\xi)\big)$ as $\partial^\gamma\big(m(\xi)\widehat{\zeta_r}(2^{-j}\xi)\big)$, where $\widehat{\zeta_r}(\xi)=\xi_r\widehat{\zeta}(\xi)$. The previous calculation with $\widehat{\zeta_r}$ replacing $\widehat{\zeta}$ can then be used to complete the proof of (6.2.16).

We now show that for all $x \neq 0$, the series $\sum_{j\in\mathbf{Z}} K_j(x)$ converges to a function, which we denote by $K(x)$. Indeed, as a consequence of (6.2.15) we have that

$$(1+2^j\delta)^{\frac{1}{4}} \int_{|x|\geq\delta} |K_j(x)|\,dx \leq \widetilde{C}_n A\,,$$

for any $\delta > 0$, which implies that the function $\sum_{j>0} |K_j(x)|$ is integrable away from the origin and satisfies $\int_{\delta\leq|x|\leq 2\delta} \sum_{j>0} |K_j(x)|\,dx < \infty$. Now note that (6.2.15) also holds with $-\frac{1}{4}$ in place of $\frac{1}{4}$. Using this observation we obtain

$$(1+2^j 2\delta)^{-\frac{1}{4}} \int_{|x|\leq 2\delta} |K_j(x)|\,dx \leq \int_{|x|\leq 2\delta} |K_j(x)|(1+2^j|x|)^{-\frac{1}{4}}\,dx \leq \widetilde{C}_n A\,,$$

and from this it follows that $\int_{\delta\leq|x|\leq 2\delta} \sum_{j\leq 0} |K_j(x)|\,dx < \infty$.

We conclude that the series $\sum_{j\in\mathbf{Z}} K_j(x)$ converges a.e. on $\mathbf{R}^n \setminus \{0\}$ to a function $K(x)$ that coincides with the distribution $W = m^\vee$ on $\mathbf{R}^n \setminus \{0\}$ and satisfies

$$\sup_{\delta>0} \int_{\delta\leq|x|\leq 2\delta} |K(x)|\,dx < \infty\,.$$

We now prove that the function $K = \sum_{j\in\mathbf{Z}} K_j$ (defined on $\mathbf{R}^n \setminus \{0\}$) satisfies Hörmander's condition. It suffices to prove that for all $y \neq 0$ we have

$$\sum_{j\in\mathbf{Z}} \int_{|x|\geq 2|y|} |K_j(x-y) - K_j(x)|\,dx \leq 2C_n' A\,. \tag{6.2.20}$$

Fix a $y \in \mathbf{R}^n \setminus \{0\}$ and pick a $k \in \mathbf{Z}$ such that $2^{-k} \leq |y| \leq 2^{-k+1}$. The part of the sum in (6.2.20) where $j > k$ is bounded by

$$\begin{aligned}
\sum_{j>k} \int_{|x|\geq 2|y|} |K_j(x-y)| + |K_j(x)|\,dx &\leq 2\sum_{j>k} \int_{|x|\geq|y|} |K_j(x)|\,dx \\
&\leq 2\sum_{j>k} \int_{|x|\geq|y|} |K_j(x)| \frac{(1+2^j|x|)^{\frac{1}{4}}}{(1+2^j|x|)^{\frac{1}{4}}}\,dx \\
&\leq \sum_{j>k} \frac{2\widetilde{C}_n A}{(1+2^j|y|)^{\frac{1}{4}}} \\
&\leq \sum_{j>k} \frac{2\widetilde{C}_n A}{(1+2^j 2^{-k})^{\frac{1}{4}}} = C_n' A\,,
\end{aligned}$$

where we used (6.2.15). The part of the sum in (6.2.20) where $j \leq k$ is bounded by

$$\begin{aligned}
&\sum_{j\leq k} \int_{|x|\geq 2|y|} |K_j(x-y) - K_j(x)|\,dx \\
&\qquad\qquad \leq \sum_{j\leq k} \int_{|x|\geq 2|y|} \int_0^1 |-y\cdot\nabla K_j(x-\theta y)|\,d\theta\,dx
\end{aligned}$$

$$\le \int_0^1 \sum_{j\le k} 2^{-k+1} \int_{\mathbf{R}^n} |\nabla K_j(x-\theta y)|(1+2^j|x-\theta y|)^{\frac{1}{4}}\, dx\, d\theta$$

$$\le \int_0^1 \sum_{j\le k} 2^{-k+1} \widetilde{C}_n A 2^j\, d\theta \le C_n' A\,,$$

using (6.2.16). Hörmander's condition is satisfied for K, and we appeal to Theorem 5.3.3 to complete the proof of (6.2.13). □

Example 6.2.8. Let m be a smooth function away from the origin that is homogeneous of imaginary order, i.e., for some fixed τ real and all $\lambda > 0$ we have

$$m(\lambda\xi) = \lambda^{i\tau} m(\xi)\,. \tag{6.2.21}$$

Then m is an L^p Fourier multiplier for $1 < p < \infty$. Indeed, differentiating both sides of (6.2.21) with respect to ∂_ξ^α we obtain

$$\lambda^{|\alpha|}\partial_\xi^\alpha m(\lambda\xi) = \lambda^{i\tau}\partial_\xi^\alpha m(\xi)$$

and taking $\lambda = |\xi|^{-1}$, we deduce condition (6.2.14) with $C_\alpha = \sup_{|\theta|=1}|\partial^\alpha m(\theta)|$. An explicit example of such a function is $m(\xi) = |\xi|^{i\tau}$. Another example is

$$m_0(\xi_1,\xi_2,\xi_3) = \frac{\xi_1^2+\xi_2^2}{\xi_1^2 + i(\xi_2^2+\xi_3^2)}$$

which is homogeneous of degree zero and also smooth on $\mathbf{R}^n \setminus \{0\}$.

Example 6.2.9. Let z be a complex numbers with $\operatorname{Re} z \ge 0$. Then the functions

$$m_1(\xi) = \left(\frac{|\xi|^2}{1+|\xi|^2}\right)^z, \qquad m_2(\xi) = \left(\frac{1}{1+|\xi|^2}\right)^z$$

defined on $\mathbf{R}^n$ are L^p Fourier multipliers for $1 < p < \infty$. To prove this assertion for m_1, we verify condition (6.2.14). To achieve this, introduce the function on $\mathbf{R}^{n+1}$

$$M_1(\xi_1,\dots,\xi_n,t) = \left(\frac{|\xi_1|^2+\cdots+|\xi_n|^2}{t^2+|\xi_1|^2+\cdots+|\xi_n|^2}\right)^z = \left(\frac{|\xi|^2}{t^2+|\xi|^2}\right)^z,$$

where $\xi = (\xi_1,\dots,\xi_n)$. Then M is homogeneous of degree 0 and smooth on $\mathbf{R}^{n+1}\setminus\{0\}$. The derivatives $\partial^\beta M_1$ are homogeneous of degree $-|\beta|$ and by the calculation in the preceding example they satisfy $|\partial^\beta M_1(\xi,t)| \le C_\beta |(\xi,t)|^{-|\beta|}$, with $C_\beta = \sup_{|\theta|=1}|\partial^\beta M_1(\theta)|$, whenever $(\xi,t)\neq 0$ and β is a multi index of $n+1$ variables. In particular, taking $\beta = (\alpha,0)$, we obtain

$$\left|\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n} M_1(\xi_1,\dots,\xi_n,t)\right| \le \frac{C_\alpha}{(t^2+|\xi|^2)^{|\alpha|/2}}\,,$$

and setting $t = 1$ we deduce that $|\partial^\alpha m_1(\xi)| \le C_\alpha(1+|\xi|^2)^{-|\alpha|/2} \le C_\alpha|\xi|^{-|\alpha|}$.

For m_2 we introduce the function

$$M_2(\xi_1,\dots,\xi_n,t)=\left(\frac{1}{t^2+|\xi_1|^2+\cdots+|\xi_n|^2}\right)^z$$

on $\mathbf{R}^{n+1}$, which is homogeneous of degree $-2z$. Then the derivative $\partial^\beta M_2$ is homogeneous of degree $-|\beta|-2z$, hence it satisfies $|\partial^\beta M_2(\xi,t)|\le C_\beta|(\xi,t)|^{-|\beta|-2\mathrm{Re}\,z}$ for all multi-indices β of $n+1$ variables. In particular, taking $\beta=(\alpha,0)$, we obtain

$$\left|\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n}M_2(\xi_1,\dots,\xi_n,t)\right|\le\frac{C_\alpha}{(t^2+|\xi|^2)^{\frac{|\alpha|}{2}+\mathrm{Re}\,z}},$$

and setting $t=1$, we deduce $|\partial^\alpha m_2(\xi)|\le C_\alpha(1+|\xi|^2)^{-|\alpha|/2}\le C_\alpha|\xi|^{-|\alpha|}$, where in the first inequality we used that $\mathrm{Re}\,z\ge 0$.

We end this section by comparing Theorems 6.2.2 and 6.2.4 with Theorem 6.2.7. It is obvious that in dimension $n=1$, Theorem 6.2.2 is stronger than Theorem 6.2.7 in view of the inequality

$$\int_{2^j<|\xi|<2^{j+1}}|m'(\xi)|\,d\xi\le 2^{j/2}\left(\int_{2^j<|\xi|<2^{j+1}}|m'(\xi)|^2\,d\xi\right)^{\frac12},$$

which implies that (6.2.2) is weaker than (6.2.12). Note also that in Theorem 6.2.2 the multiplier m is not required to be differentiable at the points $\pm 2^j$. But in higher dimensions neither theorem includes the other. In Theorem 6.2.4 the multiplier is allowed to be singular on a set of measure zero but is required to be differentiable in every variable, i.e., to be at least $\mathscr{C}^n$ in the complement of this null set. In Theorem 6.2.7, the multiplier is only allowed to be singular only at the origin, but it is assumed to be $\mathscr{C}^{[n/2]+1}$, requiring almost half the differentiability called for by condition (6.2.9). It should be noted that both theorems have their shortcomings. In particular, they are not L^p sensitive, i.e., delicate enough to detect whether m is a bounded Fourier multiplier on some L^p but not on some other L^q.

Exercises

6.2.1. Let $\psi(\xi)$ be a smooth function supported in $[3/4,2]\cup[-2,-3/4]$ and equal to 1 on $[1,3/2]\cup[-3/2,-1]$ that satisfies $\sum_{j\in\mathbf{Z}}\psi(2^{-j}\xi)=1$ for all $\xi\neq 0$. Let $1\le k\le n$. Prove that $m\in\mathscr{M}_p(\mathbf{R}^n)$ if and only if (6.2.1) is satisfied with $m_{\mathbf{j}}(\xi)$ replaced by the function $m(\xi)\psi(2^{-j_1}\xi_1)\cdots\psi(2^{-j_k}\xi_k)$.
[*Hint:* To prove one direction, partition $\mathbf{Z}^k$ in 2^k sets such that for every $\mathbf{j}=(j_1,\dots,j_k)$ in each of these sets, j_i has a fixed remainder modulo 2. For the other direction, use Theorem 6.1.6 in the variables $x_1,\dots,x_k$. Also use the inequality $\|f\|_{L^p(\mathbf{R}^n)}\le C_p\|(\sum_{\mathbf{j}\in\mathbf{Z}^k}|(\widehat{f}\chi_{R_{\mathbf{j}}})^\vee|^2)^{1/2}\|_{L^p(\mathbf{R}^n)}$, $R_{\mathbf{j}}=([-2,-\frac12]\cup[\frac12,2])^k\times\mathbf{R}^{n-k}$, which can be derived by duality from the identity $\sum_{\mathbf{j}\in\mathbf{Z}^k}\chi_{R_{\mathbf{j}}}=2^k$.]

6.2.2. Let φ be a smooth function on the real line supported in the interval $[-1,1]$. Let $\psi(t)$ be a smooth function on the real line that is equal to 1 when $|t| \ge 10$ and vanishes when $|t| \le 9$. Show that for the function $m(\xi_1,\xi_2) = e^{i\xi_2^2/\xi_1}\varphi(\xi_2)\psi(\xi_1)$ lies in $\mathscr{M}_p(\mathbf{R}^2)$, $1<p<\infty$, using Theorem 6.2.4. Also show that Theorem 6.2.7 does not apply.

6.2.3. Consider the differential operators

$$\begin{aligned} L_1 &= \partial_1 - \partial_2^2 + \partial_3^4\,, \\ L_2 &= \partial_1 + \partial_2^2 + \partial_3^2\,. \end{aligned}$$

Prove that for every $1<p<\infty$ there exists a constant $C_p<\infty$ such that for all Schwartz functions f on $\mathbf{R}^3$ we have

$$\begin{aligned} \big\|\partial_2\partial_3^2 f\big\|_{L^p} &\le C_p\big\|L_1(f)\big\|_{L^p}\,, \\ \big\|\partial_1 f\big\|_{L^p} &\le C_p\big\|L_2(f)\big\|_{L^p}\,. \end{aligned}$$

$\big[$*Hint:* Use Corollary 6.2.5 and the idea of Example 6.2.6.$\big]$

6.2.4. Suppose that $m(\xi)$ is a real-valued function that satisfies either (6.2.9) or $|\partial^\alpha m(\xi)| \le C_\alpha|\xi|^{-|\alpha|}$ for all multi-indices α with $|\alpha| \le [\frac{n}{2}]+1$ and all $\xi \in \mathbf{R}^n\setminus\{0\}$. Show that $e^{im(\xi)}$ lies in $\mathscr{M}_p(\mathbf{R}^n)$ for any $1<p<\infty$.
$\big[$*Hint:* Prove by induction and use that

$$\partial^\alpha\big(e^{im(\xi)}\big) = e^{im(\xi)} \sum_{\substack{l_j\ge 0,\beta^j\le\alpha \\ l_1\beta^1+\cdots+l_k\beta^k=\alpha}} c_{\beta^1,\dots,\beta^k}(\partial^{\beta^1}m(\xi))^{l_1}\cdots(\partial^{\beta^k}m(\xi))^{l_k}\,,$$

where the sum is taken over all partitions of the multi-index α as a linear combination of multi-indices β^j with coefficients $l_j \in \mathbf{Z}^+\cup\{0\}$.$\big]$

6.2.5. Suppose that $\varphi(\xi)$ is a smooth function on $\mathbf{R}^n$ that vanishes in a neighborhood of the origin and is equal to 1 in a neighborhood of infinity. Prove that the function $e^{i\xi_j|\xi|^{-1}}\varphi(\xi)$ is in $\mathscr{M}_p(\mathbf{R}^n)$ for $1<p<\infty$.

6.2.6. Let $\tau,\tau_1,\dots,\tau_n$ be real numbers and $\rho_1,\dots,\rho_n$ be even natural numbers. Prove that the following functions are L^p multipliers on $\mathbf{R}^n$ for $1<p<\infty$:

$$\begin{aligned} &|\xi_1|^{i\tau_1}\cdots|\xi_n|^{i\tau_n}\,, \\ &(|\xi_1|^{\rho_1}+\cdots+|\xi_n|^{\rho_n})^{i\tau}\,, \\ &(|\xi_1|^{-\rho_1}+|\xi_2|^{-\rho_2})^{i\tau}\,. \end{aligned}$$

6.2.7. Let $\widehat{\zeta}(\xi)$ be a smooth function on $\mathbf{R}^n$ is supported in a compact set that does not contain the origin and let a_j be a bounded sequence of complex numbers. Prove that the function

$$m(\xi) = \sum_{j\in\mathbf{Z}} a_j \widehat{\zeta}(2^{-j}\xi)$$

is in $\mathscr{M}_p(\mathbf{R}^n)$ for all $1 < p < \infty$.

6.2.8. Let $\widehat{\zeta}(\xi)$ be a smooth function on $\mathbf{R}^n$ supported in a compact set that does not contain the origin and let $\Delta_j^{\zeta}(f) = \big(\widehat{f}(\xi)\widehat{\zeta}(2^{-j}\xi)\big)^{\vee}$. Show that the operator

$$f \to \sup_{N\in\mathbf{Z}} \Big| \sum_{j<N} \Delta_j^{\zeta}(f) \Big|$$

is bounded on $L^p(\mathbf{R})$ when $1 < p < \infty$.
$\big[$*Hint:* Pick a Schwartz function φ satisfying $\sum_{j\in\mathbf{Z}} \widehat{\varphi}(2^{-j}\xi) = 1$ on $\mathbf{R}^n \setminus \{0\}$ with $\widehat{\varphi}(\xi)$ supported in $\frac{6}{7} \le |\xi| \le 2$. Then $\Delta_k^{\varphi}\Delta_j^{\zeta} = 0$ if $|j-k| < c_0$ and we have

$$\sum_{j<N} \Delta_j^{\zeta} = \sum_{k<N+c_0} \Delta_k^{\varphi} \sum_{j<N} \Delta_j^{\zeta} = \sum_{k<N+c_0} \Delta_k^{\varphi} \sum_{j} \Delta_j^{\zeta} - \sum_{k<N+c_0} \Delta_k^{\varphi} \sum_{j\ge N} \Delta_j^{\zeta},$$

which is a finite sum plus a term controlled by a multiple of the operator

$$f \mapsto M\Big(\sum_{j\in\mathbf{Z}} \Delta_j^{\zeta}(f)\Big),$$

where M is the Hardy–Littlewood maximal function.$\big]$

6.2.9. Let Ψ be a Schwartz function whose Fourier transform is real-valued, supported in a compact set that does not contain the origin, and satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1 \qquad \text{when } \xi \neq 0.$$

Let Δ_j be the Littlewood–Paley operator associated with Ψ. Prove that

$$\Big\| \sum_{|j|<N} \Delta_j(g) - g \Big\|_{L^p} \to 0$$

as $N \to \infty$ for all functions $g \in \mathscr{S}(\mathbf{R}^n)$. Deduce that Schwartz functions whose Fourier transforms have compact supports that do not contain the origin are dense in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.
$\big[$*Hint:* Use the result of Exercise 6.2.8 and the Lebesgue dominated convergence theorem.$\big]$

6.3 Applications of Littlewood–Paley Theory

We now turn our attention to some important applications of Littlewood–Paley theory. We are interested in obtaining bounds for singular and maximal operators. These bounds are obtained by controlling the corresponding operators by quadratic expressions.

6.3.1 Estimates for Maximal Operators

One way to control the maximal operator $\sup_k |T_k(f)|$ is by introducing a good averaging function φ and using the majorization

$$\begin{aligned} \sup_k |T_k(f)| &\le \sup_k |T_k(f) - f * \varphi_{2^{-k}}| + \sup_k |f * \varphi_{2^{-k}}| \\ &\le \Big(\sum_k |T_k(f) - f * \varphi_{2^{-k}}|^2\Big)^{\frac{1}{2}} + C_\varphi M(f) \end{aligned} \tag{6.3.1}$$

for some constant C_φ depending on φ. We apply this idea to prove the following theorem.

Theorem 6.3.1. *Let m be a bounded function on $\mathbf{R}^n$ that is $\mathscr{C}^1$ in a neighborhood of the origin and satisfies $m(0) = 1$ and $|m(\xi)| \le C|\xi|^{-\varepsilon}$ for some $C, \varepsilon > 0$ and all $\xi \neq 0$. For each $k \in \mathbf{Z}$ define $T_k(f)(x) = (\widehat{f}(\xi)m(2^{-k}\xi))^\vee(x)$. Then there is a constant C_n such that for all L^2 functions f on $\mathbf{R}^n$ we have*

$$\Big\| \sup_{k\in\mathbf{Z}} |T_k(f)| \Big\|_{L^2} \le C_n \big\| f \big\|_{L^2}. \tag{6.3.2}$$

Proof. Select a Schwartz function φ such that $\widehat{\varphi}(0) = 1$. Then there are positive constants C_1 and C_2 such that $|m(\xi) - \widehat{\varphi}(\xi)| \le C_1|\xi|^{-\varepsilon}$ for $|\xi|$ away from zero and $|m(\xi) - \widehat{\varphi}(\xi)| \le C_2|\xi|$ for $|\xi|$ near zero. These two inequalities imply that

$$\sum_k |m(2^{-k}\xi) - \widehat{\varphi}(2^{-k}\xi)|^2 \le C_3 < \infty,$$

from which the L^2 boundedness of the operator

$$f \mapsto \Big(\sum_k |T_k(f) - f * \varphi_{2^{-k}}|^2\Big)^{1/2}$$

follows easily. Using estimate (6.3.1) and the well-known L^2 estimate for the Hardy–Littlewood maximal function, we obtain (6.3.2). □

If $m(\xi)$ is the characteristic function of a rectangle with sides parallel to the axes, this result can be extended to L^p.

Theorem 6.3.2. *Let $1 < p < \infty$ and let U be the n-fold product of open intervals that contain zero. For each $k \in \mathbf{Z}$ define $T_k(f)(x) = (\widehat{f}(\xi)\chi_U(2^{-k}\xi))^\vee(x)$. Then there is a constant $C_{p,n}$ such that for all L^p functions f on $\mathbf{R}^n$ we have*

$$\big\| \sup_{k\in\mathbf{Z}} |T_k(f)| \big\|_{L^p(\mathbf{R}^n)} \le C_{p,n} \big\| f \big\|_{L^p(\mathbf{R}^n)}.$$

Proof. Let us fix an open annulus A whose interior contains the boundary of U and take a smooth function with compact support $\widehat{\psi}$ that vanishes in a neighborhood of zero and a neighborhood of infinity and is equal to 1 on the annulus A. Then the function $\widehat{\varphi} = (1-\widehat{\psi})\chi_U$ is Schwartz. Since $\chi_U = \chi_U\widehat{\psi} + \widehat{\varphi}$, it follows that for all $f \in L^p(\mathbf{R}^n)$ we have

$$T_k(f) = T_k(f) - f*\varphi_{2^{-k}} + f*\varphi_{2^{-k}} = T_k(f*\psi_{2^{-k}}) + f*\varphi_{2^{-k}}\,.$$

Taking the supremum over k and using Corollary 2.1.12 we obtain

$$\sup_{k\in\mathbf{Z}} |T_k(f)| \le \Big(\sum_k |T_k(f) - f*\varphi_{2^{-k}}|^2\Big)^{1/2} + C_\varphi M(f)\,. \tag{6.3.3}$$

The operator $T_k(f) - f*\varphi_{2^{-k}}$ is given by multiplication on the Fourier transform side by the multiplier

$$\chi_U(2^{-k}\xi) - \widehat{\varphi}(2^{-k}\xi) = \chi_U(2^{-k}\xi)\widehat{\psi}(2^{-k}\xi) = \chi_{2^kU}(\xi)\widehat{\psi}(2^{-k}\xi)\,.$$

Since $\{2^kU\}_{k\in\mathbf{Z}}$ is a measurable family of rectangles with sides parallel to the axes, Exercise 5.6.1(b) yields the following inequality:

$$\Big\| \Big(\sum_{k\in\mathbf{Z}} |T_k(f) - f*\varphi_{2^{-k}}|^2\Big)^{\frac12} \Big\|_{L^p} \le C_{p,n} \Big\| \Big(\sum_{k\in\mathbf{Z}} |f*\psi_{2^{-k}}|^2\Big)^{\frac12} \Big\|_{L^p}. \tag{6.3.4}$$

Since $f*\psi_{2^{-k}} = \Delta_j^\psi(f)$, estimate (6.1.4) of Theorem 6.1.2 yields that the expression on the right in (6.3.4) is controlled by a multiple of $\|f\|_{L^p}$. Taking L^p norms in (6.3.3) and using the L^p estimate for the square function yields the required conclusion. □

The following lacunary version of the Carleson–Hunt theorem is yet another indication of the powerful techniques of Littlewood–Paley theory.

Corollary 6.3.3. *(a) Let f be in $L^2(\mathbf{R}^n)$ and let Ω be an open set that contains the origin in $\mathbf{R}^n$. Then*

$$\lim_{k\to\infty} \int_{2^k\Omega} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi = f(x)$$

for almost all $x \in \mathbf{R}^n$.
(b) Let f be in $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$. Then

$$\lim_{k\to\infty} \int_{\substack{|\xi_1|<2^k \\ \cdots \\ |\xi_n|<2^k}} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi = f(x)$$

for almost all $x \in \mathbf{R}^n$.

Proof. Both limits exist everywhere for functions f in the Schwartz class. To obtain almost everywhere convergence for general f in L^p we appeal to Theorem 2.1.14. The required control of the corresponding maximal operator is a consequence of Theorem 6.3.1 with $m = \chi_\Omega$ in case (a) and Theorem 6.3.2 in case (b). □

6.3.2 Estimates for Singular Integrals with Rough Kernels

We now turn to another application of the Littlewood–Paley theory involving singular integrals.

Theorem 6.3.4. *Suppose that μ is a finite Borel measure on $\mathbf{R}^n$ with compact support that satisfies $|\widehat{\mu}(\xi)| \le B \min\big(|\xi|^{-b}, |\xi|^{b}\big)$ for some $b > 0$ and all $\xi \neq 0$. Define measures μ_j by setting $\widehat{\mu_j}(\xi) = \widehat{\mu}(2^{-j}\xi)$. Then the operator*

$$T_\mu(f)(x) = \sum_{j\in\mathbf{Z}} (f * \mu_j)(x)$$

is bounded on $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.

Proof. It is natural to begin with the L^2 boundedness of T_μ. The estimate on $\widehat{\mu}$ implies that

$$\sum_{j\in\mathbf{Z}} |\widehat{\mu}(2^{-j}\xi)| \le \sum_{j\in\mathbf{Z}} B \min\big(|2^{-j}\xi|^{b}, |2^{-j}\xi|^{-b}\big) \le C_b B < \infty . \tag{6.3.5}$$

The L^2 boundedness of T_μ is an immediate consequence of (6.3.5).

We now turn to the L^p boundedness of T_μ for $1 < p < \infty$. We fix a radial Schwartz function ψ whose Fourier transform is supported in the annulus $\frac{1}{2} < |\xi| < 2$ that satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\psi}(2^{-j}\xi) = 1 \tag{6.3.6}$$

whenever $\xi \neq 0$. We let $\psi_{2^{-k}}(x) = 2^{kn}\psi(2^k x)$, so that $\widehat{\psi_{2^{-k}}}(\xi) = \widehat{\psi}(2^{-k}\xi)$, and we observe that the identity

$$\mu_j = \sum_{k\in\mathbf{Z}} \mu_j * \psi_{2^{-j-k}}$$

is valid by taking Fourier transforms and using (6.3.6). We now define operators S_k by setting

$$S_k(f) \;=\; \sum_{j\in\mathbf{Z}} \mu_j * \psi_{2^{-j-k}} * f = \sum_{j\in\mathbf{Z}} (\mu * \psi_{2^{-k}})_{2^{-j}} * f .$$

Then for f in $\mathscr{S}$ we have that

$$T_\mu(f) = \sum_{j\in\mathbf{Z}} \mu_j * f = \sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \mu_j * \psi_{2^{-j-k}} * f = \sum_{k\in\mathbf{Z}} S_k(f) .$$

It suffices therefore to obtain L^p boundedness for the sum of the S_k's. We begin by investigating the L^2 boundedness of each S_k. Since the product $\widehat{\psi_{2^{-j-k}}}\,\widehat{\psi_{2^{-j'-k}}}$ is nonzero only when $j' \in \{j-1, j, j+1\}$, it follows that

$$\begin{aligned}
\big\|S_k(f)\big\|_{L^2}^2 &\le \sum_{j\in\mathbf{Z}}\sum_{j'\in\mathbf{Z}}\int_{\mathbf{R}^n} |\widehat{\mu_j}(\xi)\widehat{\mu_{j'}}(\xi)\widehat{\psi}(2^{-j-k}\xi)\widehat{\psi}(2^{-j'-k}\xi)|\,|\widehat{f}(\xi)|^2\,d\xi \\
&\le C_1\sum_{j\in\mathbf{Z}}\sum_{j'=j-1}^{j+1}\int_{|\xi|\approx 2^{j+k}} |\widehat{\mu_j}(\xi)\widehat{\mu_{j'}}(\xi)|\,|\widehat{f}(\xi)|^2\,d\xi \\
&\le C_2\sum_{j\in\mathbf{Z}}\int_{|\xi|\approx 2^{j+k}} B^2\min(|2^{-j}\xi|^b, |2^{-j}\xi|^{-b})^2|\widehat{f}(\xi)|^2\,d\xi \\
&\le C_3^2B^2 2^{-2|k|b}\sum_{j\in\mathbf{Z}}\int_{|\xi|\approx 2^{j+k}} |\widehat{f}(\xi)|^2\,d\xi \\
&= C_3^2B^2\,2^{-2|k|b}\big\|f\big\|_{L^2}^2\,.
\end{aligned}$$

We have therefore obtained that for all $k \in \mathbf{Z}$ and $f \in \mathscr{S}(\mathbf{R}^n)$ we have

$$\big\|S_k(f)\big\|_{L^2} \le C_3\,B\,2^{-b|k|}\big\|f\big\|_{L^2}\,. \tag{6.3.7}$$

We notice that for any $R > 0$ we have

$$\begin{aligned}
\int_{R\le|x|\le 2R}\sum_{j\in\mathbf{Z}}\big|\big(\mu*\psi_{2^{-k}}\big)_{2^{-j}}(x)\big|dx &= \sum_{j\in\mathbf{Z}}\int_{2^jR\le|x|\le 2^{j+1}R}\big|\big(\mu*\psi_{2^{-k}}\big)(x)\big|dx \\
&= \int_{\mathbf{R}^n}\big|\big(\mu*\psi_{2^{-k}}\big)(x)\big|dx \\
&\le \big\|\mu\big\|\,\|\psi\|_{L^1}\,,
\end{aligned}$$

thus condition (5.3.4) of Theorem 5.3.3 is satisfied.

Next we verify that the kernel of each S_k satisfies Hörmander's condition with constant at most a multiple of $(1+|k|)$. Fix $y \ne 0$. Then

$$\begin{aligned}
&\int_{|x|\ge 2|y|}\bigg|\sum_{j\in\mathbf{Z}}\Big((\mu*\psi_{2^{-k}})_{2^{-j}}(x-y)-(\mu*\psi_{2^{-k}})_{2^{-j}}(x)\Big)\bigg|dx \\
&\qquad\le \sum_{j\in\mathbf{Z}}\int_{|x|\ge 2|y|}2^{jn}\big|(\mu*\psi_{2^{-k}})(2^jx-2^jy)-(\mu*\psi_{2^{-k}})(2^jx)\big|\,dx \\
&\qquad= \sum_{j\in\mathbf{Z}} I_{j,k}(y)\,,
\end{aligned}$$

where

$$I_{j,k}(y) = \int_{|x|\ge 2^{j+1}|y|}\big|(\mu*\psi_{2^{-k}})(x-2^jy)-(\mu*\psi_{2^{-k}})(x)\big|\,dx\,.$$

We observe that $I_{j,k}(y) \le C_4\|\mu\|_{\mathscr{M}}$. Let $|\mu|$ be the total variation of μ. To obtain a more delicate estimate for $I_{j,k}(y)$ we argue as follows:

$$\begin{aligned}
I_{j,k}(y) \le & \int_{|x|\ge 2^{j+1}|y|} \int_{\mathbf{R}^n} \big|\psi_{2^{-k}}(x-2^j y - z) - \psi_{2^{-k}}(x-z)\big|\, d|\mu|(z)\, dx \\
&= \int_{\mathbf{R}^n} 2^{kn} \int_{|x|\ge 2^{j+1}|y|} \big|\psi(2^k x - 2^k z - 2^{j+k} y) - \psi(2^k x - 2^k z)\big|\, dx\, d|\mu|(z) \\
&\le C_5 \int_{|x|\ge 2^{j+1}|y|} \int_{\mathbf{R}^n} 2^{kn} 2^{j+k} |y| \big|\nabla\psi(2^k x - 2^k z - \theta)\big|\, d|\mu|(z)\, dx \\
&\le C_6 2^{j+k} \int_{\mathbf{R}^n} \int_{|x|\ge 2^{j+1}|y|} 2^{kn} |y| \big(1 + |2^k x - 2^k z - \theta|\big)^{-n-2} dx\, d|\mu|(z) \\
&= C_6 2^{j+k} |y| \int_{\mathbf{R}^n} \int_{|x|\ge 2^{j+k+1}|y|} \big(1 + |x - 2^k z - \theta|\big)^{-n-2} dx\, d|\mu|(z)\,,
\end{aligned}$$

where $|\theta| \le 2^{j+k}|y|$. Note that θ depends on j,k, and y. From this and from $I_{j,k}(y) \le C_4\|\mu\|_{\mathscr{M}}$ we obtain

$$I_{j,k}(y) \le C_7 \left\|\mu\right\|_{\mathscr{M}} \min\big(1, 2^{j+k}|y|\big), \tag{6.3.8}$$

which is valid for all j,k, and $y \ne 0$. To estimate the last double integral even more delicately, we consider the following two cases: $|x| \ge 2^{k+2}|z|$ and $|x| < 2^{k+2}|z|$. In the first case we have $|x - 2^k z - \theta| \ge \frac{1}{4}|x|$, given the fact that $|x| \ge 2^{j+k+1}|y|$. In the second case we have that $|x| \le 2^{k+2}R$, where $B(0,R)$ contains the support of μ. Applying these observations in the last double integral, we obtain the following estimate:

$$\begin{aligned}
I_{j,k}(y) \le\ & C_8 2^{j+k}|y| \int_{\mathbf{R}^n} \Bigg[\int_{\substack{|x|\ge 2^{j+k+1}|y| \\ |x|\ge 2^{k+2}|z|}} \frac{dx}{\big(1+\frac{1}{4}|x|\big)^{n+2}} + \int_{\substack{|x|\ge 2^{j+k+1}|y| \\ |x|< 2^{k+2}R}} dx \Bigg] d|\mu|(z) \\
\le\ & C_9 2^{j+k}|y| \|\mu\|_{\mathscr{M}} \left[\frac{1}{(2^{j+k}|y|)^2} + 0 \right] \\
=\ & C_9 (2^{j+k}|y|)^{-1} \left\|\mu\right\|_{\mathscr{M}},
\end{aligned}$$

provided $2^j|y| \ge 2R$. Combining this estimate with (6.3.8), we obtain

$$I_{j,k}(y) \le C_{10}\|\mu\|_{\mathscr{M}} \begin{cases} \min\big(1, 2^{j+k}|y|\big) & \text{for all } j,k \text{ and } y, \\ (2^{j+k}|y|)^{-1} & \text{when } 2^j|y| \ge 2R. \end{cases} \tag{6.3.9}$$

We now estimate $\sum_j I_{j,k}(y)$. When $2^k \ge (2R)^{-1}$ we use (6.3.9) to obtain

$$\begin{aligned}\sum_j I_{j,k}(y) \;&\le\; C_{10}\big\|\mu\big\|_{\mathscr{M}}\Big[\sum_{2^j\le\frac{1}{2^k|y|}} 2^{j+k}|y| + \sum_{\frac{1}{2^k|y|}\le 2^j\le\frac{2R}{|y|}} 1 + \sum_{2^j\ge\frac{2R}{|y|}} (2^{j+k}|y|)^{-1}\Big] \\ &\le C_{11}\big\|\mu\big\|_{\mathscr{M}}(|\log R| + |k|)\,.\end{aligned}$$

Also when $2^k < (2R)^{-1}$ we again use (6.3.9) to obtain

$$\sum_j I_{j,k}(y) \le C_{10}\big\|\mu\big\|_{\mathscr{M}}\Big[\sum_{2^j\le\frac{1}{2^k|y|}} 2^{j+k}|y| + \sum_{2^j\ge\frac{1}{2^k|y|}} (2^{j+k}|y|)^{-1}\Big] \le C_{12}\big\|\mu\big\|_{\mathscr{M}}\,,$$

since in the second sum we have $2^j|y| \ge 2^{-k} > 2R$, which justifies use of the corresponding estimate in (6.3.9). This gives

$$\sum_j I_{j,k}(y) \le C_{13}\big\|\mu\big\|_{\mathscr{M}}(1+|k|)\,, \tag{6.3.10}$$

where the constant C_{13} depends on the dimension and on R. We now use estimates (6.3.7) and (6.3.10) and Theorem 5.3.3 to obtain that each S_k maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ with constant at most

$$C_n(2^{-b|k|} + 1 + |k|)\big\|\mu\big\|_{\mathscr{M}} \le C_n(2+|k|)\big\|\mu\big\|_{\mathscr{M}}\,.$$

It follows from the Marcinkiewicz interpolation theorem (Theorem 1.3.2) that S_k maps $L^p(\mathbf{R}^n)$ to itself for $1<p<2$ with bound at most $C_{p,n}2^{-b|k|\theta_p}(1+|k|)^{1-\theta_p}$, when $\frac{1}{p} = \frac{\theta_p}{2} + 1 - \theta_p$. Summing over all $k \in \mathbf{Z}$, we obtain that T_μ maps $L^p(\mathbf{R}^n)$ to itself for $1<p<2$. The boundedness of T_μ for $p>2$ follows by duality. □

An immediate consequence of the previous result is the following.

Corollary 6.3.5. *Suppose that μ is a finite Borel measure on $\mathbf{R}^n$ with compact support that satisfies $|\widehat{\mu}(\xi)| \le B\min\big(|\xi|^{-b}, |\xi|^{b}\big)$ for some $b>0$ and all $\xi \ne 0$. Define measures μ_j by setting $\widehat{\mu_j}(\xi) = \widehat{\mu}(2^{-j}\xi)$. Then the square function*

$$G(f) = \Big(\sum_{j\in\mathbf{Z}} |\mu_j * f|^2\Big)^{\frac{1}{2}} \tag{6.3.11}$$

maps $L^p(\mathbf{R}^n)$ to itself whenever $1<p<\infty$.

Proof. To obtain the boundedness of the square function in (6.3.11) we use the Rademacher functions $r_j(t)$, introduced in Appendix C.1, reindexed so that their index set is the set of all integers (not the set of nonnegative integers). For each t we introduce the operators

$$T_\mu^t(f) = \sum_{j\in\mathbf{Z}} r_j(t)(f*\mu_j)\,.$$

Next we observe that for each t in $[0,1]$ the operators T_μ^t map $L^p(\mathbf{R}^n)$ to itself with the same constant as the operator T_μ, which is in particular independent of t. Using that the square function in (6.3.11) raised to the power p is controlled by a multiple of the quantity

$$\int_0^1 \Big|\sum_{j\in\mathbf{Z}} r_j(t)(f*\mu_j)\Big|^p dt\,,$$

a fact stated in Appendix C.2, we obtain the required conclusion by integrating over $\mathbf{R}^n$. □

6.3.3 An Almost Orthogonality Principle on L^p

Suppose that T_j are multiplier operators given by $T_j(f)=(\widehat{f}m_j)^\vee$, for some multipliers m_j. If the functions m_j have disjoint supports and they are bounded uniformly in j, then the operator

$$T=\sum_j T_j$$

is bounded on L^2. The following theorem gives an L^p analogue of this result.

Theorem 6.3.6. *Suppose that $1<p\le 2\le q<\infty$. Let m_j be Schwartz functions supported in the annuli $2^{j-1}\le|\xi|\le 2^{j+1}$ and let $T_j(f)=(\widehat{f}m_j)^\vee$. Suppose that the T_j's are uniformly bounded operators from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, i.e.,*

$$\sup_j \big\|T_j\big\|_{L^p\to L^q}=A<\infty\,.$$

Then for each $f\in L^p(\mathbf{R}^n)$, the series

$$T(f)=\sum_j T_j(f)$$

converges in the L^q norm and there exists a constant $C_{p,q,n}<\infty$ such that

$$\big\|T\big\|_{L^p\to L^q}\le C_{p,q,n}A. \tag{6.3.12}$$

Proof. Fix a radial Schwartz function φ whose Fourier transform $\widehat{\varphi}$ is real, equal to one on the annulus $\frac{1}{2}\le|\xi|\le 2$, and vanishes outside the annulus $\frac{1}{4}\le|\xi|\le 4$. We set $\varphi_{2^{-j}}(x)=2^{jn}\varphi(2^jx)$, so that $\widehat{\varphi_{2^{-j}}}$ is equal to 1 on the support of each m_j. Setting $\Delta_j(f)=f*\varphi_{2^{-j}}$, we observe that

$$T_j=\Delta_jT_j\Delta_j$$

for all $j\in\mathbf{Z}$. For a positive integer N we set

$$T^N=\sum_{|j|\le N}\Delta_jT_j\Delta_j\,.$$

Fix $f \in L^p(\mathbf{R}^n)$. Clearly for every N, $T^N(f)$ is in $L^q(\mathbf{R}^n)$. Using (6.1.21) we obtain

$$\begin{aligned}\big\|T^N(f)\big\|_{L^q} &= \Big\| \sum_{|j|\le N} \Delta_j T_j \Delta_j(f)\Big\|_{L^q} \\ &\le C'_q \Big\| \Big(\sum_{j\in\mathbf{Z}} |T_j\Delta_j(f)|^2\Big)^{\frac{1}{2}} \Big\|_{L^q} \\ &= C'_q \Big\| \sum_{j\in\mathbf{Z}} |T_j\Delta_j(f)|^2 \Big\|_{L^{q/2}}^{\frac{1}{2}} \\ &\le C'_q \Big(\sum_{j\in\mathbf{Z}} \Big\| |T_j\Delta_j(f)|^2 \Big\|_{L^{q/2}} \Big)^{\frac{1}{2}} \\ &= C'_q \Big(\sum_{j\in\mathbf{Z}} \big\|T_j\Delta_j(f)\big\|_{L^q}^2 \Big)^{\frac{1}{2}},\end{aligned}$$

where we used Minkowski's inequality, since $q/2 \ge 1$. Using the uniform boundedness of the T_j's from L^p to L^q, we deduce that

$$\begin{aligned}C'_q \Big(\sum_{j\in\mathbf{Z}} \big\|T_j\Delta_j(f)\big\|_{L^q}^2 \Big)^{\frac{1}{2}} &\le C'_q A \Big(\sum_{j\in\mathbf{Z}} \big\|\Delta_j(f)\big\|_{L^p}^2 \Big)^{\frac{1}{2}} \\ &= C'_q A \Big(\sum_{j\in\mathbf{Z}} \big\| |\Delta_j(f)|^2 \big\|_{L^{p/2}} \Big)^{\frac{1}{2}} \\ &\le C'_q A \Big(\Big\| \sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big\|_{L^{p/2}} \Big)^{\frac{1}{2}} \\ &= C'_q A \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \\ &\le C'_q C_p A \big\|f\big\|_{L^p(\mathbf{R}^n)},\end{aligned}$$

where we used the result of Exercise 1.1.5(b), since $p \le 2$, and Theorem 6.1.2. We conclude that the operators T^N are uniformly bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$.

If $\widehat{h}$ is compactly supported in a subset of $\mathbf{R}^n \setminus \{0\}$, then the sequence $T^N(h)$ becomes independent of N for N large enough and hence it is Cauchy in L^q. But in view of Exercise 6.2.9, the set of all such h is dense in $L^p(\mathbf{R}^n)$. Combining these two results with the uniform boundedness of the T^N's from L^p to L^q, a simple $\frac{\varepsilon}{3}$ argument gives that for all $f \in L^p$ the sequence $T^N(f)$ is Cauchy in L^q. Therefore, for all $f \in L^p$ the sequence $\{T^N(f)\}_N$ converges in L^q to some $T(f)$. Fatou's lemma gives

$$\big\|T(f)\big\|_{L^q} \le C'_q C_p A \big\|f\big\|_{L^p},$$

which proves (6.3.12). □

Exercises

6.3.1. (*The g-function*) Let $P_t(x) = \Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}} t(t^2+|x|^2)^{-\frac{n+1}{2}}$ be the Poisson kernel on $\mathbf{R}^n$.
(a) Use Exercise 6.1.4 with $\Psi(x) = \frac{\partial}{\partial t}P_t(x)\big|_{t=1}$ to obtain that the operator

$$f \to \left(\int_0^\infty t\left|\frac{\partial}{\partial t}(P_t * f)(x)\right|^2 dt\right)^{1/2}$$

is bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.
(b) Use Exercise 6.1.4 with $\Psi(x) = \partial_k P_1(x)$ to obtain that the operator

$$f \to \left(\int_0^\infty t|\partial_k(P_t * f)(x)|^2 dt\right)^{1/2}$$

is bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.
(c) Conclude that the g-function

$$g(f)(x) = \left(\int_0^\infty t|\nabla_{x,t}(P_t * f)(x)|^2 dt\right)^{1/2}$$

is bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

6.3.2. Suppose that μ is a finite Borel measure on $\mathbf{R}^n$ with compact support that satisfies $\widehat{\mu}(0) = 0$ and $|\widehat{\mu}(\xi)| \le C|\xi|^{-a}$ for some $a > 0$ and all $\xi \neq 0$. Define measures μ_j by setting $\widehat{\mu_j}(\xi) = \widehat{\mu}(2^{-j}\xi)$. Show that the operator

$$T_\mu(f)(x) = \sum_{j\in\mathbf{Z}} (f * \mu_j)(x)$$

is bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.
$\big[$*Hint:* Use Theorem 6.3.4$\big]$

6.3.3. ([50], [71]) (a) Suppose that μ is a finite Borel measure on $\mathbf{R}^n$ with compact support that satisfies $|\widehat{\mu}(\xi)| \le C|\xi|^{-a}$ for some $a > 0$ and all $\xi \neq 0$. Show that the maximal function

$$\mathscr{M}_\mu(f)(x) = \sup_{j\in\mathbf{Z}} \left|\int_{\mathbf{R}^n} f(x - 2^j y)\, d\mu(y)\right|$$

is bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.
(b) Let μ be the surface measure on the sphere $\mathbf{S}^{n-1}$ when $n \ge 2$. Conclude that the *dyadic spherical maximal function* $\mathscr{M}_\mu$ is bounded on $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.
$\big[$*Hint:* Part (a): Pick a $\mathscr{C}_0^\infty$ function φ on $\mathbf{R}^n$ with $\widehat{\varphi}(0) = 1$. Then the measure $\sigma = \mu - \widehat{\mu}(0)\varphi$ satisfies the hypotheses of Corollary 6.3.5. Since,

$$\mathscr{M}_\mu(f)(x) \le \left(\sum_j |(\sigma_j * f)(x)|^2\right)^{1/2} + |\widehat{\mu}(0)|M(f)(x),$$

it follows that $\mathscr{M}_\mu$ is bounded on $L^p(\mathbf{R}^n)$ whenever $1 < p < \infty$. Part (b): If $\mu = d\sigma$ is surface measure on $\mathbf{S}^{n-1}$, then $|\widehat{d\sigma}(\xi)| \le C|\xi|^{-\frac{n-1}{2}}$ (Appendices B.4 and B.7).]

6.3.4. Let Ω be in $L^q(\mathbf{S}^{n-1})$ for some $1 < q < \infty$ and define the absolutely continuous measure

$$d\mu(x) = \frac{\Omega(x/|x|)}{|x|^n}\chi_{1<|x|\le 2}\,dx\,.$$

Show that for all $a < 1/q'$ we have that $|\widehat{\mu}(\xi)| \le C|\xi|^{-a}$. Under the additional hypothesis that Ω has mean value zero, conclude that the singular integral operator

$$T_\Omega(f)(x) = \text{p.v.}\int_{\mathbf{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy = \sum_j f * \mu_j$$

is L^p bounded for all $1 < p < \infty$. This provides an alternative proof of Theorem 5.2.10 under the hypothesis that $\Omega \in L^q(\mathbf{S}^{n-1})$.

6.3.5. For a continuous function F on $\mathbf{R}$ define

$$u(F)(x) = \left(\int_0^\infty |F(x+t)+F(x-t)-2F(x)|^2\,\frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

Given $f \in L^1_{\text{loc}}(\mathbf{R})$ we denote by F_f the indefinite integral of f, that is,

$$F_f(x) = \int_0^x f(t)\,dt\,.$$

Prove that for all $1 < p < \infty$ there exist constants c_p and C_p such that for all functions $f \in L^p(\mathbf{R})$ we have

$$c_p\big\|f\big\|_{L^p} \le \big\|u(F_f)\big\|_{L^p} \le C_p\big\|f\big\|_{L^p}\,.$$

[*Hint:* Let $\varphi = \chi_{[-1,0]} - \chi_{[0,1]}$. Then

$$(\varphi_t * f)(x) = \frac{1}{t}\big(F_f(x+t)+F_f(x-t)-2F_f(x)\big)$$

and the double inequality follows from parts (b) and (c) of Exercise 6.1.4.]

6.3.6. Let m be a bounded function on $\mathbf{R}^n$ that is $\mathscr{C}^1$ in a neighborhood of zero, it satisfies $m(0) = 1$ and $|m(\xi)| \le B|\xi|^{-\varepsilon}$ for all $\xi \neq 0$, for some $B, \varepsilon > 0$. Define an operator T_t by setting $T_t(f)\widehat{\ }(\xi) = \widehat{f}(\xi)m(t\xi)$. Show that the maximal operator

$$\sup_{N>0}\left(\frac{1}{N}\int_0^N \big|T_t(f)(x)\big|^2\,dt\right)^{\frac{1}{2}}$$

maps $L^2(\mathbf{R}^n)$ to itself.

[*Hint:* Majorize this maximal operator by a constant multiple of the sum

$$M(f)(x) + \left(\int_0^\infty |T_t(f)(x) - (f * \varphi_t)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

where φ is a $\mathscr{C}_0^\infty$ function such that $\widehat{\varphi}(0) = 1$.]

6.3.7. ([150]) Let $0 < \beta < 1$ and $p_0 = (1-\beta/2)^{-1}$. Suppose that $\{f_j\}_{j\in\mathbf{Z}}$ are L^2 functions on the real line with norm at most 1. Assume that each f_j is supported in interval of length 1 and that the orthogonality relation $\left|\langle f_j \,|\, f_k \rangle\right| \le (1+|j-k|)^{-\beta}$ holds for all $j,k \in \mathbf{Z}$.
(a) Let $I \subsetneqq \mathbf{Z}$ be such that for all $j \in I$ the functions f_j are supported in a fixed interval of length 3. Show that for all p satisfying $0 < p \le 2$ there is $C_{p,\beta} < \infty$ such that

$$\Big\| \sum_{j\in I} \varepsilon_j f_j \Big\|_{L^p} \le C_{p,\beta} \, |I|^{1-\frac{\beta}{2}}$$

whenever ε_j are complex numbers with $|\varepsilon_j| \le 1$.
(b) Under the same hypothesis as in part (a), prove that for all $0 < p < p_0$ there is a constant $C'_{p,\beta} < \infty$ such that

$$\Big\| \sum_{j\in I} c_j f_j \Big\|_{L^p} \le C'_{p,\beta} \Big(\sum_{j\in\mathbf{Z}} |c_j|^p \Big)^{\frac{1}{p}}$$

for all complex-valued sequences $\{c_j\}_j$ in ℓ^p.
(c) Derive the conclusion of part (b) without the assumption that the f_j are supported in a fixed interval of length 3.
[*Hint:* Part (a): Pass from L^p to L^2 and use the hypothesis. Part (b): Assume $\sum_{j\in\mathbf{Z}} |c_j|^p = 1$. For each $k = 0,1,\dots$, set $I_k = \{ j \in \mathbf{Z} : \, 2^{-k-1} < |c_j| \le 2^{-k} \}$. Write $\big\| \sum_{j\in\mathbf{Z}} c_j f_j \big\|_{L^p} \le \sum_{k=0}^\infty 2^{-k} \big\| \sum_{j\in I_k} (c_j 2^k) f_j \big\|_{L^p}$, use part (b), Hölder's inequality, and the fact that $\sum_{k=0}^\infty 2^{-kp} |I_k| \le 2^p$. Part (c): Write $\sum_{j\in\mathbf{Z}} c_j f_j = \sum_{m\in\mathbf{Z}} F_m$, where F_m is the sum of $c_j f_j$ over all j such that the support of f_j meets the interval $[m, m+1]$. These F_m's are supported in $[m-1, m+2]$ and are almost orthogonal.]

6.4 The Haar System, Conditional Expectation, and Martingales

There is a very strong connection between the Littlewood–Paley operators and certain notions from probability, such as conditional expectation and martingale difference operators. The conditional expectation we are concerned with is with respect to the increasing σ-algebra of all dyadic cubes on $\mathbf{R}^n$.

6.4.1 Conditional Expectation and Dyadic Martingale Differences

We recall the definition of dyadic cubes.

Definition 6.4.1. A *dyadic interval* in $\mathbf{R}$ is an interval of the form

$$\big[m2^{-k},(m+1)2^{-k}\big)$$

where m,k are integers. A *dyadic cube* in $\mathbf{R}^n$ is a product of dyadic intervals of the same length. That is, a dyadic cube is a set of the form

$$\prod_{j=1}^{n}\big[m_j2^{-k},(m_j+1)2^{-k}\big)$$

for some integers $m_1,\dots,m_n,k$.

We defined dyadic intervals to be closed on the left and open on the right, so that different dyadic intervals of the same length are always disjoint sets.

Given a cube Q in $\mathbf{R}^n$ we denote by $|Q|$ its Lebesgue measure and by $\ell(Q)$ its side length. We clearly have $|Q|=\ell(Q)^n$. We introduce some more notation.

Definition 6.4.2. For $k\in\mathbf{Z}$ we denote by $\mathscr{D}_k$ the set of all dyadic cubes in $\mathbf{R}^n$ whose side length is 2^{-k}. We also denote by $\mathscr{D}$ the set of all dyadic cubes in $\mathbf{R}^n$. Then we have

$$\mathscr{D}=\bigcup_{k\in\mathbf{Z}}\mathscr{D}_k\,,$$

and moreover, the σ-algebra $\sigma(\mathscr{D}_k)$ of measurable subsets of $\mathbf{R}^n$ formed by countable unions and complements of elements of $\mathscr{D}_k$ is increasing as k increases.

We observe the fundamental property of dyadic cubes, which clearly justifies their usefulness. Any two dyadic intervals of the same side length either are disjoint or coincide. Moreover, either two given dyadic intervals are disjoint, or one contains the other. Similarly, either two dyadic cubes are disjoint, or one contains the other.

Definition 6.4.3. Given a locally integrable function f on $\mathbf{R}^n$, we denote by

$$\operatorname*{Avg}_Q f=\frac{1}{|Q|}\int_Q f(t)\,dt$$

the average of f over a cube Q.

The *conditional expectation* of a locally integrable function f on $\mathbf{R}^n$ with respect to the increasing family of σ-algebras $\sigma(\mathscr{D}_k)$ generated by $\mathscr{D}_k$ is defined as

$$E_k(f)(x)=\sum_{Q\in\mathscr{D}_k}(\operatorname*{Avg}_Q f)\,\chi_Q(x),$$

for all $k \in \mathbf{Z}$. We also define the *dyadic martingale difference operator* D_k as follows:

$$D_k(f) = E_k(f) - E_{k-1}(f),$$

also for $k \in \mathbf{Z}$.

Next we introduce the family of Haar functions.

Definition 6.4.4. For a dyadic interval $I = [m2^{-k}, (m+1)2^{-k})$ we define $I_L = [m2^{-k}, (m+\frac{1}{2})2^{-k})$ and $I_R = [(m+\frac{1}{2})2^{-k}, (m+1)2^{-k})$ to be the left and right parts of I, respectively. The function

$$h_I(x) = |I|^{-\frac{1}{2}} \chi_{I_L} - |I|^{-\frac{1}{2}} \chi_{I_R}$$

is called the *Haar function associated with the interval* I.

We remark that Haar functions are constructed in such a way that they have L^2 norm equal to 1. Moreover, the Haar functions have the following fundamental orthogonality property:

$$\int_{\mathbf{R}} h_I(x) h_{I'}(x)\, dx = \begin{cases} 0 & \text{when } I \neq I', \\ 1 & \text{when } I = I'. \end{cases} \tag{6.4.1}$$

To see this, observe that the Haar functions have L^2 norm equal to 1 by construction. Moreover, if $I \neq I'$, then I and I' must have different lengths, say we have $|I'| < |I|$. If I and I' are not disjoint, then I' is contained either in the left or in the right half of I, on either of which h_I is constant. Thus (6.4.1) follows.

We recall the notation

$$\langle f, g \rangle = \int_{\mathbf{R}} f(x) g(x)\, dx$$

valid for square integrable functions. Under this notation, (6.4.1) can be rewritten as $\langle h_I, h_{I'} \rangle = \delta_{I,I'}$, where the latter is 1 when $I = I'$ and zero otherwise.

6.4.2 Relation Between Dyadic Martingale Differences and Haar Functions

We have the following result relating the Haar functions to the dyadic martingale difference operators in dimension one.

Proposition 6.4.5. *For every locally integrable function f on $\mathbf{R}$ and for all $k \in \mathbf{Z}$ we have the identity*

$$D_k(f) = \sum_{I \in \mathscr{D}_{k-1}} \langle f, h_I \rangle h_I \tag{6.4.2}$$

and also

$$\big\|D_k(f)\big\|_{L^2}^2 = \sum_{I\in\mathscr{D}_{k-1}} \big|\langle f,h_I\rangle\big|^2. \tag{6.4.3}$$

Proof. We observe that every interval J in $\mathscr{D}_k$ is either an I_L or an I_R for some unique $I\in\mathscr{D}_{k-1}$. Thus we can write

$$\begin{aligned} E_k(f) &= \sum_{J\in\mathscr{D}_k} (\operatorname*{Avg}_J f)\,\chi_J \\ &= \sum_{I\in\mathscr{D}_{k-1}} \bigg[\bigg(\frac{2}{|I|}\int_{I_L} f(t)\,dt\bigg)\chi_{I_L} + \bigg(\frac{2}{|I|}\int_{I_R} f(t)\,dt\bigg)\chi_{I_R}\bigg]. \end{aligned} \tag{6.4.4}$$

But we also have

$$\begin{aligned} E_{k-1}(f) &= \sum_{I\in\mathscr{D}_{k-1}} (\operatorname*{Avg}_I f)\,\chi_I \\ &= \sum_{I\in\mathscr{D}_{k-1}} \bigg(\frac{1}{|I|}\int_{I_L} f(t)\,dt + \frac{1}{|I|}\int_{I_R} f(t)\,dt\bigg)\big(\chi_{I_L}+\chi_{I_R}\big). \end{aligned} \tag{6.4.5}$$

Now taking the difference between (6.4.4) and (6.4.5) we obtain

$$\begin{aligned} D_k(f) = \sum_{I\in\mathscr{D}_{k-1}} \bigg[&\bigg(\frac{1}{|I|}\int_{I_L} f(t)\,dt\bigg)\chi_{I_L} - \bigg(\frac{1}{|I|}\int_{I_R} f(t)\,dt\bigg)\chi_{I_L} \\ &+\bigg(\frac{1}{|I|}\int_{I_R} f(t)\,dt\bigg)\chi_{I_R} - \bigg(\frac{1}{|I|}\int_{I_L} f(t)\,dt\bigg)\chi_{I_R}\bigg], \end{aligned}$$

which is easily checked to be equal to

$$\sum_{I\in\mathscr{D}_{k-1}} \bigg(\int_I f(t)h_I(t)\,dt\bigg)h_I = \sum_{I\in\mathscr{D}_{k-1}} \langle f,h_I\rangle\, h_I.$$

Finally, (6.4.3) is a consequence of (6.4.1). □

Theorem 6.4.6. *Every function $f\in L^2(\mathbf{R}^n)$ can be written as*

$$f = \sum_{k\in\mathbf{Z}} D_k(f), \tag{6.4.6}$$

where the series converges almost everywhere and in L^2. We also have

$$\|f\|_{L^2(\mathbf{R}^n)}^2 = \sum_{k\in\mathbf{Z}} \|D_k(f)\|_{L^2(\mathbf{R}^n)}^2. \tag{6.4.7}$$

Moreover, when $n=1$ we have the representation

$$f = \sum_{I\in\mathscr{D}} \langle f,h_I\rangle h_I\,, \tag{6.4.8}$$

where the sum converges a.e. and in L^2 and also

$$\|f\|_{L^2(\mathbf{R})}^2 = \sum_{I\in\mathscr{D}} |\langle f, h_I\rangle|^2 . \tag{6.4.9}$$

Proof. In view of the Lebesgue differentiation theorem, Corollary 2.1.16, given a function $f \in L^2(\mathbf{R}^n)$ there is a set N_f of measure zero on $\mathbf{R}^n$ such that for all $x \in \mathbf{R}^n \setminus N_f$ we have that

$$\operatorname*{Avg}_{Q_j} f \to f(x)$$

whenever Q_j is a sequence of decreasing cubes such that $\bigcap_j \overline{Q_j} = \{x\}$. Given x in $\mathbf{R}^n \setminus N_f$ there exists a unique sequence of dyadic cubes $Q_j(x) \in \mathscr{D}_j$ such that $\bigcap_{j=0}^\infty \overline{Q_j(x)} = \{x\}$. Then for all $x \in \mathbf{R}^n \setminus N_f$ we have

$$\lim_{j\to\infty} E_j(f)(x) = \lim_{j\to\infty} \sum_{Q\in\mathscr{D}_j} (\operatorname*{Avg}_Q f)\,\chi_Q(x) = \lim_{j\to\infty} \operatorname*{Avg}_{Q_j(x)} f = f(x) .$$

From this we conclude that $E_j(f) \to f$ a.e. as $j \to \infty$. We also observe that since $|E_j(f)| \le M_c(f)$, where M_c denotes the uncentered maximal function with respect to cubes, we have that $|E_j(f) - f| \le 2M_c(f)$, which allows us to obtain from the Lebesgue dominated convergence theorem that $E_j(f) \to f$ in L^2 as $j \to \infty$.

Next we study convergence of $E_j(f)$ as $j \to -\infty$. For a given $x \in \mathbf{R}^n$ and $Q_j(x)$ as before we have that

$$|E_j(f)(x)| = \Big|\operatorname*{Avg}_{Q_j(x)} f\Big| \le \left(\frac{1}{|Q_j(x)|}\int_{Q_j(x)} |f(t)|^2\,dt\right)^{\frac{1}{2}} \le 2^{\frac{jn}{2}} \|f\|_{L^2} ,$$

which tends to zero as $j \to -\infty$, since the side length of each $Q_j(x)$ is 2^{-j}. Since $|E_j(f)| \le M_c(f)$, the Lebesgue dominated convergence theorem allows us to conclude that $E_j(f) \to 0$ in L^2 as $j \to -\infty$. To obtain the conclusion asserted in (6.4.6) we simply observe that

$$\sum_{k=M}^{N} D_k(f) = E_N(f) - E_{M-1}(f) \to f$$

in L^2 and almost everywhere as $N \to \infty$ and $M \to -\infty$.

To prove (6.4.7) we first observe that we can rewrite $D_k(f)$ as

$$\begin{aligned} D_k(f) &= \sum_{Q\in\mathscr{D}_k} (\operatorname*{Avg}_Q f)\,\chi_Q - \sum_{R\in\mathscr{D}_{k-1}} (\operatorname*{Avg}_R f)\,\chi_R \\ &= \sum_{R\in\mathscr{D}_{k-1}} \Big[\sum_{\substack{Q\in\mathscr{D}_k\\ Q\subseteq R}} (\operatorname*{Avg}_Q f)\,\chi_Q - (\operatorname*{Avg}_R f)\,\chi_R \Big] \end{aligned}$$

$$= \sum_{R\in\mathscr{D}_{k-1}} \Bigg[\sum_{\substack{Q\in\mathscr{D}_k\\ Q\subseteq R}} \big(\operatorname*{Avg}_Q f\big)\,\chi_Q - \frac{1}{2^n} \sum_{\substack{Q\in\mathscr{D}_k\\ Q\subseteq R}} \big(\operatorname*{Avg}_Q f\big)\,\chi_R \Bigg]$$

$$= \sum_{R\in\mathscr{D}_{k-1}} \sum_{\substack{Q\in\mathscr{D}_k\\ Q\subseteq R}} \big(\operatorname*{Avg}_Q f\big)\big(\chi_Q - 2^{-n}\chi_R\big). \tag{6.4.10}$$

Using this identity we obtain that for given integers $k' > k$ we have

$$\int_{\mathbf{R}^n} D_k(f)(x)\,D_{k'}(f)(x)\,dx$$
$$= \sum_{R\in\mathscr{D}_{k-1}} \sum_{\substack{Q\in\mathscr{D}_k\\ Q\subseteq R}} \big(\operatorname*{Avg}_Q f\big) \sum_{R'\in\mathscr{D}_{k'-1}} \sum_{\substack{Q'\in\mathscr{D}_{k'}\\ Q'\subseteq R'}} \big(\operatorname*{Avg}_{Q'} f\big) \int \big(\chi_Q - 2^{-n}\chi_R\big)\big(\chi_{Q'} - 2^{-n}\chi_{R'}\big)\,dx.$$

Since $k' > k$, the last integral may be nonzero only when $R' \subsetneqq R$. If this is the case, then $R' \subseteq Q_{R'}$ for some dyadic cube $Q_{R'} \in \mathscr{D}_k$ with $Q_{R'} \subsetneqq R$. See Figure 6.1.

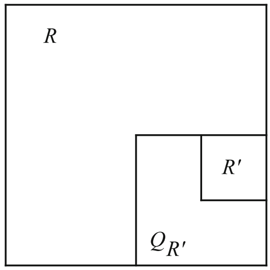

Fig. 6.1 Picture of the cubes R, R', and $Q_{R'}$.

Then the function $\chi_{Q'} - 2^{-n}\chi_{R'}$ is supported in the cube $Q_{R'}$ and the function $\chi_Q - 2^{-n}\chi_R$ is constant on any dyadic subcube Q of R (of half its side length) and in particular is constant on $Q_{R'}$. Then

$$\sum_{\substack{Q'\in\mathscr{D}_{k'}\\ Q'\subseteq R'}} \big(\operatorname*{Avg}_{Q'} f\big) \int_{Q_{R'}} \chi_{Q'} - 2^{-n}\chi_{R'}\,dx = \sum_{\substack{Q'\in\mathscr{D}_{k'}\\ Q'\subseteq R'}} \big(\operatorname*{Avg}_{Q'} f\big)\big(|Q'| - 2^{-n}|R'|\big) = 0,$$

since $|R'| = 2^n|Q'|$. We conclude that $\big\langle D_k(f), D_{k'}(f)\big\rangle = 0$ whenever $k \neq k'$, from which we easily derive (6.4.7).

Now observe that (6.4.8) is a direct consequence of (6.4.2), and (6.4.9) is a direct consequence of (6.4.3). $\square$

6.4.3 The Dyadic Martingale Square Function

As a consequence of identity (6.4.7), proved in the previous subsection, we obtain that

$$\Big\|\Big(\sum_{k\in\mathbf{Z}}|D_k(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbf{R}^n)}=\big\|f\big\|_{L^2(\mathbf{R}^n)}\,, \tag{6.4.11}$$

which says that the *dyadic martingale square function*

$$S(f)=\Big(\sum_{k\in\mathbf{Z}}|D_k(f)|^2\Big)^{\frac{1}{2}}$$

is L^2 bounded. It is natural to ask whether there exist L^p analogues of this result, and this is the purpose of the following theorem.

Theorem 6.4.7. *For any $1<p<\infty$ there exists a constant $c_{p,n}$ such that for every function f in $L^p(\mathbf{R}^n)$ we have*

$$\frac{1}{c_{p',n}}\big\|f\big\|_{L^p(\mathbf{R}^n)}\le\big\|S(f)\big\|_{L^p(\mathbf{R}^n)}\le c_{p,n}\big\|f\big\|_{L^p(\mathbf{R}^n)}\,. \tag{6.4.12}$$

The lower inequality subsumes the fact that if $\big\|S(f)\big\|_{L^p(\mathbf{R}^n)}<\infty$, then f must be an L^p function.

Proof. Let $\{r_j\}_j$ be the Rademacher functions (see Appendix C.1) enumerated in such a way that their index set is the set of integers. We rewrite the upper estimate in (6.4.12) as

$$\int_0^1\int_{\mathbf{R}^n}\Big|\sum_{k\in\mathbf{Z}}r_k(\omega)D_k(f)(x)\Big|^p\,dx\,d\omega\le C_p^p\big\|f\big\|_{L^p}^p\,. \tag{6.4.13}$$

We prove a stronger estimate than (6.4.13), namely that for all $\omega\in[0,1]$ we have

$$\int_{\mathbf{R}^n}\Big|T_\omega(f)(x)\Big|^p\,dx\le C_p^p\big\|f\big\|_{L^p}^p\,, \tag{6.4.14}$$

where

$$T_\omega(f)(x)=\sum_{k\in\mathbf{Z}}r_k(\omega)D_k(f)(x)\,.$$

In view of the L^2 estimate (6.4.11), we have that the operator T_ω is L^2 bounded with norm 1. We show that T_ω is weak type $(1,1)$.

To show that T_ω is of weak type $(1,1)$ we fix a function $f\in L^1$ and $\alpha>0$. We apply the Calderón–Zygmund decomposition (Theorem 5.3.1) to f at height α to write

$$f=g+b,\qquad\qquad b=\sum_j\big(f-\operatorname*{Avg}_{Q_j}f\big)\chi_{Q_j}\,,$$

where Q_j are dyadic cubes that satisfy $\sum_j |Q_j| \le \frac{1}{\alpha}\|f\|_{L^1}$ and g has L^2 norm at most $(2^n\alpha\|f\|_{L^1})^{\frac{1}{2}}$; see (5.3.1). To achieve this decomposition, we apply the proof of Theorem 5.3.1 starting with a dyadic mesh of large cubes such that $|Q| \ge \frac{1}{\alpha}\|f\|_{L^1}$ for all Q in the mesh. Then we subdivide each Q in the mesh by halving each side, and we select those cubes for which the average of f over them is bigger than α (and thus at most $2^n\alpha$). Since the original mesh consists of dyadic cubes, the stopping-time argument of Theorem 5.3.1 ensures that each selected cube is dyadic.

We observe (and this is the key observation) that $T_\omega(b)$ is supported in $\bigcup_j Q_j$. To see this, we use identity (6.4.10) to write $T_\omega(b)$ as

$$\sum_j \Bigg[\sum_k r_k(\omega) \sum_{R\in\mathscr{D}_{k-1}} \sum_{\substack{Q\in\mathscr{D}_k\\ Q\subseteq R}} \operatorname*{Avg}_Q[(f-\operatorname*{Avg}_{Q_j} f)\chi_{Q_j}]\,(\chi_Q - 2^{-n}\chi_R)\Bigg]. \tag{6.4.15}$$

We consider the following three cases for the cubes Q that appear in the inner sum in (6.4.15): (i) $Q_j \subseteq Q$, (ii) $Q_j \cap Q = \emptyset$, and (iii) $Q \subsetneq Q_j$. It is simple to see that in cases (i) and (ii) we have $\operatorname{Avg}_Q[(f-\operatorname{Avg}_{Q_j} f)\chi_{Q_j}] = 0$. Therefore the inner sum in (6.4.15) is taken over all Q that satisfy $Q \subsetneq Q_j$. But then we must have that the unique dyadic parent R of Q is also contained in Q_j. It follows that the expression inside the square brackets in (6.4.15) is supported in R and therefore in Q_j. We conclude that $T_\omega(b)$ is supported in $\bigcup_j Q_j$. Using Exercise 5.3.5(a) we obtain that T_ω is weak type $(1,1)$ with norm at most

$$\frac{\alpha\big|\{|T_\omega(g)| > \frac{\alpha}{2}\}\big| + \alpha\big|\bigcup_j Q_j\big|}{\|f\|_{L^1}} \le \frac{\alpha 4\alpha^{-2}\|g\|_{L^2}^2 + \|f\|_{L^1}}{\|f\|_{L^1}} \le 2^{n+2}+1\,.$$

We have now established that T_ω is weak type $(1,1)$. Since T_ω is L^2 bounded with norm 1, it follows by interpolation that T_ω is L^p bounded for all $1<p<2$. The L^p boundedness of T_ω for the remaining $p>2$ follows by duality. (Note that the operators D_k and E_k are self-transpose.) We conclude the validity of (6.4.14), which implies that of (6.4.13). As observed, this is equivalent to the upper estimate in (6.4.12).

Finally, we notice that the lower estimate in (6.4.12) is a consequence of the upper estimate as in the case of the Littlewood–Paley operators Δ_j. Indeed, we need to observe that in view of (6.4.6) we have

$$\begin{aligned}
\big|\langle f,g\rangle\big| &= \Big|\Big\langle \sum_k D_k(f), \sum_{k'} D_{k'}(g)\Big\rangle\Big| \\
&= \Big|\sum_k\sum_{k'}\big\langle D_k(f), D_{k'}(g)\big\rangle\Big| \\
&= \Big|\sum_k \big\langle D_k(f), D_k(g)\big\rangle\Big| \qquad\qquad [\text{Exercise 6.4.6(a)}] \\
&\le \int_{\mathbf{R}^n}\sum_k |D_k(f)(x)|\,|D_k(g)(x)|\,dx
\end{aligned}$$

$$\leq \int_{\mathbf{R}^n} S(f)(x)\,S(g)(x)\,dx \qquad \text{(Cauchy–Schwarz inequality)}$$
$$\leq \big\|S(f)\big\|_{L^p}\,\big\|S(g)\big\|_{L^{p'}} \qquad \text{(Hölder's inequality)}$$
$$\leq \big\|S(f)\big\|_{L^p}\,c_{p',n}\big\|g\big\|_{L^{p'}}\,.$$

Taking the supremum over all functions g on $\mathbf{R}^n$ with $L^{p'}$ norm at most 1, we obtain that f gives rise to a bounded linear functional on $L^{p'}$. It follows by the Riesz representation theorem that f must be an L^p function that satisfies the lower estimate in (6.4.12). □

6.4.4 Almost Orthogonality Between the Littlewood–Paley Operators and the Dyadic Martingale Difference Operators

Next, we discuss connections between the Littlewood–Paley operators Δ_j and the dyadic martingale difference operators D_k. It turns out that these operators are almost orthogonal in the sense that the L^2 operator norm of the composition $D_k\Delta_j$ decays exponentially as the indices j and k get farther away from each other.

For the purposes of the next theorem we define the Littlewood–Paley operators Δ_j as convolution operators with the function $\Psi_{2^{-j}}$, where

$$\widehat{\Psi}(\xi) = \widehat{\Phi}(\xi) - \widehat{\Phi}(2\xi)$$

and Φ is a fixed radial Schwartz function whose Fourier transform $\widehat{\Phi}$ is real-valued, supported in the ball $|\xi| < 2$, and equal to 1 on the ball $|\xi| < 1$. In this case we clearly have the identity

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \qquad \xi \neq 0.$$

Then we have the following theorem.

Theorem 6.4.8. *There exists a constant C such that for every k, j in $\mathbf{Z}$ the following estimate on the operator norm of $D_k\Delta_j : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is valid:*

$$\big\|D_k\Delta_j\big\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} = \big\|\Delta_j D_k\big\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \leq C\,2^{-\frac{1}{2}|j-k|}. \tag{6.4.16}$$

Proof. Since Ψ is a radial function, it follows that Δ_j is equal to its transpose operator on L^2. Moreover, the operator D_k is also equal to its transpose. Thus

$$(D_k\Delta_j)^t = \Delta_j D_k$$

and it therefore suffices to prove only that

$$\big\|D_k\Delta_j\big\|_{L^2\to L^2} \leq C2^{-\frac{1}{2}|j-k|}. \tag{6.4.17}$$

By a simple dilation argument it suffices to prove (6.4.17) when $k = 0$. In this case we have the estimate

$$\begin{aligned}\big\|D_0\Delta_j\big\|_{L^2\to L^2} &= \big\|E_0\Delta_j - E_{-1}\Delta_j\big\|_{L^2\to L^2}\\ &\le \big\|E_0\Delta_j - \Delta_j\big\|_{L^2\to L^2} + \big\|E_{-1}\Delta_j - \Delta_j\big\|_{L^2\to L^2},\end{aligned}$$

and since the D_k's and Δ_j's are self-transposes, we have

$$\begin{aligned}\big\|D_0\Delta_j\big\|_{L^2\to L^2} = \big\|\Delta_j D_0\big\|_{L^2\to L^2} &= \big\|\Delta_j E_0 - \Delta_j E_{-1}\big\|_{L^2\to L^2}\\ &\le \big\|\Delta_j E_0 - E_0\big\|_{L^2\to L^2} + \big\|\Delta_j E_{-1} - E_0\big\|_{L^2\to L^2}.\end{aligned}$$

Estimate (6.4.17) when $k = 0$ will be a consequence of the pair of inequalities

$$\big\|E_0\Delta_j - \Delta_j\big\|_{L^2\to L^2} + \big\|E_{-1}\Delta_j - \Delta_j\big\|_{L^2\to L^2} \le C2^{\frac{j}{2}} \quad \text{for } j \le 0, \tag{6.4.18}$$

$$\big\|\Delta_j E_0 - E_0\big\|_{L^2\to L^2} + \big\|\Delta_j E_{-1} - E_0\big\|_{L^2\to L^2} \le C2^{-\frac{1}{2}j} \quad \text{for } j \ge 0. \tag{6.4.19}$$

We start by proving (6.4.18). We consider only the term $E_0\Delta_j - \Delta_j$, since the term $E_{-1}\Delta_j - \Delta_j$ is similar. Let $f \in L^2(\mathbf{R}^n)$. Then

$$\begin{aligned}&\big\|E_0\Delta_j(f) - \Delta_j(f)\big\|_{L^2}^2\\ &= \sum_{Q\in\mathscr{D}_0} \Big\|f * \Psi_{2^{-j}} - \operatorname*{Avg}_Q(f * \Psi_{2^{-j}})\Big\|_{L^2(Q)}^2\\ &\le \sum_{Q\in\mathscr{D}_0} \int_Q\int_Q |(f*\Psi_{2^{-j}})(x) - (f*\Psi_{2^{-j}})(t)|^2\,dt\,dx\\ &\le 3\sum_{Q\in\mathscr{D}_0} \int_Q\int_Q \bigg(\int_{5\sqrt{n}Q} |f(y)||\Psi_{2^{-j}}(x-y)|\,dy\bigg)^2 dt\,dx\\ &\qquad + 3\sum_{Q\in\mathscr{D}_0} \int_Q\int_Q \bigg(\int_{5\sqrt{n}Q} |f(y)||\Psi_{2^{-j}}(t-y)|\,dy\bigg)^2 dt\,dx\\ &\qquad + 3\sum_{Q\in\mathscr{D}_0} \int_Q\int_Q \bigg(\int_{(5\sqrt{n}Q)^c} |f(y)|2^{jn+j}|\nabla\Psi(2^j(\xi_{x,t}-y))|\,dy\bigg)^2 dt\,dx,\end{aligned}$$

where $\xi_{x,t}$ lies on the line segment joining x and t. Applying the Cauchy-Schwarz inequality to the first two terms, we see that the last expression is bounded by

$$C2^{jn}\sum_{Q\in\mathscr{D}_0}\int_{5\sqrt{n}Q}|f(y)|^2\,dy + C_M 2^{2j}\sum_{Q\in\mathscr{D}_0}\int_Q\bigg(\int_{\mathbf{R}^n}\frac{2^{jn}|f(y)|\,dy}{(1+2^j|x-y|)^M}\bigg)^2 dx,$$

which is clearly controlled by $C(2^{jn}+2^{2j})\|f\|_{L^2}^2 \le 2C2^j\|f\|_{L^2}^2$. This proves (6.4.18).

We now turn to the proof of (6.4.19). We set $S_j = \sum_{k\le j}\Delta_k$. Since Δ_j is the difference of two S_j's, it suffices to prove (6.4.19), where Δ_j is replaced by S_j. We work only with the term $S_jE_0 - E_0$, since the other term can be treated similarly. We have

$$\begin{aligned}\big\|S_jE_0(f)-E_0(f)\big\|_{L^2}^2 &= \Big\|\sum_{Q\in\mathscr{D}_0}(\operatorname*{Avg}_Q f)\,(\Phi_{2^{-j}}*\chi_Q-\chi_Q)\Big\|_{L^2}^2\\ &\le 2\Big\|\sum_{Q\in\mathscr{D}_0}(\operatorname*{Avg}_Q f)\,(\Phi_{2^{-j}}*\chi_Q-\chi_Q)\chi_{5\sqrt{n}Q}\Big\|_{L^2}^2\\ &\qquad+2\Big\|\sum_{Q\in\mathscr{D}_0}(\operatorname*{Avg}_Q f)\,(\Phi_{2^{-j}}*\chi_Q)\chi_{(5\sqrt{n}Q)^c}\Big\|_{L^2}^2.\end{aligned}$$

Since the functions appearing inside the sum in the first term have supports with bounded overlap, we obtain

$$\Big\|\sum_{Q\in\mathscr{D}_0}(\operatorname*{Avg}_Q f)(\Phi_{2^{-j}}*\chi_Q-\chi_Q)\chi_{5\sqrt{n}Q}\Big\|_{L^2}^2\le C\sum_{Q\in\mathscr{D}_0}(\operatorname*{Avg}_Q|f|)^2\big\|\Phi_{2^{-j}}*\chi_Q-\chi_Q\big\|_{L^2}^2,$$

and the crucial observation is that

$$\big\|\Phi_{2^{-j}}*\chi_Q-\chi_Q\big\|_{L^2}^2\le C2^{-j},$$

a consequence of Plancherel's identity and the fact that $|1-\widehat{\Phi}(2^{-j}\xi)|\le\chi_{|\xi|\ge 2^j}$. Putting these observations together, we deduce

$$\Big\|\sum_{Q\in\mathscr{D}_0}(\operatorname*{Avg}_Q f)\,(\Phi_{2^{-j}}*\chi_Q-\chi_Q)\chi_{3Q}\Big\|_{L^2}^2\le C\sum_{Q\in\mathscr{D}_0}(\operatorname*{Avg}_Q|f|)^2 2^{-j}\le C2^{-j}\big\|f\big\|_{L^2}^2,$$

and the required conclusion will be proved if we can show that

$$\Big\|\sum_{Q\in\mathscr{D}_0}(\operatorname*{Avg}_Q f)\,(\Phi_{2^{-j}}*\chi_Q)\chi_{(3Q)^c}\Big\|_{L^2}^2\le C2^{-j}\big\|f\big\|_{L^2}^2. \tag{6.4.20}$$

We prove (6.4.20) by using an estimate based purely on size. Let c_Q be the center of the dyadic cube Q. For $x\notin 3Q$ we have the estimate

$$|(\Phi_{2^{-j}}*\chi_Q)(x)|\le\frac{C_M2^{jn}}{(1+2^j|x-c_Q|)^M}\le\frac{C_M2^{jn}}{(1+2^j)^{M/2}}\frac{1}{(1+|x-c_Q|)^{M/2}},$$

since both $2^j\ge1$, and $|x-c_Q|\ge1$. We now control the left-hand side of (6.4.20) by

$$\begin{aligned}&2^{j(2n-M)}\sum_{Q\in\mathscr{D}_0}\sum_{Q'\in\mathscr{D}_0}(\operatorname*{Avg}_Q|f|)(\operatorname*{Avg}_{Q'}|f|)\int_{\mathbf{R}^n}\frac{C_M\,dx}{(1+|x-c_Q|)^{\frac{M}{2}}(1+|x-c_{Q'}|)^{\frac{M}{2}}}\\ &\le 2^{j(2n-M)}\sum_{Q\in\mathscr{D}_0}\sum_{Q'\in\mathscr{D}_0}\frac{(\operatorname*{Avg}_Q|f|)(\operatorname*{Avg}_{Q'}|f|)}{(1+|c_Q-c_{Q'}|)^{\frac{M}{4}}}\int_{\mathbf{R}^n}\frac{C_M\,dx}{(1+|x-c_Q|)^{\frac{M}{4}}(1+|x-c_{Q'}|)^{\frac{M}{4}}}\\ &\le 2^{j(2n-M)}\sum_{Q\in\mathscr{D}_0}\sum_{Q'\in\mathscr{D}_0}\frac{C_M}{(1+|c_Q-c_{Q'}|)^{\frac{M}{4}}}\bigg(\int_Q|f(y)|^2\,dy+\int_{Q'}|f(y)|^2\,dy\bigg)\end{aligned}$$

$$\leq C_M 2^{j(2n-M)} \sum_{Q\in\mathscr{D}_0} \int_Q |f(y)|^2\,dy$$
$$= C_M 2^{j(2n-M)} \|f\|_{L^2}^2 .$$

By taking M large enough, we obtain (6.4.20) and thus (6.4.19). □

Exercises

6.4.1. (a) Prove that no dyadic cube in $\mathbf{R}^n$ contains the point 0 in its interior.
(b) Prove that every interval $[a,b]$ is contained in the union of three dyadic intervals of length less than $b-a$.
(c) Prove that every cube of length l in $\mathbf{R}^n$ is contained in the union of 3^n dyadic cubes, each having length less than l.

6.4.2. Let $k \in \mathbf{Z}$. Show that the set $[m2^{-k}, (m+s)2^{-k})$ is a dyadic interval if and only if $s = 2^p$ for some $p \in \mathbf{Z}$ and m is an integer multiple of s.

6.4.3. Given a cube Q in $\mathbf{R}^n$ of side length $\ell(Q) \leq 2^{k-1}$ for some integer k, prove that there is a dyadic cube D_Q of side length 2^k such that $Q \subsetneqq \sigma + D_Q$ for some $\sigma = (\sigma_1, \dots, \sigma_n)$, where $\sigma_j \in \{0, 1/3, -1/3\}$.

6.4.4. Show that the martingale maximal function $f \mapsto \sup_{k\in\mathbf{Z}} |E_k(f)|$ is weak type $(1,1)$ with constant at most 1.
[*Hint:* Use Exercise 2.1.12.]

6.4.5. (a) Show that $E_N(f) \to f$ a.e. as $N \to \infty$ for all $f \in L^1_{\text{loc}}(\mathbf{R}^n)$.
(b) Prove that $E_N(f) \to f$ in L^p as $N \to \infty$ for all $f \in L^p(\mathbf{R}^n)$ whenever $1 < p < \infty$.

6.4.6. (a) Let $k, k' \in \mathbf{Z}$ be such that $k \neq k'$. Show that for functions f and g in $L^2(\mathbf{R}^n)$ we have
$$\big\langle D_k(f), D_{k'}(g)\big\rangle = 0 .$$

(b) Conclude that for functions f_j in $L^2(\mathbf{R}^n)$ we have

$$\Big\|\sum_{j\in\mathbf{Z}} D_j(f_j)\Big\|_{L^2(\mathbf{R}^n)} = \Big(\sum_{j\in\mathbf{Z}} \big\|D_j(f_j)\big\|^2_{L^2(\mathbf{R}^n)}\Big)^{\frac12} .$$

(c) Let Δ_j and C be as in the statement of Theorem 6.4.8. Show that for any $r \in \mathbf{Z}$ we have
$$\Big\|\sum_{j\in\mathbf{Z}} D_j \Delta_{j+r} D_j\Big\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \leq C 2^{-\frac12 |r|} .$$

6.4.7. ([133]) Let D_j, Δ_j be as in Theorem 6.4.8.
(a) Prove that the operator
$$V_r = \sum_{j\in\mathbf{Z}} D_j \Delta_{j+r}$$

is bounded from $L^2(\mathbf{R}^n)$ to itself with norm at most a multiple of $2^{-\frac12 |r|}$.

(b) Show that V_r is $L^p(\mathbf{R}^n)$ bounded for all $1 < p < \infty$ with a constant depending only on p and n.
(c) Conclude that for each $1 < p < \infty$ there is a constant $c_p > 0$ such that V_r is bounded on $L^p(\mathbf{R}^n)$ with norm at most a multiple of $2^{-c_p|r|}$.
[*Hint:* Part (a): Write $\Delta_j = \Delta_j \widetilde{\Delta}_j$, where $\widetilde{\Delta}_j$ is another family of Littlewood–Paley operators and use Exercise 6.4.6 (b). Part (b): Use duality and (6.1.21).]

6.5 The Spherical Maximal Function

In this section we discuss yet another consequence of the Littlewood–Paley theory, the boundedness of the spherical maximal operator.

6.5.1 Introduction of the Spherical Maximal Function

We denote throughout this section by $d\sigma$ the normalized Lebesgue measure on the sphere $\mathbf{S}^{n-1}$. For f in $L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, we define the maximal operator

$$\mathscr{M}(f)(x) = \sup_{t>0} \left| \int_{\mathbf{S}^{n-1}} f(x-t\theta)\, d\sigma(\theta) \right| \tag{6.5.1}$$

and we observe that by Minkowski's integral inequality each expression inside the supremum in (6.5.1) is well defined for $f \in L^p$ for almost all $x \in \mathbf{R}^n$. The operator $\mathscr{M}$ is called the *spherical maximal function*. It is unclear at this point for which functions f we have $\mathscr{M}(f) < \infty$ a.e. and for which values of $p < \infty$ the maximal inequality

$$\big\| \mathscr{M}(f) \big\|_{L^p(\mathbf{R}^n)} \le C_p \big\| f \big\|_{L^p(\mathbf{R}^n)} \tag{6.5.2}$$

holds for all functions $f \in L^p(\mathbf{R}^n)$.

Spherical averages often make their appearance as solutions of partial differential equations. For instance, the spherical average

$$u(x,t) = \frac{1}{4\pi} \int_{\mathbf{S}^2} t\, f(x-ty)\, d\sigma(y) \tag{6.5.3}$$

is a solution of the *wave equation*

$$\begin{aligned} \Delta_x(u)(x,t) &= \frac{\partial^2 u}{\partial t^2}(x,t)\,, \\ u(x,0) &= 0\,, \\ \frac{\partial u}{\partial t}(x,0) &= f(x)\,, \end{aligned}$$

in $\mathbf{R}^3$. The introduction of the spherical maximal function is motivated by the fact that the related spherical average

$$u(x,t) = \frac{1}{4\pi}\int_{\mathbf{S}^2} f(x-ty)\,d\sigma(y) \tag{6.5.4}$$

solves *Darboux's equation*

$$\begin{aligned} \Delta_x(u)(x,t) &= \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{2}{t}\frac{\partial u}{\partial t}(x,t)\,, \\ u(x,0) &= f(x)\,, \\ \frac{\partial u}{\partial t}(x,0) &= 0\,, \end{aligned}$$

in $\mathbf{R}^3$. It is rather remarkable that the Fourier transform can be used to study almost everywhere convergence for several kinds of maximal averaging operators such as the spherical averages in (6.5.4). This is achieved via the boundedness of the corresponding maximal operator; the maximal operator controlling the averages over $\mathbf{S}^{n-1}$ is given in (6.5.1).

Before we begin the analysis of the spherical maximal function, we recall that

$$\widehat{d\sigma}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|)\,,$$

as shown in Appendix B.4. Using the estimates in Appendices B.6 and B.7 and the identity

$$\frac{d}{dt}J_\nu(t) = \frac{1}{2}(J_{\nu-1}(t) - J_{\nu+1}(t))$$

derived in Appendix B.2, we deduce the crucial estimate

$$|\widehat{d\sigma}(\xi)| + |\nabla\widehat{d\sigma}(\xi)| \le \frac{C_n}{(1+|\xi|)^{\frac{n-1}{2}}}\,. \tag{6.5.5}$$

Theorem 6.5.1. *Let $n \ge 3$. For each $\frac{n}{n-1} < p \le \infty$, there is a constant C_p such that*

$$\big\|\mathscr{M}(f)\big\|_{L^p(\mathbf{R}^n)} \le C_p \big\|f\big\|_{L^p(\mathbf{R}^n)} \tag{6.5.6}$$

holds for all f in $L^p(\mathbf{R}^n)$. Consequently, for all $\frac{n}{n-1} < p \le \infty$ and $f \in L^p(\mathbf{R}^n)$ we have

$$\lim_{t\to 0}\frac{1}{\omega_{n-1}}\int_{\mathbf{S}^{n-1}} f(x-t\theta)\,d\sigma(\theta) = f(x) \tag{6.5.7}$$

for almost all $x \in \mathbf{R}^n$. Here we set $\omega_{n-1} = |\mathbf{S}^{n-1}|$.

The proof of this theorem is given in the rest of this section. Before we present the proof we explain the validity of (6.5.7). Clearly this assertion is valid for functions $f \in \mathscr{S}(\mathbf{R}^n)$. Using inequality (6.5.6) and Theorem 2.1.14 we obtain that (6.5.7) holds for all functions in $f \in L^p(\mathbf{R}^n)$.

We now focus on (6.5.6). Define $m(\xi) = \widehat{d\sigma}(\xi)$ and notice that $m(\xi)$ is a $\mathscr{C}^\infty$ function. To study the maximal multiplier operator

$$\sup_{t>0} \left| \left(\widehat{f}(\xi)\, m(t\xi) \right)^\vee \right|$$

we decompose the multiplier $m(\xi)$ into radial pieces as follows: We fix a radial $\mathscr{C}^\infty$ function φ_0 in $\mathbf{R}^n$ such that $\varphi_0(\xi) = 1$ when $|\xi| \le 1$ and $\varphi_0(\xi) = 0$ when $|\xi| \ge 2$. For $j \ge 1$ we let

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{1-j}\xi) \tag{6.5.8}$$

and we observe that $\varphi_j(\xi)$ is localized near $|\xi| \approx 2^j$. Then we have

$$\sum_{j=0}^{\infty} \varphi_j = 1\,.$$

Set $m_j = \varphi_j m$ for all $j \ge 0$. The m_j's are $\mathscr{C}_0^\infty$ functions that satisfy

$$m = \sum_{j=0}^{\infty} m_j\,.$$

Also, the following estimate is valid:

$$\mathscr{M}(f) \le \sum_{j=0}^{\infty} \mathscr{M}_j(f)\,,$$

where

$$\mathscr{M}_j(f)(x) = \sup_{t>0} \left| \left(\widehat{f}(\xi)\, m_j(t\xi) \right)^\vee (x) \right|.$$

Since the function m_0 is $\mathscr{C}_0^\infty$, we have that $\mathscr{M}_0$ maps L^p to itself for all $1 < p \le \infty$. (See Exercise 6.5.1.)

We define *g-functions* associated with m_j as follows:

$$G_j(f)(x) = \left(\int_0^\infty |A_{j,t}(f)(x)|^2 \,\frac{dt}{t} \right)^{\frac{1}{2}},$$

where $A_{j,t}(f)(x) = \left(\widehat{f}(\xi)\, m_j(t\xi) \right)^\vee (x)$.

6.5.2 The First Key Lemma

We have the following lemma:

Lemma 6.5.2. *There is a constant $C = C(n) < \infty$ such that for any $j \geq 1$ we have the estimate*

$$\big\|\mathscr{M}_j(f)\big\|_{L^2} \leq C2^{(\frac{1}{2}-\frac{n-1}{2})j}\big\|f\big\|_{L^2}$$

for all functions f in $L^2(\mathbf{R}^n)$.

Proof. We define a function

$$\widetilde{m}_j(\xi) = \xi \cdot \nabla m_j(\xi)\,,$$

we let $\widetilde{A}_{j,t}(f)(x) = \big(\widehat{f}(\xi)\,\widetilde{m}_j(t\xi)\big)^\vee(x)$, and we let

$$\widetilde{G}_j(f)(x) = \left(\int_0^\infty |\widetilde{A}_{j,t}(f)(x)|^2\,\frac{dt}{t}\right)^{\frac{1}{2}}$$

be the associated g-function. For $f \in L^2(\mathbf{R}^n)$, the identity

$$s\,\frac{dA_{j,s}}{ds}(f) = \widetilde{A}_{j,s}(f)$$

is clearly valid for all j and s. Since $A_{j,s}(f) = f * (m_j^\vee)_s$ and $m_j^\vee$ has integral zero for $j \geq 1$ (here $(m_j^\vee)_s(x) = s^{-n}m_j^\vee(s^{-1}x)$), it follows from Corollary 2.1.19 that

$$\lim_{s\to 0} A_{j,s}(f)(x) = 0$$

for all $x \in \mathbf{R}^n \setminus E_f$, where E_f is some set of Lebesgue measure zero. By the fundamental theorem of calculus for $x \in \mathbf{R}^n \setminus E_f$ we deduce that

$$\begin{aligned}
(A_{j,t}(f)(x))^2 &= \int_0^t \frac{d}{ds}(A_{j,s}(f)(x))^2\,ds \\
&= 2\int_0^t A_{j,s}(f)(x)\,s\,\frac{dA_{j,s}}{ds}(f)(x)\,\frac{ds}{s} \\
&= 2\int_0^t A_{j,s}(f)(x)\widetilde{A}_{j,s}(f)(x)\,\frac{ds}{s}\,,
\end{aligned}$$

from which we obtain the estimate

$$\big|A_{j,t}(f)(x)\big|^2 \leq 2\int_0^\infty \big|A_{j,s}(f)(x)\big|\,\big|\widetilde{A}_{j,s}(f)(x)\big|\,\frac{ds}{s}\,. \tag{6.5.9}$$

Taking the supremum over all $t > 0$ on the left-hand side in (6.5.9) and integrating over $\mathbf{R}^n$, we obtain the estimate

$$\begin{aligned}\big\|\mathscr{M}_j(f)\big\|_{L^2}^2 &\le 2\int_{\mathbf{R}^n}\int_0^\infty \big|A_{j,s}(f)(x)\big|\,\big|\widetilde{A}_{j,s}(f)(x)\big|\,\frac{ds}{s}\,dx\\ &\le 2\int_{\mathbf{R}^n} G_j(f)(x)\widetilde{G}_j(f)(x)\,dx\\ &\le 2\big\|G_j(f)\big\|_{L^2}\big\|\widetilde{G}_j(f)\big\|_{L^2},\end{aligned}$$

by applying the Cauchy–Schwarz inequality twice. Next we claim that as a consequence of (6.5.5) we have for some $c, \widetilde{c} < \infty$,

$$\big\|m_j\big\|_{L^\infty} \le c\,2^{-j\frac{n-1}{2}} \quad \text{and} \quad \big\|\widetilde{m}_j\big\|_{L^\infty} \le \widetilde{c}\,2^{j(1-\frac{n-1}{2})}.$$

Using these facts together with the facts that the functions m_j and $\widetilde{m}_j$ are supported in the annuli $2^{j-1} \le |\xi| \le 2^{j+1}$, we obtain that the g-functions G_j and $\widetilde{G}_j$ are L^2 bounded with norms at most a constant multiple of the quantities $2^{-j\frac{n-1}{2}}$ and $2^{j(1-\frac{n-1}{2})}$, respectively; see Exercise 6.5.2. Note that since $n \ge 3$, both exponents are negative. We conclude that

$$\big\|\mathscr{M}_j(f)\big\|_{L^2} \le C2^{j(\frac{1}{2}-\frac{n-1}{2})}\big\|f\big\|_{L^2},$$

which is what we needed to prove. □

6.5.3 The Second Key Lemma

Next we need the following lemma.

Lemma 6.5.3. *There exists a constant $C = C(n) < \infty$ such that for all $j \ge 1$ and for all f in $L^1(\mathbf{R}^n)$ we have*

$$\big\|\mathscr{M}_j(f)\big\|_{L^{1,\infty}} \le C\,2^j\big\|f\big\|_{L^1}.$$

Proof. Let $K^{(j)} = (\varphi_j)^\vee * d\sigma = \Phi_{2^{-j}} * d\sigma$, where Φ is a Schwartz function. Setting

$$(K^{(j)})_t(x) = t^{-n}K^{(j)}(t^{-1}x)$$

we have that

$$\mathscr{M}_j(f) = \sup_{t>0}\big|(K^{(j)})_t * f\big|. \tag{6.5.10}$$

The proof of the lemma is based on the estimate:

$$\mathscr{M}_j(f) \le C\,2^j\,\mathcal{M}(f) \tag{6.5.11}$$

and the weak type $(1,1)$ boundedness of the Hardy–Littlewood maximal operator $\mathcal{M}$ (Theorem 2.1.6). To establish (6.5.11), it suffices to show that for any $M>n$ there is a constant $C_M<\infty$ such that

$$|K^{(j)}(x)| = |(\Phi_{2^{-j}} * d\sigma)(x)| \le \frac{C_M\, 2^j}{(1+|x|)^M}\,. \tag{6.5.12}$$

Then Theorem 2.1.10 yields (6.5.11) and hence the required conclusion.

Using the fact that Φ is a Schwartz function, we have for every $N>0$,

$$|(\Phi_{2^{-j}} * d\sigma)(x)| \le C_N \int_{\mathbf{S}^{n-1}} \frac{2^{nj}\, d\sigma(y)}{(1+2^j|x-y|)^N}\,.$$

We pick an N to depend on M (6.5.12); in fact, any $N>M$ suffices for our purposes. We split the last integral into the regions

$$S_{-1}(x) = \mathbf{S}^{n-1} \cap \{y\in\mathbf{R}^n:\ 2^j|x-y|\le 1\}$$

and for $r\ge 0$,

$$S_r(x) = \mathbf{S}^{n-1} \cap \{y\in\mathbf{R}^n:\ 2^r < 2^j|x-y| \le 2^{r+1}\}\,.$$

The key observation is that whenever $B(y,R)$ is a ball of radius R in $\mathbf{R}^n$ centered at $y\in\mathbf{S}^{n-1}$, then the spherical measure of the set $\mathbf{S}^{n-1}\cap B(y,R)$ is at most a dimensional constant multiple of R^{n-1}. This implies that the spherical measure of each $S_r(x)$ is at most $c_n 2^{(r+1-j)(n-1)}$, an estimate that is useful only when $r\le j$. Using this observation, together with the fact that for $y\in S_r(x)$ we have $|x|\le 2^{r+1-j}+1$, we obtain the following estimate for the expression $|(\Phi_{2^{-j}} * d\sigma)(x)|$:

$$\begin{aligned}
&\sum_{r=-1}^{j} \int_{S_r(x)} \frac{C_N 2^{nj}\, d\sigma(y)}{(1+2^j|x-y|)^N} + \sum_{r=j+1}^{\infty} \int_{S_r(x)} \frac{C_N 2^{nj}\, d\sigma(y)}{(1+2^j|x-y|)^N}\\
&\quad\le C_N' 2^{nj} \Bigg[\sum_{r=-1}^{j} \frac{d\sigma(S_r(x))\chi_{B(0,3)}(x)}{2^{rN}} + \sum_{r=j+1}^{\infty} \frac{d\sigma(S_r(x))\,\chi_{B(0,2^{r+1-j}+1)}(x)}{2^{rN}} \Bigg]\\
&\quad\le C_N' 2^{nj} \Bigg[\sum_{r=-1}^{j} \frac{c_n 2^{(r+1-j)(n-1)}\chi_{B(0,3)}(x)}{2^{rN}} + \sum_{r=j+1}^{\infty} \frac{\omega_{n-1}\,\chi_{B(0,2^{r+2-j})}(x)}{2^{rN}} \Bigg]\\
&\quad\le C_{N,n} \Bigg[2^j \chi_{B(0,3)}(x) + 2^{nj} \sum_{r=j+1}^{\infty} \frac{1}{2^{rN}} \frac{(1+2^{r+2-j})^M}{(1+|x|)^M} \Bigg]\\
&\quad\le C_{M,n}' \frac{2^j}{(1+|x|)^M} \Bigg[1 + \sum_{r=j+1}^{\infty} \frac{2^{(r-j)(M-N)}}{2^{j(N+1-n)}} \Bigg]\\
&\quad\le \frac{C_{M,n}'' 2^j}{(1+|x|)^M}\,,
\end{aligned}$$

where we used that $N>M>n$. This establishes (6.5.12). □

6.5.4 Completion of the Proof

It remains to combine the previous ingredients to complete the proof of the theorem. Interpolating between the $L^2 \to L^2$ and $L^1 \to L^{1,\infty}$ estimates obtained in Lemmas 6.5.2 and 6.5.3, we obtain

$$\big\|\mathscr{M}_j(f)\big\|_{L^p(\mathbf{R}^n)} \le C_p 2^{(\frac{n}{p}-(n-1))j}\big\|f\big\|_{L^p(\mathbf{R}^n)}$$

for all $1 < p \le 2$. When $p > \frac{n}{n-1}$ the series $\sum_{j=1}^{\infty} 2^{(\frac{n}{p}-(n-1))j}$ converges and we conclude that $\mathscr{M}$ is L^p bounded for these p's. The boundedness of $\mathscr{M}$ on L^p for $p > 2$ follows by interpolation between L^q for $q < 2$ and the estimate $\mathscr{M} : L^\infty \to L^\infty$.

Exercises

6.5.1. Let m be in $L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ that satisfies $|m^\vee(x)| \le C(1+|x|)^{-n-\delta}$ for some $\delta > 0$. Show that the maximal multiplier

$$\mathscr{M}_m(f)(x) = \sup_{t>0}\big|\big(\widehat{f}(\xi)\,m(t\xi)\big)^\vee(x)\big|$$

is L^p bounded for all $1 < p < \infty$.

6.5.2. Suppose that the function m is supported in the annulus $R \le |\xi| \le 2R$ and is bounded by A. Show that the g-function

$$G(f)(x) = \left(\int_0^\infty |(m(t\xi)\widehat{f}(\xi))^\vee(x)|^2 \,\frac{dt}{t}\right)^{\frac{1}{2}}$$

maps $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$ with bound at most $A\sqrt{\log 2}$.

6.5.3. ([302]) Let $A, a, b > 0$ with $a+b > 1$. Use the idea of Lemma 6.5.2 to show that if $m(\xi)$ satisfies $|m(\xi)| \le A\,(1+|\xi|)^{-a}$ and $|\nabla m(\xi)| \le A\,(1+|\xi|)^{-b}$ for all $\xi \in \mathbf{R}^n$, then the maximal operator

$$\mathscr{M}_m(f)(x) = \sup_{t>0}\big|\big(\widehat{f}(\xi)\,m(t\xi)\big)^\vee(x)\big|$$

is bounded from $L^2(\mathbf{R}^n)$ to itself.
$\big[$*Hint:* Use that

$$\mathscr{M}_m \le \sum_{j=0}^{\infty} \mathscr{M}_{m,j}\,,$$

where $\mathscr{M}_{m,j}$ corresponds to the multiplier $\varphi_j m$; here φ_j is as in (6.5.8). Show that

$$\big\|\mathscr{M}_{m,j}(f)\big\|_{L^2} \le C\,\big\|\varphi_j m\big\|_{L^\infty}^{\frac{1}{2}}\big\|\varphi_j \widetilde{m}\big\|_{L^\infty}^{\frac{1}{2}}\big\|f\big\|_{L^2} \le C\,2^{j\frac{1-(a+b)}{2}}\big\|f\big\|_{L^2},$$

where $\widetilde{m}(\xi) = \xi \cdot \nabla m(\xi)$.$\big]$

6.5.4. Let $A, c > 0$, $a > 1/2$, $0 < b < n$. Follow the idea of the proof of Theorem 6.5.1 to obtain the following more general result: If $d\mu$ is a finite Borel measure supported in the closed unit ball that satisfies $|\widehat{d\mu}(\xi)| \le A(1+|\xi|)^{-a}$ for all $\xi \in \mathbf{R}^n$ and $d\mu(B(y,R)) \le cR^b$ for all $R > 0$, then the maximal operator

$$f \mapsto \sup_{t>0} \left| \int_{\mathbf{R}^n} f(x-ty)\, d\mu(y) \right|$$

maps $L^p(\mathbf{R}^n)$ to itself when $p > \frac{2n-2b+2a-1}{n-b+2a-1}$.
$\big[$*Hint:* Using the notation of the preceding exercise, show that $\|\mathscr{M}_{m,j}(f)\|_{L^2} \le C2^{j(\frac{1}{2}-a)}\|f\|_{L^2}$ and that $\|\mathscr{M}_{m,j}(f)\|_{L^{1,\infty}} \le C2^{j(n-b)}\|f\|_{L^1}$ for all $j \in \mathbf{Z}^+$, where C is a constant depending on the given parameters.$\big]$

6.5.5. Show that Theorem 6.5.1 is false when $n = 1$, that is, show that the maximal operator

$$\mathscr{M}_1(f)(x) = \sup_{t>0} \frac{|f(x+t)+f(x-t)|}{2}$$

is unbounded on $L^p(\mathbf{R})$ for all $p < \infty$.

6.5.6. Show that when $n \ge 2$ and $p \le \frac{n}{n-1}$ there exists an $L^p(\mathbf{R}^n)$ function f such that $\mathscr{M}(f)(x) = \infty$ for all $x \in \mathbf{R}^n$. Hence Theorem 6.5.1 is false is this case.
$\big[$*Hint:* Choose a compactly supported and radial function equal to $|y|^{1-n}(-\log|y|)^{-1}$ when $|y| \le 1/2$.$\big]$

6.6 Wavelets and Sampling

In this section we construct orthonormal bases of $L^2(\mathbf{R})$ generated by translations and dilations of a single function. An example of such base is given by the Haar functions we encountered in Section 6.4. The Haar functions are generated by integer translations and dyadic dilations of the single function $\chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}$. This function is not smooth, and the main question addressed in this section is whether there exist smooth analogues of the Haar functions.

Definition 6.6.1. A square integrable function φ on $\mathbf{R}^n$ is called a *wavelet* if the family of functions

$$\varphi_{\nu,k}(x) = 2^{\frac{\nu n}{2}}\varphi(2^\nu x - k),$$

where ν ranges over $\mathbf{Z}$ and k over $\mathbf{Z}^n$, is an orthonormal basis of $L^2(\mathbf{R}^n)$. This means that the functions $\varphi_{\nu,k}$ are mutually orthogonal and span $L^2(\mathbf{R}^n)$, and φ is normalized to have L^2 norm equal to 1. Note that the Fourier transform of $\varphi_{\nu,k}$ is given by

$$\widehat{\varphi_{\nu,k}}(\xi) = 2^{-\frac{\nu n}{2}}\widehat{\varphi}(2^{-\nu}\xi)e^{-2\pi i 2^{-\nu}\xi\cdot k}. \tag{6.6.1}$$

Rephrasing the question posed earlier, the main issue addressed in this section is whether smooth wavelets actually exist. Before we embark on this topic, we recall that we have already encountered examples of nonsmooth wavelets.

Example 6.6.2. (The Haar wavelet) Recall the family of functions

$$h_I(x) = |I|^{-\frac{1}{2}}(\chi_{I_L} - \chi_{I_R}),$$

where I ranges over $\mathscr{D}$ (the set of all dyadic intervals) and I_L is the left part of I and I_R is the right part of I. Note that if $I = [2^{-\nu}k, 2^{-\nu}(k+1))$, then

$$h_I(x) = 2^{\frac{\nu}{2}}\varphi(2^\nu x - k),$$

where

$$\varphi(x) = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}. \tag{6.6.2}$$

The single function φ in (6.6.2) therefore generates the Haar basis by taking translations and dilations. Moreover, we observed in Section 6.4 that the family $\{h_I\}_I$ is orthonormal. Moreover, in Theorem 6.4.6 we obtained the representation

$$f = \sum_{I\in\mathscr{D}} \langle f, h_I\rangle h_I \qquad \text{in } L^2,$$

which proves the completeness of the system $\{h_I\}_{I\in\mathscr{D}}$ in $L^2(\mathbf{R})$.

6.6.1 Some Preliminary Facts

Before we look at more examples, we make some observations. We begin with the following useful fact.

Proposition 6.6.3. *Let $g \in L^1(\mathbf{R}^n)$. Then*

$$\widehat{g}(m) = 0 \qquad \text{for all } m \in \mathbf{Z}^n \setminus \{0\}$$

if and only if

$$\sum_{k\in\mathbf{Z}^n} g(x+k) = \int_{\mathbf{R}^n} g(t)\,dt$$

for almost all $x \in \mathbf{T}^n$.

Proof. We define the periodic function

$$G(x) = \sum_{k\in\mathbf{Z}^n} g(x+k),$$

which is easily shown to be in $L^1(\mathbf{T}^n)$. Moreover, we have

$$\widehat{G}(m) = \widehat{g}(m)$$

for all $m \in \mathbf{Z}^n$, where $\widehat{G}(m)$ denotes the mth Fourier coefficient of G and $\widehat{g}(m)$ denotes the Fourier transform of g at $\xi = m$. If $\widehat{g}(m) = 0$ for all $m \in \mathbf{Z}^n \setminus \{0\}$, then all the Fourier coefficients of G (except for $m = 0$) vanish, which means that the sequence $\{\widehat{G}(m)\}_{m\in\mathbf{Z}^n}$ lies in $\ell^1(\mathbf{Z}^n)$ and hence Fourier inversion applies. We conclude that for almost all $x \in \mathbf{T}^n$ we have

$$G(x) = \sum_{m\in\mathbf{Z}^n} \widehat{G}(m)e^{2\pi i m\cdot x} = \widehat{G}(0) = \widehat{g}(0) = \int_{\mathbf{R}^n} g(t)\,dt\,.$$

Conversely, if G is a constant, then $\widehat{G}(m) = 0$ for all $m \in \mathbf{Z}^n \setminus \{0\}$, and so the same holds for g. □

A consequence of the preceding proposition is the following.

Proposition 6.6.4. *Let $\varphi \in L^2(\mathbf{R}^n)$. Then the sequence*

$$\{\varphi(x-k)\}_{k\in\mathbf{Z}^n} \tag{6.6.3}$$

forms an orthonormal set in $L^2(\mathbf{R}^n)$ if and only if

$$\sum_{k\in\mathbf{Z}^n} |\widehat{\varphi}(\xi+k)|^2 = 1 \tag{6.6.4}$$

for almost all $\xi \in \mathbf{R}^n$.

Proof. Observe that either (6.6.4) or the hypothesis that the sequence in (6.6.3) is orthonormal implies that $\|\varphi\|_{L^2} = 1$. Also the orthonormality condition

$$\int_{\mathbf{R}^n} \varphi(x-j)\overline{\varphi(x-k)}\,dx = \begin{cases} 1 & \text{when } j = k, \\ 0 & \text{when } j \neq k, \end{cases}$$

is equivalent to

$$\int_{\mathbf{R}^n} e^{-2\pi i k\cdot\xi}\widehat{\varphi}(\xi)\overline{e^{-2\pi i j\cdot\xi}\widehat{\varphi}(\xi)}\,d\xi = (|\widehat{\varphi}|^2)\widehat{\ }(k-j) = \begin{cases} 1 & \text{when } j = k, \\ 0 & \text{when } j \neq k, \end{cases}$$

in view of Parseval's identity. Proposition 6.6.3 with $g(\xi) = |\widehat{\varphi}(\xi)|^2$ gives that the latter is equivalent to

$$\sum_{k\in\mathbf{Z}^n} |\widehat{\varphi}(\xi+k)|^2 = \int_{\mathbf{R}^n} |\widehat{\varphi}(t)|^2\,dt = 1$$

for almost all $\xi \in \mathbf{R}^n$. □

Corollary 6.6.5. *Let $\varphi \in L^2(\mathbf{R}^n)$ and suppose that the sequence*

$$\{\varphi(x-k)\}_{k\in\mathbf{Z}^n} \tag{6.6.5}$$

forms an orthonormal set in $L^2(\mathbf{R}^n)$. Then the measure of the support of $\widehat{\varphi}$ is at least 1*, that is,*

$$|\operatorname{supp}\widehat{\varphi}| \geq 1\,. \tag{6.6.6}$$

Moreover, if $|\operatorname{supp}\widehat{\varphi}| = 1$, then $|\widehat{\varphi}(\xi)| = 1$ for almost all $\xi \in \operatorname{supp}\widehat{\varphi}$.

Proof. It follows from (6.6.4) that $|\widehat{\varphi}| \leq 1$ for almost all $\xi \in \mathbf{R}^n$ and thus

$$|\operatorname{supp}\widehat{\varphi}| \geq \int_{\mathbf{R}^n} |\widehat{\varphi}(\xi)|^2\, d\xi = \int_{[0,1)^n} \sum_{k\in\mathbf{Z}^n} |\widehat{\varphi}(\xi+k)|^2\, d\xi = \int_{[0,1)^n} 1\, d\xi = 1\,.$$

If equality holds in (6.6.6), then equality holds in the preceding inequality, and since $|\widehat{\varphi}| \leq 1$ a.e., it follows that $|\widehat{\varphi}(\xi)| = 1$ for almost all ξ in $\operatorname{supp}\widehat{\varphi}$. □

6.6.2 Construction of a Nonsmooth Wavelet

Having established these preliminary facts, we now start searching for examples of wavelets. It follows from Corollary 6.6.5 that the support of the Fourier transform of a wavelet must have measure at least 1. It is reasonable to ask whether this support can have measure exactly 1. Example 6.6.6 indicates that this can indeed happen. As dictated by the same corollary, the Fourier transform of such a wavelet must satisfy $|\widehat{\varphi}(\xi)| = 1$ for almost all $\xi \in \operatorname{supp}\widehat{\varphi}$, so it is natural to look for a wavelet φ such that $\widehat{\varphi} = \chi_A$ for some set A. We can start by asking whether the function

$$\widehat{\varphi} = \chi_{[-\frac{1}{2},\frac{1}{2}]}$$

on $\mathbf{R}$ is an appropriate Fourier transform of a wavelet, but a moment's thought shows that the functions $\varphi_{\mu,0}$ and $\varphi_{\nu,0}$ cannot be orthogonal to each other when $\mu \neq 0$. The problem here is that the Fourier transforms of the functions $\varphi_{\nu,k}$ cluster near the origin and do not allow for the needed orthogonality. We can fix this problem by considering a function whose Fourier transform vanishes near the origin. Among such functions, a natural candidate is

$$\chi_{[-1,-\frac{1}{2})} + \chi_{[\frac{1}{2},1)}\,, \tag{6.6.7}$$

which is indeed the Fourier transform of a wavelet.

Example 6.6.6. Let $A = [-1,-\frac{1}{2}) \bigcup [\frac{1}{2},1)$ and define a function φ on $\mathbf{R}$ by setting

$$\widehat{\varphi} = \chi_A\,.$$

Then we assert that the family of functions

$$\{\varphi_{\nu,k}(x)\}_{k\in\mathbf{Z},\nu\in\mathbf{Z}} = \{2^{\nu/2}\varphi(2^\nu x-k)\}_{k\in\mathbf{Z},\nu\in\mathbf{Z}}$$

is an orthonormal basis of $L^2(\mathbf{R})$ (i.e., the function φ is a wavelet). This is an example of a wavelet with *minimally supported frequency*.

To verify this assertion, first note that $\{\varphi_{0,k}\}_{k\in\mathbf{Z}}$ is an orthonormal set, since (6.6.4) is easily seen to hold. Dilating by 2^ν, it follows that $\{\varphi_{\nu,k}\}_{k\in\mathbf{Z}}$ is also an orthonormal set for every fixed $\nu\in\mathbf{Z}$. Next, observe that if $\mu\neq\nu$, then

$$\operatorname{supp}\widehat{\varphi_{\nu,k}}\cap\operatorname{supp}\widehat{\varphi_{\mu,l}}=\emptyset\,. \tag{6.6.8}$$

This implies that the family $\{2^{\nu/2}\varphi(2^\nu x-k)\}_{k\in\mathbf{Z},\nu\in\mathbf{Z}}$ is also orthonormal.

Next, we observe that the completeness of $\{\varphi_{\nu,k}\}_{\nu,k\in\mathbf{Z}}$ is equivalent to that of $\{\widehat{\varphi_{\nu,k}}(\xi)\}_{\nu,k\in\mathbf{Z}} = \{2^{-\nu/2}e^{-2\pi ik\xi 2^{-\nu}}\chi_{2^\nu A}(\xi)\}_{\nu,k\in\mathbf{Z}}$. Let $f\in L^2(\mathbf{R})$, fix any $\nu\in\mathbf{Z}$, and define

$$h(\xi)=2^{\nu/2}f(2^\nu\xi).$$

Suppose that for all $k\in\mathbf{Z}$,

$$\begin{aligned}0=\langle f,\widehat{\varphi_{\nu,k}}\rangle &= \int_{2^\nu A} f(\xi)2^{-\nu/2}e^{-2\pi ik\xi 2^{-\nu}}\,d\xi\\ &= \int_A 2^{\nu/2}f(2^\nu\xi)e^{-2\pi ik\xi}\,d\xi\\ &= \langle\chi_A h, e^{-2\pi ik\xi}\rangle\,.\end{aligned}$$

Exercise 6.6.1(a) shows $\{e^{-2\pi ik\xi}\}_{k\in\mathbf{Z}}$ is an orthonormal basis of $L^2(A)$, and therefore $\chi_A h=0$ almost everywhere. From the definition of h it follows that $\chi_{2^\nu A}f=0$ almost everywhere. Now suppose for all $\nu,k\in\mathbf{Z}$

$$0=\langle f,\widehat{\varphi_{\nu,k}}\rangle.$$

Then $\chi_{2^\nu A}f=0$ almost everywhere for all $\nu\in\mathbf{Z}$. Since $\cup_{\nu\in\mathbf{Z}}2^\nu A=\mathbf{R}\setminus\{0\}$, it follows that $f=0$ almost everywhere. We conclude $\{\widehat{\varphi_{\nu,k}}\}_{\nu,k\in\mathbf{Z}}$ is complete.

6.6.3 Construction of a Smooth Wavelet

The wavelet basis of $L^2(\mathbf{R}^n)$ constructed in Example 6.6.6 is forced to have slow decay at infinity, since the Fourier transforms of the elements of the basis are nonsmooth. Smoothing out the function $\widehat{\varphi}$ but still expecting φ to be wavelet is a bit tricky, since property (6.6.8) may be violated when $\mu\neq\nu$, and moreover, (6.6.4) may be destroyed. These two obstacles are overcome by the careful construction of the next theorem.

Theorem 6.6.7. *There exists a Schwartz function φ on the real line that is a wavelet, that is, the collection of functions $\{\varphi_{\nu,k}\}_{k,\nu\in\mathbf{Z}}$ with $\varphi_{\nu,k}(x) = 2^{\frac{\nu}{2}}\varphi(2^\nu x - k)$ is an orthonormal basis of $L^2(\mathbf{R})$. Moreover, the function φ can be constructed so that its Fourier transform satisfies*

$$\operatorname{supp}\widehat{\varphi} \subseteq \left[-\tfrac{4}{3}, -\tfrac{1}{3}\right] \cup \left[\tfrac{1}{3}, \tfrac{4}{3}\right]. \tag{6.6.9}$$

Note that in view of condition (6.6.9), the function φ must have vanishing moments of all orders.

Proof. We start with an odd smooth real-valued function Θ on the real line such that $\Theta(t) = \frac{\pi}{4}$ for $t \geq \frac{1}{6}$ and such that Θ is strictly increasing on the interval $\left[-\frac{1}{6}, \frac{1}{6}\right]$. We set

$$\alpha(t) = \sin(\Theta(t) + \tfrac{\pi}{4}), \qquad \beta(t) = \cos(\Theta(t) + \tfrac{\pi}{4}),$$

and we observe that

$$\alpha(t)^2 + \beta(t)^2 = 1$$

and that

$$\alpha(-t) = \beta(t)$$

for all real t. Next we introduce the smooth function ω defined via

$$\omega(t) = \begin{cases} \beta(-\frac{t}{2} - \frac{1}{2}) = \alpha(\frac{t}{2} + \frac{1}{2}) & \text{when } t \in \left[-\frac{4}{3}, -\frac{2}{3}\right], \\ \alpha(-t - \frac{1}{2}) = \beta(t + \frac{1}{2}) & \text{when } t \in \left[-\frac{2}{3}, -\frac{1}{3}\right], \\ \alpha(t - \frac{1}{2}) & \text{when } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ \beta(\frac{t}{2} - \frac{1}{2}) & \text{when } t \in \left[\frac{2}{3}, \frac{4}{3}\right], \end{cases}$$

on the interval $\left[-\frac{4}{3}, -\frac{1}{3}\right] \bigcup \left[\frac{1}{3}, \frac{4}{3}\right]$. Note that ω is an even function. Finally we define the function φ by letting

$$\widehat{\varphi}(\xi) = e^{-\pi i \xi}\omega(\xi),$$

and we note that

$$\varphi(x) = \int_{\mathbf{R}} \omega(\xi) e^{2\pi i \xi (x - \frac{1}{2})} d\xi = 2\int_0^\infty \omega(\xi)\cos\left(2\pi(x - \tfrac{1}{2})\xi\right) d\xi.$$

It follows that the function φ is symmetric about the number $\frac{1}{2}$, that is, we have

$$\varphi(x) = \varphi(1 - x)$$

for all $x \in \mathbf{R}$. Note that φ is a Schwartz function whose Fourier transform is supported in the set $\left[-\frac{4}{3}, -\frac{1}{3}\right] \bigcup \left[\frac{1}{3}, \frac{4}{3}\right]$.

Having defined φ, we proceed by showing that it is a wavelet. In view of identity (6.6.1) we have that $\widehat{\varphi_{\nu,k}}$ is supported in the set $\frac{1}{3}2^\nu \leq |\xi| \leq \frac{4}{3}2^\nu$, while $\widehat{\varphi_{\mu,j}}$ is supported in the set $\frac{1}{3}2^\mu \leq |\xi| \leq \frac{4}{3}2^\mu$. The intersection of these sets has measure zero when $|\mu - \nu| \geq 2$, which implies that such wavelets are orthogonal to each other. Therefore, it suffices to verify orthogonality between adjacent scales (i.e., when $\nu = \mu$ and $\nu = \mu + 1$).

We begin with the case $\nu=\mu$, which, by a simple dilation, is reduced to the case $\nu=\mu=0$. Thus to obtain the orthogonality of the functions $\varphi_{0,k}(x)=\varphi(x-k)$ and $\varphi_{0,j}(x)=\varphi(x-j)$, in view of Proposition 6.6.4, it suffices to show that

$$\sum_{k\in\mathbf{Z}}|\widehat{\varphi}(\xi+k)|^2=1\,. \tag{6.6.10}$$

Since the sum in (6.6.10) is 1-periodic, we check that is equal to 1 only for ξ in $\left[\frac{1}{3},\frac{4}{3}\right]$. First for $\xi\in\left[\frac{1}{3},\frac{2}{3}\right]$, the sum in (6.6.10) is equal to

$$\begin{aligned}|\widehat{\varphi}(\xi)|^2+|\widehat{\varphi}(\xi-1)|^2 &=\omega(\xi)^2+\omega(\xi-1)^2\\ &=\alpha(\xi-\tfrac{1}{2})^2+\beta((\xi-1)+\tfrac{1}{2})^2\\ &=1\end{aligned}$$

from the definition of ω. A similar argument also holds for $\xi\in\left[\frac{2}{3},\frac{4}{3}\right]$, and this completes the proof of (6.6.10). As a consequence of this identity we also obtain that the functions $\varphi_{0,k}$ have L^2 norm equal to 1, and thus so have the functions $\varphi_{\nu,k}$, via a change of variables.

Next we prove the orthogonality of the functions $\varphi_{\nu,k}$ and $\varphi_{\nu+1,j}$ for general $\nu,k,j\in\mathbf{Z}$. We begin by observing the validity of the following identity:

$$\widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\tfrac{\xi}{2})}=\begin{cases}e^{-\pi i\xi/2}\beta(\frac{\xi}{2}-\frac{1}{2})\alpha(\frac{\xi}{2}-\frac{1}{2}) & \text{when } \frac{2}{3}\le\xi\le\frac{4}{3},\\ e^{-\pi i\xi/2}\alpha(\frac{\xi}{2}+\frac{1}{2})\beta(\frac{\xi}{2}+\frac{1}{2}) & \text{when } -\frac{4}{3}\le\xi\le-\frac{2}{3}.\end{cases} \tag{6.6.11}$$

Indeed, from the definition of φ, it follows that

$$\widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\tfrac{\xi}{2})}=e^{-\pi i\xi/2}\omega(\xi)\omega(\tfrac{\xi}{2})\,.$$

This function is supported in

$$\{\xi\in\mathbf{R}:\ \tfrac{1}{3}\le|\xi|\le\tfrac{4}{3}\}\cap\{\xi\in\mathbf{R}:\ \tfrac{2}{3}\le|\xi|\le\tfrac{8}{3}\}=\{\xi\in\mathbf{R}:\ \tfrac{2}{3}\le|\xi|\le\tfrac{4}{3}\},$$

and on this set it is equal to

$$e^{-\pi i\xi/2}\begin{cases}\beta(\frac{\xi}{2}-\frac{1}{2})\alpha(\frac{\xi}{2}-\frac{1}{2}) & \text{when } \frac{2}{3}\le\xi\le\frac{4}{3},\\ \alpha(\frac{\xi}{2}+\frac{1}{2})\beta(\frac{\xi}{2}+\frac{1}{2}) & \text{when } -\frac{4}{3}\le\xi\le-\frac{2}{3},\end{cases}$$

by the definition of ω. This establishes (6.6.11).

We now turn to the orthogonality of the functions $\varphi_{\nu,k}$ and $\varphi_{\nu+1,j}$ for general $\nu,k,j\in\mathbf{Z}$. Using (6.6.1) and (6.6.11) we have

$$\begin{aligned}\left\langle\varphi_{\nu,k}\,|\,\varphi_{\nu+1,j}\right\rangle &= \left\langle\widehat{\varphi_{\nu,k}}\,|\,\widehat{\varphi_{\nu+1,j}}\right\rangle\\ &=\int_{\mathbf{R}}2^{-\frac{\nu}{2}}\widehat{\varphi}(2^{-\nu}\xi)e^{-2\pi i\frac{\xi k}{2^\nu}}\,\overline{2^{-\frac{\nu+1}{2}}\widehat{\varphi}(2^{-(\nu+1)}\xi)e^{-2\pi i\frac{\xi j}{2^{\nu+1}}}}\,d\xi\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}} \int_{\mathbf{R}} \widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\tfrac{\xi}{2})} e^{-2\pi i \xi (k-\frac{j}{2})}\, d\xi \\
&= \frac{1}{\sqrt{2}} \int_{-\frac{4}{3}}^{-\frac{2}{3}} \alpha(\tfrac{\xi}{2}+\tfrac{1}{2})\beta(\tfrac{\xi}{2}+\tfrac{1}{2}) e^{-2\pi i \xi (k-\frac{j}{2}+\frac{1}{4})}\, d\xi \\
&\qquad + \frac{1}{\sqrt{2}} \int_{\frac{2}{3}}^{\frac{4}{3}} \alpha(\tfrac{\xi}{2}-\tfrac{1}{2})\beta(\tfrac{\xi}{2}-\tfrac{1}{2}) e^{-2\pi i \xi (k-\frac{j}{2}+\frac{1}{4})}\, d\xi \\
&= 0,
\end{aligned}
$$

where the last identity follows from the change of variables $\xi = \xi' - 2$ in the second-to-last integral, which transforms its range of integration to $\left[\frac{2}{3}, \frac{4}{3}\right]$ and its integrand to the negative of that of the last displayed integral.

Our final task is to show that the orthonormal system $\{\varphi_{\nu,k}\}_{\nu,k\in\mathbf{Z}}$ is complete. We show this by proving that whenever a square-integrable function f satisfies

$$
\langle f \,|\, \varphi_{\nu,k} \rangle = 0 \tag{6.6.12}
$$

for all $\nu, k \in \mathbf{Z}$, then f must be zero. Suppose that (6.6.12) holds. Plancherel's identity yields

$$
\int_{\mathbf{R}} \widehat{f}(\xi) 2^{-\frac{\nu}{2}} \overline{\widehat{\varphi}(2^{-\nu}\xi) e^{-2\pi i 2^{-\nu}\xi k}}\, d\xi = 0
$$

for all ν, k and thus

$$
\int_{\mathbf{R}} \widehat{f}(2^{\nu}\xi)\overline{\widehat{\varphi}(\xi)} e^{2\pi i \xi k}\, d\xi = \big(\widehat{f}(2^{\nu}(\cdot))\,\overline{\widehat{\varphi}}\big)\widehat{\ }(-k) = 0 \tag{6.6.13}
$$

for all $\nu, k \in \mathbf{Z}$. It follows from Proposition 6.6.3 and (6.6.13) (with $k = 0$) that

$$
\sum_{k\in\mathbf{Z}} \widehat{f}(2^{\nu}(\xi+k))\overline{\widehat{\varphi}(\xi+k)} = \int_{\mathbf{R}} \widehat{f}(2^{\nu}\xi)\,\overline{\widehat{\varphi}(\xi)}\, d\xi = \big(\widehat{f}(2^{\nu}(\cdot))\,\overline{\widehat{\varphi}}\big)\widehat{\ }(0) = 0
$$

for all $\nu \in \mathbf{Z}$.

Next, we show that the identity

$$
\sum_{k\in\mathbf{Z}} \widehat{f}(2^{\nu}(\xi+k))\overline{\widehat{\varphi}(\xi+k)} = 0 \tag{6.6.14}
$$

for all $\nu \in \mathbf{Z}$ implies that $\widehat{f}$ is identically equal to zero. Suppose that $\frac{1}{3} \le \xi \le \frac{2}{3}$. In this case the support properties of $\widehat{\varphi}$ imply that the only terms in the sum in (6.6.14) that do not vanish are $k = 0$ and $k = -1$. Thus for $\frac{1}{3} \le \xi \le \frac{2}{3}$ the identity in (6.6.14) reduces to

$$
\begin{aligned}
0 &= \widehat{f}(2^{\nu}(\xi-1))\overline{\widehat{\varphi}(\xi-1)} + \widehat{f}(2^{\nu}\xi)\overline{\widehat{\varphi}(\xi)} \\
&= \widehat{f}(2^{\nu}(\xi-1)) e^{\pi i(\xi-1)} \beta((\xi-1)+\tfrac{1}{2}) + \widehat{f}(2^{\nu}\xi) e^{\pi i \xi} \alpha(\xi-\tfrac{1}{2});
\end{aligned}
$$

hence

$$-\widehat{f}(2^\nu(\xi-1))\beta(\xi-\tfrac{1}{2})+\widehat{f}(2^\nu\xi)\alpha(\xi-\tfrac{1}{2})=0, \quad \tfrac{1}{3}\le\xi\le\tfrac{2}{3}. \tag{6.6.15}$$

Next we observe that when $\frac{2}{3}\le\xi\le\frac{4}{3}$, only the terms with $k=0$ and $k=-2$ survive in the identity in (6.6.14). This is because when $k=-1$, $\xi+k=\xi-1\in\left[-\frac{1}{3},\frac{1}{3}\right]$ and this interval has null intersection with the support of $\widehat{\varphi}$. Therefore, (6.6.14) reduces to

$$\begin{aligned}0 &= \widehat{f}(2^\nu(\xi-2))\overline{\widehat{\varphi}(\xi-2)}+\widehat{f}(2^\nu\xi)\overline{\widehat{\varphi}(\xi)}\\ &= \widehat{f}(2^\nu(\xi-2))e^{\pi i(\xi-2)}\alpha(\tfrac{\xi-2}{2}+\tfrac{1}{2})+\widehat{f}(2^\nu\xi)e^{\pi i\xi}\beta(\tfrac{\xi}{2}-\tfrac{1}{2});\end{aligned}$$

hence

$$\widehat{f}(2^\nu(\xi-2))\alpha(\tfrac{\xi}{2}-\tfrac{1}{2})+\widehat{f}(2^\nu\xi)\beta(\tfrac{\xi}{2}-\tfrac{1}{2})=0, \quad \tfrac{2}{3}\le\xi\le\tfrac{4}{3}. \tag{6.6.16}$$

Replacing first ν by $\nu-1$ and then $\frac{\xi}{2}$ by ξ in (6.6.16), we obtain

$$\widehat{f}(2^\nu(\xi-1))\alpha(\xi-\tfrac{1}{2})+\widehat{f}(2^\nu\xi)\beta(\xi-\tfrac{1}{2})=0, \quad \tfrac{1}{3}\le\xi\le\tfrac{2}{3}. \tag{6.6.17}$$

Now consider the 2×2 system of equations given by (6.6.15) and (6.6.17) with unknown $\widehat{f}(2^\nu(\xi-1))$ and $\widehat{f}(2^\nu\xi)$. The determinant of the system is

$$\det\begin{pmatrix}-\beta(\xi-1/2) & \alpha(\xi-1/2)\\ \alpha(\xi-1/2) & \beta(\xi-1/2)\end{pmatrix}=-1\neq 0.$$

Therefore, the system has the unique solution

$$\widehat{f}(2^\nu(\xi-1))=\widehat{f}(2^\nu\xi)=0,$$

which is valid for all $\nu\in\mathbf{Z}$ and all $\xi\in[\frac{1}{3},\frac{2}{3}]$. We conclude that $\widehat{f}(\xi)=0$ for all $\xi\in\mathbf{R}$ and thus $f=0$. This proves the completeness of the system $\{\varphi_{\nu,k}\}$. We conclude that the function φ is a wavelet. □

6.6.4 Sampling

Next we discuss how one can recover a band-limited function by its values at a countable number of points.

Definition 6.6.8. An integrable function on $\mathbf{R}^n$ is called *band limited* if its Fourier transform has compact support.

For every band-limited function there is a $B>0$ such that its Fourier transform is supported in the cube $[-B,B]^n$. In such a case we say that the function is band limited on the cube $[-B,B]^n$.

It is an interesting observation that such functions are completely determined by their values at the points $x = k/2B$, where $k \in \mathbf{Z}^n$. We have the following result.

Theorem 6.6.9. *(a) Let f in $L^1(\mathbf{R}^n)$ be band limited on the cube $[-B,B]^n$. Then f can be sampled by its values at the points $x = k/2B$, where $k \in \mathbf{Z}^n$. In particular, we have*

$$f(x_1,\dots,x_n) = \sum_{k\in\mathbf{Z}^n} f\Big(\frac{k}{2B}\Big) \prod_{j=1}^{n} \frac{\sin(2\pi Bx_j - \pi k_j)}{2\pi Bx_j - \pi k_j} \tag{6.6.18}$$

for almost all $x \in \mathbf{R}^n$.
(b) Suppose that f is band-limited on the cube $[-B',B']^n$ where $0 < B' < B$. Then f can be sampled by its values at the points $x = k/2B$, $k \in \mathbf{Z}^n$ as follows

$$f(x_1,\dots,x_n) = \sum_{k\in\mathbf{Z}^n} f\Big(\frac{k}{2B}\Big)\Phi(x-k), \tag{6.6.19}$$

for some Schwartz function Φ that depends on B, B'.

Proof. Since the function $\widehat{f}$ is supported in $[-B,B]^n$, we use Exercise 6.6.2 to obtain

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{(2B)^n} \sum_{k\in\mathbf{Z}^n} \widehat{\widehat{f}}\Big(\frac{k}{2B}\Big) e^{2\pi i \frac{k}{2B}\cdot\xi} \\ &= \frac{1}{(2B)^n} \sum_{k\in\mathbf{Z}^n} f\Big(-\frac{k}{2B}\Big) e^{2\pi i \frac{k}{2B}\cdot\xi}.\end{aligned}$$

Inserting this identity in the inversion formula

$$f(x) = \int_{[-B,B]^n} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi,$$

which holds for almost all $x \in \mathbf{R}^n$ since $\widehat{f}$ is continuous and therefore integrable over $[-B,B]^n$, we obtain

$$\begin{aligned} f(x) &= \int_{[-B,B]^n} \frac{1}{(2B)^n} \sum_{k\in\mathbf{Z}^n} f\Big(-\frac{k}{2B}\Big) e^{2\pi i \frac{k}{2B}\cdot\xi} e^{2\pi i x\cdot\xi}\, d\xi \\ &= \sum_{k\in\mathbf{Z}^n} f\Big(-\frac{k}{2B}\Big) \frac{1}{(2B)^n} \int_{[-B,B]^n} e^{2\pi i(\frac{k}{2B}+x)\cdot\xi}\, d\xi \qquad (6.6.20) \\ &= \sum_{k\in\mathbf{Z}^n} f\Big(-\frac{k}{2B}\Big) \prod_{j=1}^{n} \frac{\sin(2\pi Bx_j + \pi k_j)}{2\pi Bx_j + \pi k_j}. \qquad (6.6.21) \end{aligned}$$

This is exactly (6.6.18) when we change k to $-k$ and thus part (a) is proved. For part (b) we argue similarly, except that we replace $\chi_{[-B,B]^n}$ by $\widehat{\Phi}$, where $\widehat{\Phi}$ is smooth, equal to 1 on $[-B',B']^n$ and vanishes outside $[-B,B]^n$. Then we can insert the function $\widehat{\Phi}(\xi)$ in (6.6.20) and instead of (6.6.21) we obtain the expression on the right in (6.6.19). □

Remark 6.6.10. Identity (6.6.18) holds for any $B'' > B$. In particular, we have

$$\sum_{k\in\mathbf{Z}^n} f\Big(\frac{k}{2B}\Big)\prod_{j=1}^n \frac{\sin(2\pi Bx_j-\pi k_j)}{2\pi Bx_j-\pi k_j} = \sum_{k\in\mathbf{Z}^n} f\Big(\frac{k}{2B''}\Big)\prod_{j=1}^n \frac{\sin(2\pi B''x_j-\pi k_j)}{2\pi B''x_j-\pi k_j}$$

for all $x\in\mathbf{R}^n$ whenever f is band-limited in $[-B,B]^n$. In particular, band-limited functions in $[-B,B]^n$ can be sampled by their values at the points $k/2B''$ for any $B''\ge B$.

However, band-limited functions in $[-B,B]^n$ cannot be sampled by the points $k/2B'$ for any $B'<B$, as the following example indicates.

Example 6.6.11. For $0<B'<B$, let $f(x)=g(x)\sin(2\pi B'x)$, where $\widehat{g}$ is supported in the interval $[-(B-B'),B-B']$. Then f is band limited in $[-B,B]$, but it cannot be sampled by its values at the points $k/2B'$, since it vanishes at these points and f is not identically zero if g is not the zero function.

Next, we give a couple of results that relate the L^p norm of a given function with the ℓ^p norm (or quasi-norm) of its sampled values.

Theorem 6.6.12. *Let f be a tempered*[1] *function whose Fourier transform is supported in the closed ball $\overline{B(0,t)}$ for some $0<t<\infty$. Assume that f lies in $L^p(\mathbf{R}^n)$ for some $0<p\le\infty$. Then there is a constant $C(n,p)$ such that*

$$\big\|\{f(k)\}_{k\in\mathbf{Z}^n}\big\|_{\ell^p(\mathbf{Z}^n)} \le C(n,p)\,t\,(1+t^{\frac{2n}{p}})\big\|f\big\|_{L^p(\mathbf{R}^n)}\,.$$

Proof. The proof is based on the following fact, whose proof can be found in [131] (Lemma 2.2.3). Let $0<r<\infty$. Then there exists a constant $C_2=C_2(n,r)$ such that for all $t>0$ and for all $\mathscr{C}^1$ functions u on $\mathbf{R}^n$ whose distributional Fourier transform is supported in the ball $|\xi|\le t$ we have

$$\sup_{z\in\mathbf{R}^n}\frac{1}{t}\frac{|\nabla u(x-z)|}{(1+t|z|)^{\frac{n}{r}}} \le C_2 M(|u|^r)(x)^{\frac{1}{r}}\,, \tag{6.6.22}$$

where M denotes the Hardy–Littlewood maximal operator.

Notice that f is a $\mathscr{C}^\infty$ function since its Fourier transform is compactly supported. Assuming (6.6.22), for each $k\in\mathbf{Z}^n$ and $x\in[0,1]^n$ we use the mean value theorem to obtain

$$\begin{aligned}|f(k)| &\le |f(x+k)|+\sqrt{n}\sup_{z\in[0,1]^n}|\nabla f(z+k)|\\ &\le |f(x+k)|+\sqrt{n}\sup_{z\in B(x+k,\sqrt{n})}|\nabla f(z)|\,.\end{aligned}$$

We raise this inequality to the power p, we integrate over the cube $[0,1]^n$, we sum over $k\in\mathbf{Z}^n$, and then we take the $1/p$ power. Let $c_p=\max(1,2^{1/p-1})$ and $c(n,r,t)=$

[1] A function is called tempered if there are constants C,M such that $|f(x)|\le C(1+|x|)^M$ for all $x\in\mathbf{R}^n$. Tempered functions are tempered distributions.

$\sqrt{n}t(1+t\sqrt{n})^{n/r}$. The sum over k and the integral over $[0,1]^n$ yield an integral over $\mathbf{R}^n$ and thus we obtain

$$\begin{aligned}
\Big[\sum_{k\in\mathbf{Z}^n}|f(k)|^p\Big]^{\frac{1}{p}} &\le \Big[\int_{\mathbf{R}^n}|f(x)+\sqrt{n}\sup_{z\in B(x,\sqrt{n})}|\nabla f(z)|^p\,dx\Big]^{\frac{1}{p}}\\
&\le c_p\Big[\big\|f\big\|_{L^p}+\sqrt{n}\Big(\int_{\mathbf{R}^n}\sup_{z\in B(0,\sqrt{n})}|\nabla f(x-z)|^p\,dx\Big)^{\frac{1}{p}}\Big]\\
&\le c_p\Big[\big\|f\big\|_{L^p}+c(n,r,t)\Big(\int_{\mathbf{R}^n}\Big\{\sup_{z\in B(0,\sqrt{n})}\frac{|\nabla f(x-z)|}{t(1+t|z|)^{\frac{n}{r}}}\Big\}^p dx\Big)^{\frac{1}{p}}\Big]\\
&\le c_p\Big[\big\|f\big\|_{L^p}+c(n,r,t)\Big(\int_{\mathbf{R}^n}\Big\{\sup_{z\in\mathbf{R}^n}\frac{|\nabla f(x-z)|}{t(1+t|z|)^{\frac{n}{r}}}\Big\}^p dx\Big)^{\frac{1}{p}}\Big]\\
&\le c_p\Big[\big\|f\big\|_{L^p}+c(n,r,t)C_2\Big(\int_{\mathbf{R}^n}[M(|f|^r)(x)]^{\frac{p}{r}}\,dx\Big)^{\frac{1}{p}}\Big],
\end{aligned}$$

where the last step uses (6.6.22). We now select $r=p/2$ if $p<\infty$ and r to be any number if $p=\infty$. The required inequality follows from the boundedness of the Hardy-Littlewood maximal operator on L^2 if $p<\infty$ or on L^∞ if $p=\infty$. □

The next theorem could be considered a partial converse of Theorem 6.6.13

Theorem 6.6.13. *Suppose that an integrable function f has Fourier transform supported in the cube $[-(\frac{1}{2}-\varepsilon),\frac{1}{2}-\varepsilon]^n$ for some $0<\varepsilon<1/2$. Furthermore, suppose that the sequence of coefficients $\{f(k)\}_{k\in\mathbf{Z}^n}$ lies in $\ell^p(\mathbf{Z}^n)$ for some $0<p\le\infty$. Then f lies in $L^p(\mathbf{R}^n)$ and the following estimate is valid*

$$\big\|f\big\|_{L^p(\mathbf{R}^n)}\le C_{n,p,\varepsilon}\big\|\{f(k)\}_k\big\|_{\ell^p(\mathbf{Z}^n)}\,. \tag{6.6.23}$$

Proof. We fix a smooth function $\widehat{\Phi}$ supported in $[-\frac{1}{2},\frac{1}{2}]^n$ and equal to 1 on the smaller cube $[-(\frac{1}{2}-\varepsilon),\frac{1}{2}-\varepsilon]^n$. Then we may write $f=f*\Phi$, since $\widehat{\Phi}$ is equal to one on the support of $\widehat{f}$. Writing $\widehat{f}$ in terms of its Fourier series we have

$$\widehat{f}(\xi)=\sum_{k\in\mathbf{Z}^n}\widehat{\widehat{f}}(k)e^{2\pi ik\cdot\xi}\chi_{[-\frac{1}{2},\frac{1}{2}]^n}=\sum_{k\in\mathbf{Z}^n}f(-k)e^{2\pi ik\cdot\xi}\chi_{[-\frac{1}{2},\frac{1}{2}]^n} \tag{6.6.24}$$

Since f is integrable, $\widehat{f}$ is continuous and thus integrable over $[-\frac{1}{2},\frac{1}{2}]^n$. By Fourier inversion we have

$$f(x)=\int_{[-\frac{1}{2},\frac{1}{2}]^n}\widehat{f}(\xi)e^{2\pi ix\cdot\xi}d\xi=\int_{[-\frac{1}{2},\frac{1}{2}]^n}\widehat{f}(\xi)\widehat{\Phi}(\xi)e^{2\pi ix\cdot\xi}d\xi \tag{6.6.25}$$

for almost all $x\in\mathbf{R}^n$. Inserting (6.6.25) in (6.6.24) we obtain

$$f(x)=\int_{[-\frac{1}{2},\frac{1}{2}]^n}\sum_{k\in\mathbf{Z}^n}f(-k)e^{2\pi ik\cdot\xi}e^{2\pi ix\cdot\xi}\widehat{\Phi}(\xi)\,d\xi$$

$$= \sum_{k\in\mathbf{Z}^n} f(k) \int_{[-\frac{1}{2},\frac{1}{2}]^n} e^{-2\pi i k\cdot\xi} e^{2\pi i x\cdot\xi}\,\widehat{\Phi}(\xi)\,d\xi$$
$$= \sum_{k\in\mathbf{Z}^n} f(k)\Phi(x-k)\,.$$

This identity combined with the rapid decay of Φ yields (6.6.23) as follows. For $0 < p \le 1$ we have

$$\|f\|_{L^p}^p \le \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}^n} |f(k)|^p |\Phi(x-k)|^p = \big\|\{f(k)\}_k\big\|_{\ell^p(\mathbf{Z}^n)}^p \|\Phi\|_{L^p}^p$$

while for $1 < p \le \infty$, setting $Q = [-\frac{1}{2},\frac{1}{2}]^n$ we write:

$$\begin{aligned}
\|f\|_{L^p(\mathbf{R}^n)} &\le \bigg[\sum_{l\in\mathbf{Z}^n}\int_{l+Q}\Big(\sum_{k\in\mathbf{Z}^n}|f(k)||\Phi(x-k)|\Big)^p dx\bigg]^{\frac{1}{p}}\\
&\le C_{n,N}\bigg[\sum_{l\in\mathbf{Z}^n}\int_{l+Q}\Big(\sum_{k\in\mathbf{Z}^n}|f(k)|\frac{1}{(2\sqrt{n}+|x-k|)^N}\Big)^p dx\bigg]^{\frac{1}{p}}\\
&\le C'_{n,N}\bigg[\sum_{l\in\mathbf{Z}^n}\int_{l+Q}\Big(\sum_{k\in\mathbf{Z}^n}|f(k)|\frac{1}{(\sqrt{n}+|l-k|)^N}\Big)^p dx\bigg]^{\frac{1}{p}}\\
&\le C'_{n,N}\bigg[\sum_{l\in\mathbf{Z}^n}\Big(\sum_{k\in\mathbf{Z}^n}|f(k)|\frac{1}{(\sqrt{n}+|l-k|)^N}\Big)^p\bigg]^{\frac{1}{p}}.
\end{aligned}$$

The preceding expression can be viewed as the ℓ^p norm of the discrete convolution of the sequences $\{f(k)\}_k$ and $\frac{1}{(\sqrt{n}+|k|)^N}$ and thus it is bounded by a constant multiple of $\big\|\{f(k)\}_k\big\|_{\ell^p(\mathbf{Z}^n)}$, since the sequence $\frac{1}{(\sqrt{n}+|k|)^N}$ is in $\ell^1(\mathbf{Z}^n)$ if N is large enough. This completes the proof. □

Exercise 6.6.6 gives examples of functions for which Theorem 6.6.13 fails if $\varepsilon = 0$.

Exercises

6.6.1. (a) Let $A = [-1,-\frac{1}{2})\bigcup[\frac{1}{2},1)$. Show that the family $\{e^{2\pi i m x}\}_{m\in\mathbf{Z}}$ is an orthonormal basis of $L^2(A)$.
(b) Obtain the same conclusion for the family $\{e^{2\pi i m\cdot x}\}_{m\in\mathbf{Z}^n}$ in $L^2(A^n)$.
$\big[$*Hint:* To show completeness, given $f \in L^2(A)$, define h on $[0,1]$ by setting $h(x) = f(x-1)$ for $x \in [0,\frac{1}{2})$ and $h(x) = f(x)$ for $x \in [\frac{1}{2},1)$. Observe that $\widehat{h}(m) = \widehat{f}(m)$ for all $m \in \mathbf{Z}$ and expand h in Fourier series.$\big]$

6.6.2. Let g be an integrable function on $\mathbf{R}^n$.
(a) Suppose that g is supported in $[-b,b]^n$ for some $b>0$ and that the sequence $\{\widehat{g}(k/2b)\}_{k\in\mathbf{Z}^n}$ lies in $\ell^2(\mathbf{Z}^n)$. Show that

$$g(x) = (2b)^{-n}\sum_{k\in\mathbf{Z}^n}\widehat{g}(\tfrac{k}{2b})e^{2\pi i\frac{k}{2b}\cdot x}\chi_{[-b,b]^n}\,,$$

where the series converges in $L^2(\mathbf{R}^n)$ and deduce that g is in $L^2(\mathbf{R}^n)$.
(b) Suppose that g is supported in $[0,b]^n$ for some $b>0$ and that the sequence $\{\widehat{g}(k/b)\}_{k\in\mathbf{Z}^n}$ lies in $\ell^2(\mathbf{Z}^n)$. Show that

$$g(x) = b^{-n}\sum_{k\in\mathbf{Z}^n}\widehat{g}(\tfrac{k}{b})e^{2\pi i\frac{k}{b}\cdot x}\chi_{[0,b]^n}\,,$$

where the series converges in $L^2(\mathbf{R}^n)$ and deduce that g is in $L^2(\mathbf{R}^n)$.
(c) When $n=1$, obtain the same as the conclusion in part (b) for $x\in[-b,-\frac{b}{2})\bigcup[\frac{b}{2},b)$, provided g is supported in this set.
$\big[$*Hint:* Part (c): Use the result in Exercise 6.6.1.$\big]$

6.6.3. Show that the sequence of functions

$$H_k(x_1,\dots,x_n) = (2B)^{\frac{n}{2}}\prod_{j=1}^{n}\frac{\sin\big(\pi(2Bx_j-k_j)\big)}{\pi(2Bx_j-k_j)}\,,\qquad k\in\mathbf{Z}^n\,,$$

is orthonormal in $L^2(\mathbf{R}^n)$.
$\big[$*Hint:* Interpret the functions H_k as the Fourier transforms of known functions.$\big]$

6.6.4. Prove the following spherical multidimensional version of Theorem 6.6.9. Suppose that $\widehat{f}$ is supported in the ball $|\xi|\le R$. Show that

$$f(x) = \sum_{k\in\mathbf{Z}^n} f\big(-\tfrac{k}{2R}\big)\frac{1}{2^n}\frac{J_{\frac{n}{2}}(2\pi|Rx+\frac{k}{2}|)}{|Rx+\frac{k}{2}|^{\frac{n}{2}}}\,,$$

where J_a is the Bessel function of order a.

6.6.5. Let $\{a_k\}_{k\in\mathbf{Z}^n}$ be in ℓ^p for some $1<p<\infty$. Show that the partial sums

$$\sum_{\substack{k\in\mathbf{Z}^n\\|k|\le N}} a_k\prod_{j=1}^{n}\frac{\sin(2\pi Bx_j-\pi k_j)}{2\pi Bx_j-\pi k_j}$$

converge in $\mathscr{S}'(\mathbf{R}^n)$ as $N\to\infty$ to an L^p function on $\mathbf{R}^n$ whose Fourier transform is supported in $[-B,B]^n$. Here $k=(k_1,\dots,k_n)$. Moreover, the L^p norm of A is controlled by a constant multiple of the ℓ^p norm of $\{a_k\}_k$.

6.6.6. Consider the function $\prod_{j=1}^n\sin(\pi x_j)/(\pi x_j)$ on $\mathbf{R}^n$ to show that Theorem 6.6.13 fails when $\varepsilon=0$ and $p\le 1$. When $1<p\le\infty$ consider the function $x_1+\prod_{j=1}^n\sin(\pi x_j)/(\pi x_j)$.

6.6.7. (a) Let $\psi(x)$ be a nonzero continuous integrable function on $\mathbf{R}$ that satisfies $\int_{\mathbf{R}} \psi(x)\,dx = 0$ and

$$C_\psi = \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(t)|^2}{|t|}\,dt < \infty.$$

Define the *wavelet transform* of f in $L^2(\mathbf{R})$ by setting

$$W(f;a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} f(x)\overline{\psi\Big(\frac{x-b}{a}\Big)}\,dx$$

when $a \neq 0$ and $W(f;0,b) = 0$. Show that for any $f \in L^2(\mathbf{R})$ the following inversion formula holds:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{|a|^{\frac{1}{2}}}\psi\Big(\frac{x-b}{a}\Big) W(f;a,b)\,db\,\frac{da}{a^2}.$$

(b) State and prove an analogous wavelet transform inversion property on $\mathbf{R}^n$.
[*Hint:* Apply Theorem 2.2.14 (5) in the b-integral and use Fourier inversion.]

6.6.8. (*P. Casazza*) On $\mathbf{R}^n$ let e_j be the vector whose coordinates are zero everywhere except for the jth entry, which is 1. Set $q_j = e_j - \frac{1}{n}\sum_{k=1}^n e_k$ for $1 \le j \le n$ and also $q_{n+1} = \frac{1}{\sqrt{n}}\sum_{k=1}^n e_k$. Prove that

$$\sum_{j=1}^{n+1} |q_j \cdot x|^2 = |x|^2$$

for all $x \in \mathbf{R}^n$. This provides an example of a *tight frame* on $\mathbf{R}^n$.

HISTORICAL NOTES

An early account of square functions in the context of Fourier series appears in the work of Kolmogorov [196], who proved the almost everywhere convergence of lacunary partial sums of Fourier series of periodic square-integrable functions. This result was systematically studied and extended to L^p functions, $1 < p < \infty$, by Littlewood and Paley [227], [228], [229] using complex-analysis techniques. The real-variable treatment of the Littlewood and Paley theorem was pioneered by Stein [334] and allowed the higher-dimensional extension of the theory. The use of vector-valued inequalities in the proof of Theorem 6.1.2 is contained in Benedek, Calderón, and Panzone [22]. A Littlewood–Paley theorem for lacunary sectors in $\mathbf{R}^2$ was obtained by Nagel, Stein, and Wainger [264].

An interesting Littlewood–Paley estimate holds for $2 \le p < \infty$: There exists a constant C_p such that for all families of disjoint open intervals I_j in $\mathbf{R}$ the estimate $\|(\sum_j |(\widehat{f}\chi_{I_j})^\vee|^2)^{\frac{1}{2}}\|_{L^p} \le C_p\|f\|_{L^p}$ holds for all functions $f \in L^p(\mathbf{R})$. This was proved by Rubio de Francia [301], but the special case in which $I_j = (j, j+1)$ was previously obtained by Carleson [55]. An alternative proof of Rubio de Francia's theorem was obtained by Bourgain [34]. A higher-dimensional analogue of this estimate for arbitrary disjoint open rectangles in $\mathbf{R}^n$ with sides parallel to the axes was obtained by Journé [181]. Easier proofs of the higher-dimensional result were subsequently obtained by Sjölin [326], Soria [329], and Sato [311].

Part (a) of Theorem 6.2.7 is due to Mihlin [254] and the generalization in part (b) to Hörmander [159]. Theorem 6.2.2 can be found in Marcinkiewicz's article [241] in the context of one-dimensional Fourier series. Calderón and Torchinsky [45] have improved Theorem 6.2.7 in the following way: if for a suitable smooth bump η supported in an annulus the functions $m(2^k\xi)\eta(\xi)$ lie in the Sobolev space L^r_γ uniformly in $k \in \mathbf{Z}$, where $\gamma > n(\frac{1}{p} - \frac{1}{2})$, $1 < p < 2$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$, then m lies in $\mathscr{M}_p(\mathbf{R}^n)$. The power 6 in estimate (6.2.3) that appears in the statement of Theorem 6.2.2 is not optimal. Tao and Wright [357] proved that in dimension 1, the best power of $(p-1)^{-1}$ in this theorem is $\frac{3}{2}$ as $p \to 1$. An improvement of the Marcinkiewicz multiplier theorem in one dimension was obtained by Coifman, Rubio de Francia, and Semmes [69]. Weighted norm estimates for Hörmander–Mihlin multipliers were obtained by Kurtz and Wheeden [209] and for Marcinkiwiecz multipliers by Kurtz [208]. Heo, Nazarov, and Seeger [150] have obtained a very elegant characterization of radial L^p multipliers in large dimensions; precisely, they showed that for dimensions $n \ge 4$ and $1 < p < \frac{2n-2}{n+1}$, a radial function m on $\mathbf{R}^n$ is an L^p Fourier multiplier if and only if there exists a nonzero Schwartz function η such that $\sup_{t>0} t^{n/p} \|(m(\cdot)\eta(t\,\cdot))^\vee\|_{L^p} < \infty$. This characterization builds on and extends a previously obtained simple characterization by Garrigós and Seeger [124] of radial multipliers on the invariant subspace of radial L^p functions when $1 < p < \frac{2n}{n+1}$.

The method of proof of Theorem 6.3.4 is adapted from Duoandikoetxea and Rubio de Francia [102]. The method in this article is rather general and can be used to obtain L^p boundedness for a variety of rough singular integrals. A version of Theorem 6.3.6 was used by Christ [59] to obtain L^p smoothing estimates for Cantor–Lebesgue measures. When $p = q \neq 2$, Theorem 6.3.6 is false in general, but it is true for all r satisfying $|\frac{1}{r} - \frac{1}{2}| < |\frac{1}{p} - \frac{1}{2}|$ under the additional assumption that the m_j's are Lipschitz functions uniformly at all scales. This result was independently obtained by Carbery [52] and Seeger [316]. Miyachi [255] has obtained a complete characterization of the indices $a, b > 0$ such that the functions $|x|^{-b} e^{i|x|^a} \psi(x)$ are L^p Fourier multipliers; here ψ is a smooth function that is equal to 1 near infinity and vanishes near zero.

The probabilistic notions of conditional expectations and martingales have a strong connection with the Littlewood–Paley theory discussed in this chapter. For the purposes of this exposition we considered only the case of the sequence of σ-algebras generated by the dyadic cubes of side length 2^{-k} in $\mathbf{R}^n$. The L^p boundedness of the maximal conditional expectation (Doob [97]) is analogous to the L^p boundedness of the dyadic maximal function; likewise with the corresponding weak type $(1,1)$ estimate. The L^p boundedness of the dyadic martingale square function was obtained by Burkholder [39] and is analogous to Theorem 6.1.2. Moreover, the estimate $\big\| \sup_k |E_k(f)| \big\|_{L^p} \approx \big\| S(f) \big\|_{L^p}$, $0 < p < \infty$, obtained by Burkholder and Gundy [40] and also by Davis [90] is analogous to the square-function characterization of the Hardy space H^p norm. For an exposition on the different and unifying aspects of Littlewood–Paley theory we refer to Stein [337]. The proof of Theorem 6.4.8, which quantitatively expresses the almost orthogonality of the Littlewood–Paley and the dyadic martingale difference operators, is taken from Grafakos and Kalton [133].

The use of quadratic expressions in the study of certain maximal operators has a long history. We refer to the article of Stein [340] for a historical survey. Theorem 6.5.1 was first proved by Stein [339]. The proof in the text is taken from an article of Rubio de Francia [302]. Another proof when $n \ge 3$ is due to Cowling and Mauceri [76]. The more difficult case $n = 2$ was settled by Bourgain [36] about 10 years later. Alternative proofs when $n = 2$ were given by Mockenhaupt, Seeger, and Sogge [256] as well as Schlag [313]. The boundedness of maximal operators associated to more general smooth measures on compact surfaces of finite type were investigated by Iosevich and Sawyer [173]. The powerful machinery of Fourier integral operators was used by Sogge [328] to obtain the boundedness of spherical maximal operators on compact manifolds without boundary and positive injectivity radius; a simple proof for the boundedness of the spherical maximal function on the sphere was given by Nguyen [269]. Weighted norm inequalities for the spherical maximal operator were obtained by Duoandikoetxea and Vega [103]. The discrete spherical maximal function was studied by Magyar, Stein, and Wainger [237].

Much of the theory of square functions and the ideas associated with them has analogues in the dyadic setting. A dyadic analogue of the theory discussed here can be obtained. For an introduction to the area of dyadic harmonic analysis, we refer to Pereyra [276].

The idea of expressing (or reproducing) a signal as a weighted average of translations and dilations of a single function appeared in early work of Calderón [42]. This idea is in some sense a forerunner of wavelets. An early example of a wavelet was constructed by Strömberg [352] in his search for unconditional bases for Hardy spaces. Another example of a wavelet basis was obtained by Meyer [249]. The construction of an orthonormal wavelet presented in Theorem 6.6.7 is in Lemarié and Meyer [216]. A compactly supported wavelet was constructed by Daubechies [88]. Mallat [238] introduced the notion of multiresolution analysis, which led to a systematic production of wavelets. Theorem 6.6.9 is Shannon's [319] version of Nyquist's theorem [270] and is referred to as the Nyquist-Shannon sampling theorem. It is a fundamental result in telecommunications and signal processing, since it describes how to reconstruct a signal that contains no frequencies higher than B Hertz in terms of its values at a sequence of points spaced $1/(2B)$ seconds apart.

The area of wavelets has taken off significantly since its inception, spurred by these early results. A general theory of wavelets and its use in Fourier analysis was carefully developed in the two-volume monograph of Meyer [250], [251] and its successor Meyer and Coifman [253]. For further study and a deeper account of developments on the subject the reader may consult the books of Daubechies [89], Chui [64], Wickerhauser [374], Kaiser [184], Benedetto and Frazier [23], Hérnandez and Weiss [151], Wojtaszczyk [379], Mallat [239], Meyer [252], Frazier [120], Gröchenig [140], and the references therein. Theorems 6.6.12 and 6.6.13 first appeared in a combined form in the work of Plancherel and Pólya [285] for restrictions of entire functions of exponential type on the real line.