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Algebraic Monoids, Group Embeddings, and Algebraic Combinatorics

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Preface

The field of Algebraic Monoids owes a great deal to both Mohan Putcha and Lex Renner. In their hands, since the 1980s, this theory has turned into a full-fledged subject with the 2010 MSC code: 20M32. Dedicated to the 60th birthdays of Putcha and Renner, the International Workshop on *Algebraic Monoids, Group Embeddings, and Algebraic Combinatorics* took place at the Fields Institute in Toronto, Canada, from July 3 to July 6, 2012.

The purpose of the workshop was to stimulate research on the interplay between algebraic monoids, group embeddings, and algebraic combinatorics by bringing together some of the principal investigators, junior researchers, and graduate students in these three areas, as well as to contribute to the increased synthesis of these areas. The research talks given in the workshop not only reflected the current accomplishments of the invited speakers, but also outlined future directions of research. As planned, it has led to active collaborations between the participants. For example, shortly after the workshop Michel Brion and Lex Renner jointly proved that every algebraic monoid is strongly π-regular, which settled a long standing open problem.

The workshop had two main components. The first component consisted of mini-courses on introductory topics for graduate students delivered by Michel Brion, Eric Jespers, and Anne Schilling. These tutorials, staggered throughout the 4 days, introduced the necessary background for the remaining 17 research talks, which formed the second component of the workshop. The invited talks were delivered by Georgia Benkart, Nantel Bergeron, Tom Denton, Stephen Doty, Wenxue Huang, Kiumars Kaveh, Stuart Margolis, Mohan Putcha, Jan Okníński, Lex Renner, Alvaro Rittatore, Yuval Roichman, Dewey Taylor, Ryan Therkelsen, Nicholas Thiéry, Sandeep Varma, and Monica Vazirani. The topics of these talks were diverse and varied. They included structure and representation theory of reductive algebraic monoids, monoid schemes, monoids related to Lie theory, equivariant embeddings of algebraic groups, constructions and properties of monoids from algebraic combinatorics, endomorphism monoids induced from vector bundles, Hodge-Newton decompositions of reductive monoids, and applications of monoids.

Putcha and Renner originated the systematic study of algebraic monoids independently around 1978. Putcha, at North Carolina State University, first obtained many foundational results from the semigroup point of view. He investigated Green's relations, regularity, connections of regular monoids to reductive monoids in characteristic zero, semilattices, conjugacy classes of idempotents, and so on. In particular, he showed the existence of cross-section lattices for irreducible algebraic monoids, connecting Green's relations to group actions. Soon after, it became clear that this notion is closely related to Borel subgroups.

At the same time Renner began writing his thesis on the subject at the University of British Columbia. This period witnessed the first wave of applications of algebraic geometry in algebraic monoids. One pivotal result was that reductive monoids in any characteristic are (von Neumann) regular, a result that dramatically influenced the later developments. In late 1980s, using the connection of cross-section lattices to Borel subgroups as a starting point, Renner classified all normal, semisimple monoids numerically in the spirit of classical Lie theory. As a consequence, $\mathcal{J}$-irreducible algebraic monoids (monoids with a unique $\mathcal{J}$-class) appeared. He then found an analogue of the Bruhat decomposition for reductive monoids with equally striking consequences by introducing the concept of a Renner monoid, which plays the same role for monoids that the Weyl group does for groups.

Meanwhile, Putcha developed the monoid analogue of Tits's theory of groups with BN pair, monoids of Lie type. He also discovered that every reductive monoid has a type map, which is the monoid analogue of the Dynkin diagram and becomes the most important combinatorial invariant in the structure theory of reductive monoids. Around this time period, Putcha and Renner together determined explicitly the type map of $\mathcal{J}$-irreducible algebraic monoids.

In 1990s, Putcha gave a classification of monoids of Lie type, investigated highest weight categories of representations, and studied monoid Hecke algebras. Oknínski and Putcha showed that every complex representation of a finite monoid of Lie type is completely reducible, in particular proving that the complex algebra of the monoid of $n \times n$ matrices over a finite field is semisimple. Putcha and Renner systematically studied the canonical compactification of a finite group of Lie type and found that the restriction of any irreducible modular representation of a finite monoid of Lie type to its unit group is still irreducible. Furthermore, they computed the number of such irreducible modular representations. Renner introduced the concept of finite reductive monoids and studied modular representations of such monoids, showing that each finite reductive monoid is a monoid of Lie type. He obtained an analogue of the Tits system for reductive monoids by introducing a length function on the monoids.

In the first decade of the twenty-first century, Putcha explored shellability, Bruhat-Chevalley order, root semigroups in reductive monoids, and parabolic monoids. Renner investigated Betti numbers of rationally smooth group embeddings, blocks and representations of algebraic monoids, cellular decompositions (analogous to Schubert cells) of compactifications of a reductive group, descent systems for Bruhat posets, and H-polynomials.

Indeed, the theory of linear algebraic monoids has been developed significantly over the past three decades, due in large part to the efforts of Putcha and Renner. Meanwhile, it has also attracted researchers from different areas of mathematics because of its connections to algebraic group emdeddings, algebraic combinatorics, convex geometry, groups with BN-pairs, Lie theory, Kazhdan-Lusztig theory, semigroup theory, and toric varieties among others.

Algebraic group embedding theory studies compactifications of algebraic groups. It incorporates torus embeddings and reductive monoids, and it provides us with a large and important class of spherical varieties. Some aspects of representation theory are related to the geometry of group embeddings, especially through the examples of linear algebraic monoids.

Algebraic combinatorics, which is concerned with discrete objects such as posets, permutations, and polytopes, is an ever-growing field of mathematics with increasing importance in other disciplines including quantum chemistry, statistical biology, statistical physics, theoretical computer science, and so forth. Many questions in the combinatorial representation theory of algebraic monoids remain open.

This volume contains the refereed proceedings of the workshop; all the papers were strictly refereed and are previously unpublished. We thank all the 35 participants including students, research experts, and speakers from Belgium, Canada, China, France, Israel, Poland, Turkey, Uruguay, and USA, and especially the authors whose papers are included here. We also thank all the referees who spent their valuable time reviewing these papers and providing useful suggestions for their improvement. We are grateful to the Fields Institute and the National Science Foundation of USA for the funding and support of this workshop. We thank the editorial staff of the Fields Institute Communications Series, as well as that of Springer, especially Ms. Debbie Iscoe and Dr. Carl Riehm for their kind cooperation, help, and guidance in the preparation of this volume.

New Orleans, USA — Mahir Can
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Contents

On Algebraic Semigroups and Monoids

Michel Brion

Abstract We present some fundamental results on (possibly nonlinear) algebraic semigroups and monoids. These include a version of Chevalley's structure theorem for irreducible algebraic monoids, and the description of all algebraic semigroup structures on curves and complete varieties.

Keywords Algebraic semigroup • Algebraic monoid • Algebraic group

Subject Classifications: 14L10, 14L30, 20M32, 20M99

1 Introduction

Algebraic semigroups are defined in very simple terms: they are algebraic varieties endowed with a composition law which is associative and a morphism of varieties. So far, their study has focused on the class of linear algebraic semigroups, that is, of closed subvarieties of the space of $n \times n$ matrices that are stable under matrix multiplication; note that for an algebraic semigroup, being linear is equivalent to being affine. The theory has been especially developed by Putcha and Renner for linear algebraic monoids, i.e., those having a neutral element (see the books [23, 26]).

In addition, there has been recent progress on the structure of (possibly nonlinear) algebraic monoids: by work of Rittatore, the invertible elements of any irreducible algebraic monoid M form an algebraic group $G(M)$, open in M (see [28, Thm. 1]). Moreover, M is linear if and only if so is $G(M)$ (see [29, Thm. 5]). Also, the

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structure of normal irreducible algebraic monoids reduces to the linear case, as shown by Rittatore and the author: any such monoid is a homogeneous fiber bundle over an abelian variety, with fiber a normal irreducible linear algebraic monoid (see [6, Thm. 4.1], and [27] for further developments). This was extended by the author to all irreducible monoids in characteristic 0 (see [3, Thm. 3.2.1]).

In this article, we obtain some fundamental results on algebraic semigroups and monoids, that include the above structure theorems in slightly more general versions. We also describe all algebraic semigroup structures on abelian varieties, irreducible curves and complete irreducible varieties. The latter result is motivated by a remarkable theorem of Mumford and Ramanujam: if a complete irreducible variety X has a (possibly nonassociative) composition law μ with a neutral element, then X is an abelian variety with group law μ (see [19, Chap. II, §4, Appendix]).

As in [23, 26], we work over an algebraically closed field of arbitrary characteristic (most of our results extend to a perfect field without much difficulty; this is carried out in Sect. 3.5). But we have to resort to somewhat more advanced methods of algebraic geometry, as the varieties under consideration are not necessarily affine. For example, to show that every algebraic semigroup has an idempotent, we use an argument of reduction to a finite field, while the corresponding statement for affine algebraic semigroups follows from linear algebra. Also, we occasionally use some semigroup and monoid schemes (these are briefly considered in [9, Chap. II]), but we did not endeavour to study them systematically.

This text is organized as follows. Section 2 presents general results on idempotents of algebraic semigroups and on invertible elements of algebraic monoids. Both topics are fairly interwoven: for example, the fact that every algebraic monoid having no nontrivial idempotent is a group (whose proof is again more involved than in the linear case) implies a version of the Rees structure theorem for simple algebraic semigroups. In Sect. 3, we show that the Albanese morphism of an irreducible algebraic monoid M is a homogeneous fibration with fiber an affine monoid scheme. This generalization of the main result of [6] is obtained via a new approach, based on the consideration of the universal homomorphism from M to an algebraic group. In Sect. 4, we describe all semigroup structures on certain classes of varieties. We begin with the easy case of abelian varieties; as an unexpected consequence, we show that all the maximal submonoids of a given irreducible algebraic semigroup have the same Albanese variety. Then we show that every irreducible semigroup of dimension 1 is either an algebraic group or an affine monomial curve; this generalizes a result of Putcha in the affine case (see [21, Thm. 2.13] and [22, Thm. 2.9]). We also describe all complete irreducible semigroups, via another variant of the Rees structure theorem. Next, we obtain two general rigidity results; one of them implies (in loose words) that the automorphisms of a complete variety are open and closed in the endomorphisms. This has applications to complete algebraic semigroups, and yields another approach to the above theorem of Mumford and Ramanujam. Finally, we determine all families of semigroup laws on a given complete irreducible variety.

This article makes only the first steps in the study of (possibly nonlinear) algebraic semigroups and monoids, which presents many open questions. From the viewpoint of algebraic geometry, it is an attractive problem to describe all algebraic semigroup structures on a given variety. Our classes of examples suggest that the associativity condition imposes strong restrictions which might make this problem tractable: for instance, the composition laws on the affine line are of course all the polynomial functions in two variables, but those that are associative are obtained from the maps $(x, y) \mapsto 0, x, y, x + y$ or xy by a change of coordinate. From the viewpoint of semigroup theory, it is natural to investigate the structure of an algebraic semigroup in terms of its idempotents and the associated (algebraic) subgroups. Here a recent result of Renner and the author (see [5]) asserting that every algebraic semigroup S is strongly π-regular (i.e., for any $x \in S$, some power x^m belongs to a subgroup) opens the door to further developments.

Notation and conventions. Throughout this article, we fix an algebraically closed field k. A *variety* is a reduced, separated scheme of finite type over k; in particular, varieties need not be irreducible. By a *point* of a variety X, we mean a closed (or equivalently, k-rational) point; we may identify X to its set of points equipped with the Zariski topology and with the structure sheaf $\mathcal{O}_X$. *Morphisms* of varieties are understood to be k-morphisms.

The textbook [13] will be our standard reference for algebraic geometry, and [10] for commutative algebra. We will also use the books [33] and [19] for some basic results on linear algebraic groups, resp. abelian varieties.

2 Algebraic Semigroups and Monoids

2.1 Basic Definitions and Examples

Definition 1. An (abstract) *semigroup* is a set S equipped with an associative composition law $\mu : S \times S \to S$. When S is a variety and μ is a morphism, we say that (S, μ) is an *algebraic semigroup*.

A *neutral (resp. zero) element* of a semigroup (S, μ) is an element $x_o \in S$ such that $\mu(x, x_o) = \mu(x_o, x) = x$ for all $x \in S$ (resp. $\mu(x, x_o) = \mu(x_o, x) = x_o$ for all $x \in S$).

An abstract (resp. algebraic) semigroup (S, μ) equipped with a neutral element x_o is called an abstract (resp. algebraic) *monoid*.

An *algebraic group* is a group G equipped with the structure of a variety, such that the group law μ and the inverse map $\iota : G \to G, g \mapsto g^{-1}$ are morphisms.

Clearly, a neutral element x_o of a semigroup S is unique if it exists; we then denote x_o by 1_S, or just by 1 if this yields no confusion. Likewise, a zero element is unique if it exists, and we then denote it by 0_S or 0. Also, we simply denote the semigroup law μ by $(x, y) \mapsto xy$.

Definition 2. A *left ideal* of a semigroup (S, μ) is a subset I of S such that $xy \in I$ for any $x \in S$ and $y \in I$. *Right ideals* are defined similarly; a *two-sided ideal* is of course a left and right ideal.

Definition 3. Given two semigroups S and S', a *homomorphism of semigroups* is a map $\varphi : S \to S'$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. When S and S' are monoids, we say that φ is a *homomorphism of monoids* if in addition $\varphi(1_S) = 1_{S'}$.

A *homomorphism of algebraic semigroups* is a homomorphism of semigroups which is also a morphism of varieties. Homomorphisms of algebraic monoids, resp. of algebraic groups, are defined similarly.

Definition 4. An *idempotent* of a semigroup S is an element $e \in S$ such that $e^2 = e$. We denote by $E(S)$ the set of idempotents.

Idempotents yield much insight in the structure of semigroups; this is illustrated by the following:

Remark 1. (i) Let $\varphi : S \to S'$ be a homomorphism of semigroups. Then φ sends $E(S)$ to $E(S')$; moreover, the fiber of φ at an arbitrary point $x' \in S'$ is a subsemigroup of S if and only if $x' \in E(S')$.

(ii) Let S be a semigroup, and $M \subseteq S$ a submonoid with neutral element e. Then M is contained in the subset

$$\{x \in S \mid ex = xe = x\} = \{exe \mid x \in S\} =: eSe,$$

which is the largest submonoid of S with neutral element e. This defines a bijective correspondence between idempotents and maximal submonoids of S.

(iii) Let S be a semigroup, and $e \in E(S)$. Then the subset

$$Se := \{xe \mid x \in S\} = \{x \in S \mid xe = x\}$$

is a left ideal of S, and the map

$$\varphi : S \longrightarrow Se, \quad x \longmapsto xe$$

is a *retraction* (i.e., $\varphi(x) = x$ for all $x \in Se$). The fiber of φ at e,

$$S_e := \{x \in S \mid xe = e\},$$

is a subsemigroup of S. Moreover, the restriction

$$\psi := \varphi|_{eS} : eS \longrightarrow eS \cap Se = eSe$$

is a *retraction of semigroups*, that is, $\psi(x) = x$ for all $x \in eSe$ and ψ is a homomorphism (indeed, $xeye = xye$ for all $x \in S$ and $y \in eS$). The fiber of ψ at e,

$$eS_e := \{x \in S \mid ex = x \text{ and } xe = e\},$$

is a subsemigroup of S with law $(x, y) \mapsto y$ (since $xy = xey = ey = y$ for all $x, y \in eS_e$).

When S is an algebraic semigroup, $E(S)$ is a closed subvariety. Moreover, Se, S_e, eSe and eS_e are closed in S as well, and φ (resp. ψ) is a retraction of varieties (resp. of algebraic semigroups). In particular, every maximal abstract submonoid of S is closed.

Similar assertions hold for the right ideal eS and the subsemigroups

$$_eS := \{x \in S \mid ex = e\}, \quad {}_eSe := \{x \in S \mid xe = x \text{ and } ex = e\}.$$

An abstract semigroup may have no idempotent; for example, the set of positive integers equipped with the addition. Yet we have:

Proposition 1. *Any algebraic semigroup has an idempotent.*

Proof. We use a classical argument of reduction to a finite field. Consider first the case where k is the algebraic closure of a prime field $\mathbb{F}_p$. Then S and μ are defined over some finite subfield $\mathbb{F}_q$ of k, where q is a power of p. Thus, for any $x \in S$, the powers x^n, where n runs over the positive integers, form a finite subsemigroup of S. We claim that some x^n is idempotent. Indeed, we have $x^a = x^b$ for some integers $a > b > 0$. Thus, $x^b = x^b x^{a-b} = x^{b+m(a-b)}$ for all $m > 0$. In particular, $x^b = x^{b(a-b+1)}$. Multiplying by $x^{b(a-b-1)}$, we obtain $x^{b(a-b)} = x^{2b(a-b)}$, i.e., $x^{b(a-b)}$ is idempotent.

Next, consider the case of an arbitrary field k. Choose $x \in S$; then S, μ and x are defined over some finitely generated subring R of k. In the language of schemes, we have a scheme S_R together with morphisms

$$\varphi : S_R \to \mathrm{Spec}(R), \quad \mu_R : S_R \times_{\mathrm{Spec}(R)} S_R \to S_R$$

and with a section $x_R : \mathrm{Spec}(R) \to S_R$ of φ, such that $S_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ is isomorphic to S; moreover, this isomorphism identifies $\mu_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ to μ, and $x_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ to x. Also, S_R is a semigroup scheme over $\mathrm{Spec}(R)$ (that is, μ_R is associative in an obvious sense), and the morphism φ is of finite type. Denote by $E(S_R)$ the subscheme of idempotents of S_R, i.e., $E(S_R)$ is the preimage of the diagonal in $S_R \times S_R$ under the morphism $S_R \to S_R \times S_R$, $s \mapsto (\mu_R(s, s), s)$. Then $E(S_R)$ is a closed subscheme of S_R; let

$$\psi : E(S_R) \to \mathrm{Spec}(R)$$

be the restriction of φ.

We claim that the image of ψ contains all closed points of $\mathrm{Spec}(R)$. Indeed, consider a maximal ideal $\mathfrak{m}$ of R; then $R/\mathfrak{m}$ is a finite field. By the first step, the semigroup $S_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{m})$ (obtained from S_R by reduction mod $\mathfrak{m}$) contains an idempotent; this yields the claim.

Since R is Noetherian and the morphism ψ is of finite type, its image is constructible (see e.g. [13, Exer. II.3.19]). In view of the claim, it follows that this image contains the generic point of Spec(R) (see e.g. [loc. cit., Exer. II.3.18]), i.e., $E(S_R)$ has (not necessarily closed) points over the fraction field of R. Hence $E(S)$ has (closed) points over the larger algebraically closed field k. □

Combining the above proposition with Remark 1 (i), we obtain:

Corollary 1. *Let $f : S \to S'$ be a surjective homomorphism of algebraic semigroups. Then $f(E(S)) = E(S')$.*

We now present several classes of (algebraic) semigroups:

Example 1. (i) Any set X has two semigroup laws μ_l, μ_r given by $\mu_l(x, y) := x$ (resp. $\mu_r(x, y) := y$) for all $x, y \in X$. For both laws, every element is idempotent and X has no proper two-sided ideal.

Also, every point $x \in X$ defines a semigroup law μ_x by $\mu_x(y, z) := x$ for all $y, z \in X$. Then x is the zero element; it is the unique idempotent, and the unique proper two-sided ideal as well.

The maps μ_l, μ_r, μ_x ($x \in X$) will be called the *trivial semigroup laws* on X. When X is a variety, these maps are algebraic semigroup laws. Note that every morphism of varieties $f : X \to Y$ yields a homomorphism of algebraic semigroups $(X, \mu_r) \to (Y, \mu_r)$, and likewise for μ_l. Also, f yields a homomorphism $(X, \mu_x) \to (Y, \mu_y)$, where $y := f(x)$.

(ii) Let X be a set, $Y \subseteq X$ a subset, $\rho : X \to Y$ a retraction, and ν a semigroup law on Y. Then the map

$$\mu : X \times X \longrightarrow X, \quad (x_1, x_2) \longmapsto \nu(\rho(x_1), \rho(x_2))$$

is easily seen to be a semigroup law on X. Moreover, ρ is a retraction of semigroups, and $E(X) = E(Y)$. If in addition X is a variety, Y is a closed subvariety and ρ, ν are morphisms, then (X, μ) is an algebraic semigroup.

When Y consists of a single point x, we recover the semigroup law μ_x of the preceding example.

(iii) Given two semigroups (S, μ) and (S', μ'), we may define a composition law ν on the disjoint union $S \sqcup S'$ by

$$\nu(x, y) = \begin{cases} \mu(x, y) & \text{if } x, y \in S, \\ y & \text{if } x \in S \text{ and } y \in S', \\ x & \text{if } x \in S' \text{ and } y \in S, \\ \mu'(x, y) & \text{if } x, y \in S'. \end{cases}$$

One readily checks that $(S \sqcup S', \nu)$ is a semigroup; moreover, (S, μ) is a subsemigroup and (S', μ') is a two-sided ideal. Also, note that $E(S \sqcup S') = E(S) \sqcup E(S')$.

When S (resp. S') has a zero element 0_S (resp. $0_{S'}$), consider the set $S \cup_0 S'$ obtained from $S \sqcup S'$ by identifying 0_S and $0_{S'}$. One checks that $S \cup_0 S'$ has a unique semigroup law ν_0 such that the natural map $S \sqcup S' \to S \cup_0 S'$ is a homomorphism; moreover, the image of 0_S is the zero element. Here again, S is a subsemigroup of $S \cup_0 S'$, and S' is a two-sided ideal; we have $E(S \cup_0 S') = E(S) \cup_0 E(S')$.

If in addition (S, μ) and (S', μ') are algebraic semigroups, then so are $(S \sqcup S', \nu)$ and $(S \cup_0 S', \nu_0)$. This construction still makes sense when (say) S' is a scheme of finite type over k, equipped with a closed point $0 = 0_{S'}$ and with the associated trivial semigroup law μ_0. Taking for S' the spectrum of a local ring of finite dimension as a k-vector space, and for 0 the unique closed point of S', we obtain many examples of nonreduced semigroup schemes (having a fat point at their zero element).

(iv) Any finite semigroup is algebraic. In the opposite direction, the (finite) set of connected components of an algebraic semigroup (S, μ) has a natural structure of semigroup. Indeed, if C_1, C_2 are connected components of S, then $\mu(C_1, C_2)$ is contained in a unique connected component, say, $\nu(C_1, C_2)$. The resulting composition law ν on the set of connected components, $\pi_o(S)$, is clearly associative, and the canonical map $f : S \to \pi_o(S)$ is a homomorphism of algebraic semigroups. In fact, f is the universal homomorphism from S to a finite semigroup.

Next, we present examples of algebraic monoids and of algebraic groups:

Example 2. (i) Consider the set M_n of $n \times n$ matrices with coefficients in k, where n is a positive integer. We may view M_n as an affine space of dimension n^2; this is an irreducible algebraic monoid relative to matrix multiplication, the neutral element being of course the identity matrix.

The subspaces D_n of diagonal matrices, and T_n of upper triangular matrices, are closed irreducible submonoids of M_n. Note that D_n is isomorphic to the affine n-space $\mathbb{A}^n$ equipped with pointwise multiplication.

An example of a closed reducible submonoid of M_n consists of those matrices having at most one nonzero entry in each row and each column. This submonoid, that we denote by R_n, is the closure in M_n of the group of monomial matrices (those having exactly one nonzero entry in each row and each column). Note that $R_n = D_n S_n$, where S_n denotes the symmetric group on n letters, viewed as the group of permutation matrices. Thus, the irreducible components of R_n are parametrized by S_n. Each such component contains the zero matrix; in particular, R_n is connected.

(ii) A *linear* algebraic monoid is a closed submonoid M of some matrix monoid M_n. Then the variety M is affine; conversely, every affine algebraic monoid is linear (see [9, Thm. II.2.3.3]). It follows that every affine algebraic semigroup is linear as well, see [23, Cor. 3.16].

(iii) Let A be a k-algebra (not necessarily associative, or commutative, or unital) and denote by $\mathrm{End}(A)$ the set of algebra endomorphisms of A. Then $\mathrm{End}(A)$, equipped with the composition of endomorphisms, is an (abstract) monoid with zero. Its idempotents are exactly the retractions of A to subalgebras. Given such a retraction $e : A \to B$, we have

$$e\text{End}(A) \cong \text{Hom}(A, B), \quad \text{End}(A)e \cong \text{Hom}(B, A), \quad e\text{End}(A)e \cong \text{End}(B),$$

where Hom denotes the set of algebra homomorphisms. Also, $\text{End}(A)_e$ (resp. $_e\text{End}(A)$) consists of those $\varphi \in \text{End}(A)$ such that $\varphi(x) = x$ for all $x \in B$ (resp. $\varphi(x) - x \in I$ for all $x \in A$, where I denotes the kernel of e).

If A is finite-dimensional as a k-vector space, then $\text{End}(A)$ is a linear algebraic monoid; indeed, it identifies to a closed submonoid of M_n, where $n := \dim(A)$.

(iv) Examples of algebraic groups include:

- The *additive group* $\mathbb{G}_a$, i.e., the affine line equipped with the addition,
- The *multiplicative group* $\mathbb{G}_m$, i.e., the affine line minus the origin, equipped with the multiplication,
- The *elliptic curves*, i.e., the complete nonsingular irreducible curves of genus 1, equipped with a base point; then there is a unique algebraic group structure for which this point is the neutral element, see e.g. [13, Chap. II, §4].

  In fact, these examples yield all the connected algebraic groups of dimension 1, see [15, Prop. 10.7.1].

(v) A complete connected algebraic group is called an *abelian variety*; elliptic curves are examples of such algebraic groups. It is known that every abelian variety A is a commutative group and a projective variety; moreover, the group law on A is uniquely determined by the structure of variety and the neutral element (see [19, Chap. II]).

2.2 The Unit Group of an Algebraic Monoid

In this section, we obtain some fundamental results on the group of invertible elements of an algebraic monoid. We shall need the following observation:

Proposition 2. *Let (M, μ) be an algebraic monoid. Then M has a unique irreducible component containing 1: the neutral component M^o. Moreover, $M^o X = XM^o = X$ for any irreducible component X of M; in particular, M^o is a closed submonoid of M.*

Proof. Let X, Y be irreducible components of M. Then XY is the image of the restriction of μ to $X \times Y$, and hence is a constructible subset of M; moreover, its closure $\overline{XY}$ is an irreducible subvariety of M. If $1 \in X$, then $Y \subseteq XY \subseteq \overline{XY}$. Since Y is an irreducible component, we must have $Y = XY = \overline{XY}$; likewise, one obtains that $YX = Y$. In particular, $XX = X$, i.e., X is a closed submonoid. If in addition $1 \in Y$, then also $XY = YX = Y$, hence $Y = X$. This yields our assertions. □

Remark 2. Any algebraic group G is a nonsingular variety, and hence every connected component of G is irreducible. Moreover, the neutral component G^o is a closed normal subgroup, and the quotient group G/G^o parametrizes the components of G.

In contrast, there exist connected reducible algebraic monoids: for example, the monoid R_n of Example 2 (i). Also, algebraic monoids are generally singular; for example, the zero locus of $z^2 - xy$ in $\mathbb{A}^3$ equipped with pointwise multiplication.

On a more advanced level, note that any group scheme is reduced in characteristic 0 (see e.g. [9, Thm. II.6.1.1]). In contrast, there always exist nonreduced monoid schemes. For example, one may stick an arbitrary fat point at the origin of the multiplicative monoid $(\mathbb{A}^1, \times)$, by the construction of Example 1 (iii).

Definition 5. Let M be a monoid and let $x, y \in M$. Then y is a *left* (resp. *right*) *inverse* of x if $yx = 1$ (resp. $xy = 1$). We say that x is *invertible* (also called a *unit*) if it has a left and a right inverse.

With the above notation, one readily checks that the left and right inverses of any unit $x \in M$ are equal. Moreover, if $x' \in M$ is another unit with inverse y', then xy' is a unit with inverse $x'y$. Thus, the invertible elements of M form a subgroup: the *unit group*, that we denote by $G(M)$.

The following result on unit groups of algebraic monoids is due to Rittatore in the irreducible case (see [28, Thm. 1]). The proof presented here follows similar arguments.

Theorem 1. *Let M be an algebraic monoid. Then $G(M)$ is an algebraic group, open in M. In particular, $G(M)$ consists of nonsingular points of M.*

Proof. Let

$$G := \{(x, y) \in M \times M^{\mathrm{op}} \mid xy = yx = 1\},$$

where M^{op} denotes the *opposite* monoid to M, i.e., the variety M equipped with the composition law $(x, y) \mapsto yx$. One readily checks that G (viewed as a closed subvariety of $M \times M^{\mathrm{op}}$) is a submonoid; moreover, every $(x, y) \in G$ has inverse (y, x). Thus, G is a closed algebraic subgroup of $M \times M^{\mathrm{op}}$.

The first projection $p_1 : M \times M^{\mathrm{op}} \to M$ restricts to a homomorphism of monoids $\pi : G \to M$ with image being the unit group $G(M)$. In fact, G acts on M by left multiplication: $(x, y) \cdot z := xz$, and π is the orbit map $(x, y) \mapsto (x, y) \cdot 1$; in particular, $G(M)$ is the G-orbit of 1. The isotropy subgroup of 1 in G is clearly trivial as a set. We claim that this also holds as a scheme; in other words, the isotropy Lie algebra of 1 is trivial as well.

To check this, recall that the Lie algebra $\mathrm{Lie}(G)$ is the Zariski tangent space $T_{(1,1)}(G)$ and hence is contained in $T_{(1,1)}(M \times M^{\mathrm{op}}) \cong T_1(M) \times T_1(M)$. Since the differential at $(1, 1)$ of the monoid law $\mu : M \times M \to M$ is the map

$$T_1(M) \times T_1(M) \longrightarrow T_1(M), \quad (x, y) \longmapsto x + y,$$

we have

$$T_{(1,1)}(G) \subseteq \{(x, y) \in T_1(M) \times T_1(M) \mid x + y = 0\}.$$

Thus, the first projection $\mathrm{Lie}(G) \to T_1(M)$ is injective; but this projection is the differential of π at $(1, 1)$. This proves our claim.

By that claim, π is a locally closed immersion. Thus, $G(M)$ is a locally closed subvariety of M, and π induces an isomorphism of groups $G \cong G(M)$. So $G(M)$ is an algebraic group.

It remains to show that $G(M)$ is open in M; it suffices to check that $G(M)$ contains an open subset U of M (then the translates gU, where $g \in G(M)$, form a covering of $G(M)$ by open subsets of M). For this, we may replace M with its neutral component M^o (Proposition 2) and hence assume that M is irreducible. Note that

$$G(M) = \{x \in M \mid xy = zx = 1 \text{ for some } y, z \in M\}$$

(then $y = zxy = z$). In other words,

$$G(M) = p_1(\mu^{-1}(1)) \cap p_2(\mu^{-1}(1)),$$

where $p_1, p_2 : M \times M \to M$ denote the projections. Also, the set-theoretic fiber at 1 of the restriction $p_1 : \mu^{-1}(1) \to M$ consists of the single point 1. By a classical result on the dimension of fibers of a morphism (see [13, Exer. II.3.22]), it follows that every irreducible component C of $\mu^{-1}(1)$ containing 1 satisfies $\dim(C) = \dim(M)$, and that the restriction $p_1 : C \to M$ is dominant. Thus, $p_1(C)$ contains a dense open subset of M. Likewise, $p_2(C)$ contains a dense open subset of M, and hence so does $G(M)$. □

Note that the unit group of a linear algebraic monoid is linear, see [23, Cor. 3.26]. Further properties of the unit group are gathered in the following:

Proposition 3. *Let M be an algebraic monoid, and G its unit group.*

(i) *If $x \in M$ has a left (resp. right) inverse, then $x \in G$.*
(ii) *$M \setminus G$ is the largest proper two-sided ideal of M.*
(iii) *If 1 is the unique idempotent of M, then $M = G$.*

Proof. (i) Assume that x has a left inverse y. Then the left multiplication $M \to M$, $z \mapsto xz$ is an injective endomorphism of the variety M. By [1, Thm. C] (see also [2]), this endomorphism is surjective, and hence there exists $z \in M$ such that $xz = 1$. Then $y = yxz = z$, i.e., $x \in G$. The case where x has a right inverse is handled similarly.

(ii) Clearly, any proper two-sided ideal of M is contained in $M \setminus G$. We show that the latter is a two-sided ideal: let $x \in M \setminus G$ and $y \in M$. If $xy \in G$, then

$y(xy)^{-1}$ is a right inverse of x. By (i), it follows that $x \in G$, a contradiction. Thus, $(M \setminus G)M \subseteq M \setminus G$. Likewise, $M(M \setminus G) \subseteq M \setminus G$.

(iii) By Theorem 1, $M \setminus G$ is closed in M; also, $M \setminus G$ is a subsemigroup of M by (ii). Thus, if $M \neq G$ then $M \setminus G$ contains an idempotent by Proposition 1. □

2.3 The Kernel of an Algebraic Semigroup

In this subsection, we show that every algebraic semigroup has a smallest two-sided ideal (called its *kernel*) and we describe the structure of that ideal, thereby generalizing some of the known results about the kernel of a linear algebraic semigroup (see [14, 23]).

First, recall that the idempotents of any (abstract) semigroup S are in bijective correspondence with the maximal submonoids of S, via $e \mapsto eSe$, and hence with the maximal subgroups of S, via $e \mapsto G(eSe)$. Thus, when S is an algebraic semigroup, its maximal subgroups are all locally closed in view of Theorem 1. They are also pairwise disjoint: if $e \in E(S)$ and $x \in G(eSe)$, then there exists $y \in eSe$ such that $xy = yx = e$. Thus, $xSx \supseteq xySyx = eSe$. But $xSx \subseteq eSeSeSe \subseteq eSe$, and hence $xSx = eSe$. So xSx is a closed submonoid of S with neutral element e.

Next, we recall the classical definition of a partial order on the set of idempotents of any (abstract) semigroup:

Definition 6. Let S be a semigroup and let $e, f \in E(S)$. Then $e \leq f$ if we have $e = ef = fe$.

Note that $e \leq f$ if and only if $e \in fSf$; this is also equivalent to the condition that $eSe \subseteq fSf$. Thus, $\leq$ is indeed a partial order on $E(S)$ (this fact may of course be checked directly). Also, note that $\leq$ is preserved by every homomorphism of semigroups. For an algebraic semigroup, the partial order $\leq$ satisfies additional finiteness properties:

Proposition 4. *Let S be an algebraic semigroup.*

(i) *Every subset of $E(S)$ has a minimal element with respect to the partial order $\leq$, and also a maximal element.*
(ii) *$e \in E(S)$ is minimal among all idempotents if and only if eSe is a group.*
(iii) *If S is commutative, then $E(S)$ is finite and has a smallest element.*

Proof. (i) Note that the eSe, where $e \in E(S)$, form a family of closed subsets of the Noetherian topological space S; hence any subfamily has a minimal element. For the existence of maximal elements, consider the family

$$S \times_{Se} S := \{(x, y) \in S \times S \mid xe = ye\}$$

of closed subsets of $S \times S$. Let $f \in E(S)$ such that $e \leq f$. Then $S \times_{Sf} S \subseteq S \times_{Se} S$, since the equality $xf = yf$ implies that $xe = xfe = yfe = ye$. Moreover, if $S \times_{Se} S = S \times_{Sf} S$, then $(x, xe) \in S \times_{Sf} S$ for all $x \in S$, i.e., $xf = xef$. Hence $xf = xe$, and $f = f^2 = fe = e$. Thus, a minimal $S \times_{Se} S$ (for e in a given subset of $E(S)$) yields a maximal e.

(ii) Let e be a minimal idempotent of S. Then e is the unique idempotent of the algebraic monoid eSe. By Proposition 3 (iii), it follows that eSe is a group. The converse is immediate (and holds for any abstract semigroup).

(iii) Let e, f be idempotents. Then $ef = fe$ is also idempotent, and $ef = e(ef)e = f(ef)f$ so that $ef \leq e$ and $ef \leq f$. Thus, any two minimal idempotents are equal, i.e., $E(S)$ has a smallest element.

To show that $E(S)$ is finite, we may replace S with its closed subsemigroup $E(S)$, and hence assume that every element of S is idempotent. Then every connected component of S is a closed subsemigroup in view of Example 1 (iv). So we may further assume that S is connected. Let $x \in S$; then xS is a connected commutative algebraic monoid with neutral element x, and consists of idempotents. Thus, $G(xS) = \{x\}$. By Theorem 1, it follows that x is an isolated point of xS. Hence $xS = \{x\}$, i.e., $xy = x$ for all $y \in S$. Since S is commutative, we must have $S = \{x\}$. □

As a consequence of the above proposition, every algebraic semigroup admits minimal idempotents. These are of special interest, as shown by the following:

Proposition 5. *Let S be an algebraic semigroup, $e \in S$ a minimal idempotent, and eSe the associated closed subgroup of S.*

(i) *The map*

$$\rho = \rho_e : S \longrightarrow S, \quad s \longmapsto s(ese)^{-1}s$$

is a retraction of varieties of S to SeS. In particular, SeS is a closed two-sided ideal of S.

(ii) *The map*

$$\varphi : {}_eSe \times eSe \times eS_e \longrightarrow S, \quad (x, g, y) \longmapsto xgy$$

yields an isomorphism of varieties to its image, SeS.

(iii) *Via the above isomorphism, the semigroup law on SeS is identified to that on ${}_eSe \times eSe \times eS_e$ given by*

$$(x, g, y)(x', g', y') = (x, g\pi(y, x')g', y'),$$

where $\pi : eS_e \times {}_eSe \to eSe$ denotes the map $(y, z) \mapsto yz$. This identifies the idempotents of SeS to the triples $(x, \pi(y, x)^{-1}, y)$, where $x \in {}_eSe$ and $y \in eS_e$. In particular, $E(SeS) \cong {}_eSe \times eS_e$ as a variety.

(iv) *The semigroup SeS has no proper two-sided ideal.*

(v) *SeS is the smallest two-sided ideal of S; in particular, it does not depend on the minimal idempotent e.*
(vi) *The minimal idempotents of S are exactly the idempotents of SeS.*

Proof. (i) Clearly, ρ is a morphism; also, since $s(ese)^{-1}s \in SeSeS$ for all $s \in S$, the image of ρ is contained in SeS. Let $z \in SeS$ and write $z = set$, where $s, t \in S$; then

$$\rho(z) = set(esete)^{-1}set = sete(ete)^{-1}(ese)^{-1}eset = set = z.$$

This yields the assertions.

(ii) Clearly, φ takes values in SeS. Moreover, for any s, t as above, we have

$$set = se(ese)^{-1}esete(ete)^{-1}t = xgy,$$

where $x := se(ese)^{-1}$, $g := esete$ and $y := (ete)^{-1}et$. Furthermore, $x \in {}_eSe$, $g \in eSe$ and $y \in eS_e$. In particular, the image of φ is the whole SeS. Also, the map

$$\psi : SeS \longrightarrow {}_eSe \times eSe \times eS_e, \quad set \longmapsto (x, g, y)$$

(where x, g, y are defined as above) is a morphism of varieties and satisfies $\varphi \circ \psi = \mathrm{id}$. Thus, it suffices to check that $\psi \circ \varphi = \mathrm{id}$. Let $x \in {}_eSe$, $g \in eSe$, $y \in eS_e$ and put $s := xgy$. Then $se = xg$ and $es = gy$. Hence $g = ese$, $x = se(ese)^{-1}$ and $y = (ese)^{-1}es$, which yields the desired assertion.

(iii) For the first assertion, just write $(xgy)(x'g'y') = x(g(yx')g')y'$, and note that $yx' \in eS_e\,{}_eSe \subseteq eSe$. The assertions on idempotents follow readily.

(iv) Let $z \in SeS$ and write $z = \varphi(x, g, y)$. Then the subset SzS of SeS is identified with that of ${}_eSe \times eSe \times eS_e$ consisting of the triples $(x_1, g_1\pi(y_1, x)g\pi(y, x_2)g_2, y_2)$, where $x_1, x_2 \in {}_eSe$, $g_1, g_2 \in eSe$ and $y_1, y_2 \in eS_e$. It follows that $SzS = SeS$; in particular, SzS contains z. Hence SeS is the smallest two-sided ideal containing z.

(v) Let I be a two-sided ideal of S. Then SeI is a two-sided ideal of S contained in SeS; hence $SeI = SeS$ by (iv). But $SeI \subseteq I$; this yields our assertions.

(vi) If $f \in E(S)$ is minimal, then $SfS = SeS$ by (v). Thus, $f \in SeS$.

For the converse, let $f \in E(SeS)$. Then $SfS = SeS$ by (iv), and hence $fSf = fSfSf = f(SeS)f$. Identifying f to a triple $(x, \pi(y, x)^{-1}, y)$, one checks as in the proof of (iv) that $f(SeS)f$ is identified to the set of triples (x, g, y), where $g \in eSe$. But $(x, \pi(y, x)^{-1}, y)$ is the unique idempotent of this set. Thus, f is the unique idempotent of fSf, i.e., f is minimal. □

In view of these results, we may set up the following:

Definition 7. The *kernel* of an algebraic semigroup S is the smallest two-sided ideal of S, denoted by $\ker(S)$.

Remark 3. (i) As a consequence of Proposition 5, we see that any algebraic semigroup having no proper closed two-sided ideal is *simple*, i.e., has no proper two-sided ideal at all. Moreover, any simple algebraic semigroup S, equipped with an idempotent e, is isomorphic (as a variety) to the product $X \times G \times Y$, where $X := {}_eSe$ and $Y := eS_e$ are varieties, and $G := eSe$ is an algebraic group. This identifies e to a point of the form $(x_o, 1, y_o)$, where $x_o \in X$ and $y_o \in Y$; moreover, the semigroup law of S is identified to that as in Proposition 5 (iii), where $\pi : Y \times X \to G$ is a morphism such that $\pi(x_o, y) = \pi(x, y_o) = 1$ for all $x \in X$ and $y \in Y$.

Conversely, any tuple (X, Y, G, π, x_o, y_o) satisfying the above conditions defines an algebraic semigroup law on $S := X \times G \times Y$ such that $e := (x_o, 1, y_o)$ is idempotent and ${}_eSe = X \times \{(1, y_o)\}$, $eSe = \{x_o\} \times G \times \{y_o\}$, $eS_e = \{(x_o, 1)\} \times Y$.

This description of algebraic semigroups having no proper closed two-sided ideal is a variant of the classical Rees structure theorem for those (abstract) semigroups that are *completely simple*, that is, simple and having a minimal idempotent (see e.g. [23, Thm. 1.9]).

(ii) By analogous arguments, one shows that every algebraic semigroup S contains minimal left ideals, and these are exactly the subsets Sf, where f is a minimal idempotent. In particular, the minimal left ideals are all closed. Also, given a minimal idempotent e, these ideals are exactly the subsets $X \times G \times \{y\}$ of $\ker(S)$, where $X := {}_eSe$, $G := eSe$ and $y \in Y := eS_e$ as above. Similar assertions hold of course for the minimal right ideals; it follows that the intersections of minimal left and minimal right ideals are exactly the subsets $\{x\} \times G \times \{y\}$, where $x \in X$ and $y \in Y$.

2.4 Unit Dense Algebraic Monoids

In this subsection, we introduce and study the class of unit dense algebraic monoids. These include the irreducible algebraic monoids, and will play an important role in their structure.

Let M be an algebraic monoid, and $G(M)$ its unit group. Then the algebraic group $G(M) \times G(M)$ acts on M via left and right multiplication: $(g, h) \cdot x := gxh^{-1}$. Moreover, the orbit of 1 under this action is just $G(M)$, and the isotropy subgroup scheme of 1 equals the diagonal, $\mathrm{diag}(G(M)) := \{(g, g) \mid g \in G(M)\}$.

Definition 8. An algebraic monoid M is *unit dense* if $G(M)$ is dense in M.

For instance, every irreducible algebraic monoid is unit dense. An example of a reducible unit dense algebraic monoid consists of $n \times n$ matrices having at most one nonzero entry in each row and each column (Example 2 (i)).

Any unit dense monoid may be viewed as an equivariant embedding of its unit group, in the sense of the following:

Definition 9. Let G be an algebraic group. An *equivariant embedding* of G is a variety X equipped with an action of $G \times G$ and with a point $x \in X$ such that the orbit $(G \times G) \cdot x$ is dense in X, and the isotropy subgroup scheme $(G \times G)_x$ is the diagonal, $\mathrm{diag}(G)$.

Note that the law of a unit dense monoid is uniquely determined by its structure of equivariant embedding, since that structure yields the law of the unit group. Also, given an *affine* algebraic group G, every *affine* equivariant embedding of G has a unique structure of algebraic monoid such that G is the unit group, by [28, Prop. 1]. Conversely, every unit dense algebraic monoid with unit group G is affine by Theorem 2 below. For an arbitrary connected algebraic group G, the equivariant embeddings of G that admit a monoid structure are characterized in Theorem 4 below.

Proposition 6. *Let M be an algebraic monoid, and G its unit group. Then the unit group of the neutral component M^o is the neutral component G^o of G.*

If M is unit dense, then its irreducible components are exactly the subsets gM^o, where $g \in G$; they are indexed by G/G^o, the group of components of G.

Proof. Note that $G(M^o)$ is contained in G, and open in M^o by Theorem 1. Hence $G(M^o)$ contains an open neighborhood of 1 in M^o, or equivalently in G. Using the group structure, it follows that $G(M^o)$ is open in G; also, $G(M^o)$ is irreducible since so is M^o. But the algebraic group G contains a unique open irreducible subgroup: its neutral component. Thus, $G(M^o) = G^o$.

Clearly, gM^o is an irreducible component of M for any $g \in G$, and this component depends only on the coset gG^o. If M is unit dense, then any irreducible component X of M contains a unit, say g. Since $g^{-1}X$ is an irreducible component containing 1, it follows that $X = gM^o$. If $X = hM^o$ for some $h \in G$, then $g^{-1}h \in G \cap M^o$. Thus, $g^{-1}hG^o$ is an open subset of M^o, and hence meets G^o; so $g^{-1}h \in G^o$, i.e., $gG^o = hG^o$. □

Proposition 7. *Let M be a unit dense algebraic monoid, and G its unit group. Then the kernel,* $\ker(M)$*, is the unique closed orbit of $G \times G$ acting by left and right multiplication. Moreover,* $\ker(M) = GeG$ *for any minimal idempotent e of M.*

Proof. We may choose a closed $G \times G$-orbit Y in M. Then

$$MYM = \overline{G}Y\overline{G} \subseteq \overline{GYG} = \overline{Y} = Y.$$

Thus, Y is a two-sided ideal of M. Moreover, if Z is another two-sided ideal, then Z is stable by $G \times G$, and meets Y since $YZ \subseteq Y \cap Z$. Thus, Z contains Y; this shows that $Y = \ker(M)$. In particular, Y is the unique closed $G \times G$-orbit; this proves the first assertion. The second one follows from Proposition 5. □

Proposition 8. *Let M be a unit dense algebraic monoid with unit group G. Then the following conditions are equivalent for any $x \in M$:*

(i) *The orbit Gx (for the G-action by left multiplication) is closed in M.*

(ii) $Gx = Mx$.
(iii) $x \in \ker(M)$.

Moreover, all closed G-orbits in M are equivariantly isomorphic; in other words, the isotropy subgroup schemes G_x, where $x \in \ker(M)$, are all conjugate. Also, each closed orbit contains a minimal idempotent. For any such idempotent e, the algebraic group eMe equals eGe.

Proof. (i)⇒(ii) Since Gx is closed in M, we have $Mx = \overline{G}x \subseteq \overline{Gx} = Gx$ and hence $Mx = Gx$.

(ii)⇒(iii) We have $Gx = Mx \supset \ker(M)x$ and the latter subset is stable under left multiplication by G. Hence $Gx = \ker(M)x$ is contained in $\ker(M)$.

(iii)⇒(i) Let e be a minimal idempotent of M. Since $\ker(M) = GeG$, the G-orbits in $\ker(M)$ are exactly the orbits Geg, where $g \in G$. Since the right multiplication by g is an automorphism of the variety M commuting with left multiplications, these orbits are all isomorphic as G-varieties. In particular, they all have the same dimension; hence they are closed in $\ker(M)$, and thus in M. Also, the orbit Geg contains $g^{-1}eg$, which is a minimal idempotent since the map $M \to M$, $x \mapsto g^{-1}xg$ is an automorphism of algebraic monoids. Finally, we have $Ge = Me$ by (i); likewise, $eG = eM$ and hence $eGe = eMe$. □

Note that the closed orbits for the left G-action are exactly the minimal left ideals (considered in Remark 3 (ii) in the setting of algebraic semigroups).

2.5 The Normalization of an Algebraic Semigroup

In this subsection, we begin by recalling some background results on the normalization of an arbitrary variety (see e.g. [10, §4.2, §11.2]). Then we discuss the normalization of algebraic semigroups and monoids; as in the previous subsection, this construction will play an important role in their structure.

A variety X is *normal* at a point x if the local ring $\mathcal{O}_{X,x}$ is integrally closed in its total quotient ring; X is normal if it is so at any point. The normal points of a variety form a dense open subset, which contains the nonsingular points. The irreducible components of a normal variety are pairwise disjoint, and each of them is normal.

An arbitrary variety X has a *normalization*, i.e., a normal variety $\tilde{X}$ together with a finite surjective morphism $\eta : \tilde{X} \to X$ which satisfies the following universal property: for any normal variety Y and any morphism $\varphi : Y \to X$ which is dominant (i.e., the image of φ is dense in X), there exists a unique morphism $\tilde{\varphi} : Y \to \tilde{X}$ such that $\varphi = \eta \circ \tilde{\varphi}$. Then $\tilde{X}$ is uniquely determined up to unique isomorphism, and η is an isomorphism above the open subset of normal points of X; in particular, η is birational (i.e., an isomorphism over a dense open subset of X).

Proposition 9. *Let (S, μ) be an algebraic semigroup and let $\eta : \tilde{S} \to S$ be the normalization.*

(i) *If the morphism* $\mu : S \times S \to S$ *is dominant, then* $\tilde{S}$ *has a unique algebraic semigroup law* $\tilde{\mu}$ *such that* η *is a homomorphism. Moreover,* $\eta(E(\tilde{S})) = E(S)$.
(ii) *If* S *is an algebraic monoid (so that* μ *is surjective), then* $\tilde{S}$ *is an algebraic monoid as well, with neutral element the unique preimage of* 1_S *under* η. *Moreover,* η *induces an isomorphism* $G(\tilde{S}) \cong G(S)$.

Proof. (i) By the assumption on μ, the morphism $\mu \circ (\eta \times \eta) : \tilde{S} \times \tilde{S} \to S$ is dominant. Since $\tilde{S} \times \tilde{S}$ is normal, there exists a unique morphism $\tilde{\mu} : \tilde{S} \times \tilde{S} \to \tilde{S}$ such that $\eta \circ \tilde{\mu} = \mu \circ (\eta \times \eta)$. Then $\tilde{\mu}$ is associative, since it coincides with μ on the dense open subset of normal points; moreover, η is a homomorphism by construction. The assertion on idempotents is a consequence of Corollary 1.

(ii) The neutral element 1_S is a nonsingular point of S by Theorem 1; thus, it has a unique preimage $1_{\tilde{S}}$ under η. Moreover, we have for any $\tilde{x} \in \tilde{S}$:

$$\eta(\tilde{\mu}(\tilde{x}, 1_{\tilde{S}})) = \mu(\eta(\tilde{x}), \eta(1_{\tilde{S}})) = \eta(\tilde{x}) = \eta(\tilde{\mu}(1_{\tilde{S}}, \tilde{x})).$$

Thus, $\tilde{\mu}(\tilde{x}, 1_{\tilde{S}}) = \tilde{\mu}(1_{\tilde{S}}, \tilde{x}) = \tilde{x}$ for all $\tilde{x}$ such that $\eta(\tilde{x})$ is a normal point of S. By density of these points, it follows that $1_{\tilde{S}}$ is the neutral element of $(\tilde{S}, \tilde{\mu})$. Finally, the assertion on unit groups follows from the inclusion $G(\tilde{S}) \subseteq \eta^{-1}(G(S))$ and from the fact that η is an isomorphism above the nonsingular locus of S. □

Remark 4. (i) For an arbitrary algebraic semigroup S, there may exist several algebraic semigroup laws on the normalization $\tilde{S}$ that lift μ. For example, let $x \in S$ and consider the trivial semigroup law μ_x of Example 1 (i). Then $\mu_{\tilde{x}}$ lifts μ for any $\tilde{x} \in \tilde{X}$ such that $\eta(\tilde{x}) = x$. In general, such a point $\tilde{x}$ is not unique, e.g., when S is a plane curve and x an ordinary multiple point.

(ii) With the above notation, there may also exist no algebraic semigroup law on $\tilde{S}$ that lifts μ. To construct examples of such algebraic semigroups, consider a normal irreducible affine variety X and a complete nonsingular irreducible curve C, and choose a finite surjective morphism $\varphi : C \to \mathbb{P}^1$. Let $Y := C \setminus \{\varphi^{-1}(\infty)\}$; then Y is an affine nonsingular irreducible curve equipped with a finite surjective morphism $\varphi : Y \to \mathbb{A}^1$. Choose a point $x_o \in X$ and let $\gamma : Y \to X \times Y$, $y \mapsto (x_o, y)$; then γ is a section of the second projection $p_2 : X \times Y \to Y$. By [11, Thm. 5.1], there exists a unique irreducible variety S that sits in a co-cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & \mathbb{A}^1 \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\iota} \\ X \times Y & \xrightarrow{\eta} & S. \end{array}$$

Then ι is a closed immersion, and η is a finite morphism that restricts to an isomorphism $(X \setminus \{x_o\}) \times Y \cong S \setminus \iota(\mathbb{A}^1)$ and to the (given) morphism

$\{x_o\} \times Y \to \mathbb{A}^1$, $(x_o, y) \mapsto \varphi(y)$. In particular, η is the normalization; S is obtained by "pinching $X \times Y$ along $\{x_o\} \times Y$ via φ". Since the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & \mathbb{A}^1 \\ \gamma\big\downarrow & & \mathrm{id}\big\downarrow \\ X \times Y & \xrightarrow{\varphi \circ p_2} & \mathbb{A}^1 \end{array}$$

commutes, it yields a unique morphism $\rho : S \to \mathbb{A}^1$ such that $\rho \circ \iota = \mathrm{id}$ and $\rho \circ \eta = \varphi \circ p_2$. The retraction ρ defines in turn an algebraic semigroup law μ on S by $\mu(s, s') := \iota(\rho(s)\rho(s'))$ as in Example 1 (ii).

We claim that μ does not lift to any algebraic semigroup law on $X \times Y$, if the curve C is nonrational. Indeed, any such lift $\tilde{\mu}$ satisfies

$$\eta(\tilde{\mu}((x, y), (x', y'))) = \mu(\eta(x, y), \eta(x', y'))$$

$$= \iota(\rho(\eta(x, y)\rho(\eta(x', y')))) = \iota(\varphi(y)\varphi(y'))$$

for any $x, x' \in X$ and any $y, y' \in Y$. As a consequence, $\tilde{\mu}((x, y), (x', y'))$ only depends on (y, y'), and this yields an algebraic semigroup law on Y such that φ is a homomorphism. But such a law does not exist, as follows e.g. from Theorem 5 below.

3 Irreducible Algebraic Monoids

3.1 Algebraic Monoids with Affine Unit Group

The aim of this subsection is to prove the following result, due to Rittatore for irreducible algebraic monoids (see [29, Thm. 5]). The proof presented here follows his argument closely, except for an intermediate step (Proposition 10).

Theorem 2. *Let M be a unit dense algebraic monoid, and G its unit group. If G is affine, then so is M.*

Proof. Let $\eta : \tilde{M} \to M$ denote the normalization. Then $\tilde{M}$ is an algebraic monoid with unit group isomorphic to G, by Proposition 9. Moreover, G is dense in $\tilde{M}$ since it is so in M. If $\tilde{M}$ is affine, then M is affine by a result of Chevalley: the image of an affine variety by a finite morphism is affine (see [13, Exer. II.4.2]). Thus, we may assume that M is normal. Then M is the disjoint union of its irreducible components, and each of them is isomorphic (as a variety) to the neutral component M^o (Proposition 6). So we may assume in addition that M is irreducible.

By Proposition 7, the connected algebraic group $G \times G$ acts on M with a unique closed orbit. In view of a result of Sumihiro (see [34]), it follows that M is quasiprojective; in other words, there exists a locally closed immersion $\iota : M \to \mathbb{P}^n$ for some positive integer n. (We may further assume that ι is equivariant for some action of G on $\mathbb{P}^n$; we will not need that fact in this proof). Then the pull-back $L := \iota^* O_{\mathbb{P}^n}(1)$ is an ample line bundle on M. The associated principal $\mathbb{G}_m$-bundle $\pi : X \to M$ (where X is the complement of the zero section in L) is the pull-back to M of the standard principal $\mathbb{G}_m$-bundle $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Thus, X is a locally closed subvariety of $\mathbb{A}^{n+1}$, and hence is quasi-affine.

By Proposition 10 below (a version of [29, Thm. 4]), X has a structure of algebraic monoid such that π is a homomorphism. Since that monoid is quasi-affine, it is in fact affine by a result of Renner (see [25, Thm. 4.4]). Moreover, the map $\pi : X \to M$ is the categorical quotient by the action of $\mathbb{G}_m$; hence M is affine. □

Proposition 10. *Let M be a normal irreducible algebraic monoid, and assume that its unit group G is affine. Let $\varphi : L \to M$ be a line bundle, and $\pi : X \to M$ the associated principal $\mathbb{G}_m$-bundle. Then X has a structure of a normal irreducible algebraic monoid such that π is a homomorphism.*

Proof. By [16, Lem. 4.3], the preimage $Y := \pi^{-1}(G)$ has a structure of algebraic group such that the restriction of π is a homomorphism; we then have an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow Y \stackrel{\pi}{\longrightarrow} G \longrightarrow 1,$$

where $\mathbb{G}_m$ is contained in the center of Y. Thus, the group law $\mu_G : G \times G \to G$ sits in a cartesian square

$$\begin{array}{ccc} Y \times^{\mathbb{G}_m} Y & \stackrel{\mu_Y}{\longrightarrow} & Y \\ {\scriptstyle \pi\times\pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ G \times G & \stackrel{\mu_G}{\longrightarrow} & G, \end{array}$$

where $Y \times^{\mathbb{G}_m} Y$ denotes the quotient of $Y \times Y$ by the action of $\mathbb{G}_m$ via $t \cdot (y, z) = (ty, t^{-1}z)$, and μ_Y stands for the group law on Y. Via the correspondence between principal $\mathbb{G}_m$-bundles and line bundles, this translates into a cartesian square

$$\begin{array}{ccc} p_1^*(L|_G) \otimes p_2^*(L|_G) & \longrightarrow & L|_G \\ {\scriptstyle \varphi\times\varphi}\downarrow & & {\scriptstyle \varphi}\downarrow \\ G \times G & \stackrel{\mu_G}{\longrightarrow} & G, \end{array}$$

where $p_1, p_2 : G \times G \to G$ denote the projections. In other words, we have an isomorphism

$$p_1^*(L|_G) \otimes p_2^*(L|_G) \stackrel{\cong}{\longrightarrow} \mu_G^*(L|_G)$$

of line bundles over $G \times G$.

Over $M \times M$, this yields an isomorphism

$$p_1^*(L) \otimes p_2^*(L) \xrightarrow{\cong} \mu^*(L) \otimes \mathcal{O}_{M\times M}(D),$$

where D is a Cartier divisor with support in $(M \times M) \setminus (G \times G)$. Since G is affine, the irreducible components $E_1, \ldots, E_n$ of $M \setminus G$ are divisors of M. Thus, the irreducible components of $(M \times M) \setminus (G \times G)$ are exactly the divisors $E_i \times M$ and $M \times E_j$, where $i, j = 1, \ldots, n$. Hence

$$D = p_1^*(D_1) + p_2^*(D_2)$$

for some Weil divisors D_1, D_2 with support in $M \setminus G$. In particular, the pullback of D to $M \times G$ is $D_1 \times G$. Since D is Cartier, so is D_1; likewise, D_2 is Cartier. We thus obtain an isomorphism

$$p_1^*(L) \otimes p_2^*(L) \xrightarrow{\cong} \mu^*(L) \otimes p_1^*(\mathcal{O}_M(D_1)) \otimes p_2^*(\mathcal{O}_M(D_2))$$

of line bundles over $M \times M$. We now pull back this isomorphism to $M \times \{1\}$. Note that $\mu^*(L)|_{M\times\{1\}} = L = p_1^*(L)|_{M\times\{1\}}$; also, $p_1^*(\mathcal{O}_M(D_1))|_{M\times\{1\}} = \mathcal{O}_M(D_1)$, and both $p_2^*(L)|_{M\times\{1\}}$, $p_2^*(\mathcal{O}_M(D_2))|_{M\times 1}$ are trivial. Thus, $\mathcal{O}_M(D_1)$ is trivial; one shows similarly that $\mathcal{O}_M(D_2)$ is trivial. Hence we have in fact an isomorphism

$$p_1^*(L) \otimes p_2^*(L) \xrightarrow{\cong} \mu^*(L).$$

As above, this translates into a cartesian square

$$\begin{array}{ccc} X \times^{\mathbb{G}_m} X & \longrightarrow & X \\ {\scriptstyle \pi\times\pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ M \times M & \xrightarrow{\mu} & M. \end{array}$$

In turn, this yields a morphism $\nu : X \times X \to X$ which lifts $\mu : M \times M \to M$ and extends the group law $Y \times Y \to Y$. It follows readily that ν is associative and has 1_Y as a neutral element. □

A noteworthy consequence of Theorem 2 is the following sufficient condition for an algebraic monoid to be affine, which slightly generalizes [6, Cor. 3.3]:

Corollary 2. *Let M be a unit dense algebraic monoid having a zero element. Then M is affine.*

Proof. Consider the action of the unit group G on M via left multiplication. This action is faithful, and fixes the zero element. It follows that G is affine (see e.g. [7, Prop. 2.1.6]). Hence M is affine by Theorem 2. □

3.2 Induction of Algebraic Monoids

In this subsection, we show that any unit dense algebraic monoid has a universal homomorphism to an algebraic group, and we study the fibers of this homomorphism.

Proposition 11. *Let M be a unit dense algebraic monoid, and G its unit group.*

(i) *There exists a homomorphism of algebraic monoids $\varphi : M \to \mathscr{G}(M)$, where $\mathscr{G}(M)$ is an algebraic group, such that every homomorphism of algebraic monoids $\psi : M \to \mathscr{G}$, where $\mathscr{G}$ is an algebraic group, factors uniquely as φ followed by a homomorphism of algebraic groups $\mathscr{G}(M) \to \mathscr{G}$.*
(ii) *We have $\mathscr{G}(M) = \varphi(M) = \varphi(G) = G/H$, where H denotes the smallest normal subgroup scheme of G containing the isotropy subgroup scheme G_x for some $x \in \ker(M)$.*

Proof. We show both assertions simultaneously. Let $\psi : M \to \mathscr{G}$ be a homomorphism as in the statement. Then $\psi|_G$ is a homomorphism of algebraic groups, and hence its image is a closed subgroup of $\mathscr{G}$. Since M is unit dense, it follows that $\psi(M) = \psi(G)$. Let K be the scheme-theoretic kernel of $\psi|_G$. Then K is a normal subgroup scheme of G, and ψ induces an isomorphism from G/K to $\psi(G)$; we may thus view ψ as a G-equivariant homomorphism $M \to G/K$. In particular, for any $x \in M$, the map $g \mapsto \psi(g \cdot x)$ yields a morphism $G \to G/K$ which is equivariant under the action of G by left multiplication, and invariant under the action of the isotropy subgroup scheme G_x by right multiplication. Thus, K contains G_x; hence K contains H, and $\psi|_G$ factors as the quotient homomorphism $\gamma : G \to G/H$ followed by the canonical homomorphism $\pi : G/H \to G/K$.

Next, choose $x \in \ker(M)$; then $Gx = Mx$ by Proposition 8. Thus, the morphism $M \to Mx$, $y \mapsto yx$ may be viewed as a morphism $M \to Gx \cong G/G_x$. Composing with the morphism $G/G_x \to G/H$ induced by the inclusion of G_x in H, we obtain a morphism $\varphi : M \to G/H$. Clearly, φ is G-equivariant, and $\varphi(1)$ is the neutral element of G/H. Thus, the restriction $\varphi|_G$ is the quotient homomorphism γ. By density, φ is a homomorphism of monoids, and $\psi = \pi \circ \varphi$. So φ is the desired homomorphism. □

Remark 5. (i) As a consequence of the above proposition, the smallest subgroup scheme of G containing G_x is independent of the choice of $x \in \ker(M)$. This also follows from the fact that the subgroup schemes G_x, where $x \in \ker(M)$, are all conjugate in G (Proposition 8). By that proposition, we may take for x any minimal idempotent of M.

(ii) As another consequence, any irreducible semigroup S has a universal homomorphism to an algebraic group (in the sense of the above proposition). Indeed, choose an idempotent e in S, and consider a homomorphism of semigroups $\psi : S \to \mathscr{G}$, where $\mathscr{G}$ is an algebraic group. Then $\psi(x) = \psi(exe)$ for all $x \in S$; moreover, eSe is an irreducible monoid with neutral element e. Thus, there exists a unique homomorphism $\pi : \mathscr{G}(eSe) \to \mathscr{G}$ such that $\psi(x) = \pi(\phi(exe))$ for all $x \in S$, where $\phi : eSe \to \mathscr{G}(eSe)$ denotes the universal homomorphism. Then we must have $\pi(\phi(exye)) = \pi(\phi(exe)\phi(eye))$ for all $x, y \in S$. Let H denote the smallest normal subgroup scheme of $\mathcal{G}(eSe)$ containing the image of the morphism

$$S \times S \longrightarrow \mathcal{G}(eSe), \quad (x, y) \longmapsto \phi(exye)\phi(exe)^{-1}\phi(eye)^{-1},$$

and let $\varphi : S \to \mathcal{G}(eSe)/H$ denote the homomorphism that sends every x to the image of exe. Then π factors as φ followed by a unique homomorphism of algebraic groups $\mathcal{G}(eSe)/H \to \mathcal{G}$, i.e., φ is the desired homomorphism. Note that $\varphi : S \to \mathcal{G}(S)$ is surjective by construction; in particular, $\mathcal{G}$ is connected.

Proposition 12. *Keep the notation and assumptions of Proposition 11.*

(i) *If H is an algebraic group (e.g., if* $\mathrm{char}(k) = 0$*), then the scheme-theoretic fibers of φ are reduced.*
(ii) *If M is normal, then H is a connected algebraic group; moreover, the scheme-theoretic fibers of φ are reduced and irreducible.*

Proof. (i) Denote by $\gamma : G \to G/H$ the quotient homomorphism and form the cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\varphi'} & G \\ \gamma' \downarrow & & \downarrow \gamma \\ M & \xrightarrow{\varphi} & G/H. \end{array}$$

Since γ and φ are equivariant for the actions of G by left multiplication, X is equipped with a G-action such that γ' and φ' are equivariant. Denote by N the (scheme-theoretic) fiber of φ' at the neutral element 1_G. Then the morphism

$$G \times N \longrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

is an isomorphism with inverse given by $x \mapsto (\varphi'(x), \varphi'(x)^{-1} \cdot x)$. Moreover, the fiber of φ' at every $g \in G$ is $g \cdot N \cong N$. If H is an algebraic group (i.e., if H is smooth; this holds when $\mathrm{char}(k) = 0$), then the morphism γ is smooth as well; hence so is γ'. It follows that X is reduced. But $X \cong G \times N$ and hence N is reduced. If in addition H is connected, then the fibers of γ are irreducible; hence the same holds for γ', and X is irreducible. As above, it follows that N is irreducible.

(ii) Consider the reduced neutral component $H^o_{\mathrm{red}} \subseteq H$; then H^o_{red} is a closed normal subgroup of G. Moreover, the natural map $\delta : G/H^o_{\mathrm{red}} \to G/H$ is a finite morphism and sits in a commutative square

$$\begin{array}{ccc} G & \xrightarrow{\iota} & M \\ \epsilon \downarrow & & \downarrow \varphi \\ G/H^o_{\mathrm{red}} & \xrightarrow{\delta} & G/H, \end{array}$$

where ι denotes the inclusion. Let $\Gamma \subseteq M \times G/H^o_{\mathrm{red}}$ be the closure of the graph of ϵ. Then the projection $p_1 : \Gamma \to M$ is a finite morphism, and an isomorphism over the dense open subset G of M. Since M is normal, it follows that p_1 is an isomorphism, i.e., ϵ extends to a morphism $\psi : M \to G/H^o_{\mathrm{red}}$. As ϵ is a homomorphism of algebraic groups, ψ must be a homomorphism of monoids. Thus, ψ factors through φ, and hence $H^o_{\mathrm{red}} = H$. In other words, H is a connected algebraic group. □

But in general, the scheme-theoretic fibers of the homomorphism $\varphi : M \to \mathcal{G}(M)$ are reducible; also, these fibers are nonreduced in prime characteristics, as shown by the following:

Example 3. Consider the monoid $\mathbb{A}^3$ equipped with pointwise multiplication, and the locally closed subset

$$M := \{(x, y, z) \mid z^n = xy^n \text{ and } x \neq 0\},$$

where n is a positive integer. Then M is an irreducible commutative algebraic monoid with unit group

$$G = \{(x, y, z) \mid z^n = xy^n \text{ and } z \neq 0\},$$

isomorphic to $\mathbb{G}_m^2$ via the projection $(x, y, z) \mapsto (y, z)$. Moreover, $\ker(M) = Me$, where $e := (1, 0, 0)$ is the unique minimal idempotent. Since M is commutative, the isotropy subgroup scheme H of Proposition 11 is just G_e; the latter is the scheme-theoretic kernel of the homomorphism $x : G \to \mathbb{G}_m$. Thus,

$$H \cong \{(y, z) \in \mathbb{G}_m^2 \mid y^n = z^n\} \cong \mathbb{G}_m \times \mu_n,$$

where μ_n denotes the subgroup scheme of nth roots of unity. The universal homomorphism $\varphi : M \to G/H$ is identified to $x : M \to \mathbb{G}_m$, and this identifies the fiber of φ at 1 to the submonoid scheme $(y^n = z^n)$ of $(\mathbb{A}^2, \times)$. The latter scheme is reducible when $n \geq 2$, and nonreduced when n is a multiple of $\mathrm{char}(k)$.

We keep the notation and assumptions of Proposition 11, and denote by N the scheme-theoretic fiber of φ at 1. Assume in addition that *H is an algebraic group* (this holds e.g. if M is normal or if $\mathrm{char}(k) = 0$). Then N is reduced by Proposition 12; also, N is a closed submonoid of M, containing H and stable under the action of G on M by conjugation (via $g \cdot x := gxg^{-1}$). Moreover, the map

$$\pi : G \times N \longrightarrow M, \quad (g, y) \longmapsto gy$$

is a homomorphism of algebraic monoids, where $G \times N$ is equipped with the composition law

$$(g_1, y_1)\,(g_2, y_2) := (g_1 g_2, g_2^{-1} y_1 g_2 y_2)$$

with unit $(1_G, 1_N)$ (this defines the *semi-direct product of* G *with* N). Finally, π is the quotient morphism for the action of H on $G \times N$ via

$$h \cdot (g, y) := (gh^{-1}, hy).$$

In other words, $\varphi : M \to G/H$ identifies M to the fiber bundle $G \times^H N \to G/H$ associated to the principal H-bundle $G \to G/H$ and to the variety N on which H acts by left multiplication. We say that the algebraic monoid M is *induced* from N.

If we no longer assume that H is an algebraic group, then N is just a submonoid scheme of M, and the above properties hold in the setting of monoid schemes. We now obtain slightly weaker versions of these properties in the setting of algebraic monoids.

Proposition 13. *Let* M *be a unit dense algebraic monoid,* G *its unit group,* $\varphi : M \to G/H$ *the universal homomorphism to an algebraic group, and* N *the scheme-theoretic fiber of* φ *at* 1*. Denote by* H_{red} *(resp.* N_{red}*) the largest reduced scheme of* H *(resp.* N*).*

(i) H_{red} *is a closed normal subgroup of* G *and* N_{red} *is a closed submonoid of* M*, stable under the action of* G *by conjugation.*
(ii) $G \times^{H_{\mathrm{red}}} N_{\mathrm{red}}$ *is an algebraic monoid, and the natural map*

$$\psi : G \times^{H_{\mathrm{red}}} N_{\mathrm{red}} \longrightarrow M$$

is a finite bijective homomorphism of algebraic monoids.
(iii) N_{red} *is unit dense and its unit group is* H_{red}*.*
(iv) ψ *is birational.*

Proof. (i) The assertion on H_{red} is well-known. That on N_{red} follows readily from the fact that N is a closed submonoid scheme of M, stable under the G-action by conjugation.

(ii) The natural map $G/H_{\mathrm{red}} \to G/H$ is a purely inseparable homomorphism of algebraic groups, and hence is finite and bijective. Also, $G \times^{H_{\mathrm{red}}} N$ is the fibered product of $M = G \times^H N$ and G/H_{red} over G/H. Thus, $G \times^{H_{\mathrm{red}}} N$ is a monoid scheme; moreover, the natural morphism $G \times^{H_{\mathrm{red}}} N \to M$ is finite, and bijective on closed points. As $G \times^{H_{\mathrm{red}}} N_{\mathrm{red}} = (G \times^{H_{\mathrm{red}}} N)_{\mathrm{red}}$, this yields our assertions.

(iii) Since M is unit dense with unit group G and ψ is a homeomorphism, we see that $G \times^{H_{\mathrm{red}}} N_{\mathrm{red}}$ is unit dense with unit group G as well. It follows that $G \times N_{\mathrm{red}}$ is unit dense with unit group $G \times H_{\mathrm{red}}$. Thus, H_{red} is the unit group of N_{red} and is dense there.

(iv) Just note that ψ restricts to the natural isomorphism $G \times^{H_{\mathrm{red}}} H_{\mathrm{red}} \stackrel{\cong}{\to} G$; moreover, $G \times^{H_{\mathrm{red}}} H_{\mathrm{red}}$ is a dense open subset of $G \times^{H_{\mathrm{red}}} N_{\mathrm{red}}$. □

Example 4. Assume that $\mathrm{char}(k) = p > 0$. Consider the monoid

$$M := \{(x, y, z) \in \mathbb{A}^3 \mid z^p = xy^p \text{ and } x \neq 0\}$$

relative to pointwise multiplication, as in Example 3. Recall from this example that $G \cong \mathbb{G}_m^2$ and that the universal homomorphism $\varphi : M \to G/H$ is just $x : M \to \mathbb{G}_m$, with scheme-theoretic fiber N at 1 being the submonoid scheme $(z^p = y^p)$ of $(\mathbb{A}^2, \times)$. It follows that $N_{\mathrm{red}} \cong (\mathbb{A}^1, \times)$, $H_{\mathrm{red}} \cong \mathbb{G}_m$ and $G \times^{H_{\mathrm{red}}} N_{\mathrm{red}} \cong \mathbb{G}_m \times \mathbb{A}^1$, where the right-hand side is equipped with pointwise multiplication. One checks that $\psi : G \times^{H_{\mathrm{red}}} N_{\mathrm{red}} \to M$ is identified with the map $(t, u) \mapsto (t^p, t, u)$.

Returning to the general setting, we now relate the idempotents and kernel of M with those of N_{red}:

Proposition 14. *Keep the notation and assumptions of Proposition 13.*

(i) $E(M) = E(N_{\mathrm{red}})$.
(ii) *The assignment* $I \mapsto I \cap N_{\mathrm{red}}$ *defines a bijection between the two-sided ideals of* M *and those two-sided ideals of* N_{red} *that are stable under conjugation by* G. *The inverse bijection is given by* $J \mapsto GJ$.
(iii) *We have* $\ker(M) \cap N_{\mathrm{red}} = \ker(N_{\mathrm{red}})$ *and* $\ker(M) = G \ker(N_{\mathrm{red}})$.

Proof. (i) Clearly, $E(N_{\mathrm{red}}) \subseteq E(M)$. Moreover, if $e \in E(M)$ then $\varphi(e) = 1_{G/H}$ and hence $e \in N$, i.e., $e \in N_{\mathrm{red}}$.

(ii) Consider a two-sided ideal I of M. Then $J := I \cap N_{\mathrm{red}}$ is a two-sided ideal of N_{red}, stable under conjugation by G (since so are I and N_{red}). Moreover, $I = GJ$, since $M = GN_{\mathrm{red}}$ and $I = GI$.

Conversely, let J be a two-sided ideal of N_{red}, stable under conjugation by G. Then $I := GJ$ is closed in M and satisfies $I \cap N_{\mathrm{red}} = J$, as follows easily from the fact that $\psi : G \times^{H_{\mathrm{red}}} N_{\mathrm{red}} \to M$ is a homeomorphism. Moreover, $GIG = GJG = GJ = I$ by stability of J under conjugation. Since M is unit dense and I is closed in M, it follows that $MIM = I$; in other words, I is a two-sided ideal.

(iii) Since N_{red} is stable under conjugation by G, so is $\ker(N_{\mathrm{red}})$. In view of (ii), it follows that $\ker(M) \cap N_{\mathrm{red}} = \ker(N_{\mathrm{red}})$. Together with Proposition 7, this yields $\ker(M) = G \ker(N_{\mathrm{red}})G = G \ker(N_{\mathrm{red}})$. □

Remark 6. (i) If N is reduced, then any homomorphism of algebraic monoids from N to an algebraic group is trivial.

Indeed, let $\kappa : N \to \mathscr{G}$ be such a homomorphism. We may assume that κ is the universal morphism $N \to H/K$ of Proposition 11, where K is a normal subgroup scheme of H. Then the G-action on N by conjugation yields a G-action on H/K, compatible with the conjugation action on H; thus, K is a normal subgroup scheme of G. Moreover, the H-equivariant morphism κ induces a G-equivariant morphism

$$\psi : M = G \times^H N \longrightarrow G \times^H H/K \cong G/K, \quad (g, y)H \longmapsto (g, \kappa(y))H.$$

Also, $\psi(1)$ is the neutral element of G/K, since $\kappa(1)$ is the neutral element of H/K. It follows that $\psi(xy) = \psi(x)\psi(y)$ for all $x \in G$ and $y \in M$, and hence for all $x, y \in M$. By Proposition 11, ψ factors through φ, and hence $K = H$.

We do not know if the analogous statement holds for N_{red} when N is nonreduced.

(ii) Let M be a unit dense algebraic monoid, and M^o its neutral component. Then M^o is stable under the action of the unit group G by conjugation on M; also, $M = GM^o$ by Proposition 6. Thus, $G \times^{G^o} M^o$ is an algebraic monoid, the disjoint union of the irreducible components of M. Moreover, the map

$$\varphi : G \times^{G^o} M^o \longrightarrow M, \quad (g, x)G^o \longmapsto gx$$

is a homomorphism of algebraic monoids, which is readily seen to be finite and birational. Hence φ is an isomorphism whenever M is normal.

For an arbitrary (unit dense) M, it follows that $E(M) = E(M^o)$. Indeed, φ is surjective, and hence restricts to a surjection $E(G \times^{G^o} M^o) \to E(M)$ by Corollary 1. Also, $E(G \times^{G^o} M^o) = E(M^o)$, since the unique idempotent of G/G^o is the coset of 1_G.

3.3 Structure of Irreducible Algebraic Monoids

We begin this subsection by presenting some classical results on the structure of an arbitrary connected algebraic group G. By Chevalley's structure theorem, G has a largest connected affine normal subgroup G_{aff}; moreover, the quotient group G/G_{aff} is an abelian variety. In other words, G sits in a unique exact sequence of connected algebraic groups

$$1 \longrightarrow G_{\text{aff}} \longrightarrow G \xrightarrow{\alpha} A \longrightarrow 1,$$

where G_{aff} is linear, and $A := G/G_{\text{aff}}$ is an abelian variety. This exact sequence is generally nonsplit; yet G has a smallest closed subgroup H such that $\alpha|_H$ is surjective. Moreover, H is connected, contained in the center of G, and satisfies $\mathcal{O}(H) = k$. In fact, H is the largest closed subgroup of G satisfying the latter property, which defines the class of *anti-affine* algebraic groups; we denote H by G_{ant}. Finally, we have the *Rosenlicht decomposition*: $G = G_{\text{aff}} G_{\text{ant}}$, and $(G_{\text{ant}})_{\text{aff}}$ is the connected neutral component of the scheme-theoretic intersection $G_{\text{aff}} \cap G_{\text{ant}}$. In other words, we have an isomorphism of algebraic groups

$$G \cong (G_{\text{aff}} \times G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}},$$

and the quotient group scheme $(G_{\text{aff}} \cap G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}}$ is finite. We refer to [7] for an exposition of these results and of further developments.

We shall obtain a similar structure result for an arbitrary irreducible algebraic monoid; then the unit group is a connected algebraic group by Theorem 1. Our starting point is the following:

Proposition 15. *Let M be an irreducible algebraic monoid, G its unit group, $\varphi : M \to \mathscr{G}(M) = G/H$ the universal homomorphism to an algebraic group, and N the scheme-theoretic fiber of φ at 1. Then H and N are affine.*

Proof. Recall from Proposition 11 that H is the normal subgroup scheme of G generated by G_x, where x is an arbitrary point of $\ker(M)$. Since G_x is the isotropy subgroup scheme of a point for a faithful action of G (the action on M by left multiplication), it follows that G_x is affine (see e.g. [7, Cor. 2.1.9]). The image of G_x under the homomorphism $\alpha : G \to A$ is affine (as the image of an affine group scheme by a homomorphism of group schemes) and proper (as a subgroup scheme of the abelian variety G/G_{aff}), hence finite. But $\alpha(G_x) = \alpha(H)$ by the definition of H and the commutativity of A; hence $\alpha(H)$ is finite. Also, the kernel of the homomorphism $\alpha|_H$ is a subgroup scheme of G_{aff}, and hence is affine. Thus, the reduced scheme H_{red} is an extension of a finite group by an affine algebraic group, and hence is affine. Thus, so is N_{red} in view of Theorem 2 and of Proposition 13. It follows that N is affine, by [13, Exer. III.3.1]. □

Remark 7. If $\mathrm{char}(k) = 0$, then N is reduced and any homomorphism from N to an algebraic group is trivial by Remark 6 (i). If in addition N is irreducible (e.g., if M is normal), then $\ker(N)$ is generated by the minimal idempotents. Indeed, the unit group H of N is generated by the conjugates of the isotropy group H_e for some idempotent $e \in \ker(N)$, by Proposition 11 (ii). So the assertion follows from [24, Thm. 2.1].

A noteworthy consequence of Proposition 15 is the following:

Corollary 3. *Any irreducible algebraic monoid is quasiprojective.*

Proof. With the notation of the above proposition, the morphism φ is affine, since $M = G \times^H N$ where N is affine. Moreover, G/H is quasiprojective since so is any algebraic group (see e.g. [7, Prop. 3.1.1]). Thus, M is quasiprojective as well. □

Another consequence is a version of Chevalley's structure theorem for an irreducible algebraic monoid; it generalizes [6, Thm. 1.1], where the monoid is assumed to be normal.

Theorem 3. *Let M be an irreducible algebraic monoid, G its unit group, and M_{aff} the closure of G_{aff} in M.*

(i) *M_{aff} is an irreducible affine algebraic monoid with unit group G_{aff}.*
(ii) *The action of G_{aff} on M_{aff} extends to an action of $G = G_{\mathrm{aff}} G_{\mathrm{ant}}$, where G_{ant} acts trivially.*
(iii) *The natural map $G_{\mathrm{ant}} \times^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}} M_{\mathrm{aff}} \to G \times^{G_{\mathrm{aff}}} M_{\mathrm{aff}}$ is an isomorphism of irreducible algebraic monoids. Moreover, the natural map*

$$\kappa : G \times^{G_{\mathrm{aff}}} M_{\mathrm{aff}} \to M$$

is a finite birational homomorphism of algebraic monoids.
(iv) *$E(M) = E(M_{\mathrm{aff}})$ and $\ker(M) = G \ker(M_{\mathrm{aff}})$.*

(v) *M is normal if and only if M_{aff} is normal and κ is an isomorphism. Then the assignment $I \mapsto I \cap M_{\mathrm{aff}}$ defines a bijection between the two-sided ideals of M and those of M_{aff}; the inverse bijection is given by $J \mapsto GJ$. In particular,* $\ker(M) \cap M_{\mathrm{aff}} = \ker(M_{\mathrm{aff}})$.

Proof. (i) Clearly, M_{aff} is an irreducible submonoid of M, and $G(M_{\mathrm{aff}})$ contains G_{aff} as an open subgroup. Since $G(M_{\mathrm{aff}})$ is connected, it follows that $G(M_{\mathrm{aff}}) = G_{\mathrm{aff}}$. Hence M_{aff} is affine by Theorem 2.

(ii) The assertion follows readily from the Rosenlicht decomposition: since $G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$ is contained in the center of G_{aff}, its action on M_{aff} by conjugation is trivial. Thus, the G_{aff}-action by conjugation on M_{aff} extends to an action of $G \cong (G_{\mathrm{aff}} \times G_{\mathrm{ant}})/(G_{\mathrm{aff}} \cap G_{\mathrm{ant}})$, where G_{ant} acts trivially.

(iii) The first assertion follows from the Rosenlicht decomposition again, since that decomposition yields an isomorphism $G \cong G_{\mathrm{ant}} \times^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}} G_{\mathrm{aff}}$ of principal G_{aff}-bundles over $G/G_{\mathrm{aff}} \cong G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap G_{\mathrm{aff}})$. For the second assertion, note first that κ restricts to the natural isomorphism $G \times^{G_{\mathrm{aff}}} G_{\mathrm{aff}} \to G$, and hence is a birational homomorphism of algebraic monoids. To show that κ is finite, we use the isomorphism $M \cong G \times^{H} N$ of Sect. 3.2. Here H and N are affine by Proposition 15; also, the natural map $G \times^{H_{\mathrm{red}}} N_{\mathrm{red}} \to M$ is finite and bijective by Proposition 13. It follows that the analogous map

$$\gamma : G \times^{H^o_{\mathrm{red}}} N^o_{\mathrm{red}} \longrightarrow M$$

is finite and surjective. But $H^o_{\mathrm{red}} \subseteq G_{\mathrm{aff}}$ since H is affine. Thus,

$$G \times^{H^o_{\mathrm{red}}} N^o_{\mathrm{red}} \cong G \times^{G_{\mathrm{aff}}} (G_{\mathrm{aff}} \times^{H^o_{\mathrm{red}}} N^o_{\mathrm{red}}).$$

Moreover, $N^o_{\mathrm{red}} \subseteq M_{\mathrm{aff}}$, since N^o_{red} is the closure in M of $H^o_{\mathrm{red}} \subseteq G_{\mathrm{aff}}$; hence $G_{\mathrm{aff}} N^o_{\mathrm{red}} \subseteq M_{\mathrm{aff}}$. So γ factors as the natural map

$$\beta : G \times^{G_{\mathrm{aff}}} (G_{\mathrm{aff}} \times^{H^o_{\mathrm{red}}} N^o_{\mathrm{red}}) \to G \times^{G_{\mathrm{aff}}} M_{\mathrm{aff}}$$

(induced from the map $\delta : G_{\mathrm{aff}} \times^{H^o_{\mathrm{red}}} N^o_{\mathrm{red}} \to M_{\mathrm{aff}}$), followed by κ. Now δ is the restriction of γ to a closed subvariety, and hence is finite; thus, its image $G_{\mathrm{aff}} N^o_{\mathrm{red}}$ is closed in M_{aff}. But $M_{\mathrm{aff}} = \overline{G_{\mathrm{aff}}}$, and hence δ is surjective. Hence β is finite and surjective. Since $\gamma = \kappa \circ \beta$, it follows that κ is finite and surjective as well.

(iv) Let $e \in E(M)$. By Corollary 1, we may lift e to an idempotent f of $G \times^{G_{\mathrm{aff}}} M_{\mathrm{aff}}$. Then the image of f in G/G_{aff} is the neutral element, and hence $f \in M_{\mathrm{aff}}$ so that $e \in E(M_{\mathrm{aff}})$. The converse is obvious.

Next, choose e minimal. Then $\ker(M) = GeG$ by Proposition 7, and hence $\ker(M) = G(G_{\mathrm{aff}} e G_{\mathrm{aff}})$ in view of the Rosenlicht decomposition. But $G_{\mathrm{aff}} e G_{\mathrm{aff}} = \ker(M_{\mathrm{aff}})$ since e is a minimal idempotent of G_{aff}.

(v) Assume that M is normal. By (iii) and Zariski's Main Theorem, it follows that κ is an isomorphism. In particular, $G \times^{G_{\mathrm{aff}}} M_{\mathrm{aff}}$ is normal. Since the natural morphism $G \times M_{\rm aff} \to G \times^{G_{\rm aff}} M_{\rm aff}$ is smooth, it follows that $G \times M_{\rm aff}$ is normal (e.g., by Serre's criterion); hence so is $M_{\rm aff}$. The converse is straightforward. This proves the first assertion.

The second assertion is proved by the argument of Proposition 14 (ii); note that any two-sided ideal of $M_{\rm aff}$ is stable under conjugation by G, in view of (ii) above. □

Example 5. Let n be a positive integer, μ_n the group scheme of nth roots of unity, and A an abelian variety containing μ_n as a subgroup scheme (any ordinary elliptic curve will do). As in Example 3, let N be the submonoid scheme $(z^n = y^n)$ of $(\mathbb{A}^2, \times)$, and H the unit subgroup scheme of N; then $H \cong \mu_n \times \mathbb{G}_m$. Next, let $G := A \times \mathbb{G}_m$; this is a connected commutative algebraic group containing H as a subgroup scheme. Finally, let

$$M := G \times^H N = A \times^{\mu_n} N.$$

Then one checks that M is an irreducible algebraic monoid with unit group G. Clearly, $G_{\rm aff} = \mathbb{G}_m$ and $A(G) = A$; also, one checks that $M_{\rm aff} = (\mathbb{A}^1, \times)$ and hence $G \times^{G_{\rm aff}} M_{\rm aff} \cong A \times \mathbb{A}^1$. The morphism $\kappa : G \times^{G_{\rm aff}} M_{\rm aff} \to M$ sends the closed subscheme $\mu_n \times \{0\}$ to 0, and restricts to an isomorphism over the complement. So M is obtained from $A \times \mathbb{A}^1$ by pinching $\mu_n \times \{0\}$ to a point.

In view of Theorem 3, we may transfer information from affine algebraic monoids (about which much is known, see [23, 26]) to general ones. For example, the minimal idempotents of any irreducible algebraic monoid are all conjugate under the unit group, since this holds in the affine case by [23, Prop. 6.1, Cor. 6.8]. Another noteworthy corollary is the following relation between the partial order on idempotents and limits of one-parameter subgroups:

Corollary 4. *Let (S, μ) be an irreducible algebraic semigroup, and $e, f \in E(S)$. Then $e \leq f$ if and only if there exists a homomorphism of algebraic semigroups $\lambda : (\mathbb{A}^1, \times) \to (S, \mu)$ such that $\lambda(0) = e$ and $\lambda(1) = f$.*

Proof. The "if" implication is obvious (and holds in every algebraic semigroup). For the converse, assume that $e \leq f$. Then $e \in fSf$ and the latter is an irreducible algebraic monoid. Thus, we may assume that S itself is an irreducible algebraic monoid, and f is the neutral element. In view of Theorem 3, we may further assume that S is affine. Then the assertion follows from [22, Thm. 2.9, Thm. 2.10]. □

3.4 The Albanese Morphism

By [30, Sec. 4], every irreducible variety X admits a universal morphism to an abelian variety: the *Albanese morphism*,

$$\alpha : X \longrightarrow A(X).$$

The group $A(X)$ is generated by the differences $\alpha(x)-\alpha(y)$, where $x, y \in X$. Also, X admits a universal rational map to an abelian variety: the *Albanese rational map*,

$$\alpha_{\mathrm{rat}} : X - \to A(X)_{\mathrm{rat}}.$$

The maps α and α_{rat} are uniquely determined up to translations and isomorphisms of the algebraic group $A(X)$. Moreover, there exists a unique morphism

$$\beta : A(X)_{\mathrm{rat}} \longrightarrow A(X)$$

such that $\alpha = \beta \circ \alpha_{\mathrm{rat}}$. The morphism β is always surjective; when X is nonsingular, it is an isomorphism. For an arbitrary X, we have $A(X)_{\mathrm{rat}} = A(U)$, where $U \subseteq X$ denotes the nonsingular locus; in particular, α_{rat} is defined at any nonsingular point of X.

When X is equipped with a base point x, we may assume that $\alpha(x)$ is the origin of $A(X)$. If X is nonsingular at x, then we may further assume that $\alpha_{\mathrm{rat}}(x)$ is the origin of $A(X)_{\mathrm{rat}}$. Then α and α_{rat} are unique up to isomorphisms of algebraic groups.

Next, observe that the Albanese morphism of a connected linear algebraic group G is constant: indeed, G is generated by rational curves, and any morphism from such a curve to an abelian variety is constant. For a connected algebraic group G (not necessarily linear), it follows that $\alpha = \alpha_{\mathrm{rat}}$ is the quotient homomorphism by the largest connected affine subgroup G_{aff}. This determines the Albanese rational map of an irreducible algebraic monoid M, which is just the Albanese morphism of its unit group. Some properties of the Albanese morphism of M are gathered in the following:

Proposition 16. *Let M be an irreducible algebraic monoid with unit group G. Then the map $\alpha : M \to A(M)$ is a homomorphism of algebraic monoids, and an affine morphism. Moreover, the map $\beta : A(M)_{\mathrm{rat}} = A(G) \to A(M)$ is an isogeny. If M is normal, then β is an isomorphism.*

Proof. The monoid law $\mu : M \times M \to M$, $(1_M, 1_M) \mapsto 1_M$ induces a morphism of varieties $A(\mu) : A(M \times M) \to A(M), 0 \mapsto 0$. Since $A(M \times M) = A(M) \times A(M)$, it follows that $A(\mu)$ is a homomorphism; hence so is α. In particular, α factors through the universal homomorphism $\varphi : M \to G/H$ of Proposition 11. Hence $A(M) = A(G/H) = G/G_{\mathrm{aff}}H$, where $G_{\mathrm{aff}}H$ is a normal subgroup scheme of G such that the quotient $G_{\mathrm{aff}}H/G_{\mathrm{aff}} \cong H/(H \cap G_{\mathrm{aff}})$ is finite. Write $M = G \times^H N$ as in Proposition 15; then

$$M \cong G \times^{G_{\mathrm{aff}}H} (G_{\mathrm{aff}}H \times^H N)$$

and this identifies α with the natural map to $G/G_{\mathrm{aff}}H$, with fiber $G_{\mathrm{aff}}H \times^H N$. But that fiber is affine, since so are N and $G_{\mathrm{aff}}H/H \cong G_{\mathrm{aff}}/(G_{\mathrm{aff}} \cap H)$. It follows that the morphism α is affine. Also, β is identified with the natural homomorphism $G/G_{\mathrm{aff}} \to G/G_{\mathrm{aff}}H$; hence the kernel of β is isomorphic to $G_{\mathrm{aff}}H/G_{\mathrm{aff}}$, a finite group scheme.

If M is normal, then $M \cong G \times^{G_{\mathrm{aff}}} M_{\mathrm{aff}}$ by Theorem 3. Thus, the natural map $M \to G/G_{\mathrm{aff}}$ is the Albanese morphism. □

Consider for instance the monoid M constructed in Example 5. Then $A(M) \cong A/\mu_n$ and $A(G) \cong A$; this identifies β to the quotient morphism $A \to A/\mu_n$.

Returning to our general setting, recall that every irreducible algebraic monoid may be viewed as an equivariant embedding of its unit group. For an arbitrary equivariant embedding X of a connected algebraic group G, we may again identify $A(X)_{\mathrm{rat}}$ with $A(G)$; when X is normal, we still have $A(X) = A(X)_{\mathrm{rat}}$ as a consequence of [4, Thm. 3]. But the morphism α is generally nonaffine, and the finiteness of β is an open question in this setting.

We now characterize algebraic monoids among equivariant embeddings:

Theorem 4. *Let X be an equivariant embedding of a connected algebraic group G. Then X has a structure of algebraic monoid with unit group G if and only if the Albanese morphism $\alpha : X \to A(X)$ is affine.*

Proof. In view of Proposition 16, it suffices to show that X is an algebraic monoid if α is affine. Note that α is $G \times G$-equivariant for the given action of $G \times G$ on X, and a transitive action on $A(X)$. It follows that $A(X) \cong (G \times G)/(K \times K)\mathrm{diag}(G)$ for a unique normal subgroup scheme K of G; then $A(X) \cong G/K$ equivariantly for the left (or right) action of G. Moreover, α is a fiber bundle of the form

$$G \times G \times^{(K\times K)\mathrm{diag}(G)} Y \longrightarrow (G \times G)/(K \times K)\mathrm{diag}(G),$$

where Y is a scheme equipped with an action of $(K \times K)\mathrm{diag}(G)$; for the left (or right) G-action, this yields the fiber bundle $G \times^K Y \to G/K$. Since α is affine, so is Y. Also, Y meets the open orbit $G \cong (G \times G)/\mathrm{diag}(G)$ along a dense open subscheme isomorphic to K, where $K \times K$ acts by left and right multiplication, and $\mathrm{diag}(G)$ by conjugation. Thus, the group scheme K is quasi-affine, and hence is affine.

We now show that the group law $\mu_K : K \times K \to K$ extends to a morphism $\mu_Y : Y \times Y \to Y$, by following the argument of [28, Prop. 1]. The left action $K \times Y \to Y$ and the right action $Y \times K \to Y$ restrict both to μ_K on $K \times K$, and hence yield a morphism $(K \times Y) \cup (Y \times K) \to Y$. Since Y is affine, it suffices to show the equality

$$\mathscr{O}((K \times Y) \cup (Y \times K)) = \mathscr{O}(Y \times Y).$$

But $\mathscr{O}(Y \times Y) = \mathscr{O}(Y) \otimes \mathscr{O}(Y) \subseteq \mathscr{O}(K) \otimes \mathscr{O}(K) = \mathscr{O}(K \times K)$, since K is dense in Y. Moreover,

$$\mathscr{O}((K \times Y) \cup (Y \times K)) = (\mathscr{O}(K) \otimes \mathscr{O}(Y)) \cap (\mathscr{O}(Y) \otimes \mathscr{O}(K)),$$

where the intersection is considered in $\mathscr{O}(K) \otimes \mathscr{O}(K)$. Now for any vector space V and subspace W, we easily obtain the equality $(W \otimes V) \cap (V \otimes W) = W \otimes W$ as subspaces of $V \otimes V$. When applied to $\mathscr{O}(Y) \subseteq \mathscr{O}(K)$, this yields the desired equality.

Since μ_Y is associative on the dense subscheme K, it is associative everywhere; likewise, μ_Y admits 1_K as a neutral element. Thus, μ_Y is an algebraic monoid law on Y. We may now form the induced monoid $G \times^K Y$ as in Sect. 3.2, to get the desired structure on X. □

3.5 Algebraic Semigroups and Monoids over Perfect Fields

In this subsection, we extend most of the above results to the setting of algebraic semigroups and monoids defined over a perfect field. We use the terminology and results of [33], especially Chapter 11 which discusses basic rationality results on varieties.

Let F be a subfield of the algebraically closed field k. We assume that F is *perfect*, i.e., every algebraic extension of F is separable; we denote by $\bar{F}$ the algebraic closure of F in k, and by Γ the Galois group of $\bar{F}$ over F.

We say that an algebraic semigroup (S, μ) (over k) is *defined over* F, or an *algebraic* F*-semigroup*, if S is an F-variety and the morphism μ is defined over F. Then the set of $\bar{F}$-points, $S(\bar{F})$, is a subsemigroup of S equipped with an action of Γ by semigroup automorphisms, and the fixed point subset $S(\bar{F})^{\Gamma}$ is the semigroup of F-points, $S(F)$.

Note that an algebraic F-semigroup may well have no F-point; for example, an F-variety without F-point equipped with the trivial semigroup law μ_l or μ_r. But this is the only obstruction to the existence of F-idempotents, as shown by the following:

Proposition 17. *Let* (S, μ) *be an algebraic* F*-semigroup.*

(i) $E(S)$ *and* $\ker(S)$ *(viewed as closed subsets of* S*) are defined over* F.
(ii) *If* S *is commutative, then its smallest idempotent is defined over* F.
(iii) *If* S *has an* F*-point, then it has an idempotent* F*-point.*

Proof. (i) Clearly, $E(S)$ and $\ker(S)$ are defined over $\bar{F}$ and their sets of $\bar{F}$-points are stable under the action of Γ on $S(\bar{F})$. Thus, $E(S)$ and $\ker(S)$ are defined over F by [33, Prop. 11.2.8(i)].

(ii) Is proved similarly.

(iii) Let $x \in S(F)$ and denote by $\langle x \rangle$ the closure in S of the set $\{x^n,\ n \geq 1\}$. Then $\langle x \rangle$ is a closed commutative subsemigroup of S, defined over F by [33, Lem. 11.2.4]. In view of (ii), $\langle x \rangle$ contains an idempotent defined over F.

□

We do not know if any algebraic F-semigroup S has a minimal idempotent defined over F. This holds if S is irreducible, as we will see in Proposition 19. First, we record two rationality results on algebraic monoids:

Proposition 18. *Let* $(M, \mu, 1_M)$ *be an algebraic monoid with unit group* G *and neutral component* M^o. *If* M *and* μ *are defined over* F, *then so are* 1_M, G *and* M^o. *Moreover, the inverse map* $\iota : G \to G$ *is defined over* F.

Proof. Observe that 1_M is the unique point $x \in M$ such that $xy = yx = y$ for all $y \in M(\bar{F})$ (since $M(\bar{F})$ is dense in M). It follows that $1_M \in M(\bar{F})$; also, 1_M is Γ-invariant by uniqueness. Thus, $1_M \in M(F)$.

The assertion on G follows from [33, Prop. 11.2.8(ii)]. It implies that G^o is defined over F by [loc. cit., Prop. 12.1.1]. Since M^o is the closure of G^o in M, it is also defined over F in view of [loc. cit., Prop. 11.2.8(i)].

It remains to show that ι is defined over F; equivalently, its graph is an F-subvariety of $G \times G$. But this graph equals

$$\{(x, y) \in G \times G \mid xy = 1_M\} = \mu_G^{-1}(1_M),$$

where $\mu_G : G \times G \to G$ denotes the restriction of μ, and $\mu_G^{-1}(1_M)$ stands for the set-theoretic fiber. Moreover, this fiber is defined over F in view of [33, Cor. 11.2.14]. □

Proposition 19. *Let* $(M, \mu, 1_M)$ *be an irreducible algebraic monoid with unit group* G. *If* (M, μ) *is defined over* F, *then the universal homomorphism to an algebraic group,* $\varphi : M \to \mathscr{G}(M)$, *is defined over* F *as well. Moreover,* G_{aff} *and* M_{aff} *are defined over* F.

Proof. The first assertion follows from the uniqueness of φ by a standard argument of Galois descent, see [31, Chap. V, §4]. The (well-known) assertion on G_{aff} is proved similarly; it implies the assertion on M_{aff} by [33, Prop. 11.2.8(i)]. □

Returning to algebraic semigroups, we obtain the promised:

Proposition 20. *Let* (S, μ) *be an irreducible algebraic* F*-semigroup. If* S *has an* F*-point, then some minimal idempotent of* S *is defined over* F.

Proof. By Proposition 17, we may choose $e \in E(S(F))$. Then eSe is a closed irreducible submonoid of S, and is defined over S in view of [33, Prop. 11.2.8(i)] again. Moreover, any minimal idempotent of eSe is a minimal idempotent of S. So we may assume that S is an irreducible monoid, M. In view of Theorem 3 and Proposition 19, we may further assume that M is affine. Then the unit group of M contains a maximal torus T defined over F, by Proposition 18 and [33, Thm. 13.3.6, Rem. 13.3.7]. The closure $\bar{T}$ of T in M is defined over F, and meets $\ker(M)$ in view of [23, Cor. 6.10]. So the (set-theoretic) intersection $N := \bar{T} \cap \ker(M)$ is a commutative algebraic semigroup, defined over F by [33, Thm. 11.2.13]. Now applying Proposition 17 to N yields the desired idempotent. □

Remark 8. The above observations leave open all the rationality questions for an algebraic semigroup S over a field F, not necessarily perfect. In fact, S has an idempotent F-point if it has an F-point, as follows from the main result of [5]. But some results do not extend to this setting: for example, the kernel of an algebraic F-monoid may not be defined over F, as shown by a variant of the standard example of a linear algebraic F-group whose unipotent radical is not defined over F (see [32, Exp. XVII, 6.4.a)] or [33, 12.1.6]; specifically, replace the multiplicative group $\mathbb{G}_m$ with the monoid $(\mathbb{A}^1, \times)$ in the construction of this example). Also, note that Chevalley's structure theorem fails over any imperfect field F (see [32, Exp. XVII, App. III, Cor.], and [35] for recent developments). Thus, G_{aff} may not be defined over F with the notation and assumptions of Proposition 19. Yet the Albanese morphism still exists for any F-variety equipped with an F-point (see [36, App. A]) and hence for any algebraic F-semigroup equipped with an F-idempotent.

4 Algebraic Semigroup Structures on Certain Varieties

4.1 Abelian Varieties

In this subsection, we begin by describing all the algebraic semigroup laws on an abelian variety. Then we apply the result to the study of the Albanese morphism of an irreducible algebraic semigroup.

Proposition 21. *Let A be an abelian variety with group law denoted additively, μ an algebraic semigroup law on A, and e an idempotent of (A, μ); choose e as the neutral element of $(A, +)$.*

(i) *There exists a unique decomposition of algebraic semigroups*

$$(A, \mu) = (A_0, \mu_0) \times (A_l, \mu_l) \times (A_r, \mu_r) \times (B, +)$$

where A_0, A_l, A_r and B are abelian varieties, and μ_0 (resp. μ_l, μ_r) the trivial semigroup law on A_0 (resp. A_l, A_r) defined in Example 1 (i).

(ii) *The corresponding projection $\varphi : A \to B$ is the universal homomorphism of (A, μ) to an algebraic group. Moreover, we have $E(S) = \{e\} \times A_l \times A_r \times \{e\}$ and $\ker(S) = \{e\} \times A_l \times A_r \times B$.*

Proof. (i) By [19, Chap. II, §4, Cor. 1], the morphism $\mu : A \times A \to A$ satisfies

$$\mu(x, y) = \varphi(x) + \psi(y) + x_0,$$

where $x_0 \in A$ and φ, ψ are endomorphisms of the algebraic group A. Since $\mu(e, e) = e$ and $\varphi(e) = \psi(e) = e$, we have $x_0 = e$, i.e., $\mu(x, y) = \varphi(x) + \psi(y)$. Now the associativity of μ is equivalent to the equality

$$\varphi \circ \varphi(x) + \varphi \circ \psi(y) + \psi(z) = \varphi(x) + \psi \circ \varphi(y) + \psi \circ \psi(z),$$

that is, to the equalities

$$\varphi \circ \varphi = \varphi, \quad \varphi \circ \psi = \psi \circ \varphi, \quad \psi \circ \psi = \psi.$$

This easily yields the desired decomposition, where $A_0 := \mathrm{Ker}(\varphi) \cap \mathrm{Ker}(\psi)$, $A_l := \mathrm{Im}(\varphi) \cap \mathrm{Ker}(\psi)$, $A_r := \mathrm{Ker}(\varphi) \cap \mathrm{Im}(\psi)$, and $B := \mathrm{Im}(\varphi) \cap \mathrm{Im}(\psi)$, so that φ (resp. ψ) is the projection of A to $A_l \times B$ (resp. $A_r \times B$). The uniqueness of this decomposition follows from that of φ and ψ.

(ii) Let $\gamma : (A, \mu) \to \mathcal{G}$ be a homomorphism to an algebraic group. Then the image of γ is a complete irreducible variety, and hence generates an abelian subvariety of $\mathcal{G}$. Thus, we may assume that $\mathcal{G}$ is an abelian variety, with group law also denoted additively. As above, we have $\gamma(x) = \pi(x) + x_0'$, where $\pi : A \to \mathcal{G}$ is a homomorphism of algebraic groups and $x_0' \in \mathcal{G}$. Since $\gamma(e)$ is idempotent, we obtain $x_0' = 0$, i.e., $\gamma : (A, +) \to \mathcal{G}$ is also a homomorphism. It follows readily that γ sends $A_0 \times A_l \times A_r \times \{e\}$ to 0. So γ factors as φ followed by a unique homomorphism $\gamma' : B \to \mathcal{G}$. This proves the assertion on φ; those on $E(S)$ and $\ker(S)$ are easily checked. □

Proposition 22. *Let (S, μ) be an irreducible algebraic semigroup, $e \in E(S)$, and $\alpha : S \to A(S)$ the Albanese morphism; assume that $\alpha(e) = 0$.*

(i) *There exists a unique algebraic semigroup law $A(\mu)$ on $A(S)$ such that α is a homomorphism.*

(ii) *Let $\varphi : (A(S), A(\mu)) \to B(S)$ be the universal homomorphism to an algebraic group. Then the map $eSe \to B(S)$, $x \mapsto \varphi(\alpha(x))$ is the Albanese morphism of eSe.*

Proof. (i) The assertion follows from the functorial properties of the Albanese morphism (see [30, Sec. 2]) by arguing as in the beginning of the proof of Proposition 16.

(ii) Consider the inclusions $eSe \subseteq eS \subseteq S$. Each of them admits a retraction, $x \mapsto xe$ (resp. $x \mapsto ex$). Thus, the corresponding morphisms of Albanese varieties, $A(eSe) \to A(eS) \to A(S)$, also admit retractions, and hence are closed immersions. So we may identify $A(eSe)$ with the subgroup of $A(S)$ generated by the differences $\alpha(exe) - \alpha(eye)$, where $x, y \in S$. But $\alpha(exe) = A(\mu)(\alpha(e), A(\mu)(\alpha(x), \alpha(e)))$ and $\alpha(e)$ is of course an idempotent of $(A(S), A(\mu))$. Hence $\alpha(e) = (e, a_l, a_r, e)$ in the decomposition of Proposition 21. Using that decomposition, we obtain $\alpha(exe) = (e, a_l, a_r, b(x))$, where $b(x)$ denotes the projection of $\alpha(x)$ to $B(S)$. As a consequence, $\alpha(exe) - \alpha(eye) = (e, e, e, b(x) - b(y))$; this yields the desired identification of $A(eSe)$ to $B(S)$. □

Combined with Proposition 16, the above result yields:

Corollary 5. *Let S be an irreducible algebraic semigroup.*

(i) *All the maximal submonoids of S have the same Albanese variety, and all the maximal subgroups have isogenous Albanese varieties.*

(ii) *The irreducible monoid eSe is affine for all $e \in E(S)$ if eSe is affine for some $e \in E(S)$.*

Remark 9. (i) With the notation and assumptions of Proposition 22, the morphism $\varphi \circ \alpha : S \to B(S)$ is the universal homomorphism to an abelian variety. Also, recall from Remark 5 that there exists a universal homomorphism to an algebraic group, $\psi : S \to \mathscr{G}(S)$, and that $\mathscr{G}(S)$ is connected. It follows that $B(S)$ is the Albanese variety of $\mathscr{G}(S)$.

(ii) Consider an irreducible algebraic semigroup (S, μ) and its rational Albanese map $\alpha_{\text{rat}} : S - \to A(S)_{\text{rat}}$. If the image of $\mu : S \times S \to S$ meets the domain of definition of α_{rat}, then there exists a unique algebraic semigroup structure $A(\mu)$ on $A(S)_{\text{rat}}$ such that α_{rat} is a 'rational homomorphism', i.e., $\alpha_{\text{rat}}(\mu(x, y)) = A(\mu)(\alpha_{\text{rat}}(x), \alpha_{\text{rat}}(y))$ whenever α_{rat} is defined at $x, y \in S$ and at $\mu(x, y)$ (as can be checked by the argument of Proposition 22). But this does not hold for an arbitrary (S, μ); for example, if $S \subseteq \mathbb{A}^3$ is the affine cone over an elliptic curve $E \subseteq \mathbb{P}^2$ and if $\mu = \mu_0$. Here 0, the origin of $\mathbb{A}^3$, is the unique singular point of S, and α_{rat} is the natural map $S \setminus \{0\} \to E$.

4.2 Irreducible Curves

In this subsection, we classify the irreducible algebraic semigroups of dimension 1; those having a nontrivial law (as defined in Example 1 (i)) turn out to be algebraic monoids.

Such semigroups include of course the connected algebraic groups of dimension 1, presented in Example 2 (iv). We now construct further examples: let $(a_1, \ldots, a_n)$ be a strictly increasing sequence of positive integers having no nontrivial common divisor, and consider the map

$$\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^n, \quad x \longmapsto (x^{a_1}, \ldots, x^{a_n}).$$

Then φ is a homomorphism of algebraic monoids, where $\mathbb{A}^1$ and $\mathbb{A}^n$ are equipped with pointwise multiplication. Also, one checks that the morphism φ is finite; hence its image is a closed submonoid of $\mathbb{A}^n$, containing the origin as its zero element. We denote this monoid by $M(a_1, \ldots, a_n)$, and call it an *affine monomial curve*; it only depends on the abstract submonoid of $(\mathbb{Z}, +)$ generated by $a_1, \ldots, a_n$. One may check that φ restricts to an isomorphism $\mathbb{A}^1 \setminus \{0\} \xrightarrow{\cong} M(a_1, \ldots, a_n) \setminus \{0\}$; also, $M(a_1, \ldots, a_n)$ is singular at the origin unless φ is an isomorphism, i.e., unless $a_1 = 1$.

Theorem 5. *Let S be an irreducible curve, and μ a nontrivial algebraic semigroup structure on S. Then (S, μ) is either an algebraic group or an affine monomial curve.*

Proof. As the arguments are somewhat long and indirect, we divide them into four steps.

Step 1: we show that every idempotent of S is either a neutral or a zero element. Let $e \in E(S)$. Since Se is a closed irreducible subvariety of S, it is either the whole S or a single point; in the latter case, $Se = \{e\}$. Thus, one of the following cases occurs:

(i) $Se = eS = S$. Then any $x \in S$ satisfies $xe = ex = x$, i.e., e is the neutral element.

(ii) $Se = \{e\}$ and $eS = S$. Then for any $x, y \in S$, we have $xe = e$ and $ey = y$. Thus, $xy = xey = ey = y$. So $\mu = \mu_r$ in the notation of Example 1 (i), a contradiction since μ is assumed to be nontrivial.

(iii) $eS = \{e\}$ and $Se = S$. This case is excluded similarly.

(iv) $Se = eS = \{e\}$. Then e is the zero element of S.

Step 2: we show that if S is complete, then it is an elliptic curve.
For this, we first reduce to the case where S has a zero element. Otherwise, S has a neutral element by Step 1. Hence S is a monoid with unit group G being $\mathbb{G}_a$, $\mathbb{G}_m$ or an elliptic curve, in view of the classification of connected algebraic groups of dimension 1. In the latter case, G is complete and hence $G = S$. On the other hand, if $G = \mathbb{G}_a$ or $\mathbb{G}_m$, then $S \setminus G$ is a nonempty closed subsemigroup of S in view of Proposition 3. Hence $S \setminus G$ contains an idempotent, which must be the zero element of S by Step 1. This yields the desired reduction.
The semigroup law $\mu : S \times S \to S$ sends $S \times \{0\}$ to the point 0. By the rigidity lemma (see e.g. [19, Chap. II, §4]), it follows that $\mu(x, y) = \varphi(y)$ for some morphism $\varphi : S \to S$. The associativity of μ yields

$$\varphi(z) = (xy)z = x(yz) = \varphi(yz) = \varphi(\varphi(z))$$

for all $x, y, z \in S$; hence φ is a retraction to its image. Since S is an irreducible curve, either $\varphi = \mathrm{id}$ or the image of φ consists of a single point x. In the former case, $\mu = \mu_r$, whereas $\mu = \mu_x$ in the latter case. Thus, the law μ is trivial, a contradiction.

Step 3: we show that if S is an affine monoid, then it is isomorphic to $\mathbb{G}_a$, $\mathbb{G}_m$ or an affine monomial curve.
We may view S as an equivariant embedding of its unit group G, and that group is either $\mathbb{G}_a$ or $\mathbb{G}_m$. Since $\mathbb{G}_a \cong \mathbb{A}^1$ as a variety, any affine equivariant embedding of $\mathbb{G}_a$ is $\mathbb{G}_a$ itself. So we may assume that $G = \mathbb{G}_m$. Then the coordinate ring $\mathcal{O}(S)$ is a subalgebra of $\mathcal{O}(\mathbb{G}_m) = k[x, x^{-1}]$, stable under the natural action of $\mathbb{G}_m$. It follows that $\mathcal{O}(S)$ has a basis consisting of Laurent monomials, and hence that

$$\mathcal{O}(S) = \bigoplus_{n \in \mathcal{M}} x^n,$$

where $\mathcal{M}$ is a submonoid of $(\mathbb{Z}, +)$. Moreover, since $\mathbb{G}_m$ is open in S, the fraction field of $\mathcal{O}(S)$ is the field of rational functions $k(x)$; it follows that $\mathcal{M}$ generates the group $\mathbb{Z}$. Thus, either $\mathcal{M} = \mathbb{Z}$ or $\mathcal{M}$ is generated by finitely many integers, all of the same sign and having no nontrivial common divisor. In the former case, $S = \mathbb{G}_m$; in the latter case, S is an affine monomial curve.

Step 4: in view of Step 2, we may assume that the irreducible curve S is noncomplete, and hence is affine. Then it suffices to show that S has a nonzero idempotent: then S is an algebraic monoid by Step 1, and we conclude by Step 3. We may further assume that S is nonsingular: indeed, by the nontriviality assumption, the semigroup law $\mu : S \times S \to S$ is dominant. Using Proposition 9, it follows that the normalization $\tilde{S}$ (an irreducible nonsingular curve) has a compatible algebraic semigroup structure; then the image of a nonzero idempotent of $\tilde{S}$ is a nonzero idempotent of S.

So we assume that S is an affine irreducible nonsingular semigroup of dimension 1, having a zero element 0, and show that S has a neutral element. We use the "right regular representation" of S, i.e., its action on the coordinate ring $\mathcal{O}(S)$ by right multiplication; specifically, an arbitrary point $x \in S$ acts on $\mathcal{O}(S)$ by sending a regular function f on S to the regular function $x \cdot f : y \mapsto f(yx)$. This yields a map

$$\varphi : S \longrightarrow \mathrm{End}(\mathcal{O}(S)), \quad x \longmapsto x \cdot f$$

which is readily seen to be a homomorphism of abstract semigroups. Moreover, the action of S on $\mathcal{O}(S)$ stabilizes the maximal ideal $\mathfrak{m}$ of 0, and all its powers $\mathfrak{m}^n$. This defines compatible homomorphisms of abstract semigroups

$$\varphi_n : S \longmapsto \mathrm{End}(\mathfrak{m}/\mathfrak{m}^n) \quad (n \geq 1).$$

Since S is a nonsingular curve, we have compatible isomorphisms of k-algebras

$$\mathfrak{m}/\mathfrak{m}^n \cong k[t]/t^n k[t],$$

where t denotes a generator of the maximal ideal $\mathfrak{m}\mathcal{O}_{S,0}$ of the local ring $\mathcal{O}_{S,0}$; the right-hand side is the algebra of truncated polynomials at the order n. Thus, an endomorphism γ of $\mathfrak{m}/\mathfrak{m}^n$ is uniquely determined by $\gamma(\bar{t})$, where $\bar{t}$ denotes the image of t mod $t^n k[t]$. Moreover, the assignment $\gamma \mapsto \gamma(\bar{t})$ yields compatible isomorphisms of abstract semigroups

$$\mathrm{End}(\mathfrak{m}/\mathfrak{m}^n) \xrightarrow{\cong} tk[t]/t^n k[t],$$

where the semigroup law on the right-hand side is the composition of truncated polynomials. Thus, we obtain compatible homomorphisms of abstract semigroups

$$\psi_n : S \longrightarrow tk[t]/t^n k[t].$$

Clearly, the right-hand side is an algebraic semigroup. Moreover, ψ_n is a morphism: indeed, for any $f \in \mathcal{O}(S)$, we have $f(yx) = \sum_{i \in I} f_i(x) g_i(y)$ for some finite collection of functions $f_i, g_i \in \mathscr{O}(S)$ (since the semigroup law is a morphism). In other words, $x \cdot f = \sum_{i \in I} f_i(x) g_i$. Thus, the matrix coefficients of the action of x in $\mathscr{O}(S)/\mathfrak{m}^n$, and hence in $\mathfrak{m}/\mathfrak{m}^n$, are regular functions of x.

We claim that there exists $n \geq 1$ such that $\psi_n \neq 0$. Otherwise, we have $\varphi_n(x) = 0$ for all $n \geq 1$ and all $x \in S$. Since $\bigcap_n \mathfrak{m}^n = \{0\}$, it follows that $\varphi(x)$ sends $\mathfrak{m}$ to 0. But $\mathscr{O}(S) = k \oplus \mathfrak{m}$, where the line k of constant functions is fixed pointwise by $\varphi(x)$. Hence $\varphi(x) = \varphi(0)$ for all x, i.e., $f(yx) = f(0)$ for all $f \in \mathscr{O}(S)$ and all $x, y \in S$. Thus, $yx = 0$, i.e., $\mu = \mu_0$; a contradiction.

Now let n be the smallest integer such that $\psi_n \neq 0$. Then ψ_n sends S to the quotient $t^{n-1}k[t]/t^n k[t]$, i.e., to the semigroup of endomorphisms of the algebra $k[t]/t^n k[t]$ given by $\bar{t} \mapsto c\bar{t}^{n-1}$, where $c \in k$. If $n \geq 3$, then the composition of any two such endomorphisms is 0, and hence $\psi_n(xy) = 0$ for all $x, y \in S$. Thus, xy belongs to the fiber of ψ_n at 0, a finite set containing 0. Since S is irreducible, it follows that $xy = 0$, i.e., $\mu = \mu_0$; a contradiction. Thus, we must have $n = 2$, and we obtain a nonconstant morphism $\psi = \psi_2 : S \to \mathbb{A}^1$, where the semigroup law on $\mathbb{A}^1$ is the multiplication. The image of ψ contains 0 and a nonempty open subset U of the unit group $\mathbb{G}_m$. Then $UU = \mathbb{G}_m$ and hence ψ is surjective. By Proposition 1, it follows that there exists an idempotent $e \in S$ such that $\psi(e) = 1$. Then e is the desired nonzero idempotent. □

Remark 10. One may also deduce the above theorem from the description of algebraic semigroup structures on abelian varieties (Proposition 21), when the irreducible curve S is assumed to be nonsingular and nonrational. Then the Albanese morphism of S is a locally closed embedding in its Jacobian variety A. It follows that A has no trivial summand A_0, A_l or A_r (otherwise, the projection to that summand is constant since μ is nontrivial; as the differences of points of S generate the group A, this yields a contradiction). In other words, the inclusion of S into A is a homomorphism for a suitable choice of the origin of A. This implies that $S = A$, and we conclude that S is an elliptic curve equipped with its group law.

4.3 Complete Irreducible Varieties

In this subsection, we obtain a description of all complete irreducible algebraic semigroups, analogous to that of the kernels of algebraic semigroups presented in Proposition 5:

Theorem 6. *There is a bijective correspondence between the following objects:*

- *The triples (S, μ, e), where S is a complete irreducible variety, μ an algebraic semigroup structure on S, and e an idempotent of (S, μ),*
- *The tuples $(X, Y, G, \iota, \rho, x_o, y_o)$, where X (resp. Y) is a complete irreducible variety equipped with a base point x_o (resp. y_o), G is an abelian variety, $\iota : X \times G \times Y \to S$ is a closed immersion, and $\rho : S \to X \times G \times Y$ a retraction of ι.*

This correspondence assigns to any such tuple, the algebraic semigroup structure ν *on* $X \times G \times Y$ *defined by*

$$\nu((x, g, y), (x', g', y')) := (x, gg', y')$$

and then the algebraic semigroup structure μ *on* S *defined by*

$$\mu(s, s') := \iota(\nu(\rho(s), \rho(s'))).$$

The idempotent is $e := \iota(x_o, 1_G, y_o)$. *Moreover,* ι *and* ρ *are homomorphisms of algebraic semigroups.*

The inverse correspondence will be constructed at the end of the proof. We begin that proof with three preliminary results.

Lemma 1. *Let* $\varphi : X \to Y$ *be a morphism of varieties, where* X *is complete and irreducible; assume that* φ *has a section (for example,* φ *is a retraction of* X *to a subvariety* Y*). Then* Y *is complete and irreducible as well. Moreover, the map* $\varphi^{\#} : \mathcal{O}_Y \to \varphi_*(\mathcal{O}_X)$ *is an isomorphism; in particular, the fibers of* φ *are connected.*

Proof. Note that φ is surjective, since it admits a section. This readily yields the first assertion.

Next, consider the Stein factorization of φ as the composition

$$X \xrightarrow{\varphi'} X' \xrightarrow{\psi} Y,$$

where φ' is the natural morphism to the Spec of the sheaf of $\mathcal{O}_Y$-algebras $\varphi_*(\mathcal{O}_X)$, and ψ is finite (see [13, Cor. III.11.5]). Then φ' is surjective, and hence X' is a complete irreducible variety. Also, given a section σ of φ, the map $\varphi' \circ \sigma$ is a section of ψ. In view of the irreducibility of X' and the finiteness of ψ, it follows that ψ is an isomorphism; this yields the second assertion. □

Lemma 2. *Let* S *be a complete irreducible algebraic semigroup, and* e *an idempotent of* S. *Then* $xy = xey$ *for all* $x, y \in S$.

Proof. Recall that the map $\varphi : S \to eS$, $x \mapsto ex$ is a retraction. Thus, its fibers are connected by Lemma 1. Let F be a (set-theoretic) fiber. Then the morphism $\mu : S \times S \to S$, $(x, y) \mapsto xy$ sends $\{e\} \times F$ to a point. By the rigidity lemma (see e.g. [19, Chap. II, §4]), $\mu(\{x\} \times F)$ consists of a single point for any $x \in S$. Thus, the map $y \mapsto xy$ is constant on the fibers of φ. Since $\varphi(y) = \varphi(ey)$ for all $y \in S$, this yields the statement. □

Lemma 3. *Keep the assumptions of the above lemma.*

(i) *The closed submonoid* eSe *of* S *is an abelian variety.*
(ii) *The map* $\varphi : S \to eSe$, $x \mapsto exe$ *is a retraction of algebraic semigroups.*
(iii) *The above map* φ *is the universal homomorphism to an algebraic group.*

Proof. (i) By Proposition 3 (iii), it suffices to show that e is the unique idempotent of eSe. But if $f \in E(eSe)$, then $xy = xfy$ for all $x, y \in S$, by Lemma 2. Taking $x = y = e$ yields $e = efe = f$.

(ii) By Lemma 2 again, we have $exye = exeye = (exe)(eye)$ for all $x, y \in S$.

(iii) Let $\mathscr{G}$ be an algebraic group and let $\psi : S \to \mathscr{G}$ be a homomorphism of algebraic semigroups. Then $\psi(e) = 1$ and hence $\psi(x) = \psi(exe)$ for all $x \in S$. Thus, ψ factors uniquely as the homomorphism φ followed by some homomorphism of algebraic groups $eSe \to \mathscr{G}$. □

Remark 11. By Lemma 3, every idempotent e of a complete irreducible algebraic semigroup (S, μ) is minimal. Moreover, by Lemma 2, the image of the morphism μ is exactly the kernel of S; this is a simple algebraic semigroup in view of Proposition 5. One may thus deduce part of Theorem 6 from the structure of simple algebraic semigroups presented in Remark 3 (i). Yet we will provide a direct, self-contained proof by adapting the arguments of Proposition 5.

Proof of Theorem 6. One readily checks that the map ν (resp. μ) as in the statement yields an algebraic semigroup structure on $X \times G \times Y$ (resp. on S); compare with Example 1 (ii).

Conversely, given (S, μ, e) as in the statement, consider

$$X := {}_eSe, \quad G := eSe, \quad Y := eS_e$$

with the notation of Remark 1 (ii). Then G is an abelian variety by Lemma 3. Let $\iota : X \times G \times Y \to S$ denote the multiplication map: $\iota(x, g, y) = xgy$. Finally, define a map $\rho : S \to S \times G \times S$ by

$$\rho(s) = (s(ese)^{-1}, ese, (ese)^{-1}s).$$

Then $s(ese)^{-1} \in X$, since $es(ese)^{-1} = ese(ese)^{-1} = e$ and $s(ese)^{-1}e = s(ese)^{-1}$. Likewise, $(ese)^{-1}s \in Y$. So the image of ρ is contained in $X \times G \times Y$.

We claim that $\rho \circ \iota$ is the identity of $X \times G \times Y$. Indeed, $(\rho \circ \iota)(x, g, y) = \rho(xgy)$. Moreover, $exgye = g$ so that

$$\rho(xgy) = (xgyg^{-1}, g, g^{-1}gy).$$

Now $xgyg^{-1} = xgyeg^{-1} = xgeg^{-1} = xe = x$ and likewise, $g^{-1}xgy = y$. This proves the claim.

By that claim, ι is a closed immersion, and ρ a retraction of ι. Also, we have for any $x, x' \in X$, $g, g' \in G$ and $y, y' \in Y$:

$$xgyx'g'y' = xgyex'g'y' = xgex'g'y' = xgeg'y'.$$

In other words, ι is a homomorphism of algebraic semigroups, where $X \times G \times Y$ is given the semigroup structure ν as in the statement.

We next claim that ρ is a homomorphism of algebraic semigroups as well. Indeed,

$$\rho(ss') = (ss'(ess'e)^{-1}, ess'e, (ess'e)^{-1}ss')$$

and hence, using Lemma 3,

$$\rho(ss') = (ss'(es'e)^{-1}(ese)^{-1}, eses'e, (es'e)^{-1}(ese)^{-1}ss').$$

Moreover,

$$ss'(es'e)^{-1}(ese)^{-1} = ses'e(es'e)^{-1}(ese)^{-1} = se(ese)^{-1} = s(ese)^{-1}$$

by Lemma 2, and likewise $(es'e)^{-1}(ese)^{-1}ss' = (es'e)^{-1}s'$. Thus,

$$\rho(ss') = (s(ese)^{-1}, eses'e, (es'e)^{-1}s') = \nu(\rho(s), \rho(s'))$$

as required.

Finally, we claim that $ss' = \iota(\nu(\rho(s), \rho(s')))$. Indeed, the right-hand side equals

$$s(ese)^{-1}eses'e(es'e)^{-1}s' = ses' = ss'$$

in view of Lemma 2 again.

Remark 12. (i) The description of algebraic semigroup laws on a given abelian variety A (Proposition 21) may of course be deduced from Theorem 6: with the notation of that theorem, the inclusion ι and retraction ρ yield a decomposition $A \cong A_0 \times A_l \times A_r \times B$, where $A_l := X$, $A_r := Y$, $B := G$ and A_0 denotes the fiber of ρ at 0. Yet the original proof of Proposition 21 is simpler and more direct.

(ii) As a direct consequence of Theorem 6, every algebraic semigroup law on a complete irreducible curve is either trivial or the group law of an elliptic curve. This yields an alternative proof of part of the classification of irreducible algebraic semigroups of dimension 1 (Theorem 5); but in fact, both arguments make a similar use of the rigidity lemma.

(iii) As another consequence of Theorem 6, for any complete irreducible algebraic semigroup (S, μ), the closed subset $E(S)$ of idempotents is an irreducible subsemigroup. Indeed, choosing $e \in E(S)$, we have with the notation of that theorem

$$E(S) = \iota(X \times \{1_G\} \times Y) \cong X \times Y.$$

Moreover, $\mu(\iota(x, 1_G, y), \iota(x', 1_G, y')) = \iota(x, 1_G, y')$ with an obvious notation.

(iv) In fact, some of the ingredients of Theorem 6 only depend of (S, μ), but not of the choice of $e \in E(S)$. Specifically, note first that the projections $\varphi : E(S) \to X$, $\psi : E(S) \to Y$ are independent of e. Indeed, let $f \in E(S)$ and write $f = \iota(x, 1_G, y)$. Then $fE(S) = \iota(\{(x, 1_G)\} \times Y)$ and hence $fE(S)$ is the fiber of φ at f; likewise, the fiber of ψ at f is $E(S)f$.

As seen in Remark 11, $\ker(S) = SeS$ is isomorphic to $X \times G \times Y$ via ι. The resulting projection $\gamma : \ker(S) \to G$ is the universal homomorphism to an algebraic group by Lemma 3, and hence is also independent of e; its fiber at 1_G is $E(S)$. In particular, the algebraic group $G = \mathcal{G}(S)$ is independent of e. Note however that $G = eSe$, viewed as a subgroup of S, does depend of the choice of the idempotent e. Indeed, $eSe = \iota(\{x_o\} \times G \times \{y_o\})$ with the notation of Theorem 6, while $fSf = \iota(\{x\} \times G \times \{y\})$ for f as above.

The map $\rho : S \to X \times G \times Y$ satisfies $\iota \circ \rho = \rho_e$, where $\rho_e : S \to \ker(S)$ denotes the retraction $s \mapsto s(ese)^{-1}s$ of Proposition 5. We check that ρ_e is independent of e (this also follows from Proposition 23 below). Let $f \in E(S)$, $s \in S$, and write $f = \iota(x, 1_G, y)$, $\rho(s) = (x_s, g_s, y_s)$. Then $fsf = \iota(x, g_s, y)$ and hence $(fsf)^{-1} = \iota(x, g_s^{-1}, y)$. Thus,

$$\rho_f(s) = s(fsf)^{-1}s = \iota(x_s, g_s, y_s) = \rho_e(s).$$

Finally, consider the action of the abelian variety G on $\ker(S) \cong X \times G \times Y$ via translation on the second factor:

$$g' \cdot \iota(x, g, y) := \iota(x, gg', y).$$

We check that this action lifts to an action of G on S such that $\rho : S \to \ker(S)$ is equivariant. For any $s, s' \in S$, define

$$s' \cdot s := s(ese)^{-1}s's.$$

Then we have

$$s' \cdot s = s(ese)^{-1}\, es'e\, es = s(ese)^{-1}\, ese\, es'e\, (ese)^{-1}s.$$

It follows that $s' \cdot s = es'e \cdot s = es'e \cdot \rho(s)$. Moreover, $s' \cdot S = \ker(S)$ and the endomorphism $s \mapsto s' \cdot s$ of $\ker(S)$ is just the translation by $es'e \in G$ on $G = eSe$; we have

$$s' \cdot s_1 s_2 = s_1 s' s_2$$

for all $s_1, s_2 \in S$. Also, one may check as above that $s' \cdot s$ is independent of the choice of e.

(v) Theorem 6 extends readily to those irreducible algebraic semigroups that are defined over a perfect subfield F of k, and that have an F-point; indeed, this implies the existence of an idempotent F-point by Proposition 17.

Likewise, the results of Sects. 4.1 and 4.2 extend readily to the setting of perfect fields. In view of Theorem 5, every nontrivial algebraic semigroup law μ on an irreducible curve S is commutative; by Proposition 17 again, it follows that S has an idempotent F-point whenever S and μ are defined over F.

4.4 Rigidity

In this subsection, we obtain two rigidity results (both possibly known, but for which we could not locate adequate references) and we apply them to the study of endomorphisms of complete varieties.

Our first result is a scheme-theoretic version of a classical rigidity lemma for irreducible varieties (see [8, Lem. 1.15]; further versions can be found in [20, Prop. 6.1]).

Lemma 4. *Let $f : X \to Y$ and $g : X \to Z$ be morphisms of schemes of finite type over k, satisfying the following assumptions:*

(i) *f is proper and the map $f^\# : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is an isomorphism.*
(ii) *There exists a k-rational point $y_o \in Y$ such that g maps the scheme-theoretic fiber $f^{-1}(y_o)$ to a single point.*
(iii) *f has a section, s.*
(iv) *X is irreducible.*

Then g factors through f; specifically, $g = h \circ f$, where $h := g \circ s$.

Proof. We first treat the case where y_o is the unique closed point of Y. We claim that X is the unique open neighborhood of $f^{-1}(y_o)$. Indeed, given such a neighborhood U with complement $F := X \setminus U$, the image $f(F)$ is closed, since f is proper. If F is nonempty, then $f(F)$ contains y_o, a contradiction.

Let $z_o \in Z$ be the point $g(f^{-1}(y_o))$, and choose an open affine neighborhood W of z_o in Z. Then $g^{-1}(W) = X$ by the claim together with (iii); thus, we may assume that $Z = W$ is affine. Then g is uniquely determined by the homomorphism of algebras $g^\# : \mathcal{O}(Z) \to \mathcal{O}(X)$. But the analogous map $f^\# : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an isomorphism in view of (iv). Thus, there exists a morphism $h' : Y \to Z$ such that $g = h' \circ f$. Then $h = g \circ s = h' \circ f \circ s = h'$; this completes the proof in that case (note that the assumptions (i) and (ii) suffice to conclude that g factors through f).

Next, we treat the general case. The scheme $Y' := \mathrm{Spec}(\mathcal{O}_{Y,y_o})$ has a unique closed point y'_o and comes with a flat morphism $\psi : Y' \to Y$, $y'_o \mapsto y_o$. Moreover, $X' := X \times_Y Y'$ is equipped with morphisms $f' : X' \to Y'$, $g' = g \circ p_1 : X' \to Z$ that satisfy (i) (since taking the direct image commutes with flat base extension, see [13, Prop. III.9.3]) and (ii). Also, note that f' has a section s' given by the morphism $(s \circ \psi) \times \mathrm{id} : Y' \to X \times Y'$. By the preceding step, we thus have $g' = h' \circ f'$, where $h' := g' \circ s'$. It follows that there exists an open neighborhood V of y_0 in Y such that $g = h \circ f$ over $f^{-1}(V)$.

We now consider the largest subscheme W of X over which $g = h \circ f$, i.e., W is the preimage of the diagonal in $Z \times Z$ under the morphism $g \times (h \circ f)$. Then W is closed in X and contains $f^{-1}(V)$. Since X is irreducible, it follows that $W = X$. □

Remark 13. The assertion of Lemma 4 still holds under the assumptions (ii), (iii) and the following (weaker but more technical) versions of (i), (iv):

(i)′ f is proper, and for any irreducible component Y' of Y, the scheme-theoretic preimage $X' := f^{-1}(Y')$ is an irreducible component of X. Moreover, the map $f'^{\#} : \mathcal{O}_{Y'} \to f'_*(\mathcal{O}_{X'})$ is an isomorphism, where $f' : X' \to Y'$ denotes the restriction of f.

(iv)′ X is connected.

Indeed, let Y_o be an irreducible component of Y containing y_o; then $X_o := f^{-1}(Y_o)$ is an irreducible component of X. Moreover, the restrictions $f_o : X_o \to Y_o$, $g_o : X_o \to Z$, and $s_o : Y_o \to X_o$ satisfy the assumptions of Lemma 4. By that lemma, it follows that $g_o = g_o \circ s_o \circ f_o$, i.e., $g = h \circ f$ on X_o. In particular, g maps the scheme-theoretic fiber of f at any point of Y_o to a single point.

Next, let Y_1 be an irreducible component of Y intersecting Y_o. Then again, $X_1 := f^{-1}(Y_1)$ is an irreducible component of X; moreover, the restrictions $f_1 : X_1 \to Y_1$, $g_1 : X_1 \to Z$ and $s_1 : Y_1 \to X_1$ satisfy the assumptions of the above lemma, for any point y_1 of $Y_o \cap Y_1$. Thus, $g = h \circ f$ on $X_o \cup X_1$. Iterating this argument completes the proof in view of the connectedness of X.

As a first application of the above lemma and remark, we present a rigidity result for retractions; further applications will be obtained in the next subsection.

Proposition 23. *Let X be a complete irreducible variety, and φ a retraction of X to a subvariety Y. Let T be a connected scheme of finite type over k, equipped with a k-rational point t_o, and let $\Phi : X \times T \to X$ be a morphism such that the morphism $\Phi_{t_o} : X \to X$, $x \mapsto \Phi(x, t_o)$ equals φ.*

(i) *There exists a unique morphism $\Psi : Y \times T \to X$ such that $\Phi(x,t) = \Psi(\varphi(x), t)$ on $X \times T$.*

(ii) *If Φ is a family of retractions to Y (i.e., $\Phi(y,t) = y$ on $Y \times T$), then Φ is constant (i.e., $\Phi(x,t) = \varphi(x)$ on $X \times T$).*

Proof. Consider the morphisms

$$f : X \times T \longrightarrow Y \times T, \quad (x,t) \mapsto (\varphi(x), t),$$

$$g : X \times T \longrightarrow X \times T, \quad (x,t) \mapsto (\Phi(x,t), t).$$

Then the assumption (i)′ of Remark 13 holds, since $\varphi_*(\mathcal{O}_X) = \mathcal{O}_Y$ in view of Lemma 1. Also, the assumption (ii) of Lemma 4 holds for any point (y, t_o), where $y \in Y$, and the assumption (iii) of that lemma holds with s being the inclusion of $Y \times T_o$ in $X \times T_o$. Finally, the assumption (iv)′ of Remark 13 is satisfied, since T is connected. By that remark, we thus have $g = g \circ s \circ f$ on $X \times T$. Hence there exists a unique morphism $\Psi : Y \times T \to X$ such that $\Phi(x,t) = \Psi(\varphi(x),t)$ on $X \times T$, namely, $\Psi(y,t) := \Phi(y,t)$. If Φ is a family of retractions, then we get that $\Phi(x,t) = \varphi(x)$ on $X \times T$. □

Remark 14. The preceding result has a nice interpretation when X is projective. Then there exists a quasiprojective k-scheme, $\mathrm{End}(X)$, which represents the endomorphism functor of X, i.e., for any Noetherian k-scheme T, the set of T-points $\mathrm{End}(X)(T)$ is naturally identified with the set of endomorphisms of $X \times T$ over T; equivalently,

$$\mathrm{End}(X)(T) = \mathrm{Hom}(X \times T, X).$$

Moreover, each connected component of $\mathrm{End}(X)$ is of finite type. These results hold, more generally, for the similarly defined functor $\mathrm{Hom}(X,Y)$ of morphisms from a projective scheme X to another projective scheme Y (see [12, p. 21]). The composition of morphisms yields a morphism of Hom functors, and hence of Hom schemes by Yoneda's lemma. In particular, $\mathrm{End}(X)$ is a monoid scheme; its idempotent k-points are exactly the retractions with source X.

Returning to the setting of an irreducible projective variety X together with a retraction $\varphi : X \to Y$, we may identify φ with the idempotent endomorphism e of X with image Y. Now Proposition 23 yields that the connected component of e in $\mathrm{End}(X)$ is isomorphic to the connected component of the inclusion $Y \to X$ in $\mathrm{Hom}(Y,X)$, by assigning to any $\phi \in \mathrm{Hom}(Y,X)(T) = \mathrm{Hom}(Y \times T, X)$, the composition $\psi \circ (\varphi \times \mathrm{id}) \in \mathrm{Hom}(X \times T, X)$. Moreover, this isomorphism identifies the connected component of e in

$$\mathrm{End}(E)_e := \{\Phi \in \mathrm{End}(X) \mid \Phi \circ e = e\}$$

to the (reduced) point e.

Next, we obtain our second rigidity result:

Lemma 5. *Let X be a complete variety, T a connected scheme of finite type over k, and*

$$\Phi : X \times T \longrightarrow X \times T, \quad (x,t) \longmapsto (\varphi(x,t),t)$$

an endomorphism of $X \times T$ over T. Assume that T has a point t_o such that $\Phi_{t_o} : X \to X$, $x \mapsto \varphi(x,t_o)$ is an automorphism. Then Φ is an automorphism.

Proof. Note that Φ is proper, as the composition of the closed immersion $X \times T \to X \times X \times T$, $(x,t) \mapsto (x, \varphi(x,t), t)$ and of the projection $X \times X \times T \to X \times T$, $(x,y,t) \mapsto (y,t)$.

We now show that the fibers of Φ are finite. Assuming the contrary, we may find a complete irreducible curve $C \subseteq X$ and a point $t_1 \in T$ such that $\varphi : X \times T \to X$ sends $C \times \{t_1\}$ to a point. By the rigidity lemma, it follows that the restriction of φ to $C \times T$ factors through the projection $C \times T \to T$. Taking $t = t_o$, we get a contradiction.

The morphism Φ is finite, since it is proper and its fibers are finite; it is also surjective, since for any $t \in T$, the map $\Phi_t : X \to X$, $x \mapsto \varphi(x,t)$ is a finite endomorphism of X and hence is surjective.

We now claim that Φ restricts to an automorphism of $X \times V$, for some open neighborhood V of t_o in T. This claim is proved in [17, Lem. I.1.10.1]; we recall the argument for completeness. Since Φ is proper, the sheaf $\Phi_*(\mathcal{O}_{X\times T})$ is coherent; it is also flat over T, since Φ lifts the identity of T. Moreover, the map $\Phi^\# : \mathcal{O}_{X\times T} \to \Phi_*(\mathcal{O}_{X\times T})$ induces an isomorphism $\Phi^\#_{t_o} : \mathcal{O}_X \to (\Phi_{t_o})_*(\mathcal{O}_X)$. In view of a version of Nakayama's lemma (see [17, Prop. I.7.4.1]), it follows that $\Phi^\#$ is an isomorphism over a neighborhood of t_o. This yields the claim.

By that claim, the points $t \in T$ such Φ_t is an isomorphism form an open subset of T. Since T is connected, it suffices to show that this subset is closed. For this, we may assume that T is an irreducible curve; replacing T with its normalization, we may further assume that T is nonsingular. By shrinking T, we may finally assume that it has a point s such that φ_t is an automorphism for all $t \in T \setminus \{s\}$; we have to show that φ_s is an automorphism as well.

If X is normal, then so is $X \times T$; moreover, the above endomorphism Φ is finite and birational, and hence an automorphism. Thus, every φ_t is an automorphism.

For an arbitrary X, consider the normalization $\eta : \tilde{X} \to X$. Then Φ lifts to an endomorphism $\tilde{\Phi} : \tilde{X} \times T \to \tilde{X} \times T$, which is an automorphism by the above step. In particular, φ_s lifts to an automorphism $\tilde{\varphi}_s$ of $\tilde{X}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\varphi}_s} & \tilde{X} \\ {\scriptstyle\eta}\downarrow & & {\scriptstyle\eta}\downarrow \\ X & \xrightarrow{\varphi_s} & X \end{array}$$

and hence a commutative diagram of morphisms of sheaves

$$\begin{array}{ccc} \mathcal{O}_X & \longrightarrow & (\varphi_s)_*(\mathcal{O}_X) \\ \downarrow & & \downarrow \\ \eta_*(\mathcal{O}_{\tilde{X}}) & \longrightarrow & \eta_*(\tilde{\varphi}_s)_*(\mathcal{O}_{\tilde{X}}). \end{array}$$

Moreover, the bottom horizontal arrow in the latter diagram is the identity (as $(\tilde{\varphi}_s)_*(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_{\tilde{X}}$), and the other maps are all injective. Thus, $\mathcal{O}_X \subseteq (\varphi_s)_*(\mathcal{O}_X) \subseteq \eta_*(\mathcal{O}_{\tilde{X}})$, and hence the iterates $(\varphi_s^n)_*(\mathcal{O}_X)$ form an increasing sequence of subsheaves of $\eta_*(\mathcal{O}_{\tilde{X}})$. As the latter sheaf is coherent, we get

$$(\varphi_s^n)_*(\mathcal{O}_X) = (\varphi_s^{n+1})_*(\mathcal{O}_X) \quad (n \gg 0).$$

Since φ_s is finite and surjective, it follows that $\mathcal{O}_X = (\varphi_s)_*(\mathcal{O}_X)$ and hence that φ_s is an isomorphism. $\square$

A noteworthy consequence of Lemma 5 is the following:

Corollary 6. (i) *Let M be a complete algebraic monoid. Then $G(M)$ is a union of connected components of M. In particular, if M is connected then it is an abelian variety.*

(ii) *Let S be a complete algebraic semigroup and let e, f be distinct idempotents such that $e \leq f$. Then e and f belong to distinct connected components of S. In particular, if S is connected then every idempotent is minimal.*

Proof. (i) Let T be a connected component of M containing a unit t_o. Applying Lemma 5 to the morphism $M \times T \to M \times T$, $(x,t) \mapsto (xt,t)$, we see that the map $x \mapsto xt$ is an isomorphism for any $t \in T$. Likewise, the map $x \mapsto tx$ is an isomorphism as well. Thus, t has a left and a right inverse in M, and hence is a unit. So T is contained in $G(M)$.

Alternatively, we may deduce the statement from Theorem 5: indeed, $G(M)$ contains no subgroup isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$, since the latter do not occur as unit groups of complete irreducible monoids. By Chevalley's structure theorem, it follows that the reduced neutral component of $G(M)$ is an abelian variety. Thus, $G(M)$ is complete, and hence closed in M. But $G(M)$ is open in M, hence the assertion follows.

(ii) Assume that e and f belong to the same connected component T of S. Then T is a closed subsemigroup, and hence we may assume that S is connected. Now fSf is a complete connected algebraic monoid, and hence an abelian variety. It follows that $e = f$, a contradiction. □

Remark 15. (i) Like for Proposition 23, the statement of Lemma 5 has a nice interpretation when X is projective. Then its functor of automorphisms is represented by an open subscheme $\mathrm{Aut}(X)$ of $\mathrm{End}(X)$ (see [12, p. 21]); in fact, $\mathrm{Aut}(X)$ is the unit group scheme of the monoid scheme $\mathrm{End}(X)$. Now Lemma 5 implies that $\mathrm{Aut}(X)$ is also closed in $\mathrm{End}(X)$. In other words, $\mathrm{Aut}(X)$ is a union of connected components of $\mathrm{End}(X)$.

For an arbitrary complete variety X, the automorphism functor defined as above is still represented by a group scheme $\mathrm{Aut}(X)$; moreover, each connected component of $\mathrm{Aut}(X)$ is of finite type (see [18, Thm. 3.7] for these results). We do not know if $\mathrm{End}(X)$ is representable in this generality; yet the above interpretation of Lemma 5 still makes sense in terms of functors.

(ii) Let X and T be complete varieties, where T is irreducible, and let $\mu : X \times T \to X$ be a morphism such that $\mu(x, t_o) = x$ for some $t_o \in T$ and all $x \in X$. Then by Lemma 5, the map $\mu_t : x \mapsto \mu(x,t)$ is an automorphism for any $t \in T$. This yields a morphism of schemes

$$\varphi : T \longrightarrow \mathrm{Aut}(X), \quad t \longmapsto \mu_t$$

such that $\varphi(t_o)$ is the identity. Hence φ sends T to the neutral component $\mathrm{Aut}^o(X)$. Consider the subgroup G of $\mathrm{Aut}^o(X)$ generated by the image of T;

then G is closed and connected by [9, Prop. II.5.4.6], and hence is an abelian variety. In loose words, the morphism μ arises from an action of an abelian variety on X.

(iii) Let X be a complete irreducible variety, and $\mu : X \times X \to X$ a morphism such that $\mu(x, x_o) = \mu(x_o, x) = x$ for some $x_o \in X$ and all $x \in X$. Then the above morphism $\varphi : X \to \mathrm{Aut}^o(X)$ satisfies $\varphi(x)(x_o) = x$, and hence is a closed immersion; we thus identify X to its image in $\mathrm{Aut}^o(X)$. As seen above, X generates an abelian subvariety G of $\mathrm{Aut}^o(X)$. The natural action of G on X is transitive, since the orbit Gx_o contains $Xx_o = X$. Thus, X itself is an abelian variety on which G acts by translations. Moreover, since $Gx_o = Xx_o$, we have $G = XG_{x_o}$, where G_{x_o} denotes the isotropy subgroup scheme of x_o. As G is commutative and acts faithfully and transitively on X, this isotropy subgroup scheme is trivial, i.e., $G = X$. In conclusion, X is an abelian variety with group law μ and neutral element x_o. This result is due to Mumford and Ramanujam, see [19, Chap. II, §4, Appendix].

4.5 Families of Semigroup Laws

Definition 10. Let S be a variety, and T a k-scheme. A *family of semigroup laws on S parameterized by T* is a morphism $\mu : S \times S \times T \to S$ such that the associativity condition

$$\mu(s, \mu(s', s'', t), t) = \mu(\mu(s, s', t), s'', t)$$

holds on $S \times S \times S \times T$.

Such a family yields a structure of semigroup scheme on $S \times T$ over T: to any scheme T' equipped with a morphism $\theta : T' \to T$, one associates the (abstract) semigroup consisting of all morphisms $\sigma : T' \to S$, equipped with the law μ_θ defined by

$$\mu_\theta(\sigma, \sigma') = \mu(\sigma, \sigma', \theta).$$

In particular, the choice of a k-rational point t_o of T yields an algebraic semigroup structure on S,

$$\mu_{t_o} : S \times S \longrightarrow S, \quad (s, s') \longmapsto \mu(s, s', t_o).$$

This sets up a bijective correspondence between families of semigroup laws on S parameterized by T, and structures of T-semigroup scheme on $S \times T$.

For example, every algebraic semigroup law $S \times S \to S$, $(s, s') \mapsto ss'$ defines a family of semigroup laws on $S \times S$ parameterized by S, via

$$\mu : S \times S \times S \longrightarrow S, \quad (s, s', t) \longmapsto sts'.$$

If S is irreducible and complete, and $e \in S$ is idempotent, then $\mu_e(s, s') = ss'$ in view of Lemma 2. More generally, for any $t \in S$, we have $\mu_t(s, s') = t \cdot ss'$, with the notation of Remark 12 (iv). In other words, the family μ arises from the action of the abelian variety $G = eSe$ on S, defined in that remark.

We now generalize this construction to obtain all families of semigroup structures on a complete irreducible variety, under a mild assumption on the parameter scheme.

Theorem 7. *Let S be a complete irreducible variety, T a connected scheme of finite type over k, and $\mu : S \times S \times T \to S$ a family of semigroup laws. Choose a k-point $t_o \in T$ and denote by $\ker(S)$ the kernel of (S, μ_{t_o}), by $\rho : S \to \ker(S)$ the associated retraction, and by G the associated abelian variety; recall that G acts on $\ker(S)$ by translations.*

Then there exist unique morphisms $\varphi : \ker(S) \times T \to S$ and $\gamma : T \to G$ such that

$$\mu(s, s', t) = \varphi(\mu_{t_o}(s, s'), t)$$

on $S \times S' \times T$, and that the composition $\rho \circ \varphi : \ker(S) \times T \to \ker(S)$ is the translation $(s, t) \mapsto \gamma(t) \cdot s$.

Conversely, given $\varphi : \ker(S) \times T \to S$ such that there exists $\gamma : T \to G$ satisfying the preceding condition, the assignment $(s, s', t) \mapsto \varphi(\mu_{t_o}(s, s'), t)$ yields an algebraic semigroup law over T. Moreover, $\varphi(s, t_o) = s$ on $\ker(S)$, and $\gamma(t_o) = 1_G$.

Proof. Denote for simplicity $\mu_{t_o}(s, s')$ by ss'. We begin by showing that there exists a unique morphism $\varphi : \ker(S) \times T \to S$ such that $\mu(s, s', t) = \varphi(ss', t)$. For this, we apply Lemma 4 and the subsequent Remark 13 to the morphisms

$$\mu_{t_o} \times \mathrm{id} : S \times S \times T \to \ker(S) \times T, \quad \mu \times \mathrm{id} : S \times S \times T \to S \times T.$$

To check the corresponding assumptions, note first that μ_{t_o} has a section

$$\sigma : \ker(S) \longrightarrow S \times S, \quad s \longmapsto (s, s(ese)^{-2}s),$$

where e denotes a fixed idempotent of (S, μ_{t_o}). (Indeed, let $\iota : X \times G \times Y \to S$ be the associated closed immersion with image $\ker(S)$. Then

$$\sigma(\iota(x, g, y)) = (\iota(x, g, y), \iota(x, 1_G, y))$$

as an easy consequence of Theorem 6. Thus, $\mu_{t_o}(\sigma(\iota(x, g, y))) = \iota(x, g, y)$.) By Lemma 1, it follows that the map $\mu_{t_o}^\# : \mathscr{O}_{\ker(S)} \to (\mu_{t_o})_*(\mathscr{O}_{S \times S})$ is an isomorphism. Thus, μ_{t_o} satisfies the assumption (i)' of Remark 13; hence so does $\mu_{t_o} \times \mathrm{id}$. Also, the assumption (ii) of Lemma 4 holds for any point (s, t_o) with $s \in \ker(S)$, and the assumption (iii) of that lemma holds as well, since $\sigma \times \mathrm{id}$ is a section of $\mu_{t_o} \times \mathrm{id}$. Finally, $S \times S \times T$ is connected, i.e., the assumption (iv)' of Remark 13 is satisfied. Hence that remark yields the desired morphism φ.

In particular, $ss' = \mu(s, s', t_o) = \varphi(ss', t_o)$ for all $s, s' \in S$. Since the image of μ_{t_o} equals $\ker(S)$, it follows that $\varphi(s, t_o) = s$ for all $s \in \ker(S)$.

Next, consider the morphism

$$\Psi := (\rho \circ \varphi) \times \mathrm{id} : \ker(S, \mu_{t_o}) \times T \longrightarrow \ker(S, \mu_{t_o}) \times T.$$

Then Ψ_{t_o} is the identity by the preceding step; thus, Ψ is an automorphism in view of Lemma 5. In other words, Ψ arises from a morphism

$$\pi : T \longrightarrow \mathrm{Aut}(\ker(S)), \quad t_o \longmapsto \mathrm{id}.$$

Since T is connected, the image of π is contained in $\mathrm{Aut}^o(\ker(S))$. We identify $\ker(S)$ with $X \times G \times Y$ via ι. Then the natural map

$$\mathrm{Aut}^o(X) \times \mathrm{Aut}^o(G) \times \mathrm{Aut}^o(Y) \longrightarrow \mathrm{Aut}^o(\ker(S))$$

is an isomorphism by [7, Cor. 4.2.7]. Moreover, $\mathrm{Aut}^o(G) \cong G$ via the action of G on itself by translations, see e.g. [loc. cit.,Prop. 4.3.2]. Thus, we have

$$\Psi(x, g, y, t) = (\alpha(x, t), g + \gamma(t), \beta(y, t), t)$$

for uniquely determined morphisms $\alpha : X \times T \to X$, $\beta : Y \times T \to Y$ and $\gamma : T \to G$ such that $\alpha \times \mathrm{id}$ is an automorphism of $X \times T$ over T, and likewise for $\beta \times \mathrm{id}$.

We now use the assumption that μ is associative. This is equivalent to the condition that

$$\varphi(s\varphi(s's'', t), t) = \varphi(\varphi(ss', t)s'', t)$$

on $S \times S \times S \times T$. Let $\psi := \rho \circ \varphi$, then

$$\psi(s\psi(s's'', t), t) = \psi(\psi(ss', t)s'', t)$$

on $\ker(S) \times \ker(S) \times \ker(S) \times T$, since $ss' = \rho(s)\rho(s') = \rho(ss')$ on $S \times S$. In view of the equalities $\psi(x, g, y) = (\alpha(x, t), g + \gamma(t), \beta(y, t))$ and $(x, g, y)(x', g', y') = (x, gg', y')$, the above associativity condition for ψ yields that $\alpha(x, t) = \alpha(\alpha(x, t), t)$ on $X \times T$, and $\beta(y, t) = \beta(\beta(y, t), t)$ on $Y \times T$. As $\alpha \times \mathrm{id}$ and $\beta \times \mathrm{id}$ are automorphisms, it follows that $\alpha(x, t) = x$ and $\beta(y, t) = y$. Thus,

$$\psi(x, g, y, t) = (x, g + \gamma(t), y),$$

that is, $\rho \circ \varphi$ is the translation by γ.

For the converse, let φ, γ be as in the statement. Then the morphism

$$\mu : S \times S \times T \longrightarrow S \times T, \quad (s, s', t) \longmapsto \varphi(ss', t)$$

satisfies the associativity condition, since

$$\varphi(s\varphi(s's'',t),t) = \varphi(\rho(s)\rho(\varphi(s's'',t)),t) = \varphi(\gamma(t)\cdot\rho(s)\rho(s')\rho(s''),t)$$

and the right-hand side is clearly associative. Moreover, as already checked, $\varphi(s,t_o) = s$ on $\ker(S)$; it follows that

$$\gamma(t_o)\cdot s = (\rho\circ\varphi)(s,t_o) = \rho(s) = s$$

for all $s \in \ker(S)$. Thus, $\gamma(t_o) = 1_G$. □

Remark 16. (i) With the notation and assumptions of Theorem 7, one can easily obtain further results on the semigroup scheme structure of $S \times T$ over T that corresponds to μ, along the lines of Theorem 6 and of Remark 12. For example, one may check that the idempotent sections of the projection $S \times T \to T$ are exactly the morphisms

$$T \longrightarrow S, \quad t \longmapsto \varphi(\gamma(t)^{-2}\cdot\varepsilon(t)),$$

where $\varepsilon : T \to E(S,\mu_{t_o})$ is a morphism. In particular, any such semigroup scheme has an idempotent section.

(ii) Consider the functor of composition laws on a variety S, i.e., the contravariant functor from schemes to sets given by $T \mapsto \mathrm{Hom}(S \times S \times T, S)$; then the families of algebraic semigroup laws yield a closed subfunctor (defined by the associativity condition). When S is projective, the former functor is represented by a quasiprojective k-scheme,

$$\mathrm{CL}(S) := \mathrm{Hom}(S \times S, S);$$

moreover, each connected component of $\mathrm{CL}(S)$ is of finite type over k (as mentioned in Remark 14). Thus, the latter subfunctor is represented by a closed subscheme,

$$\mathrm{SL}(S) \subseteq \mathrm{CL}(S).$$

In particular, $\mathrm{SL}(S)$ is quasi-projective, and its connected components are of finite type.

By Theorem 7, the connected component of μ_{t_o} in $\mathrm{SL}(S)$ is identified with the closed subscheme of $\mathrm{Hom}(\ker(S),S) \times G$ consisting of those pairs (φ,γ) such that $\rho\circ\varphi$ is the translation by γ. Via the assignment $(\varphi,\gamma) \mapsto (\gamma^{-1}\cdot\varphi,\gamma)$ (where $\gamma^{-1}\cdot\varphi$ is defined as in Remark 12 (iv)), the above component of $\mathrm{SL}(S)$ is identified with the closed subscheme of $\mathrm{Hom}(\ker(S),S) \times G$ consisting of those pairs (σ,γ) such that $\rho\circ\sigma = \mathrm{id}$, that is, σ is a section of ρ. This identifies the universal semigroup law on the above component, with the morphism

$$(s,s') \mapsto \gamma\cdot\sigma(\mu_{t_o}(s,s')).$$

Note that the scheme of sections of ρ is isomorphic to an open subscheme of the Hilbert scheme, $\mathrm{Hilb}(S)$, by assigning to every section its image (see [12, p. 21]). This open subscheme is generally nonreduced, as shown by a classical example where S is a ruled surface over an elliptic curve C. Specifically, S is obtained as the projective completion of a nontrivial principal $\mathbb{G}_a$-bundle over C, and $\rho : S \to C$ is the ruling; then the section at infinity of ρ yields a fat point of $\mathrm{Hilb}(S)$, as follows from obstruction theory (see e.g. [17, Sec. I.2]). As a consequence, the scheme $\mathrm{SL}(S)$ is generally nonreduced as well.

(iii) The families of semigroup laws on further classes of varieties are worth investigating. Following the approach of deformation theory, one may consider those families of semigroup laws μ on a prescribed variety S that are parameterized by the spectrum of a local artinian k-algebra R with residue field k, and that have a prescribed law μ_{t_o} at the closed point. Then the first-order deformations (i.e., those parameterized by $\mathrm{Spec}(k[t]/(t^2))$) form a k-vector space which may well be infinite-dimensional; this happens when S is the affine line, and μ_{t_o} the multiplication.

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Algebraic Semigroups Are Strongly π-Regular

Michel Brion and Lex E. Renner

Abstract Let S be an algebraic semigroup (not necessarily linear) defined over a field F. We show that there exists a positive integer n such that x^n belongs to a subgroup of $S(F)$ for any $x \in S(F)$. In particular, the semigroup $S(F)$ is strongly π-regular.

Keywords Algebraic semigroup • Strong π-regularity

Subject Classifications: 20M14, 20M32, 20M99

1 Introduction

A fundamental result of Putcha (see [2, Thm. 3.18]) states that any *linear* algebraic semigroup S over an algebraically closed field k is strongly π-regular. The proof follows from the corresponding result for $M_n(k)$ (essentially the Fitting decomposition), combined with the fact that S is isomorphic to a closed subsemigroup of $M_n(k)$, for some $n > 0$. At the other extreme it is easy to see that any *complete* algebraic semigroup is strongly π-regular. It is therefore natural to ask whether *any* algebraic semigroup S is strongly π-regular. The purpose of this note is to provide an affirmative answer to this question, over an arbitrary field F; then the set $S(F)$

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of points of S over F is an abstract semigroup (we shall freely use the terminology and results of [3, Chap. 11] for algebraic varieties defined over a field).

2 The Main Results

Theorem 1. *Let S be an algebraic semigroup defined over a subfield F of k. Then $S(F)$ is strongly π-regular, that is for any $x \in S(F)$, there exists a positive integer n and an idempotent $e \in S(F)$ such that x^n belongs to the unit group of $eS(F)e$.*

Proof. We may replace S with any closed subsemigroup defined over F and containing some power of x. Denote by $\langle x \rangle$ the smallest closed subsemigroup of S containing x, that is, the closure of the subset $\{x^m, m > 0\}$; then $\langle x \rangle$ is defined over F by [3, Lem. 11.2.4]. The subsemigroups $\langle x^n \rangle, n > 0$, form a family of closed subsets of S, and satisfy $\langle x^{mn} \rangle \subseteq \langle x^m \rangle \cap \langle x^n \rangle$. Thus, there exists a smallest such semigroup, say $\langle x^{n_0} \rangle$. Replacing x with x^{n_0}, we may assume that $S = \langle x \rangle = \langle x^n \rangle$ for all $n > 0$.

Lemma 1. *With the above notation and assumptions, xS is dense in S. Moreover, S is irreducible.*

Proof. Since $S = \langle x^2 \rangle$, the subset $\{x^n, n \geq 2\}$ is dense in S. Hence xS is dense in S by an easy observation (Lemma 3) that we will use repeatedly.

Let $S_1, \ldots, S_r$ be the irreducible components of S. Then each xS_i is contained in some component S_j. Since xS is dense in S, we see that xS_i is dense in S_j. In particular, j is unique and the map $\sigma : i \mapsto j$ is a permutation. By induction, $x^n S_i$ is dense in $S_{\sigma^n(i)}$ for all n and i; thus $x^n S_i$ is dense in S_i for some n and all i. Choose i such that $x^n \in S_i$. Then it follows that $x^{mn} \in S_i$ for all m. Thus, $\langle x^n \rangle \subseteq S_i$, and $S = S_i$ is irreducible. □

Lemma 2. *Let S be an algebraic semigroup and let $x \in S$. Assume that $S = \langle x \rangle$ (in particular, S is commutative), xS is dense in S, and S is irreducible. Then S is a monoid and x is invertible.*

Proof. For $y \in S$, consider the decreasing sequence

$$\cdots \subseteq \overline{y^{n+1}S} \subseteq \overline{y^n S} \subseteq \cdots \subseteq \overline{yS} \subseteq S$$

of closed, irreducible ideals of S. We claim that

$$\overline{y^d S} = \overline{y^{d+1} S} = \cdots ,$$

where $d := \dim(S) + 1$. Indeed, there exists $n \leq d$ such that $\overline{y^{n+1}S} = \overline{y^n S}$, that is, $y^{n+1}S$ is dense in $\overline{y^n S}$. Multiplying by y^{m-n} and using Lemma 3, it follows that $y^{m+1}S$ is dense in $\overline{y^m S}$ for all $m \geq n$ and hence for $m \geq d$. This proves the claim.

We may thus set

$$I_y := \overline{y^d S} = \overline{y^{d+1} S} = \cdots$$

Then we have for all $y, z \in S$,

$$\overline{y^d I_z} = I_{yz} \subseteq I_z,$$

since $y^d(z^d S) = (yz)^d S \subseteq z^d S$. Also, note that $I_x = S$, and $I_e = eS$ for any idempotent e of S. By [1, Sec. 2.3], S has a smallest idempotent e_S, and $e_S S$ is the smallest ideal of S. In particular, $e_S S \subseteq I_y$ for all y. Define

$$\mathcal{I} = \{I \subseteq S \mid I = I_y \text{ for some } y \in S\}.$$

This is a set of closed, irreducible ideals, partially ordered by inclusion, with smallest element $e_S S$ and largest element S. If $S = e_S S$, then S is a group and we are done. Otherwise, we may choose $I \in \mathcal{I}$ which covers $e_S S$ (since $\mathcal{I} \setminus \{e_S S\}$ has minimal elements under inclusion). Consider

$$T = \{y \in S \mid yI \text{ is dense in } I\}.$$

If $y, z \in T$ then $\overline{yzI} = \overline{y\overline{zI}} = I$ and hence T is a subsemigroup of S. Also, note that $T \cap e_S S = \emptyset$, since $e_S zI \subseteq e_S S$ is not dense in I for any $z \in S$. Furthermore $x \in T$. (Indeed, xS is dense in S and hence $xy^d S$ is dense in $\overline{y^d S}$ for all $y \in S$. Thus, $x\ \overline{y^d S}$ is dense in $\overline{y^d S}$; in particular, xI is dense in I).

We now claim that

$$T = \{y \in S \mid y^d I \not\subseteq e_S S\}.$$

Indeed, if $y \in T$ then $y^d I$ is dense in I and hence not contained in $e_S S$. Conversely, assume that $y^d I \not\subseteq e_S S$ and let $z \in S$ such that $I = I_z$. Since $\overline{y^d I} = \overline{y^d I_z} = I_{yz} \in \mathcal{I}$ and $\overline{y^d I} \subseteq I$, it follows that $\overline{y^d I} = I$ as I covers $e_S S$.

By that claim, we have

$$S \setminus T = \{y \in S \mid y^d I \subseteq e_S S\} = \{y \in S \mid e_S y^d z = y^d z \text{ for all } z \in I\}.$$

Hence $S \setminus T$ is closed in S. Thus, T is an open subsemigroup of S; in particular, T is irreducible. Moreover, since $x \in T$ and xS is dense in S, it follows that xT is dense in T; also note that $\{x^n, n > 0\}$ is dense in T.

Let $e_T \in T$ be the minimal idempotent, then $e_T \notin e_S S$ and hence the closed ideal $e_T S$ contains strictly $e_S S$. Since both are irreducible, we have $\dim(e_T T) = \dim(e_T S) > \dim(e_S S)$. Now the proof is completed by induction on $\kappa(S) := \dim(S) - \dim(e_S S)$. Indeed, if $\kappa(S) = 0$, then $S = e_S S$ is a group. In the general case, we have $\kappa(T) < \kappa(S)$. By the induction assumption, T is a monoid and x is invertible in T. As T is dense in S, the neutral element of T is also neutral for S, and hence x is invertible in S. □

By Lemmas 1 and 2, there exists n such that $\langle x^n \rangle$ is a monoid defined over F, and x^n is invertible in that monoid. To complete the proof of Theorem 1, it suffices to show that the neutral element e of $\langle x^n \rangle$ is defined over F. For this, consider the morphism

$$\phi : S \times S \longrightarrow S, \quad (y, z) \longmapsto x^n yz.$$

Then ϕ is the composition of the multiplication

$$\mu : S \times S \longrightarrow S, \quad (y, z) \longmapsto yz$$

and of the left multiplication by x^n; the latter is an automorphism of S, defined over F. So ϕ is defined over F as well, and the fiber $Z := \phi^{-1}(x^n)$ is isomorphic to $\mu^{-1}(e)$, hence to the unit group of S. In particular, Z is smooth. Moreover, Z contains (e, e), and the tangent map

$$d\phi_{(e,e)} : T_{(e,e)}(S \times S) \longrightarrow T_{x^n} S$$

is surjective, since

$$d\mu_{(e,e)} : T_{(e,e)}(S \times S) = T_e S \times T_e S \longrightarrow T_e S$$

is just the addition. So Z is defined over F by [3, Cor. 11.2.14]. But Z is sent to the point e by μ. Since that morphism is defined over F, so is e. □

Lemma 3. *Let X be a topological space, and $f : X \to X$ a continuous map. If $Y \subseteq X$ is a dense subset then $f(Y) \subseteq \overline{f(X)}$ is a dense subset.*

Proof. Let $U \subseteq \overline{f(X)}$ be a nonempty open subset. Then $f^{-1}(U) \subseteq X$ is open, and nonempty since $f(X)$ is dense in $\overline{f(X)}$. Hence $Y \cap f^{-1}(U) \neq \emptyset$. If $y \in Y \cap f^{-1}(U)$ then $f(y) \in f(Y) \cap U$. Hence $f(Y) \cap U \neq \emptyset$. □

Remark 1. Given $x \in S$, there exists a *unique* idempotent $e = e(x) \in S$ such that x^n belongs to the unit group of eSe for some $n > 0$. Indeed, we then have $x^n S x^n \subseteq eSe$; moreover, since there exists $y \in eSe$ such that $x^n y = yx^n = e$, we also have $eSe = x^n y S y x^n e \subseteq x^n S x^n$. Thus, $x^n S x^n = eSe$. It follows that $x^{mn} S x^{mn}$ is a monoid with neutral element e for any $m > 0$, which yields the desired uniqueness.

In particular, if $x \in S(F)$ then the above idempotent $e(x)$ is an F-point of the closed subsemigroup $\langle x \rangle$. We now give some details on the structure of the latter semigroup. For x, e, n as above, we have $x^n = ex^n = (ex)^n$, and $y(ex)^n = e$ for some $y \in H_e$ (the unit group of $e\langle x \rangle$). But then $ex \in H_e$ since $(y(ex)^{n-1})(ex) = e$. Thus, $ex^m = (ex)^m \in H_e$ for all $m > 0$. But if $m \geq n$ then $x^m = ex^m$. Thus, if $x \notin H_e$ then there exists an unique $r > 0$ such that $x^r \notin H_e$ and $x^m \in H_e$ for any $m > r$. In particular, $x^r \in e\langle x \rangle$ for all $m \geq r$. Thus *we can write*

$$\langle x \rangle = e\langle x \rangle \sqcup \{x, x^2, \ldots, x^s\}$$

for some $s < r$. Notice also that *these* x^i*'s, with* $i \le s$*, are all distinct* (if $x^i = x^j$ with $1 \le i < j \le s$, then $x^{i+s+1-j} = x^{s+1} \in e\langle x\rangle$, a contradiction). Moreover, a similar decomposition holds for the semigroup of F-rational points.

The set $\{ex^m, m > 0\}$ is dense in $e\langle x\rangle$ by Lemma 3. But $ex^m = (ex)^m$, and $ex \in H_e$. So $e\langle x\rangle$ *is a unit-dense algebraic monoid*. Furthermore, if $\langle x^{m_0}\rangle$ is the smallest subsemigroup of $\langle x\rangle$ of the form $\langle x^m\rangle$, for some $m > 0$, then $\langle x^{m_0}\rangle$ *is the neutral component of* $e\langle x\rangle$ (the unique irreducible component containing e). Indeed, $\langle x^{m_0}\rangle$ is irreducible by Lemma 1, and $y^{m_0} \in \langle x^{m_0}\rangle$ for any $y \in \langle x\rangle$ in view of Lemma 3. Thus, the unit group of $\langle x^{m_0}\rangle$ has finite index in the unit group of $\langle x\rangle$, and hence in that of $e\langle x\rangle$.

Finally, we show that Theorem 1 is self-improving by obtaining the following stronger statement:

Corollary 1. *Let* S *be an algebraic semigroup. Then there exists* $n > 0$ *(depending only on* S*) such that* $x^n \in H_{e(x)}$ *for all* $x \in S$*, where* $e : x \mapsto e(x)$ *denotes the above map. Moreover, there exists a decomposition of* S *into finitely many disjoint locally closed subsets* U_j *such that the restriction of e to each* U_j *is a morphism.*

Proof. We first show that for any irreducible subvariety X of S, there exists a dense open subset U of X and a positive integer $n = n(U)$ such that $x^n \in H_{e(x)}$ for all $x \in U$, and $e|_U$ is a morphism. We will consider the semigroup $S(k(X))$ of points of S over the function field $k(X)$, and view any such point as a rational map from X to S; the semigroup law on $S(k(X))$ is then given by pointwise multiplication of rational maps. In particular, the inclusion of X in S yields a point $\xi \in S(k(X))$ (the image of the generic point of X). By Theorem 1, there exist a positive integer n and points $e, y \in S(k(X))$ such that $e^2 = e$, $\xi^n e = e\xi^n = \xi^n$, $ye = ey = y$ and $\xi^n y = y\xi^n = e$. Let U be an open subset of X on which both rational maps e, y are defined. Then the above relations are equalities of morphisms $U \to S$, where ξ is the inclusion. This yields the desired statements.

Next, start with an irreducible component X_0 of S and let U_0 be an open subset of X_0 such that $e|_{U_0}$ is a morphism. Now let X_1 be an irreducible component of $X_0 \setminus U_0$ and iterate this construction. This yields disjoint locally closed subsets $U_0, U_1, \ldots, U_j, \ldots$ such that $e|_{U_j}$ is a morphism for all j, and $X \setminus (U_0 \cup \cdots \cup U_j)$ is closed for all j. Hence $U_0 \cup \cdots \cup U_j = X$ for $j \gg 0$. □

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Rees Theorem and Quotients in Linear Algebraic Semigroups

Mohan S. Putcha

Abstract Let S be an irreducible linear algebraic semigroup over an algebraically closed field k. We analyze the Rees theorem for a regular $\mathscr{J}$-class J of S. We define the support of J to be $\mathbb{X} = J/\mathscr{H}$. We show that $\mathbb{X}$ is a quasi-projective variety that is isomorphic to the direct product of two geometric quotients of algebraic group actions. When J is completely simple, $\mathbb{X}$ is an affine variety, while in reductive monoids, $\mathbb{X}$ is always a projective variety. We define the support of S to be that of the maximum regular $\mathscr{J}$-class of S. We study closed irreducible regular subsemigroups S of the full linear monoid $M_n(k)$ with projective supports $\mathbb{X}$. We determine the possible $\mathbb{X}$ and for a given $\mathbb{X}$, all the possible S. Along the way, we pose some open problems, the chief among which is the conjecture that any irreducible regular linear algebraic semigroup S with zero has projective support. We prove this in the simplest case of when S is a completely 0-simple semigroup.

Keywords Linear algebraic semigroups • Rees theorem • Quotients • Projective support

Subject Classifications: Primary 20M32, Secondary 20G99

1 Introduction

The purpose of this paper is to study varieties related to the Rees theorem, when applied to a regular $\mathscr{J}$-class J of an irreducible linear algebraic semigroup S. For reductive monoids, this topic has been touched upon by the author [11, Section 5],

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where the primary focus was to find the coordinatizing sets of the Rees theorem for a $\mathscr{J}$-class J within the reductive unit group G. The time seems ripe to revisit the subject, 30 years later. We work in the much more general situation of irreducible semigroups. Let e be an idempotent in J, and let R, L, H be the respective $\mathscr{R}, \mathscr{L}, \mathscr{H}$-classes of e. It is easy to see that R and L are quasi-affine varieties and that H is a connected algebraic group. We begin by showing that J too is a quasi-affine variety (Theorem 1). For an algebraic variety X with an equivalence relation $\sim$, $X/\sim$ is generally not a variety. We show however that $\mathbb{X}_l(J) = J/\mathscr{R}, \mathbb{X}_r(J) = J/\mathscr{L}$ and $\mathbb{X} = J/\mathscr{H}$ all have natural structures of quasi-projective varieties. We further show (Theorem 2) that $\mathbb{X}_r(J)$ is the geometric quotient R/H of the left action of the algebraic group H on R, $\mathbb{X}_l(J)$ is the geometric quotient L/H of the right action of H on L, and that $\mathbb{X} \cong \mathbb{X}_l \times \mathbb{X}_r$. When J is completely simple, we show that $\mathbb{X}(J)$ is an affine variety. In the situation of irreducible linear algebraic monoids, $\mathbb{X}(J)$ is affine in solvable monoids and projective in reductive monoids. The big conjecture is that $\mathbb{X}(J)$ is always projective in irreducible regular linear algebraic semigroups S with 0. We easily dispose of the simplest case when S is completely 0-simple.

Let S be a regular irreducible linear algebraic semigroup. Then it has a maximum $\mathscr{J}$-class J. We define the support $\mathbb{X}(S)$ to be $\mathbb{X}(J)$. We tackle the problems of: (1) determining all the possible projective supports $\mathbb{X}$, and (2) finding all semigroups S with a given support $\mathbb{X}$. We work within a fixed linear monoid $M_n(k)$. We find all possible supports $\mathbb{X}$ within the product of the the relevant Grassmannian spaces. We find that the only restriction on $\mathbb{X}$ is a non-degeneracy condition coming from the Rees theorem. For a given $\mathbb{X}$, we find (Theorem 10) that the semigroups S are classified by their cores ($\mathscr{H}$-class of a maximal idempotent). We find the precise conditions on H, and this allows us to list all S in some examples.

Even without the assumption that S has projective support, we are able to prove (Theorem 6) a conjecture of Renner that greatly elucidates the structure of S. As a consequence (Theorem 7), we are able to find a Rees matrix cover for any irreducible completely regular linear algebraic semigroup.

2 Semigroups

Many years ago, when told that I worked in semigroup theory, an algebraic geometer quipped: "Isn't that like studying sets?" If this view is correct, then the work of Green [6] truly represents creating something from nothing. Let S be a semigroup with idempotent set $E(S)$. $E(S)$ is ordered as:

$$e \leq f \text{ if } e = ef = fe$$

For $X \subseteq S$, let $E(X) = X \cap E(S)$. As usual, let $S^1 = S$ or $S \cup \{1\}$, depending on whether S has an identity element. The Green's relations $\mathscr{J}, \mathscr{R}, \mathscr{L}, \mathscr{H}$ on S, [4,6] are defined as:

$a \mathscr{J} b$ if $S^1aS^1=S^1bS^1$, $a \mathscr{R} b$ if $aS^1=bS^1$, $a \mathscr{L} b$ if $S^1a=S^1b$, $\mathscr{H}=\mathscr{R} \cap \mathscr{L}$

$S/\mathscr{J}$ is ordered as:

$$J \leq J' \text{ if } J \subseteq S^1J'S^1$$

In addition, the Green's relation $\mathscr{D}$ is defined as, $\mathscr{D} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}$. This yields an egg box picture for each $\mathscr{D}$-class D of S. If R, R' are $\mathscr{R}$-classes and L, L' $\mathscr{L}$-classes of D, then by Green's lemma [4, Lemma 2.2], there exist $a, b \in S^1$, such that $aR \cap R' \neq \emptyset$ and $Lb \cap L' \neq \emptyset$ and for any such a, b:

$$\text{the map} : x \to ax \text{ is an } \mathscr{L}\text{- preserving bijection between } R \text{ and } R' \tag{1}$$

and

$$\text{the map} : y \to yb \text{ is an } \mathscr{R}\text{- preserving bijection between } L \text{ and } L' \tag{2}$$

These maps play a pivotal role in semigroup theory. A $\mathscr{D}$-class D is called *regular* if $E(D) \neq \emptyset$. This implies that each $\mathscr{R}$-class and each $\mathscr{L}$-class of D contains an idempotent. S is *regular* if each $\mathscr{D}$-class of S is regular. Equivalently for all $a \in S$, there exists $x \in S$ such that $a = axa$. At the other end of the spectrum (generalizing commutativity) we have semigroups that satisfy the condition that $a|b$ $(b \in S^1aS^1)$ implies that $a^2|b^i$ for some positive integer i. Such semigroups are semilattice unions of Archimedean semigroups (for all $a, b \in S, a|b^i$ for some positive integer i), cf. [9] (undergraduate research of the author). This is analogous to the spectrum between reductive groups and solvable groups in the theory of algebraic groups. Indeed in the context of irreducible linear algebraic monoids with zero, this analogy is precise, cf. [12].

In this paper, we only consider *strongly π-regular* semigroups. This means that for all $a \in S$, a^n lies in a subgroup of S. The first example is when S is a union of groups. These semigroups are called *completely regular*. Their study was initiated by Clifford [3], and is a major area of study in semigroup theory. Finite semigroups, periodic semigroups, the monoid $M_n(F)$ over a field, and linear algebraic semigroups are all further examples of strongly π-regular semigroups. Then by Munn [8], $\mathscr{J} = \mathscr{D}$. For $J \in S/\mathscr{J}$, we define the *local semigroup* at J to be, $J^0 = J \cup \{0\}$ where for $a, b \in J$,

$$a \circ b = \begin{cases} ab \text{ if } ab \in J \\ 0 \text{ if } ab \notin J \end{cases}$$

If $E(J) = \emptyset$, then J^0 is a null semigroup. Otherwise, by Munn [8], J^0 is a completely 0-simple semigroup. So by Rees [13], J^0 is a Rees matrix semigroup, providing local coordinates for J. We elaborate. Let $e \in E(J)$ and let R, L, H denote respectively the $\mathscr{R}$-class, $\mathscr{L}$-class and $\mathscr{H}$-class of e. Then H is a group. Let $\Lambda = R/\mathscr{H}$, $\Gamma = L/\mathscr{H}$. For $i \in \Lambda$, let $r_i \in R$ denote an $\mathscr{H}$-class representative, and for $j \in \Gamma$, let $l_j \in L$ denote an $\mathscr{H}$-class representative. Then by the Rees theorem [4, 13],

$$J = \bigsqcup_{i \in \Lambda, j \in \Gamma} l_j H r_i \text{ and } P(i,j) = r_i l_j \in H^0 = H \cup \{0\}, i \in \Lambda, j \in \Gamma$$

The map $P : \Lambda \times \Gamma \longrightarrow H^0$ is called a sandwich map or a sandwich matrix. Then

$$P(i,j) \neq 0 \text{ if and only if } H_{ji} = l_j H r_i \text{ is a group} \tag{3}$$

Moreover P is *non-degenerate*:

$$i \in \Lambda \Longrightarrow P(i,j) \neq 0 \text{ for some } j \in \Gamma; \quad j \in \Gamma \Longrightarrow P(i,j) \neq 0 \text{ for some } i \in \Lambda \tag{4}$$

Thus if we view P as a $\Lambda \times \Gamma$-matrix over H^0 and the elements of J as $\Gamma \times \Lambda$ matrices over H^0 with exactly one non-zero entry, then the multiplication in J^0 is given by:

$$A \circ B = APB, \quad A, B \in J$$

A special case is when P has no non-zero entries, in which case J is a semigroup and is called *completely simple*. By [3], every completely regular semigroup is a semilattice union of completely simple semigroups.

For the purposes of this paper, we introduce some new notation. The group H acts on the left on R. Let $\mathbb{X}_r = R/H$ denote the orbit space and for $a \in R$, let $[a] = Ha$ denote the orbit of a. We will call $\mathbb{X}_r = \mathbb{X}_r(J)$ the *right support* of J. Let $\lambda_r : R \longrightarrow \mathbb{X}_r$, denote the orbit map given by $\lambda_r(a) = [a]$. Also H acts on the right on L. Let $\mathbb{X}_l = L/H$ denote the orbit space and for $b \in L$, let $[b] = bH$ denote the orbit of b. We will call $\mathbb{X}_l = \mathbb{X}_l(J)$ the *left support* of J. Let $\lambda_l : L \longrightarrow \mathbb{X}_l$, denote the orbit map given by $\lambda_l(b) = [b]$. We will call $\mathbb{X} = \mathbb{X}(J) = \mathbb{X}_l \times \mathbb{X}_r$ the *support* of J. We call H the *core* of J. Thus as sets, by (1) and (2),

$$J/\mathscr{L} \cong \mathbb{X}_r \cong \Lambda; \quad J/\mathscr{R} \cong \mathbb{X}_l \cong \Gamma; \quad J/\mathscr{H} \cong \mathbb{X} \cong \Gamma \times \Lambda$$

Thus we have a natural map $\lambda : J \longrightarrow \mathbb{X}$, whose fibers are the $\mathscr{H}$-classes of J. In particular λ is 1-1 on the idempotent set $E(J)$ and by (3), its image in $\mathbb{X}$ is given by:

$$E(J) \cong O = \{([b],[a]) \mid ab \neq 0\}$$

For the multiplicative monoid $M_n(k)$ of all $n \times n$ matrices over an algebraically closed field k, the Rees theorem is naturally connected with some classical algebraic geometry. We begin with $n = 2$. Let J denote the $\mathscr{J}$-class of rank 1 matrices, $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We can think of R and L as being:

$$R = \{(x,y) \mid x \neq 0 \text{ or } y \neq 0\}; \quad L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, x \neq 0 \text{ or } y \neq 0 \right\}$$

Accordingly

$$\mathbb{X}_r = \{[x, y] \mid x \neq 0 \text{ or } y \neq 0\} \cong \mathbb{P}^1; \quad \mathbb{X}_l = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x \neq 0 \text{ or } y \neq 0 \right\} \cong \mathbb{P}^1$$

In this case the support $\mathbb{X} \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a projective variety and $\lambda : J \longrightarrow \mathbb{X}$ is a morphism, given by:

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\begin{bmatrix} a \\ c \end{bmatrix}, [a, b] \right) & \text{if } a \neq 0 \\ \left(\begin{bmatrix} b \\ d \end{bmatrix}, [a, b] \right) & \text{if } b \neq 0 \\ \left(\begin{bmatrix} a \\ c \end{bmatrix}, [c, d] \right) & \text{if } c \neq 0 \\ \left(\begin{bmatrix} b \\ d \end{bmatrix}, [c, d] \right) & \text{if } d \neq 0 \end{cases}$$

Also O is an affine open subset of $\mathbb{X}$ given by:

$$O = \left\{ \left(\begin{bmatrix} x \\ y \end{bmatrix}, [z, w] \right) \in \mathbb{X} \middle| \, xz + yw \neq 0 \right\}$$

Moreover, $\lambda|_{E(J)} : E(J) \longrightarrow O$ is an isomorphism of varieties since its inverse $\mathfrak{e}$ is given by:

$$\mathfrak{e} \left(\begin{bmatrix} x \\ y \end{bmatrix}, [z, w] \right) = \frac{1}{xz + yw} \begin{pmatrix} xz & xw \\ yz & yw \end{pmatrix}$$

Now let us look at the general situation. In $M_n(k)$, let $G = GL_n(k)$ denote its unit group, $e = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$. Let $J(m), R(m), L(m)$ denote respectively the $\mathscr{J}, \mathscr{R}$ and $\mathscr{L}$-classes of e. Then $J(m)$ is the variety of of rank m matrices. Also

$$P = P(e) = \{a \in G \mid ae = eae\}, \qquad P^- = P^-(e) = \{a \in G \mid ea = eae\} \quad (5)$$

are opposite parabolic subgroups of G consisting respectively of block upper triangular and block lower triangular matrices, Now the $\mathscr{H}$-class of e is naturally isomorphic to $H(m) = GL_m(k)$. We can think of $R(m)$ as the variety of $m \times n$ matrices of rank m and $L(m)$ as the variety of $n \times m$ matrices of rank m. The variety $\mathbb{X}_r = R(m)/H(m) \cong G/P^-$ is the Grassmannian variety $\mathbb{G}r(m) = \mathbb{G}r(m, n)$ of m-dimensional subspaces of $k^{n \times 1}$. The map $\lambda_r(m) : R(m) \longrightarrow \mathbb{G}r(m)$ given by:

$$\lambda_r(m)(A) = \text{Row space of } A \tag{6}$$

is a surjective morphism of varieties. If $B \in L(m)$, then

$$\mathbb{G}r(m)_B = \{[A] \mid A \in R(m), \det AB \neq 0\} \tag{7}$$

is an affine open subset of $\mathbb{G}r(m)$, isomorphic to

$$R(m)_B = \{A \in k^{m\times n} \mid AB = I\} \tag{8}$$

with the isomorphism $\lambda_r^{-1}(m) : \mathbb{G}r(m)_B \longrightarrow R(m)_B$ given by:

$$\lambda_r^{-1}(m)([A]) = (AB)^{-1}A \tag{9}$$

The space $\mathbb{X}_l = L(m)/H(m) \cong G/P$ is the Grassmannian variety $\mathbb{G}r^*(m) = \mathbb{G}r^*(m,n)$ of m-dimensional subspaces of $k^{1\times n}$. The map $\lambda_l(m) : L \longrightarrow \mathbb{G}r^*(m)$ given by:

$$\lambda_l(m)(A) = \text{Column space of } A \tag{10}$$

is a surjective morphism of varieties. Thus $\mathbb{X} = \mathbb{G}r^*(m) \times \mathbb{G}r(m)$ is a projective variety and $\lambda(m) : J(m) \longrightarrow \mathbb{X}$ given by:

$$\lambda(m)(A) = (\text{Rowspace of } A, \text{Column space of } A) \tag{11}$$

is a surjective morphism of varieties. The affine open subset $O(m) = \lambda(m)(E(J(m)))$ of $J(m)$ is given by:

$$O(m) = \{([B],[A]) \in \mathbb{X} \mid \det AB \neq 0\} \tag{12}$$

Then $\lambda(m)|_{E(J(m))} : E(J(m)) \longrightarrow O(m)$ is an isomorphism of varieties since its inverse $\mathfrak{e}$, that picks the idempotent in the $\mathscr{H}$-class, is given by:

$$\mathfrak{e}([B],[A]) = B(AB)^{-1}A \tag{13}$$

Let $X_r(m)$ and $X_l(m)$ denote respectively the sets of reduced row echelon matrices in R and the set of reduced column echelon matrices of of L. Then by the Rees theorem

$$J(m) = X_l(m) \cdot GL_m(k) \cdot X_r(m) \tag{14}$$

This is the full rank factorization of matrix theory. Now $S(m) = \overline{J(m)}$ is the regular semigroup of matrices of rank $\leq m$ and we have:

$$S(m) = X_l(m) \cdot M_m(k) \cdot X_r(m) \tag{15}$$

Since $X_r(m)X_l(m) \subseteq M_m(k)$, we see that $S(m)$ is a homomorphic image of an $X_l(m) \times X_r(m)$ Rees matrix semigroup over $M_m(k)$. We will generalize (15) to any irreducible regular linear algebraic semigroup (Theorem 6 and Remark 3).

3 Linear Algebraic Semigroups

Let k be an algebraically closed field and $M_n(k)$ the full linear monoid of all $n \times n$ matrices. We will use round parentheses for writing matrices, reserving square parentheses for the images in the appropriate Grassmannian spaces. A linear algebraic semigroup is a semigroup S such that the underlying set is an affine variety and the product map is a morphism of varieties. Equivalently S is isomorphic to a closed subsemigroup of some $M_n(k)$. Our interest is when S is irreducible as an affine variety. Irreducible linear algebraic monoids have been much studied, cf. [12, 15]. See [7] for an introduction to the classical theory of linear algebraic groups. We note that irreducible linear algebraic semigroups have not been studied beyond the initial work of the author [10] on irreducible (regular) linear algebraic semigroups.

Fix an irreducible linear algebraic semigroup S and $e \in E(S)$. Let J, R, L, H denote respectively the $\mathscr{J}$-class, $\mathscr{R}$-class, $\mathscr{L}$-class and $\mathscr{H}$-class of e. So H is the unit group of the irreducible algebraic monoid eSe, and is hence a connected algebraic group. Also $eS = \{a \in S \mid ea = a\}$ and $Se = \{a \in S \mid ae = a\}$ are closed irreducible subsemigroups of S. However SeS in general is not even a variety:

Example 1. Let $S = \mathbb{A}^3$ with

$$(a, b, c)(a', b', c') = (aa', ab', ca')$$

Then S is an irreducible algebraic semigroup with zero $0 = (0, 0, 0)$. Let $e = (1, 0, 0)$. Then

$$S^2 = SeS = \{(aa', ab', ca') \mid a, b, c \in k\} = J \cup \{0\}, S = \overline{SeS},$$

$$eS = \{(a, 0, c) \mid a, c \in k\}, Se = \{(a, b, 0) \mid a, b \in k\}$$

Then SeS is not a variety since for the projection onto the x-axis, the inverse image of 0 has dimension 1.

Theorem 1. *R, L, J are respectively open subsets of eS, Se and $\overline{SeS}$. In particular, R, L, J are quasi-affine varieties.*

Proof. We may assume that S is a closed subsemigroup of some $M_n(k)$. Let rk $(e) = m$. Then

$$U = \{a \in S \mid \text{rk}\, a \geq m\}$$

is an open subset of S. If $a \in \overline{SeS}$, then rk $a \leq m$. So,

$$V = \{a \in \overline{SeS} \mid \text{rk}\, a = m\} = \overline{SeS} \cap U$$

is an open subset of $\overline{SeS}$. Clearly $J \subseteq V$. If $e' \in E(\overline{SeS})$, then $e' \in SeS$ by [11, Corollary 3.30]. Let J' denote the $\mathscr{J}$-class of e' in S. If $J' < J$, then $e > e_1$ for some $e_1 \in E(J')$. So rk $e' =$ rk $e_1 <$ rk $e = m$. Hence:

$$E(\overline{SeS}) \cap V = E(J)$$

Clearly

$$V' = \{a \in \overline{SeS} \mid (ax)^2 \in V \text{ for some } x \in S\}$$

is an open subset of $\overline{SeS}$. If $a \in J$, then $a\mathscr{R}f$ for some $f \in E(J)$. Thus rk $f = m$ and $ax = f$ for some $x \in S$. So $(ax)^2 = f \in V$. So $a \in V'$ and $J \subseteq V'$.

Now let $a \in V'$. Then for some $x \in S$, rk $(ax)^2 = m$. So rk ax =rk $(ax)^2$. Hence $ax\mathscr{H}f$ for some $f \in E(M_n(k))$, rk $f = m$. Since S is strongly π-regular, $f \in E(S)$ and $ax\mathscr{H}f$ in S. Hence $f \in E(J)$. Now $ay = f$ for some $y \in S$. Since rk $a = m =$ rk f, $fa = f$. So $a\mathscr{R}f$ in S. Hence $a \in J$. Thus $J = V'$ is open $\overline{SeS}$. So $R = eS \cap J$ is open in eS and $L = Se \cap J$ is open Se. □

Let X be an algebraic variety and $\sim$ an equivalence relation on X. Let Y be an algebraic variety and $\pi : X \longrightarrow Y$ a surjective morphism such that the fibres of π are the $\sim$-classes and for any morphism $\mu : X \longrightarrow Z$ of algebraic varieties, that is constant on on $\sim$-classes, there exists a morphism $\nu : Y \longrightarrow Z$ such that $\mu = \nu \circ \pi$. Such a Y, if it exists, is clearly unique and we *define* $X/\sim$ to be Y and say that $X/\sim$ *exists*. We note that this is the bare minimum of what one wants in a quotient. When studying algebraic group actions, cf. [1, Definition 1.18], one generally wants the orbit space to be a geometric quotient. This means the following. Suppose an algebraic group H is acting on X and Y is the orbit space X/H. Let $U \subseteq Y$, $V = \pi^{-1}(U)$ with V open in X. This should imply that U is open in Y and that $k[V]^H \cong k[U]$.

Let S be a closed irreducible subsemigroup of $M_n(k)$ and let $e = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \in E(S)$. Let J, R, L, H denote the respective $\mathscr{J}, \mathscr{R}, \mathscr{L}, \mathscr{H}$-classes of e in S. Let λ_r denote the restriction of the map $\lambda_r(m)$ in (6) to R and let $\mathbb{X}_r = \lambda_r(R) \subseteq \mathbb{G}r(m)$. Let λ_l denote the restriction of the map $\lambda_l(m)$ in (10) to L and let $\mathbb{X}_l = \lambda_l(L) \subseteq \mathbb{G}r^*(m)$.

Lemma 1. *(i)* $\mathbb{X}_r$ *is a quasi-projective variety covered by affine open subsets* $U_1, \cdots, U_t$ *such that there are closed subsets* $R_1, \cdots, R_t$ *of* S *contained in* R *and isomorphisms* $\eta_j : U_j \longrightarrow R_j$, *with* $\eta_j^{-1} = \lambda_r|_{R_j}$, $j = 1, \cdots, t$.

(ii) $\mathbb{X}_l$ *is a quasi-projective variety covered by affine open subsets* $U_1', \cdots, U_s'$ *such that there are closed subsets* $L_1, \cdots, L_s$ *of* S *contained in* L *and isomorphisms* $\eta_i' : U_i' \longrightarrow L_i$*, with* $\eta_i'^{-1} = \lambda_l|_{L_i}, i = 1, \cdots, s$.

Proof. Let $y \in L$. Analogous to (6) and (7), let

$$V_y = \{[x] \in \mathbb{G}r(m) \mid \text{rk}\, xy = m\}$$

Then V_y is an open subset of $\mathbb{G}r(m)$ and hence $U_y = V_y \cap \mathbb{X}_r$ is open in $\mathbb{X}_r$. Clearly $U_y \neq \emptyset$ by (4). By (9), we have a morphism $\eta : V_y \longrightarrow M_n(k)$, given by: $\eta[x] = (xy)^{-1}x$. Then clearly $\eta(U_y) \subseteq S$. Let $[x] \in V_y \cap \eta^{-1}(S)$. Then $x' = (xy)^{-1}x \in S, [x'] = [x]$ and $x'y = e$. So $x'\mathscr{R}e$ in S and $x' \in R$. So $[x] = [x'] \in U_y$, Thus

$$U_y = V_y \cap \eta^{-1}(S) \text{ is closed in } V_y \text{ and hence in } \overline{\mathbb{X}_r} \cap V_y \tag{16}$$

Now

$$R_y = \{x \in eS \mid xy = e\}$$

is closed in S, contained in R. Let $\eta_y = \eta|_{U_y} : U_y \longrightarrow X_y$. Then $\lambda_r(\eta_y[x]) = [(xy)^{-1}x] = [x]$ for $[x] \in U_y$. Also for $x \in X_y$, $[x] \in U_y$ and $\eta_y[x] = (xy)^{-1}x = ex = x$. Hence $\eta_y^{-1} = \lambda_r|_{R_y}$ and $U_y \cong X_y$. Since R is irreducible, we see by [7, Theorem 4.4] that $\mathbb{X}_r = \lambda_r(R)$ contains a non-empty open subset O of $\overline{\mathbb{X}_r}$. Since $U_y \neq \emptyset$, $\overline{\mathbb{X}_r} \cap V_y$ is also a non-empty open subset of the irreducible variety $\overline{\mathbb{X}_r}$. Hence $O \cap V_y = O \cap \overline{\mathbb{X}_r} \cap V_y$ is a non-empty open subset of $\overline{\mathbb{X}_r}$. Hence $O \cap V_y$ is an open and hence dense subset of the irreducible variety $\overline{\mathbb{X}_r} \cap V_y$. Now $O \cap V_y \subseteq \mathbb{X}_r \cap V_y = U_y$. Hence by (16),

$$U_y = \overline{\mathbb{X}_r} \cap V_y, \;\; y \in L \tag{17}$$

If $x \in R$, then by (4), $xy \in H$ for some $y \in L$. Hence $[x] \in U_y$. So by (17),

$$\mathbb{X}_r = \overline{\mathbb{X}_r} \cap \bigcup_{y \in L} V_y$$

is a locally closed subset of the projective variety $\mathbb{G}r(m)$, and is hence a quasi-projective variety. Now $\mathbb{X}_r$ is covered by the open subsets $U_y, y \in L$. By the Hilbert basis theorem, a finite number of them suffice. This completes the proof of (i). (ii) follows dually. □

Theorem 2. *(i) The left action of* H *on* R *has a geometric quotient* R/H*, that is isomorphic to the quasi-projective variety* $\mathbb{X}_r$.

(ii) $J/\mathscr{L}$ *exists and is isomorphic to* $\mathbb{X}_r$.

(iii) The right action of H *on* L *has a geometric quotient* L/H*, that is isomorphic to the quasi-projective variety* $\mathbb{X}_l$.

(iv) *$J/\mathscr{R}$ exists and is isomorphic to $\mathbb{X}_l$.*
(v) *$J/\mathscr{H} \cong \mathbb{X} = \mathbb{X}_l \times \mathbb{X}_r$ exists and is a quasi-projective variety.*
(vi) *$O_{\mathbb{X}} = \{([b],[a]) \in \mathbb{X} \,|\, ab \in H\}$ is an open subset of $\mathbb{X}$ and $\mathfrak{e} : O_{\mathbb{X}} \longrightarrow E(J)$ given by $\mathfrak{e}([b],[a]) = b(ab)^{-1}a$, is an isomorphism of varieties.*

Proof. We continue with the notation of Lemma 1 and of (6)–(13). Let $\emptyset \neq U \subseteq \mathbb{X}_r$ and let $V = \lambda_r^{-1}(U)$. Suppose V is open in R. So $V \cap R_j$ is open in R_j. So by Lemma 1, $U \cap U_j = \lambda_r(V \cap R_j)$ is open in U_j and hence in $\mathbb{X}_r$, $j = 1, \cdots, t$. Hence U is open in $\mathbb{X}_r$. Next let $f \in k[V]^H$. Then f factors through a map $\tilde{f}$ on U. We need to show that $\tilde{f} \in k[U]$. Now $f|_{V \cap R_j} \in k[V \cap R_j] \cong k[U \cap U_j]$ by Lemma 1. Hence $\tilde{f}|_{U \cap U_j} \in k[U \cap U_j], j = 1, \cdots, t$. Thus $\tilde{f} \in k[U]$ and $k[V]^H \cong k[U]$ and $R /\!/ H \cong \mathbb{X}_r$, proving (i).

Now by (11), λ_r extends to a morphism $\tilde{\lambda}_r : J \longrightarrow \mathbb{X}_r$ with the fibres being the $\mathscr{L}$-classes of J. Let $\pi : J \longrightarrow Y$ be a morphism that is constant on $\mathscr{L}$-classes. Then clearly π factors through a unique map $\theta : \mathbb{X}_r \longrightarrow Y$. We need to show that θ is a morphism. This is clear since by Lemma 1, $\mathbb{X}_r$ is covered by open sets $U_1, \cdots, U_t$ and

$$\theta[x] = \pi \circ \eta_i[x], \text{ if } [x] \in U_j, j = 1, \cdots, t$$

This shows that $J/\mathscr{L}$ exists and is isomorphic to $\mathbb{X}_r$. This proves (ii). (iii) and (iv) are proved analogously.

We now prove (v). By (11), we have a surjective morphism $\lambda : J \longrightarrow \mathbb{X} = \mathbb{X}_l \times \mathbb{X}_r$, with the fibres being the $\mathscr{H}$-classes of J. By Lemma 1, $\mathbb{X}$ is covered by open sets $U_i' \times U_j, i = 1, \cdots, s,\ j = 1, \cdots, t$. So if $\mu : J \longrightarrow Z$ is a morphism of varieties that is constant on $\mathscr{H}$-classes, then $\mu = \nu \circ \lambda$ where the morphism $\nu : \mathbb{X} \longrightarrow Z$ is given by:

$$\nu([b],[a]) = \mu(\eta_i'[b] \cdot \eta_j[a]) \text{ if } ([b],[a]) \in U_i' \times U_j, i = 1, \cdots, s,\ j = 1, \cdots, t$$

Thus $J/\mathscr{H}$ exists and is the quasi-projective variety $\mathbb{X}$. This proves (v). (vi) now follows from (12) and (13). □

Michel Brion points out that in the language of algebraic group actions, Lemma 1 and Theorem 2 show that R is a principal H-bundle which is locally trivial for the Zariski topology and with the sets R_y being the sections of this bundle.

There is no natural action of H on J withe orbits being the $\mathscr{H}$-classes. We now show however that $J/\mathscr{H}$ has properties similar to a geometric quotient:

Corollary 1. *The surjective morphism $\lambda : J \longrightarrow \mathbb{X} = J/\mathscr{H}$ has the following properties:*

(i) *λ is open.*
(ii) *If $U \subseteq \mathbb{X}$ is open and $V = \lambda^{-1}(U)$, then $k[V]^{\mathscr{H}} \cong k[U]$, where $k[V]^{\mathscr{H}}$ consisits of $f \in k[V]$ that are constant on $\mathscr{H}$-classes.*

Proof. By Lemma 1, there exist $y_1, \cdots, y_t \in L$ such that $\mathbb{X}_r$ is covered by open subsets:

$$U_j = \{[a] \in \mathbb{X}_r \mid ay_j \in H\}, j = 1, \cdots, t$$

with morphisms $\eta_j : \mathbb{X}_r \longrightarrow J$ given by

$$\eta_j[a] = (ay_j)^{-1}a \in Ha, j = 1, \cdots, t$$

Also there exist $x_1, \cdots, x_s \in R$ such that $\mathbb{X}_l$ is covered by open subsets:

$$U_i' = \{[b] \in \mathbb{X}_r \mid x_i b \in H\}, i = 1, \cdots, s$$

with morphisms $\eta_i' : \mathbb{X}_r \longrightarrow J$ given by

$$\eta_i'[b] = b(x_i b)^{-1} \in bH, i = 1, \cdots, t$$

(i) Let V be an open subset of J and let $c_0 \in V$. Then $c_0 = b_0 a_0$ for some $a_0 \in R$ and $b_0 \in L$. Then $[b_0] \in U_i'$ and $[a_0] \in U_j$ for some i, j. Let $h' = x_i b_0, h = a_0 y_j \in H$. Define $\xi : U_i' \times U_j \longrightarrow J$ as:

$$\xi([b], [a]) = \eta_i'[b] \cdot h'h \cdot \eta_j[a] \mathscr{H} ba$$

Then $U(c_0) = \xi^{-1}(V)$ is open in $U_i' \times U_j$ and hence in $\mathbb{X}$. Since $\lambda \circ \xi$ is the identity map on $U_i' \times U_j$, we see that $U(c_0) \subseteq \lambda(V)$. Clearly

$$\xi([b_0], [a_0]) = \eta_i'[b_0] \cdot h'h \cdot \eta_j[a_0] = b_0(x_i b_0)^{-1} h'h(a_0 y_j)^{-1} a_0 = b_0 a_0 = c_0$$

Hence $\lambda(c_0) = ([b_0], [a_0]) \in U(c_0)$. So $\lambda(V)$ is open in $\mathbb{X}$

(ii) Let U be an open subset of $\mathbb{X}$ and let $V = \lambda^{-1}(U)$. Let $f \in k[V]$ that is constant on $\mathscr{H}$-classes. This yields a map $\tilde{f}$ on U. Define morphisms $\xi_{ij} : U_i' \times U_j \longrightarrow J$ as:

$$\xi_{ij}([b], [a]) = \eta_i'[b] \cdot \eta_j[a], i = 1, \cdots, s,\ j = 1. \cdots, t$$

Then

$$\tilde{f}|_{U \cap (U_i' \times U_j)} = f \circ \xi_{ij} \in k[U \cap (U_i' \times U_j)], i = 1, \cdots, s,\ j = 1. \cdots, t$$

Hence $\tilde{f} \in k[U]$. This completes the proof. □

Since $\tilde{\lambda}_r : J \longrightarrow \mathbb{X}_r$ is given by $\tilde{\lambda}_r = \rho \circ \lambda$ where ρ is the projection map from $\mathbb{X} = \mathbb{X}_l \times \mathbb{X}_r$ to $\mathbb{X}_r$, we have:

Corollary 2. *The surjective morphism* $\tilde{\lambda}_r : J \longrightarrow \mathbb{X}_r = J/\mathscr{L}$ *has the following properties:*

(i) $\tilde{\lambda}_r$ *is open.*
(ii) If $U \subseteq \mathbb{X}_r$ *is open and* $V = \lambda_r^{-1}(U)$*, then* $k[V]^{\mathscr{L}} \cong k[U]$*, where* $k[V]^{\mathscr{L}}$ *consisits of* $f \in k[V]$ *that are constant on* $\mathscr{L}$*-classes.*

If J is completely simple, then by Theorem 2(vi), $\mathbb{X} = O_{\mathbb{X}} \cong E(J)$. Hence we have:

Corollary 3. *If* J *is completely simple, then* $\mathbb{X}(J)$ *is an affine variety.*

The converse is an open question:

Problem 1. If $\mathbb{X}(J)$ is an affine variety, then is J necessarily completely simple?

Remark 1. Suppose S has an identity element and G is its unit group. If G is solvable, then by [12, Theorem 6.32], J is completely simple and hence by Corollary 3, has affine support $\mathbb{X}$. If G is reductive, then by [11, Section 5], there is a bijective morphism from $G/P \times G/P^-$ to $\mathbb{X}$ and hence J has a projective support $\mathbb{X}$.

The following result points out the significant impact that $\mathbb{X}_r, \mathbb{X}_l$ or $\mathbb{X}$ being projective have on the respective closures of R, L and J.

Theorem 3. *(i) If* $\mathbb{X}_r$ *is projective, then* $\overline{SeS} = SeS = SeR$ *and* $eS = eSe \cdot R$*.*
(ii) If $\mathbb{X}_l$ *is projective, then* $\overline{SeS} = SeS = LeS$ *and* $Se = L \cdot eSe$*.*
(iii) If $\mathbb{X}$ *is projective, then* $\overline{SeS} = SeS = L \cdot eSe \cdot R$*.*

Proof. It suffices to prove (i). Let $G = GL_n(k)$ and let $P^- = P^-(e)$ be as in (5). Since $\mathbb{X}_r$ is closed in $\mathbb{G}r(m)$,

$$X = \{x \in G \mid [ex] \in \mathbb{X}_r\}$$

is closed in G and $P^-X \subseteq X$. Let G/P^- denote the projective variety of right cosets of P^- in G. The natural map from $M_n(k) \times G$ to $M_n(k) \times G/P^- = ((M_n(k), +) \times G)/(\{0\} \times P^-)$ is open. Now

$$Z = \{(a, x) \mid a \in S, x \in X, ax^{-1}e = ax^{-1}\}$$

is closed in $M_n(k) \times G$. If $(a, x) \in Z$ and $q \in P^-$, then $qx \in X$ and

$$a(qx)^{-1}e = ax^{-1}q^{-1}e = ax^{-1}eq^{-1}e = ax^{-1}eq^{-1} = ax^{-1}q^{-1} = a(qx)^{-1}$$

So $(a, qx) \in Z$. It follows that the image of Z in $M_n(k) \times G/P^-$,

$$\tilde{Z} = \{(a, P^-x) \mid a \in S, x \in X, ax^{-1}e = ax^{-1}\}$$

is closed in $M_n(k) \times G/P^-$. Since G/P^- is a projective variety, we see that the projection of $\tilde{Z}$ in $M_n(k)$,

$$S_1 = \{a \in S \mid ax^{-1}e = ax^{-1} \text{ for some } x \in X\}$$

is closed in $M_n(k)$ and hence in S. Let $a \in S_1$. Then for some $x \in X$, $ax^{-1}e = ax^{-1}$. Let $b = ax^{-1}$. Then $bx = a$ and $be = b$. Since $x \in X, ex = ur$ for some $r \in R$ and some u in the $\mathscr{H}$-class of e in $M_n(k)$. Hence $a = bur$. Let $c = bu$. Then $a = cr$ and $ce = bue = bu = c$. Now $rl = e$ for some $l \in L$. So $c = crl = al \in S$. Hence $c = ce \in Se$ and $a = cr \in SeR$. So $S_1 \subseteq SeR$. Conversely let $a \in SeR$. Then $a = br$ for some $b \in Se$ and $r \in R$. Now $r = ex$ for some $x \in X$. Then

$$ax^{-1} = brx^{-1} = be = bee = brx^{-1}e = ax^{-1}e$$

and $a \in S_1$. Thus $SeR = S_1$ is closed in S. Now $J = LR \subseteq SeR$ and $\overline{J} = \overline{SeS}$ by Theorem 1. Hence $\overline{SeS} = SeR$. This completes the proof. □

Example 2. Let $S = \mathbb{A}^2$ with $(a,b)(a'.b') = (aa', ab'), e = (1,0), J = R = \{(a,b) \mid a \neq 0\}$. Then $S = eS \neq eSe \cdot R = R \cup \{0\}$. Of course, here $\mathbb{X} = \mathbb{X}_r \cong \mathbb{A}^1$ is affine.

Corollary 4. *Suppose that J has projective support and let $\theta : S \longrightarrow S'$ be a homomorphism of algebraic semigroups. Then $\tilde{J} = \theta(J)$ is a $\mathscr{J}$-class of $\tilde{S} = \overline{\theta(S)}$ and $\tilde{J}$ has projective support.*

Proof. Let $\tilde{J}, \tilde{R}, \tilde{L}, \tilde{H}$ denote respectively the $\mathscr{J}, \mathscr{R}, \mathscr{L}, \mathscr{H}$-classes of $\theta(e)$ in $\tilde{S}$ and let $\tilde{\mathbb{X}} = \mathbb{X}(\tilde{J})$. Then $\theta(H) = \tilde{H}$ and θ induces a dominant morphism $\tilde{\theta}$ from $\mathbb{X}$ to $\tilde{\mathbb{X}}$. Since $\mathbb{X}$ is complete, $\tilde{\mathbb{X}} = \tilde{\theta}(\mathbb{X})$ is complete. Hence $\tilde{R} = \theta(R)$ and $\tilde{L} = \theta(L)$. Hence $\tilde{J} = \tilde{L}\tilde{R} = \theta(LR) = \theta(J)$. This completes the proof. □

If dim $H > 1$, then there is no way to make the local semigroup $J^0 = J \cup \{0\}$ into an algebraic semigroup. We now show that if J has projective support, then we can do the next best thing.

Corollary 5. *Suppose that J has projective support and $J \neq H$. Then there is an irreducible completely 0-simple linear algebraic semigroup $S' = J' \cup \{0\}$ and a surjective homomorphism $\theta : \overline{J} = SeS \longrightarrow S'$ of algebraic semigroups, such that θ is 0 on $SeS \setminus J$ and induces isomorphisms $E(J) \cong E(J')$ and $\mathbb{X}(J) \cong \mathbb{X}(J')$.*

Proof. Let θ denote the mth exterior power homomorphism from $M_n(k)$ into $M_{\binom{n}{m}}(k)$. Let $J(m)$ denote the rank m $\mathscr{J}$-class of $M_n(k)$. By Theorem 3, SeS is closed in $M_n(k)$ and $J = SeS \cap J(m)$. If $J = SeS$, then by Corollary 3, J has affine support and hence $J = H$, a contradiction. So $J \neq SeS$. Thus $\theta(eSe) = k\theta(e)$, Hence $J' = \theta(J)$ is closed under the action of k^*. So to show that $S' = \theta(S)$ is closed in $M_{\binom{n}{m}}(k)$, it suffices to show that the images of J' in $\mathbb{G}r(1, \binom{n}{m})$ and $\mathbb{G}r^*(1, \binom{n}{m})$ are closed. But these images are the respective isomorphic copies of the projective varieties $\mathbb{X}_r(J)$ and $\mathbb{X}_l(J)$, by the very definition of the Grassmannian varieties. By Theorem 2(vi), this also establishes an isomorphism between $E(J)$ and $E(J')$. This completes the proof. □

We have seen in Theorem 2(vi) that the support $\mathbb{X}(J)$ determines the idempotent set $E(J)$ of J. We analyze this further.

Theorem 4. *(i)* $S_0 = \overline{\langle E(J)\rangle}$ *is an irreducible algebraic semigroup.*
(ii) $J_0 = J \cap \langle E(J)\rangle$ *is a regular* $\mathscr{J}$*-class of* S_0 *with* $\mathscr{R}$*-class* $R_0 = R \cap \langle E(J)\rangle$*,* $\mathscr{L}$*-class* $L_0 = L \cap \langle E(J)\rangle$ *and* $\mathscr{H}$*-class* $H_0 = H \cap \langle E(J)\rangle$*.*
(iii) $\mathbb{X}(J_0) \cong \mathbb{X}(J)$, $R = HR_0$, $L = L_0H$, $J = L_0HR_0$.

Proof. By [12, Theorem 5.9], We see that

$$f \in E(J) \Longrightarrow e\mathscr{R}e_1\mathscr{L}e_2\mathscr{R}f \text{ for some } e_1, e_2 \in E(J) \tag{18}$$

Since $E(J)$ is irreducible, we have an ascending chain of closed irreducible sets:

$$E(J) \subseteq \overline{E(J)^2} \subseteq \overline{E(J)^3} \subseteq \cdots$$

So for some positive integer i,

$$S_0 = \overline{\langle E(J)\rangle} = \overline{E(J)^i} = \overline{E(J)^{i+1}} = \cdots \tag{19}$$

is an irreducible algebraic semigroup. This proves (i).

By (18), $J_0 = J \cap S_0$, $R_0 = R \cap S_0$, $L_0 = L \cap S_0$, $H_0 = H \cap S_0$ are respectively the $\mathscr{J}, \mathscr{R}, \mathscr{L}, \mathscr{H}$-classes of e in S_0. By (19), $E(J)^i$ contains a non-empty open subset V of S_0. So $\overline{eVe} = eS_0e$ and eVe contains a non-empty open subset V_0 of eS_0e. So $U = H_0 \cap V_0$ is a non-empty open subset of H_0 and $U \subseteq eE(J)^ie$. Since H_0 is a connected group, $U^2 = H_0$. So $H_0 \subseteq \langle E(J)\rangle$ and $H_0 = H \cap \langle E(J)\rangle$. Let $a \in R_0$. Then for some $f \in E(J), a\mathscr{L}f$ in S_0. By (18), $e\mathscr{R}e_1\mathscr{L}e_2\mathscr{R}f$ for some $e_1, e_2 \in E(J)$. So $e_1 f \mathscr{H} a$ in S_0. Hence $a \in H_0 e_1 f \subseteq \langle E(J)\rangle$. So $R_0 = R \cap \langle E(J)\rangle$. Similarly $L_0 = L \cap \langle E(J)\rangle$. Since $J_0 = L_0R_0$, we see that $J_0 = J \cap \langle E(J)\rangle$. This proves (ii).

Let $a \in R$. Then $a\mathscr{L}f$ for some $f \in E(J) = E(J_0)$. So for some $a' \in R_0, a'\mathscr{L}f$ in S_0. Hence $a\mathscr{H}a'$ in S and $a \in Ha'$. So $R = HR_0$. Similarly $L = L_0H$. Hence $J = LR = L_0HR_0$. This proves (iii). ◻

By [12, Theorem 5.10], S has a maximum regular $\mathscr{J}$-class. If J is that $\mathscr{J}$-class, then we define the *right support* $\mathbb{X}_r(S)$, *left support* $\mathbb{X}_l(S)$ and *support* $\mathbb{X}(S)$ to be respectively $\mathbb{X}_r(J), \mathbb{X}_l(J)$ and $\mathbb{X}(J)$. Then by Theorem 4, $\mathbb{X}_r(S) \cong \mathbb{X}_r(\overline{\langle E(J)\rangle}), \mathbb{X}_l(S) \cong \mathbb{X}_l(\overline{\langle E(J)\rangle})$ and $\mathbb{X}(S) \cong \mathbb{X}(\overline{\langle E(J)\rangle})$. We also define the core H of J to be the *core* of S.

Example 3. Let S be the subsemigroup of $M_3(k)$ consisting of matrices of rank ≤ 1 and having last column 0. Then

$$I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix} \middle| \, a, b \in k \right\}$$

is an ideal of S and $J = S \setminus I$ is the only non-zero regular $\mathscr{J}$-class of S. Then

$$\mathbb{X}_r = \{[a,b,0] \mid a \neq 0 \text{ or } b \neq 0\} \cong \mathbb{P}^1$$

is a projective variety. However

$$\mathbb{X}_l = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| a \neq 0 \text{ or } b \neq 0 \right\}$$

is neither projective nor affine. Thus S has a projective right support but not a projective left support.

Our fundamental expectation is:

Conjecture 1. Any irreducible regular linear algebraic semigroup with 0 has projective support.

Remark 2. (1) The restriction on S having 0 is essential because any irreducible affine variety can be made into a completely simple semigroup trivially by defining $ab = b$ for $a, b \in S$.

(2) One consequence of Conjecture 1 would be that if S is an irreducible regular algebraic semigroup with 0, and if $\theta : S \longrightarrow S'$ is a homomorphism of linear algebraic semigroups, then $\tilde{S} = \overline{\theta(S)}$ is also an irreducible regular linear algebraic semigroup with 0. First of of all $\tilde{S}$ and $\overline{\theta(H)}$ have zero $\theta(0)$. Since H is reductive, so is $\theta(H)$ and hence $\overline{\theta(H)}$ is a regular monoid. By Theorem 3 and Corollary 4, $\tilde{S}$ has projective support and $\tilde{S} = \theta(L)\overline{\theta(H)}\theta(R)$ is regular.

We now prove Conjecture 1 in the simplest situation.

Theorem 5. *Any irreducible completely 0-simple linear algebraic semigroup S has projective support.*

Proof. Since $eS = R \cup \{0\}$, R is a closed subset of $X = k^{m \times n} - \{0\}$. Now H is a one dimensional torus and $eSe = H \cup \{0\}$. By [11, Chapter 8], we may assume that there exist positive integers $i_1, \cdots, i_m$ such that $H = \{\text{diag}\,(\alpha^{i_1}, \cdots, \alpha^{i_m}) \mid \alpha \in k^*\}$. Hence the orbit space X/H is a weighted projective space, and by [5, chapter 1] is a projective variety. Since R is H-invariant, R/H is a closed subset of this projective variety. Hence $\mathbb{X}_r$ is a projective variety. Similarly $\mathbb{X}_l$ is a projective variety. Hence $\mathbb{X} = \mathbb{X}_l \times \mathbb{X}_r$ is also a projective variety. This completes the proof. □

4 Renner's Conjecture

Let S be an irreducible regular linear algebraic semigroup with maximum $\mathscr{J}$-class J. Fix $e \in E(J)$ and let H, R, L denote the respective $\mathscr{H}, \mathscr{R}, \mathscr{L}$-class of e. Then eSe is an irreducible regular algebraic monoid with unit group H. If S has a 0, then

H is a reductive group. In connection with trying to prove Conjecture 1, Lex Renner has conjectured that $eS = eSe \cdot R$. Of course this is true by Theorem 3, if $\mathbb{X}(S)$ is assumed to be projective. The conjecture says that this is true without this *a priori* assumption. Example 2 shows that the conjecture is not true without the assumption of regularity. We prove the conjecture (Theorem 6) in full generality, whereby it also applies to when S is completely regular (Theorem 7).

For $e_1, e_2 \in E(S)$, J' a $\mathscr{J}$-class of S, let

$$e_1 \star J' = e_1 S \cap J', \quad e_1 \star J' \star e_2 = e_1 S e_2 \cap J'$$

Thus for instance, $R = e \star J$.

Lemma 2. *Let $e_0 \in E(S), e_0 < e, J_0$ the $\mathscr{J}$-class of $e_0, H_0 = e_0 C_H(e_0)$ the $\mathscr{H}$-class of e_0. Then:*

(i) $e \star J_0 = (He_0)(e_0 \star J_0)$.
(ii) *dim* $e \star J_0$= *dim* He_0+ *dim* $e_0 \star J_0$− *dim* H_0.
(iii) *If $f_0 \in E(J_0)$, then dim eSf_0 = dim He_0 and dim e_0Sf_0 = dim H_0.*
(iv) $e_0 \star J_0 = e_0 R$.

Proof. He_0 is the $\mathscr{L}$-class of e_0 in the algebraic monoid eSe and $e_0 \star J_0$ is the $\mathscr{R}$-class of e_0 in S. Hence $(He_0)(e_0 \star J_0) \subseteq e \star J_0$. Let $a \in e \star J_0$. Then $a \in J_0, ea = a$. So $e_0 \mathscr{L} b \mathscr{R} a, e_0 \mathscr{R} c \mathscr{L} a$ for some $b, c \in J_0$. Since $ea = a, eb = b$. So $b \in eSe$ and $e_0 \mathscr{L} b$ in eSe. Hence $b \in He_0$. Clearly $c \in e_0 \star J_0$. By the Rees Theorem for J_0,

$$\mathscr{H}\text{-class of } a = (\mathscr{H}\text{-class of } b)(\mathscr{H}\text{-class of } c) \tag{20}$$

Hence $a \in (He_0)(e_0 \star J_0)$. This proves (i). Further we have the product map $p : He_0 \times e_0 \star J_0 \longrightarrow e \star J_0$. By (20), the inverse image of any $\mathscr{H}$-class in $e \star J_0$ is of the form $H_1 \times H_2$, where H_1 is an $\mathscr{H}$-class in He_0 and H_2 is an $\mathscr{H}$-class in $e_0 \star J_0$. The dimension theorem [7, Theorem 4.1] then yields (ii).

(iii) $e \star J_0 \star f_0 = J_0 \cap eSf_0$ is an open dense subset of eSf_0. Now $e_0 \mathscr{R} a \mathscr{L} f_0$ for some $a \in J_0$. Now $e_0 \star J_0 \star f_0$ is the $\mathscr{H}$-class of a and is a dense open subset of e_0Sf_0. Hence

$$\dim eSf_0 = \dim e \star J_0 \star f_0, \ \ \dim e_0Sf_0 = \dim e_0 \star J_0 \star f_0 = \dim H_0$$

Now $ab = e_0$ for some $b \in J_0$. Then $aba = a$. By (2), the map: $x \rightarrow xa$ is a bijective map from the $\mathscr{L}$-class of e_0 in S to the $\mathscr{L}$-class of a in S with inverse map: $y \rightarrow yb$. These bijections restrict to bijective morphisms between He_0 and $e \star J_0 \star f_0$. Hence the two varieties have the same dimension.

(iv) First assume that e covers e_0. Then $J \cap eSe$ covers $J_0 \cap eSe$ in the irreducible algebraic monoid eSe and hence J covers J_0 in S. Define $\theta : eS \longrightarrow e_0S$ as, $\theta(x) = e_0x$. Let $a \in e_0 \star J_0$. Then $e_0 \mathscr{R} a$. Now for some $f_0 \in E(J_0), a \mathscr{L} f_0$. Then $e_0 \star J_0 \star f_0$ is the $\mathscr{H}$-class of a and hence

$$e_0 \star J_0 \star f_0 = H_0 a = e_0 C_H(e_0) a = C_H(e_0) e_0 a = C_H(e_0) a \tag{21}$$

Now $\theta(eSf_0) = e_0Sf_0$. Let X be an irreducible component of $\theta^{-1}(e_0Sf_0)$ containing eSf_0. Suppose $X \cap J = \emptyset$. Then $X \subseteq \overline{J'}$ for some $\mathscr{J}$-class J' of S, $J' \neq J$. Since $a \in J_0 \cap X$, $J_0 \leq J' < J$. Since J covers J_0, $J' = J_0$. Hence $X = \overline{J_0}$. Now $O = \{x \in X \mid e_0x \in J_0\} \subseteq J_0$ is an open subset of X and $a \in O$. Let $x \in O$. Then $e_0x \in J_0$ and $e_0 \mathscr{R} e_0 x \mathscr{L} x$. Then $e_0x = \theta(x) \in e_0Sf_0$. So $x \in Se_0x \subseteq Sf_0$. Hence $O \subseteq Sf_0$. Hence $X = \overline{O} \subseteq Sf_0$. Thus $X = eSf_0$ By the dimension theorem, cf. [7, Theorem 4.1], $\dim X \geq \dim eS - \dim e_0S + \dim e_0Sf_0$. So

$$\begin{aligned}\dim eS &\leq \dim X + \dim e_0S - \dim e_0Sf_0\\ &= \dim He_0 + \dim e_0 \star J_0 - \dim H_0, \text{ by } (iii)\\ &= \dim e \star J_0, \text{ by } (ii)\end{aligned}$$

So $eS = e\overline{J_0}$, a contradiction. So $X \cap J \neq \emptyset$. Hence $X = \overline{X \cap R}$. So $\theta(X \cap R)$ is dense in e_0Sf_0. Hence for some $z \in X \cap R$, $e_0z = \theta(z) \in e_0 \star J_0 \star f_0 = C_H(e_0)a$, by (21). So $e_0z = ga$ for some $h \in C_H(e_0)$. So $a = e_0(h^{-1}z)$ and $h^{-1}z \in HR = R$.

Now assume that e does not cover e_0. Let

$$e_0 < e_1 < \cdots < e_t = e$$

be a maximal chain of idempotents from e_0 to e. Let J_i denote the $\mathscr{J}$-class of $e_i, i = 0, \cdots, t$. We prove by induction on t, the length of any maximal chain of idempotents from e to e_0. Let $a \in e_0 \star J_0$. Considering the monoid $\overline{J_1}$, we see by above that there exists $a_1 \in e_1 \star J_1$, such that $e_0a_1 = a$. By the induction hypothesis, there exists $a_t \in R$, such that $e_1a_t = a_1$. Then $e_0a_t = e_0e_1a_t = e_0a_1 = a$. So $e_0R = e_0 \star J_0$, completing the proof. □

Theorem 6. *Let S be an irreducible regular linear algebraic semigroup. Then $eS = eSe \cdot R, Se = L \cdot eSe$ and $S = L \cdot eSe \cdot R$.*

Proof. Let $a \in eS$. Let J_0 denote the $\mathscr{J}$-class of a. Then $a\mathscr{R}e_0'$ for some $e_0' \in E(J_0)$. Since, $ea = a, ee_0' = e_0'$. Let $e_0 = e_0'e \in E(J_0)$. Then $e_0' \mathscr{R} e_0 \leq e$. Then $a\mathscr{R}e_0$. So by Lemma 2 (iv), $a \in e_0R \subseteq eSeR$. Hence $eS = eSe \cdot R$. By duality $Se = L \cdot eSe$. Hence $S = Se \cdot eS = L \cdot eSe \cdot R$. This completes the proof. □

Remark 3. Note that eSe is an irreducible regular linear algebraic monoid, which if $0 \in S$, is reductive monoid. Further $\tilde{S} = L \times M \times R$ is a Rees matrix semigroup over M with the product sandwich map : $R \times L \longrightarrow M$. Clearly $\tilde{S}$ maps homomorphically onto S. We note that $\tilde{S}$ is an irreducible algebraic semigroup in the sense of Brion and Rittatore, cf. [2]. It is in general not a linear algebraic semigroup since R and L need not be affine. We further note that the construction of $\tilde{S}$ fits in well with the program of John Rhodes [16] of finding Rees matrix covers for semigroups. If J is completely regular, then we can replace L by $E(L)$ and R

by $E(R)$. In general, trimming down L and R will result in reducible sets, such as in (15).

Theorem 7. *Suppose S is completely regular. Let $\tilde{S} = E(L) \times eSe \times E(R)$ be the Rees matrix semigroup over the completely regular monoid eSe with sandwich map being the product map from $E(R) \times E(L)$ to H. Then $\tilde{S}$ is a completely regular irreducible linear algebraic semigroup and the product map φ from $\tilde{S}$ to S is a surjective homomorphism that is finite to one.*

Proof. We only need to show that φ is finite to one. By [10, Theorem 3.7], for all $f \in E(S)$,

$$\{f' \in E(S) \mid f' \geq f\} \text{ is finite} \tag{22}$$

Since S is completely regular, φ preserves $\mathscr{J}$-classes. Fix $a \in S$. Then $a = e_1'be_1$ for some $e_1' \in E(L)$ and $e_1 \in E(R)$. Then $b = eae$. Now $b\mathscr{H}h$ for some $h \in E(eSe)$. Then

$$he_1' = (he)e_1' = he = h = eh = e_1(eh) = e_1h$$

So $h_1' = e_1'h, h_1 = he_1 \in E(S)$. So

$$ea = be_1 = bh_1,\ ae = e_1'b = h_1'b \tag{23}$$

Let $e_2 \in E(R), e_2' \in E(L)$. such that $a = e_2'be_2$. Then by (23), $bh_1 = be_2$. So $h_1 = he_2$ and hence $h_1e_2 = h_1$. Also $e_2h_1 = e_2(eh_1) = eh_1 = h_1$. Hence $h_1 \leq e_2$. Similarly $h_1' \leq e_2'$. So by (22), the number of possible e_2 and e_2' is finite. This completes the proof. □

Remark 4. (1) We note that Renner [14] has classified completely regular algebraic monoids with solvable unit groups.

(2) Let M be an irreducible completely regular linear algebraic monoid with unit group H. Let X, Y be a irreducible affine varieties and $p : X \times Y \longrightarrow H$ any morphism. Then $Y \times M \times X$ is an irreducible completely regular linear algebraic semigroup, if we define:

$$(y, a, x) \cdot (y', a', x') = (y, ap(x, y')a', x')$$

$\tilde{S}$ in Theorem 7 is such an example of a Rees matrix semigroup over a completely regular monoid.

(3) It is natural to wonder about the homomorphism φ in Theorem 7. Is it always an isomorphism of algebraic semigroups? If not, is it a finite morphism? Michel Brion points out that if S is a normal variety, then φ is indeed an isomorphism.

We now determine all closed irreducible regular subsemigroups of S, having the same support as S.

Theorem 8. *Let $R_0 = \langle E(J)\rangle \cap R, L_0 = \langle E(J)\rangle \cap L, H_0 = \langle E(J)\rangle \cap H$. Then there is a 1-1 correspondence between the closed irreducible regular subsemigroups of S containing $E(J)$ and the closed connected subgroups H_1 of H containing H_0 such that $\overline{H_1}$ is a regular monoid. The semigroup associated with H_1 is $S_{H_1} = L_0\overline{H_1}R_0$.*

Proof. Let $S_0 = \overline{\langle E(J)\rangle}$. Then by Theorem 4, R_0, L_0 and H_0 are respectively the $\mathscr{R}, \mathscr{L}$ and $\mathscr{H}$-classes of e in S_0, and $J_0 = \langle E(J)\rangle \cap J$ is the $\mathscr{J}$-class of e in S_0. Let H_1 be a closed connected subgroup of H containing H_0 such that $\overline{H_1}$ is a regular monoid. Let $S_1 = S_{H_1} = L_0\overline{H_1}R_0$. Since $R_0L_0 \subseteq e_0S_0e = \overline{H_0} \subseteq \overline{H_1}$, we see that S_1 is a subsemigroup of S. Let

$$S_1' = \{a \in S \mid R_0aL_0 \subseteq \overline{H_1}\}$$

Then clearly $S_1 \subseteq S_1'$. Let $a \in S_1'$. By Theorems 4 and 6, $S = L_0eSeR_0$. So $a = lbr$ for some $l \in L_0, r \in R_0, b \in eSe$. By (4) applied to J_0, there exist $r' \in R_0$ and $l' \in L_0$ such that $h_1 = r'l, h_2 = rl' \in H_0$. Since $a \in S_1'$,

$$b = ebe = h_1^{-1}h_1bh_2h_2^{-1} = h_1^{-1}r'lbrl'h_2^{-1} = h_1^{-1}r'al'h_2^{-1} \in h_1^{-1}\overline{H_1}h_2^{-1} \subseteq H_0\overline{H_1}H_0 = \overline{H_1}$$

Hence $a = lbr \in L_0\overline{H_1}R_0 = S_1$. Thus $S_1 = S_1'$ is closed. Also clearly $S_1 = \overline{L_0H_1R_0}$ is irreducible. Let $a \in S_1$. Then $a = lbr$ for some $l \in L_0, r \in R_0, b \in \overline{H_1}$. By (4), there exist $r' \in R_0$ and $l' \in L_0$ such that $h_1 = r'l, h_2 = rl' \in H_0$. So $a \mathscr{J} h_1bh_2$ in S_1. Since $h_1bh_2 \in \overline{H_1}$ and $\overline{H_1}$ is regular, we see that S_1 is regular.

Assume conversely that S_1 is a closed irreducible regular subsemigroup of S containing $E(J)$. Let H_1 denote the core of S_1. Then H_1 is a closed connected subgroup of H containing H_0 and $\overline{H_1} = eS_1e$ is a regular monoid. So by Theorems 4 and 6, $S_1 = S_{H_1}$. This completes the proof. □

Example 4. Let

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; a, b, c, d \in k, a \neq 0 \right\},$$

which is an irreducible completely regular linear algebraic semigroup with

$$J = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; ad \neq 0 \right\}, E(J) = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; c \in k \right\}, H \cong \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \; ad \neq 0 \right\}$$

We list below the proper closed irreducible subsemigroups of S, not contained in J and containing $E(J)$:

$$H_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \middle| \, d \neq 0 \right\}, \quad S_{H_1} = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, c, d \in k \right\}$$

$$H_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| \, ad \neq 0 \right\}, \quad S_{H_2} = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a \neq 0 \right\}$$

$$H_3 = \left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \middle| \, d \neq 0 \right\}, \quad S_{H_3} = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, b, c, d \in k \right\}.$$

We want to classify the closed irreducible regular subsemigroups of $M_n(k)$.

Theorem 9. *Let S, S' be closed irreducible regular subsemigroups of $M_n(k)$, having the same core and support. Then $S = S'$.*

Proof. Let J, J' be the respective top $\mathcal{J}$-classes of S and S'. Since J and J' have the same support, we see by Theorem 2(vi), that $E(J) = E(J')$. Let $e \in E(J)$. Then S, S' have the same $\mathcal{H}$-class H of e. Let R_0, L_0 denote the respectively the $\mathcal{R}$ and $\mathcal{L}$-classes of e in $\overline{\langle E(J) \rangle}$. Then by Theorems 4 and 6, $S = L_0 \overline{H} R_0 = S'$. This completes the proof. □

Here is the problem going forward. Given a possible support $\mathbb{X}$, we can by (13), construct $E(J)$ and hence L_0, R_0 and H_0. Now given a potential core H containing H_0 and with $\overline{H}$ regular, how do know if S exists? The problem is that $S = L_0 H R_0$ may or may not be closed in $M_n(k)$. See for instance, Example 2. We will see that this problem does not arise when the support $\mathbb{X}$ is projective.

5 Projective Support

Let $M = M_n(k), G = GL_n(k), m \leq n, e = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, M_0 = eMe$. We follow the notation of Sect. 2. Let $\mathbb{X}_r \subseteq \mathbb{G}r(m)$ and $\mathbb{X}_l \subseteq \mathbb{G}r^*(m)$ be closed non-empty irreducible subsets and let $\mathbb{X} = \mathbb{X}_l \times \mathbb{X}_r$. Let

$$O_{\mathbb{X}} = \mathbb{X} \cap O(m) = \{([b], [a]) \in \mathbb{X} \mid \det ab \neq 0\}$$

and let $E_{\mathbb{X}} = \mathfrak{e}(O_{\mathbb{X}})$. Then by (13), $\mathfrak{e} : O_{\mathbb{X}} \longrightarrow E_{\mathbb{X}}$ is an isomorphism of varieties. $O_{\mathbb{X}}$ is closed in $O(m)$ and hence $E_{\mathbb{X}}$ is closed in $E(J(m))$. $O_{\mathbb{X}}$ is an open subset of the irreducible variety $\mathbb{X}$ and is hence irreducible. Thus $E_{\mathbb{X}}$ is irreducible. We may assume without loss of generality that $e \in E_{\mathbb{X}}$. We will assume that $\mathbb{X}$ satisfies the *non-degeneracy condition*:

$$[a] \in \mathbb{X}_r \Longrightarrow \det ab \neq 0 \text{ for some } [b] \in \mathbb{X}_l; \ [b] \in \mathbb{X}_l \Longrightarrow \det ab \neq 0 \text{ for some } [a] \in \mathbb{X}_r \quad (24)$$

By (4), this is a necessary condition for $\mathbb{X}$ to be the support of a semigroup. We will prove that this is also sufficient. Consider the morphisms, (6) and (10), $\lambda_r(m) : R(m) \longrightarrow \mathbb{G}r(m), \lambda_l(m) : L(m) \longrightarrow \mathbb{G}r^*(m)$ and let:

$$\tilde{R} = \lambda_r^{-1}(m)(\mathbb{X}_r), \quad \tilde{L} = \lambda_l^{-1}(m)(\mathbb{X}_l)$$

Then $\tilde{R}, \tilde{L}$ are irreducible, $H(m) = GL_m(k)$ acts on the left on $\tilde{R}$ and on the right on $\tilde{L}$. Our first task is to show that

$$\tilde{S} = \tilde{S}_{\mathbb{X}} = \tilde{L}M_0\tilde{R} \quad (25)$$

is closed in M. Let $P = P(e)$, $P^- = P^-(e)$ be as in (5). Now

$$X = \{x \in G \mid ex \in \tilde{R}\}$$

is closed in G and $P^-X \subseteq X$. Let G/P^- denote the projective variety of right cosets of P^- in G. Now

$$Z = \{(a, x) \mid a \in eM, x \in X, ax^{-1} \in M_0\}$$

is closed in $M \times G$. If $(a, x) \in Z$ and $q \in P^-$, then $qx \in X$ and

$$a(qx)^{-1} = ax^{-1}q^{-1} \in M_0q^{-1} = M_0eq^{-1} = M_0eq^{-1}e \subseteq M_0$$

So $(a, qx) \in Z$. We can view $M \times G/P^-$ as $((M, +) \times G)/(\{0\} \times P^-)$, and hence the natural map from $M \times G$ to $M \times G/P^-$ is open. It follows that the image of Z in $M \times G/P^-$,

$$\tilde{Z} = \{(a, P^-x \mid a \in eM, x \in X, ax^{-1} \in M_0\}$$

is closed in $M \times G/P^-$. Since G/P^- is a projective variety, we see that the projection of $\tilde{Z}$ in M,

$$M_1 = \{a \in eM \mid ax^{-1} \in M_0 \text{ for some } x \in X\}$$

is closed in M. Let $a \in M_1$. Then for some $x \in X, a \in M_0x = M_0ex \subseteq M_0\tilde{R}$. Hence $M_1 \subseteq M_0\tilde{R}$. Conversely, let $a \in M_0\tilde{R}$. Then $a \in M_0r$ for some $r \in \tilde{R}$. Now $r = ex$ for some $x \in X$. Hence $ax^{-1} \in M_0rx^{-1} = M_0e = M_0$. So $a \in M_1$. Thus $M_1 = M_0\tilde{R}$ is closed in M.

Now

$$Y = \{y \in G \mid ye \in L\}$$

is closed in G and $YP \subseteq Y$. Let G/P denote the projective variety of left cosets P in G. Now

$$A = \{(a, y) \mid a \in M, y \in Y, y^{-1}a \in M_1\}$$

is closed in $M \times G$. Let $(a, y) \in A$ and $p \in P$, then $yp \in Y$ and

$$(yp)^{-1}a = p^{-1}y^{-1}a \in p^{-1}M_0\tilde{R} = p^{-1}eM_0\tilde{R} = ep^{-1}eM_0\tilde{R} \subseteq M_0\tilde{R} = M_1$$

Hence $(a, py) \in A$. Since the natural map from $M \times G$ to $M \times G/P$ is open, it follows that the image of A in $M \times G/P$,

$$\tilde{A} = \{(a, yP) \mid a \in M, y \in Y, y^{-1}a \in M_1\}$$

is closed in $M \times G/P$. Since G/P is a projective variety, we see that the projection of $\tilde{A}$ in M,

$$S_1 = \{a \in M \mid y^{-1}a \in M_1 \text{ for some } y \in Y\}$$

is closed in M. If $a \in S_1$, then $y^{-1}a \in M_1$ for some $y \in Y$. So

$$a \in yM_1 = yM_0\tilde{R} = yeM_0R \subseteq \tilde{L}M_0\tilde{R} = \tilde{S}$$

Conversely let $a \in \tilde{S}$. Then for some $l \in \tilde{L}, a \in lM_0\tilde{R} = lM_1$. Now $l = ye$ for some $y \in Y$. So $y^{-1}a \in eM_0\tilde{R} = M_1$. Hence $a \in S_1$ and $\tilde{S} = S_1$ is closed M.

Since $\tilde{R}\tilde{L} \subseteq M_0$, we see that $\tilde{S}$ is a closed subsemigroup of M. Since $\tilde{R}$ and $\tilde{L}$ are irreducible, $\tilde{S}$ is irreducible. Let $a \in \tilde{S}$. Then $a = lbr$ for some $l \in \tilde{L}, r \in \tilde{R}, b \in M_0$. By the non-degenracy condition (24), there exist $r' \in \tilde{R}, l' \in \tilde{L}$ such that $r'l, lr' \in H(m)$. It follows that $a \mathscr{J} b$ in $\tilde{S}$. Hence $\tilde{S}$ is an irreducible regular linear algebraic semigroup with top $\mathscr{J}$-class $\tilde{J} = \tilde{L}\tilde{R}$ and $E(\tilde{J}) = E_{\mathbb{X}}$. Let

$$H_{\mathbb{X}} = H(m) \cap \langle E_{\mathbb{X}} \rangle, \ R_{\mathbb{X}} = \tilde{R} \cap \langle E_{\mathbb{X}} \rangle, \ L_{\mathbb{X}} = \tilde{L} \cap \langle E_{\mathbb{X}} \rangle \tag{26}$$

By Theorems 4 and 8, we now have our main result that determines all closed irreducible regular subsemigroups of $M_n(k)$ with maximum $\mathscr{J}$-class of rank m.

Theorem 10. *There is a 1-1 correspondence between the closed irreducible regular subsemigroups of subsemigroups of $M_n(k)$ with support $\mathbb{X}$ and the closed connected subgroups H of $GL_m(k)$ containing $H_{\mathbb{X}}$ such that $\overline{H}$ is a regular monoid. The semigroup associated with H is $S_H = L_{\mathbb{X}}\overline{H}R_{\mathbb{X}}$.*

When $\mathbb{X}$ is degenerate, $\tilde{S}_{\mathbb{X}}$ is still an irreducible linear algebraic semigroup, but it just will not be regular, and of course not have $\mathbb{X}$ as its support.

Example 5. In $M_3(k)$, let $m = 1$ and let:

$$\mathbb{X}_r = \{[x, y, z] \mid y = 0]\}, \quad \mathbb{X}_l = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = 0 \right\}$$

Then $\mathbb{X} = \mathbb{X}_l \times \mathbb{X}_r$ is degenerate. Accordingly

$$\tilde{S}_{\mathbb{X}} = \left\{ \begin{pmatrix} a & 0 & b \\ c & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \middle| ad = bc \right\}$$

is not regular. The only non-zero regular $\mathscr{J}$-class of $\tilde{S}_{\mathbb{X}}$ is:

$$\tilde{J} = \left\{ \begin{pmatrix} a & 0 & b \\ c & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \middle| a \neq 0,\ ad = bc \right\}$$

which is completely simple. Hence $\mathbb{X}(S) = \mathbb{X}(J)$ is affine and not equal to $\mathbb{X}$.

Example 6. In $M_3(k)$, let $m = 1$ and let:

$$\mathbb{X}_r = \{[x, y, z] \mid z^2 = xy]\}, \quad \mathbb{X}_l = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = 0 \right\}$$

Then $\mathbb{X}$ is non-degenerate and

$$\tilde{S}_{\mathbb{X}} = \left\{ \begin{pmatrix} a & b & c \\ a' & b' & c' \\ 0 & 0 & 0 \end{pmatrix} \middle| c^2 = ab, c'^2 = a'b', ab' = ba' = cc', ac' = ca', bc' = cb' \right\}$$

is the only irreducible regular semigroup in $M_3(k)$ with support $\mathbb{X}$.

Example 7. In $M_3(k)$, let $m = 1$ and let $\mathbb{X}_r = \{[x, y, z] \mid z^2 = xy]\}, \mathbb{X}_l = \mathbb{G}r^*(1)$. Then again $\mathbb{X} = \mathbb{X}_l \times \mathbb{X}_r$ is non-degenerate. So again $\tilde{S}_{\mathbb{X}}$ is the only irreducible regular linear algebraic semigroup with support $\mathbb{X}$. Note also that dim $\mathbb{X}_r = 1$, while dim $\mathbb{X}_l = 2$, a phenomenon that does not occur in irreducible regular linear algebraic monoids.

Example 8. In $M_3(k)$, let $m = 2$. Let $\mathbb{X}_r \subseteq \mathbb{G}r(2), \mathbb{X}_l \subseteq \mathbb{G}r^*(2)$ consist of the planes containing the x-axis:

$$\mathbb{X}_r = \left\{ \begin{bmatrix} x & y & z \\ x' & y' & z' \end{bmatrix} \in \mathbb{G}r(2) \,\middle|\, yz' = y'z \right\}, \quad \mathbb{X}_l = \left\{ \begin{bmatrix} x & x' \\ y & y' \\ z & z' \end{bmatrix} \in \mathbb{G}r^*(2) \,\middle|\, yz' = y'z \right\}$$

It is easily verified that $\mathbb{X} = \mathbb{X}_l \times \mathbb{X}_r$ is non-degenerate. Accordingly

$$\tilde{S}_{\mathbb{X}} = \left\{ \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \,\middle|\, bc' = b'c, bc'' = b''c, b'c'' = b''c', a'b'' = b'a'', a'c'' = c'a'' \right\}$$

is an irreducible regular linear algebraic semigroup of matrices with second and third rows linearly dependent, and the second and third columns linearly dependent. In $H(2) = GL_2(k)$, we compute:

$$H_{\mathbb{X}} = \langle E_{\mathbb{X}} \rangle \cap H(2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \,\middle|\, \alpha \neq 0 \right\},$$

$$R_{\mathbb{X}} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \end{pmatrix} \,\middle|\, \alpha \neq 0 \text{ or } \beta \neq 0 \right\}, \quad L_{\mathbb{X}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \alpha \\ 0 & \beta \end{pmatrix} \,\middle|\, \alpha \neq 0 \text{ or } \beta \neq 0 \right\},$$

and

$$S_{\mathbb{X}} = L_{\mathbb{X}} H_{\mathbb{X}} R_{\mathbb{X}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \,\middle|\, \det A = 0 \right\}$$

There are three closed connected subgroups between $H(2)$ and $H_{\mathbb{X}}$ that have regular closures. The first is:

$$H_1 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \,\middle|\, \alpha\beta \neq 0 \right\}$$

with

$$S_1 = S_{H_1} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & A \end{pmatrix} \,\middle|\, \alpha \in k, \ \det A = 0 \right\}$$

The second is:

$$H_2 = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \,\middle|\, \beta \neq 0 \right\}$$

with

$$S_2 = S_{H_2} = L_{\mathbb{X}}\overline{H_2}R_{\mathbb{X}} = \left\{ \begin{pmatrix} 1 & a & a' \\ 0 & b & b' \\ 0 & c & c' \end{pmatrix} \middle| \; ab' = ba', ac' = ca', bc' = cb' \right\}$$

The third is:

$$H_3 = \left\{ \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \middle| \; \beta \neq 0 \right\}$$

with

$$S_3 = S_{H_3} = L_{\mathbb{X}}\overline{H_3}R_{\mathbb{X}} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & b & c \\ a' & b' & c' \end{pmatrix} \middle| \; ab' = ba', ac' = ca', bc' = cb' \right\}$$

We note that S_2 has only two $\mathscr{J}$-classes with the bottom $\mathscr{J}$-class being:

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; a, b \in k \right\}$$

which is completely simple and has affine support. Similarly S_3 has only two $\mathscr{J}$-classes with the bottom $\mathscr{J}$-class being:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \middle| \; a, b \in k \right\}$$

which is also completely simple and has affine support.

We close with an open problem:

Problem 2. Is $\overline{H_{\mathbb{X}}}$ always regular? Is $H_{\mathbb{X}}$ always equal to $\langle E(\tilde{R})E(\tilde{L}) \cap H(m) \rangle$?

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Representations of Reductive Normal Algebraic Monoids

Stephen Doty

Dedicated to Lex Renner and Mohan Putcha

Abstract The rational representation theory of a reductive normal algebraic monoid (with one-dimensional center) forms a highest weight category, in the sense of Cline, Parshall, and Scott. This is a fundamental fact about the representation theory of reductive normal algebraic monoids. We survey how this result was obtained, and treat some natural examples coming from classical groups.

Keywords Algebraic monoids • Normal • Representation theory

Subject Classifications: 20M32, 20G05, 16T10

1 Introduction

Let M be an affine algebraic monoid over an algebraically closed field K. See [9, 11, 12] for general surveys and background on algebraic monoids. Assuming that M is reductive (its group G of units is a reductive group) what can be said about the representation theory of M over K?

Recall that any affine algebraic group is smooth and hence normal (as a variety). The normality of the algebraic group plays a significant role in its representation theory, for instance in the proof of Chevalley's theorem classifying the irreducible representations. Thus it seems reasonable in trying to extend (rational)

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representation theory from reductive groups to reductive monoids to look first at the case when the monoid M is normal. Furthermore, even in cases where a given reductive algebraic monoid is not normal, one may always pass to its normalization, which should be closely related to the original object.

Renner [10] has obtained a classification theorem for reductive normal algebraic monoids under the additional assumptions that the center $Z(M)$ is 1-dimensional and that M has a zero element. Renner's classification theorem depends on an algebraic monoid version of Chevalley's big cell, which holds for any reductive affine algebraic monoid (with no assumptions about its center or a zero). As a corollary of its construction, Renner derives a very useful extension principle [10, (4.5)] which is a key ingredient in the analysis.

2 Reductive Normal Algebraic Monoids

Let M be a *linear algebraic monoid* over an algebraically closed field K. In other words, M is a monoid (with unit element $1 \in M$) which is also an affine algebraic variety over K, such that the multiplication map $\mu : M \times M \to M$ is a morphism of varieties. We assume that M is irreducible as a variety. Hence the unit group $G = M^\times$ (the subgroup of invertible elements of M) is a connected linear algebraic group over K and G is Zariski dense in M.

Associated with M is its affine coordinate algebra $K[M]$, the ring of regular functions on M. There exist K-algebra homomorphisms

$$\Delta : K[M] \to K[M] \otimes_K K[M], \quad \varepsilon : K[M] \to K$$

called comultiplication and counit, respectively. For a given $f \in K[M]$, we have $\varepsilon(f) = f(1)$; furthermore, if $\Delta(f) = \sum_{i=1}^r f_i \otimes f_i'$ then $f(m_1 m_2) = \sum_{i=1}^r f_i(m_1) f_i'(m_2)$, for all $m_1, m_2 \in M$. The maps Δ, ε make $K[M]$ into a bialgebra over K. This means that they satisfy the bialgebra axioms:

1. $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$,
2. $(\varepsilon \mathbin{\overline{\otimes}} id) \circ \Delta = id = (id \mathbin{\overline{\otimes}} \varepsilon) \circ \Delta$

where $\varphi \mathbin{\overline{\otimes}} \varphi'$ denotes the map $a \otimes a' \mapsto \varphi(a)\varphi'(a')$.

We note that the commutative bialgebra $(K[M], \Delta, \varepsilon)$ determines M, as the set $\mathrm{Hom}_{K-\mathrm{alg}}(K[M], K)$ of K-algebra homomorphisms from $K[M]$ into K. The multiplication on this set is defined by $\varphi \cdot \varphi' = (\varphi \mathbin{\overline{\otimes}} \varphi') \circ \Delta$ and the identity element is just the counit ε. One easily verifies that this reconstructs M from its coordinate bialgebra $K[M]$.

More generally, given any commutative bialgebra (A, Δ, ε) over K, one defines on the set $M(A) = \mathrm{Hom}_{K-\mathrm{alg}}(A, K)$ an algebraic monoid structure with multiplication $\mu(\varphi, \varphi') = \varphi \cdot \varphi' = (\varphi \mathbin{\overline{\otimes}} \varphi') \circ \Delta$. This gives a functor

$$\{\text{commutative bialgebras over } K\} \to \{\text{algebraic monoids over } K\}$$

which is quasi-inverse to the functor $M \mapsto K[M]$. Thus the two categories are antiequivalent.

Since G is dense in M, the restriction map $K[M] \to K[G]$ (given by $f \mapsto f_{|G}$) is injective, so we may identify $K[M]$ with a subbialgebra of the Hopf algebra $K[G]$ of regular functions on G.

Assume that M is reductive; i.e., its unit group $G = M^\times$ is reductive as an algebraic group. Fix a maximal torus T in G. (Up to conjugation T is unique.) Let $X(T) = \mathrm{Hom}(T, K^\times)$ be the character group of T; this is the abelian group of morphisms from T into the multiplicative group $K^\times$ of K. Let $X^\vee(T) = \mathrm{Hom}(K^\times, T)$ be the abelian group of cocharacters into T. Let $R \subset X(T)$ be the root system for the pair (G, T) and $R^\vee \subset X^\vee(T)$ the system of coroots. According to the classification of reductive algebraic groups, the reductive group G is uniquely determined up to isomorphism by its root datum $(X(T), R, X^\vee(T), R^\vee)$.

We now add the assumption that M is *normal* as a variety. Let $D = \overline{T}$ be the Zariski closure of T in M. Then $T \subset D$ is an affine torus embedding. Let $X(D) = \mathrm{Hom}(D, K)$ be the monoid of algebraic monoid homomorphisms from D into K. The restriction $\chi_{|T}$ of any $\chi \in X(D)$ is an element of $X(T)$, so restriction defines a homomorphism $X(D) \to X(T)$. Since T is dense in D, this map is injective, and thus we may identify $X(D)$ with a submonoid of $X(T)$. Renner has shown that the additional datum $X(D)$ is all that is needed to determine M up to isomorphism, under the additional hypotheses (probably unnecessary) that the center

$$Z(M) = \{z \in M : zm = mz, \text{ for all } m \in M\}$$

is 1-dimensional and that M has a zero element. (One can always add a zero formally, so the last requirement is insubstantial.)

It turns out that the set $X(D)$ also determines the rational representation theory of the reductive normal algebraic monoid M, in a sense made precise in Sect. 3.

Note that it is easy to construct reductive algebraic monoids. Start with a rational representation $\rho : G \to \mathrm{End}_K(V)$ of a reductive group G in some vector space V with $\dim_K V = n < \infty$. The image $\rho(G)$ is a reductive affine algebraic subgroup of $\mathrm{End}_K(V) \simeq \mathrm{M}_n(K)$, the monoid of all $n \times n$ matrices under ordinary matrix multiplication. Desiring our monoid to have a center of at least dimension 1, we include the scalars $K^\times$ as scalar matrices, defining G_0 to be the subgroup of $\mathrm{End}_K(V)$ generated by $\rho(G)$ and $K^\times$. Now we set $M = \overline{G_0}$, the Zariski closure of G_0 in $\mathrm{End}_K(V) \simeq M_n(K)$. This is a reductive algebraic monoid.

For example, if $G = \mathrm{SL}_n(K)$ and V is its natural representation then $G_0 \simeq \mathrm{GL}_n(K)$ and $M = \mathrm{M}_n(K)$. (In general, to obtain a monoid M closely related to the starting group G, one should pick V to be a faithful representation.) There is no guarantee that this procedure will always produce a normal reductive monoid, but if not then one can always pass to its normalization.

3 Examples: Symplectic and Orthogonal Monoids

The paper [4] considered some more substantial examples of reductive algebraic monoids coming from other classical groups. Let $V = K^n$ with its standard basis $\{e_1, \dots, e_n\}$. Put $i' = n + 1 - i$ for any $i = 1, \dots, n$.

3.1 The Orthogonal Monoid

Assume the characteristic of K is not 2. Define a symmetric nondegenerate bilinear form $\langle\ ,\ \rangle$ on V by putting

$$\langle e_i, e_j \rangle = \delta_{i,j'} \qquad \text{for any } 1 \le i, j \le n. \tag{a}$$

Here δ is Kronecker's delta function. Let J be the matrix of $\langle\ ,\ \rangle$ with respect to the basis $\{e_1, \dots, e_n\}$. Then the orthogonal group $\mathrm{O}(V)$ is the group of linear operators $f \in \mathrm{End}_K(V)$ preserving the form:

$$\mathrm{O}(V) = \{f \in \mathrm{End}_K(V) : \langle f(v), f(v') \rangle = \langle v, v' \rangle, \text{ all } v, v' \in V\}. \tag{b}$$

Let A be the matrix of f with respect to the basis $\{e_1, \dots, e_n\}$. Then we may identify $\mathrm{O}(V)$ with the matrix group

$$\mathrm{O}_n(K) = \{A \in \mathrm{M}_n(K) : A^{\mathsf{T}} J A = J\}. \tag{c}$$

This is contained in the larger group $\mathrm{GO}_n(K)$, the group of orthogonal similitudes (see e.g., [8]) defined by

$$\mathrm{GO}_n(K) = \{A \in \mathrm{M}_n(K) : A^{\mathsf{T}} J A = cJ, \text{ some } c \in K^{\times}\}. \tag{d}$$

Note that $\mathrm{GO}_n(K)$ is generated by $\mathrm{O}_n(K)$ and $K^{\times}$. We define the *orthogonal monoid* $\mathrm{OM}_n(K)$ to be

$$\mathrm{OM}_n(K) = \overline{\mathrm{GO}_n(K)}, \tag{e}$$

the Zariski closure in $\mathrm{M}_n(K)$. These monoids (for n odd) were studied by Grigor'ev [7]. In [4] the following result was proved.

Proposition 1. *The orthogonal monoid* $\mathrm{OM}_n(K)$ *is the set of all* $A \in \mathrm{M}_n(K)$ *such that* $A^{\mathsf{T}} J A = cJ = AJA^{\mathsf{T}}$*, for some* $c \in K$.

Note that the scalar $c \in K$ in the above is allowed to be zero, and the "extra" condition $cJ = AJA^{\mathsf{T}}$ is necessary. If $c \neq 0$ then it is easy to see that $A^{\mathsf{T}} J A = cJ$ is equivalent to $cJ = AJA^{\mathsf{T}}$, but when $c = 0$ this equivalence fails. The equivalence means that we could just as well have defined $\mathrm{GO}_n(K)$ by

$$\mathrm{GO}_n(K) = \{A \in \mathrm{M}_n(K) : A^{\mathsf{T}} J A = cJ = AJA^{\mathsf{T}},\ \text{some } c \in K^{\times}\}$$

which is perhaps more suggestive for the description of $\mathrm{OM}_n(K)$ given above.

3.2 The Symplectic Monoid

Assume that $n = \dim_K V$ is even, say $n = 2m$. Define an antisymmetric nondegenerate bilinear form $\langle\ ,\ \rangle$ on V by putting

$$\langle e_i, e_j \rangle = \varepsilon_i \delta_{i,j'} \qquad \text{for any } 1 \le i, j \le n. \tag{a}$$

where ε_i is 1 if $i \le m$ and -1 otherwise. Let J be the matrix of $\langle\ ,\ \rangle$ with respect to the basis $\{e_1, \dots, e_n\}$. Then the symplectic group $\mathrm{Sp}(V)$ is the group of linear operators $f \in \mathrm{End}_K(V)$ preserving the bilinear form:

$$\mathrm{Sp}(V) = \{f \in \mathrm{End}_K(V) : \langle f(v), f(v') \rangle = \langle v, v' \rangle,\ \text{all } v, v' \in V\}. \tag{b}$$

Let A be the matrix of f with respect to the basis $\{e_1, \dots, e_n\}$. Then we may identify $\mathrm{Sp}(V)$ with the matrix group

$$\mathrm{Sp}_n(K) = \{A \in \mathrm{M}_n(K) : A^{\mathsf{T}} J A = J\}. \tag{c}$$

This is contained in the larger group $\mathrm{GSp}_n(K)$, the group of symplectic similitudes, defined by

$$\mathrm{GSp}_n(K) = \{A \in \mathrm{M}_n(K) : A^{\mathsf{T}} J A = cJ,\ \text{some } c \in K^{\times}\}. \tag{d}$$

Note that $\mathrm{GSp}_n(K)$ is generated by $\mathrm{Sp}_n(K)$ and $K^{\times}$. As in the orthogonal case, we could just as well have defined $\mathrm{GSp}_n(K)$ by

$$\mathrm{GSp}_n(K) = \{A \in \mathrm{M}_n(K) : A^{\mathsf{T}} J A = cJ = AJA^{\mathsf{T}},\ \text{some } c \in K^{\times}\}$$

thanks to the equivalence of the conditions $A^{\mathsf{T}} J A = cJ$ and $cJ = AJA^{\mathsf{T}}$ in case $c \neq 0$. We define the *symplectic monoid* $\mathrm{SpM}_n(K)$ to be

$$\mathrm{SpM}_n(K) = \overline{\mathrm{GSp}_n(K)}, \tag{e}$$

the Zariski closure in $\mathrm{M}_n(K)$. In [4] the following was proved.

Proposition 2. *The symplectic monoid* $\mathrm{SpM}_n(K)$ *is the set of all* $A \in \mathrm{M}_n(K)$ *such that* $A^{\mathsf{T}} J A = cJ = AJA^{\mathsf{T}}$, *for some* $c \in K$.

Note that the scalar $c \in K$ in the above is allowed to be zero, and the condition $cJ = AJA^{\mathsf{T}}$ is necessary, just as it was in the orthogonal case.

3.3 Sketch of Proof

I want to briefly sketch the ideas involved in the proof of Propositions 1 and 2. Full details are available in [4]. The method of proof works for any infinite field K (except that characteristic 2 is excluded in the orthogonal case). We continue to assume that $n = 2m$ is even in the symplectic case.

Let $G = \mathrm{GO}_n(K)$ or $\mathrm{GSp}_n(K)$ and let $M = \mathrm{OM}_n(K)$ or $\mathrm{SpM}_n(K)$, respectively. Let T be the maximal torus of diagonal elements of G. Then we have inclusions

$$\overline{T} \subset \overline{G} \subset M \tag{a}$$

and we desire to prove that the latter inclusion is actually an equality. To accomplish this, we consider the action of $G \times G$ on M given by $(g, h) \cdot m = gmh^{-1}$. Suppose that we can show that every $G \times G$ orbit is of the form GaG, for some $a \in \overline{T}$. Then it follows that

$$M = \bigcup_{a \in \overline{T}} GaG \subset \overline{G} \tag{b}$$

and this gives the opposite inclusion that proves Propositions 1 and 2. In fact, as it turned out, the distinct $a \in \overline{T}$ in the above decomposition can be taken to be certain idempotents in $\overline{T}$.

This suggests the program that was carried out in [4], which in the end leads to additional structural information on M:

(i) Classify all idempotents in $\overline{T}$.
(ii) Obtain an explicit description of $\overline{T}$.
(iii) Determine the $G \times G$ orbits in M.

Part (i) is easy. For part (ii) one exploits the action of T on $\overline{T}$ by left multiplication and determines the orbits of that action. Part (iii) involves developing orthogonal and symplectic versions of classical Gaussian elimination.

3.4 The Normality Question

It is clear from the equalities in Propositions 1 and 2 that $\mathrm{OM}_n(K)$ and $\mathrm{SpM}_n(K)$ both have one-dimensional centers and contain zero. What is not clear, and not addressed in [4], is whether or not they are normal as algebraic varieties.

This question was recently settled in [5], where it is shown that $\mathrm{SpM}_n(K)$ is always normal, while $\mathrm{OM}_n(K)$ is normal only in case n is even. More precisely, it is shown in [5] that when $n = 2m$ is even, $\mathrm{OM}_n^+(K)$ and $\mathrm{OM}_n^-(K)$ are both normal varieties. Here

$$\mathrm{OM}_n(K) = \mathrm{OM}_n^+(K) \cup \mathrm{OM}_n^-(K) \tag{a}$$

is the decomposition into irreducible components, where $\mathrm{OM}_n^+(K)$ is the component containing the unit element 1.

4 Representation Theory

From now on we assume that M is an arbitrary reductive normal algebraic monoid, with unit group $G = M^{\times}$. We wish to describe some results of [4]. The main result is that the category of rational M-modules is a highest weight category in the sense of Cline et al. [1].

We work with a fixed maximal torus $T \subset G$, and set $D = \overline{T}$. We assume that $\dim Z(M) = 1$ and $0 \in M$. Recall that restriction induces an injection $X(D) \to X(T)$, so we may identify $X(D)$ with a submonoid of $X(T)$. We fix a Borel subgroup B with $T \subset B \subset G$ and let the set R^- of *negative* roots be defined by the pair (B, T). We set $R^+ = -(R^-)$, the set of positive roots. We have $R = R^+ \cup R^-$. Let

$$X(T)^+ = \{\lambda \in X(T) : (\alpha^\vee, \lambda) \geq 0, \text{ for all } \alpha \in R^+\}$$

be the usual set of dominant weights. We define

$$X(D)^+ = X(T)^+ \cap X(D).$$

By a (left) rational M-module we mean a linear action $M \times V \to V$ such that the coefficient functions $M \to K$ of the action are all in $K[M]$. This is the same as having a (right) $K[M]$-comodule structure on V. This means that we have a comodule structure map

$$\Delta_V : V \to V \otimes_K K[M]. \tag{a}$$

Since $K[M] \subset K[G]$ the map Δ_V induces a corresponding map $V \to V \otimes_K K[G]$ making V into a $K[G]$-comodule; i.e., a rational G-module. Thus, rational M-modules may also be regarded as rational G-modules. Any rational M-module is semisimple when regarded as a rational D-module, with corresponding weight space decomposition

$$V = \bigoplus_{\lambda \in X(D)} V_\lambda \tag{b}$$

where $V_\lambda = \{v \in V : d \cdot v = \lambda(d)\, v, \text{ all } d \in D\}$.

Recall that any rational G-module V is semisimple when regarded as a rational T-module, with corresponding weight space decomposition

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda \tag{c}$$

where $V_\lambda = \{v \in V : t \cdot v = \lambda(t)\, v, \text{ all } t \in T\}$. If V is a rational M-module then the weight spaces relative to T are the same as the weight spaces relative to D. So the weights of a rational M-module all belong to $X(D)$. Conversely, we have the following.

Lemma 1. *If V is a rational G-module such that*

$$\{\lambda \in X(T) : V_\lambda \neq 0\} \subset X(D)$$

then V extends uniquely to a rational M-module.

This is proved as an application of Renner's extension principle, which is a version of Chevalley's big cell construction for algebraic monoids.

Remark 1. A special case of the lemma (for the case $M = \mathrm{M}_n(K)$) can be found in [6].

Next one needs a notion of induction for algebraic monoids, i.e., a left adjoint to restriction. The usual definition of induced module for algebraic groups does not work for algebraic monoids. Instead, we use the following definition. Let V be a rational L-module where L is an algebraic submonoid of M. We define $\mathrm{ind}_L^M V$ by

$$\mathrm{ind}_L^M V = \{f \in \mathrm{Hom}(M, V) : f(lm) = l \cdot f(m),\ \text{all}\ l \in L, m \in M\}.$$

This is viewed as a rational M-module via right translation. One can check that in case L, M are algebraic groups then this is isomorphic to the usual induced module.

It is well known that the Borel subgroup B has the decomposition $B = TU$, where U is its unipotent radical. Given a character $\lambda \in X(T)$ one regards K as a rational T-module via λ; this is often denoted by K_λ. One extends K_λ to a rational B-module by letting U act trivially. Similarly, we have the decomposition $\overline{B} = DU$. If $\lambda \in X(D)$ then we have K_λ as above, and again we may regard this as a rational $\overline{B}$-module by letting U act trivially.

Now we can formulate the classification of simple rational M-modules.

Theorem 1. *Let M be a reductive normal algebraic monoid. Let $\lambda \in X(D)$ and let K_λ be the rational $\overline{B}$-module as above. Then*

(a) $\mathrm{ind}_{\overline{B}}^M K_\lambda \neq 0$ *if and only if* $\lambda \in X(D)^+$.
(b) *If* $\mathrm{ind}_{\overline{B}}^M K_\lambda \neq 0$ *then its socle is a simple rational M-module (denoted by $L(\lambda)$).*
(c) *The set of $L(\lambda)$ with $\lambda \in X(D)^+$ gives a complete set of isomorphism classes of simple rational M-modules.*

Let $\lambda \in X(T)$. Let $\nabla(\lambda) = \mathrm{ind}_B^G K_\lambda$. It is well known that $\nabla(\lambda) \neq 0$ if and only if $\lambda \in X(T)^+$. The following is a key fact.

Lemma 2. *If* $\lambda \in X(D)^+$ *then* $\mathrm{ind}_{\overline{B}}^M K_\lambda = \nabla(\lambda) = \mathrm{ind}_B^G K_\lambda$.

Now we consider truncation. Let $\pi \subset X(T)^+$. Given a rational G-module V, let $\mathcal{O}_\pi V$ be the unique largest rational submodule of V with the property that the highest weights of all its composition factors belong to π. The (left exact) truncation functor $\mathcal{O}_\pi$ was defined by Donkin [2].

Recall that $X(T)$ is partially ordered by $\lambda \leq \mu$ if $\mu - \lambda$ can be written as a sum of positive roots; this is sometimes called the dominance order. A subset π of

$X(T)^+$ is said to be *saturated* if it is predecessor closed under the dominance order on $X(T)$. In other words, π is saturated if for any $\mu \in \pi$ and any $\lambda \in X(T)^+$, $\lambda \leq \mu$ implies that $\lambda \in \pi$.

In order to show that the category of rational M-modules is a highest weight category, we are going to take $\pi = X(D)^+$. We need the following observation.

Lemma 3. *The set $\pi = X(D)^+$ is a saturated subset of $X(T)^+$.*

For $\lambda \in X(T)^+$, let $I(\lambda)$ be the injective envelope of $L(\lambda)$ in the category of rational G-modules. For $\lambda \in X(D)^+$ let $Q(\lambda)$ be the injective envelope of $L(\lambda)$ in the category of rational M-modules. The following records the effect of truncation on various classes of rational G-modules.

Theorem 2. *Let $\pi = X(D)^+$. For any $\lambda \in X(T)^+$ we have the following:*

(a) $\mathscr{O}_\pi \nabla(\lambda) = \begin{cases} \nabla(\lambda) & \text{if } \lambda \in \pi \\ 0 & \text{otherwise.} \end{cases}$

(b) $\mathscr{O}_\pi I(\lambda) = \begin{cases} Q(\lambda) & \text{if } \lambda \in \pi \\ 0 & \text{otherwise.} \end{cases}$

(c) $\mathscr{O}_\pi K[G] = K[M]$.

Note that $K[M]$ is regarded as a rational M-module via right translation. A ∇-filtration for a rational G-module V is an ascending series

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = V$$

of rational submodules such that for each $j = 1, \ldots, r$, the quotient V_j / V_{j-1} is isomorphic to some $\nabla(\lambda_j)$. Whenever V is a rational G-module with a ∇-filtration, let $(V : \nabla(\lambda))$ be the number of λ_j for which $\lambda = \lambda_j$. This number is independent of the filtration.

The proof of the above theorem, which relies on results of [3], also shows the following facts.

Corollary 1. (a) *Let $\lambda \in \pi = X(D)^+$. The module $Q(\lambda)$ has a ∇-filtration. Furthermore, it satisfies the reciprocity property*

$$(Q(\lambda) : \nabla(\mu)) = [\nabla(\mu) : L(\lambda)]$$

for any $\mu \in X(D)^+$, where $[V : L]$ stands for the multiplicity of a simple module L in a composition series of V.

(b) *The module $K[M]$ has a ∇-filtration. Moreover, $(K[M] : \nabla(\lambda)) = \dim_K \nabla(\lambda)$ for each $\lambda \in X(D)^+$.*

From these results one obtains the important fact that the category of rational M-modules is a highest weight category, in the sense of [1]. In particular, one also sees that $\dim_K Q(\lambda)$ is finite, for any $\lambda \in X(D)^+$. (In contrast, it is well known that $\dim_K I(\lambda)$ is infinite.)

It is also shown in [4], exploiting the assumption that $Z(M)$ is one-dimensional, that the category of rational M-modules splits into a direct sum of 'homogeneous' subcategories each of which is controlled by a finite saturated subset of $X(D)^+$. From the results of [1] it then follows that there is a finite dimensional quasihereditary algebra in each homogeneous degree, whose module category is precisely the homogeneous subcategory in that degree. Details are given in [4], where it is also shown that the quasihereditary algebras in question are in fact generalized Schur algebras in the sense of Donkin [2].

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On Linear Hodge Newton Decomposition for Reductive Monoids

Sandeep Varma

To Professors Putcha and Renner, with admiration

Abstract Let F be the field of fractions of a complete discrete valuation ring $\mathfrak{o}$. Let $\bar{\mathbf{G}}$ be an irreducible linear reductive monoid over F, such that its group $\mathbf{G}$ of units is split over $\mathfrak{o}$. When $\bar{\mathbf{G}}$ is either a connected reductive $\mathfrak{o}$-split linear algebraic group over F or the monoid of $n \times n$ matrices over F, Kottwitz and Viehmann had proved a relation between the Hodge point and the Newton point associated to an element $\gamma \in \bar{\mathbf{G}}(F)$. Suppose F has characteristic zero. In [7], we had given a monoid theoretic generalization of this phenomenon. On the way, we had applied the Putcha-Renner theory of linear algebraic monoids over algebraically closed fields to study $\bar{\mathbf{G}}(F)$ by generalizing various results for linear algebraic groups over F such as the Iwasawa, Cartan and affine Bruhat decompositions. In this article we give an exposition of these results.

Keywords Hodge-Newton decomposition • Linear algebraic monoids • Mazur's inequality

Subject Classifications: Primary 11F85; Secondary 20Mxx, 20G25

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1 An Introductory Example

1.1 Preliminary Notation

Let $\mathfrak{o}$ be a complete discrete valuation ring, F its quotient field and $\bar{F}$ an algebraic closure of F. The discrete valuation val : $F \to \mathbb{Z} \cup \{\infty\}$ can be uniquely extended to a valuation map from $\bar{F}$ to $\mathbb{Q} \cup \{\infty\}$, which we will still denote by val. Let ϖ be a uniformizer in $\mathfrak{o}$.

1.2 The Newton Point Attached to T

Let V be a finite dimensional vector space over F of dimension $n \in \mathbb{N}$, and $T : V \to V$ an invertible linear transformation. Then we can attach elements $\nu_1, \nu_2, \ldots, \nu_n \in \mathbb{Q}$ to T as follows. Recall that the roots of the characteristic polynomial $f(x) = \det(xI - T)$ of T (lying in $\bar{F}$) are known as the generalized eigenvalues of T. Let $\lambda_1, \ldots, \lambda_n \in \bar{F}^\times$ be the generalized eigenvalues of T, and set $\nu_i = \mathrm{val}(\lambda_i)$ for $i = 1, \ldots, n$. We may and do rearrange the λ_i's to ensure $\nu_1 \leq \cdots \leq \nu_n$.

Definition 1. The n-tuple $\nu(T) := (\nu_1, \ldots, \nu_n) \in \mathbb{Q}^n$ is known as the *Newton point* attached to $T : V \to V$. The elements $\nu_1, \nu_2, \ldots, \nu_n$ are called the *slopes* of T.

1.3 The Hodge Point Attached to (T, Λ)

Now let Λ be a lattice (i.e., a free $\mathfrak{o}$-module of maximal rank) in V. Then one can associate a point $(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \subset \mathbb{Q}^n$ to the pair (T, Λ), with $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, as follows. $T\Lambda$ and Λ are both lattices in V, so one can choose a basis $e_1, \ldots, e_n$ for Λ and $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{Z}$ with $\mu_1 \leq \cdots \leq \mu_n$ such that $\varpi^{\mu_1} e_1, \ldots, \varpi^{\mu_n} e_n$ is a basis for $T\Lambda$. Thus, $(\mu_1, \ldots, \mu_n)$, which we will also denote by $\mathrm{inv}(T\Lambda, \Lambda)$ (it does not depend on the choice of $e_1, \ldots, e_n$), measures the relative position between Λ and $T\Lambda$.

Definition 2. $\mu(T, \Lambda) := (\mu_1, \ldots, \mu_n)$ is known as the *Hodge point* attached to the pair (T, Λ).

1.4 A Theorem Relating the Newton Point and the Hodge Point

Definition 3. For $\mu, \nu \in \mathbb{Q}^n$, we say that $\mu \geq \nu$ if :

$$\mu_1 + \cdots + \mu_i \leq \nu_1 + \cdots + \nu_i \text{ for } i = 1, \ldots, n-1, \tag{1}$$

and

$$\mu_1 + \cdots + \mu_n = \nu_1 + \cdots + \nu_n. \tag{2}$$

In the following theorem, (a) is a special case of Corollary 3.6 in [2], while (b) is a special case of Theorem 3.5 (2) of loc. cit. Alternatively, one may view the theorem as a special case of Theorem 4.2 of [2].

Theorem 1. *Let V be an n-dimensional F-vector space, let $T : V \to V$ be an invertible linear transformation, and let Λ be a lattice in V. Then :*

(a) The Newton point associated to T and the Hodge point associated to (T, Λ) are related by :

$$\nu(T) \leq \mu(T, \Lambda).$$

(b) Suppose $V = U \oplus W$, with $T(U) = U$ and $T(W) = W$. Assume in addition that every slope (cf. Definition 1) of T on U is strictly less than every slope of T on W. Further, suppose $\mu_1 + \cdots + \mu_r = \nu_1 + \cdots + \nu_r$, where $r = \dim U$. Then Λ decomposes as

$$\Lambda = (\Lambda \cap U) \oplus (\Lambda \cap W).$$

In the above theorem, part (a) is a linear analog of Mazur's inequality, and part (b) a linear analog of Katz' Hodge-Newton decomposition.

Two generalizations of the above results were given by R. Kottwitz and E. Viehmann in [2], as will be explained in the next section. In [7] a monoid theoretic generalization was sought for, and this write up reports on the results obtained therein.

In Sect. 2 we describe the two generalizations of Kottwitz and Viehmann alluded to above. We briefly comment on the proofs of one of these generalizations in Sect. 3. In Sect. 4 we review some basic results on reductive monoids defined over F, whose underlying unit group is split over F. Section 5 is devoted to explaining how a crucial ingredient, known as the Newton homomorphism, is generalized to algebraic monoids. The main results of [7] (under a simplifying hypothesis) are stated and briefly commented on in Sect. 6.

This is an expanded version of the talk the author gave on this material at the Fields Institute workshop on Algebraic Monoids, Group Embeddings and Algebraic Combinatorics, held in honor of the sixtieth birthdays of Professors Putcha and Renner.

I am very thankful to the organizers of the workshop, Professors M. Can, Z. Li, B. Steinberg and Q. Wang, for having given me an opportunity to speak and also for putting the conference and these proceedings together. The work of [7] which I am reporting on was suggested by Professor Kottwitz, to whom I am extremely grateful for his guidance and encouragement. Finally, the work was only possible because I

could stand on the shoulders of Professors Putcha and Renner, and appeal to their elegant machinery.

2 Two Generalizations of Theorem 1

2.1 Possible Directions to Generalize

Theorem 1 is a statement about elements $T \in \mathbf{GL}(V)(F)$, and [2] features two generalizations of this theorem. One, namely Theorem 3.5 and Corollary 3.6 of [2], generalizes Theorem 1 to a statement about elements $\gamma \in \mathbf{G}(F)$, where $\mathbf{G}$ is a split reductive group over F. Another, namely Theorem 4.2 of [2], generalizes it to a statement about endomorphisms $T \in \mathrm{End}(V)$ of the vector space V that are not necessarily invertible.

2.2 The Generalization from $\mathbf{GL}(V)$ *to* $\mathbf{End}(V)$

Let us state the latter generalization first. The notions of Newton point and Hodge point generalize in an obvious fashion to endomorphisms $T : V \to V$. Before stating these definitions, note that the linear order on $\mathbb{Q}$ naturally extends to one on $\mathbb{Q} \cup \{\infty\}$, where ∞ is declared to be greater than every rational number.

Definition 4. For an endomorphism $T : V \to V$ of an n-dimensional vector space over F, the Newton point $\nu(T)$ of T is defined to be $(\nu_1, \ldots, \nu_n) \in (\mathbb{Q} \cup \{\infty\})^n$, where the ν_i's are the valuations of the generalized eigenvalues of T, arranged so that $\nu_1 \leq \cdots \leq \nu_n$.

Thus, the only difference here is that some of the generalized eigenvalues can be 0, and consequently some of the ν_i's can be ∞.

Given a lattice $\Lambda \subset V$, $T\Lambda$ is no longer necessarily a lattice in V but only a finitely generated $\mathfrak{o}$-submodule of V. We can still define the Hodge point associated to (T, Λ) as follows.

Definition 5. Choose $e_1, \ldots, e_n \in \Lambda$ that form a basis for Λ, such that for some $r \leq n$ and $\mu_1, \ldots, \mu_r \in \mathbb{Z}$ with $\mu_1 \leq \cdots \leq \mu_r$, $\varpi^{\mu_1} e_1, \ldots, \varpi^{\mu_r} e_r$ form a basis for $T\Lambda$. Set $\mu_i = \infty$ for $r + 1 \leq i \leq n$. Then $(\mu_1, \ldots, \mu_n)$, which we will also denote by $\mathrm{inv}(T\Lambda, \Lambda)$ is defined to be the Hodge point associated to (T, Λ).

Theorem 4.2 of [2] tells us that:

Theorem 2. *With the conventions above, in the statement of Theorem 1, one can take T to be any linear transformation, and not just an invertible one.*

2.3 Notations Pertaining to Our Split Reductive Group G

We next state the generalization of Theorem 1 to split reductive groups. Let $\mathbf{G}$ be a connected reductive group defined and split over F. Then we may and do choose a Chevalley basis and assume that $\mathbf{G}$ is defined and split over $\mathfrak{o}$. Let $\mathbf{A}$ be an $\mathfrak{o}$-split maximal torus of $\mathbf{G}$, and $W = W(\mathbf{G}, \mathbf{A})$ the Weyl group of $\mathbf{A}$ in $\mathbf{G}$. We also fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{A}$, and write $\mathbf{U}$ for its unipotent radical. We write $\mathbf{B}^-$ for the Borel subgroup of $\mathbf{G}$ opposite to $\mathbf{B}$ and containing $\mathbf{A}$. We will usually write $G = \mathbf{G}(F)$, $A = \mathbf{A}(F)$ etc.

Set $K := \mathbf{G}(\mathfrak{o})$, which then becomes a hyperspecial maximal compact subgroup of $\mathbf{G}$. For any algebraic group $\mathbf{H}$ we will write $X_*(\mathbf{H})$ for algebraic homomorphisms from the multiplicative group $\mathbb{G}_m$ into $\mathbf{H}$, and $X^*(\mathbf{H})$ for algebraic homomorphisms from $\mathbf{H}$ into $\mathbb{G}_m$.

Notation. Whenever $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$, $\mathbf{R}_u(\mathbf{P})$ will denote the unipotent radical of $\mathbf{P}$.

2.4 Reinterpreting the Newton Point for GL(V)

We will follow [2] in describing the generalizations of Newton point and Hodge point to general reductive groups. So how do we attach a Newton point to an element $\gamma \in G = \mathbf{G}(F)$ (generalizing the situation of $T \in GL(V)$)? It turns out to be easier to first generalize a finer invariant of (T, V) than $\nu(T)$, namely the *slope decomposition*:

$$V = \bigoplus_{a \in \mathbb{Q}} V_a \tag{3}$$

of V induced by T. Here V_a is the subspace of V consisting of all the generalized eigenspaces of T corresponding to generalized eigenvalues with valuation a. In other words, T induces a $\mathbb{Q}$-grading of V, and $\nu(T) = (\nu_1, \ldots, \nu_n)$ is merely the coarser invariant listing in non-decreasing order the various a such that $V_a \neq 0$, and with multiplicity equal to $\dim V_a$. Just as giving a $\mathbb{Z}$-grading of a vector space is equivalent to giving a homomorphism from the multiplicative group $\mathbb{G}_m = \operatorname{Spec} F[\mathbb{Z}]$ to $\mathbf{GL}(V)$, giving a $\mathbb{Q}$-grading, such as the slope decomposition, is equivalent to giving a homomorphism from $\mathbb{D} := \operatorname{Spec} F[\mathbb{Q}]$ to $\mathbf{GL}(V)$. $\mathbb{D}$ is not an algebraic group but a pro-algebraic group. Recall that $\mathbb{Q}$ is the direct limit of '$\mathbb{N}$'-many copies of $\mathbb{Z}$ with 'multiplication by n' maps between appropriate copies, and similarly (in fact consequently) $\mathbb{D} = \operatorname{Spec} F[\mathbb{Q}]$ is the projective limit of '$\mathbb{N}$'-many copies of $\mathbb{G}_m$, with 'n-th power' maps between the appropriate copies. The slope decomposition (3) thus corresponds to an element ν_T in:

$$\operatorname{Hom}(\mathbb{D}, \mathbf{GL}(V)) = \operatorname{Hom}(\mathbb{G}_m, \mathbf{GL}(V)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The Newton point, on the other hand, corresponds to the *conjugacy class* of this homomorphism, as the invariant $(\nu_1, \dots, \nu_n)$ picks out precisely the conjugacy class of the slope decomposition. Indeed, one can conjugate a homomorphism as above to take values in a split maximal torus $\mathbf{A}_V$ of $\mathbf{GL}(V)$ and, denoting the Weyl group of $\mathbf{A}_V$ in $\mathbf{GL}(V)$ by, say $W_V \cong S_n$, get a well defined element in $X_*(\mathbf{A}_V) \otimes \mathbb{Q}/S_n$ which may then be identified with $(\mathbb{Z}^n \otimes \mathbb{Q})/S_n = \mathbb{Q}^n/S_n$. If $V = F^n$ so that $\mathbf{GL}(V) = \mathbf{GL}_n$, and $\mathbf{A}_V$ is taken to be the standard maximal torus, then our requirement $\nu_1 \leq \cdots \leq \nu_n$ corresponds to picking out an element of this W_V-orbit in $X_*(\mathbf{A}_V) \otimes \mathbb{Q}$ that is dominant with respect to the Borel subgroup of *lower triangular* matrices.

Thus, what we are seeking to do is to attach to every $\gamma \in G$ a homomorphism $\nu_\gamma : \mathbb{D} \to \mathbf{G}$, which generalizes the homomorphism ν_T above in an appropriate sense. The Newton point of γ will then be the W-orbit in $\mathrm{Hom}(\mathbb{D}, \mathbf{A}) = X_*(\mathbf{A}) \otimes \mathbb{Q}$ obtained by conjugating ν_γ so as to have image in $\mathbf{A}$, or a $\mathbf{B}$-dominant representative for the same.

2.5 Review of Some Very Basic Tannakian Formalism

Attaching ν_γ is done via the Tannakian formalism, which in some sense reduces the question to the case of a general linear group. Note that any homomorphism $\nu : \mathbb{D} \to \mathbf{G}$ induces a functor $\nu^* : \mathrm{Rep}_0\,\mathbf{G} \to \mathrm{Rep}_0\,\mathbb{D}$ from the category of finite dimensional representations of $\mathbf{G}$ to that for $\mathbb{D}$. This functor has two obvious properties:

(a) It respects the tensor product structures on $\mathrm{Rep}_0\,\mathbb{D}$ and $\mathrm{Rep}_0\,\mathbf{G}$; and
(b) It is strictly compatible with the fiber functors on the categories $\mathrm{Rep}\,\mathbf{G}$ and $\mathrm{Rep}\,\mathbb{D}$, i.e., takes a representation of $\mathbb{D}$ of the form (ρ, V), where $\rho : \mathbb{D} \to \mathbf{GL}(V)$, to one of the form (ρ', V), with the same V.

The Tannakian theory says that the converse is true, or more precisely, that any functor from $\mathrm{Rep}_0\,\mathbf{G}$ to $\mathrm{Rep}_0\,\mathbb{D}$ that satisfies properties (a) and (b) above is ν^* for a unique homomorphism $\nu : \mathbb{D} \to \mathbf{G}$.

2.6 Generalizing the Newton Point to G

However, the observations in the case of $\mathbf{GL}(V)$ above tell us just how to construct such a functor $\mathrm{Rep}_0\,\mathbf{G} \to \mathrm{Rep}_0\,\mathbb{D}$ from a given $\gamma \in \mathbf{G}(F)$. Indeed, given any representation (ρ, V) of $\mathbf{G}$, γ gives rise to an element $\rho(\gamma)$ of $\mathbf{GL}(V)$, whose slope decomposition yields a homomorphism $\nu_{\rho,\gamma} : \mathbb{D} \to \mathbf{GL}(V)$, i.e., a representation of $\mathbb{D}$ over the same space V that ρ acted on. It is straight forward to see that $\rho \mapsto \nu_{\rho,\gamma}$ is compatible with tensor products too. Thus, the Tannakian theory gives us a homomorphism $\nu_\gamma : \mathbb{D} \to \mathbf{G}$ as desired, and then we have the Newton point $[\nu_\gamma] \in X_*(\mathbf{A}) \otimes \mathbb{Q}$, as discussed earlier.

The Newton homomorphism ν_γ can be characterized a bit more explicitly as follows, by slightly unraveling the above description. Let R be any F-algebra, and $d \in \mathbb{D}(R) = \mathrm{Hom}_{F-\mathrm{alg}}(F[\mathbb{Q}], R) = \mathrm{Hom}(\mathbb{Q}, R^\times)$. Write this homomorphism as $a \mapsto d_a$. We wish to characterize $\nu_\gamma(d) \in \mathbf{G}(R)$. By the Tannakian theory, it is enough to characterize $\rho(\nu_\gamma(d)) \in \mathbf{GL}(V)(R) = \mathbf{GL}(V \otimes R)$ for every representation (ρ, V) of $\mathbf{G}$. We have a slope decomposition as in Eq. 3 for $\rho(\gamma)$, and with the notation of that equation, for every $a \in \mathbb{Q}$, $\rho(\nu_\gamma(d))$ acts on the subspace $V_a \otimes_F R$ of $V \otimes_F R$ by multiplication by d_a.

2.7 Generalizing 'inv' *to* G

Now we come to the question of generalizing the notion of a Hodge point. In order to do that, we must discuss the generalization of the construct $\mathrm{inv}(\Lambda_1, \Lambda_2)$ that measured the relative position between lattices $\Lambda_1, \Lambda_2 \subset V$. Fixing a lattice $\Lambda_0 \subset V$ gives rise to a hyperspecial compact subgroup $K_0 \subset GL(V)$, namely the stabilizer of Λ_0 in $GL(V) = \mathbf{GL}(V)(F)$. Then $g \mapsto g \cdot \Lambda_0$ realizes the lattices in V as in one-one correspondence with $GL(V)/K_0$. Thus, $\mathrm{inv}(\cdot, \cdot)$ is a function on $GL(V)/K_0 \times GL(V)/K_0$, bi-invariant under the diagonal action of $GL(V)$. Similarly, in the context of our group $\mathbf{G}$ that we have fixed, inv is a function on $G/K \times G/K$ that is bi-invariant under G, namely a function on $G\backslash(G/K \times G/K)$, which is in one-one correspondence with $K\backslash G/K$. The Cartan decomposition says that the map $\mu \mapsto K\mu(\varpi)K$ induces a one-one correspondence between $X_*(\mathbf{A})/W$ and $K\backslash G/K$. Thus, we now have the function:

$$\mathrm{inv} : G/K \times G/K \to X_*(\mathbf{A})/W,$$

taking (g_1K, g_2K) to the W-orbit of any $\mu \in X_*(\mathbf{A})$ satisfying $g_2^{-1}g_1 \in K\mu(\varpi)K$.

2.8 The Hodge Point in the Context of G

Therefore, the Hodge point of $\gamma \in G = \mathbf{G}(F)$, with respect to $gK \in G/K$ (which generalizes the set of all lattices), should be $\mathrm{inv}(\gamma gK, gK)$, namely the W-orbit of any $\mu \in X_*(\mathbf{A})$ such that $g^{-1}\gamma g \in K\mu(\varpi)K$. Indeed, suppose that $\mathbf{G} = \mathbf{GL}_n = \mathbf{GL}(F^n)$, $\mathbf{A}$ is the torus that scales the elements of the standard basis of F^n and K the stabilizer of the standard lattice Λ_0 (the $\mathfrak{o}$-span of the standard basis elements of F^n). In this case, $X_*(\mathbf{A})$ can be identified with $\mathbb{Z}^n$ in an obvious fashion, and under this identification one can check that the dominant $\mu^{\mathrm{dom}} \in X_*(\mathbf{A})$ in the W-orbit associated to γ and gK as above, the notion of dominance defined with respect to the lower triangular Borel subgroup, corresponds exactly to what we defined earlier as the Hodge point associated to the linear transformation $\gamma : F^n \to F^n$ and the lattice $g\Lambda_0 \subset F^n$. This checking is simplified by the appropriate G-invariance and K-invariance properties of either definitions, which reduces the checking to the case where $gK = K$ and $\gamma \in \mathbf{A}(F)$.

2.9 Relating Newton and Hodge Points for G

Given $T \in GL(V)$, to say that $(\mu_1, \dots, \mu_n)$ is the Hodge point associated to T and *some lattice* in V, generalizes therefore to saying that $X_\mu^{\mathbf{G}}(\gamma) \neq \emptyset$, where:

$$X_\mu^{\mathbf{G}}(\gamma) = \{x \in G/K \mid x^{-1}\gamma x \in K\mu(\varpi)K\}.$$

This explains why the following result of Kottwitz and Viehmann, found in Corollary 3.6 of [2], generalizes Theorem 1 to our reductive group $\mathbf{G}$ (that is split over F).

Theorem 3. *Let μ be a (*$\mathbf{B}$*-)dominant coweight, and let $\gamma \in G$ so that we have $[\nu_\gamma] \in X_*(\mathbf{A}) \otimes \mathbb{Q}$ as above. If in addition $X_\mu^{\mathbf{G}}(\gamma)$ is nonempty, then $[\nu_\gamma] \leq \mu$ in the sense that $\mu - [\nu_\gamma]$ is a non-negative rational linear combination of simple coroots (with respect to* $\mathbf{B}$*).*

2.10 Notation Towards Linear Hodge-Newton Decomposition for G

To discuss the linear Hodge-Newton decomposition, on the other hand, we introduce some more notation. First, one has the quotient $\Lambda_{\mathbf{G}}$ of $X_*(\mathbf{A})$ by the coroot lattice of $\mathbf{G}$. It is then well known that $\Lambda(\mathbf{G})$ is the cocharacter lattice of the torus $\mathbf{T} := \mathbf{G}/\mathbf{G}_{\mathrm{sc}}$, where $\mathbf{G}_{\mathrm{sc}}$ is the simply connected cover of the derived group of $\mathbf{G}$. Note that there exists a unique map $T = \mathbf{T}(F) \to X_*(\mathbf{T})$, such that for all $\mu \in X_*(\mathbf{T})$ and $t \in \mathbf{T}(\mathfrak{o})$, $\mu(\varpi)t \mapsto \mu$. The map $\mathbf{G} \to \mathbf{T}$ may be viewed as a generalization of the determinant map $\det : \mathbf{GL}(V) \to \mathbb{G}_m$. The resulting map $w_{\mathbf{G}} : \mathbf{G} \to \Lambda_{\mathbf{G}}$, obtained by composing the map from G to T with the aforementioned map $T \to X_*(\mathbf{T})$, may be viewed as a generalization of $\mathrm{val} \circ \det : GL(V) \to \mathbb{Z}$. One also has a natural map $X_*(\mathbf{A}) \to \Lambda_{\mathbf{G}}$, which will be denoted $p_{\mathbf{G}}$. Analogous constructs for a Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ will be denoted $\Lambda_{\mathbf{M}}, p_{\mathbf{M}}, w_{\mathbf{M}}$ etc.

2.11 The Hodge-Newton Decomposition for $\mathbf{G}$

Now we can state Theorem 3.5 from [2], of which part (2) is the linear Hodge Newton decomposition for reductive groups:

Theorem 4. *Let $\mu \in X_*(\mathbf{A})$ be dominant (with respect to $\mathbf{B}$), $\mathbf{P} = \mathbf{MN}$ the Levi decomposition of a standard parabolic subgroup and $\gamma \in \mathbf{M}(F)$. Then:*

(a) If $X_\mu^{\mathbf{G}}(\gamma)$ is nonempty, then $w_{\mathbf{M}}(\gamma) \overset{\mathbf{P}}{\leq} p_{\mathbf{M}}(\mu)$, in the sense that $p_{\mathbf{M}}(\mu) - w_{\mathbf{M}}(\gamma)$ is the image in $\Lambda_{\mathbf{M}}$ of a non-negative integral linear combination of coroots $\check{\alpha}$, as α varies over the roots of $\mathbf{A}$ in $\mathbf{N}$.
(b) Suppose that $w_{\mathbf{M}}(\gamma) = p_{\mathbf{M}}(\mu)$ and that every slope (valuation of generalized eigenvalue) of $\mathrm{Ad}(\gamma)$ on the Lie algebra $\mathrm{Lie}\,\mathbf{N}$ of $\mathbf{N}$ is strictly positive. Then the natural injection $X_\mu^{\mathbf{M}}(\gamma) \hookrightarrow X_\mu^{\mathbf{G}}(\gamma)$ is a bijection.

2.12 Why Theorems 3 and 4 Generalize Theorem 1

In the special case where $\mathbf{G} = \mathbf{GL}(V)$, $V = U \oplus W$ is a decomposition of V into γ-invariant subspaces, $\mathbf{P}$ is the parabolic subgroup stabilizing W and $\mathbf{M}$ is the Levi subgroup preserving U and W, part (b) of the above theorem generalizes Theorem 1 (b). To see this one chooses $\mathbf{A}$ to be determined by a basis obtained by putting together a basis of U with one of W, and K to be the stabilizer of the lattices $\mathfrak{o}$-spanned by that basis. To say that every slope of $\mathrm{Ad}(\gamma)$ on $\mathrm{Lie}\,\mathbf{N}$ is strictly positive is, in this case, to say that every slope of γ on U is strictly less than every slope of γ on W. To say that $w_{\mathbf{M}}(\gamma) = p_{\mathbf{M}}(\mu)$ is to say that, in the notation of Theorem 1, $\nu_1 + \cdots + \nu_r = \mu_1 + \cdots + \mu_r$. To say that the natural injection $X_\mu^{\mathbf{M}}(\gamma) \hookrightarrow X_\mu^{\mathbf{G}}(\gamma)$ is bijective is to say that any lattice Λ such that the Hodge point of γ and Λ is μ, has to split as $(\Lambda \cap U) \oplus (\Lambda \cap W)$ – this follows since the lattices stabilized by K and hence also all lattices stabilized by compact open subgroups of the form mKm^{-1}, where $m \in \mathbf{M}$, split in exactly such a fashion.

2.13 Comments on the Problem of Generalizing to Monoids

Theorem 3, and Theorem 4 of which Theorem 3 is a corollary, are thus natural generalizations to reductive groups of corresponding statements for $\mathbf{GL}_n$. Conceptual and elegant proofs of these theorems have been given in [2]. However, the formulation of Theorem 2, in contrast, does not seem to highlight entirely transparently what the structures involved are, for instance whether this result on $\mathbf{M}_n(F)$ is a monoid-theoretic phenomenon or if the additive structure on $\mathbf{M}_n$ plays an indispensable role. The proof that has been given for Theorem 2 in [2] employs slightly indirect means, namely by first *additively* perturbing an element of $\mathbf{M}_n(F)$ by an element of the center of $\mathbf{M}_n(F)$ to get an element of $\mathbf{GL}_n(F)$, and then using the Hodge-Newton decomposition for $\mathbf{GL}_n(F)$.

Thus, one seeks to find a formulation of Theorem 2 for general reductive monoids, together with a monoid theoretic proof. This was partially done in [7]. Namely, Theorem 3 was generalized to reductive monoids, as also part (1) of Theorem 4. Part (2) of Theorem 4 might perhaps generalize, the author just has not attempted it enough to form an opinion.

3 On the Proofs of Theorems 3 and 4

Before getting into these monoid theoretic formulations and proofs, we will recall some inputs into the proof of [2], which [7] sought to generalize to monoids.

3.1 The Bruhat-Tits Inequality

In [2], Theorem 3 is deduced from Theorem 4. One of the key inputs into the proof of Theorem 4 is what is known as the Bruhat-Tits inequality, which we proceed to review. This is an inequality relating the Cartan and the Iwasawa decompositions of G. Recall that the Iwasawa decomposition says that, for any Borel subgroup $\mathbf{B}'$ of $\mathbf{G}$ containing $\mathbf{A}$, the map $\mu \mapsto \mu(\varpi)$ from $X_*(\mathbf{A})$ to G gives a bijection between $X_*(\mathbf{A})$ and $U\mathbf{A}(\mathfrak{o})\backslash G/K$. This gives us a map $r_{\mathbf{B}'} : G \to X_*(\mathbf{A})$. These maps $r_{\mathbf{B}'}$ will be referred to as retractions. Now, given any $g \in G$, the Cartan decomposition gives us an element $\mu^{\mathrm{dom}} \in X_*(\mathbf{A})$, which is $\mathbf{B}$-dominant. We also have the element $r_{\mathbf{B}}(g) \in X_*(\mathbf{A})$. The Bruhat-Tits inequality says that $r_{\mathbf{B}}(g) \leq \mu^{\mathrm{dom}}$, in the sense that $\mu^{\mathrm{dom}} - r_{\mathbf{B}}(g)$ is a non-negative integral linear combination of coroots corresponding to simple roots of $\mathbf{A}$ in $\mathbf{B}$.

3.2 The Retractions and $w_{\mathbf{G}}$

One can also consider the map $p_{\mathbf{G}} \circ r_{\mathbf{B}}$ from G to $\Lambda_{\mathbf{G}}$. In fact, this map turns out to be precisely the homomorphism $w_{\mathbf{G}}$ that we discussed earlier. In particular, $p_{\mathbf{G}} \circ r_{\mathbf{B}}$ is independent of $\mathbf{B}$. This also follows from a well known property of the retractions $r_{\mathbf{B}'}$. Namely if $\mathbf{B}', \mathbf{B}''$ are Borel subgroups containing $\mathbf{A}$ that are adjacent, then $r_{\mathbf{B}'}(g) - r_{\mathbf{B}''}(g)$ is a non-negative integral multiple of $\alpha^\vee$, where α is the unique root that is positive for $\mathbf{B}'$ and negative for $\mathbf{B}''$. This implies that the various $r_{\mathbf{B}'}$ all differ by elements in the coroot lattice, so that $p_{\mathbf{G}} \circ r_{\mathbf{B}'}$ is independent of $\mathbf{B}'$.

We will not comment any more on the proof of Theorem 4. We hope that the assertion "$w_{\mathbf{M}}(\gamma) \overset{\mathbf{P}}{\leq} p_{\mathbf{M}}(\mu)$" in Theorem 4 (a) does not look too strange now, since the left hand side of this inequality is related to $r_{\mathbf{B}}(\gamma)$ and the right hand side is related to the Cartan decomposition for $g^{-1}\gamma g$, so it should at least begin to look like the Bruhat-Tits inequality should play a role.

3.3 The Newton Point and $w_{\mathbf{G}}$

Let us now comment on the relationship of Theorem 3 with Theorem 4 from which it is derived. For this we need to explain how the Newton point $[\nu_\gamma] \in X_*(\mathbf{A}) \otimes \mathbb{Q}$ of γ is related to the $w_{\mathbf{G}}(\gamma) \in \Lambda_{\mathbf{G}}$. Note that $X_*(\mathbf{A}) \otimes \mathbb{Q}$ and $\Lambda_{\mathbf{G}}$ both map to $\Lambda_{\mathbf{G}} \otimes \mathbb{Q}$. We claim that the image of $[\nu_\gamma]$ in $\Lambda_{\mathbf{G}} \otimes \mathbb{Q}$ equals that of $w_{\mathbf{G}}(\gamma)$. Indeed, when $\mathbf{G}$ is a torus this is easily verified. One then deduces the result for general $\mathbf{G}$ by considering the quotient $\mathbf{C} := \mathbf{G}/\mathbf{G}_{\text{der}}$ of $\mathbf{G}$ by its derived subgroup, and using that the Newton homomorphism and $w_{\mathbf{G}}$ both behave well with respect to homomorphisms of algebraic groups. More precisely, Newton homomorphisms are functorial (see Lemma 2.1 of [2]), and we have a commutative diagram:

$$\begin{array}{ccc} G & \longrightarrow & C \\ {\scriptstyle w_{\mathbf{G}}}\downarrow & & \downarrow{\scriptstyle w_{\mathbf{C}}} \\ \Lambda_{\mathbf{G}} & \longrightarrow & \Lambda_{\mathbf{C}} \end{array}$$

in which the bottom row becomes an isomorphism upon tensoring with $\mathbb{Q}$.

3.4 Detecting the Newton Point from the $w_{\mathbf{M}}$'s

However, this does not yet let us detect the Newton point from $w_{\mathbf{G}}$, precisely because the passage from $X_*(\mathbf{A}) \otimes \mathbb{Q}$ to $\Lambda_{\mathbf{G}} \otimes \mathbb{Q}$ results in loss of information. Nevertheless note that, if $\mathbf{A_G} \subset \mathbf{A}$ denotes the identity component of the center of $\mathbf{G}$, then the map $X_*(\mathbf{A_G}) \otimes \mathbb{Q} \to \Lambda_{\mathbf{G}} \otimes \mathbb{Q}$ is an isomorphism. This is because $X_*(\mathbf{A_G})$ and the coroot lattice span complementary subspaces of $X_*(\mathbf{A}) \otimes \mathbb{Q}$. Therefore, in the situation where $[\nu_\gamma] \in \text{Hom}(\mathbb{D}, \mathbf{A_G}) \subset \text{Hom}(\mathbb{D}, \mathbf{A})$, we can recover $[\nu_\gamma]$ from $w_{\mathbf{G}}(\gamma)$.

Definition 6. $\gamma \in G$ is said to be basic if $\nu_\gamma : \mathbb{D} \to \mathbf{G}$ factors through the center of $\mathbf{G}$ (and hence, through $\mathbf{A_G}$).

3.5 On Deriving Theorem 3 from Theorem 4

Though an arbitrary $\gamma \in G$ will not be basic in general, one can always find a Levi subgroup $\mathbf{M}_\gamma$ of $\mathbf{G}$ for which γ is basic. Indeed, the image of $\nu_\gamma : \mathbb{D} \to \mathbf{G}$ is a split torus in $\mathbf{G}$, whose centralizer is necessarily a Levi subgroup $\mathbf{M}_\gamma$ of $\mathbf{G}$. It is an easy consequence of the definition of ν_γ that replacing γ by $h\gamma h^{-1}$, $h \in G$, replaces ν_γ by $\mathrm{Int}(h) \circ \nu_\gamma$, where $\mathrm{Int}(h)$ denotes the inner automorphism defined by h. This forces $\gamma \in M_\gamma$, so that γ is a basic element of M_γ. Hence, one can always conjugate γ and assume that it is basic for a Levi subgroup containing $\mathbf{A}$, in fact (say by making ν_γ dominant) for a Levi subgroup of a standard parabolic subgroup. From this point, it is rather straightforward to derive Theorem 3, by applying Theorem 4 to the element $\gamma \in M_\gamma$ (with $\mathbf{M}_\gamma$ taking the role of $\mathbf{M}$), and relating the partial order $\leq$ on $X_*(\mathbf{A})$ with the partial order $\overset{\mathbf{P}}{\leq}$ on $\Lambda_{\mathbf{M}} \otimes \mathbb{Q}$.

4 Basics on Split Reductive Monoids over *F*

4.1 Notation for Our Reductive Monoid

Let $\bar{\mathbf{G}}$ be an irreducible reductive monoid over F, with unit group $\mathbf{G}$. $\mathbf{G}$ will continue to be split over $\mathfrak{o}$ as before. Further, henceforth we assume F to be of characteristic zero. Note that $\mathbf{G} \times \mathbf{G}$ acts on $\bar{\mathbf{G}}$ by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$.

Hypothesis. In this particular article, for simplicity of exposition, we assume $\bar{\mathbf{G}}$ to be normal.

For a subset $\mathbf{X}$ of $\mathbf{G}$ (or $\bar{\mathbf{G}}$), $\bar{\mathbf{X}}$ will denote its Zariski closure in $\bar{\mathbf{G}}$. Thus, $\bar{\mathbf{A}}$ will be the toric variety obtained as the Zariski closure of $\mathbf{A}$ in $\bar{\mathbf{G}}$ (since $\bar{\mathbf{G}}$ is normal, by Exercise 4.6.2 (8) of [5], $\bar{\mathbf{A}}$ is normal too).

Notation. For any $\bar{\mathbf{X}} \subset \bar{\mathbf{G}}$, let $E(\bar{\mathbf{X}})$ denote the set of idempotents in $\bar{\mathbf{X}}(\bar{F})$. If $\bar{\mathbf{X}}$ is a subsemigroup of $\bar{\mathbf{G}}$ that is a monoid in its own right, we write $\mathbf{G}(\bar{\mathbf{X}})$ for the group of unit elements in $\bar{\mathbf{X}}$. For any algebraic monoid $\bar{\mathbf{H}}$ we will also write $X_*(\bar{\mathbf{H}})$ for algebraic semigroup homomorphisms from $\mathbb{G}_m$ into $\bar{\mathbf{H}}$. Such a homomorphism should take the identity element of $\mathbb{G}_m$ to an idempotent, and we have in particular:

$$X_*(\bar{\mathbf{A}}) = \bigsqcup_{e \in E(\bar{\mathbf{A}})} X_*(e\mathbf{A}),$$

where $e\mathbf{A}$ is viewed as a torus in its own right.

4.2 Some Groups and Homomorphisms Attached to $e \in E(\bar{\mathbf{A}})$

Recall that the notion of a Hodge Point in the context of the group $\mathbf{G}$, the map $r_{\mathbf{B}}$ etc. depended on facts from the structure theory for reductive p-adic groups, such as the Iwasawa decomposition and the Cartan decomposition. We will need similar results for monoids. These will be obtained from results on the structure theory of $\bar{\mathbf{G}}$ over the algebraic closure $\bar{F}$ of F. We refer the reader to the books [3] and [5] for the results that we are going to recall, and for a comprehensive treatment of algebraic monoids.

Definition 7. Let $e \in E(\bar{\mathbf{A}})$. Then (following the convention of [1]; [3] follows a slightly different convention) the centralizer, the left centralizer and the right centralizer of e are defined to be respectively the groups:

$$\mathbf{F}_e := \mathbf{C}_{\mathbf{G}}(e) := \{x \in \mathbf{G} \mid xe = ex\},$$

$$\mathbf{F}_e^l := \mathbf{C}_{\mathbf{G}}^l(e) := \{x \in \mathbf{G} \mid xe = exe\},$$

and

$$\mathbf{F}_e^r := \mathbf{C}_{\mathbf{G}}^r(e) := \{x \in \mathbf{G} \mid ex = exe\}.$$

One also associates to e the group $\mathbf{H}_e := \mathbf{G}(e\bar{\mathbf{G}}e)$ and the groups:

$$\mathbf{G}_e^l = \{x \in \mathbf{G} \mid xe = e\}, \text{ and } \mathbf{G}_e^r = \{x \in \mathbf{G} \mid ex = e\}.$$

Here, note that $e\bar{\mathbf{G}}e$ is a subsemigroup of $\bar{\mathbf{G}}$ that is a monoid in its own right, with e as the identity element.

Definition 8. Let $e \in E(\bar{\mathbf{A}})$. One defines the maps (that are readily checked to be well defined homomorphisms):

$$\rho_e : \mathbf{F}_e^l \to \mathbf{H}_e, \lambda_e : \mathbf{F}_e^r \to \mathbf{H}_e \text{ and } \tau_e : \mathbf{F}_e \to \mathbf{H}_e$$

by:

$$\rho_e(x) = xe = exe, \quad \lambda_e(x) = ex = exe, \quad \tau_e(x) = ex = xe = exe.$$

Thus, $\mathbf{G}_e^l = \ker \rho_e, \mathbf{G}_e^r = \ker \lambda_e$.

4.3 Properties of Groups and Homomorphisms Attached to $e \in E(\bar{\mathbf{A}})$

Some early results from the theory of Putcha and Renner have been collected into the following theorem. One can find their proofs in either [3] or [5].

Theorem 5. *Let $e \in E(\bar{\mathbf{A}})$. Then we have:*

(a) $\mathbf{F}_e^l$ *and* $\mathbf{F}_e^r$ *are opposite parabolic subgroups of* $\mathbf{G}$*, whose common Levi subgroup is* $\mathbf{F}_e$ *(which contains* $\mathbf{A}$*).*
(b) $\mathbf{R}_u(\mathbf{F}_e^l) \cdot e = e \cdot \mathbf{R}_u(\mathbf{F}_e^r) = \{e\}$*. In other words,* $\mathbf{R}_u(\mathbf{F}_e^l) \subset \mathbf{G}_e^l, \mathbf{R}_u(\mathbf{F}_e^r) \subset \mathbf{G}_e^r$*.*
(c) λ_e, ρ_e *and* τ_e *are surjective, so that we have an exact sequence:*

$$1 \to \mathbf{G}_e^l \to \mathbf{F}_e^l \xrightarrow{\rho_e} \mathbf{H}_e \to 1,$$

and similarly with $\mathbf{G}_e^r, \mathbf{F}_e^r, \lambda_e, \mathbf{H}_e$ *and* $\mathbf{G}_e, \mathbf{F}_e, \tau_e, \mathbf{H}_e$*.*
(d) The stabilizer $(\mathbf{G} \times \mathbf{G})_e$ *of e in* $\mathbf{G} \times \mathbf{G}$ *is:*

$$(\mathbf{G} \times \mathbf{G})_e = \{(x, y) \in \mathbf{F}_e^l \times \mathbf{F}_e^r \mid \rho_e(x) = \lambda_e(y)\}.$$

4.4 Renner's Bruhat Decomposition for $\bar{\mathbf{G}}(\bar{F})$

Definition 9. The monoid $R := \overline{\mathbf{N}_{\mathbf{G}}(\mathbf{A})}(\bar{F})/\mathbf{A}(\bar{F})$, namely the quotient by $\mathbf{A}(\bar{F})$ of the Zariski closure $\overline{\mathbf{N}_{\mathbf{G}}(\mathbf{A})}(\bar{F})$ of the normalizer $\mathbf{N}_{\mathbf{G}}(\mathbf{A})(\bar{F})$ of $\mathbf{A}(\bar{F})$, is known as the Renner monoid of $\mathbf{A}$ in $\bar{\mathbf{G}}$.

Theorem 8.8 of [5] gives a Bruhat decomposition for $\bar{\mathbf{G}}(\bar{F})$:

Theorem 6.

$$\bar{\mathbf{G}}(\bar{F}) = \bigsqcup_{x \in R} \mathbf{B}(\bar{F}) \cdot x \cdot \mathbf{B}(\bar{F}).$$

4.5 Putcha's Description of $\mathbf{G}(\bar{F}) \times \mathbf{G}(\bar{F})$*-Orbits in* $\bar{\mathbf{G}}(\bar{F})$

On the other hand, the $\mathbf{G}(\bar{F}) \times \mathbf{G}(\bar{F})$-orbits in $\bar{\mathbf{G}}(\bar{F})$ have a description too. First we define:

Definition 10. Let:

$$\Lambda = \{e \in E(\bar{\mathbf{A}}) \mid \mathbf{B} \subset \mathbf{F}_e^l\}, \quad \Lambda^- = \{e \in E(\bar{\mathbf{A}}) \mid \mathbf{B}^- \subset \mathbf{F}_e^l\}.$$

Then (see Theorem 4.5 (c) of [5] for part (a); part (b) of this theorem is Exercise 8.9.3 in [5]):

Theorem 7. *Then* Λ *(and similarly,* Λ^-*) forms a representatives for the* $W(\mathbf{G}, \mathbf{A})$*-orbits in* $E(\bar{\mathbf{A}})$ *and:*

$$\bar{\mathbf{G}}(\bar{F}) = \bigsqcup_{e \in \Lambda} \mathbf{G}(\bar{F}) \cdot e \cdot \mathbf{G}(\bar{F}) = \bigsqcup_{e \in \Lambda^-} \mathbf{G}(\bar{F}) \cdot e \cdot \mathbf{G}(\bar{F}).$$

Further,

$$\bar{\mathbf{G}}(\bar{F}) = \bigsqcup_{e \in E(\bar{\mathbf{A}})} \mathbf{G}(\bar{F}) \cdot e \cdot \mathbf{B}(\bar{F}).$$

4.6 Relative Bruhat Decomposition and G × G Orbits on $\bar{\mathbf{G}}$

Now, since F has characteristic zero, it turns out to be possible to use the above Bruhat decomposition (Theorem 6) together with Galois cohomology, and get various decompositions for $\bar{G} = \bar{\mathbf{G}}(F)$. Indeed, Theorem 6 turns out to be more convenient for this than Theorem 7, since Galois cohomology is easier to handle for solvable groups. This way, one gets the following 'relative' Bruhat decomposition as well as decompositions of $\bar{G}$ into $G \times G$-orbits and into $G \times B$-orbits (for details see [7], Sect. 3):

Theorem 8. *Let* $R_F = \overline{\mathbf{N_G}(\mathbf{A})(F)/\mathbf{A}(F)}$, *the relative Renner monoid of* $\mathbf{A}$ *in* $\bar{\mathbf{G}}$. *Then:*

$$\bar{G} = \bigsqcup_{x \in R_F} BxB = \bigsqcup_{w \in W(\mathbf{G},\mathbf{A})} \bigsqcup_{e \in E(\bar{\mathbf{A}})} BweB.$$

Then:

$$\bar{G} = \bigsqcup_{e \in \Lambda} \bigsqcup_{\dot{t} \in (e\mathbf{A})(F)/\mathbf{A}(F)} (G \times G) \cdot \dot{t} = \bigsqcup_{e \in \Lambda} (G \times G) \cdot e.$$

Finally,

$$\bar{G} = \bigsqcup_{e \in E(\bar{\mathbf{A}})} \bigsqcup_{\dot{t} \in (e\mathbf{A})(F)/\mathbf{A}(F)} (G \times B) \cdot \dot{t} = \bigsqcup_{e \in E(\bar{\mathbf{A}})} (G \times B) \cdot e.$$

4.7 Clarifications on the Above Result

Remark. Here are some clarifications and explanations regarding the above result.

(a) In the latter equality of the relative Bruhat decomposition, the w of $BweB$ stands for a representative in G of the Weyl group element w. On the other hand, any $e \in E(\bar{\mathbf{A}})$ automatically belongs to $\bar{G}$ since $\mathbf{A}$ is defined over F.

(b) In each of the three decompositions in Theorem 8, the second equality uses the fact that $\bar{\mathbf{G}}$ is normal, while the first does not. Indeed, one gets a counterexample to the second equality in each of the assertions above by considering, following Example 6.12 in [3],

$$\bar{\mathbf{G}} = \bar{\mathbf{B}} = \bar{\mathbf{A}} = \{(a, b, c) \in \bar{\mathbb{G}}_m^3 \mid a^2 b = c^2\}.$$

Here $\bar{\mathbb{G}}_m$ denotes the affine line with its usual monoid structure (which has the multiplicative group $\mathbb{G}_m$ as its group of units). One can see that in this case $(0, b, 0)$ with $b \in F^\times \setminus (F^\times)^2$ does not lie in GeG, or equivalently in any $BweB$ or GeB.

(c) The results in Theorem 8 are all valid for an arbitrary field of characteristic zero, not just our field F that is the quotient field of a complete discrete valuation ring.

(d) Since the 'second equalities' in the various decompositions above are nicer, F-points of normal monoids are much better behaved than F-points of non-normal monoids. In fact, $\bar{G} = \bar{\mathbf{G}}(F)$ is a 'split reductive monoid' (see [4]; this is a monoid theoretic generalization of the notion of BN-pairs) for normal $\bar{\mathbf{G}}$ (as we are assuming), but this is not true without this assumption of normality.

(e) However, in [7] $\bar{\mathbf{G}}$ was not assumed to be normal, since that was not necessary for the purposes of that paper. However, we will assume so here, since that makes the statements simpler and nicer.

4.8 **B**-*dominance for Elements of* $X_*(\bar{\mathbf{A}})$

Using the decomposition of $\bar{G}$ into $G \times G$-orbits, one can then prove monoid-theoretic versions of the Iwasawa and the Cartan decompositions. We also wish to state the affine Bruhat decomposition for monoids, to which end we define the Iwahori subgroup I to be the preimage of $\mathbf{B}(k_F)$ under the reduction map $\mathbf{G}(\mathfrak{o}) \to \mathbf{G}(k_F)$, where k_F denotes the residue field of F. A few decompositions including the Iwasawa decomposition and the Cartan decomposition are stated in Theorem 9 below. Before stating the theorem, we note that for any Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{A}$ and for any $e \in E(\bar{\mathbf{A}})$, $\mathbf{B}_{\mathbf{F}_e} := \mathbf{B} \cap \mathbf{F}_e$ is a Borel subgroup of $\mathbf{F}_e$, and consequently $\mathbf{B}_{\mathbf{H}_e} := \tau_e(\mathbf{B}_{\mathbf{F}_e})$ is a Borel subgroup of $\mathbf{H}_e$. Thus, we may (and shall) use the following notions of dominance.

Definition 11. Let $\mu \in X_*(e\mathbf{A}) \subset X_*(\bar{\mathbf{A}})$ where $e \in E(\bar{\mathbf{A}})$. We say that μ is dominant if $e \in \Lambda^-$ (cf. Definition 10) and μ is a dominant element of $X_*(e\mathbf{A})$ (with respect to $\mathbf{B}_{\mathbf{H}_e}$). We denote by $X_*(\bar{\mathbf{A}})^{\mathrm{dom}}$ the set of dominant elements of $X_*(\bar{\mathbf{A}})$.

Let us briefly discuss the motivation for this definition. Recall that if $\mu \in X_*(\mathbf{A})$ and $\lambda \in X^*(\mathbf{A})$ are $\mathbf{B}$-dominant we have, letting w_0 denote the long element of the Weyl group $W(\mathbf{G}, \mathbf{A})$,

$$\min_{w \in W(\mathbf{G},\mathbf{A})} \langle w\lambda, \mu \rangle = \langle w_0 \lambda, \mu \rangle. \tag{4}$$

The pairing $X^*(\mathbf{A}) \times X_*(\mathbf{A}) \to \mathbb{Z}$ admits a monoid theoretic variant:

$$X^*(\bar{\mathbf{A}}) \times X_*(\bar{\mathbf{A}}) \to \mathbb{Z} \cup \{\infty\},$$

sending the pair (λ, μ) to ∞ if $\lambda \circ \mu = 0$, and to $n \in \mathbb{Z}$ if $\lambda \circ \mu : \mathbb{G}_m \to \mathbb{G}_a$ is given by $t \mapsto t^n$. With these definitions, Lemma 27 of [7] says that Eq. 4 holds for $\lambda \in X^*(\bar{\mathbf{A}}) \subset X^*(\mathbf{A})$ and $\mu \in X_*(\bar{\mathbf{A}})$ provided λ is $\mathbf{B}$-dominant (as an element of $X^*(\mathbf{A})$ in the usual sense) and μ is $\mathbf{B}$-dominant as per Definition 11.

Theorem 9. *We have the following decompositions:*

(a) The Iwasawa decomposition : The map $\mu \mapsto \mu(\varpi)$ from $X_*(\bar{\mathbf{A}})$ into $\bar{G}$ induces a one-to-one correspondence between $X_*(\bar{\mathbf{A}})$ and $U\backslash\bar{G}/K$ (and, equivalently, between $X_*(\bar{\mathbf{A}})$ and $U\mathbf{A}(\mathfrak{o})\backslash\bar{G}/K$).
(b) The affine Bruhat decomposition: The inclusion of $\overline{\mathbf{N_G}(\mathbf{A})}(F)$ into $\bar{G}$ induces a one-to-one correspondence between $\overline{\mathbf{N_G}(\mathbf{A})}(F)/\mathbf{A}(\mathfrak{o})$ and $I\backslash\bar{G}/I$.
(c) The Cartan decomposition : The map $\mu \mapsto \mu(\varpi)$ induces a one-to-one correspondence between $X_*(\bar{\mathbf{A}})^{\mathrm{dom}}$ and $K\backslash G/K$.
(d) The inclusion of $\overline{\mathbf{N_G}(\mathbf{A})}(F)$ into $\bar{G}$ induces a one-to-one correspondence between $\overline{\mathbf{N_G}(\mathbf{A})}(F)/\mathbf{A}(\mathfrak{o})$ and $U\backslash\bar{G}/I$ (or equivalently, between $\overline{\mathbf{N_G}(\mathbf{A})}(F)/\mathbf{A}(\mathfrak{o})$ and $U\mathbf{A}(\mathfrak{o})\backslash\bar{G}/I$).

The proof of Theorem 9 is relatively easy. They all start from the decomposition of $\bar{G}$ into the double cosets GeG (e varying over Λ or equivalently Λ^-), and use the corresponding decompositions for groups associated to the idempotents e that show up, combined with the early results of Putcha-Renner theory that we have recalled. The simplest of these, the Cartan decomposition, follows from the computation below:

$$\begin{aligned} K\backslash(GeG)/K &= (K\backslash G) \times (K\backslash G)/(G \times G)_e \\ &= (K \cap F_e^l\backslash F_e^l) \times (K \cap F_e^r\backslash F_e^r)/(F_e^l \times F_e^r)_e \\ &= (K \cap F_e\backslash F_e) \times (K \cap F_e\backslash F_e)/(F_e \times F_e)_e \\ &= \tau_e(K \cap F_e)\backslash H_e/\tau_e(K \cap F_e) \\ &= X_*(e\mathbf{A})^{\mathrm{dom}}, \end{aligned}$$

where we have used in the second step that $KF_e^l = KF_e^r = G$ (a consequence of the Iwasawa decomposition for $\mathbf{G}$), and in the final step that, for $\bar{\mathbf{G}}$ normal, $\tau_e(K \cap F_e)$ can be shown to be a hyperspecial compact subgroup of $\mathbf{H}_e$. The other decompositions need more work, but not much more.

4.9 The Retractions $r_{\mathbf{B}'}$ and the Maps $p_{\bar{\mathbf{G}}}$ and $w_{\bar{\mathbf{G}}}$

The Iwasawa decomposition naturally gives the following extension of the retractions $r_{\mathbf{B}'}$ from G to $\bar{G}$:

Definition 12. Let $\mathbf{B}' = \mathbf{A}\mathbf{U}'$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{A}$. Then $r_{\mathbf{B}'} : \bar{G} \to X_*(\bar{\mathbf{A}}) = \bigsqcup_{e \in E(\bar{\mathbf{A}})(\bar{F})} X_*(e\mathbf{A})$ is defined to be the map that assigns to each $g \in \bar{G}$ the element $\mu \in X_*(\bar{\mathbf{A}})$ satisfying $g \in U' \cdot \mu(\varpi)K$ (the existence and uniqueness of μ being guaranteed by the Iwasawa decomposition).

We need to define a monoid-theoretic analogue of the map $w_{\mathbf{G}} : G \to \Lambda_{\mathbf{G}}$ now. Recall that $w_{\mathbf{G}}$ involved quotienting by $\mathbf{G}_{\mathrm{sc}}$ and then taking the cocharacter lattice. In the first step, if we had the derived subgroup $\mathbf{G}_{\mathrm{der}}$ of $\mathbf{G}$ instead of $\mathbf{G}_{\mathrm{sc}}$, we already would have an analogue – thanks to the abelization of $\bar{\mathbf{G}}$, due to Vinberg ([8]). While the abelization is not what we need, the description in [8] does motivate the following definition for our normal monoid $\bar{\mathbf{G}}$:

Definition 13.

$$\Lambda_{\bar{\mathbf{G}}} := \bigsqcup_{e \in E(\bar{\mathbf{Z}})} \Lambda_{\mathbf{H}_e},$$

where $\bar{\mathbf{Z}}$ denotes the center of $\bar{\mathbf{G}}$.

There is an obvious monoid structure on $\Lambda_{\bar{\mathbf{G}}}$. Now we need to generalize the map $p_{\mathbf{G}}$ (see Sect. 2.10) to a map $p_{\bar{\mathbf{G}}} : X_*(\bar{\mathbf{A}}) \to \Lambda_{\bar{\mathbf{G}}}$. Given $\mu \in X_*(e'\mathbf{A}) \subset X_*(\bar{\mathbf{A}})$, we define $e := \prod we'w^{-1}$, the product running over $w \in W(\mathbf{G}, \mathbf{A})$. Then the centralizer of e is a Levi subgroup of $\mathbf{G}$ containing $\mathbf{A}$ and having $W(\mathbf{G}, \mathbf{A})$ in its Weyl group, forcing $e \in E(\bar{\mathbf{Z}})$ and allowing:

Definition 14. With notation as above, we define $p_{\bar{\mathbf{G}}}(\mu) = p_{\mathbf{H}_e}(e\mu) \in \Lambda_{\mathbf{H}_e} \subset \Lambda_{\bar{\mathbf{G}}}$.

Definition 15. We define $w_{\bar{\mathbf{G}}} = p_{\bar{\mathbf{G}}} \circ r_{\mathbf{B}'}$, where $\mathbf{B}'$ is any Borel subgroup of $\mathbf{G}$ containing $\mathbf{A}$.

Of course, one then needs to verify that this definition is independent of the choice of $\mathbf{B}'$. Recall that in the group case one could see this from the fact that for any two Borel subgroups $\mathbf{B}', \mathbf{B}''$ containing $\mathbf{A}$, $r_{\mathbf{B}'}(g) - r_{\mathbf{B}''}(g)$ belonged to the coroot lattice of $\mathbf{A}$ in $\mathbf{G}$, and satisfied an appropriate inequality depending on the relative positions of $\mathbf{B}'$ and $\mathbf{B}''$ with respect to $\mathbf{B}$. In the case of monoids one can find a similar relationship, though it is a bit more complicated. For instance, we can have $r_{\mathbf{B}'}(g) \in X_*(e'\mathbf{A})$ and $r_{\mathbf{B}''}(g) \in X_*(e''\mathbf{A})$ with $e' \neq e''$. This turns out not to matter because such e' and e'' will always lie in the same $W(\mathbf{G}, \mathbf{A})$-orbit. One can indeed show that $w_{\bar{\mathbf{G}}}$ does not depend on the choice of $\mathbf{B}'$, and also find a natural relation between $r_{\mathbf{B}'}$ and $r_{\mathbf{B}''}$ for Borel subgroups $\mathbf{B}', \mathbf{B}''$ containing $\mathbf{A}$, generalizing the one for groups. We refer to Lemma 29 of [7] for the details.

One can also show, cf. Lemma 33 of [7], that $w_{\bar{\mathbf{G}}}$ is a homomorphism of monoids.

5 Generalizing the Newton Homomorphism

Recall from around Sects. 2.5 and 2.6 that the group theoretic version of the Newton homomorphism involved Tannakian formalism. We can take a similar approach to defining Newton points for monoids too.

5.1 *Newton Homomorphism When* $\bar{\mathbf{G}} = \mathbf{End}(V)$

Let us briefly take stock of what we should expect a Newton homomorphism to be. Suppose we are in the case where $\bar{\mathbf{G}} = \mathbf{End}(V)$, and want to attach a Newton homomorphism to $T \in \mathrm{End}(V)$. We have a natural analogue of the slope decomposition of Eq. 3, with the exception that the summation should be taken not over $a \in \mathbb{Q}$, but over $a \in \mathbb{Q} \cup \{\infty\}$. V_∞ will then correspond to the subspace of V where the given linear endomorphism T of V acts nilpotently. Let $V_{\neq\infty}$ denote the complement of V_∞ obtained as the direct sum of all the V_a, $a \in \mathbb{Q}$ (i.e., $a \neq \infty$). Then the slope decomposition is captured by the decomposition $V = V_\infty \oplus V_{\neq\infty}$, together with a homomorphism $\mathbb{D} \to \mathbf{GL}(V_{\neq\infty})$. We thus get a composite homomorphism

$$\nu_T : \mathbb{D} \to \mathrm{End}(V_{\neq\infty}) \hookrightarrow \mathrm{End}(V)$$

of algebraic semigroups (where $\mathrm{End}(V_{\neq\infty})$ sits inside $\mathrm{End}(V)$ as endomorphisms that annihilate V_∞). Further, the decomposition $V \to V_\infty \oplus V_{\neq\infty}$ can be recovered from this homomorphism as corresponding to the idempotent $\nu_T(1)$. Thus, for a general monoid $\bar{\mathbf{G}}$, and each $\gamma \in \bar{G}$, one expects an algebraic semigroup homomorphism $\mathbb{D} \to \bar{\mathbf{G}}$ to take the role of the Newton homomorphism.

5.2 *The Tannakian Formalism for Monoids*

Fortunately, almost all of the work needed to extend the Tannakian formalism so as to make it work in the monoid theoretic context has been done by N. Saavedra Rivano in [6]. The only difference in our situation is that a semigroup homomorphism $\mathbb{D} \to \bar{\mathbf{G}}$ will take the identity element of $\mathbb{D}$ to only an idempotent in $\bar{\mathbf{G}}$, and not necessarily the identity element of $\bar{\mathbf{G}}$. This forces us to work with algebraic semigroup representations of $\mathbb{D}$ and $\bar{\mathbf{G}}$ (i.e., homomorphisms $\mathbb{D} \to \mathbf{End}(V)$ or $\bar{\mathbf{G}} \to \mathbf{End}(V)$ of semigroups which may not be homomorphisms of monoids). The precise form of the result we need for our purposes is nevertheless an easy consequence of (part of) [6], and is discussed in Section 5 of [7]. We proceed to recall this consequence that we need. Note that any semigroup homomorphism $\nu : \mathbb{D} \to \bar{\mathbf{G}}$ induces a functor $\nu^* : \mathrm{Rep}_0^{\mathrm{sg}} \bar{\mathbf{G}} \to \mathrm{Rep}_0^{\mathrm{sg}} \mathbb{D}$ from the category of finite dimensional semigroup representations of $\bar{\mathbf{G}}$ to that for $\mathbb{D}$. This functor has three properties:

(a) It respects the tensor product structures on $\mathrm{Rep}_0^{\mathrm{sg}}\,\mathbb{D}$ and $\mathrm{Rep}_0^{\mathrm{sg}}\,\bar{\mathbf{G}}$; and
(b) It is strictly compatible with the fiber functors on the categories $\mathrm{Rep}_0^{\mathrm{sg}}\,\bar{\mathbf{G}}$ and $\mathrm{Rep}_0^{\mathrm{sg}}\,\mathbb{D}$, i.e., takes a representation of $\mathbb{D}$ of the form (ρ, V), where $\rho : \mathbb{D} \to \mathbf{End}(V)$, to one of the form (ρ', V), with the same V; and
(c) It takes the trivial representation of $\bar{\mathbf{G}}$ to the trivial representation of $\mathbb{D}$ and the zero representation of $\bar{\mathbf{G}}$ to the zero representation of $\mathbb{D}$.

Fortunately, it turns out that the converse is true. More precisely, any functor from $\mathrm{Rep}_0^{\mathrm{sg}}\,\bar{\mathbf{G}}$ to $\mathrm{Rep}_0^{\mathrm{sg}}\,\mathbb{D}$ that satisfies properties (a), (b) and (c) above is ν^* for a unique semigroup homomorphism $\nu : \mathbb{D} \to \bar{\mathbf{G}}$.

5.3 Newton Homomorphisms in the Monoid Setting

Now let $\gamma \in \bar{G}$. It is easy to see, following the construction in the group case, that the slope decomposition attached to γ induces a functor from $\mathrm{Rep}_0^{\mathrm{sg}}\,\bar{\mathbf{G}}$ to $\mathrm{Rep}_0^{\mathrm{sg}}\,\mathbb{D}$, which is easily shown to satisfy properties (a), (b) and (c) above, leading to the desired Newton homomorphism ν_γ.

There turns out to be a well known result in Putcha-Renner theory that lets us access this homomorphism ν_γ more conveniently. Page 35 of [5] associates to γ an idempotent $e_\gamma \in \bar{G}$, canonically determined by γ, such that for large $n \in \mathbb{N}$ we have $\gamma^n \in H_{e_\gamma}$. Further, it satisfies that $\gamma e_\gamma \in H_{e_\gamma}$, so that one can consider the Newton homomorphism $\nu_{\gamma e_\gamma} : \mathbb{D} \to \mathbf{H}_{e_\gamma}$ associated to $\gamma e_\gamma \in H_{e_\gamma}$. It turns out that $\nu_\gamma = \nu_{\gamma e_\gamma}$. This description seems more convenient in practice to access ν_γ.

6 The Main Results

6.1 Bruhat Tits Inequality

As with groups, the final proof will crucially use a monoid theoretic variant of the Bruhat-Tits inequality. It was Kottwitz who told the author how to formulate the inequality in the case $\bar{\mathbf{G}} = \mathbf{End}(V)$, and also showed him its proof in this case. It was then rather straight forward to interpret this proof in terms of constructs from Bruhat-Tits theory, and to then unravel it to express it more plainly, without any of those constructs. The statement of the Bruhat-Tits inequality for monoids turns out to be as follows.

Lemma 1. *Let $g \in \bar{G}$, and set $\nu = r_{\mathbf{B}}(g)$. Let $e \in \Lambda^-$ and $\mu \in X_*(e\mathbf{A})^{\mathrm{dom}} \subset X_*(\bar{\mathbf{A}})^{\mathrm{dom}}$ be such that $g \in K\mu(\varpi)K$. Note that $\nu^{\mathrm{dom}} \in X_*(e\mathbf{A})$ by Theorem 7.*

We then have that $\mu - \nu$ is an integral linear combination of terms of the form $e\alpha^\vee$, with α running over positive roots of $\mathbf{A}$ in $\mathbf{G}$ (i.e., the roots of $\mathbf{A}$ in $\mathbf{B}$).

6.2 Generalization of Theorem 4(1)

As mentioned earlier, [7] does not generalize part (2) of Theorem 4. As for generalizing part (1) of that theorem, recall that in the group case we had a partial order $\overset{\mathbf{P}}{\leq}$ on $\Lambda_{\mathbf{M}}$ when $\mathbf{P} = \mathbf{MN}$ was the Levi decomposition of a parabolic subgroup of $\mathbf{G}$, with $\mathbf{M} \supset \mathbf{A}$. We first need a generalization of this notion, to a reflexive transitive relation on $\Lambda_{\bar{\mathbf{M}}}$ where $\mathbf{P} = \mathbf{MN}$ as above. Recall that:

$$\Lambda_{\bar{\mathbf{M}}} = \oplus_{e \in E(\bar{\mathbf{Z}}_{\bar{\mathbf{M}}})} \Lambda_{\mathbf{H}_{e,\mathbf{M}}},$$

where $\bar{\mathbf{Z}}_{\bar{\mathbf{M}}}$ is the center of $\mathbf{M}$ and $\mathbf{H}_{e,\mathbf{M}}$ is the group of units in $e\bar{\mathbf{M}}e$ (i.e., just like $\mathbf{H}_e$ but defined for $\mathbf{M}$ as opposed to $\mathbf{G}$).

Definition 16. Let for $i = 1, 2$ $\mu_i \in \Lambda_{\mathbf{H}_{e_i,\mathbf{M}}} \subset \Lambda_{\bar{\mathbf{M}}}$, where $e_i \in E(\bar{\mathbf{Z}}_{\bar{\mathbf{M}}})$. Then $\mu_1 \leq \mu_2$ if and only if $e_1 = e_2$ and $\mu_2 - \mu_1$ is the image in $\Lambda_{\bar{\mathbf{M}}}$ of a non-negative integral linear combination of terms of the form $e\alpha^\vee$, as α runs over the roots of $\mathbf{A}$ in $\mathbf{N}$ (or equivalently, over the roots of $\mathbf{A}$ in $\mathbf{B}$).

This relation is easily seen to be reflexive and transitive, but it is easy to see that anti-symmetry can fail if $e \notin \Lambda^-$. For $e \in \Lambda^-$, the author does not know whether this is a partial order in general, though this can be checked for 'flat reductive monoids' in the sense of Vinberg ([8]); see Lemmas 39, 40 and 41 of [7].

Definition 17. For $\mu \in X_*(\bar{\mathbf{A}})$ and $\gamma \in \bar{G}$, define:

$$X_\mu^{\mathbf{G}}(\gamma) := \{x \in G/K \mid x^{-1}\gamma x \in K\mu(\varpi)K\}.$$

Here is the generalization of Theorem 4(1).

Theorem 10. *Let $\mathbf{M}$ be a standard Levi subgroup of $\mathbf{G}$, and suppose $\gamma \in \bar{M}$ satisfies $\gamma \in M \cdot e \cdot M$ for some $e \in \Lambda^-$. If $\mu \in X_*(e\mathbf{A})^{\text{dom}}$ (dominant with respect to $\bar{\mathbf{G}}$, not just $\bar{\mathbf{M}}$) with $X_\mu^{\mathbf{G}}(\gamma) \neq 0$, then:*

$$w_{\bar{\mathbf{M}}}(\gamma) \leq p_{\bar{\mathbf{M}}}(\mu).$$

We do not go into the details of the proof here, but merely note that the key input is the Bruhat-Tits inequality for monoids, namely Lemma 1.

6.3 Generalization of Theorem 3

Finally, here is the generalization of Theorem 3 to monoids.

Theorem 11. *Let $\mu \in X_*(e\mathbf{A}) \subset X_*(\bar{\mathbf{A}})$ be $\mathbf{B}$-dominant (in particular $e \in \Lambda^-$). Let $\gamma \in \bar{G}$ and write $[\nu_\gamma]$ for the unique element in the G-conjugacy class of the Newton homomorphism associated to γ that is $\mathbf{B}$-dominant - so there exists $e_1 \in \Lambda^-$ such that $[\nu_\gamma] \in X_*(e_1\mathbf{A})$ (and in fact belongs to the cone generated by $X_*(e_1\mathbf{A})^{\mathrm{dom}}$). Then $[\nu_\gamma] \leq \mu$ in the sense that $ee_1 = e_1$ and $e_1\mu - [\nu_\gamma]$ is a non-negative $\mathbb{Q}$-linear combination of terms of the form $e_1\alpha^\vee$ corresponding to simple roots α of $\mathbf{A}$ in $\mathbf{B}$.*

We will not recall the proof of this theorem, but merely mention that it is obtained from Theorem 10, just as Theorem 3 was a corollary of Theorem 4. In the situation of Theorem 3, recall that given $\gamma \in G$ one chose a Levi subgroup $\mathbf{M}$ of $\mathbf{G}$, containing γ, for which γ was basic, so that $w_{\mathbf{M}}(\gamma)$ captured ν_γ. Similarly, given $\gamma \in \bar{G}$, one comes up with a Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ such that $\gamma \in \bar{M}$ satisfies the conditions of Theorem 10, and such that $w_{\bar{\mathbf{M}}}(\gamma)$ captures ν_γ. One then applies Theorem 10.

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The Structure of Affine Algebraic Monoids in Terms of Kernel Data

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Abstract We describe the structure of affine algebraic monoids M in terms of kernel data, especially the interconnection between $\ker(M)$ or the isotropy groups at minimal idempotents under the natural actions and the unipotent radical $R_u(G)$ of the unit group G of M.

Keywords Algebraic groups • Algebraic monoids • Kernel • Minimal idempotents • Unipotent radicals

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1 Introduction

An affine (or linear) algebraic monoid (or semigroup) M is both an affine algebraic variety over an algebraically closed field K and a monoid (or semigroup) for which the product map $M \times M \to M$ is a morphism of varieties. When M is an algebraic monoid, its unit group G is an (affine) algebraic group; and when M is irreducible, $M = \overline{G}$. By [18, Theorem 3.15], an algebraic monoid is isomorphic to a (Zariski) closed submonoid of total n by n matrix monoid $M_n(K)$ for some n.

Unlike a general abstract or even a linear semigroup (e.g., $\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$ has no kernel) but like a compact semigroup, an algebraic semigroup M always has a kernel, $\ker(M)$, that is, the minimum two-sided semigroup-theoretic ideal ([18, Theorem 3.28]), which is the intersection of all ideals of the semigroup.

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If an algebraic semigroup has zero, then the kernel of the semigroup is zero. The next simplest kernel is a group. A nilpotent or reductive algebraic monoid has a group kernel. There are algebraic monoids whose kernels are non-group. For example, the kernel of a non-nilpotent regular solvable algebraic monoid is not a group. More generally, an algebraic monoid with more than one minimal idempotent is not a group.

The structure of the kernel of an algebraic monoid M can be of considerable impact on that of M or its unit group G. As we will see later that there is close connection between the kernel and the unipotent radical of G.

We now assemble some notions and notations. All algebraic groups and monoids involved are affine. Let M be an algebraic monoid over an algebraically closed field K. By convention, throughout this article, we denote by J_e, L_e, R_e and H_e the $\mathscr{J}$-, $\mathscr{L}$-, $\mathscr{R}$- and $\mathscr{H}$-classes of e in M under Green's relations, respectively (see [18, Chapter 1]). Denote by $E(M)$ the idempotent set of M. An algebraic monoid is referred to as *irreducible* if it is so as a variety. We call an algebraic monoid M *regular* (respectively, *unit regular, completely regular*) if it is so as a semigroup, i.e. , $M = \cup_{e\in E(M)} J_e$ (respectively, $M = E(M)G$, $M = \cup_{e\in E(M)} H_e$). Recall that H_e is a maximal subgroup of M and also the unit group of the algebraic monoid eMe. Let M be an irreducible algebraic monoid with unit group G. We call M *reductive* (respectively, *semisimple*, *solvable*, *nilpotent*, *toric*) if its unit group is reductive (respectively, reductive with 1-dimensional center, solvable, nilpotent, a torus). For a linear algebraic group G, we denote by $R(G)$ (respectively, $R_u(G)$) the radical (respectively, unipotent radical) of G. We call the dimension of a maximal torus of G the rank of G, denoted rank(G). For a subset V of an algebraic monoid M, denote by $\overline{V}$ the Zariski closure of V in M. An algebraic submonoid (respectively, subsemigroup) of an algebraic monoid M is a (topologically) closed submonoid (respectively, subsemigroup) of M. If N is an algebraic submonoid of M, we denote by N^c the identity component of N. For any subset X of M,

$$C^r_X(e) = \{a \in X | ae = eae\}, \quad C^l_X(e) = \{a \in M | ea = eae\},$$
$$C_X(e) = \{x \in X | xe = ex\}.$$

The one-sided annihilators $M^r(e)$ and $M^l(e)$ are defined as

$$M^r(e) = \{a \in M | ae = e\}, \quad M^l(e) = \{a \in M | ea = e\}, \quad M(e) = M^l(e) \cap M^r(e)$$

and

$$M^l_e := M^l(e)^c, \quad M^r_e = M^r(e)^c, \quad M_e = M(e)^c.$$

For any idempotent e, we write G_e for the unit group of M_e. Recall that if M is irreducible with unit group G, then both $M(e)$ and $M^r(e)$ have dense unit groups (see [18, Theorem 6.11]) but may be reducible (see [18, Example 6.12]); by [3], they are connected. For a semigroup S, we write $C(S)$ for the center of S.

References [18, 22] are our primary references for algebraic monoids, and [1, 12] for algebraic groups.

We will first describe the general structure of the kernels of linear algebraic semigroups in Sect. 2; in Sect. 3, we discuss the properties of kernels of irreducible linear algebraic monoids; Sect. 4 is for regularity conditions in terms of minimal idempotents; Sect. 5 is for specific types of algebraic monoids and their characterizations in terms of kernel data; Sect. 6 is for the unipotent radicals of unit groups of irreducible monoids and the unipotent radicals of one- and two-sided centralizers of minimal idempotents; Sect. 7 is to address the interactions between the kernels and unipotent radicals; Sect. 8 is for applications to the structure of parabolic subgroups of the unit groups of linear algebraic monoids; Sect. 9 is for the structure of an irreducible algebraic monoid which is the union of the unit group and its kernel.

2 The Kernel of a Linear Algebraic Semigroup

The following are about general structure of the kernel of an affine algebraic semigroup.

Proposition 1 ([9, Fact 1. 1]). *Let S be an algebraic semigroup. Then*

- $E(\ker(S)) \neq \emptyset$ *and* $\ker(S) = J_e$ *for any* $e \in E(\ker(S))$*;*
- $\ker(S)$ *is a completely simple semigroup;*
- $\ker(S)$ *is a closed subsemigroup of* S*;*
- *(Clark) if* S *is a closed subsemigroup of* $M_n(K)$, $\ker(S)$ *consists of the elements of* S *with the minimal matrix rank;*
- *As an abstract semigroup,* S *is simple if and only if* S *is completely simple if and only if* $S = \ker(S)$.

We denote by $\mathfrak{m}_l(S)$ (respectively, $\mathfrak{m}_r(S)$) the set of minimal left (respectively, right) ideals of S. We shall see shortly that for any affine algebraic semigroup S, $\mathfrak{m}_l(S) \neq \emptyset$.

Proposition 2 ([9, Lemma 1.3]). *Let S be an algebraic semigroup. Then*

- $\ker(S) = \cup_{A \in \mathfrak{m}_l(S)} A$*, and the union is disjoint;*
- $\mathfrak{m}_l(S) = \{Ss \mid s \in \ker(S)\} = \{Se \mid e \in E(\ker(S))\}$*;*
- *If* $a \in A \in \mathfrak{m}_l(S)$*, then* $A = Aa = Sa$*;*
- $\mathfrak{m}_l(S)$ *is invariant under right translation, that is, for each* $A \in \mathfrak{m}_l(S)$ *and each* $s \in S$*, we have* $As \in \mathfrak{m}_l(S)$*;*
- *For any* $A \in \mathfrak{m}_l(S)$ *and* $B \in \mathfrak{m}_r(S)$*, there exists* $e \in E(\ker(S))$ *such that* $A \cap B = H_e$.

Minimal idempotents play key roles in the study of algebraic monoids. As the following proposition shows, they can be used to characterize the kernel of an algebraic semigroup.

Proposition 3 ([9, Proposition 1.4]). *Let S be an algebraic semigroup, $e \in E(S)$. Then the following are equivalent:*

- *e is a minimal idempotent of S;*
- *$H_e = eSe$;*
- *$Se \in \mathfrak{m}_l(S)$;*
- *$e \in \ker(S)$;*
- *$Se = L_e$;*
- *$SeS = J_e$.*
- *$SeS = \ker(S)$.*
- *J_e is a minimal regular J−class in S.*

The following theorem characterizes $\ker(S)$ in terms of algebraic group and shows that $\ker(S)$ is a completely regular semigroup.

Theorem 1 ([9, Theorem 2.1]). *Let S be an algebraic semigroup. Then, for any $e, f \in E(\ker(S))$, there exists $g \in E(\ker(S))$ such that*

- *$e\mathscr{L}g\mathscr{R}f$ and $Se \cap fS = H_g$, where the H_hs are the maximal algebraic subgroups of $\ker(S)$. Moreover, $\ker(S) = \cup_{h\in E(\ker(S))} H_h$, and the union is disjoint. Thus $\ker(S)$ as a semigroup is completely regular.*
- *$H_e \to H_f$ defined by $x \mapsto gxf$ is an algebraic group isomorphism from H_e onto H_f, whose inverse morphism is given by $y \mapsto eyg$.*

Recall that a subset X of an affine algebraic variety V is a *retract* of V if there is a variety morphism $r : V \to V$ such that $r(V) = X$ and $r|_X = 1_X$. For an algebraic semigroup S and $e \in E(S)$, eSe, eS and Se are always retracts of S (in the sense of algebraic variety); the corresponding retractions can be chosen as

$$a \mapsto eae, \quad a \mapsto ea \text{ and } a \to ae,$$

respectively. We have the following result:

Theorem 2 ([9, Theorem 2.3]). *Let S be a linear algebraic semigroup. Then $\ker(S)$ is a retract of S in the sense of algebraic variety.*

Actually, for each idempotent $e \in E(\ker(S))$,

$$\delta_e : a \mapsto a(eae)^{-1}a$$

is a retraction $S \twoheadrightarrow \ker(S)$. So when $\dim E(\ker(S)) > 0$, there are infinitely many retractions. In general, $\delta_e \neq \delta_f$ when $e \neq f$. In fact, if in Theorem 2.3, S is an algebraic monoid, then there is a natural one-to-one correspondence between $\{\delta_e \mid e \in E(\ker(S))\}$ and $E(\ker(S))$. As we shall see later, retractions δ_e turn out to be useful in the study of the relation between the kernel of an irreducible algebraic monoid M and the unipotent radical of the unit group of M.

For any algebraic semigroup S without zero, by [18, Theorem 1.9] and [9, Fact 1.1], $\ker(S)$ as an abstract semigroup is isomorphic to a Rees matrix semigroup without zero over a group. On the other hand, given an algebraic group H, affine varieties X and Y and a vaiety morphism $\phi : Y \times X \to H$, let $V = X \times H \times Y$ with multiplication (Rees construction)

$$(x_1, h_1, y_1)(x_2, h_2, y_2) = (x_1, h_1\phi(y_1, x_2)h_2, y_2).$$

Then by [17, Example 3.9], V is an algebraic semigroup, which as an abstract semigroup is completely simple [17, Theorem 1.9]. Let S' be an algebraic semigroup with zero 0 and $S = S' \times V$, the direct product of algebraic semigroups S' and V. Then $\ker(S) = \{0\} \times V$, which as an algebraic semigroup is canonically isomorphic to V.

Two questions arise naturally: for an algebraic semigroup S (without zero), is $\ker(S)$ as algebraic semigroup isomorphic to a Rees matrix semigroup without zero of the above form? If so, how to realize the Rees construction? The following theorem answers these two questions completely.

Theorem 3 ([9, Theorem 2.4 and Corollary 2.5]). *Let S be an algebraic semigroup with $e \in E(\ker(S))$. Then*

- $\ker(S) = E(Se)H_eE(eS)$.
- *Let $R := E(Se) \times H_e \times E(eS)$ be the algebraic semigroup with the canonical Rees product*

$$(f_1, x_1, g_1)(f_2, x_2, g_2) = (f_1, x_1g_1f_2x_2, g_2).$$

The map $\psi : R \to \ker(S)$ defined by $(f, x, g) \mapsto fxg$ is an algebraic semigroup isomorphism.

In particular, any semigroup-theoretically simple algebraic semigroup is obtained by the above Rees construction. Under the Rees construction $\ker(S) = E(Se) \times H_e \times E(eS)$,

$$E(\ker(S)) = \{(f, (gf)^{-1}, g) | f \in E(Se), g \in E(eS)\},$$

which as an algebraic variety is isomorphic to $E(Se) \times E(eS)$.

Remark 1. Even if $e \in E(\ker(S))$, $E(J_e)(= E(\ker(S)))$ is not necessarily a subsemigroup of S. Let

$$S = \begin{pmatrix} 1 & K & K \\ 0 & K & K \\ 0 & 0 & 1 \end{pmatrix} := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & d & c \\ 0 & 0 & 1 \end{pmatrix} | a, b, c, d \in K \right\}.$$

Then $\ker(S) = \begin{pmatrix} 1 & K & K \\ 0 & 0 & K \\ 0 & 0 & 1 \end{pmatrix}$, and

$$E(\ker(S)) = \left\{ \begin{pmatrix} 1 & a & -ac \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in K \right\},$$

For any $e = \begin{pmatrix} 1 & a & -ac \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix} \in E(\ker(S))$, $H_e = eSe = \begin{pmatrix} 1 & a & K \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix} \cong (K, +)$, the one dimensional unipotent group.

$$E(\ker(S)) = \left\{ \begin{pmatrix} 1 & a & -ab \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in K \right\}$$

is not a subsemigroup of S, for

$$\begin{pmatrix} 1 & a & -ab \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & f & -fg \\ 0 & 0 & g \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & -ab - fg + ag \\ 0 & 0 & g \\ 0 & 0 & 1 \end{pmatrix}.$$

The following theorem answers precisely when $E(\ker(S))$ is a subsemigroup of S.

Proposition 4 ([9, Theorem 2.7]). *Let S be an algebraic semigroup. Then the following are equivalent:*

- *The minimal idempotents form a (closed) subsemigroup of S;*
- *For some (thus every) $e \in E(\ker(S))$, the map $S \twoheadrightarrow H_e$ defined by $a \mapsto eae$ is an (algebraic) semigroup homomorphism;*
- *For some (thus every) $e \in E(\ker(S))$, the map $\ker(S) \twoheadrightarrow H_e$ defined by $a \mapsto eae$ is an (algebraic) semigroup homomorphism.*

Most of the above structural properties of $\ker(S)$ have been extended to non-affine algebraic semigroups by Brion recently, see his article [2] in these proceedings.

3 Kernels of Irreducible Algebraic Monoids

An algebraic semigroup S may not contain identity element. If so, $S^1 := \{1\} \cup S$ is then an algebraic monoid and $\ker(S^1) = \ker(S)$. Unlike algebraic group instances, an algebraic monoid M's irreducible identity component M^c may not equal its connected identity component. For example, if N is an irreducible algebraic monoid with unit group $F \neq N$, then $M := N_e \cup \ker(N)$ is connected algebraic submonoid of N; but M is reducible when $\dim \ker(N) > 0$. Besides, for an algebraic monoid M, unlike an algebraic group, the identity component M^c may have very loose relation with other components. Nevertheless, unit dense algebraic monoids, in particular, irreducible algebraic monoids, have better behaviors. In this and subsequent sections, we consider irreducible linear algebraic monoids M over an algebraically closed field K. We have seen that the kernel of an arbitrary affine algebraic monoid has many nice properties. What about the kernels of irreducible monoids?

3.1 Basic Properties

Proposition 5 ([9, Proposition 3.2]). *Let M be an irreducible algebraic monoid with unit group G. Then* $\ker(M)$ *is an irreducible smooth closed subset of M with*

$$\dim\ker(M) = \dim G - \dim G_e = \dim H_e + \dim E(\ker(M)),$$

where e is a minimal idempotent of M.

So we have

$$\dim M = \dim M_e + \dim\ker(M).$$

This equality is informative and suggests that M be "geometrically spanned" by M_e (or G_e) and $\ker(M)$.

In subsemigroup generating, $\langle M_e, \ker(M)\rangle = M_e \cup \ker(M)$, which is also a smallest algebraic submonoid of M containing M_e and $\ker(M)$. However, this algebraically generated submonoid is reducible when the kernel is nonzero. This inspires my recent initial work on irreducible algebraic submonoid generating problems, especially with kernel data (cf. [4]) when $\ker(M)$ is not a group. There can be infinitely many minimal irreducible algebraic submonoids N containing $\ker(M)$ and 1 (for example, $M = K \times K$ with the usual multiplication. There are infinitely many minimal irreducible submonoids containing $\{1, 0\}$).

Suppose N is an irreducible submonoid of M containing $\{1\}$ and $\ker(N)$. Then, by the above proposition, the smallest irreducible submonoid of M containing both M_e and N is $M = \overline{\langle M_e, N\rangle}$.

There is a natural regular group action of $G \times G$ on M

$$(x, y) * a := xay^{-1} \text{ for } x, y \in G \text{ and } a \in M. \quad (1)$$

$\ker(M)$ is the "bottom" of the algebraic monoid M and carries a lot of structural information about M as well as the unit group G.

Proposition 6. *Assume $M \supsetneq G$ is irreducible and $e \in E(\ker(M))$.*

(1) $\ker(M) = GeG$ *thus* $\ker(M)$ *is a homogeneous space under the restriction to* $\ker(M)$ *of the action (1), which implies* $\ker(M)$ *is an irreducible smooth subvariety;*

(2) [10, Lemma 3.2] *The retraction* $\delta : M \twoheadrightarrow \ker(M)$ *is an open map and* $\ker(M) = \delta_e(G)$*;*

(3) [23, Theorem 1] $\ker(M)$ *is the unique closed* $G \times G$*-orbit under the action (1), which implies that the group embedding* $G \hookrightarrow M$ *is simple;*

(4) $\ker(M) = \cup_{f \in E(\ker(M))} fGf = \cup_{f \in E(\ker(M))} Gf = \cup_{f \in E(\ker(M))} fG$*;*

(5) $\ker(M) = E(Me)GE(eM) \cong E(Me) \times eGe \times E(eM)$ *as affine varieties.*

Remark 2. There is a matrix rank interpretation to Rittatore's characterization that $\ker(M)$ is the unique closed orbit of the two-sided action: assume $M \subseteq M_n(K)$. For each $a \in M$, all elements in the orbit GaG has the same matrix rank rank(a). While $\overline{GaG} = \overline{MaM}$ contains certain elements in M with lower matrix rank than rank(a). If the orbit GaG is closed then all the elements in orbit $GaG = MaM = \overline{MaM}$ are of constant matrix rank. This can only happen that $a \in \ker(M)$, due to Clark's theorem that $\ker(M)$ consists of elements in M with lowest matrix rank. As above mentioned, $\ker(M) = J_a = GaG = MaM$ for any $a \in \ker(M)$.

The following properties of an irreducible algebraic monoid (see [18, Theorem 6.30]), due to Putcha [16, Theorem 2.13] and [17, Theorem 2.3], are useful.

$$E(\ker(M)) \subset E(\overline{R(G)}) = \{e \in E(M) | J_e = J_e^2\}. \quad (2)$$

So

$$E(\ker(M)) = E(\ker(\overline{R(G)})). \quad (3)$$

3.2 One- and Two-Sided Centralizers of Minimal Idempotents of Algebraic Monoids

If an irreducible algebraic monoid M has a group kernel, then M has a unique minimal idempotent e so $\ker(M) = eGe = H_e$, and vise versa. Here our main interest is the case that the kernel is non-group. In an algebraic group G, usually, the normal subgroups assure the inheritance of many properties of G. If $e \in E(\ker(M))$, then $N_G(G_e) = C_G(e)$. We have the following

Proposition 7 ([11, Proposition 2.1]). *Let* M *be an irreducible algebraic monoid with unit group* G *and a minimal idempotent* e*. The following are equivalent:*

(i) $C_G(e) \lhd G$*;*
(ii) e *is central in* M*, that is,* $\ker(M) = H_e$ *is a group;*
(iii) $C(\ker(M)) = \ker(C(M))$*;*
(iv) $C(\ker(M)) \neq \emptyset$*;*
(v) $\dim E(\ker(M)) = 0$*;*
(vi) $C_B(e) = B$ *for some (or any) Borel subgroup* B *of* G*;*
(vii) $C_P(e) = P$ *for some (or any) parabolic subgroup* P *of* G*;*

(viii) $R_u(G) \subset C_G(e)$;
(ix) $G_e \lhd G$.

Moreover, if M *is regular with kernel a group, then* $\dim \ker(C(M)) \geq \dim R_u(C(G))$.

Although for any $e \in E(M)$, the algebraic subgroup $C_G(e)$ as well as $C_G^l(e)$ is irreducible [18, Theorem 6.16], $C_M(e)$ can be reducible [18, Example 6.15]. However, we have the following

Theorem 4 ([9, Theorem 3.8]). *Assume* M *is an irreducible algebraic monoid with unit group* G *and* $e \in E(\ker(M))$.

- $C_M(e)$ *is irreducible and* $E(C_M(e)) = E(M_e)$.
- $C_M^r(e)$ *is irreducible and* $E(C_M^r(e)) = E(M_e^r)$.
- *If* M *is regular, then* $C_M(e) = C_{E(M)}(e)C_G(e) = E(M_e)C_G(e)$.
- *If* M *is completely regular,* $C_M(e) = E(\overline{T})C_G(e)$*, where* T *is a maximal torus of* $C_G(e)$*. (We remind the reader (cf. [18, Theorem 7.4]) that a regular solvable algebraic monoid is completely regular.)*

When M is an irreducible algebraic monoid and $e \in E(\ker(M))$, by a theorem of Putcha (cf. [18, Theorem 6.11]), $M(e)$ is unit dense; by Brion [3], if $\mathrm{char}(K) = 0$, all $M(e)$, $C_M(e)$ and $C_M^l(e)$ are connected algebraic submonoids. Brion [3] also proved that M (as an algebraic variety) is normal if and only M_e (or $C_M(e)$) is so. Renner [21] proved this when M is regular. So $M(e) = M_e$ when M is normal. Actually, Brion's arguments for these properties work for arbitrary characteristic of the ground field K. Thus we have the following

Theorem 5. *When* M *is an irreducible algebraic monoid with* $e \in E(\ker(M))$*,*

(1) All $M(e)$*,* $C_M(e)$ *and* $C_M^l(e)$ *are connected dense unit algebraic submonoids;*
(2) M *(as an algebraic variety) is normal if and only if* M_e *(or* $C_M(e)$*) is so;*
(3) $M(e) = M_e$ *when* M *is normal.*

Remark 3. Property (3) does not hold in general for irreducible non-normal algebraic monoids. The following counter-example is suggested by [18, Example 6.12]

$$M = \{(a, b, c) \in K^3 | a^2 b = c^2, b \neq 0\}$$

with pointwise multiplication. Then M is a non-normal toric monoid with $\ker(M) = (0, K^*, 0)$ and with unit group $G = \{(a, b, c) \in K^3 | a^2 b = c^2 \neq 0\}$. Take $e = (0, 1, 0)$. Then $e \in E(\ker(M))$ and

$$G(e) = \{x \in G | xe = e\} = \{(a, 1, a) | a \in K^*\} \cup \{(b, 1, -b) | b \in K^*\}.$$

Thus $G(e)$ is not connected, but

$$M(e) = \{x \in M | xe = e\} = \{(a, 1, a) | a \in K\} \cup \{(b, 1, -b) | b \in K\}$$

is a connected monoid.

4 Regularity Conditions

Regular algebraic monoids are of central position in the Putcha-Renner theory of algebraic monoids. By the author [10, Theorem 5.1], every connected algebraic group G with nontrivial characters can be realized as the proper unit group of an irreducible regular (normal) algebraic monoid. On the other hand, every unit dense regular algebraic monoid M with unit group G is decomposed into a disjoint union of finite $G \times G$ orbits, of which the kernel $\ker(M)$ is the unique closed orbit; and each orbit is exactly a regular $\mathscr{J}$-class J_e, where $e \in E(M)$. Since the normalization of a regular algebraic monoid is regular, Renner [21] (or [22, Theorem 4.13]) determines all regular normal algebraic monoids with a given unit group as follows.

Theorem 6 ([22, Theorem 4.13]). *If M is irreducible, normal and regular, and $e \in E(\ker(M))$, then $\overline{G_e R_u(G)}$ is normal and regular and*

$$M = \overline{G_e R_u(G)} \times^{G_e R_u(G)} G.$$

All irreducible normal regular algebraic monoids are of this form.

Since a normalization of an irreducible affine regular monoid is an irreducible normal regular affine algebraic monoid, by the above theorem, an irreducible regular algebraic monoid is birationally equivalent to a normal algebraic monoid of the form given in the above theorem.

Most regularity conditions found are with kernel data.

Theorem 7. *Let M be an irreducible algebraic monoid with G its unit group, $e \in E(\ker(M))$. Then the following are equivalent:*

(1) *M is regular;*
(2) [18, Theorem 7.4] *M_e is regular (equivalently, one thus any of M_e^l, $C_M(e)$ and $C_M^l(e)$, is so);*
(3) [18, Theorem 7.4] *G_e is reductive;*
(4) [10, Theorem 5.5] $\dim R_u(G) = \dim E(\ker(M)) + \dim R_u(eGe)$*;*
(5) [10, Theorem 6.3] *the product map*

$$\psi : R_u(G_e^l) \times R_u(C_G(e)) \times R_u(G_e^r) \to R_u(G)$$

is a variety isomorphism.

In Theorem 7's last condition, the constraint that $\mathrm{Char}(K) = 0$ in [10, Theorem 6.3] is removed: the corresponding argument is replaced by using Renner's proof for [21, Proposition 2.5].

Characterization (3) is of fundamental impact on and the start of the development of the Putcha-Renner theory of reductive monoids.

Remark 4. The kernel of an irreducible regular algebraic monoid M also plays a determining role in semigroup algebra $F[M]$. Indeed, by [13], if M is regular, for any field F of characteristic zero, the semigroup algebra $F[M]$ is semisimple if and only if $F[\ker(M)]$ is semisimple.

5 Nilpotency, Solvability, Reductivity and Semisimpleness

The concepts of nilpotency, solvability and reductivity are of basic importance in linear algebraic groups, Lie groups and Lie algebras.

Theorem 8. *Let M be an irreducible algebraic monoid with G its unit group, $e \in E(\ker(M))$.*

- [6] *If M is nilpotent, then $E(M)$ is finite and* $\ker(M)$ *is a nilpotent (algebraic) group; when M is regular, the converse is true.*
- [11, Proposition 2.3] *M is solvable if and only if $C_M(e)$ is so.*
- [8] *M is reductive if and only if M is regular and* $\ker(M)$ *is a reductive group if and only if both M_e and* $\ker(M)$ *are reductive.*
- [8] *M is semisimple if and only if M_e is semisimple monoid and* $\ker(M)$ *is a semisimple group if and only if M is regular with $|C(E(M))| = 2$ and* $\ker(M)$ *is a semisimple group.*
- *M is toric if and only if both M_e and* $\ker(M)$ *are toric.*

We also have the following relevant results about nilpotency, solvability, reductivity and semisimpleness with kernel data.

Remark 5. Let M be an irreducible algebraic monoid with G its unit group, $e \in E(\ker(M))$.

- If M is not regular, that $|E(M)| < \infty$ and $\ker(M)$ being a nilpotent algebraic group does not imply the nilpotency of G: for example,

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix} a, b \in K \right\}.$$

- [18, Proposition 6.24] If eGe is solvable and $|E(M)| < \infty$, then M is solvable.
- Renner [20] classifies all normal (irreducible) regular solvable algebraic monoids with a given unit group G and proves that if M is normal, regular and solvable, then $M = R_u(G)_e^l C_M(T) R_u(G)_e^r$. His construction for group embedding is generalized by the author for general group embedding [10, Theorem 5.1]: when (and only when) $\mathrm{rank}(R(G)) > 0$ (equivalently, $\chi(G) \neq 0$), there is an irreducible regular normal algebraic monoid $M \supsetneq G$ such that G is the unit group of M. Indeed, by Renner's construction [20], there is a normal regular solvable monoid $\overline{R(G)}$; let $M = \overline{R(G)} \times^{R(G)} G$. Then M is irreducible normal regular monoid with G its unit group.

- It follows from the above theorem that, if M is reductive, then M_e is a reductive monoid and $\ker(M)$ is reductive group. Vinberg [24] proves that when $\mathrm{char}(K) = 0$, a normal reductive monoid is an almost direct product of a reductive monoid with zero and a reductive group. It shows how a normal reductive monoid is determined by M_e and $\ker(M)$.

6 Unipotent Radicals and Their Decompositions with Kernel Data

In this section, we describe the relations between $R_u(G)$ and algebraic subgroups $R_u(G_e^l)$, $R_u(G_e^r)$, $R_u(C_G^l(e))$, $R(C_G^r(e))$, $R(G_e)$, $R(C_G(e))$, where e is a minimal idempotent.

6.1 ker(M), *Minimal Idempotents and Unipotent Radicals*

In an irreducible algebraic monoid M with unit group G, there is very close interconnection between $\ker(M)$ and $R_u(G)$ as well as its one- and two-sided centralizers of minimal idempotents (cf. [10, 20, 21]).

Proposition 8 ([10, Corollary 4.5]). *Let M be an irreducible algebraic monoid, G its unit group and $e \in E(\ker(M))$. Then*

$$\ker(M) = R_u(G_e^l)eGeR_u(G_e^r).$$

For any $e \in E(\ker(M))$, by [10, Lemma 3.7],

$$E(\ker(M)) = \{vev^{-1} | v \in R_u(G)\}.$$

It is well known that $E(\ker(M)) \cong E(Me) \times E(eM)$ as algebraic varieties, and that $E(Me)$ and $E(eM)$ are irreducible closed subvarieties of M (thus so is $E(\ker(M))$). Since $E(Me) = \ker(M_e^l)$ and $E(eM) = \ker(M_e^r)$, by [9, Corollary 2.5 and Proposition 3.2], all $E(Me)$, $E(eM)$ and $E(\ker(M))$ are smooth varieties.

We have the following decomposition of the unit group of an irreducible monoid, which plays a key role in the study of interconnection between unipotent radicals and the kernel data.

Proposition 9 ([10, Lemma 3.6]). *Let M be an irreducible algebraic monoid with unit group G, e a minimal idempotent of M. Then*

$$G = C_G(e)R_u(G) = G_e^l C_G(e) G_e^r = R_u(G_e^l)C_G(e)R_u(G_e^r). \quad (4)$$

Remark 6. Intriguingly, following counterexample shows that in general $M \supseteq C_M(e)R_u(G) \neq R_u(G)C_M(e) \subsetneq M$ even if M is regular.

Let

$$M = \left\{ \begin{pmatrix} x & bx & 0 & 0 \\ 0 & ax & 0 & 0 \\ 0 & 0 & a^2x & cx \\ 0 & 0 & 0 & x \end{pmatrix} \mid a, b, c \in K, x \in P \right\} \subset M_{4n}(K),$$

where $P \subseteq \mathrm{GL}_n(K)$ is a nontrivial connected algebraic group. Then $M = G \cup \ker(M)$ thus M is regular, where

$$G = \left\{ \begin{pmatrix} x & bx & 0 & 0 \\ 0 & ax & 0 & 0 \\ 0 & 0 & a^2x & cx \\ 0 & 0 & 0 & x \end{pmatrix} \mid a \neq 0, b, c \in K, x \in P \right\},$$

$$\ker(M) = \left\{ \begin{pmatrix} x & bx & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & cx \\ 0 & 0 & 0 & x \end{pmatrix} \mid b, c \in K, x \in P \right\};$$

$$E(\ker(M)) = \left\{ \begin{pmatrix} I_n & bI_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & cI_n \\ 0 & 0 & 0 & I_n \end{pmatrix} \mid b, c \in K, x \in P \right\}.$$

Take $e = \left\{ \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix} \in E(\ker(M)) \right\}$. We have that

$$M \supsetneq C_M(e)R_u(G) \neq R_u(G)C_M(e) \subsetneq M$$

and

$$M \supsetneq C_M^l(e) \neq C_M^r(e) \subsetneq M.$$

As we shall see shortly, when $e \in E(\ker(M))$, all $R_u(G_e^l)$, $R_u(G_e^r)$, $R_u(C_G^l(e))$, $R_u(C_G^r(e))$, $R_u(G_e)$, $R_u(C_G(e))$ are subgroups of $R_u(G)$. We remind the reader that in general if e is not minimal, $R_u(G_e^l)$, $R_u(G_e^r)$, $R_u(C_G^l(e))$, $R_u(C_G^r(e))$, $R_u(G_e)$, $R_u(C_G(e))$ may not be subgroups of $R_u(G)$. For example, let $M = M_3(K)$ (whose unit group $G = GL_3(K)$) and $e = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. We have

$$C_G(e) = \begin{pmatrix} K^* & 0 \\ 0 & GL_2(K) \end{pmatrix}, C_G^r(e) = \begin{pmatrix} K^* & K & K \\ 0 & GL_2(K) & \end{pmatrix},$$

$$G_e = \begin{pmatrix} 1 & 0 \\ 0 & GL_2(K) \end{pmatrix}, \qquad G_e^r = \begin{pmatrix} 1 & K & K \\ 0 & GL_2(K) & \end{pmatrix},$$

$$R_u(C_G^r(e)) = \begin{pmatrix} 1 & K & K \\ 0 & I_2 & \end{pmatrix}, R_u(G_e^r) = \begin{pmatrix} 1 & K & K \\ 0 & I_2 & \end{pmatrix},$$

$$R(G) = K^* I_3, \qquad R_u(G) = \{I_3\},$$

where I_j is the identity matrix of degree j and $K^* = K \setminus \{0\}$.

We shall show how $R_u(G)$ is decomposed into product of unipotent radicals of those featured subgroups. The following results are basic but also serve as convenience tools.

Proposition 10 ([10, Lemmas 4.2 and 4.4]). *Let M be an irreducible algebraic monoid, G its unit group and $e \in E(\ker(M))$.*

(i) $R_u(G_e) \subseteq R_u(C_G(e)) \subseteq R_u(C_G^r(e)) \subseteq R_u(G)$ *and* $R_u(G_e) \subseteq R_u(G_e^r) \subseteq R_u(C_G^r(e))$;
(ii) $R_u(G_e) = R_u(G)_e \lhd C_G(e)$;
(iii) $R_u(G_e^r) = R_u(G)_e^r \lhd C_G^r(e)$;
(iv) $R_u(C_G(e)) = C_{R_u(G)}(e)$;
(v) $R_u(C_G^r(e)) = C_{R_u(G)}^r(e) = R_u(C_G(e)) R_u(G_e^r)$;
(vi) $R_u(G)$ *has the following decompositions:*

$$R_u(G) = R_u(G_e^l) R_u(C_G(e)) R_u(G_e^r) = R_u(C_G^l(e)) R_u(C_G^r(e)).$$

By Theorem 7 (5) below, the first decomposition is a direct product (as varieties) when M is regular.

Corresponding to the above properties of unipotent radicals, we have the following properties of radicals of algebraic subgroups of the unit group of an irreducible algebraic monoid.

Proposition 11 ([10, Lemma 4.1]). *Let M be an irreducible algebraic monoid, G its unit group and $e \in E(\ker(M))$.*

(i) $R(H_e) = \overline{R(H_e)} \subseteq \overline{R(C_G(e))}$;
(ii) $R(G_e) = R(G)_e \lhd C_G(e)$;
(iii) $R(G_e^r) = R(G)_e^r = R(G_e) R_u(G_e^r) \lhd C_G^r(e)$;
(iv) $R(C_G(e)) = C_{R(G)}(e)$;
(v) $R(C_G^r(e)) = C_{R(G)}^r(e) = R(C_G(e)) R_u(G_e^r)$;
(vi) $R(G_e) \subseteq R(C_G(e)) \subseteq R(C_G^r(e)) \subseteq R(G)$ *and* $R(G_e) \subseteq R(G_e^r) \subseteq R(C_G^r(e))$.

7 Relations Between the Kernel and Unipotent Radical

There are close relations between $\ker(M)$ (or minimal idempotents) and $R_u(G)$. Since the dimension of an algebraic variety carries lots of structural information of the variety, the following relation is perhaps the deepest and of intrinsic interest to structure of algebraic monoids.

Theorem 9 ([10, Theorem 5.5]). *Let M be an irreducible algebraic monoid, G the unit group of M and $e \in E(\ker(M))$. Then*

$$\begin{aligned}\dim R_u(G) &= \dim E(\ker(M)) + \dim R_u(G_e) + \dim R_u(eGe)\\ &= \dim E(\ker(M)) + \dim R_u(C_G(e)).\end{aligned}$$

Now the reductivity condition of M that M_e is a reductive monoid and $\ker(M)$ is a reductive group is a direct consequence of the above theorem.

A closely relevant dimensional relation is the following [10, Theorem 5.5]

$$\begin{aligned}\dim R(G) &= \dim E(\ker(M)) + \dim R(C_G(e))\\ &= \dim E(\ker(M)) + \dim R(G_e) + \dim R(eGe).\end{aligned}$$

It is well known that if $\mathrm{char}(K) = 0$, then a connected algebraic group G over K admits a Levi decomposition $G = G_0 \ltimes R_u(G)$. On the other hand, whenever an algebraic group G with nontrivial $\chi(G)$ admits a Levi decomposition, Renner [21] shows that there is a simple group embedding $G = G(M) \subsetneq M$ such that $G_0 = G_e$ for a minimal idempotent e of M. The following result shows that given an irreducible algebraic monoid M when G admits a Levi decomposition $G = G_e \ltimes R_u(G)$.

Proposition 12 ([10, Lemma 6.1]). *Let M be an irreducible algebraic monoid with unit group G and $e \in E(\ker(M))$. Then the following are equivalent:*

(i) $rank(G_e) = rank(G)$;
(ii) $G = G_e R_u(G)$;
(iii) $\ker(M) = R_u(G)eR_u(G)$;
(iv) $\ker(M) = \delta_e(R_u(G))$;
(v) $H_e = eR_u(C_G(e))$;
(vi) H_e *is unipotent.*

Then a natural question arises: is there any easily controllable relations between $\ker(M)$ and $R_u(G)$ featuring the Levi decomposition? Via the above mentioned retraction $\delta_e : M \twoheadrightarrow \ker(M)$, we have the following result.

Theorem 10 ([10, Theorem 6.4 and Corollary 6.5]). *Let M be an irreducible regular algebraic monoid with G its unit group, $e \in E(\ker(M))$ and $char(K) = 0$. Then the following are equivalent:*

(i) $G = G_e \ltimes R_u(G)$ *is a Levi decomposition;*
(ii) $G = G_e R_u(G)$*;*
(iii) $R_u(G) \cong \ker(M)$ *as algebraic varieties;*
(iv) $\dim R_u(G) = \dim \ker(M)$.

When one thus all of the above (equivalent) conditions are satisfied, the restriction to $R_u(G)$ of δ_e gives rise to an algebraic variety isomorphism from $R_u(G)$ onto $\ker(M)$.

Proposition 13 ([10, Corollary 6.6]). *If G is a connected algebraic group admitting a Levi decomposition with $\chi(G) \neq 0$ and $char(K) = 0$, then there is an irreducible regular algebraic monoid M with $M = \overline{G}$ such that $R_u(G)$ as an affine variety is isomorphic to* $\ker(M)$.

When $\ker(M)$ consists of (minimal) idempotents, what will happen?

Proposition 14 ([10, Corollary 6.7]). *Let M be an irreducible algebraic monoid with unit group G, $e \in E(\ker(M))$ and $char(K) = 0$. If $\ker(M) = E(\ker(M))$, then the following are equivalent.*

(i) *M is regular;*
(ii) *$G = G_e \ltimes R_u(G)$, where G_e is a Levi subgroup;*
(iii) *$R_u(G) \cong E(\ker(M))$ as algebraic varieties.*

Theorem 11 ([10, Theorem 6.8]). *Let M be an irreducible algebraic monoid with G its unit group, $e \in E(\ker(M))$ and $char(K) = 0$. Then the following are equivalent:*

(i) *$R_u(G) \cong E(\ker(M))$ as algebraic varieties;*
(ii) $\dim R_u(G) = \dim E(\ker(M))$*;*
(iii) *$C_G(e)$ is a reductive group.*

Problem 1. We do not know yet if the constraint $\mathrm{char}(K) = 0$ in the above theorems is removable.

8 Parabolic Subgroups

The parabolic subgroups play a crucial role in the structure of the algebraic groups, especially in the structure of reductive groups. Analogues of parabolic subgroups of algebraic groups are parabolic subgroups in complex Lie groups [14, §6.1], parabolic subalgebras in Lie algebras, parabolic subgroups of finite groups of Lie type, block-triangularizable sub-algebras in linear associative algebras (cf. [5]).

Parabolic subgroups also play an important role in the Putcha-Renner theory of (algebraic) reductive monoids. The parabolic subgroups for reductive monoids with zero have been well studied by Putcha et al. [18, 19, 22], where parabolic subgroups are realized as one-sided centralizers of idempotents in a cross-section lattice [18, Theorem 10.20]; while parabolic subgroups for general algebraic monoids are yet to be studied. In [11], the author proved that there exists a one-to-one correspondence between the parabolic subgroups of G and their counterparts of $C_G(e)$, where e is a minimal idempotent of M. In many situations, $\ker(M)$ is not a group, in other words, $C_G(e)$ is a proper (closed connected) subgroup of G (we shall see that this is equivalent to that $C_P(e)$ is proper in P for any/some parabolic subgroup P of G). Notice that when M is normal (respectively, smooth) if and only if the irreducible algebraic submonoid $C_M(e)$ is so; and $G/C_G(e)$ is an affine variety (cf. [3, Lemma 1.2.3 and Corollary 2.3.3]).

It is well known that if $\mathrm{char}(K) = 0$ or G being reductive, parabolic subgroups of a reductive group have a Levi decomposition. Here we shall show that, for an irreducible algebraic monoid M over an algebraically closed field and with unit group G, any parabolic subgroup P of G is uniquely determined by the proper subgroup $C_P(e)$; precisely, P can be properly decomposed into a product of the centralizer $C_P(e)$ of a minimal idempotent e and the unipotent radical $R_u(G)$ of G. In the important particular case that $C_G(e)$ is reductive, we shall see that the product $P = C_P(e)R_u(G)$ is semi-direct.

Recall that if M is an irreducible algebraic monoid with unit group G and $e \in E(\ker(M))$, then $E(\ker(M)) \subset \overline{R(G)}$. Thus $E(\ker(M)) \subset \overline{P}$ for any parabolic subgroup P of G.

Theorem 12 ([11, Theorem 2.4]). *Let M be an irreducible algebraic monoid with unit group G, $e \in E(\ker(M))$ and P a parabolic subgroup of G. Then*

(i) $P = C_P(e)R(G) = C_P(e)R_u(G)$;
(ii) $P = C_P(e) \ltimes R_u(G) \iff \dim R_u(G) = \dim E(\ker(M))$;
(iii) *if M is regular with* $\ker(M) = E(\ker(M))$, *then* $P = P_e \ltimes R_u(G)$.

Let's consider the case (iii) of the above theorem: M is regular with $\ker(M) = E(\ker(M))$ and a more general case that $G = G_e R_u(G)$. On the one hand, every connected group over an algebraically closed field of characteristic zero admits a Levi decomposition $G = G_0 \ltimes R_u(G)$. In this case, by Renner's construction [22, Chapter 4], there exists an irreducible regular algebraic monoid $M = \overline{G}$ such that $G_0 = G_e$ for some $e \in E(\ker(M))$. On the other hand, if $M = \overline{G_e R_u(G)}$ is regular with $e \in E(\ker(M))$, then the decomposition $G = G_e R_u(G)$ is a Levi decomposition.

Recall that a Putcha lattice of cross-sections Λ is a subset of $E(\overline{T})$, where T is a maximal torus of G, such that $|\Lambda \cap J| = 1$ for each regular $\mathscr{J}$-class J and if $e, f \in \Lambda$ then $J_e \geq J_f \implies e \geq f$.

We have the following determination of the parabolic subgroups.

Theorem 13. *Assume that M is an irreducible regular algebraic monoid with unit group G and $e \in E(\ker(M))$. If $G = G_e R_u(G)$ (in particular, if* $\ker(M) = E(\ker(M))$*), then each parabolic subgroup P is of the form $C^r_{G_e}(\Gamma) \ltimes R_u(G)$ or $C^l_{G_e}(\Gamma) \ltimes R_u(G)$, where Γ is a nonempty subset of a Putcha lattice of cross-sections Λ of M_e.*

Proof. We first prove that $P = P_e R_u(G)$. Indeed, $P_e R_u(G)$ is a connected closed subgroup of P. By the above theorem, $P = C_P(e)R_u(G)$. So

$$\begin{aligned}\dim P &= \dim C_P(e) + \dim R_u(G) - \dim R_u(C_P(e))\\ &= \dim P_e + \dim R_u(G) + \dim ePe - \dim R_u(P_e) - \dim R_u(ePe)\\ &= \dim P_e R_u(G) + (\dim ePe - \dim R_u(ePe))\\ &= \dim P_e R_u(G) (\text{ by [10, Lemma 6.1]}).\end{aligned}$$

Thus $P = P_e R_u(G)$. The rest then follows from [18, Theorem 10.20].

Proposition 15 ([11, Corollary 2.6]). *Let M be an irreducible algebraic monoid with unit group G, $e \in E(\ker(M))$ and P a parabolic subgroup of G. The following are equivalent:*

(i) $P = C_P(e) \ltimes R_u(G)$
(ii) $G = C_G(e) \ltimes R_u(G)$;
(iii) $B = C_B(e) \ltimes R_u(G)$ *for some Borel subgroup B of G;*
(iv) *M is regular with eGe reductive.*

When one thus all above conditions are satisfied, $G = C_G(e)R_u(G)$ is a Levi decomposition of G.

Example 1. Let

$$M = \begin{pmatrix}1 & K^n & 0\\ 0 & M_n(K) & 0\\ 0 & 0 & H\end{pmatrix},$$

where H is a nontrivial connected reductive group. Choose $e = \begin{pmatrix}1 & 0 & 0\\ 0 & 0_{M_n(k)} & 0\\ 0 & 0 & 1_H\end{pmatrix}$.

Then M is an irreducible regular algebraic monoid with

$$C_G(e) = \begin{pmatrix}1 & 0 & 0\\ 0 & \mathrm{GL}_n(K) & 0\\ 0 & 0 & H\end{pmatrix}, \qquad C_B(e) = \begin{pmatrix}1 & 0 & 0\\ 0 & T_n(K) & 0\\ 0 & 0 & B_H\end{pmatrix}$$

and

$$R_u(G) = \begin{pmatrix}1 & K^n & 0\\ 0 & 1_{M_n(K)} & 0\\ 0 & 0 & 1_H\end{pmatrix}.$$

Consider a Borel subgroup B of G, the unit group of M, given by

$$B = \begin{pmatrix} 1 & K^n & 0 \\ 0 & T_n(K) & 0 \\ 0 & 0 & B_H \end{pmatrix}$$

where $T_n(K)$ is the upper nonsingular matrices in $M_n(K)$ and B_H is a Borel subgroup of H. We have the following decompositions:

$$\begin{aligned} B &= C_B(e)R_u(G) = C_B(e) \ltimes R_u(G) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & T_n(K) & 0 \\ 0 & 0 & B_H \end{pmatrix} \ltimes \begin{pmatrix} 1 & K^n & 0 \\ 0 & 1_{M_n(K)} & 0 \\ 0 & 0 & 1_H \end{pmatrix}; \end{aligned}$$

$$\begin{aligned} G &= C_G(e)R_u(G) = C_G(e) \ltimes R_u(G) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{GL}_n(K) & 0 \\ 0 & 0 & H \end{pmatrix} \ltimes \begin{pmatrix} 1 & K^n & 0 \\ 0 & 1_{M_n(K)} & 0 \\ 0 & 0 & 1_H \end{pmatrix}. \end{aligned}$$

The corresponding Levi decomposition of B is

$$\begin{aligned} B &= T_B R_u(B) = T_B \ltimes R_u(B) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & D_n(K) & 0 \\ 0 & 0 & T_H \end{pmatrix} \ltimes \begin{pmatrix} 1 & K^n & 0 \\ 0 & U_n(K) & 0 \\ 0 & 0 & R_u(B_H) \end{pmatrix}, \end{aligned}$$

where $D_n(K)$ is the set of nonsingular diagonal matrices in $M_n(K)$, $U_n(K)$ the strictly upper triangular matrices in $\mathrm{GL}_n(K)$ and $T_H \subseteq B_H$ is a maximal torus of H.

Notice that $C_G(e)$ is reductive. Thus in this case the decomposition $C_G(e)R_u(G)$ is a Levi decomposition of G.

We denote by $\mathfrak{B}_G$ the set of Borel subgroups of G and by $\mathfrak{P}_B^G$ the set of parabolic subgroup of G containing the Borel subgroup B. The following result as we shall see is a direct consequence of Theorem 12.

Theorem 14 ([11, Theorem 3.1]). *Let M be an irreducible algebraic monoid with unit group G, $e \in E(\ker(M))$.*

(i) $\mathfrak{B}_{C_G(e)} = \{C_B(e) | B \in \mathfrak{B}_G\}$ *and the map*

$$\mathfrak{B}_G \to \mathfrak{B}_{C_G(e)} \text{ by } B \mapsto C_B(e)$$

is a bijection.

(ii) *For a given* $B \in \mathfrak{B}_G$, $\mathfrak{P}_{C_B(e)}^{C_G(e)} = \{C_P(e) | P \in \mathfrak{P}_B^G\}$ *and the map*

$$\mathfrak{P}_B^G \to \mathfrak{P}_{C_B(e)}^{C_G(e)} \text{ by } P \mapsto C_P(e)$$

is a bijection.

(iii) *If* P *is a parabolic subgroup of* G *and* $char(K) = 0$, *then*

$$G/P \cong C_G(e)/C_P(e)$$

as projective varieties.

(iv) *If* P *is a parabolic subgroup of* G, *then* $G/P = \{xP | x \in C_G(e)\}$.

Remark 7. A direct consequence of Theorems 12 and 14 is the following codimension equations:

$$\dim G - \dim P = \dim C_G(e) - \dim C_P(e), \text{ and}$$
$$\dim P - \dim R(G) = \dim C_P(e) - \dim C_{R(G)}(e).$$

The first immediately follows from Theorem 14 (ii). The second results from Theorem 12 (i) and [10, Lemma 3.4].

9 Group with Kernel

Putcha [15, Theorem 2.13] proved that if an irreducible algebraic monoid M is one dimensional with $M \neq G$ then $M = G \cup \{0\}$. Conversely, it is easy to show that if $M = G \cup \{0\}$ then M is one dimensional. The author [7, Proposition 4.7] proved that if M is an irreducible algebraic monoid with unit group G and $M = G \cup E(M)$, then M is solvable with $\text{rank}(G) = 1$, where $e \in E(\ker(M))$. Furthermore, we see that, in this case, $M = R_u(G)M_eR_u(G)$. In fact, by [11], M is solvable and $G = C_G(e)R_u(G) = G_eR_u(G)$. Then

$$\begin{aligned} M &= \overline{G_eR_u(G)} = G_eR_u(G) \cup \ker(\overline{G_eR_u(G)}) \\ &= G_eR_u(G) \cup R_u(G)eR_u(G) = R_u(G)M_eR_u(G). \end{aligned}$$

We now study the structure of an algebraic monoid which is a group with the kernel.

Proposition 16 ([4, Proposition 5.1]). *Let* M *be an irreducible algebraic monoid with unit group* G.

(i) $M = G \cup \ker(M)$ *if and only if* $\dim M = 1 + \dim \ker(M)$;
(ii) *if* $M = G \cup \ker(M)$, *then* M *is the smallest algebraic monoid containing* $\ker(M)$.

An irreducible algebraic monoid M which is a group with kernel has a simple structure of M_e (group with zero), where e is a minimal idempotent, while its kernel can be of quite arbitrary structure: $\ker(M)$ can be a zero, can be any connected group, or can be of a general Rees structure.

Example 2. Let

$$M = \begin{pmatrix} 1 & K \\ 0 & K \end{pmatrix} \times P,$$

where P is a connected algebraic group. Let $e = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 1_P)$. Then $e \in E(\ker(M))$, and

$$G = \begin{pmatrix} 1 & K \\ 0 & K^* \end{pmatrix} \times P; \qquad \ker(M) = \begin{pmatrix} 1 & K \\ 0 & 0 \end{pmatrix} \times P;$$

$$M = G \cup \ker(M); \qquad M_e = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix} \times \{1_P\} = M_e^r;$$

$$C_M(e) = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix} \times P = C_M^r(e); \qquad R_u(G) = \begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix} \times R_u(P);$$

$$G_e = \begin{pmatrix} 1 & 0 \\ 0 & K^* \end{pmatrix} \times \{1_P\}; \qquad H_e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times P;$$

$$H_e R_u(G) = \begin{pmatrix} 1 & K \\ 0 & 0 \end{pmatrix} \times P = \ker(M); \quad R_u(G) H_e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times P \subsetneq \ker(M);$$

$$M_e^l = \begin{pmatrix} 1 & K \\ 0 & K \end{pmatrix} \times \{1_P\};$$

$$M_e^l R_u(G) = R_u(G) M_e^l = M_e R_u(G) = \begin{pmatrix} 1 & K \\ 0 & K \end{pmatrix} \times R_u(P);$$

$$C_M^l(e) = C_M(e) R_u(G) = M = R_u(G) C_M(e).$$

We saw from the counterexample in (6) that it can happen that $M = G \cup \ker(M)$ but

$$M \supsetneq C_M(e) R_u(G) \neq R_u(G) C_M(e) \subsetneq M$$

and

$$M \supsetneq C_M^l(e) \neq C_M^r(e) \subsetneq M.$$

Problem 2. If M has no proper irreducible closed submonoid containing $\ker(M)$, is M a group with kernel?

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Algebraic Monoids and Renner Monoids

Zhenheng Li, Zhuo Li, and You'an Cao

Abstract We collect some necessary concepts and principles in the theory of linear algebraic monoids which apply to further investigation on other topics such as the classification of reductive monoids, representations of algebraic monoids, monoids of Lie type, cell decompositions, monoid Hecke algebra, and monoid schemes. We use classical monoids as examples to demonstrate notions.

Keywords Algebraic monoid • Renner monoid • Classical monoid • Rook monoid

Subject Classifications: 20M32

1 Introduction

The Putcha-Renner theory of linear algebraic monoids is a big subject, which is built on linear algebraic groups, torus embeddings, and semigroups [61, 82]. Over the last three decades the theory has made significant progress in different fields: reductive monoids, Renner monoids, finite monoids of Lie type, monoids on groups with BN-pairs, group embeddings, monoid schemes, semisimple monoids,

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$\mathscr{J}$-irreducible monoids, combinatorics, and classical algebraic monoids [4, 12, 36, 40, 51, 61, 62, 64, 71, 75, 77, 80]. Unfortunately, the theory has a marketing problem as Solomon mentioned in [85], which is a very engaging introduction to the theory.

The aim of this survey paper is two-fold. We first give an introduction to the theory of linear algebraic monoids, and then focus on the recent developments in Renner monoids, with the intent to attract readers with interests in algebraic groups, combinatorics, Lie theory, and semigroup theory. We state the main theorems and provide sources instead of giving proofs. Occasionally, for some statements of conclusions we give short arguments.

Classical algebraic monoids are a special class of linear algebraic monoids. Throughout the paper, classical algebraic monoids are used as examples extensively to demonstrate important concepts.

The following section is devoted to algebraic monoids in general, including definitions, methods to construct algebraic monoids, classical monoids, $\mathscr{J}$-class structures, irreducible algebraic monoids, Putcha lattices, and classical rook monoids. In the next section we describe reductive monoids, with emphasis on Jordan decomposition, parabolic subgroups, type maps, and $\mathscr{J}$-irreducible monoids. The finial section records various recent results on Renner monoids such as definitions and properties, classical Renner monoids, standard form of elements in a Renner monoid, reduced row echelon form, length function, generators and defining relations, orders, conjugacy classes, generating functions, and generalized Renner monoids.

2 Algebraic Monoids

Let M be an affine variety over an algebraically closed field K together with the structure of a semigroup. We call M an *affine algebraic semigroup*, or simply *algebraic semigroup*, if the associative operation in M is a morphism of varieties. An affine algebraic monoid is an affine algebraic semigroup with an identity. The *unit group* of an algebraic monoid M is the set of elements of M with an inverse in M. We are concerned mainly with algebraic monoids, though we sometimes state some results on algebraic semigroups.

There are so many interesting examples of algebraic monoids. Every algebraic group is an algebraic monoid; every finite monoid is an algebraic monoid. Viewed as an affine space of dimension n^2, the set $\mathbf{M}_n$ of all $n \times n$ matrices over K is an algebraic monoid under matrix multiplication, called the *general linear monoid*. The unit group of $\mathbf{M}_n$ is the general linear group $\mathbf{GL}_n$. The monoid $\mathbf{D}_n$ of diagonal matrices is algebraic with the group $\mathbf{T}_n$ of invertible diagonal matrices as its unit group. Let $\overline{\mathbf{B}}_n$ be the monoid of all upper triangular matrices. Then $\overline{\mathbf{B}}_n$ is an algebraic monoid with unit group $\mathbf{B}_n$ consisting of all invertible upper triangular matrices.

A Zariski closed submonoid of $\mathbf{M}_n$ is called a *linear algebraic monoid*. The following theorem shows that every affine algebraic monoid is isomorphic to a linear algebraic monoid.

Theorem 1 ([11, II, §2, Theorem 3.3]; [51, Corollary 1.3]). *Every affine algebraic semigroup is isomorphic to a closed subsemigroup of some* $\mathbf{M}_n$. *In particular, every affine algebraic monoid is isomorphic to a closed submonoid of some* $\mathbf{M}_n$.

Just as the closed embedding of an algebraic group into some $\mathbf{GL}_n$ in algebraic group theory reduces the study of algebraic groups to that of closed subgroups in $\mathbf{GL}_n$, this theorem reduces the study of algebraic monoids to that of closed submonoids in $\mathbf{M}_n$. From now on, we identify an affine algebraic monoid with its closed embedding in $\mathbf{M}_n$, and simply refer to it as an *algebraic monoid*. Every algebraic monoid M has a dimension, which is the dimension of M as an algebraic variety [24]. If M is a point then its dimension is zero; if M is a curve then its dimension is one; if M is a surface then its dimension is two. Also $\dim \mathbf{M}_n = n^2$, $\dim \mathbf{D}_n = n$ and $\dim \overline{\mathbf{B}}_n = \frac{n(n+1)}{2}$.

The unit group of an algebraic monoid M determines the structure of M to some extent, and it has been of primary interest in finding connections between the structures of an algebraic monoid and its unit group [60]. Theorem 2 shows that the unit group is an open subgroup in the monoid and is equal to the intersection of the monoid with the general linear group.

Theorem 2 ([11, II, §2, Corollary 3.5]; [74, Corollary 2.2.3]). *Let* M *be an algebraic monoid. Then its unit group* $G = M \cap \mathbf{GL}_n$. *Furthermore,* G *is an algebraic group and there is a morphism* $\alpha : M \to K$ *such that* $G = \alpha^{-1}(K^*)$, *where* $K^* = K \setminus \{0\}$. *In particular,* G *is open in* M.

The set $E(M)$ of idempotents of M contains certain controlling structural information about M. This set carries the partial order

$$e \leq f \Leftrightarrow fe = e = ef.$$

In what follows, we assume that the partial order on any subset of $E(M)$ is inherited from this one.

Proposition 1 ([61, Corollary 3.26]; [82, Proposition 3.12]). *Let* M *be an algebraic monoid and* $e \in E(M)$. *Then* eMe *is an algebraic monoid; its unit group is precisely the* $\mathscr{H}$*-class of* e. *This unit group is an algebraic group and is open in* eMe.

The Zariski closures of subsets of M are fundamental in the theory of algebraic monoids. Lemma 1 below is useful technically in dealing with these closures. If X is a subset of M, we use $\overline{X}$ to denote the Zariski closure of X in M. In particular, if $M = \mathbf{M}_n$, then $\overline{\mathbf{GL}}_n = \mathbf{M}_n$, $\overline{\mathbf{T}}_n = \mathbf{D}_n$ and $\overline{\mathbf{B}}_n$ is the Zariski closure of $\mathbf{B}_n$.

Lemma 1 ([53, Lemma 1.2]). *Let* X *and* Y *be subsets of an algebraic monoid* M *with unit group* G. *Then*

(1) $\overline{XY} = \overline{\overline{X}\,\overline{Y}}$.
(2) If $a, b \in G$, *then* $\overline{aXb} = a\overline{X}b$.

How to construct algebraic monoids? It is an easy task, based on the obvious fact that a closed submonoid of an algebraic monoid is again an algebraic monoid. The following corollary provides us with a great deal of examples of algebraic monoids.

Corollary 1. *If S is a submonoid of $\mathbf{M}_n$, then $\overline{S}$ is an algebraic monoid.*

Indeed, it follows from Lemma 1 that $\overline{S}\,\overline{S} \subseteq \overline{\overline{S}\,\overline{S}} = \overline{S\,S} = \overline{S}$. Thus $\overline{S}$ is a closed submonoid of $\mathbf{M}_n$.

Corollary 2. *Let G be a subgroup of $\mathbf{M}_n$. Then $\overline{G} \subseteq \mathbf{M}_n$ is an algebraic monoid. If $G \subseteq \mathbf{GL}_n$, then the unit group of $\overline{G}$ is the Zariski closure of G in $\mathbf{GL}_n$. Furthermore, if G is an algebraic group then the unit group of $\overline{G}$ is G.*

Algebraic monoids are special semigroups, of which *Green relations* $\mathscr{J}, \mathscr{L}, \mathscr{R}$, and $\mathscr{H}$ are fundamental structure elements. Let S be a semigroup and $a, b \in S$. Then by definition

$$\begin{aligned} a\mathscr{J}b &\quad \text{if } S^1aS^1 = S^1bS^1; \\ a\mathscr{L}b &\quad \text{if } S^1a = S^1b; \\ a\mathscr{R}b &\quad \text{if } aS^1 = bS^1; \\ a\mathscr{H}b &\quad \text{if } a\mathscr{L}b \text{ and } a\mathscr{R}b. \end{aligned}$$

where $S^1 = S$ if S is a monoid and $S^1 = S \cup \{1\}$ with obvious multiplication if S is not a monoid. We use J_a and H_a to denote the $\mathscr{J}$-classes and $\mathscr{H}$-classes of a, respectively.

Algebraic semigroups are special kinds of strongly π-regular semigroups. A semigroup S is *strongly π-regular* if for any $a \in S$ there exists a positive integer k such that a^k lies in H_e for some idempotent $e \in E(S)$. A strongly π-regular semigroup is also refereed to as an *epigroup* or a *group-bound semigroup* in the literature of semigroup theory [14, 17, 27–30]. Every finite semigroup is strongly π-regular; so is the full matrix monoid consisting of all square matrices over a field. The concept of strongly π-regular captures the semigroup essence of algebraic semigroups [82]. Putcha [51, 52, 61] and Brion and Renner [5] in this proceedings contain more information on algebraic semigroups and strongly π-regular semigroups. Okninski [48] is a very comprehensive reference on strongly π-regular matrix semigroups.

A nonempty subset I of S is an *ideal* if $S^1IS^1 \subseteq I$. Clearly S is an ideal of S. If S is strongly π-regular and S is its only ideal then we say that S is *completely simple*. The minimal ideal, if it exists, is called the kernel of S. The reader who is interested in kernel of linear algebraic monoids can find useful results in Huang [20–22]. The following two theorems confirm that every algebraic semigroup and hence algebraic monoid is strongly π-regular, and contains a closed completely simple kernel.

Theorem 3 ([51, Corollary 1.4]). *Let S be an algebraic semigroup. Then there exists a positive integer n such that a^n lies in a subgroup of S for all $a \in S$. In particular, every algebraic monoid is strongly π-regular.*

Theorem 4 ([51, Corollary 1.5]). *Every algebraic semigroup has a kernel which is closed and completely simple.*

2.1 Some Classical Monoids

We introduce some families of algebraic monoids, called classical monoids, which are closely related to classical groups. These monoids play an important role in the theory of algebraic monoids [31–33, 36]. The parameter l in each case is 1 less than the dimension of the closed subgroup of diagonal matrices in the unit group G of the monoid under discussion. This l is also the dimension of the Cartan subalgebra of the Lie algebra of G.

A_l: The *general linear monoid* $\mathbf{M}_n$ with $n = l + 1$: Let $G = K^*\mathbf{SL}_n$ where $\mathbf{SL}_n$ is the special linear group consisting of the matrices of determinant 1 in $\mathbf{GL}_n$. Then $G = \mathbf{GL}_n$, and $\mathbf{M}_n = \overline{G}$.

C_l: The *symplectic monoid* $\mathbf{MSp}_n$ with $n = 2l$: The symplectic group is

$$\mathbf{Sp}_n = \{A \in \mathbf{GL}_n \mid A^\top JA = J\}$$

where $J = \begin{pmatrix} 0 & J_l \\ -J_l & 0 \end{pmatrix}$ with $J_l = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ of size l. Let $G = K^*\mathbf{Sp}_n$. Then $G \subseteq \mathbf{GL}_n$. The monoid $\overline{G}$ is called the symplectic monoid which will be denoted by $\mathbf{MSp}_n$. It is usually hard to give a concrete algebraic description of the Zariski closure of a subset of an algebraic monoid. It follows, however, from Doty [12] that

$$\mathbf{MSp}_n = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ \text{ for some } c \in K\}.$$

B_l: The *odd special orthogonal monoid* $\mathbf{MSO}_n$ with $n = 2l + 1$: If the characteristic of K is not 2, then the odd special orthogonal group is by definition

$$\mathbf{SO}_n = \{A \in \mathbf{SL}_n \mid A^\top JA = J\}$$

where $J = \begin{pmatrix} 0 & 0 & J_l \\ 0 & 1 & 0 \\ J_l & 0 & 0 \end{pmatrix}$. Let $G = K^*\mathbf{SO}_n \subseteq \mathbf{GL}_n$. The monoid $\overline{G}$ is called the odd special orthogonal monoid, denoted by $\mathbf{MSO}_n$. By [12] we have

$$\mathbf{MSO}_n = \{x \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ \text{ for some } c \in K\}.$$

D_l: This is the *even special orthogonal monoid* $\mathbf{MSO}_n$ with $n = 2l$, defined by taking the Zariski closure of $K^*\mathbf{SO}_n$ in which $\mathbf{SO}_n$ is given by the same condition as B_l: $A^\top JA = J$, where the matrix J now is $\begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix}$ (if the characteristic of K is not 2). Notice that the set

$$M = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ \text{ for some } c \in K\}$$

is an algebraic monoid. Naturally we ask: whether $\mathbf{MSO}_n$ equals M? Unfortunately, no. In fact, M is reducible, and $\mathbf{MSO}_n$ is its identity component. More information about this M will be provided in Sect. 2.3.

The symplectic and special orthogonal algebraic monoids arise geometrically as monoids of linear transformations that dilate certain skew-symmetric and symmetric bilinear forms, respectively.

2.2 Monoids Induced from Representations

To construct further examples of algebraic monoids, we start with rational representations of algebraic groups. A rational representation of an algebraic group G_0 is a group homomorphism $\rho : G_0 \to \mathbf{GL}_n$ which is also a morphism of varieties [1,24]. The image $\rho(G_0)$ is an algebraic group. Let

$$G = K^*\rho(G_0) = \{c\rho(g) \mid c \in K^* \text{ and } g \in G_0\}.$$

Then G is an algebraic group by [24, Corollary 7.4]. However, G is not a closed subset of $\mathbf{M}_n$ since the zero matrix is in $\overline{G}$ but not in G. Write

$$M(\rho) = \overline{G}.$$

It follows from Corollary 2 that $M(\rho)$ is an algebraic monoid with unit group G. In addition, if G_0 is irreducible, so are $M(\rho)$, G and $\rho(G_0)$. Clearly, $\rho(G_0)$ is a subgroup of G.

Why do we multiple $\rho(G_0)$ by K^* and then take the Zariski closure of the product? We note that if $\rho(G_0)$ is closed in $\mathbf{M}_n$, then the monoid $\overline{\rho(G_0)} = \rho(G_0)$ is a group, nothing new. This is the case if G_0 is the *special linear group*. To make sure that $\overline{G}$ is a monoid which includes G properly, G must contain at least one matrix whose determinant is not 1. Renner [74, Theorem 3.3.6] and Waterhouse [91] provide conditions under which an algebraic group G can be embedded as the unit group of an algebraic monoid which are not a group. Huang [21, Theorem 5.1] refines the above result and states that under the same conditions, the group G may be embedded properly into a normal regular algebraic monoid. We refer to the above references for more details.

The classical monoids can be constructed via certain representations of classical groups. Let $V = K^n$, and G_0 be the special linear group, symplectic group (n is even), or special orthogonal group. Then G_0 acts naturally on V by their very definition, and we obtain the *natural representation* $\rho : G_0 \to \mathbf{GL}_n$ with $\rho(g) = g$. The monoid $M(\rho)$ is the general linear monoid, symplectic monoid, and special orthogonal monoid, respectively.

Let's explore two more examples obtained by representations. They are taken from [85] and the latter is a variant of Example 8.5 of [61].

Example 1. Let $G_0 = \mathbf{SL}_m$ and $V = K^m \otimes K^m$ with basis $\{v_i \otimes v_j \mid 1 \leq i < j \leq m\}$. Define a rational representation $\rho : G_0 \to \mathbf{GL}_n$ by $\rho(g)(v \otimes v') = gv \otimes gv'$, where $n = m^2$. The monoid $M(\rho) = \{a \otimes a \mid a \in \mathbf{M}_m\}$ is isomorphic to $\mathbf{M}_m$. In particular, $G = \{g \otimes g \mid g \in \mathbf{GL}_m\}$, isomorphic to $\mathbf{GL}_m$.

Example 2. Let $V = K^m \otimes K^m$ be as in Example 1 and let $G_0 = \mathbf{SL}_m$. Define a rational representation $\rho : G_0 \to \mathbf{GL}_n$ by $\rho(g)(v \otimes v') = gv \otimes (g^{-1})^\top v'$, where $n = m^2$. Though the monoid $M(\rho)$ is hard to describe algebraically, we however know that the unit group of $M(\rho)$ is closely related to $\mathbf{SL}_m$. But $M(\rho)$ is different dramatically from $\mathbf{M}_m$ since $E(M(\rho))$ and $E(\mathbf{M}_m)$ are not isomorphic.

2.3 Irreducible Algebraic Monoids

An algebraic monoid is *irreducible* if it is irreducible as an affine algebraic variety, that is, it is not a union of proper Zariski closed subsets. The monoids $\mathbf{M}_n$, $\mathbf{D}_n$, and $\overline{\mathbf{B}}_n$ are irreducible. The classical monoids of types A_l, B_l, C_l, and D_l above are all irreducible. The monoid in Example 8 is irreducible since it is isomorphic to the affine space K^{n+1} as varieties.

An algebraic monoid $M \subseteq \mathbf{M}_n$ is connected if it is connected as a subset of $\mathbf{M}_n$ in the Zariski topology. Irreducible algebraic monoids are connected, but not conversely. For example, the monoid

$$M = \{(a, b) \in K^2 \mid a^2 = b^2\}$$

is connected but not irreducible. The monoid in the following example is another instance of connected monoids even though not irreducible.

Example 3 ([12, Section 6]). Assume that the characteristic of K is not 2. Let

$$M = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ \text{ for some } c \in K\}$$

where $J = \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix}$. The unit group of M is

$$G = \{A \in \mathbf{M}_n \mid AJA^\top = cJ \text{ for some } c \in K^*\}.$$

The subgroup T of G consisting of invertible diagonal matrices in G is a maximal torus of G. The orthogonal group in $\mathbf{GL}_n$ is

$$O_n = \{A \in \mathbf{GL}_n \mid AJA^\top = J\}.$$

Let $O_n^+ = \{A \in O_n \mid \det A = 1\}$ and $O_n^- = \{A \in O_n \mid \det A = -1\}$. Then $O_n = O_n^+ \cup O_n^-$. Denote by G^+ the subgroup of G generated by T and O_n^+. Then $G^+ = K^* SO_n$ is a closed and connected subgroup of G. So the Zariski closure of G^+ in $\mathbf{M}_n$ is the even special orthogonal monoid $\mathbf{MSO}_n$ with unit group G^+. Therefore, $\mathbf{MSO}_n$ is the irreducible identity component of M.

If $n = 2$, then the monoid in Example 3 is

$$M = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \middle| a, b \in K \right\} \cup \left\{ \begin{pmatrix} & c \\ d & \end{pmatrix} \middle| c, d \in K \right\}.$$

This monoid has two irreducible components, and so is reducible, but is connected. However, its unit group G is not connected since G has two connected components.

Theorem 5 ([61, Proposition 6.1]). *Suppose that M is an irreducible algebraic monoid with unit group G, and $a, b \in M$. Then*

(1) $a \mathscr{J} b$ if and only $a \in GbG$.
(2) $a \mathscr{L} b$ if and only if $a \in Gb$.
(3) $a \mathscr{R} b$ if and only if $a \in bG$.

This theorem allows us to interpret $\mathscr{J}$, $\mathscr{L}$ and $\mathscr{R}$-classes in M using group actions of G on M. Each $\mathscr{J}$, $\mathscr{L}$, and $\mathscr{R}$-class is an orbit of a group action, which sometimes indicates connections with geometry such as orbits and closures.

Since $GM \subseteq M$ and $MG \subseteq M$, we have the *left action* of G given by $g{\cdot}a = ga$, and the *right action* given by $a{\cdot}g = ag^{-1}$. Theorem 5 shows that if M is irreducible, then a, b lie in the same $\mathscr{L}$-class if and only if they lie in the same left G orbit, and that a, b are in the same $\mathscr{R}$-class if and only if they are in the same right G orbit. The $\mathscr{L}$-class of a is thus the orbit Ga, and the $\mathscr{R}$-class of a is the orbit aG. The left and right G orbits are closely related to row and column echelon forms of M, respectively, which will be described in Sect. 4.4.

Consider the group action of $G \times G$ on M by $(g, h) \cdot a = gah^{-1}$ for $g, h \in G$ and $a \in M$. Let $G \backslash M / G$ denote the set of orbits GaG for this action. It follows from Theorem 5 that if M is irreducible, then a, b lie in the same $\mathscr{J}$-class if and only if they lie in the same $G \times G$ orbit. Moreover, the $\mathscr{J}$-class $J_a = GaG$. We give $G \backslash M / G$ the partial order

$$J_a \le J_b \Leftrightarrow MaM \subseteq MbM \Leftrightarrow GaG \subseteq \overline{GbG},$$

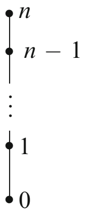
henceforth $(G \backslash M / G, \le)$ is a poset. We examine this poset for different irreducible monoids.

Example 4. Let $M = \mathbf{M}_n$. Then $G = \mathbf{GL}_n$. If $a, b \in M$ then $GaG = GbG$ if and only if a and b are of the same rank. There is a bijection of $G \backslash M/G$ onto $\{0, 1, \cdots, n\}$ given by $GaG \mapsto \text{rank } a$. The partial order is the natural linear order on $\{0, 1, \cdots, n\}$ as illustrated in the first figure below. Clearly, the number of $G \times G$ orbits in M is $n + 1$.

Example 5. Let $M = \mathbf{MSp}_n$ with unit group G and $n = 2l$. If $a, b \in M$ then $GaG = GbG$ if and only if rank $a =$ rank b. There is a bijection of $G \backslash M/G$ onto $\{0, 1, \cdots, l, n\}$ given by $GaG \mapsto \text{rank } a$. The partial order is the natural linear order on $\{0, 1, \cdots, l, n\}$ as illustrated in the second figure. Note that there are no elements of rank greater than l but less than n in M. The number of $G \times G$ orbits in M is $l + 2$.

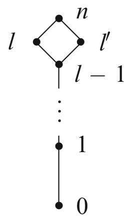
Example 6. The lattice of the $G \times G$ orbits of the odd special orthogonal monoid $\mathbf{MSO}_n$ with $n = 2l + 1$ is isomorphic to that of the symplectic monoid $\mathbf{MSp}_{2l}$.

Example 7. Let $M = \mathbf{MSO}_n$ with unit group G and $n = 2l$. If $a, b \in M$ then $GaG = GbG$ if and only if rank $a =$ rank $b = 0, 1, \cdots, l - 1, n$. However, there are two $G \times G$ orbits of rank l whose representatives are, respectively, $\text{diag}(1, \cdots, 1, 0, \cdots, 0)$ and $\text{diag}(1, \cdots, 1, 0, 1, \cdots, 0)$ each with l copies of 1. Let l' be a symbol. Then there is a bijection of $G \backslash M/G$ onto $\{0, 1, \cdots, l, l', n\}$ whose partial order is given in the third figure below. There are no elements of rank greater than l but less than n in M. The number of $G \times G$ orbits in M is $l + 3$.



$G \times G$ orbitsof $\mathbf{M}_n$

$G \times G$ orbitsof $\mathbf{MSp}_n$



$G \times G$ orbitsof $\mathbf{MSO}_n$

Example 8 ([54], Example 15). Let $M \subseteq \mathbf{M}_{n+1}$ consist of matrices

$$\begin{pmatrix} a & a_1 & a_2 & \cdots & a_n \\ & a & 0 & \cdots & 0 \\ & & a & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & a \end{pmatrix}$$

where $a, a_1, \ldots, a_n \in K$. The unit group of M consists of matrices in M whose diagonal element a is not zero. There are infinitely many $G \times G$ orbits if $n > 1$. In fact, if we denote by $(a, a_1, \cdots, a_n)$ the matrix above, then the $G \times G$ orbits are G, $\{0\}$, and orbits which contain matrices $(0, a_1, \cdots, a_n)$ with at least one a_i not zero. Moreover, for the latter we have that two elements $(0, a_1, \cdots, a_n)$ and $(0, b_1, \cdots, b_n)$ lie in the same orbit if and only if there is $c \in K^*$ such that $b_i = ca_i$ for all i. So these orbits are in bijection with points in $\mathbf{P}^{n-1}(K)$, the projective space of dimension $n-1$. More specifically, M has n orbits of form $(0, 0, \cdots, 0, a_i, 0, \cdots, 0)$ where $a_i \neq 0$ and $1 \leq i \leq n$, and M has infinitely many orbits $(0, a_1, \cdots, a_n)$ with at least two nonzero entries. Let $a_{i_1}, \cdots, a_{i_k}$ be all the nonzero entries in orbit $(0, a_1, \cdots, a_n)$. Then

$$(0, a_1, \cdots, a_n) \leq (0, b_1, \cdots, b_n) \text{ if and only if none of } b_{i_1}, \cdots, b_{i_k} \text{ is zero.}$$

Clearly, 0, 1 are idempotents of M. Check that they are the only idempotents of M. This leads to the following important definition.

Definition 1. Let M be an algebraic monoid. A $\mathscr{J}$-class J is regular if $E(J) \neq \emptyset$. Define

$$\mathscr{U}(M) = \{J \subseteq M \mid J \text{ is a regular } \mathscr{J}\text{-class}\}.$$

If M is irreducible, then $\mathscr{U}(M) = \{J \in G\backslash M/G \mid J \cap E(M) \neq \emptyset\}$ and is a finite lattice. A key result of [54] is that idempotents e, f are in the same $G \times G$ orbit if and only if they are conjugate under G. This result is useful throughout the theory of algebraic monoids. In particular, it plays a critical role in describing certain monoids with exactly one nonzero minimal $G \times G$ orbit.

Our intention below is to introduce height function on $E(M)$ and $\mathscr{U}(M)$ for irreducible algebraic monoids M. We begin by collecting results about idempotents of M.

Theorem 6 ([61, Corollaries 6.8 and 6.10 and Proposition 6.25]). *Let M be an irreducible algebraic monoid M with unit group G. Let T be a maximal torus, and W the Weyl group of G. Then*

(1) $E(M) = \cup_{g \in G}\, gE(\overline{T})g^{-1}$.
(2) Two elements $e, f \in E(\overline{T})$ are conjugate under G if and only if they are conjugate under W.

We observe from the previous theorem that there are as many G-orbits as W-orbits in $E(\overline{T})$, and that $E(M)$ is not only stable under the conjugation action of G on M

$$a \mapsto gag^{-1} \quad \text{for } a \in M \text{ and } g \in G,$$

but also completely determined by the G-orbits of the idempotents in $\overline{T}$. Theorem 7 below describes the lengths of chains of idempotents in $E(M)$. A *chain of idempotents* is a linearly ordered subset $\Gamma = \{e_0 < e_1 < e_2 < \cdots < e_k\}$ of the poset $E(M)$, and the *length* of Γ is k. A chain is *maximal* if it is properly contained in no other chain.

Theorem 7 ([61, Corollary 6.10 and Theorem 6.20]). *Let M be an irreducible algebraic monoid with unit group G. Then every chain of idempotents is contained in a maximal torus T of G. Furthermore, the lengths of the maximal chains in $E(\overline{T})$, $E(M)$, and $\mathscr{U}(M)$ are all the same. If M has a zero, then this number is equal to* $\dim T$.

We now define height function on $\mathscr{U}(M)$ and $E(M)$ for any irreducible algebraic monoid M with kernel J_0.

Definition 2. Define $\mathrm{ht}(J_0) = 0$ and $\mathrm{ht}(J) = \mathrm{ht}(J') + 1$ if $J, J' \in \mathscr{U}(M)$ and J covers J'. If $e \in J \in \mathscr{U}(M)$, then $\mathrm{ht}(e) = \mathrm{ht}(J)$. If $\mathrm{ht}(1) = p$, then $\mathrm{ht}(M) = \mathrm{ht}(E(M)) = p$.

This function is a powerful tool to prove and obtain useful results using induction on height of regular $\mathscr{J}$-classes of the monoid. This approach has been employed extensively in [61].

We can extend height function from $\mathscr{U}(M)$ to M if M is an irreducible regular algebraic monoid. A monoid M is *regular* if for each $a \in M$, there is $b \in M$ such that $a = aba$. A monoid M with unit group G is *unit regular* if for each $a \in M$, there is $b \in G$ such that $a = aba$.

Theorem 8 ([54, Theorem 1.3]; [85, Proposition 3.2]). *Suppose that M is an irreducible algebraic monoid with unit group G. The following are equivalent.*

(1) M is regular.
(2) M is unit regular.
(3) $M = GE(M)$.
(4) $G\backslash M/G = \mathscr{U}(M)$.

By Theorem 8 if M is an irreducible regular algebraic monoid then $\mathscr{U}(M)$ is equal to the set of all $\mathscr{J}$-classes. Thus height function on $\mathscr{U}(M)$ can be extended to M by

$$\mathrm{ht}(a) = \mathrm{ht}(J_a)$$

for all $a \in M$.

The height functions on classical monoids are consistent with the usual rank functions. If $a \in \mathbf{M}_n$, then $\mathrm{ht}(a) \in \{0, \cdots, n\}$. If M is a classical monoid of type B_l, C_l or D_l as defined in Sect. 2.1, then $\mathrm{ht}(a) \in \{0, \cdots, l, n\}$.

2.4 Putcha Lattice

The *Putcha lattice of cross sections*, for short *Putcha lattice*, of an irreducible algebraic monoid M with unit group G was initially introduced in [57]. Let T be a maximal torus of G.

Definition 3. A subset $\Lambda \subseteq E(\overline{T})$ is called a Putcha lattice of M if $|J \cap \Lambda| = 1$ for all $J \in \mathscr{U}(M)$, and for all $e, f \in \Lambda$, $e \leq f \Leftrightarrow J_e \leq J_f$.

A Putcha lattice Λ is indeed a sublattice of $E(\overline{T})$. We agree that Λ inherits the partial order on $E(\overline{T})$ which in turn inherits the partial order on $E(M)$. By definition Λ is a set of representatives for the $G \times G$ orbits. Thus M is a disjoint union of $G \times G$ orbits GeG with Λ as the index set

$$M = \bigsqcup_{e \in \Lambda} GeG,$$

and the bijection $\Lambda \to G\backslash M/G$ is order preserving. In addition, the lattice Λ is a set of representatives for the orbits of the conjugation action of W on $E(\overline{T})$. Thus $E(\overline{T}) = \bigsqcup_{e \in \Lambda} \{wew^{-1} \mid w \in W\}$.

Putcha lattices exist for irreducible algebraic monoids [57, Theorem 6.2]. The following theorem describes Putcha lattices making use of $\mathscr{R}$ relation and Borel subgroups of M with a zero.

Theorem 9 ([61, Theorem 9.3]). *Let M be an irreducible algebraic monoid with a zero and unit group G. Let B be a Borel subgroup of G containing a maximal torus T. Then*

$$\Lambda = \{e \in E(\overline{T}) \mid \textit{for all } f \in E(M),\ \textit{if } e\mathscr{R}f \textit{ then } f \in \overline{B}\}$$

is a Putcha lattice of M.

We describe Putcha lattices of classical algebraic monoids. Let e_i denote the diagonal matrix $\mathrm{diag}(1, \cdots, 1, 0, \cdots, 0)$ with i-copies of 1 for $i = 0, \cdots, n$, and let E_{ij} be the matrix unit of size n whose (i, j)-entry is 1 and others are all 0. So, e_n is the identity matrix of $\mathbf{M}_n$. The Putcha lattice of $\mathbf{M}_n$ is

$$\Lambda = \{e_i \mid i = 0, \cdots, n\}.$$

The Putcha lattice of $\mathbf{MSp}_n$ with $n = 2l$ is

$$\Lambda = \{e_i \mid i = 0, \cdots, l, n\},$$

which is formally the Putcha lattice of $\mathbf{MSO}_n$ where $n = 2l + 1$. The Putcha lattice of $\mathbf{MSO}_n$ with $n = 2l$ is

$$\Lambda = \{e_i \mid i = 0, \cdots, l, n\} \cup \{e_{l+1,l+1} - E_{l,l}\}.$$

2.5 Rook Monoids

Our objective here is to introduce the rook monoid and its relatives. These monoids are finite, and hence algebraic. They are vital in determining the structure of classical algebraic monoids.

2.5.1 The General Rook Monoid

A matrix of size n is a *rook matrix* if its entries are 0 or 1 and there is at most one 1 in each row and each column. Viewing each 1 as a rook, we can identify a rook matrix of rank r with an arrangement of r non-attacking rooks on an $n \times n$ chess board. Let

$$\mathbf{R}_n = \{A \in \mathbf{M}_n \mid A \text{ is a rook matrix}\}.$$

Then $\mathbf{R}_n$ is a monoid with respect to the multiplication of matrices. We call this monoid the *general rook monoid*, for short *rook monoid*. Its unit group is the *permutation group* P_n consisting of *permutation matrices* whose each row and each column have exactly one 1. The order of $\mathbf{R}_n$ is $|\mathbf{R}_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i!$. In particular,

$$\mathbf{R}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

A *partial injective transformation* σ of $\mathbf{n} = \{1, 2, \cdots, n\}$ is a one to one correspondence from a subset X of $\mathbf{n}$ onto a subset Y of $\mathbf{n}$. We call X the domain of σ, denoted by $I(\sigma)$, and Y the range of σ, denoted by $J(\sigma)$. Let $\mathbf{I}_n$ be the set of all injective partial transformations of $\mathbf{n}$. Then $\mathbf{I}_n$ is a monoid with respect to the composition of partial transformations, and is called *symmetric inverse semigroup*. The zero element of $\mathbf{I}_n$ is the empty function whose domain and range are the empty set. The unit group of $\mathbf{I}_n$ is the symmetric group S_n on n letters.

Let $A = (a_{ji}) \in \mathbf{R}_n$, and let $I(A)$ and $J(A)$ denote the sets of indices of nonzero columns and rows of A, respectively. Then A induces a partial injective transformation $\sigma_A : I(A) \to J(A)$ with $\sigma_A : i \mapsto j$, if $a_{ji} = 1$. It follows that the rook monoid is isomorphic to the symmetric inverse semigroup $\mathbf{I}_n$ via the isomorphism,

$$\zeta : \quad \mathbf{R}_n \to \mathbf{I}_n, \quad A \mapsto \sigma_A .$$

2.5.2 The Symplectic Rook Monoid

To introduce symplectic rook monoids we need some preparations. Define an involution θ of $\mathbf{n} = \{1, 2, \cdots, n\}$ by $\theta(i) = n+1-i$. A subset I of $\mathbf{n}$ is *admissible* if whenever $i \in I$, then $\theta(i) \notin I$. The empty set ∅ and the whole set **n** are considered admissible. A proper subset I of **n** is admissible if and only if $I \cap \theta(I) = \emptyset$ if and only if $\theta(I)$ is admissible. Write

$$\bar{i} = \theta(i).$$

Clearly $\{i, \bar{i}\}$ is not admissible. If $n = 4$, then the admissible subsets of **n** are

$$\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{4\},\ \{1,2\},\ \{1,3\},\ \{2,4\},\ \{3,4\},\ \{1,2,3,4\}.$$

Notice the difference of these admissible subsets from those for $n = 5$ below

$$\emptyset,\ \{1\},\ \{2\},\ \{4\},\ \{5\},\ \{1,2\},\ \{1,4\},\ \{2,5\},\ \{4,5\},\ \{1,2,3,4,5\}.$$

The centralizer C of θ in S_n consists of those elements $\sigma \in S_n$ that map any admissible subset of $\mathbf{n} = \{1, 2, \cdots, n\}$ to an admissible subset. Indeed, if $\sigma \in C$ and I is an admissible subset of **n**, then for $i \in I$ we have $\overline{\sigma(i)} = \sigma(\bar{i}) \notin \sigma(I)$ since $\bar{i} \notin I$. Thus $\sigma(I)$ is admissible. Next, if $\sigma \in S_n$ and it maps all admissible subsets to admissible subsets, so is σ^{-1}. We show that $\theta\sigma = \sigma\theta$ by contradiction. Suppose that there is $i \in \mathbf{n}$ such that $\overline{\sigma(i)} \neq \sigma(\bar{i})$. Then $\{\sigma(i), \sigma(\bar{i})\}$ is admissible. But then $\sigma^{-1}\{\sigma(i), \sigma(\bar{i})\} = \{i, \bar{i}\}$ is admissible, which is a contradiction.

Thus, C acts on the set of all admissible subsets of **n**. From [36, Theorem 3.1.7] it follows that the orbits of this action are

$$\emptyset,\ \{1, \cdots, i\},\quad \text{where } i = 1, \cdots, l, n.$$

Next, let $n = 2l$ and W the preimage of C under ζ. Then W is a subgroup of $\mathbf{R}_n$, and is referred to as the *symplectic rook group*. A rook matrix A is *symplectic* if both $I(A)$ and $J(A)$ are proper admissible subsets of **n**, or if $A \in W$.

The set of all symplectic rook matrices is a submonoid of $\mathbf{R}_n$, called the *symplectic rook monoid*, and will be denoted by $\mathbf{RSp}_n$. The unit group of $\mathbf{RSp}_n$ is W. The zero element of $\mathbf{RSp}_n$ is the zero matrix of size n.

Theorem 10 ([36, Corollary 3.1.9 and Theorem 3.1.10]; [39, Corollary 2.3]). *The symplectic rook monoid is*

$$\begin{aligned}
\mathbf{RSp}_n &= \left\{ A \in \mathbf{R}_n \;\middle|\; A = \sum_{i \in I,\, w \in W}^{n} E_{wi,i} \text{ where } I \text{ is admissible} \right\} \\
&= \left\{ A \in \mathbf{R}_n \;\middle|\; A \text{ is singular and } I(A) \text{ and } J(A) \text{ are admissible} \right\} \cup W \\
&\simeq \{ A \in \mathbf{R}_n \mid AJA^\top = A^\top JA = 0 \text{ or } J \}.
\end{aligned}$$

where J is as in the definition of $\mathbf{MSp}_n$ *for $n = 2l$.*

2.5.3 The Even Special Orthogonal Rook Monoid

Let $n = 2l \geq 2$. An admissible subset is referred to as *r-admissible* if its cardinality is r. There are no r-admissible subsets for $r > l$ except the whole set $\mathbf{n}$. A subset I of $\mathbf{n}$ is r-admissible if and only if $\theta(I)$ is r-admissible. Let C be the centralizer of θ in S_n. Denote by C_1 the subgroup of C generated by

$$(1\bar{1})(2\bar{2}), (2\bar{2})(3\bar{3}), \cdots, (l-1\ \overline{l-1})(l\bar{l}),$$

and let

$$C_2 = \{\sigma \in S_n \mid \sigma \text{ stablizes } \{1, \ldots, l\} \text{ and } \sigma(\bar{i}) = \overline{\sigma(i)}\}.$$

Then $C' = C_1C_2$ is a subgroup of C. It follows from [32, Lemmas 5.2 and 5.4] that the orbits of the restriction to C' of the action of C on the set of all admissible subsets of $\mathbf{n}$ are

$$\emptyset,\ \{1, \cdots, l-1, l+1\},\ \text{ and } \{1, \cdots, i\},\ \text{where } i = 1, \cdots, l, n.$$

An admissible subset I is called **type I** if there exists w in W such that $wI = \{1, \cdots, l-1, l\}$; **type II** if $wI = \{1, \cdots, l-1, l+1\}$. Such admissible sets contain l elements.

Let $W = \zeta^{-1}(C')$. Then W is a subgroup of $\mathbf{R}_n \cap \mathbf{SO}_n$ and is isomorphic to $(Z_2)^{l-1} \rtimes S_l$. In addition, $|W| = 2^{l-1}l!$. We call W the *even special orthogonal rook group*. A rook matrix A is *even special orthogonal* if $I(A)$ is admissible and there is $w \in C'$ such that $J(A) = w(I(A))$, or if $A \in W$. The set of all even special orthogonal rook matrices is a submonoid of $\mathbf{R}_n$, called the *even special orthogonal rook monoid*, and will be denoted by $\mathbf{RSO}_n$. The unit group of $\mathbf{RSO}_n$ is W.

Theorem 11 ([32, Corollary 5.8 and Theorem 5.9]). *The even special orthogonal rook monoid is*

$$\begin{aligned}\mathbf{RSO}_n &= \left\{A \in \mathbf{R}_n \;\middle|\; A = \sum_{i \in I, w \in W}^{n} E_{wi,i} \text{ where } I \text{ is admissible,} \right\} \\ &= \left\{A \in \mathbf{R}_n \;\middle|\; \begin{array}{l} A \text{ is singular, } I(A) \text{ and } J(A) \text{ are admissible} \\ \text{and of the same type if } |I(x)| = |J(x)| = l \end{array}\right\} \cup W \\ &= \{A \in \mathbf{R}_n \mid AJA^\top = A^\top JA = 0 \text{ or } J\}.\end{aligned}$$

where J is as in the definition of $\mathbf{MSO}_n$ *for $n = 2l$.*

2.5.4 The Odd Special Orthogonal Rook Monoid

Let $n = 2l + 1 \geq 3$ and W the preimage of C under ζ, where C is the centralizer of θ in S_n. Then W is a subgroup of $\mathbf{R}_n$, and is referred to as the *odd special orthogonal* *rook group*. A rook matrix A is *odd special orthogonal* if both $I(A)$ and $J(A)$ are proper admissible subsets of $\mathbf{n}$, or if $A \in W$. The set of all odd special orthogonal rook matrices is a submonoid of $\mathbf{R}_n$, called the *odd special orthogonal rook monoid*, and will be denoted by $\mathbf{RSO}_n$. The unit group of $\mathbf{RSO}_n$ is W. Combining [33, Theorem 3.10] and Theorem 10, we have the following conclusion.

Theorem 12. *The odd special orthogonal rook monoid* $\mathbf{RSO}_n$ *is isomorphic to the symplectic rook monoid* $\mathbf{RSp}_{2l}$*, where* $n = 2l + 1 \geq 3$.

3 Reductive Monoids

An irreducible algebraic monoid is *reductive* if its unit group is a reductive algebraic group. The monoids $\mathbf{M}_n$ and $\mathbf{D}_n$ are reductive, but $\overline{\mathbf{B}}_n$ is not for $n \geq 2$. The classical monoids of types A_l, B_l, C_l, and D_l are all reductive. The monoid in Example 3 is not reductive if $n \geq 2$, since its unit group is not connected and so not reductive. The monoid in Example 8 is not reductive for $n \geq 1$ because the unipotent radical of its unit group is

$$\{(1, a_1, \cdots, a_n) \mid a_i \in K \text{ for } i = 1, \cdots, n\}.$$

Reductive monoids are central to the theory of algebraic monoids; regular semigroups form an eminent class in semigroup theory. At a glance, reductive monoids have nothing to do with regular semigroups. But, the two notions are connected very closely. The following result is a summary of [57, Theorem 2.11], [58, Theorem 2.4], [59, Theorem 1.1], [74, Theorem 4.4.15], and [76, Theorem 3.1].

Theorem 13. *Every reductive algebraic monoid is regular. Moreover, an irreducible algebraic monoid with a zero is reductive if and only if it is regular.*

It follows from Theorems 8 and 13 that every reductive monoid is unit regular. A complete description of the reductivity of an irreducible algebraic monoid is given in [19].

Theorem 14 ([19, Theorem 2.1]). *Suppose that* M *is an irreducible algebraic monoid. Then* M *is reductive if and only if* M *is regular and the semigroup kernel of* M *is a reductive group.*

Reductive monoids are regular and unit dense monoids, which are distinguished from irreducible algebraic monoids in that they have finite number of $G \times G$ orbits, and each $G \times G$ orbit contains an idempotent. This, however, is not the case for all irreducible algebraic monoids. If $n > 1$, then the monoid in Example 8, again, is not reductive since it has infinitely many $G \times G$ orbits, but only two orbits $\{0\}$ and G have idempotents 0 and 1, respectively.

3.1 Jordan Decomposition

Every element x in an algebraic group G has its *Jordan decomposition*

$$x = su = us$$

where s is semisimple (diagonalizable) and u is unipotent (sole eigenvalue 1). This decomposition is unique. Is there an analogue of such decomposition in algebraic monoids? Putcha [71] shows that each element in a reductive monoid M is a product of a semisimple element and a quasi-unipotent element.

The unit group H_e of eMe for $e \in E(M)$ is an algebraic group. If $a \in M$, it follows from Theorem 3 that there is a positive integer k such that $a^k \in H_e$ for some $e \in E(M)$. Such e is uniquely determined by a, since if $a^k \in H_f$ for some $f \in E(M)$ then $e\mathscr{H}f$ and hence $e = ef = f$. By [27, Corollary 1] we have $ae = ea \in H_e$. The element ae is called the *invertible part* of a. An element $a \in M$ is *completely regular* if $a \in H_e$ for some $e \in E(M)$, and H_e is called the *bubble group* of a. Clearly, for any $a \in M$, the invertible part of a is always completely regular.

An element $s \in M$ is *semisimple* if s is completely regular and is semisimple in its bubble group. If $s \in M = \mathbf{M}_n$ is semisimple then s is diagonalizable. The set of all semisimple elements of M is denoted by M_s. An element u of M is *quasi-unipotent* if its invertible part ue is unipotent in its bubble group H_e. If M has a zero, then every nilpotent element is quasi-unipotent. The set of all quasi-unipotent elements will be denoted by M_u. Then

$$M_s \cap M_u = E(M).$$

If M is a closed monoid of $\mathbf{M}_n$, then M_u is the zero set of the polynomial $X^n(X - I)^n$. Renner studies the conjugacy classes of semisimple elements in algebraic monoids [78]; Winter investigates quasi-unipotent elements in a different name in [92]. Theorem 15 below shows that Jordan decomposition exists for reductive monoids.

Theorem 15 ([71, Theorem 2.2]). *Let M be a reductive monoid and $a \in M$. Then $a = su = us$ for some invertible semisimple element s and quasi-unipotent element u.*

Such decomposition is not unique. For example (cf. Example 2.3 of [71]), in $\mathbf{M}_2$, for any $b \in K$,

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & b/\alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b/\alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

where $\alpha \in K^*$.

Our next objective is to study the structure of a reductive monoid M in terms of root semigroups, which are analogues of root groups U_α for reductive groups. We fix notation. Let G be the unit group of M, let T be a maximal torus of G and B a Borel subgroup containing T, and let Φ be the roots of G relative to T. Denote by B^- the unique Borel subgroup such that $B \cap B^- = T$. Then G is generated by the root groups U_α along with T where $\alpha \in \Phi$ [24, Theorem 26.3 d)].

Is there a monoid analogue of this result for M? Putcha confirms this matter in [71]. The key is to find a monoid analogue $\widetilde{U_\alpha}$ of the one-dimensional root subgroup U_α associated with a root $\alpha \in \Phi$. Let $\widetilde{U_\alpha} = (\overline{TU_\alpha})_u$, the set of quasi-unipotent elements of $\overline{TU_\alpha}$. Then $\widetilde{U_\alpha}$ is referred to as the *root semigroup* associated with α. It is easy to see that $U_\alpha \subseteq \widetilde{U_\alpha}$. Denote by $\tilde{U}$ the set of quasi-unipotents of $\overline{B}$. Imbedding M into $\mathbf{M}_n$ in such a way that every element of B is upper triangular and every element of B^- is lower triangular, we can define a map $\phi : \overline{B} \to \overline{T}$ such that $\phi(b)$ is the diagonal matrix of the diagonal of $b \in \overline{B}$. Then ϕ is an epimorphism and $\phi|_{\overline{T}}$ is the identity.

Theorem 16 ([71, Theorem 2.6 and Corollary 4.4]). *Let M be a reductive monoid and let Φ^+ be the set of positive roots. Then*

(1) $\tilde{U}$ is an algebraic monoid and equal to $\phi^{-1}(E(\overline{T}))$.
(2) $\overline{B}$ is generated by T and $\widetilde{U_\alpha}$ for $\alpha \in \Phi^+$, and $\overline{B} = T\tilde{U} = \tilde{U}T$.
(3) $M_u = \bigcup_{g \in G} g\tilde{U}g^{-1}$.

The following corollary is from [61, Proposition 6.3] and Theorem 16.

Corollary 3. *Let M be a reductive monoid, T a maximal torus of the unit group of M, and Φ the set of roots relative to T. Then M is generated by T and $\tilde{U}_\alpha$, $\alpha \in \Phi$.*

3.2 Parabolic Subgroups

The aim here is to describe parabolic subgroups of G in terms of idempotents of a reductive monoid M. When M has a zero, these subgroups are completely determined by the chains in $E(M)$ [57, 60]. Recall that a chain of idempotents is a linearly ordered subset $\Gamma = \{e_0 < e_1 < e_2 < \cdots < e_k\}$ of the poset $E(M)$. In view of [61, Corollary 6.10], every chain of idempotents is contained in a maximal torus T of G. If $\Gamma \subseteq E(M)$, define the *left centralizer* and the *right centralizer* of Γ by

$$P(\Gamma) = \{x \in G \mid xe = exe\} \quad \text{and} \quad P^-(\Gamma) = \{x \in G \mid ex = exe\}.$$

As Brion did in [3], we switched Putcha's notation for left and right centralizers to comply with standard conventions in algebraic geometry and algebraic groups. The *centralizer* of Γ is by definition

$$C_G(\Gamma) = \{x \in G \mid xe = ex\}.$$

More information on local structures such as stabilizers, centralizers, and kernels of algebraic monoids can be found in [3, 20, 22, 61, 70, 82].

Theorem 17 ([57, Theorem 4.6]; [60, Theorem 2.7]). *Let M be a reductive monoid and let Γ be a chain in $E(M)$. Then $P(\Gamma)$ and $P^-(\Gamma)$ are a pair of opposite parabolic subgroups with common Levi factor $C_G(\Gamma)$. Furthermore, if M has a zero, then every parabolic subgroup P of G is of the form $P = P(\Gamma)$ for some chain $\Gamma \subseteq \Lambda$, where Λ is a Putcha lattice of M.*

When the chain Γ in the above theorem is maximal, its left and right centralizers are Borel subgroups as described below.

Theorem 18 ([57, Theorem 4.5]; [61, Theorem 7.1]). *Let M be a reductive monoid with a zero and let Γ be a maximal chain of $E(M)$. Then*

(1) $P(\Gamma)$ is a Borel subgroup of G whose opposite Borel subgroup is $P^-(\Gamma)$. Moreover, every Borel subgroup of G can be obtained this way.
(2) $C_G(\Gamma)$ is a maximal torus of G and every maximal torus of G is obtainable in this manner.

The set of Borel subgroups containing a maximal torus T is in one to one correspondence with the set of Putcha lattices in $E(\overline{T})$.

Theorem 19 ([60, Lemma 1.1]; [61, Theorem 7.1]). *Let M be a reductive algebraic monoid with a zero and unit group G. Let B be a Borel subgroup of G containing a maximal torus T. Then*

$$\Lambda(B) = \{e \in E(\overline{T}) \mid Be = eBe\}$$

is a Putcha lattice of M. Moreover, the map $B \mapsto \Lambda(B)$ is a bijection from the set of all Borel subgroups containing T onto the set of Putcha lattices in $E(\overline{T})$.

3.3 The Type Map

Let M be a reductive monoid with unit group G and let $W = N_G(T)/T$ be the Weyl group. Denote by Δ the set of simple roots relative to T and B, and by $S = \{s_\alpha \mid \alpha \in \Delta\}$ the set of simple reflections that generate the Weyl group. Let Λ be the cross-section lattice of M.

Definition 4. The type map of M is defined by

$$\lambda : \Lambda \to 2^{\Delta}; \quad \lambda(e) = \{\alpha \in \Delta \mid s_\alpha e = e s_\alpha\}.$$

As Renner mentions in his book [82], the type map is the most important combinatorial invariant in the structure theory of reductive monoids. In some sense, it is the monoid analogue of the Coxeter-Dynkin diagram. Especially, for $\mathscr{J}$-irreducible monoids, Putcha and Renner [72] give a very precise recipe to completely determine the type map using the Coxeter-Dynkin diagram associated with the monoids. We consider the type maps of classical algebraic monoids, and refer the reader to [34, 35, 72, 82] for further details about type maps of reductive monoids.

Example 9. The type maps of $\mathbf{M}_n$ with $n = l + 1$, $\mathbf{MSp}_n$ with $n = 2l$, and $\mathbf{MSO}_n$ with $n = 2l + 1$. Let $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ be the simple roots of type A_l, B_l and C_l, and let $e_i = \text{diag}\ (1, \cdots, 1, 0, \cdots, 0)$ with i-copies of 1, for $i = 1, \cdots, l$. Then $\Lambda = \{0, e_1, \cdots, e_l, 1\}$, and the type map is determined by $\lambda(0) = \lambda(1) = \Delta$, $\lambda(e_1) = \{\alpha_2, \cdots, \alpha_l\}$, and for $2 \le i \le l$,

$$\lambda(e_i) = \{\alpha_1, \cdots, \alpha_{i-1}\} \cup \{\alpha_{i+1}, \cdots, \alpha_l\}.$$

Example 10. The type map of $\mathbf{MSO}_n$ with $n = 2l$. Let $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ be the simple roots of SO_n, and let $e_i = \text{diag}\ (1, \cdots, 1, 0, \cdots, 0)$ with i-copies of 1, for $i = 1, \cdots, l$. Let $e_l' = \text{diag}\ (1, \cdots, 1, 0, 1, \cdots, 0)$ with l-copies of 1. Then $\Lambda = \{0, e_1, \cdots, e_l, e_l', 1\}$, and the type map is determined by $\lambda(0) = \lambda(1) = \Delta$, $\lambda(e_1) = \{\alpha_2, \cdots, \alpha_l\}$, $\lambda(e_{l-1}) = \{\alpha_1, \cdots, \alpha_{l-2}\}$, $\lambda(e_l) = \{\alpha_1, \cdots, \alpha_{l-1}\}$, $\lambda(e_l') = \{\alpha_1, \cdots, \alpha_{l-2}, \alpha_l\}$, and for $2 \le i \le l-2$ with $l \ge 4$,

$$\lambda(e_i) = \{\alpha_1, \cdots, \alpha_{i-1}\} \cup \{\alpha_{i+1}, \cdots, \alpha_l\}.$$

In general, associated with the type map of a reductive monoid are some parabolic subgroups of the Weyl group. Let $\lambda^*(e) = \{\alpha \in \Delta \mid s_\alpha e = e s_\alpha \neq e\}$ and $\lambda_*(e) = \{\alpha \in \Delta \mid s_\alpha e = e s_\alpha = e\}$. Then $\lambda(e) = \lambda^*(e) \sqcup \lambda_*(e)$. Denote by $W(e) = W_{\lambda(e)}$, $W^*(e) = W_{\lambda^*(e)}$ and $W_*(e) = W_{\lambda_*(e)}$ the parabolic subgroups of W associated with $\lambda(e), \lambda^*(e)$ and $\lambda_*(e)$, respectively. These subgroups are useful in determining the orders, conjugacy classes, and representations of Renner monoids [37, 40, 41]. Descriptions and applications of these subgroups can be found in the books [61, 82] and the references there.

Proposition 2. *Let e be an element of the Putcha lattice of a reductive monoid M. Then*

(1) $W(e) = \{w \in W \mid we = ew\}$
(2) $W^*(e) = \cap_{f \ge e} W(f)$.
(3) $W_*(e) = \cap_{f \le e} W(f) = \{w \in W \mid we = ew = e\}$.
(4) $W(e) = W^*(e) \times W_*(e)$.

3.4 $\mathscr{J}$-Irreducible Monoids

Renner introduces the concept of $\mathscr{J}$-irreducible monoids in his work on the classification of semisimple algebraic monoids in [75]. A reductive monoid M with a zero is *$\mathscr{J}$-irreducible* if its Putcha lattice has a unique minimal nonzero idempotent. A reductive monoid M with a zero and unit group G is *semisimple* if the dimension of the center $C(G)$ is one. In view of [75, Lemma 8.3.2], each $\mathscr{J}$-irreducible algebraic monoid is semisimple. The classical monoids defined in Sect. 2.1 are $\mathscr{J}$-irreducible and hence semisimple. The following results give alterative descriptions of $\mathscr{J}$-irreducible monoids.

Theorem 20 ([61, Corollary 15.3]; [75, Corollary 8.3.3]). *Let M be a reductive monoid with a zero and let W be the Weyl group of the unit group of M. Then the following are equivalent.*

(a) M is $\mathscr{J}$-irreducible.
(b) W acts transitively on the set of minimal nonzero idempotents of $E(\overline{T})$.
(c) There is an irreducible rational representation $\rho : M \to \mathbf{M}_n$ which is finite as a morphism of algebraic varieties.

Our intention next is to confirm that all $\mathscr{J}$-irreducible algebraic monoids can be obtained, up to finite morphism, from irreducible representations of semisimple algebraic groups. This is a known result given in Renner [83].

Theorem 21. *Let G be a semisimple algebraic group and ρ be an irreducible rational representation of G. Then $M(\rho) = \overline{K^*\rho(G)}$ is a $\mathscr{J}$-irreducible algebraic monoid. Furthermore, one can construct, up to finite morphism, all $\mathscr{J}$-irreducible algebraic monoids from irreducible representations of a semisimple algebraic group.*

Recall that $M(\rho)$ is the Zariski closure of $K^*\rho(G)$. Suppose that ρ is an irreducible representation of a semisimple group G, then the inclusion map $M(\rho) \to \mathbf{M}_n$ is a faithful representation of $M(\rho)$. Thus $M(\rho)$ is $\mathscr{J}$-irreducible. Now suppose that M is $\mathscr{J}$-irreducible and let H be the unit group of M. Then $M = \overline{H}$ since M is irreducible. The radical $R(H)$ of H is the identity component of the center $C(H)$ of H, and $\dim R(H) = 1$, since $C(H)$ is one dimensional. Thanks to [87, Proposition 6.15] and [61, Theorem 4.32], we have $H = R(H)G$ where G is the semisimple commutator group of H. By [61, Corollary 10.13], there exists a finite morphism $\rho : M \to \mathbf{M}_n$ of algebraic varieties such that $\rho(R(H)) = K^*$. We obtain that $\rho(H) = K^*\rho(G)$, and hence $\rho(M) = \rho(\overline{H}) \subseteq \overline{\rho(H)} = \overline{K^*\rho(G)}$. On the other hand, it is clear that $\overline{K^*\rho(G)} \subseteq \rho(M)$. Therefore, $\rho(M) = \overline{K^*\rho(G)}$ is $\mathscr{J}$-irreducible.

The Putcha lattice of a $\mathscr{J}$-irreducible monoid is completely determined by its type $J_0 = \lambda(e_0)$ where e_0 is the unique nonzero minimal element of Λ. Putcha and Renner determine the Putcha lattices of $\mathscr{J}$-irreducible monoids associated with an arbitrary dominant weight by using the following theorem, which is a summary of [72, Corollary 4.11 and Theorem 4.16].

Theorem 22. *Let M be a $\mathscr{J}$-irreducible monoid associated with a dominant weight μ and $J_0 = \{\alpha \in \Delta \mid \langle \mu, \alpha \rangle = 0\}$ where $\langle\ ,\ \rangle$ is defined as in ([23], p42). Then*

(1) $\lambda^*(\Lambda \setminus \{0\}) = \{X \subseteq \Delta \mid X$ has no connected component lying in $J_0\}$.
(2) $\lambda_*(e) = \{\alpha \in J_0 \setminus \lambda^*(e) \mid s_\alpha s_\beta = s_\beta s_\alpha$ for all $\beta \in \lambda^*(e)\}$, for $e \in \Lambda \setminus \{0\}$.

4 Renner Monoids

The Bruhat decomposition and Tits system are among the gems in the structure theory of reductive algebraic groups G. This makes it possible to reduce many questions about G to questions about the Weyl group. Renner [77, 80] finds an analogue of such decomposition for reductive algebraic monoids with many useful consequences, resulting in the Bruhat-Renner decomposition. This decomposition is now central in the structure theory of reductive monoids.

Let M be a reductive monoid with unit group G, $B \subseteq G$ a Borel subgroup, and $T \subseteq B$ a maximal torus of G. Denote by N the normalizer of T in G and $\overline{N}$ the Zariski closure of N in M. Thus $\overline{N}$ is an algebraic monoid and has N as its unit group, and $\overline{T}$ is an algebraic monoid with unit group T. The Weyl group $W = N/T$ is a finite reflection group.

Recall that an *inverse monoid* is a monoid M such that for $a \in M$, there is a unique $b \in M$ that satisfies $a = aba$ and $b = bab$. A regular monoid with commutative idempotents is an inverse monoid. An irreducible regular monoid M is inverse if and only if M have finitely many idempotents. In particular, by [18, Theorem 3.1] a regular irreducible algebraic monoid with nilpotent unit group is an inverse monoid.

Lemma 2 ([61, Proposition 11.1]; [77, Proposition 3.2.1]). *$\overline{N} = N\overline{T}$ is a unit regular inverse monoid with unit group N and idempotent set $E(\overline{T})$. Furthermore, $\overline{N} = NE(\overline{T})$.*

To show that $\overline{N} = N\overline{T}$, note that W is finite. Let $k = |W|$. Then there exists $y_i \in N$ such that $N = \bigcup_{i=1}^{k} y_i T$. It follows from Lemma 1 that

$$\overline{N} = \cup_{i=1}^{k} \overline{y_i T} = \cup_{i=1}^{k} y_i \overline{T} \subseteq N\overline{T}.$$

By Corollary 2, the unit group of $\overline{N}$ is N.

Next, we show that an idempotent of $\overline{N}$ is in $E(\overline{T})$. Let $x \in \overline{N}$. Then $x \in y\overline{T}$ for some $y \in N$. Since $yT = Ty$, we obtain that $y\overline{T} = \overline{T}y$ by Lemma 1. As $y^k \in T$, we have

$$x^k \in (y\overline{T})^k = y^k(\overline{T})^k \subseteq T\overline{T} \subseteq \overline{T}.$$

If x is an idempotent in $\overline{N}$, then $x = x^2 = x^k \in \overline{T}$, that is, $x \in E(\overline{T})$.

Finally, in view of $\overline{T} = TE(\overline{T})$, we have $\overline{N} = NE(\overline{T})$, which shows that $\overline{N}$ is unit regular. Since $E(\overline{T})$ is commutative, $\overline{N}$ is an inverse monoid.

Lemma 3. *Let $\sim$ be the relation on $\overline{N}$ given by*

$$x \sim y \text{ if and only if } x \in yT.$$

Then $\sim$ is a congruence, and the quotient set $R = \overline{N}/\sim$ is a monoid.

It is straightforward that $\sim$ is an equivalent relation. It suffices to show that for $x, y, u, v \in \overline{N}$, if $x \sim y$ and $u \sim v$ then $xu \sim yv$. Assume that $x = yt_1$, $u = vt_2$ where $t_1, t_2 \in T$. Then $xu = yt_1vt_2$. By Lemma 2, we have $v = nt_3$ for some $n \in N$ and $t_3 \in \overline{T}$. Then

$$xu = yt_1nt_3t_2 = yn(n^{-1}t_1n)t_3t_2.$$

But $n^{-1}t_1n \in T$. Hence

$$xu = ynt_3(n^{-1}t_1n)t_2 = yv(n^{-1}t_1n)t_2 \in yvT.$$

Therefore, $xu \sim yv$. Write $R = \overline{N}/T$.

Definition 5. The monoid R is called the *Renner monoid* of M, and an element of R is called a Renner element.

4.1 Classical Renner Monoids

The Renner monoids of classical algebraic monoids are called *classical Renner monoids*. More specifically, the Renner monoids of the general, symplectic, and special orthogonal algebraic monoids are referred to as *general, symplectic* and *special orthogonal* Renner monoids, respectively. We describe these monoids below.

Example 11. The **general Renner monoid.** In this case $M = \mathbf{M}_n$. Then $T = \mathbf{T}_n$ and $\overline{N}$ consists of matrices with *at most* one nonzero entry in each row and each column. The unit group of $\overline{N}$ comprises matrices which have *exactly* one nonzero entry in each row and each column. Let E_{ji} for $1 \leq i, j \leq n$ be the matrix units whose (j, i) entry is 1 and the rest are all 0. Thus

$$\overline{N} = \{\Sigma_{i=1}^{n} t_i E_{\sigma i,i} \mid t_i \in K \text{ and } \sigma \in S_n\}$$

and

$$N = \{\Sigma_{i=1}^{n} t_i E_{\sigma i,i} \mid t_i \in K^* \text{ and } \sigma \in S_n\}.$$

The map $\sum_{i=1}^{n} t_i E_{\sigma i,i} \mapsto \sum_{i=1}^{n} b_i E_{\sigma i,i}$ is an epimorphism from $\overline{N}$ onto $\mathbf{R}_n$ with kernel T, where $b_i = 0$ if $t_i = 0$, and $b_i = 1$ if $t_i \neq 0$. Thus we have proved the following result.

Theorem 23 ([77, Section 7]). *The general Renner monoid $R = \overline{N}/T$ is isomorphic to the general rook monoid $\mathbf{R}_n$, and its unit group is isomorphic to the symmetric group S_n. The order of R is $|\mathbf{R}_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i!$.*

In what follows we identify the general Renner monoid with the general rook monoid $\mathbf{R}_n$.

Example 12. The **symplectic Renner monoid.** Here $M = \mathbf{MSp}_n$ where $n = 2l$ and $l \geq 1$. Recall that

$$\mathbf{MSp}_n = \bigsqcup_{c \in K} M_c$$

where $M_c = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ\}$ with $J = \begin{pmatrix} 0 & J_l \\ -J_l & 0 \end{pmatrix}$ and $J_l = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ of size l. We are led to the following map

$$\chi : \mathbf{MSp}_n \to K, \quad A \mapsto c \quad \text{if} \quad A \in M_c.$$

By [12], $T = \{t = \sum t_i E_{i\,i} \mid t_i \in K^* \text{ and } t_i t_{\bar{i}} = \chi(t)\}$. From [7] it follows that $\overline{N} = N \cup N'$ in which

$$N = \left\{ \omega = \sum_{i=1}^{n} t_i E_{\sigma i, i} \mid \sigma \in C, t_i \in K^* \text{ and } t_i t_{\bar{i}} = \varepsilon_i \varepsilon_{\sigma i} \chi(\omega) \right\},$$

where C is as in Sect. 2.5.2, and N' consists of matrices of the form

$$\omega' = \sum_{i=1}^{l} a_i E_{j_i, k_i}$$

where $a_i \in K$, $1 \leq i \leq l$, and $\{j_1, \cdots, j_l\}$ and $\{k_1, \cdots, k_l\}$ are admissible. The map of $\overline{N}$ onto the symplectic rook monoid $\mathbf{RSp}_n$, defined by

$$\omega = \sum_{i=1}^{n} t_i E_{\sigma i, i} \mapsto \sum_{i=1}^{n} E_{\sigma i, i} \text{ with } \sigma \in C, \text{ and}$$

$$\omega' = \sum_{i=1}^{l} a_i E_{j_i, k_i} \mapsto \sum_{i=1}^{l} b_i E_{j_i, k_i},$$

where $b_i = 0$ if $a_i = 0$, and $b_i = 1$ if $a_i \neq 0$, is a homomorphism of monoids with kernel T. We conclude:

Theorem 24 ([7, Proposition 2.3]; [36, Corollary 3.1.9]). *The symplectic Renner monoid $R = \overline{N}/T$ is isomorphic to the symplectic rook monoid* $\mathbf{RSp}_n$. *Its unit group is isomorphic to the symplectic rook group. The order of R is*

$$|\mathbf{RSp}_n| = \sum_{i=0}^{l} 4^i \binom{l}{i}^2 i! + 2^l l!.$$

In what follows we identify the symplectic Renner monoid with the symplectic rook monoid, and denote them by $\mathbf{RSp}_n$.

Example 13. The **odd special orthogonal Renner monoid** is the Renner monoid of the odd special orthogonal algebraic monoid $\mathbf{MSO}_n$ with $n = 2l + 1 \geq 3$. A similar discussion to that of Example 12 gives rise to the following result.

Theorem 25 ([13, Theorem 4.2]; [33, Corollary 3.12]). *The odd special orthogonal Renner monoid R is isomorphic to the odd special orthogonal rook monoid* $\mathbf{RSO}_n$ *where* $n = 2l + 1$*; its unit group is isomorphic to the odd special orthogonal rook group. The order of R is*

$$|\mathbf{RSO}_n| = \sum_{i=0}^{l} 4^i \binom{l}{i}^2 i! + 2^l l!.$$

Example 14. The **even special orthogonal Renner monoid** is the Renner monoid of $\mathbf{MSO}_n$ where $n = 2l$ with $l \geq 1$. Recall that

$$\mathbf{MSO}_n = \bigsqcup_{c \in K} M_c,$$

where $M_c = \{A \in \mathbf{M}_n \mid A^\top JA = AJA^\top = cJ\}$ with $J = \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix}$. We have the following homomorphism of algebraic monoids

$$\chi : \mathbf{MSp}_n \to K, \quad A \mapsto c \quad \text{if} \quad A \in M_c.$$

By [12], $T = \{t = \sum t_i E_{i\,i} \mid t_i \in K^* \text{ and } t_i t_{\bar{i}} = \chi(t)\}$. From [13] we obtain that

$$\overline{N} = \bigcup_{\sigma \in A_n} M_\sigma,$$

where $M_\sigma = \bigcup_{c \in K} \{a_i E_{i,\sigma i} \mid a_i \in K \text{ and } a_i a_{\bar{i}} = c\}$ and A_n is the *alternating group* on n letters. We have the result below.

Theorem 26 ([13, Theorem 4.4]; [32, Corollary 5.12]). *The even special orthogonal Renner monoid R is isomorphic to the even special orthogonal rook monoid* $\mathbf{RSO}_n$*; its unit group is isomorphic to the even special orthogonal rook group. The order of R is*

$$|\mathbf{RSO}_n| = \sum_{i=0}^{l} 4^i \binom{l}{i}^2 i! + (1 - 2^l)2^{l-1} l!$$

We will not distinguish the even special orthogonal Renner monoid from the even special orthogonal rook monoid, and will use $\mathbf{RSO}_n$ to denote them.

4.2 Basic Properties

Now return to the general theory of a Renner monoid R. Summarizing some primary properties of R from [77], we first describe the unit group, the idempotent set $E(R)$, relations with Putcha lattices, and the Bruhat-Renner decomposition.

Proposition 3 ([77, Proposition 3.2.1, Theorem 5.7 and Corollary 5.8]). *Let M be a reductive monoid with unit group G. Let $T \subseteq G$ be a maximal torus and $\Lambda \subseteq E(\overline{T})$ be a Putcha lattice. Then*

(1) R is a finite inverse monoid.
(2) The unit group of R is the Weyl group W, and $R = WE(R)$. So R is unit regular.
(3) The idempotent set $E(R) \cong E(\overline{T}) = \bigcup_{w \in W} w\Lambda w^{-1}$.
(4) $R = \bigsqcup_{e \in \Lambda} WeW$, and $WeW = WfW \Rightarrow e = f$.
(5) $M = \bigsqcup_{\sigma \in R} B\sigma B$, and $B\sigma B = B\tau B \Rightarrow \sigma = \tau$.
(6) If $s \in S$ is a Coxeter generator then $BsB \cdot B\sigma B \subseteq Bs\sigma B \cup B\sigma B$.

We observe from (1) and (2) of Proposition 3 that Renner monoids form a special class of inverse monoids and they are closely connected to the Weyl group, indicating that Renner monoids are by themselves extremely important discrete invariants for reductive monoids. The results (3) and (4) of Proposition 3 show that R is a disjoint union of $W \times W$ double cosets with a Putcha lattice as its index set, and that the idempotent set $E(\overline{T})$ of R is completely determined by the conjugation action of W on the Putcha lattice. From (5) and (6), the Renner monoid plays the same role for reductive monoids that the Weyl group does for reductive groups. Many questions about M may be reduced to questions about R.

The idempotent set $E(\overline{T})$ is closely connected to convex geometry and torus embeddings. We characterize this connection in Proposition 4. Solomon [85] elaborates on the connection in detail by using many interesting examples. Putcha and Renner [55, 61, 75] have more conclusions and further examples. The theory of torus embeddings can be found in [26].

Proposition 4 ([26, Theorem 2]; [56, Theorem 3.6]; [61, Theorem 8.7]). *Let M be a reductive monoid with unit group G. Suppose that T is a maximal torus of G. Then there is a rational convex polytope whose face lattice is isomorphic to $E(\overline{T})$.*

4.3 Standard Form

Let $D(e)$ be the set of minimal length representatives of left cosets $wW(e)$ and $D_*(e)$ be the set of minimal length representatives of left cosets $wW_*(e)$ where $e \in \Lambda$. Then $D(e)^{-1} = \{u^{-1} \mid u \in D(e)\}$ is the set of minimal length representatives of right cosets $W(e)w$. Now $R = W\Lambda W$ with $W = \bigsqcup_{e \in \Lambda} D_*(e)W_*(e)$ and $W = \bigsqcup_{e \in \Lambda} W(e)D(e)^{-1}$. Each element $\sigma \in R$ can be uniquely written as

$$\sigma = xey, \quad x \in D_*(e),\ e \in \Lambda,\ \text{and } y \in D(e)^{-1}. \tag{1}$$

We call (1) the *standard form* of the Renner element σ.

The standard form of Renner elements is useful to determine R^+, the index set of the decomposition of $\overline{B}$ into double cosets $B\sigma B$, that is,

$$\overline{B} = \bigsqcup_{\sigma \in R^+} B\sigma B.$$

The set R^+ is a submonoid of R, and by [70]

$$R^+ = \{\sigma \in R \mid \sigma = xey \text{ with } x \le y^{-1}\}.$$

If R is the general rook monoid, then R^+ consists of upper triangular rook matrices.

The standard form of Renner elements plays a role in describing parabolic submonoids obtained by taking the Zariski closures of parabolic subgroups of G. Let S be the set of simple reflections that generate the Weyl group W. For $I \subseteq S$, denote by W_I the subgroup of W generated by I, and call $P_I = BW_IB$ and $P_I^- = B^-W_IB^-$ opposite parabolic subgroups of G with common Levi factors $L_I = P_I \cap P_I^-$. Define

$$R_I^+ = \{xey \mid e \in \Lambda,\ x \in W,\ y \in D(e)^{-1},\ ux \le y^{-1} \text{ for some } u \in W_I\},$$

$$R_I^- = \{xey \mid e \in \Lambda,\ x \in D(e),\ y \in W,\ yu \le x^{-1} \text{ for some } u \in W_I\},$$

$$\Lambda_I^+ = \{xex^{-1} \mid e \in \Lambda,\ x \in D_I^{-1}\} = \{xex^{-1} \mid e \in \Lambda,\ x \in D_I^{-1} \cap D(e)\}.$$

Theorem 27 ([70, Theorem 2.3]). *Let I be a subset of S. Then*

(1) R_I^+ and R_I^- are submonoids of R.
(2) $\overline{P_I} = BR_I^+B$ and $\overline{P_I^-} = B^-R_I^+B^-$. In particular, $\overline{P_S} = \overline{P_S^-} = M$.
(3) $\overline{L_I} = \overline{P_I} \cap \overline{P_I^-} = L_I\Lambda_IL_I$ is a reductive group.
(4) Λ_I is the Putcha lattice of $\overline{L_I}$ and $R_I = R_I^+ \cap R_I^- = W_I\Lambda_IW_I$ is the Renner monoid of $\overline{L_I}$.

The standard form of elements in R can be used to describe the Bruhat-Chevalley order on R.

Definition 6. Let $\sigma, \tau \in R$. We say that $\sigma \le \tau$ if $B\sigma B \subseteq \overline{B\tau B}$.

The Renner monoid is a poset with this partial order. This poset is characterized in Theorem 28 below using $(\Lambda, \le)$ and $(W, \le)$, where

$$e \le f \text{ in } \Lambda \quad \Leftrightarrow \quad fe = e = ef,$$

and

$$u \le v \text{ in } W \quad \Leftrightarrow \quad BuB \subseteq \overline{BvB}.$$

Theorem 28 ([50, Corollary 1.5]; [82, Corollary 8.35]). *Let $\sigma = xey$ and $\tau = ufv$ be in standard form. Then $\sigma \leq \tau$ if and only if $e \leq f$ and there is $w \in W(f)W_*(e)$ such that $x \leq uw$ and $w^{-1}v \leq y$.*

4.4 Reduced Row Echelon Form

Any matrix A over K may be changed to a matrix in *reduced row echelon form* by the Gauss-Jordan procedure, a finite sequence of elementary row operations. The set of all reduced row echelon forms of matrices in $\mathbf{M}_n$ is the set of well chosen representatives of the orbits of the left multiplication action of $\mathbf{GL}_n$ on $\mathbf{M}_n$. Row reduced echelon form in linear algebra can be generalized to any reductive monoid M with unit group G. This generalization [77] solves the orbit classification problem of the left multiplication action of G on M

$$G \times M \to M$$
$$(g, x) \mapsto gx.$$

We wish to describe the Gauss-Jordan elements of M. We begin by defining the Gauss-Jordan elements of R. The set

$$\mathscr{GJ} = \{\sigma \in R \mid B\sigma \subseteq \sigma B\}$$

is called the set of Gauss-Jordan elements of R. The Gauss-Jordan elements of R are useful to index the orbits in the conjugacy decomposition of M [69, 90]. Putcha gives a description of $\mathscr{GJ}$ using the standard form of Renner elements.

Theorem 29 ([70, Lemma 3.1]). $\mathscr{GJ} = \{ey \in R \mid e \in \Lambda,\ y \in D(e)^{-1}\}$.

The set $\mathscr{GJ}$ is a poset with respect to the following partial order

$$\sigma \leq \tau \quad \text{if and only if} \quad \sigma B \subseteq \overline{\tau B}.$$

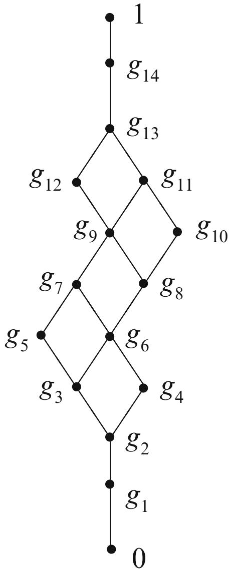
Combining [77, Theorem 9.6] and [82, Proposition 8.9], we conclude that the Renner monoid is the product of its unit group and the Gauss-Jordan elements, and that for each $\sigma \in R$ the orbit $W\sigma$ intersects the set of Gauss-Jordan elements at exactly one element g_σ. On the other hand, the orbit $W\sigma$ contains exactly one idempotent $e_\sigma \in E(\overline{T})$ (cf. [40, Lemma 3.2]). We obtain a one to one correspondence between $\mathscr{GJ}$ and $E(\overline{T})$

$$g_\sigma \mapsto e_\sigma.$$

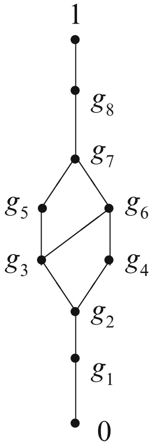
The Gauss-Jordan elements of $\mathbf{R}_n$ are the usual reduced row echelon form. If $n = 4$, we have

$$\mathscr{GJ} = \{0,\ g_1,\ \cdots,\ g_{14},\ 1\}$$

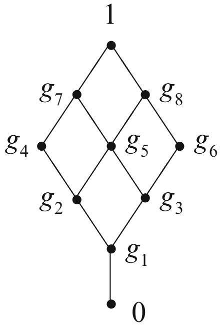
where $g_1 = E_{14},\ g_2 = E_{13},\ g_3 = E_{12},\ g_4 = E_{13} + E_{24},\ g_5 = E_{11},\ g_6 = E_{12} + E_{24},\ g_7 = E_{11} + E_{24},\ g_8 = E_{12} + E_{23},\ g_9 = E_{11} + E_{23},\ g_{10} = E_{12}+E_{23}+E_{34},\ g_{11} = E_{11}+E_{23}+E_{34},\ g_{12} = E_{11}+E_{22},\ g_{13} = E_{11}+E_{22}+E_{34},\ g_{14} = E_{11} + E_{22} + E_{33}$. The poset structure of these elements is shown in the first figure below. The idempotent e_i corresponding to g_i can be obtained by positioning the 1 in each column of g_i to the diagonal for $1 \le i \le 14$.



$(\mathscr{G}\mathscr{J},\le)$ of $R4$



$(\mathscr{G}\mathscr{J},\le)$ of $RSp4$



$(\mathscr{G}\mathscr{J},\le)$ of $RSO4$

The Gauss-Jordan elements of the symplectic Renner monoid $\mathbf{RSp}_n$ with $n = 2l$ are the usual reduced row echelon form. There are, however, no reduced row echelon form of symplectic matrices of rank i for $l < i < n$. The Hasse diagram for poset $(\mathscr{G}\mathscr{J}, \le)$ of RSp_4 is given in the middle above. Note that $B_0 = \mathbf{B}_n \cap \mathbf{Sp}_n$ is a Borel subgroup of $\mathbf{Sp}_n$, and $B = K^* B_0$ is a Borel subgroup of the unit group of $\mathbf{MSp}_n$. If $n = 4$, then B_0 consists of the invertible upper triangular matrices

$$\begin{pmatrix} a & c & d & e \\ & b & f & \frac{bd-cf}{a} \\ & & \frac{1}{b} & -\frac{c}{ab} \\ & & & \frac{1}{a} \end{pmatrix}$$

where $a, b \in K^*$ and $c, d, e, f \in K$. A simple calculation yields that

$$\mathscr{G}\mathscr{J} = \{0, g_1, \ldots, g_8, 1\}$$

where $g_0 = 0,\ g_1 = E_{14},\ g_2 = E_{13},\ g_3 = E_{12},\ g_4 = E_{13} + E_{24},\ g_5 = E_{11},$ $g_6 = E_{12} + E_{24},\ g_7 = E_{11} + E_{23},$ and $g_8 = E_{11} + E_{22}$.

The third diagram above illustrates the poset $(\mathscr{GJ}, \leq)$ of RSO_4. Let $B_0 = \mathbf{B}_n \cap \mathbf{SO}_n$ be a Borel subgroup of $\mathbf{SO}_n$. Then $B = K^*B_0$ is a Borel subgroup of the unit group of $\mathbf{MSO}_n$. If $n = 4$, then B_0 consists of the following invertible upper triangular matrices

$$\begin{pmatrix} a & c & d & -\frac{cd}{a} \\ & b & 0 & -\frac{bd}{a} \\ & & \frac{1}{b} & -\frac{c}{ab} \\ & & & \frac{1}{a} \end{pmatrix}$$

where $a, b \in K^*$ and $c, d \in K$. Thus

$$\mathscr{GJ} = \{0, g_1, \cdots, g_8, 1\}$$

where $g_0 = 0,\ g_1 = E_{14},\ g_2 = E_{13},\ g_3 = E_{12},\ g_4 = E_{13} + E_{24},\ g_5 = E_{11},$ $g_6 = E_{12} + E_{34},\ g_7 = E_{11} + E_{22},$ and $g_8 = E_{11} + E_{33}$.

The Gauss-Jordan elements of the even special orthogonal Renner monoid are not the usual reduced row echelon form. For instance,

$$g_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now describe the Gauss-Jordan elements of M. An element x of M is in *reduced form* if $x \in \sigma B$ and $x\sigma^{-1} \in \Lambda$ for some $\sigma \in \mathscr{GJ}$. The requirement $x \in \sigma B$ tells us that x is in row echelon form; the condition $x\sigma^{-1} \in \Lambda$ means roughly that it is reduced.

Theorem 30 ([82, Theorem 8.13]). *Let $x \in M$. Then $Gx \cap \sigma B \neq \emptyset$ for some unique $\sigma \in \mathscr{GJ}$. Moreover, there is a unique T-orbit in $Gx \cap \sigma B$ such that each element of the orbit is in reduced form.*

4.5 The Length Function on R

Identifying successfully the elements of length 0, Renner [80] introduces a length function on the Renner monoid of a reductive monoid. Each WeW has a unique element of length 0. Since $R = \bigsqcup_{e \in \Lambda} WeW$, there are totally $|\Lambda|$ such elements in R.

Theorem 31 ([80, Proposition 1.2]). *There is a unique element $\nu \in WeW$ such that $B\nu = \nu B$.*

Definition 7. Define the length function $l : R \to \mathbb{N}$ by $l(\sigma) = \dim(B\sigma B) - \dim(B\nu B)$ where $\nu \in W\sigma W$ with $B\nu = \nu B$.

Thus $l(\sigma) = 0$ if and only if $\sigma B = B\sigma$ if and only if $\sigma = \nu$ by Theorem 31. If $s \in S$ and $\sigma \in W$, then $l(s\sigma) = l(\sigma) \pm 1$ ([24, 29.3, Lemma A]). If $\sigma \in R$, there is a possibility that $l(s\sigma) = l(\sigma)$. By [65], if $\sigma, \tau \in WeW$, then $\sigma \leq \tau$ implies $l(\sigma) \leq l(\tau)$.

There is another description of this length function using the standard form of elements in R. If w_0, v_0 are respectively the longest elements of W and $W(e)$, then $w_0 v_0$ is the longest element of $D(e)$. It is shown in [67] that

$$l(e) = l(w_0 v_0),$$

and for $\sigma = xey$ in standard form,

$$l(\sigma) = l(x) + l(e) - l(y).$$

The length function is useful in many different topics of algebraic monoids. First, we show that it is useful to study the decomposition of nonidempotents of R^+ into positive root elements, where

$$R^+ = \{\sigma \in R \mid \sigma = xey \text{ with } x \leq y^{-1}\}.$$

For $\alpha \in \Phi^+$ where Φ^+ is the set of positive roots, let

$$R_\alpha = \{es \mid e \in \Lambda, es \neq se\}, \quad R_{-\alpha} = \{se \mid e \in \Lambda, es \neq se\}.$$

We call the elements of R_α *positive root elements*, and the elements of $R_{-\alpha}$ *negative root elements* of R.

Theorem 32 ([71, Theorem 4.2]). *Let $\sigma = xey \in R^+ \setminus E(R)$ be in standard form. Then σ is a product of $l(y) - l(x)$ positive root elements in WeW.*

Next, we characterize the product of $B \times B$ orbits of M using the length function.

Theorem 33 ([80, Theorem 1.4]).

$$BsB\sigma B = \begin{cases} B\sigma B, & \text{if } l(s\sigma) = l(\sigma) \\ Bs\sigma B, & \text{if } l(s\sigma) = l(\sigma) + 1 \\ Bs\sigma B \cup B\sigma B, & \text{if } l(s\sigma) = l(\sigma) - 1. \end{cases}$$

Our aim below is to introduce finite monoids of Lie type, and then show that the length function can also be used to describe the Iwahori-Hecke algebras associated with these monoids. Let G be a finite group of Lie type defined over F_q, a finite field with q elements. A finite regular monoid M with unit group G is a *monoid of Lie type* [64] if M is generated by $E(M)$ and G, and

1. For $e \in E(M)$, the *left centralizer* $P = \{x \in G \mid xe = exe\}$ and the *right centralizer* $P^- = \{x \in G \mid ex = exe\}$ of e in G are opposite parabolic subgroups of G, and $eP_u^- = P_u e = \{e\}$.
2. For $e \in E(M)$, if $e \mathscr{L} f$ or $e \mathscr{R} f$ then xex^{-1} for some $x \in G$.

Finite monoids of Lie type are a large class of finite regular monoids, and there are many examples of such monoids. For instance, the finite reductive monoids introduced by Renner [79] are finite monoids of Lie type [82, Section 10.5]. We elaborate briefly on finite reductive monoids now. Let $\mathbf{M}_n$ be the monoid of all $n \times n$ matrices over the algebraic closure of F_q, and let $\sigma : \mathbf{M}_n \to \mathbf{M}_n$ be the Frobenius map defined by $\sigma : [a_{ij}] \mapsto [a_{ij}^q]$. If $\underline{M} \subseteq \mathbf{M}_n$ is a reductive monoid with a zero and is stable under σ, then

$$M = \{a \in \underline{M} \mid \sigma(a) = a\}$$

is a finite monoid of fixed points, and is called a *finite reductive monoid*. For example, if $\underline{M} = \mathbf{M}_n$, then $M = \mathbf{M}_n(F_q)$. If $\underline{M} = \mathbf{MSp}_n$, then $M = \mathbf{MSp}_n(F_q)$. If $\underline{M} = \mathbf{MSO}_n$, then $M = \mathbf{MSO}_n(F_q)$. If $\underline{M} = \mathbf{D}_n$, then M is the monoid of diagonal matrices with coefficients in F_q. If $\underline{M} = \overline{\mathbf{B}}_n$, then M is the monoid of upper triangular matrices with coefficients in F_q.

Iwahori [25] initiates the study of the Iwahori-Hecke algebra associated with a Chevalley group G. Let B be a Borel subgroup of G, and W the Weyl group of G with generating set S of simple reflections. Let

$$\epsilon = \frac{1}{|B|}\Sigma_{b\in B} b \in \mathbb{C}[G].$$

The Iwahori-Hecke algebra

$$H_{\mathbb{C}}(G) = H_{\mathbb{C}}(G, B) = \epsilon\mathbb{C}[G]\epsilon.$$

is semisimple and is isomorphic to $\mathbb{C}[W]$ [9, 10]. The set $\{A_w = \epsilon w \epsilon \mid w \in W\}$ is a basis of $H_{\mathbb{C}}(G, B)$, which is normalized as $\{T_w = q^{l(\sigma)} A_w \mid w \in W\}$. With respect to this base, Iwahori found that the structure constants are integer polynomials in q, depending only on W.

Using the length function, Putcha [67] studies the monoid Iwahori-Hecke algebra of a finite monoid M of Lie type. He introduces a Putcha lattice Λ for the $G \times G$ orbits, and an analogue of the Renner monoid $R = \langle W, \Lambda\rangle$ such that

$$M = \bigsqcup_{\sigma \in R} B\sigma B.$$

The complex monoid algebra $\mathbb{C}[M]$ of M is semisimple [49]. Let

$$\epsilon = \frac{1}{|B|}\Sigma_{b\in B} b \in \mathbb{C}[G].$$

The monoid Hecke algebra of M is by definition

$$H_{\mathbb{C}}(M) = H_{\mathbb{C}}(M, B) = \epsilon\mathbb{C}[M]\epsilon.$$

It is a semisimple algebra with a natural basis

$$A_\sigma = \epsilon\sigma\epsilon, \quad \sigma \in R.$$

This basis can be normalized as

$$T_\sigma = q^{l(\sigma)} A_\sigma, \quad \sigma \in R.$$

Theorem 34 ([67, Theorem 2.1]). *The structure constants of $H_{\mathbb{C}}(M, B)$ with respect to the basis $\{A_\sigma \mid \sigma \in R\}$, and hence with respect to the normalized basis $\{T_\sigma \mid \sigma \in R\}$ are integer Laurent polynomials in q, depending only on R.*

Using Kazhdan-Lusztig polynomials and "R-polynomials", Putcha obtains the following result.

Theorem 35 ([63, Theorem 4.1]). *The Iwahori-Hecke algebra $H_{\mathbb{C}}(M, B)$ is isomorphic to the complex monoid algebra $\mathbb{C}[R]$ of the Renner monoid.*

Here are some historical notes on the length function and Iwahori-Hecke algebra. Solomon [84] first finds Theorems 31 and 33 for the Renner monoid $\mathbf{R}_n$ of $M = \mathbf{M}_n(F_q)$. He defines a length function on $\mathbf{R}_n$ in a different approach, but it agrees with Definition 7. Furthermore, he introduces the Iwahori-Hecke algebra associated with this M

$$H(M, B) = \bigoplus_{x \in \mathbf{R}_n} \mathbb{Z} \cdot T_x$$

with multiplication defined by

$$T_s T_x = \begin{cases} qT_x, & \text{if} \quad l(sx) = l(x) \\ T_{sx}, & \text{if} \quad l(sx) = l(x) + 1 \\ qT_{sx} + (q-1)T_x, & \text{if} \quad l(sx) = l(x) - 1. \end{cases}$$

$$T_x T_s = \begin{cases} qT_x, & \text{if} \quad l(xs) = l(x) \\ T_{xs}, & \text{if} \quad l(xs) = l(x) + 1 \\ qT_{xs} + (q-1)T_x, & \text{if} \quad l(xs) = l(x) - 1. \end{cases}$$

$$T_\nu T_x = q^{l(x) - l(\nu x)} T_{\nu x}$$

$$T_x T_\nu = q^{l(x) - l(x\nu)} T_{x\nu}$$

where

$$\nu = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

4.6 Presentation of R

Let G be the unit group of a reductive monoid M. Then the commutator group (G, G) is semisimple. The root system Φ and the Weyl group W of (G, G) may be identified with those of G [24, 27.1]. Since each semisimple algebraic group is a product of simple algebraic groups corresponding to the decomposition of Φ into its irreducible components [24, 27.5], without loss of generality, we may assume that G is a simple algebraic group. Denote by $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ a base of Φ and let $A = (a_{ij})$ be the Cartan matrix associated with Δ. Then W is generated by $S = \{s_1, \cdots, s_l\}$ with defining relations

$$s_i^2 = 1 \quad \text{and} \quad (s_i s_j)^{m_{ij}} = 1, \quad i, j = 1, \cdots, l,$$

where $m_{ij} = 2, 3, 4$ or 6 according to $a_{ij}a_{ji} = 0, 1, 2$ or 3, respectively. Let $\mathscr{E} = \{(s_i, s_j, m_{ij}) \mid i, j = 1, \cdots, l\}$. For $(s, t, m) \in \mathscr{E}(\Gamma)$, denote by $|s, t\rangle^m$ the word $sts\cdots st$ of length m or the word $sts\cdots s$ of length m.

Let $e, f \in \Lambda_0 = \Lambda \setminus \{1\}$ and $w \in D(e)^{-1} \cap D(f)$. Thanks to [15, Proposition 1.21], there exist a unique $h \in \Lambda_0$, and $w \in W_*(h)$ such that $h \leq e \wedge f$ and $ewf = hw = h$; this unique element h will be denoted by $e \wedge_w f$. We fix a reduced word representative $\underline{w}$ for each $w \in W$.

Theorem 36 ([15, Proposition 1.24]). *The Renner monoid has the following monoid presentation with generating set $S \cup \Lambda_0$ and defining relations*

$$\begin{aligned}
&s^2 = 1, && s \in S;\\
&|s, t\rangle^m = |t, s\rangle^m, && (s, t, m) \in \mathscr{E};\\
&se = es, && e \in \Lambda_0, s \in \lambda^*(e);\\
&se = es = e, && e \in \Lambda_0, s \in \lambda_*(e);\\
&e\underline{w} f = e \wedge_w f, && e, f \in \Lambda_0, w \in D(e)^{-1} \cap D(f).
\end{aligned}$$

4.7 Orders of Renner Monoids

The orders of Renner monoids provide numerical information about their structures. The information can sometimes be used to study the generating functions associated with the orders, indicating connections between Renner monoids and combinatorics.

Theorem 37 ([38, Theorem 2.1]). *The order of the Renner monoid R of a reductive monoid is*

$$|R| = \sum_{e\in\Lambda} \frac{|W|^2}{|W_{\lambda(e)}| \times |W_{\lambda_*(e)}|} = \sum_{e\in\Lambda} \frac{|W|^2}{|W_{\lambda^*(e)}| \times |W_{\lambda_*(e)}|^2}.$$

Consider the action of $W \times W$ on R defined by $(w_1, w_2)r = w_1 r w_2^{-1}$. The isotropic group of $e \in \Lambda$ is

$$(W \times W)_e = \{(w, ww_*) \in W \times W \mid w \in W_{\lambda(e)} \text{ and } w_* \in W_{\lambda_*(e)}\}.$$

Thus $|WeW| = |W|^2/(|W_{\lambda(e)}| \times |W_{\lambda_*(e)}|)$, and the theorem follows.

4.8 Group Conjugacy Classes

Two elements σ, τ in a Renner monoid R are *group conjugate*, denoted by $\sigma \sim \tau$, if $\tau = w\sigma w^{-1}$ for some $w \in W$. Let $W/W_*(e)$ be the set of left cosets of $W_*(e)$ in W and let $We = \{we \mid w \in W\}$.

Lemma 4 ([41, Lemmas 3.1 and 3.3]). *Each element in a Renner monoid R is group conjugate to an element in $\{we \mid w \in D_*(e)\} \subseteq We$ for some $e \in \Lambda$. Furthermore, if $f \in \Lambda$ and $f \neq e$, then no element of Wf is group conjugate to an element of We.*

Let $W(e)$ act on $W/W_*(e)$ by conjugation

$$w \cdot uW_*(e) = wuw^{-1}W_*(e),$$

where $w \in W(e)$ and $u \in W$. The normality of $W_*(e)$ in $W(e)$ shows that the action is well defined. The following theorem gives a necessary and sufficient condition for two elements to be group conjugate.

Theorem 38 ([41, Theorem 3.4]). *Let $e \in \Lambda$. Two elements ue, ve in We are group conjugate if and only if the two cosets $uW_*(e)$ and $vW_*(e)$ lie in the same $W(e)$-orbit of $W/W_*(e)$.*

Thus, there is a one-to-one correspondence between the group conjugacy classes of a Renner monoid and the orbits of the conjugation action of $W(e)$ on $W/W_*(e)$

for $e \in \Lambda$. Let n_e be the number of $W(e)$-orbits in $W/W_*(e)$. Then the number of the group conjugacy classes in a Renner monoid is $\sum_{e\in\Lambda} n_e$.

From now on, we identify the general rook monoid $\mathbf{R}_n$ with the symmetric inverse semigroup $\mathbf{I}_n$. Our purpose is to describe the group conjugacy classes of classical Renner monoids. First we collect some standard results about the conjugacy classes of the rook monoid.

Theorem 39 ([47, Theorem 1.1]). *Every injective partial transformation in the rook monoid $\mathbf{R}_n$ may be expressed uniquely as a join of disjoint cycles and links up to the order of cycles and links, where cycles and links of length 1 cannot be omitted.*

We explain the concepts used in the theorem. A cycle $(i_1 i_2 \dots i_m)$ of *length m* is an injective partial transformation with domain and range $\{i_1, i_2, \dots, i_m\}$ given by $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_m \mapsto i_1$. This is different from the usual meaning of $(i_1 i_2 \dots i_m)$ in S_n whose domain and range are $\mathbf{n}$. A link $[j_1 j_2 \dots j_m]$ of *length m* is an injective partial transformation determined by $j_1 \mapsto j_2 \mapsto \cdots \mapsto j_m$ with j_m going to nowhere; its domain is $\{j_1, \dots, j_{m-1}\}$ and range is $\{j_2, \dots, j_m\}$. Note that a cycle (i_1) of length 1 means i_1 is mapped to itself, and a link $[j_1]$ of length 1 means that j_1 is neither in its domain nor in its range, i.e., $[j_1]$ is the zero element of $\mathbf{R}_n$. A cycle of length m has m distinct expressions: $(i_1 \dots i_m) = (i_2 \dots i_m i_1) = \cdots = (i_m i_1 \dots i_{l-1})$; A link of length m has only one expression $[j_1 \dots j_m]$ since the starting point j_1 and the terminal point j_l are fixed.

Two elements $\sigma, \tau \in \mathbf{R}_n$ are *disjoint* if $(I(\sigma) \cup J(\sigma)) \cap (I(\tau) \cup J(\tau)) = \emptyset$. If $\sigma, \tau \in \mathbf{R}_n$ are disjoint, then the *join* of σ and τ is defined to be the map $\eta : I(\sigma) \cup I(\tau) \to J(\sigma) \cup J(\tau)$ given by

$$\eta(i) = \begin{cases} \sigma(i) & \text{if} \quad i \in I(\sigma), \\ \tau(i) & \text{if} \quad i \in I(\tau). \end{cases}$$

This join is denoted by $\eta = \sigma\tau$. It is clear that $\sigma\tau = \tau\sigma$.

A *signed partition* of a positive integer n is a tuple of positive integers

$$\lambda = (\lambda_1, \dots, \lambda_s \,|\, \mu_1, \dots, \mu_t),$$

where $\sum_{i=1}^{s} \lambda_i + \sum_{j=1}^{t} \mu_j = n$ with $\lambda_1 \geq \cdots \geq \lambda_s$ and $\mu_1 \geq \cdots \geq \mu_t$. Let $\sigma \in \mathbf{R}_n$ be the join of s cycles of lengths $\lambda_1, \dots, \lambda_s$ with $\lambda_1 \geq \cdots \geq \lambda_s$ and t links of lengths $\mu_1, \dots, \mu_t$ with $\mu_1 \geq \cdots \geq \mu_t$. Then σ corresponds uniquely to a signed partition of n

$$(\lambda_1, \dots, \lambda_s \,|\, \mu_1, \dots, \mu_t).$$

This partition is called the *cycle-link type* of σ.

Theorem 40 ([42, Theorem 63.5]). *Two partial injective transformations are group conjugate if and only if their cycle-link types are the same. Moreover, the number of conjugacy classes in $\mathbf{R}_n$ is*

$$\sum_{0 \le k \le n} p(k)p(n-k)$$

where $p(k)$ is the number of usual partitions of k.

The orders of conjugacy classes in $\mathbf{R}_n$ are given in [6]. Writing the cycle-link type of $\sigma \in \mathbf{R}_n$ as

$$(\lambda_1^{p_1}, \ldots, \lambda_u^{p_u} \mid \mu_1^{q_1}, \ldots, \mu_v^{q_v}), \tag{2}$$

where $\lambda_1, \ldots, \lambda_u$ are distinct positive integers and so are $\mu_1, \ldots, \mu_v$, we have the following

Theorem 41 ([6, Proposition 2.4]). *The order of the conjugacy class of σ is equal to*

$$\frac{n!}{p_1! \ldots p_u! q_1! \ldots q_v! \lambda_1^{p_1} \ldots \lambda_u^{p_u}}.$$

The conjugacy class of an injective partial transformation in $\mathbf{R}_n$ corresponds to a unique signed partition of n. Is there a similar result for the conjugacy classes of symplectic transformations? A result of [6] answers this question affirmatively. We need some preparation to state the result.

Strictly disjoint symplectic transformations are introduced in [6]. Let $\hat{I} = I \cup \bar{I}$ where $\bar{I} = \{\bar{i} \mid i \in I\}$ for $I \subseteq \mathbf{n}$. It is clear that if $I, J \subseteq \mathbf{n}$, then

$$I \cap \hat{J} = \emptyset \quad \Leftrightarrow \quad \hat{I} \cap J = \emptyset \quad \Leftrightarrow \quad \hat{I} \cap \hat{J} = \emptyset.$$

Two symplectic transformations $\sigma, \tau \in \mathbf{RSp}_n$ are *strictly disjoint* if

$$\Big(I(\sigma) \cup J(\sigma)\Big) \bigcap \Big(\widehat{I(\tau)} \cup \widehat{J(\tau)}\Big) = \emptyset.$$

Strictly disjoint symplectic transformations are disjoint, but not the other way around. Let $V = \mathbf{RSp}_n \setminus W$ be the submonoid of all singular symplectic transformations. We will describe conjugacy classes in V first and then those in W.

There is an epimorphism $\varphi : \mathbf{RSp}_n \to \mathbf{R}_l$ with $\varphi : \sigma \mapsto \tilde{\sigma}$ defined by

$$\tilde{\sigma}(|i|) = |\sigma(i)| \quad \text{for} \quad i \in I(\sigma),$$

where $|i| = i$ if $1 \le i \le l$, and $|i| = n + 1 - i$ if $l < i \le n$. The following definition will be used in Theorem 42.

Definition 8. (a) The join $\varsigma = [i_{11} \ldots i_{1t_1}][i_{21} \ldots i_{2t_2}] \ldots [i_{u1} \ldots i_{ut_u}]$ of disjoint links in V is called a *string* if $t_i \ge 2$ for $1 \le i \le u$ and

$$\bar{i}_{1t_1} = i_{21},\ \bar{i}_{2t_2} = i_{31},\ \ldots,\ \bar{i}_{u-1,t_{u-1}} = i_{u1}.$$

(b) If $\bar{i}_{ut_u} = i_{11}$, then $\tilde{\varsigma}$ is a cycle in $\mathbf{R}_l$ and ς is referred to as *positive*, otherwise, $\tilde{\varsigma}$ is a link in $\mathbf{R}_l$ and ς is called *negative*.
(c) The *length* of ς is the length of $\tilde{\varsigma}$ in $\mathbf{R}_l$.
(d) The link $[j]$ of length one is considered a negative string.

Theorem 42 ([6, Theorem 3.5]). *Every symplectic transformation $\sigma \in V$ can be expressed uniquely as a join of strictly disjoint cycles and strings up to the order in which they occur.*

For clarity, we now state the main result of [6], and provide necessary concepts needed after it.

Theorem 43 ([6, Theorem 4.5]). *There is a one-to-one correspondence between conjugacy classes in V and symplectic partitions of l.*

What is a symplectic partition of a positive integer l? Well, the story is quite long. Let m be a positive integer. A *composition* of m of *length* s is an ordered sequence of s positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ such that $\sum_{i=1}^{s} \lambda_i = m$. We agree that 0 has one composition, the empty sequence. It is also regarded as the only partition of 0. Define an equivalence relation on the set of compositions of m of length s: λ and λ' are equivalent if λ' is a cycle-permutation of λ. For instance, $(1, 3, 5)$ and $(3, 5, 1)$ are equivalent, but $(1, 3, 5)$ and $(1, 5, 3)$ are not equivalent. The equivalence class of a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ in the set of compositions of m of length s is called a *positive composition*, and will be denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ if there is no confusion. A *negative composition* is a composition itself.

A weak composition of a positive integer m is similar to a composition of m, but allowing parts of the sequence to be zero. For example, $(4, 0, 2)$ is a weak composition of 6 of length 3.

We can now define the symplectic partition of a positive integer l. Let (m_1, m_2, m_3) be a weak composition of l of length 3. If $m_1 > 0$, let $(\lambda_1, \ldots, \lambda_s)$ be a partition of m_1 where $\lambda_1 \geq \cdots, \geq \lambda_s \geq 1$. If $m_2 > 0$, let $(f_1, \cdots, f_t)$ with $f_1 \geq \cdots, \geq f_t \geq 1$ be a partition of m_2. If $m_3 > 0$, let $(g_1, \cdots, g_u, 1^{(v)})$ be a partition of m_3 where $v \geq 0$, and $g_1 \geq \cdots, \geq g_u \geq 2$ if $u \geq 1$. A *symplectic partition* of l is a set of non-negative integers

$$\left(\lambda_1, \ldots, \lambda_s \,\middle|\, \mu_{11}, \ldots, \mu_{1p_1}; \ldots; \mu_{t1}, \ldots, \mu_{tp_t} \,\middle|\, \nu_{11}, \ldots, \nu_{1q_1}; \ldots; \nu_{u1}, \ldots, \nu_{uq_u}; 1^{(v)}\right) \quad (3)$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition of m_1, $(\mu_{j1}, \mu_{j2}, \ldots, \mu_{jp_j})$ is a positive composition of f_j for $1 \leq j \leq t$, $(\nu_{k1}, \nu_{k2}, \ldots, \nu_{kq_k})$ with $\nu_{kq_k} \geq 2$ is a negative composition of g_k for $1 \leq k \leq u$, and $1^{(v)}$ is v negative compositions of length one, such that $m_1 + m_2 + m_3 = l$. We agree that if $m_i = 0$ for $i = 1, 2, 3$ then the corresponding part in the symplectic partition is empty.

The concept of the cycle-link type of a symplectic transformation $\sigma \in V$ plays a crucial role in determining conjugacy classes.

Definition 9. Let $\varsigma = [i_{11} \ldots i_{1t_1}][i_{21} \ldots i_{2t_2}] \ldots [i_{u1} \ldots i_{ut_u}]$ be a string. If ς is positive, the positive composition $(t_1 - 1, t_2 - 1, \cdots, t_{u-1} - 1, t_u - 1)$ is referred to as the *string type* of ς. If ς is negative, the negative composition $(t_1 - 1, t_2 -$

$1, \cdots, t_{u-1} - 1, t_u)$ is called the *string type* of ς. Moreover, we say that the negative string $[j]$, consisting of only one link of length 1, has the negative composition (1) as its *string type*.

If $\sigma \in V$ corresponds to the symplectic partition (3), then (3) is referred to as the *cycle-string type* of σ. For completeness we state the traditional result, taken from [8], about conjugacy classes of the unit group W of $\mathbf{RSp}_n$.

Theorem 44. *There is a bijection between the conjugacy classes in W and the signed partitions of l.*

We refer the reader to [6] for the formulas for calculating the number of conjugacy classes and the order of each class.

What are the group conjugacy classes of the even special orthogonal Renner monoid $\mathbf{RSO}_n$. Let W be the unit group of the symplectic Renner monoid $\mathbf{RSp}_n$ and A_n the unit group of the even rook monoid $\mathbf{RSO}_n$ with $n = 2l$. Thus A_n is the subgroup of W consisting of all even permutations in W. Let $V = \mathbf{RSp}_n \setminus W$ and $V' = \mathbf{RSO}_n \setminus A_n$. Then V' is the submonoid of V consisting of even special orthogonal injective partial transformations in V.

We consider the restriction to V' of the conjugation action of W on V. A simple calculation yields that if $\sigma \in V'$ and $\rho \in W$, then $\rho\sigma\rho^{-1} \in V'$. So the restriction to V' of the conjugation action of W on V induces an action of W on V', the conjugation action of W on V'. For now, let C be a W conjugacy class in V. It follows from [13, Lemma 6.3] that two elements of C are A_n conjugate if and only if there is $\sigma \in C$ that commutes with an odd permutation in W. We also know that if there are two elements in C not A_n conjugate, then C is a disjoint union of two A_n conjugacy classes with equal cardinality.

We define a class function c on V'. Let $\sigma \in V'$ with domain $I(\sigma) = \{i_1, \cdots, i_r\}$ and range $J(\sigma) = \{j_1, \cdots, j_r\}$. Define $c(\sigma)$ to be the cardinality of the set $\{|i_1|, \cdots, |i_r|, |j_1|, \cdots, |j_r|\}$. For example, if $n = 8$ and σ maps 3 to 5 and 7 to 2 and leaves the rest unchanged, then $c(\sigma) = 3$ since $I(\sigma) = \{3, 7\}$ and $J(\sigma) = \{5, 2\}$ and $\{|3|, |7|, |5|, |2|\} = \{2, 3, 4\}$. Clearly, if $\sigma, \tau \in V'$ are W conjugate, then $c(\sigma) = c(\tau)$.

Theorem 45 ([13, Theorem 6.8]). *Let C be a W conjugacy class in V'. If $c(C) < l$, then C is an A_n conjugacy class. If $c(C) = l$, then C is a disjoint union of two A_n conjugacy classes with equal number of elements.*

How can one determine if an element of V is in V' using its cycle-link type?

Theorem 46 ([13, Theorem 6.9]). *If $\sigma \in V$ has cycle-link type*

$$\left(\lambda_1, \ldots, \lambda_s \,\middle|\, \mu_{11}, \ldots, \mu_{1p_1}; \ldots; \mu_{t1}, \ldots, \mu_{tp_t} \,\middle|\, \nu_{11}, \ldots, \nu_{1q_1}; \ldots; \nu_{u1}, \ldots, \nu_{uq_u}; 1^{(v)}\right)$$

then $\sigma \in V'$ if and only if $u + v > 0$, or u=v=0 and $p_1 + \cdots + p_t$ is even.

4.9 Munn Conjugacy

The set of $i \in I(\sigma)$ such that $\sigma^k(i)$ is defined for all $k \geq 1$ is called the *stable domain* of $\sigma \in R$, and is denoted by $I^\circ(\sigma)$. That is,

$$I^\circ(\sigma) = \bigcap_{k\geq 1}^{\infty} I(\sigma^k).$$

The restriction of σ to $I^\circ(\sigma)$ induces a permutation σ° of $I^\circ(\sigma)$. This permutation is an element of R. If $\sigma^\circ \in WeW$ for some $e \in \Lambda$, then e is referred to as the subrank of σ.

Definition 10. Two elements $\sigma, \tau \in R$ are called Munn *conjugate*, denoted by $\sigma \approx \tau$, if there exists $w \in W$ such that $w^{-1}\sigma^\circ w = \tau^\circ$.

The Munn conjugacy class of σ is denoted by $[\sigma]$. All elements of $[\sigma]$ have the same subrank, and $[\sigma]$ meets one and only one parabolic subgroup of the form $\{W^*(f) \mid f \in \Lambda\}$. More specifically, $[\sigma]$ meets $W^*(e)$ where e is the subrank of σ.

Theorem 47 ([41, Theorems 4.16 and 4.17]). *There is a bijection between the set of Munn conjugacy classes of a Renner monoid R and the set of all group conjugacy classes of $W^*(e)$ for all $e \in \Lambda$.*

As a consequence, a Renner monoid has as many Munn conjugacy classes as inequivalent irreducible representations over an algebraically closed field of characteristic zero.

Theorem 48 ([6, Theorem 7.2]). *Let W be the unit group of $\mathbf{RSp}_n$ and $V = \mathbf{RSp}_n \setminus W$. Then two elements in V are Munn conjugate if and only if they have the same cycle part in their cycle-string types. Furthermore, the number of Munn classes is $\sum_{r=0}^{m} p(r)$.*

We describe the relationship between Munn conjugacy and other conjugacies in semigroup theory. Notice that there are different conjugacy relations in semigroups. We are interested in semigroup conjugacy, action conjugacy, character conjugacy, and McAlister conjugacy.

Let S be a semigroup. Then elements $\sigma, \tau \in S$ are called *primary S-conjugate* if there are $x, y \in S$ for which $\sigma = xy$ and $\tau = yx$. This latter relation is reflexive and symmetric, but not transitive. Let $\equiv$ be its transitive closure, called *semigroup conjugacy*. In general, group conjugacy is finer than semigroup conjugacy. But, in a group they are the same, equal to the usual group conjugacy.

Kudryavtseva and Mazorchuk [30] study action conjugacy and character conjugacy. To define action conjugacy, consider the partial action of S^1 on S

$$\sigma \cdot x = \begin{cases} \sigma x \sigma^{-1}, & \text{if } \sigma^{-1}\sigma \geq e_x; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

It follows from [30, Lemma 1] that if $\sigma, \tau \in S^1$ and $x \in S$ then $\tau\sigma \cdot x$ is defined if and only if $\sigma \cdot x$ and $\tau \cdot (\sigma \cdot x)$ are both defined, in which case $\tau\sigma \cdot x = \tau \cdot (\sigma \cdot x)$. We call $x, y \in S$ *primary action conjugate* if there is $\sigma \in S^1$ for which $y = \sigma \cdot x$ or $x = \sigma \cdot y$. This relation is reflexive and symmetric, but not necessarily transitive. Its transitive closure is called *action conjugacy*.

Two elements x, y in a semigroup S are referred to as *character conjugate* if for every finite-dimensional complex representation ϕ of S we have $\chi_\phi(x) = \chi_\phi(y)$, where χ_ϕ is the character of ϕ.

McAlister [43] introduces a conjugacy. Let a be an element in a finite semigroup S and $\bar{a} = ae$, where e is the unique idempotent in the subgroup $\langle a \rangle$ generated by a. Then a, b are conjugate if $\bar{b} = x'\bar{a}x$ and $\bar{a} = x\bar{b}x'$ for some regular element x with inverse x'.

Theorem 49 ([41, Corollary 4.5]). *The action conjugacy, character conjugacy, McAlister conjugacy, Munn conjugacy, and semigroup conjugacy are all the same in a Renner monoid.*

4.10 Representations

What can we say about the representations of the Renner monoid R? We will state the main result of [40] first, and then provide some related information on the representation theory of finite monoids. For any $e \in \Lambda$, let B_e be the group algebra of $W^*(e)$ over F, a field of characteristic 0.

Theorem 50 ([40, Theorem 3.1]). *The inequivalent irreducible representations of R over F are completely determined by those of B_e, where $e \in \Lambda$.*

We briefly elaborate on how to achieve the above result. Let

$$FR = \left\{ \sum_{\sigma \in R} \alpha_\sigma \sigma \;\middle|\; \alpha_\sigma \in F \right\}$$

be the monoid algebra of R over F. The key is to show that FR is isomorphic to the direct sum $\bigoplus_{e \in \Lambda} A_e$, where $A_e = M_{d_e}(B_e)$ in which $d_e = |W|/|W(e)|$. Therefore, FR is a semisimple algebra. To this end, an explicit description of the Möbius function of R is found and a precise formula for Solomon central idempotents is obtained. We refer the reader to [40] for the details.

The work of [40] is a generalization of Munn [47] and Solomon [86] from representations of rook monoids to all Renner monoids. Munn initiates the study of irreducible representations of rook monoid $\mathbf{R}_n$ in terms of irreducible representations of certain symmetric groups contained in the monoid. Solomon [86] investigates these representations using central idempotents of $F\mathbf{R}_n$, and then studies many other aspects related to these representations as well. Steinberg [88,89] discusses representations of finite inverse semigroups S and shows that there is an algebra isomorphism between the monoid algebra of S and the groupiod algebra of S.

Putcha [63, 66–68] has developed a systematic representation theory of finite monoids, including representations of any finite monoid, irreducible characters of full transformation semigroups, highest weight categories and blocks of the complex algebra of the full transformation semigroups. In particular, he provides an explicit isomorphism between the monoid algebra of the Renner monoid and the monoid Hecke algebra introduced by Solomon [84]. Putcha and Oknínski describe complex representations of matrix semigroups in [49]. Putcha and Renner study irreducible modular representations of M in [73, 81]. Munn [45, 46] investigates semigroup algebras and matrix representations of semigroups.

4.11 Generating Functions

We investigate the generating functions associated with the orders of classical Renner monoids. Let $r_n = |\mathbf{R}_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i!$. It follows from [2] that the generating function $r(x) = \sum_{n=0}^{\infty} \frac{r_n}{n!} x^n$ is convergent to the solution of the differential equation

$$\frac{r'(x)}{r(x)} = \frac{2-x}{(1-x)^2}.$$

This result is generalized in [37] to study the generating functions of the orders of the symplectic and orthogonal Renner monoids.

Let $s_n = \sum_{i=0}^{n} a^i \binom{n}{i}^2 i!$, where a is a nonzero real number. The following recursive formula for s_n, taken from [37], is a variant of [2]. It allows us to calculate the generating function of s_n. Clearly, $s_0 = 1$ and $s_1 = a + 1$.

$$s_n = [a(2n-1)+1]s_{n-1} - a^2(n-1)^2 s_{n-2}, \quad \text{for } n \geq 2. \tag{4}$$

Theorem 51 ([37, Theorem 3.1]). *Let $s(x) = \sum_{n=0}^{\infty} \frac{s_n}{a^n n!} x^n$. If $a \geq 1$, then $s(x)$ converges for $|x| < 1$ to the function $\frac{1}{1-x} e^{x/a(1-x)}$. Also, $s(x)$ satisfies the differential equation*

$$\frac{s'(x)}{s(x)} = \frac{a+1-ax}{a(1-x)^2}.$$

The generating function of s_n is closely related to the Laguerre polynomials. Let $l_n(t)$ be the nth Laguerre polynomial. Then

$$\frac{s_n}{a^n} - \frac{ns_{n-1}}{a^{n-1}} = l_n\Big(\frac{1}{a}\Big), \text{ for } a \geq 1.$$

Corollary 4. *Let r_n be the order of the symplectic Renner monoid, and let $r(x) = \sum_{n=0}^{\infty}\big(\frac{r_n}{4^n n!}\big)x^n$, the generating function of r_n. Then $r(x)$ converges for $|x| < 1$ to the function $\frac{1}{1-x}e^{x/4(1-x)} + \frac{2}{2-x}$.*

Corollary 5. *Let d_n be the order of the even special orthogonal Renner monoid, and let $d(x) = \sum_{n=0}^{\infty}\big(\frac{d_n}{4^n n!}\big)x^n$, the generating function of d_n. Then $d(x)$ converges to the function $\frac{1}{1-x}\big[e^{x/4(1-x)} - \frac{x}{2(2-x)}\big]$ for $|x| < 1$.*

Remark. The generating function of the order of the odd special orthogonal Renner monoid is the same as that of the symplectic Renner monoid.

4.12 Generalized Renner Monoids

It is convenient, for the moment, to let R be temporally a *factorizable monoid* with unit group W acting on the set $E(R)$ of idempotents by conjugation. Denote by Λ a transversal of $E(R)$ for this action. For each $e \in E(R)$ let

$$W(e) = \{w \in W \mid we = ew\}$$
$$W_*(e) = \{w \in W \mid we = ew = e\}.$$

Godelle introduces the concept of *generalized Renner monoids*, a class of factorizable monoids.

Definition 11. A generalised Coxeter-Renner system is a triple (R, W, S) such that

(1) R is a factorizable monoid and (W, S) is Coxeter system.
(2) Λ a sub-semilattice of $E(R)$.
(3) For each pair $e_1 \leq e_2$ in $E(R)$ there exists $w \in W$ and $f_1 \leq f_2$ in Λ such that $wf_iw^{-1} = e_i$ for $i = 1, 2$.
(4) For every $e \in \Lambda$ the subgroups $W(e)$ and $W_*(e)$ are standard Coxeter subgroups of W.
(5) The map $\Lambda \to 2^S : e \mapsto \lambda^*(e) = \{s \in S \mid se = es \neq e\}$ satisfies: if $e \leq f$ then $\lambda^*(e) \leq \lambda^*(f)$.

The monoid R in a Coxeter-Renner system is referred to as a generalized Renner monoid. Godelle [16] introduces a different length function on R, and he used this function to investigate the generic Hecke algebra $H(R)$ over $Z[q]$, which are deformations of the monoid Z-algebra of R. If M is a finite reductive monoid with a Borel subgroup B and Renner monoid R, he then finds the associated Iwahori-Hecke algebra $H(M, B)$ by specialising q in $H(R)$ and tensoring by C over Z. The Renner monoid of a reductive monoid and the Renner monoid of a finite monoid of Lie type are examples of generalized Renner monoids. Mokler [44] studies a different type of discrete monoids constructed from Kac-Moody Lie groups and algebras, called Weyl monoids. The Weyl monoids are generalized Renner monoids [16].

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Conjugacy Decomposition of Canonical and Dual Canonical Monoids

Ryan K. Therkelsen

Abstract Putcha's theory of conjugacy classes in a reductive monoid culminates in a decomposition of the monoid in terms of these classes, which we call the conjugacy decomposition. With this decomposition, we have a partially ordered set, with partial order analogous to the Bruhat-Chevalley order for the Bruhat-Renner Decomposition of a reductive monoid. We outline the development of the conjugacy decomposition, paying attention to the cases of canonical and dual canonical monoids. These monoids appear in the literature as $\mathscr{J}$-irreducible and $\mathscr{J}$-coirreducible, respectively, of type $\emptyset$. We conclude with a summary of new results, describing the order between classes in the conjugacy decomposition for canonical and dual canonical monoids.

Keywords Reductive monoid • Conjugacy decomposition • Conjugacy order • Canonical monoid • Dual canonical monoid

Subject Classifications: Primary 20M32; Secondary 06A06

Introduction

The development of Putcha's theory of conjugacy classes in reductive monoids, initiated in [10], may be followed over a series of papers, most notably [11, 13, 14], and [17]. These results lead to the description of a decomposition of the monoid in terms of conjugacy classes, indexed by a certain collection of its idempotents and group elements. This so-called conjugacy decomposition, along with an associated

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partial order, is the subject of this contribution. We begin with a summary of the relevant background material for reductive monoids. In the following section, we go on to describe the early developments relating to conjugacy classes in a reductive monoid, summarizing the main points from [11, 13], and [14]. From there we describe more recent developments, including new results relating to two special classes: canonical and dual canonical monoids. In the final section, we briefly remark on an application of this decomposition, identifying the irreducible components of the variety consisting of nilpotent elements in the monoid.

1 Preliminaries

Our objects of study in this paper are reductive monoids. We begin with a brief summary of the important points. For additional details, we refer the reader to [12, 21, 22].

1.1 Reductive Monoids, Cross-Section Lattices, and the Renner Monoid

Let k be an algebraically closed field. By a **reductive monoid**, we mean an irreducible linear algebraic monoid, over k, whose unit group is a reductive group. Every reductive monoid is the Zariski closure of a reductive group. That is, for a closed group G of $GL_n(k)$, $M = \overline{G} \subseteq M_n(k)$ is a reductive group with unit group G. By Green's $\mathscr{J}$-relation, [6], and the fact that M is irreducible, we have

$$a \mathscr{J} b \iff MaM = MbM \iff GaG = GbG$$

as shown in [12]. The $\mathscr{J}$-classes of M are therefore exactly the $G \times G$ orbits. In [9], Putcha showed that there exists an idempotent cross-section Λ of $G \times G$ orbits such that for $e, f \in \Lambda$,

$$e \leq f \iff GeG \subseteq \overline{GfG} \iff MeM \subseteq \overline{MfM}$$

where $e \leq f$ means $ef = fe = e$, as usual [4], and the closure is with respect to the Zariski topology. Λ is a finite lattice, called a **cross-section lattice** of M. This gives us a nice decomposition of M in terms of its $\mathscr{J}$-classes (that is, $G \times G$ orbits)

$$M = \bigsqcup_{e \in \Lambda} GeG. \tag{1}$$

Let T be a maximal torus in G and B a Borel subgroup of G containing T. It turns out that for a fixed T, the cross-section lattices of M are in one-to-one correspondence with the Borel subgroups of G containing T, [9]. In particular, we have

$$\Lambda = \{e \in E(\overline{T}) \big| Be = eBe\}. \tag{2}$$

Example 1. Let $M = M_n(k)$. If B is the group of invertible upper triangular matrices with $T \subseteq B$ the invertible diagonal matrices, then

$$\Lambda = \left\{ \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \middle| 0 \leq r \leq n \right\}$$

where I_r is the $r \times r$ identity matrix. In this case, Λ is a chain and each $G \times G$ orbit consists of matrices of a particular rank.

Let $W = N_G(T)/T$ denote the Weyl group of G. The reductive group G then has the **Bruhat decomposition**

$$G = \bigsqcup_{w \in W} BwB \tag{3}$$

with the **Bruhat-Chevalley order** on W defined in terms of the $B \times B$ orbits as

$$x \leq y \iff BxB \subseteq \overline{ByB} \tag{4}$$

for $x, y \in W$ and closure with respect to the Zariski topology, [3]. In [20], Renner extends this decomposition to reductive monoids as

$$M = \bigsqcup_{\sigma \in R} B\sigma B \tag{5}$$

where $R = \overline{N_G(T)}/T$ is a finite inverse monoid, called the **Renner monoid**. We note that the Weyl group W is the unit group of the Renner monoid R and so we have

$$R = \bigsqcup_{e \in \Lambda} WeW. \tag{6}$$

The Bruhat-Chevalley order also has a natural generalization to the monoid setting, with

$$\sigma \leq \theta \iff B\sigma B \subseteq \overline{B\theta B} \tag{7}$$

for $\sigma, \theta \in R$.

Example 2. For $M = M_n(k)$, the Renner monoid is the set of $n \times n$ partial permutation matrices (the **rook monoid**), denoted R_n. By this we mean the set of $\{0, 1\}$-matrices having at most one 1 in each column and each row. The unit group of R_n is the set of $n \times n$ permutation matrices.

1.2 Parabolic Subgroups and Quotients

In studying the structure of a reductive monoid, it is useful to describe elements of the Renner monoid R in terms of parabolic subgroups of W and the resulting quotient elements. We briefly describe these objects now.

Let W be the Weyl group of G, the unit group of a reductive monoid M. For $e, f \in \Lambda$ we have

$$W(e) = \left\{x \in W \middle| ex = xe\right\} \tag{8}$$

$$W^*(e) = \bigcap_{f \geq e} W(f) \tag{9}$$

$$W_*(e) = \bigcap_{f \leq e} W(f) \tag{10}$$

with

$$W(e) = W^*(e) \times W_*(e) \tag{11}$$

as shown in Chapter 10 of [12]. To clarify, the elements of $W(e)$ are the Weyl group elements that commute with the idempotent $e \in \Lambda$, $W_*(e)$ is the set of commuting elements of W which are absorbed by e, and $W^*(e)$ consists of those elements of W that commute with e where nothing is absorbed. Occasionally, we will find it convenient to describe these subgroups of W using the more common notation for parabolic subgroups. That is, for the Weyl group W with S the set of generating simple reflections, W_I denotes the parabolic subgroup generated by $I \subseteq S$.

Example 3. If $M = M_n(k)$ with B and $T \subseteq B$ as in Example 1, $e \in \Lambda$ is of the form $e = \begin{bmatrix} I_j & 0 \\ 0 & 0 \end{bmatrix}$ for some j, $0 \leq j \leq n$. Denoting this idempotent by e_j, we have

$$W(e_j) = \left\{\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\right\}$$

and

$$W^*(e_j) = \left\{\begin{bmatrix} P & 0 \\ 0 & I_{n-j} \end{bmatrix}\right\}, \qquad W_*(e_j) = \left\{\begin{bmatrix} I_j & 0 \\ 0 & Q \end{bmatrix}\right\}$$

where P and Q are permutation matrices. Identifying W with the symmetric group S_n, we have $W(e_j) = W_I$, $W_*(e_j) = W_K$, and $W^*(e_j) = W_{I \setminus K}$, where

$$I = \{(1\ 2), (2\ 3), \ldots, (n-1\ n)\} \setminus \{(j\ \ j+1)\}$$
$$K = \{(j+1\ \ j+2), \ldots, (n-1\ n)\}$$

Let ℓ denote the usual length function on W, [1, 7]. Given a parabolic subgroup $W(e)$, we have the following important subsets:

$$D(e) = \{x \in W \big| \ell(xw) = \ell(x) + \ell(w) \text{ for all } w \in W(e)\}$$
$$D(e)^{-1} = \{x \in W \big| \ell(wx) = \ell(w) + \ell(x) \text{ for all } w \in W(e)\}.$$

We call elements of $D(e)$ or $D(e)^{-1}$ **quotients** and note that $D(e)$ consists of the unique minimal length element from each coset $xW(e)$, with $D(e)^{-1}$ likewise for $W(e)x$. If $W_I = W(e)$, we will sometimes write D_I for $D(e)$ and D_I^{-1} for $D(e)^{-1}$. These sets are denoted W^I and ${}^I W$, respectively, in [1].

1.3 Bruhat-Chevalley Order and Gauss-Jordan Elements

Every Renner monoid element may be expressed uniquely in terms of an idempotent and quotient elements of parabolic subgroups related to the idempotent. If $\sigma \in R$, then by (6), $\sigma \in WeW$ for a unique idempotent $e \in \Lambda$. This means $\sigma = w_1 e w_2$ for some $w_1, w_2 \in W$. If we factor w_2 in terms of $W(e)$ and $D(e)^{-1}$, commute the $W(e)$ term with e, and factor the resulting element on the left of e in terms of $W_*(e)$, we have

$$\sigma = xey \tag{12}$$

for unique $y \in D(e)^{-1}$ and $x \in D_K$ where $W_*(e) = W_K$. An element of the Renner monoid is said to be in **standard form** if it is written this way.

In [20], Renner identifies the reductive monoid elements analogous to matrices in reduced row-echelon form. We're interested in such elements that are also in the Renner monoid R. We call these the **Gauss-Jordan elements** of R, formally defined as $\mathcal{GJ} = \{x \in R | Bx \subseteq xB\}$, where B is a Borel subgroup of G as usual. We are especially interested in the Gauss-Jordan elements as we will later show that they may be used to index the orbits in the conjugacy decomposition of a reductive monoid M. For now, we give a more convenient description of $\mathcal{GJ}$, in terms of quotients of parabolic subgroups of W:

$$\mathcal{GJ} = \{ey \in R \bigm| e \in \Lambda,\ y \in D(e)^{-1}\}. \tag{13}$$

Additionally, by $\mathcal{GJ}(e)$ we mean the Gauss-Jordan elements in WeW.

With this new description of Renner monoid elements we briefly look back at the Bruhat-Chevalley order on R. For the Weyl group W, it is well known that the condition $BxB \subseteq \overline{ByB}$ is equivalent to x being a subword of a reduced expression of y. The situation for Renner monoids is more complicated, though the approach is similar. The following description for the Bruhat-Chevalley order on R, (7), was introduced in [8]:

Theorem 1. *Let $\sigma, \theta \in R$, with $\sigma = xey$ and $\theta = ufv$ be in standard form. Then*

$$\sigma \leq \theta \iff e \leq f \text{ and } x \leq uw,\ w^{-1}v \leq y \tag{14}$$

for some $w \in W(f)W_*(e)$.

Corollary 1. *Let ey and fy' be Gauss-Jordan elements of R. Then $ey \leq fy'$ if and only if $e \leq f$ and there exists $w \in W_*(e)W(f)$ such that $wy' \leq y$.*

Comparing $\sigma, \theta \in R$ is easier if both elements are in the same $W \times W$ orbit since the w required in the theorem is from the subgroup $W(e)$. For Gauss-Jordan elements the comparison is easier as well since now $ey \leq ey'$ if and only if $y' \leq y$. Alternate descriptions of the Bruhat-Chevalley order within WeW are found in [15] and [23]. The order between the $W \times W$ orbits is trickier, however an effective approach for this situation is given in [16]. Here Putcha defines his **projection maps**, $p_{e,f} : WeW \to WfW$. We refer the reader to [16] for the formal description of the maps and focus instead on their especially nice properties, outlined in the main result of the paper:

Theorem 2. *Let $e, f \in \Lambda$, $e \leq f$. Then*

1. *$p_{e,f} : WeW \to WfW$ is order-preserving and $\sigma \leq p_{e,f}(\sigma)$ for all $\sigma \in WeW$.*
2. *If $\sigma \in WeW$, $\theta \in WfW$, then $\sigma \leq \theta$ if and only if $p_{e,f}(\sigma) \leq \theta$.*
3. *If $h \in \Lambda$ with $e \leq h \leq f$, then $p_{e,f} = p_{h,f} \circ p_{e,h}$.*

The essential point is this: with the projection maps, we turn a "between classes" order problem into a "within classes" order problem which, as noted, is easier to solve.

2 Conjugacy Classes in a Reductive Monoid

As previously noted, the theory of conjugacy classes in reductive monoids is developed over a series of papers, [11, 13, 14], and [17]. In this section, we outline the main results of the first three of these entries. For additional details on the topics, Chapter 12 of Renner's monograph, [21], is highly recommended. The most recent paper in this series, [17], is the launching point for the primary topics of this exposition. This paper appeared after Renner's monograph and we will devote our attention to its results separately in the subsequent section.

2.1 The Development of the Conjugacy Decomposition

Let M be a reductive group with unit group G, as usual. Two elements $x, y \in M$ are **conjugate**, denoted $x \sim y$, if $x = y^g = g^{-1}yg$ for some $g \in G$. For $X, Y \subseteq M$, we write $X \sim Y$ if every element in X is conjugate to an element in Y and every element in Y is conjugate to an element in X.

The study of conjugacy classes in M began in earnest with [11]. The main result here is the following:

Theorem 3. *There exist affine subsets $M_1, \ldots, M_k$ of M, reductive groups $G_1, \ldots, G_k$ with respective automorphisms α_i of G_i, and surjective maps $\xi_i : M_i \to G_i$ such that:*

1. *Every element of M is conjugate to an element of some M_i, and*
2. *If $x, y \in M_i$, then $x \sim y$ if and only if there exists $g \in G_i$ such that $g\xi_i(x)\alpha_i(g)^{-1} = \xi_i(y)$.*

The subsets M_i were defined in terms of a diagonal idempotent and a Weyl group element. Namely, for $e \in E(\overline{T})$ and $w \in W$,

$$M_{e,w} = eC_G(e^v \big| v \in \langle w \rangle)w$$

where $C_G(X)$ is the centralizer of X in G and $\langle w \rangle$ is the cyclic subgroup generated by w. This was a wonderful result, however an open problem remained as it is possible that an element in M_i is conjugate to an element in M_j for $i \neq j$. This issue was resolved in [13] with the identification of a certain closed subset N_{ew} of $M_{e,w}$. With these subsets identified, Putcha goes on to prove the following:

Theorem 4. *Let M be a reductive monoid with Renner monoid R.*

1. *If $\sigma, \theta \in R$ with $N_\sigma \cap N_\theta \neq \emptyset$, then $N_\sigma = N_\theta$.*
2. *If $u \in W(e^v \big| v \in \langle w \rangle)$, then $N_{euw} \subseteq M_{e,w}$ and $N_{euw} = N_{ew}^\pi$ for some $\pi \in W(e^v \big| v \in \langle w \rangle)$*
3. *Any element of $M_{e,w}$ is conjugate to some element of N_{ew}.*
4. *Any element of M is conjugate to an element of N_{ew} for some $e \in \Lambda$, $w \in W$.*
5. *The map $\xi : M_{e,w} \to G_{e,w}$ remains surjective when restricted to N_{ew}.*

Theorem 5. *The following are equivalent for $e \in \Lambda$ and $u, v \in W$.*

1. *There exists an element of $M_{e,u}$ that is conjugate to an element of $M_{e,v}$.*
2. $\bigcup_{g \in G} gM_{e,u}g^{-1} = \bigcup_{g \in G} gM_{e,v}g^{-1}$.
3. *There exists $w \in W$, with $eu \sim ew$ in R, such that $\bigcap_{i \geq 0} w^i W(e)u^{-1} \neq \emptyset$.*
4. *$N_{eu}^\pi = N_{ev}$ for some $\pi \in W(e)$.*

These two theorems are the main results of [13]. All that remains in the analysis of the decomposition of M in terms of conjugacy classes is a description of the representatives of the $\bigcup_{g \in G} gM_{e,u}g^{-1}$ classes. This feat is accomplished in the next paper in the series.

In [14], Putcha identifies a set of elements $R^* \subseteq R$ such that

$$M = \bigsqcup_{r \in R^*} X(r)$$

where $X(r) = \bigcup_{g \in G} gM_{e,u}g^{-1}$. It turns out that R^* consists of Gauss-Jordan elements. Thus for $r \in R^* \subseteq \mathscr{G}\mathscr{J}$, $r = ey$ for some $e \in \Lambda$, $y \in D(e)^{-1}$. For a fixed idempotent $e \in \Lambda$, we denote the set of quotients corresponding to elements in R^* as follows:

$$D^*(e) = \{y \in D(e)^{-1} \big| ey \in R^*\}. \tag{15}$$

Much effort is spent leading up to the definition of the $D^*(e)$ sets and the reader is encouraged to see [14] for the details. For now, we give a more succinct description of the elements of $D^*(e)$: if $y \in D^*(e)$ and $ey \sim ez$ for some $z \in D(e)^{-1}$, then $\ell(y) \leq \ell(z)$ with equality if and only if $z = y$.

Example 4. Let $M = M_4(k)$. For $e = e_2$, using the notation from Example 3, $\mathscr{G}\mathscr{J}(e)$ consists of six elements:

$$e = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \; ey_1 = \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \; ey_2 = \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

$$ez_1 = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \; ez_2 = \begin{bmatrix} 0&1&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \; ev = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

The conjugate elements in this set are $ey_1 \sim ey_2$ and $ez_1 \sim ez_2$. We note that $1 = \ell(y_1) < \ell(y_2) = 2$ and $2 = \ell(z_1) < \ell(z_2) = 3$, and hence

$$D^*(e) = \{1, y_1, z_1, v\}.$$

Theorem 6. *Let $e \in \Lambda$.*

1. *If $y \in D(e)^{-1}$, then $X(ey) = \bigcup_{g \in G} gBeyBg^{-1}$*
2. *$GeG = \bigsqcup_{y \in D^*(e)} X(ey)$.*
3. *$M = \bigsqcup_{r \in R^*} X(r)$.*

The decomposition from the last part of the above theorem is called the **conjugacy decomposition** (or **Putcha decomposition**, as in [21]) of the reductive monoid M.

We next define a transitive relation $\preceq$ on R, generated by

(a) If $r_1 \leq r_2$ in the Bruhat-Chevalley order, then $r_1 \preceq r_2$,
(b) If $y \in D(e)^{-1}$ and $w \in W$, then $eyw \preceq wey$.

Theorem 7. *1.* $(R^*, \preceq)$ *is a partially ordered set.*
2. If $r_1, r_2 \in R^*$*, then* $X(r_1) \subseteq \overline{X(r_2)}$ *if and only if* $r_1 \preceq r_2$*.*
3. If $r \in R^*$*, then* $\overline{X(r)} = \bigsqcup_{s \preceq r, s \in R^*} X(s)$*.*

We call the partial order on R^* the **conjugacy order** and the partially ordered set in the theorem the **conjugacy poset** (or **Putcha poset**, as in [21]) of the reductive monoid M.

Remark 1. The motivating example for the conjugacy decomposition comes from the multiplicative monoid $M_n(k)$. For $A, B \in M_n(k)$, define $A \equiv B$ if $\text{rank}(A^i)$ equals $\text{rank}(B^i)$ for all i, $1 \le i \le n$, and denote the $\equiv$-class of A by $[A]$. A natural partial order exists on the set of these classes, defined in terms of the ranks of successive powers of matrix representatives as follows:

$$[A] \le [B] \iff \text{rank}(A^i) \le \text{rank}(B^i), \text{ for all } 1 \le i \le n. \tag{16}$$

It turns out that the $\equiv$-classes are exactly the classes in the conjugacy decomposition and the partial order (16) corresponds to the conjugacy order. For additional details, the reader is referred to Section 1 of [17] for the motivation and Section 3 of [17] for the proofs.

Example 5. Consider the rank 2 elements of $M_4(k)$. Given a matrix A, it is straightforward to construct a matrix from the rook monoid R_n in row echelon form that has the same sequence of ranks of successive powers as A. In other words, in every $\equiv$-class we can find a Gauss-Jordan element. We therefore consider the rank 2 Gauss-Jordan elements of $M_4(k)$, as presented in Example 4, and observe the following:

Element	Successive ranks
e	$2 \to 2 \to 2$
ey_1	$2 \to 1 \to 1$
ey_2	$2 \to 1 \to 1$
ez_1	$2 \to 1 \to 0$
ez_2	$2 \to 1 \to 0$
ev	$2 \to 0 \to 0$

There are four classes here, as noted in Example 4, with $[ev] < [ez_1] < [ey_1] < [e]$ by (16).

Recall that our description of the conjugacy order was in terms of R^*. Since our idempotent is fixed, we therefore consider $eD^*(e) = \{e, ey_1, ez_1, ev\}$. Now, by Part 2 of Theorem 1.5 from [2] we have

$$ev < ez_1 < ey_1 < e$$

in the Bruhat-Chevalley order and hence

$$ev \preceq ez_1 \preceq ey_1 \preceq e.$$

3 Recent Developments

Renner's monograph [21] outlines the development of Putcha's theory of conjugacy classes over [11, 13], and [14], as described in the previous section. In this section, we track the more recent developments relating to the conjugacy poset.

In [17], Putcha continues his analysis of the conjugacy decomposition of a reductive monoid. After a description of the motivating example of $M_n(k)$ in terms of partitions, briefly noted in Remark 1 above, he gives a refined description of the decomposition for a general reductive monoid. To begin, we're reminded of the definition of $X(\sigma)$. That is, for $\sigma \in R$,

$$X(\sigma) = \bigcup_{g \in G} gB\sigma Bg^{-1}.$$

Now, since G/B is a projective variety,

$$Y(\sigma) = \overline{X(\sigma)} = \bigcup_{\sigma' \leq \sigma} X(\sigma')$$

is a closed irreducible subset of M. For $\sigma, \theta \in R$, we define

$$\sigma \approx \theta \text{ if } Y(\sigma) = Y(\theta)$$

and

$$\sigma \preceq \theta \text{ if } Y(\sigma) \subseteq Y(\theta).$$

Putcha then goes on to describe the decomposition in terms of Gauss-Jordan elements. Again from [17]:

Theorem 8. *Let R be the Renner monoid of a reductive monoid.*

1. *If $\sigma \in R$, then $\sigma \approx \sigma'$ for some $\sigma' \in \mathscr{G}\mathscr{J}$.*
2. *If $\sigma, \theta \in \mathscr{G}\mathscr{J}$, then*

$$\sigma \approx \theta \iff \sigma \sim \theta \text{ in } R \iff X(\sigma) = X(\theta).$$

Let $\mathscr{P} = \mathscr{G}\mathscr{J}/\sim$, where $\sim$ is conjugacy in R (in [17], $\mathscr{P}$ is denoted $\tilde{R}$) and for $\sigma \in \mathscr{G}\mathscr{J}$, define

$$[\sigma] = \{\sigma' \in \mathscr{G}\mathscr{J} \big| \sigma' \sim \sigma\}. \tag{17}$$

Additionally, it will be useful to consider the elements of $\mathscr{P}$ in terms of the idempotents from Λ and so we define $\mathscr{P}(e) = \{[ey] \big| y \in D(e)^{-1}\}$.

For $\sigma \in R$, let $p(\sigma) = [\sigma']$, where $\sigma \approx \sigma' \in \mathscr{G}\mathscr{J}$ as described in the previous theorem. Now, $\preceq$ from the theorem induces a partial order $\leq$ on $\mathscr{P}$, where $[ey] \leq [fz]$ if $ey \preceq fz$. $(\mathscr{P}, \leq)$ is the conjugacy poset of M, with a slightly different description than before, as the following theorem from [17] shows.

Theorem 9. *Let R be the Renner monoid of a reductive monoid M.*

1. $p : R \to \mathscr{P}$ *is an order-preserving map.*
2. $M = \bigsqcup_{[ey]\in\mathscr{P}} X(ey)$.
3. *If* $[ey] \in \mathscr{P}$, *then* $Y(ey) = \bigsqcup_{[fz]\leq[ey]} X(fz)$.

We next describe the conjugacy order on elements of $\mathscr{P}$ in terms of the Bruhat-Chevalley order on R, (14). The following, from [24], is a refinement of a result in [17], in which the description of the a required in the theorem is more precise.

Theorem 10. *Let $[\sigma] \in \mathscr{P}(e)$, $[\theta] \in \mathscr{P}(f)$. Then $[\sigma] \leq [\theta]$ if and only if $a\sigma a^{-1} \leq \theta$ for some $a \in W(f)W_*(e)$.*

With this new description, Putcha goes on to prove some new results on the conjugacy poset, beginning with the order in $\mathscr{P}(e)$ for $e \in \Lambda$. Comparing two elements is easier in this case, since by the previous theorem we check if $a\sigma a^{-1} \leq \theta$ for some $a \in W(e)W_*(e) = W(e)$. The next result, from [24], is a generalization of Theorem 2.9 from [17]. In the theorem, we write $w = \hat{w}\check{w}$ for $w \in W(e)$, where $\hat{w} \in W^*(e)$ and $\check{w} \in W_*(e)$. By (11), $\hat{w}$ and $\check{w}$ are unique and $\ell(\hat{w}\check{w}) = \ell(\hat{w}) + \ell(\check{w})$.

Theorem 11. *Let $e \in \Lambda$ and $y, z \in D(e)^{-1}$. Then the following conditions are equivalent:*

1. $[ey] \leq [ez]$.
2. $\hat{w}z \leq yw$ *for some* $w \in W(e)$.
3. $\hat{w}zw^{-1} \leq y$ *for some* $w \in W(e)$.

With this theorem, we are able to turn a conjugacy order problem into an order problem involving only Weyl group elements. Since the Bruhat-Chevalley order on R is more complicated than the order on W, this is an especially welcome result.

4 The Conjugacy Poset for Canonical and Dual Canonical Monoids

Theorem 11 gives us a nice description of the conjugacy order within the $\mathscr{P}(e)$ classes. What remains is to examine the order between these classes. In [16], the analogous problem was resolved for R by defining order-preserving maps with some nice properties. In particular, for WeW and WfW with $e \leq f$, a map $p : WeW \to WfW$ is defined such that for $\sigma \in WeW$ and $\theta \in WfW$, $\sigma \leq \theta$ if and only if $p(\sigma) \leq \theta$. The existence of similar maps for $\mathscr{P}(e)$ to $\mathscr{P}(f)$ was conjectured in [17]. In this section, we give an affirmative answer to the conjecture for the case that M is either a canonical or dual canonical monoid. These results originally appeared in [24].

4.1 Canonical and Dual Canonical Monoids

Let Λ be a cross-section lattice for the reductive monoid M. As Λ is a finite lattice, it contains a minimum and maximum element, which we denote 0 and 1, respectively. Suppose $\Lambda \setminus \{0\}$ contains a unique minimal element, say e_0, with $W(e_0) = W_I$ for $I \subseteq S$. M is then called a **$\mathscr{J}$-irreducible monoid of type I**. A **canonical monoid** is a $\mathscr{J}$-irreducible monoid of type $\emptyset$. Canonical monoids were first studied in [19]. Their construction was modeled by the canonical compactification of a reductive group, as in [5].

Example 6. Let $G_0 = \{A \otimes (A^{-1})^t \mid A \in SL_3(k)\}$ and let $M = \overline{kG_0} \subseteq M_9(k)$. Then M is a canonical monoid with $W = S_3$. This example appears in [12, 16], and [18].

Suppose instead that $\Lambda \setminus \{1\}$ contains a unique maximal element, say e_0, with $W(e_0) = W_I$ for $I \subseteq S$. M is then called a **$\mathscr{J}$-coirreducible monoid of type I**. A **dual canonical monoid** is a $\mathscr{J}$-coirreducible monoid of type $\emptyset$. Dual canonical monoids also arise naturally, as the following example shows.

Example 7. Let $G_0 = \{A \oplus (A^{-1})^t \mid A \in SL_3(k)\}$ and let $M = \overline{kG_0} \subseteq M_6(k)$. Then M is a dual canonical monoid with $W = S_3$. This example appears in [12,16], and [18].

Remark 2. The following points are useful to keep in mind when considering canonical and dual canonical monoids:

1. If M is a canonical monoid, then $W(e) = W^*(e)$ and $W_*(e) = \{1\}$ for all $e \in \Lambda \setminus \{0\}$.
2. If M is a canonical monoid, then $D^*(e) = D(e)^{-1}$ for all $e \in \Lambda \setminus \{0\}$, where $D^*(e)$ is as in (15).
3. If M is a dual canonical monoid, then $W(e) = W_*(e)$ and $W^*(e) = \{1\}$ for all $e \in \Lambda \setminus \{1\}$.
4. If M is a dual canonical monoid, then $D^*(e) = D(e) \cap D(e)^{-1}$ for all $e \in \Lambda \setminus \{1\}$, where $D^*(e)$ is as in (15).

4.2 Conjugacy Order for Dual Canonical Monoids

The order within $\mathscr{P}(e)$ is described in Theorem 11. If M is a dual canonical monoid, then $W^*(e) = \{1\}$ and the conditions in the Theorem are simplified.

Corollary 2. *Let M be a dual canonical monoid with $e \in \Lambda \setminus \{1\}$ and $y, z \in D(e)^{-1}$. Then the following conditions are equivalent:*

1. $[ey] \leq [ez]$.
2. $z \leq yw$ *for some* $w \in W(e)$.
3. $zw \leq y$ *for some* $w \in W(e)$.

As noted in Remark 2, if M is a dual canonical monoid then for $e \in \Lambda \setminus \{1\}$ we have $D^*(e) = D(e) \cap D(e)^{-1}$. By Corollary 2, $[ey] \leq [ez]$ if and only if $z \leq yw$ for some $w \in W(e)$. However, if $y, z \in D^*(e)$ then $y, z \in D(e)$ and so $z \leq yw$ if and only if $z \leq y$. Hence for dual canonical monoids, we have the following description for $\mathscr{P}(e)$:

$$[ey] \leq [ez] \iff z \leq y. \tag{18}$$

For the order between $\mathscr{P}(e)$ classes in a dual canonical monoid, there exist maps analogous to those between the $W \times W$ orbits from Theorem 2. What's more, these maps are defined in terms of the $W \times W$ maps. The trick here is to pick the correct representative from $[ey]$.

Let M be a dual canonical monoid and suppose $[\sigma] \in \mathscr{P}(e)$. Choose the unique ey from $[\sigma]$ such that $y \in D^*(e)$. We now let $W_K = W(e) \bigcap \left(y^{-1} W(e) y\right)$ with w_I, w_K the longest elements in $W(e)$, W_K, respectively.

Proposition 1. *For y, w_K, and w_I as described, $y w_K w_I \in D(e)^{-1}$.*

Proof. $y w_K w_I = w y'$ for some $w \in W(e)$ and $y' \in D(e)^{-1}$. Since $ey \sim ey'$, we have $y' = yv$ for some $v \in W_I$. Let $w_K = y^{-1} v_K y$ and so

$$y w_K w_I = w y' = w y v. \tag{19}$$

Keeping in mind that $y \in D^*(e) = D(e) \cap D(e)^{-1}$ and $yv \in D(e)^{-1}$, we observe the following:

$$\begin{aligned} \ell(y w_K w_I) &= \ell(w y v) \\ \ell(y) + \ell(w_K w_I) &= \ell(w) + \ell(y v) \\ \ell(y) + \ell(w_K w_I) &= \ell(w) + \ell(y) + \ell(v) \\ \ell(w_K w_I) &= \ell(w) + \ell(v) \end{aligned}$$

On the other hand, we may rewrite (19) as

$$v w_I = y^{-1} w^{-1} y w_K = y^{-1} w^{-1} y y^{-1} v_K y = y^{-1} w^{-1} v_K y.$$

Now, since $y^{-1} w^{-1} v_K y \in W_K$, by definition $y^{-1} w^{-1} v_K y \leq w_K$. Both elements are in W_I as well, so $v = y^{-1} w^{-1} v_K y w_I \geq w_K w_I$. Putting everything together, we see that $\ell(w) + \ell(v) = \ell(w_K w_I) \leq \ell(v)$ and hence $\ell(w) = 0$. Therefore $w = 1$ and we have $y w_K w_I = w y' = y' \in D(e)^{-1}$. That is, $y w_K w_I \in D(e)^{-1}$. ◻

Proposition 1 tells us that each set of conjugate Gauss-Jordan elements has a minimum element, with respect to (14). For $[ey]$, we denote this element by ey_m. For $e, f \in \Lambda$, $e \leq f$, we now define the projection of $[ey]$ in $\mathscr{P}(f)$ as:

$$\tilde{p}_{e,f}([ey]) = \left[p_{e,f}(ey_m)\right]. \tag{20}$$

The following theorem shows that these maps satisfies the conditions of the conjecture from [17].

Theorem 12. *Let $e, f \in \Lambda$, with $e \leq f$. Then*

1. *$\tilde{p}_{e,f} : \mathscr{P}(e) \to \mathscr{P}(f)$ is order-preserving.*
2. *$[ey] \leq \tilde{p}_{e,f}([ey])$.*
3. *$[ey] \leq [fz]$ if and only if $\tilde{p}_{e,f}([ey]) \leq [fz]$.*

Proof. 1. Let $[ey] \leq [ey']$. Then $w^{-1}ey_mw \leq ey'_m$ for some $w \in W(e)$ and so $ey_mw \leq ey'_m$ or $ey_mw = eu_1y_1 = ey_1 \leq ey'_m$, where $u_1 \in W(e)$, $y_1 \in D(e)^{-1}$. However, $ey_m \sim ey_1$ and so $ey_m \leq ey_1 \leq ey'_m$. This means $p_{e,f}(ey_m) \leq p_{e,f}(ey'_m)$ and hence $[p_{e,f}(ey_m)] \leq [p_{e,f}(ey'_m)]$ or $\tilde{p}_{e,f}([ey]) \leq \tilde{p}_{e,f}([ey'])$.

2. By Theorem 2, $ey_m \leq p_{e,f}(ey_m)$. Then, by Theorem 10, we have

$$[ey] = [ey_m] \leq [p_{e,f}(ey_m)] = \tilde{p}_{e,f}([ey]).$$

3. Suppose $[ey] \leq [fz]$ and let $\tilde{p}_{e,f}([ey]) = [fy']$. Then $wey_mw^{-1} \leq fz_m$ and $ey_m \leq fy'_m$. However, $ey_m \sim wey_mw^{-1} = ey_2$ so $ey_m \leq ey_2$ and hence $ey_m \leq fz_m$. Thus $p_{e,f}(ey_m) \leq fz_m$, from which it follows that $\tilde{p}_{e,f}([ey]) \leq [fz]$. The other direction follows from the previous part. □

Example 8. Let M be a dual canonical monoid with Weyl group of type A2 (see Example 7), with $S = \{A, B\}$ the set of simple reflections. Figure 1 shows the Hasse diagram for the conjugacy poset $\mathscr{P}$ of this monoid, with the idempotents corresponding to $\emptyset$, $\{A\}$, $\{B\}$, and S denoted by $e_\emptyset$, e_A, e_B, and e_S, respectively.

4.3 Conjugacy Order for Canonical Monoids

In [17], Theorem 2.9 describes the order within $\mathscr{P}(e)$ when M is a canonical monoid. This theorem is a special case of Theorem 11 above—in fact, it motivated the general result.

As noted in Remark 2, $D^*(e) = D(e)^{-1}$ for a canonical monoid and so there is a one-to-one correspondence between vertices in $\mathscr{G}\mathscr{J}(e)$ under the Bruhat-Chevalley order and $\mathscr{P}(e)$ under the conjugacy order. The partial orders, however, do not coincide and so the posets are not isomorphic. In particular, $ey \leq ez$ implies $[ey] \leq [ez]$, though not necessarily conversely. The following example, originally from [17], makes this clear.

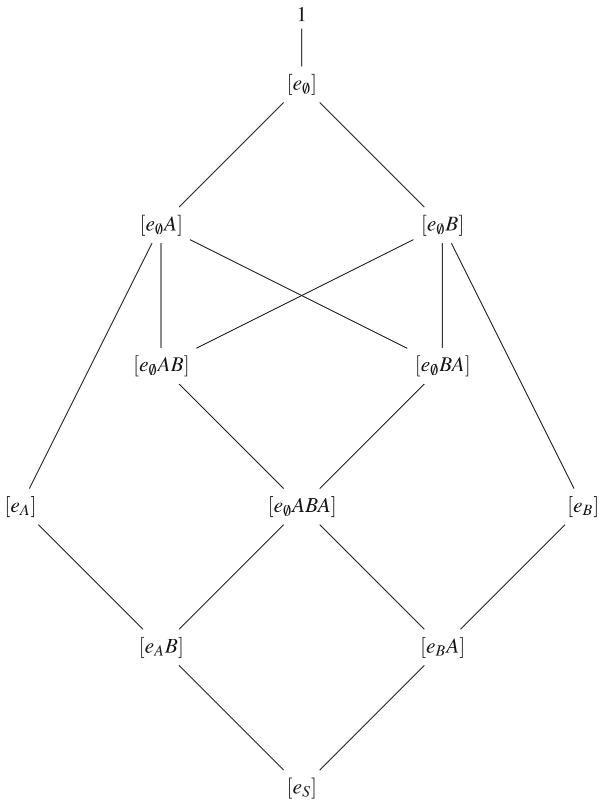

Fig. 1 $\mathscr{P}$ for the dual canonical monoid with W of Type A2

Example 9. Let M be a canonical monoid with Weyl group of type A3, with Coxeter graph $A - B - C$. Let $e \in \Lambda$ be the idempotent such that $W(e) = W_I$ for $I = \{B\}$. Then $\mathscr{G}\mathscr{J}(e)$ under Bruhat-Chevalley order is isomorphic to a (weak) subposet of $\mathscr{P}(e)$ under conjugacy order. The additional relations in $\mathscr{P}(e)$ are $[eABC] \leq [eCB]$ and $[eCBA] \leq [eAB]$, though $eABC \not\leq eCB$ and $eCBA \not\leq eAB$ in $\mathscr{G}\mathscr{J}(e)$.

For the order between the idempotent indexed classes, in [24] the author shows that the projection maps conjectured in [17] also exist for canonical monoids. These maps are, however, more difficult to describe than for the dual canonical case. We begin by noting two operations that will be crucial in the process of constructing the maps. If $a, b \in W$, then

$$a \circ b = \max\{ab' \mid b' \leq b\}$$

$$a * b = \begin{cases} ab & \text{if } \ell(ab) = \ell(a) + \ell(b) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We also will need a new relation, denoted $\leq_I$, where for $z \in D_I^{-1}$ and $y \in W$ we define

$$z \leq_I y \text{ if } wzw^{-1} \leq y \text{ for some } w \in W_I.$$

We may now describe the projection maps for M a canonical monoid. Suppose $e, f \in \Lambda$ with $e \leq f$ and let $[ey] \in \mathscr{P}(e)$, with $y \in D(e)^{-1}$ as usual. We factor y as $y = u_1 y_1$ with $u_1 \in W(f)$ and $y_1 \in D(f)^{-1}$. Now $y_1 \circ u_1 = y_1 * u_1'$ for some $u_1' \leq u_1$ and $y_1 * u_1' = u_2 y_2$ for some $u_2 \in W(f)$, $y_2 \in D(f)^{-1}$. We next consider $y_2 \circ u_2$ and repeat this process. In [24], it is shown that

$$\ell(u_1) > \ell(u_2) > \cdots > \ell(u_{m-1}) = \ell(u_m) = \ell(u_{m+1}) \cdots$$

for some $m \in \mathbb{Z}$. Note that while u_j and u_{j+1} have the same length for $j \geq m$, they need not be the same element. What is important here is that y_j *is* the same for all $j \geq m$. For $e, f \in \Lambda$, $e \leq f$, we now define the projection of $[ey]$ in $\mathscr{P}(f)$ as:

$$\tilde{p}_{e,f}([ey]) = [f y_m] \tag{21}$$

where $y = u_1 y_1$ with $u_1 \in W(f)$ and $y_1 \in D(f)^{-1}$ and y_m is as described above.

To prove that these maps satisfy the conditions we seek, we first note three helpful results. Theorem 13 is similar to Theorem 11, but between the $\mathscr{P}(e)$ classes. The proof relies on the fact that for canonical monoids, $W(e) = W^*(e)$ for all $e \in \Lambda$. This result does not hold for reductive monoids in general. For additional details, the reader is referred to [24].

Theorem 13. *Let $e, f \in \Lambda$ with $e \leq f$, $y \in D(e)^{-1}$, and $z \in D(f)^{-1}$. If $W(e) = W^*(e)$ and $W(f) = W^*(f)$, then the following conditions are equivalent:*

1. $[ey] \leq [fz]$.
2. $wz \leq yw$ *for some* $w \in W(f)$.
3. $wzw^{-1} \leq y$ *for some* $w \in W(f)$.

Lemma 1. *Let $y \in D_I^{-1}$, $u \in W_I$, and $z \in W$. Then $z \leq_I uy$ if and only if $z \leq_I y \circ u$.*

Lemma 2. *Let $W_K = \bigcap_{i \geq 0} \left(y^{-i} W_I y^i\right)$. Let $y, z \in D_I^{-1}$ and $u \in W_K$. Then $z \leq_I uy$ if and only if $z \leq_I y$.*

Finally, we prove the following for $\tilde{p}_{e,f} : \mathscr{P}(e) \to \mathscr{P}(f)$ from (21):

Theorem 14. *Let $e, f \in \Lambda$, with $e \leq f$. Then*

1. $[ey] \leq \tilde{p}_{e,f}([ey])$.
2. $[ey] \leq [fz]$ *if and only if* $\tilde{p}_{e,f}([ey]) \leq [fz]$.
3. $\tilde{p}_{e,f} : \mathscr{P}(e) \to \mathscr{P}(f)$ *is order-preserving.*

Proof. 1. Note that for $y = u_1 y_1$, $y_1 \circ u_1 = u_2 y_2$ for some $u_2 \in W(f)$ and $y_2 \in D(f)^{-1}$, and in general $y_{j-1} \circ u_{j-1} = u_j y_j$. By Lemma 2, $y_m \leq_J u_m y_m$, where J is such that $W(f) = W_J$. Now, applying Lemma 1, we have

$$
\begin{aligned}
& y_m \leq_J u_m y_m = y_{m-1} \circ u_{m-1} \\
\Longleftrightarrow\ & y_m \leq_J u_{m-1} y_{m-1} = y_{m-2} \circ u_{m-2} \\
& \vdots \\
\Longleftrightarrow\ & y_m \leq_J u_2 y_2 = y_1 \circ u_1 \\
\Longleftrightarrow\ & y_m \leq_J y = u_1 y_1 .
\end{aligned}
$$

Hence $y_m \leq_J y$, and so by Theorem 13, $[ey] \leq [f y_m] = \tilde{p}_{e,f}([ey])$.

2. Suppose $[ey] \leq [fz]$ and let $\tilde{p}_{e,f}([ey]) = [f y_m]$. By Theorem 13, $wzw^{-1} \leq y$ for some $w \in W(f)$. That is, $z \leq_J y$, where $W(f) = W_J$. Using Lemma 1, we have

$$
\begin{aligned}
& z \leq_J y = u_1 y_1 \\
\Longleftrightarrow\ & z \leq_J y_1 \circ u_1 = u_2 y_2 \\
& \vdots \\
\Longleftrightarrow\ & z \leq_J y_{m-2} \circ u_{m-2} = u_{m-1} y_{m-1} \\
\Longleftrightarrow\ & z \leq_J y_{m-1} \circ u_{m-1} = u_m y_m
\end{aligned}
$$

and, by Lemma 2,

$$
z \leq_J y_m
$$

Hence $z \leq_J y_m$, and so by Theorem 13, $[f y_m] = \tilde{p}_{e,f}([ey]) \leq [fz]$. The other direction follows from the previous part.

3. Suppose $[ey] \leq [ey']$. By the first part above, $[ey'] \leq \tilde{p}_{e,f}([ey'])$ and hence $[ey] \leq \tilde{p}_{e,f}([ey'])$. By the second part above, $\tilde{p}_{e,f}([ey]) \leq \tilde{p}_{e,f}([ey'])$. □

Example 10. Let M be a canonical monoid with Weyl group of type A2 (see Example 6), with $S = \{A, B\}$ the set of simple reflections. Figure 2 shows the Hasse diagram for the conjugacy poset $\mathscr{P}$ of this monoid, using the same notation for the idempotents as in Example 8.

5 Concluding Remarks

In this final section, we note a useful application of the conjugacy poset. For a reductive monoid M, the **nilpotent variety** is

$$
M_{nil} = \{a \in M \,|\, a^k = 0 \text{ for some } k\}. \tag{22}
$$

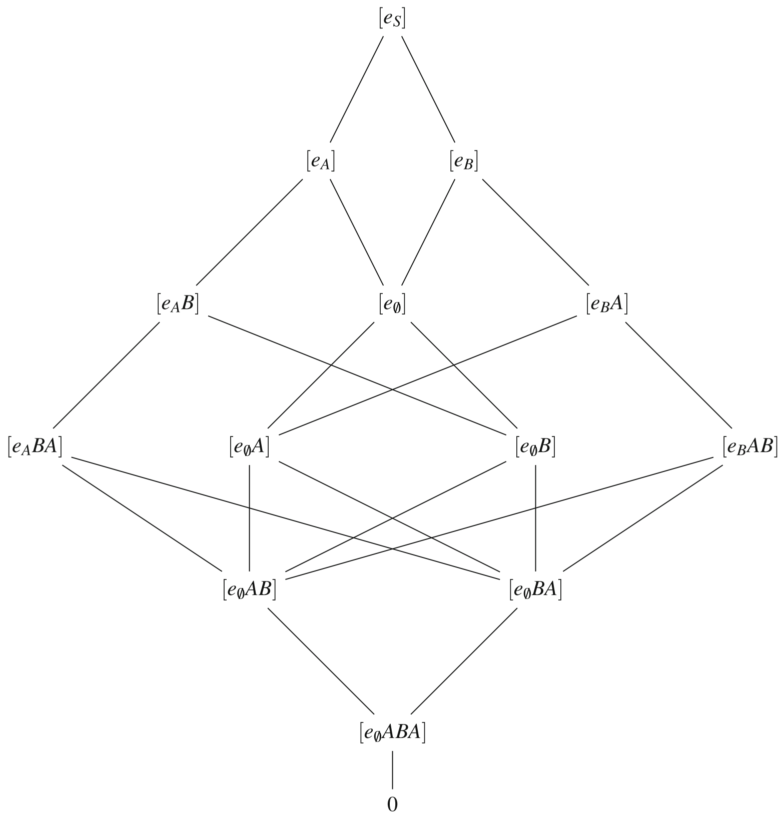

Fig. 2 $\mathscr{P}$ for the canonical monoid with W of Type A2

The study of this variety began in [14] and connections were made with the conjugacy decomposition of M. M_{nil} is in general not irreducible, however a description of the irreducible components is obtained in [14] (with $\mathscr{P}$ denoted by $\mathscr{C}$ in the original).

Theorem 15. *The variety M_{nil} decomposes as*

$$M_{nil} = \bigsqcup_{[ey] \in \mathscr{P}_{nil}} X(ey)$$

where $\mathscr{P}_{nil} = \{[ey] \in \mathscr{P} \big| (ey)^k = 0 \text{ for some } k\}$. The irreducible components of M_{nil} are $\overline{X(ey)}$, where $[ey]$ is a maximal element of $\mathscr{P}_{nil}$ with respect to the conjugacy order.

With this theorem in hand, the problem is then to identify the maximal elements of $\mathscr{P}_{nil}$. In [18], Putcha solves this problem for $\mathscr{J}$-irreducible and

$\mathscr{J}$-coirreducible monoids of type I. We conclude with two examples from [18], considering the case that $I = \emptyset$. As previously noted, these are exactly the canonical and dual canonical monoids.

Example 11 (Example 5.6,[18]). Let M be a canonical monoid. For $\alpha \in S$, let $e_\alpha \in \Lambda$ be such that $W(e_\alpha) = W_{S\setminus\{\alpha\}}$. Then the irreducible components of M_{nil} are $\overline{X(e_\alpha\alpha)}$, $\alpha \in S$.

Example 12 (Example 6.5,[18]). Let M be a dual canonical monoid. Let e_0 be the (unique) maximal element in $\Lambda \setminus \{1\}$. Then the irreducible components of M_{nil} are $\overline{X(e_0c)}$, where c is a Coxeter element of W.

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The Endomorphisms Monoid of a Homogeneous Vector Bundle

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Abstract Let E be a homogeneous vector bundle over the abelian variety A and let $\mathrm{Aut}_A(E)$ be the (algebraic) group of automorphims of E as a vector bundle. Then the fiber over 0 is a $\mathrm{Aut}_A(E)$-module. We prove that E is the induced space of this action to the whole group of automorphims of the homogeneous vector bundle. The principal significance of this result is that it allows one to obtain results about the structure of E and it provides some insight into the structure of its endomorphisms monoid.

Keywords Homogeneous vector bundles • Algebraic monoids

Subject Classifications: 14M17, 20M32, 14J60

1 Introduction

Let A be an abelian variety over an algebraically closed field of arbitrary characteristic $\Bbbk$. A vector bundle, which will be denoted by $\xi = (E, \rho, A)$ or $\rho : E \to A$, is called *homogeneous* if for all $a \in A$, $E \cong t_a^* E$, where $t_a : A \to A$ is the translation by a. A *homomorphism* $\lambda : \xi \to \xi'$ between two homogeneous vector bundles $\xi = (E, \rho, A)$ and $\xi' = (E', \rho', A)$ is a pair $\lambda := (f, t_a)$ such that the following diagram

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$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \rho \big\downarrow & & \big\downarrow \rho' \\ A & \xrightarrow[t_a]{} & A \end{array}$$

commutes and is linear in the fibers. We say that λ is an *isomorphism* if $f : E \to E'$ is an isomorphism of algebraic varieties; it is clear that in this case the pair $(f^{-1}, t_{-a}) : \xi' \to \xi$ is a homomorphism of homogeneous vector bundles.

We denote by $\mathrm{Hom}_{hb}(E, E')$ the set of homomorphisms between $\xi = (E, \rho, A)$ and $\xi' = (E', \rho', A)$ and by $\mathrm{Hom}_A(E, E')$ the set of those homomorphisms that fix the base, i.e. $t_a = Id_A$. If $E = E'$, then $\mathrm{End}_{hb}(E) := \mathrm{Hom}_{hb}(E, E)$ and $\mathrm{End}_A(E) := \mathrm{Hom}_A(E, E)$. The group of automorphisms, i.e. those endomorphisms where $f : E \to E$ is an isomorphism, is denoted by $\mathrm{Aut}_{hb}(E)$.

It is of interest to have a natural and intrinsic characterization of $\mathrm{End}_{hb}(E)$. One of the main reasons is the generalization of a well known result of affine monoids to normal algebraic monoids. More precisely, an affine monoid M can be embedded in $\mathrm{End}_{\Bbbk}(\Bbbk^n)$ as a closed submonoid, for $n \gg 0$. In [11, Theorem 5.3] it was proved that any normal algebraic monoid M can be embedded as a closed submonoid of the endomorphisms monoid $\mathrm{End}_{hb}(E)$ of an indecomposable homogeneous vector bundle E over the Albanese variety $A(M)$ of M.

The aim of this paper is to describe the geometric and algebraic structure of $\mathrm{End}_{hb}(E)$ and investigate the relation between these structures and the structure of E as a vector bundle.

In order to state our results we first recall some known results. The idea to study the relation of the algebraic structure of $\mathrm{Aut}_{hb}(E)$ to the structure of E as a vector bundle goes back at least as far as [15], where Miyanishi considers homogeneous bundles. In [16], Mukai describes the category of homogeneous vector bundles over an abelian variety. Brion and the second author proved in [11] that $\mathrm{End}_{hb}(E)$ is an algebraic monoid with unit group $\mathrm{Aut}_{hb}(E)$ and they showed that the Albanese morphism $\pi : \mathrm{End}_{hb}(E) \to A$ is a morphism of algebraic monoids, with Kernel $\mathrm{End}_A(E)$. In particular, the fiber $\mathrm{End}_A(E)$, over the unit element $0 \in A$, is an irreducible affine smooth algebraic monoid, with unit group $\mathrm{Aut}_A(E) := \pi^{-1}(0) \cap \mathrm{Aut}_{hb}(E)$.

It follows from [11] that $\pi : \mathrm{End}_{hb}(E) \to A$ is a homogeneous vector bundle with fiber isomorphic to $\mathrm{End}_A(E)$. For any indecomposable homogeneous vector bundle $\rho : E \to A$ of rank r we prove that $\mathrm{End}_{hb}(E) \to A$ has rank at most $1 + \frac{r(r-1)}{2}$ (see Theorem 1). Moreover, $\mathrm{End}_{hb}(E) \to A$ is obtained by successive extensions of a line bundle L associated with E (see Theorem 8). In particular, if L is a homogeneous line bundle, then $\mathrm{End}_{hb}(L) \cong L$ (see Lemma 1 and Corollary 6).

In order to describe the structure of $\mathrm{End}_{hb}(E)$ as a vector bundle denote by E_0 the fiber over $0 \in A$. Recall that *the induced space* $\mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0$ is defined as the geometric quotient of $\mathrm{Aut}_{hb}(E) \times E_0$ under the diagonal action of $\mathrm{Aut}_A(E)$ (see Definition 1 below). In Theorem 5 we prove the following statement.

An indecomposable homogeneous vector bundle $\rho : E \to A$ *is the induced space from the action of* $\mathrm{Aut}_A(E)$ *on* E_0 *to the action of the automorphisms group* $\mathrm{Aut}_{hb}(E)$ *on* E*, i.e.*

$$E \cong \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0.$$

The advantage of using the above description lies on the fact that it allows us to describe the structure of $\mathrm{End}_{hb}(E)$ also when E is decomposable. That is, if $E = \bigoplus_{i,j} L_i \otimes F_{i,j}$, where L_i is a homogeneous line bundle and $F_{i,j}$ an unipotent bundle, then

$$\mathrm{End}_{hb}(E) \cong \bigoplus_i L_i \otimes \big(\oplus_{j,k} \mathrm{Hom}_{hb}(F_{i,j}, F_{i,k})\big),$$

and as algebraic monoid $\mathrm{End}_{hb}(E)$ decomposes as

$$\mathrm{End}_{hb}(E) = \mathrm{Aut}_{hb}(E) \sqcup \mathscr{N}_{hb}(E),$$

where $\mathscr{N}_{hb}(E)$ is the ideal of pseudo-nilpotent elements (see Theorems 6 and 7). Moreover, the kernel $\mathrm{Ker}\big(\mathrm{End}_{hb}(E)\big)$ of the $\mathrm{End}_{hb}(E)$ is the zero section

$$\mathrm{Ker}\big(\mathrm{End}_{hb}(E)\big) = \big\{\theta_a : E \to E : \theta_a(v_x) = 0_{x+a} \ \forall\, v_x \in E_x\big\}.$$

In particular, $\mathrm{Ker}\big(\mathrm{End}_{hb}(E)\big)$ is isomorphic to the abelian variety A (see Proposition 4).

According to the above results, it is also shown that $\mathscr{N}_{hb}(E)$ is also a homogeneous vector bundle, obtained by successive extensions of L (see Theorem 8) and there exists an exact sequence of vector bundles

$$0 \longrightarrow \mathscr{N}_{hb}(E) \longrightarrow \mathrm{End}_{hb}(E) \xrightarrow{\ \rho\ } \mathrm{End}_{hb}(L) \cong L \longrightarrow 0$$

Moreover, the morphisms in the above sequence are compatible with the structures of semigroup, and if $E \ncong L$, then the sequence is non-trivial (see Theorem 9).

The paper is organized as follows: in Sect. 2 we set up notation and terminology, and review some of the standard facts on algebraic monoids and homogeneous vector bundles. In Sect. 3 we are concerned with the structure of $\mathrm{Hom}_{hb}(E, E')$. Our main results are stated and proved in Sect. 4. Section 5 is devoted to the study of $\mathrm{End}_{hb}(E)$ and $\mathrm{End}_A(E)$ when E is a homogeneous vector bundle of small rank.

2 General Results

In this section we recall basic results on algebraic monoids and homogeneous vector bundles over A. For a deeper discussion of the theory of algebraic monoids we refer the reader to [10,11,17,18] and to [1–3,15,16] for the theory of homogeneous vector bundles.

2.1 Algebraic Monoids

Let M be an *algebraic monoid* and $G(M)$ the *unit group* of M. It is well known that $G(M)$ is an algebraic group, open in M (see for example [17]). If M is an irreducible algebraic monoid, then its *Kernel*, denoted by $\mathrm{Ker}(M)$, is the minimum closed ideal and always exists. Indeed, if M is an affine algebraic monoid its $\mathrm{Ker}(M)$ is the unique closed $\big(G(M) \times G(M)\big)$-orbit (see [11, 17]).

Let M, N be algebraic monoids. A morphism of algebraic varieties $\varphi : M \to N$ is a *morphism of algebraic monoids* if $\varphi(ab) = \varphi(a)\varphi(b)$ for any $a, b \in M$ and $\varphi(1_M) = 1_N$. If $\varphi : M \to N$ is an isomorphism of algebraic monoids we write $M \cong_{am} N$.

Definition 1. Let $H \subset G$ be an algebraic subgroup of an algebraic group G such that H acts on an algebraic variety X. The *induced space* $G *_H X$ is defined as the geometric quotient of $G \times X$ under the H-action $h \cdot (g, x) = (gh^{-1}, h \cdot x)$. The class of (g, x) in $G *_H X$ is denoted as $[g, x]$.

Under mild conditions on X (e.g. X is covered by quasi-projective H-stable open subsets), the induced space always exists. Clearly, $G *_H X$ is a G-variety, for the action induced by $a \cdot (g, x) = (ag, x)$. The morphism $\pi : G *_H X \to G/H$ induced by $(g, x) \mapsto [g] = gH$ is a fiber bundle over G/H with fiber isomorphic to X. If moreover X is an H-module, then $G *_H X \to G/H$ is a vector bundle. For more information about induced spaces we refer the reader to [4] and [19].

Remark 1. Chevalley's Structure Theorem for an algebraic group G states that if $A(G)$ is the Albanese group of G, then the Albanese morphism $p : G \to A(G)$ fits into an exact sequence of algebraic groups

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \xrightarrow{\ p\ } A(G) \longrightarrow 0$$

where G_{aff} is a normal connected affine algebraic group.

In [10, 11], Brion and Rittatore generalize Chevalley's decomposition to irreducible normal algebraic monoids. In this case they prove that if G is the unit group, then M admits a Chevalley's decomposition:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M_{\mathrm{aff}} = \overline{G_{\mathrm{aff}}} & \longrightarrow & M = \overline{G} & \xrightarrow{\ p\ } & A(G) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & G_{\mathrm{aff}} & \longrightarrow & G & \xrightarrow[\ p|_G\ ]{} & A(G) & \longrightarrow & 0
\end{array}$$

where $p : M \to A(G) = G/G_{\mathrm{aff}}$ (respectively $p|_G : G \to A(G)$) is the Albanese morphism of M (respectively G). If Z^0 denotes the connected center of G, then

$A(G) \cong_{am} Z^0/(Z^0 \cap G_{\mathrm{aff}})$ and $M = G \cdot M_{\mathrm{aff}} = Z^0 \cdot M_{\mathrm{aff}}$. Moreover, $M \cong G *_{G_{\mathrm{aff}}} M_{\mathrm{aff}} \cong Z^0 *_{Z^0 \cap G_{\mathrm{aff}}} M_{\mathrm{aff}}$. Here if $A, B \subset M$, then $A \cdot B = \{ab : a \in A, b \in B\}$.

Let us mention a consequence of this Chevalley's decomposition for algebraic monoids that will prove to be extremely useful in Sect. 4.

Corollary 1. *Let M be an irreducible algebraic monoid, with unit group G. Then* $\mathrm{Ker}(M) = G\,\mathrm{Ker}(M_{\mathrm{aff}})G = G \cdot \mathrm{Ker}(M_{\mathrm{aff}}) = Z^0 \cdot \mathrm{Ker}(M_{\mathrm{aff}})$ *where Z^0 is the connected center of G.*

Proof. Since $M = Z^0 \cdot M_{\mathrm{aff}}$, it follows that $\mathrm{Ker}(M_{\mathrm{aff}}) \subset \mathrm{Ker}(M)$. Hence, $G \cdot \mathrm{Ker}(M_{\mathrm{aff}}) \cdot G \subset \mathrm{Ker}(M)$. Since both terms in the last inclusion are $\big(G(M) \times G(M)\big)$-orbits, the first equality follows.

It is clear that $\big(G\,\mathrm{Ker}(M_{\mathrm{aff}})G\big) \cap M_{\mathrm{aff}} = \mathrm{Ker}(M_{\mathrm{aff}})$ and from the decompositions $M = Z^0 \cdot M_{\mathrm{aff}} = M_{\mathrm{aff}} \cdot Z^0$ and $G = Z^0 \cdot G_{\mathrm{aff}}$, we deduce that

$$G \cdot \mathrm{Ker}(M_{\mathrm{aff}}) \cdot G = Z^0 \cdot \mathrm{Ker}(M_{\mathrm{aff}}) = G \cdot \mathrm{Ker}(M_{\mathrm{aff}}). \qquad \square$$

2.2 Homogeneous Vector Bundles

Recall that a vector bundle $\rho : E \to A$ is called *homogeneous* if for any $a \in A$, $E \cong t_a^* E$, where t_a is the translation by a. A line bundle L is homogeneous if and only if it is algebraically equivalent to zero (see [14, Sect. 9]). In particular, the trivial bundle $\mathcal{O}_A$ is homogeneous.

Let $\xi = (E, \rho, A)$ and $\xi' = (E', \rho', A)$ be two homogeneous vector bundles over A. If $(f, t_a) : \xi \to \xi'$ is a homomorphism, then the morphism $t_a : A \to A$ is determined by $f : E \to E'$. Thus, when no confusion can arise, we will write (f, t_a) simply as f. If E, E' are isomorphic as vector bundles we write $E \cong_{vb} E'$. It is well known that $\mathrm{Hom}_A(E, E') \cong H^0\big(A, \mathrm{Hom}(E, E')\big)$ and $\mathrm{End}_A(E) = H^0(A, E^* \otimes E)$. If E is indecomposable, the *algebra of endomorphisms* $\mathrm{End}_A(E)$ is a finite-dimensional $\Bbbk$-algebra and E is called *simple* if $\mathrm{End}_A(E) = \Bbbk$. A homogeneous vector bundle $E \to A$ is indecomposable if and only if $\mathrm{End}_A(E) = \Bbbk \cdot 1_E \oplus N_A(E)$, where $N_A(E) \subset \mathrm{End}_A(E)$ is the ideal of all the nilpotent endomorphisms (see [2]). Moreover, $\mathrm{Aut}_A(E) \cong G_m \times U_A(E)$, where $G_m = \Bbbk^*$ and $U_A(E)$ is the unipotent affine subgroup $\mathrm{Id} + N_A(E)$.

Remark 2. In [15, Lemma 1.1] Miyanishi described the algebraic structure of $\mathrm{Aut}_{hb}(E)$. In particular, he proved that, as an algebraic group, $\mathrm{Aut}_{hb}(E)$ is an extension of A by $\mathrm{Aut}_A(E)$, that is, we have the exact sequence of algebraic groups

$$1 \longrightarrow \mathrm{Aut}_A(E) \longrightarrow \mathrm{Aut}_{hb}(E) \longrightarrow A \longrightarrow 0$$

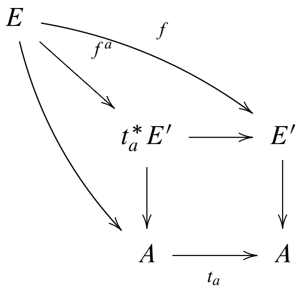
From the Chevalley's decomposition of $\mathrm{End}_{hb}(E)$ as an algebraic monoid (see Remark 1 and [11]) we have that $\mathrm{End}(E)_{\mathrm{aff}} = \mathrm{End}_A(E)$ fits in the following exact sequence of algebraic monoids

$$1 \longrightarrow \mathrm{End}_A(E) \longrightarrow \mathrm{End}_{hb}(E) \longrightarrow A \longrightarrow 0.$$

Moreover, if $\mathrm{Z}^0_{hb}(E)$ is the connected center of $\mathrm{End}_{hb}(E)$ and $\mathrm{Z}^0_A(E) = \mathrm{Z}^0_{hb}(E) \cap \mathrm{End}_A(E)$, then we have the following isomorphisms of algebraic monoids

$$\begin{aligned} \mathrm{End}_{hb}(E) &= \mathrm{Aut}_{hb}(E) \cdot \mathrm{End}_A(E) = \mathrm{Z}^0_{hb}(E) \cdot \mathrm{End}_A(E) \\ &\cong_{am} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{End}_A(E) \cong_{am} \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} \mathrm{End}_A(E). \end{aligned}$$

Remark 3. The canonical morphism $\pi : \mathrm{Hom}_{hb}(E, E') \to A$, $\pi(f, t_a) = a$, is a fibration over A with fiber $\mathrm{Hom}_A(E, t_a^* E')$ for any $a \in A$. Indeed, given $(f, t_a) \in \mathrm{Hom}_{hb}(E, E')$ there is a homomorphism $f^a : E \to t_a^* E'$ of vector bundles over A, such that the following diagram



commutes. That is, for any $a \in A$ there is a natural bijection between $\pi^{-1}(a)$ and $\mathrm{Hom}_A(E, t_a^* E')$. In particular, $\mathrm{Hom}_A(E, E') = \pi^{-1}(0)$.

Definition 2. We say that a vector bundle E of rank $r > 1$ is obtained by *successive extensions of a vector bundle* R, of length s, if there exists a filtration

$$R = E_0 \subset E_1 \subset E_2 \subset E_3 \subset \cdots \subset E_{s-1} \subset E_s = E$$

such that $E_i / E_{i-1} \cong R$ for $i = 1, \ldots, s$. In other words, there exist extensions

$$\begin{array}{lllll} \rho_1 : 0 \longrightarrow E_0 \cong R & \xrightarrow{i_1} & E_1 & \xrightarrow{p_1} & R \longrightarrow 0 \\ \rho_2 : 0 \longrightarrow E_1 & \xrightarrow{i_2} & E_2 & \xrightarrow{p_2} & R \longrightarrow 0 \\ \vdots & & \vdots & & \\ \rho_s : 0 \longrightarrow E_{s-1} & \xrightarrow{i_s} & E_s \cong E & \xrightarrow{p_s} & R \longrightarrow 0 \end{array}$$

If R is the trivial bundle $\mathcal{O}_A$, then E is called *unipotent*. We call $(\rho_1, \dots, \rho_s)$ the *extensions associated with* E. Note that $gr(E) = \oplus E_i/E_{i-1}$, the graded bundle associated with this filtration, is isomorphic to $\oplus^s R$. In particular, E is unipotent if $gr(E) = \oplus^s \mathcal{O}_A$.

Proposition 1. *If E is a vector bundle of rank r obtained by successive extensions of a vector bundle R, of length s, then* $2 \leq \dim_{\Bbbk} \mathrm{End}_A(E) \leq 1 + r(r-1)/2$.

Proof. Let

$$\begin{array}{llcccc} \rho_1 : & 0 \longrightarrow R \xrightarrow{i_1} & E_1 & \xrightarrow{p_1} R \longrightarrow 0 \\ \vdots & & \vdots & \\ \rho_s : & 0 \longrightarrow E_{s-1} \xrightarrow{i_s} & E_s \cong E & \xrightarrow{p_s} R \longrightarrow 0 \end{array}$$

be the extensions associated with E. The composition $\varphi = i_s \circ \cdots \circ i_2 \circ i_1 \circ p_s \neq 0$ defines a non invertible endomorphism of E. Therefore, $2 \leq \dim_{\Bbbk} \mathrm{End}_A(E)$.

As in [5, Prop. 1.1.9] we have that $\dim \mathrm{End}_A(E) \leq 1 + r(r-1)/2$. Note that $1 + r(r-1)/2$ is the dimension of the upper triangular matrices in $\mathrm{End}_{\Bbbk}(E_a)$. Indeed, the fiber E_a has a flag invariant under $e_a(\mathrm{End}_A(E))$ where $e_a : \mathrm{End}_A(E) \to \mathrm{End}_{\Bbbk}(E_a)$, $e_a(f) = f|_{E_a}$ is the restriction to the fiber $a \in A$. Hence,

$$\dim \mathrm{End}_A(E) \leq 1 + r(r-1)/2. \qquad \square$$

Remark 4. Let $\rho : E \to A$ be a vector bundle. From [15, 16] we have:

1. If E is an unipotent vector bundle, then E is homogeneous.
2. E is an indecomposable homogeneous vector bundle if and only if E is obtained by successive extensions of a homogeneous line bundle L. Moreover, one can choose the associated filtration $0 \subsetneq E_1 = L \subsetneq \cdots \subsetneq E_i \subsetneq \cdots \subsetneq E_n = E$ in such a way that $f(E_i) \subset E_i$ for all $f \in \mathrm{Aut}_{hb}(E)$, i.e. the filtration is $\mathrm{Aut}_{hb}(E)$-stable.

   Recall that in this case $E \cong L \otimes F$, where $L \in \mathrm{Pic}^0(A)$ and F is an indecomposable unipotent vector bundle.
3. E is homogeneous if and only if E decomposes as a direct sum $E = \bigoplus L_i \otimes F_i$, where $L_i \in \mathrm{Pic}^0(A)$ and F_i is a unipotent vector bundle.

Theorem 1. *Let $E \to A$ be a homogeneous vector bundle of rank r. Then π : $\mathrm{End}_{hb}(E) \to A$ is a homogeneous vector bundle with fiber isomorphic to* $\mathrm{End}_A(E)$. *Moreover, if E is indecomposable, then* $\mathrm{rk}(\mathrm{End}_{hb}(E)) \leq 1 + r(r-1)/2$.

Proof. Recall that $\mathrm{End}_{hb}(E) \cong_{am} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{End}_A(E)$ (see Remark 2). From general properties of the induced action and the fact that $\mathrm{End}_A(E)$ is a finite dimensional algebra, it follows that $\mathrm{End}_{hb}(E) \to A$ is a vector bundle (see for example [19]). Moreover, since $\mathrm{Aut}_{hb}(E)$ acts by left multiplication on $\mathrm{End}_{hb}(E)$

we have that $\mathrm{End}_{hb}(E)$ is homogeneous. Indeed, given $a \in A$, there exists $(f, t_a) \in \mathrm{Aut}_{hb}(E)$ and, if $\ell_f : \mathrm{End}_{hb}(E) \to \mathrm{End}_{hb}(E)$ denotes the isomorphism $\ell_f(h) = f \circ h$ for $h \in \mathrm{End}_A(E)$, then $\alpha(\ell_f) = t_a$.

The second part follows from Proposition 1. □

Remark 5. In [16, Proposition 6.13] Mukai proved that homogeneous vector bundles are Gieseker-semistable (see [12]). Moreover, a homogeneous vector bundle E is Gieseker-stable if and only if it is simple. It follows from Proposition 1 that a homogeneous vector bundle E is Gieseker-stable if and only if it is a homogeneous line bundle.

3 The Endomorphisms Monoid of a Homogeneous Vector Bundle

The affine algebraic group $\mathrm{Aut}_A(E)$ acts in at least two different ways on $\mathrm{End}_A(E)$, either by post-composing, $f \cdot h = f \circ h$, or by pre-composing, $f \cdot h = h \circ f^{-1}$, with $f \in \mathrm{Aut}_A(E)$ and $h \in \mathrm{End}_A(E)$. This allow us to endow $\mathrm{End}_{hb}(E)$ with two structures of homogeneous vector bundle. However, since

$$\mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{End}_A(E) \cong_{vb} \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} \mathrm{End}_A(E),$$

one can prove that, in fact, these structures are the same. Instead of proving this in full details, we give slightly more general results relating the structures of vector bundle of $\mathrm{Hom}_{hb}(E, E')$ (see Theorems 2 and 3).

Proposition 2. *Let E and E' be two vector bundles over A. Suppose E' is homogeneous. The inclusion $\mathrm{Z}^0_{hb}(E') \hookrightarrow \mathrm{Aut}_{hb}(E')$ induces an isomorphism of the homogeneous vector bundles*

$$\mathrm{Z}^0_{hb}(E') *_{\mathrm{Z}^0_A(E')} \mathrm{Hom}_A(E, E') \cong_{vb} \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E'),$$

where $\mathrm{Z}^0_A(E')$ and $\mathrm{Aut}_A(E')$ act on $\mathrm{Hom}_A(E, E')$ by post-composing.

Proof. Recall that the induced space $P = \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E')$ is a vector bundle over $\mathrm{Aut}_{hb}(E')/\mathrm{Aut}_A(E') = A$ and that a vector bundle R is homogeneous if the restriction map $Aut(R) \to A$ is surjective. It is clear that the canonical action of $\mathrm{Aut}_{hb}(E')$ over P induces a morphism of groups $\varphi : \mathrm{Aut}_{hb}(E') \to \mathrm{Aut}_{hb}(P)$, $\varphi(f, t_a) = (\tilde{f}, t_a)$, where $\tilde{f}\big(\big[(h, t_b), h'\big]\big) = \big[(f \circ h, t_{b+a}), h'\big]$. Since the projection $\mathrm{Aut}_{hb}(E') \to A$ defined as $(f, t_a) \mapsto a$, is surjective, it follows that the canonical projection $\mathrm{Aut}_{hb}(P) \to A$ is also surjective. In other words, the vector bundle P is homogeneous. The same conclusion can be drawn for $Q = \mathrm{Z}^0_{hb}(E') *_{\mathrm{Z}^0_A(E')} \mathrm{Hom}_A(E, E')$. The inclusion $\mathrm{Z}^0_{hb}(E') \hookrightarrow \mathrm{Aut}_{hb}(E')$ induces a morphism of homogeneous vector bundles $Q \to P$ which is bijective, this clearly forces $Q \cong_{vb} P$. □

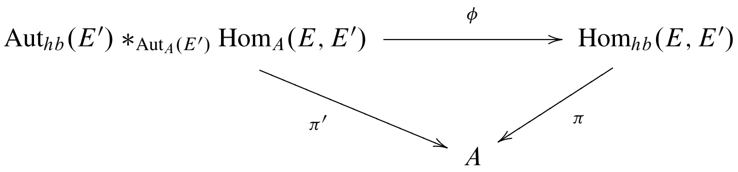
Theorem 2. *Under the same hypothesis of Proposition 2, consider the projection* $\pi : \mathrm{Hom}_{hb}(E, E') \to A$, $\pi(f, t_a) = a$, *and let*

$$\pi' : \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \to A$$
$$\pi'(g, h) = [g] \in \mathrm{Aut}_{hb}(E')/\mathrm{Aut}_A(E') \cong A$$

be the canonical projection. Then there exists a bijection

$$\phi : \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \to \mathrm{Hom}_{hb}(E, E')$$

such that the following diagram

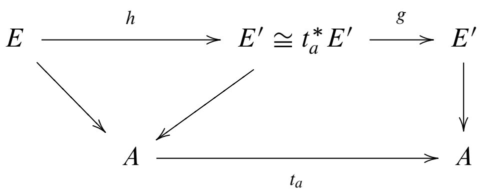


(1)

is commutative and ϕ *is linear in the fibers. Moreover,* ϕ *induces the structure of a homogeneous vector bundle on* $\mathrm{Hom}_{hb}(E, E')$ *and*

$$\mathrm{Hom}_{hb}(E, E') \cong_{vb} \mathrm{Z}^0_{hb}(E') *_{\mathrm{Z}^0_A(E')} \mathrm{Hom}_A(E, E').$$

Proof. From Proposition 2 it is sufficient to prove the existence of ϕ. Note that given $(g, t_a) \in \mathrm{Aut}_{hb}(E')$ and $h \in \mathrm{Hom}_A(E, E')$ the following diagram is commutative:



If $\varphi : \mathrm{Aut}_{hb}(E') \times \mathrm{Hom}_A(E, E') \to \mathrm{Hom}_{hb}(E, E')$ is given by $\varphi\big((g, t_a), h\big) = (g \circ h, t_a)$, then φ is constant on the $\mathrm{Aut}_A(E')$-orbits, and hence induces a homomorphism

$$\phi : \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \to \mathrm{Hom}(E, E').$$

By construction, ϕ makes the diagram (1) commutative.

To prove the surjectivity of ϕ let $(f, t_a) \in \mathrm{Hom}(E, E')$. Since E' is homogeneous, there exists $(g, t_{-a}) \in \mathrm{Aut}_{hb}(E')$ with $a \in A$ such that the following diagram is commutative:

$$\begin{array}{ccccc} E & \xrightarrow{f} & E' \cong t_a^* E' & \xrightarrow{g} & E' \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{t_a} & A & \xrightarrow{t_{-a}} & A \end{array} \qquad A \xrightarrow{t_0 = \mathrm{Id}} A$$

Hence, the composition $(g, t_{-a}) \circ (f, t_a) = (g \circ f, t_0)$ defines a homomorphism $g \circ f : E \to E'$ of vector bundles over A. Moreover,

$$\phi\big(\big[(g^{-1}, t_a), g \circ f\big]\big) = (g^{-1} \circ g \circ f, t_a) = (f, t_a),$$

and hence ϕ is surjective.

We claim that ϕ is injective. Indeed, if

$$\big[(g_1, t_{a_1}), h_1\big], \big[(g_2, t_{a_2}), h_2\big] \in \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E')$$

are such that $\phi\big(\big[(g_1, t_{a_1}), h_1\big]\big) = \phi\big(\big[(g_2, t_{a_2}), h_2\big]\big)$, then, by definition of ϕ, we have that $g_1 \circ h_1 = g_2 \circ h_2$ and $t_{a_1} = t_{a_2}$. Therefore, $a_1 = a_2$ and hence $g_2^{-1} \circ g_1 \in \mathrm{Aut}_A(E)$. This completes the proof. □

Theorem 2 states that if E' is homogeneous, then $\mathrm{Hom}_{hb}(E, E')$ is homogeneous. Under the assumptions of Proposition 2 with "E' homogeneous" replaced with "E is homogeneous" and "post-composing" by "pre-composing with the inverse" we obtain an analogue of Theorem 2 which may be proved in much the same way.

Theorem 3. *Let E, E' be vector bundles over A. If E is homogeneous, then there exists a bijection*

$$\psi : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E') \longrightarrow \mathrm{Hom}(E, E')$$

such that the following diagram is commutative and ψ is linear when restricted

$$\begin{array}{ccc} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E') & \xrightarrow{\psi} & \mathrm{Hom}(E, E') \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{-\mathrm{Id}} & A \end{array}$$

to the fibers. Moreover, ψ induces a structure of homogeneous vector bundle on $\mathrm{Hom}_{hb}(E, E')$ and

$$\mathrm{Hom}_{hb}(E, E') \cong_{vb} \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} \mathrm{Hom}_A(E, E').$$

If E, E' are both homogeneous vector bundles, it is not clear a priori that the two vector bundle structures on $\mathrm{Hom}_{hb}(E, E')$ given in Theorems 2 and 3 are the same. In order to prove that the structures coincide, we first deal with the case of $\mathrm{End}_{hb}(E)$, for E a homogeneous vector bundle.

Consider the vector bundle $P = \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} \mathrm{End}_A(E)$, where $\mathrm{Z}^0_A(E)$ acts on $\mathrm{End}_A(E)$ by post-composing and let Q be the vector bundle $Q = \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} \mathrm{End}_A(E)$, where $\mathrm{Z}^0_A(E)$ acts on $\mathrm{End}_A(E)$ by pre-composing with the inverse. An easy calculation shows that the morphism $\xi : \mathrm{Z}^0_{hb}(E) \times \mathrm{End}_A(E) \to Q$ given by $\xi\big((f, t_a), h\big) = \big[(f^{-1}, t_{-a}), h\big]$, with $h \in \mathrm{End}_A(E)$, induces an isomorphism of vector bundles $\tilde{\xi} : P \to (\text{-Id})^* Q$ and hence we have the following corollary.

Corollary 2. *If E is a homogeneous vector bundle, then the structures of homogeneous vector bundle defined on* $\mathrm{End}_{hb}(E)$ *by ϕ in Theorem 2 and ψ in Theorem 3 are isomorphic.*

Remark 6. Suppose now that $E = \bigoplus_i E_i$ and $E' = \bigoplus_j E'_j$ are two homogeneous vector bundles. Consider $\mathrm{Hom}_{hb}(E, E')$ and $\mathrm{Hom}_{hb}(E_i, E'_j)$, along with their structure as homogeneous vector bundles coming from either Theorem 2 or Theorem 3. Then

$$\mathrm{Hom}_{hb}(E, E') \cong_{vb} \bigoplus_{i,j} \mathrm{Hom}_{hb}(E_i, E'_j).$$

In particular,

$$\mathrm{End}_{hb}(E) \cong_{vb} \bigoplus_{i,j} \mathrm{Hom}_{hb}(E_i, E_j).$$

Indeed, the canonical inclusions $\varphi_{ij} : \mathrm{Hom}_{hb}(E_i, E'_j) \hookrightarrow \mathrm{Hom}_{hb}(E, E')$ are morphisms of vector bundles and induce an isomorphism

$$\varphi : \bigoplus_{i,j} \mathrm{Hom}_{hb}(E_i, E'_j) \to \mathrm{Hom}_{hb}(E, E').$$

Theorem 4. *Let E and E' be homogeneous vector bundles. The structures of vector bundle on* $\mathrm{Hom}_{hb}(E, E')$ *given in Theorem 2 and in Theorem 3 are isomorphic.*

Proof. Consider the vector bundle $E \oplus E'$. From Remark 6 we have that

$$\mathrm{End}_{hb}(E \oplus E') \cong_{vb} \mathrm{Hom}_{hb}(E, E') \oplus \mathrm{Hom}_{hb}(E', E) \oplus \mathrm{End}_{hb}(E) \oplus \mathrm{End}_{hb}(E').$$

Applying Corollary 2 we obtain an isomorphism between the structures of $\mathrm{Hom}_{hb}(E, E')$ and $\mathrm{Hom}_{hb}(E', E)$ given in Theorems 2 and 3. ☐

4 Relationship Between the Structure of a Homogeneous Bundle and Its Endomorphisms Monoid

4.1 The Homogeneous Vector Bundle as an Induced Space

We begin this section by showing that a homogeneous vector bundle $E \to A$ is obtained as an extension of the fiber E_0 over 0 by the principal bundle $\mathrm{Aut}_{hb}(E) \to A$.

Theorem 5. *Let $E \to A$ be a homogeneous vector bundle. Then, as vector bundles over A,*

$$E \cong_{vb} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0 \cong_{vb} \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} E_0.$$

Proof. Recall that $A \cong \mathrm{Aut}_{hb}(E)/\,\mathrm{Aut}_A(E)$. Define

$$\phi : \mathrm{Aut}_{hb}(E) \times E_0 \to E$$

as $\big((f, t_a), v\big) \mapsto f(v) \in E_a$, where $(f, t_a) \in \mathrm{Aut}_{hb}(E)$ and $v \in E_0$. Clearly, ϕ is constant on the $\mathrm{Aut}_A(E)$-orbits, and hence induces a homomorphism

$$\varphi : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0 \to E.$$

In fact ϕ is an isomorphism of vector bundles. Indeed, given $a \in A$, consider $(f, t_a) \in \mathrm{Aut}_{hb}(E)$. Then

$$\big(\mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0\big)_a = \big\{\big((f, t_a), v\big) : v \in E_0\big\},$$

and $f|_{E_0} : E_0 \to E_a$ is a linear isomorphism. It follows that the restriction $\varphi_a : \big(\mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0\big)_a \to E_A$, $\varphi_a\big((f, t_a), v\big) = f(v)$, is a linear isomorphism and φ is an isomorphism of vector bundles as claimed.

Since $\mathrm{Z}^0_{hb}(E) \to A$ is surjective, it is clear that we can apply the same argument to prove that $E \cong \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} E_0$. □

Theorem 5 allows us to provide some insight into the structure of homogeneous vector bundles.

Corollary 3. *Let $E \to A$ be an indecomposable homogeneous vector bundle. Then:*

(i) E_0 is an indecomposable $\mathrm{Z}^0_A(E)$-module.
(ii) E_0 is an indecomposable $\mathrm{End}_A(E)$-module;

Proof. We prove (i) by way of contradiction. So assume that $E_0 \cong V_1 \oplus V_2$ as $\mathrm{Z}^0_A(E)$-modules. Then, we have the isomorphisms

$$\big(\mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} V_1\big) \oplus \big(\mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} V_2\big) \cong_{vb} \big(\mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} (V_1 \oplus V_2)\big) \cong_{vb} E$$

as vector bundles over A, where the last isomorphism is given by Theorem 5. It follows that E is decomposable, a contradiction. It is clear that (i) implies (ii). □

The converse of Corollary 3 is false in general, as the following example shows.

Example 1. Let $E = L \oplus L$ where L is a homogeneous line bundle over A. Then E is a decomposable homogeneous vector bundle. However, E_0 is an indecomposable $\mathrm{Aut}_A(E)$-module, since $\mathrm{Aut}_A(E) \cong \mathrm{GL}_2(\Bbbk)$.

Denote by $I_r = \bigoplus_{i=1}^r \mathcal{O}_X = X \times \Bbbk^r$, the trivial homogeneous vector bundle over X of rank r, where X is a complete homogeneous space. In [15, Lemma 1.4], Miyanishi gives a characterization of $I_r \to X$ in terms of the existence of schematic sections for certain fibrations. Theorem 5 allows us to characterize I_r over an abelian variety in a simpler way in terms of their endomorphisms monoid.

Recall that a schematic section of a fibration $\pi : \mathrm{Aut}_{hb}(E) \to A$ is a morphism $\sigma : A \to \mathrm{Aut}_{hb}(E)$ such that $\pi \circ \sigma = \mathrm{Id}_A$.

Corollary 4. *Let $E \to A$ be a homogeneous vector bundle of rank r. Then the following assertions are equivalent:*

1. $E \cong_{vb} I_r$.
2. $\mathrm{End}_{hb}(E) \cong_{am} A \times \mathrm{End}(\Bbbk^n)$.
3. $\pi : \mathrm{Aut}_{hb}(E) \to A$ *has a schematic section.*

Proof. It is clear that the endomorphisms monoid $\mathrm{End}_{hb}(I_r)$ of I_r, satisfies $\mathrm{End}_{hb}(I_r) \cong_{am} A \times \mathrm{End}(\Bbbk^n)$, so (1) implies (2) and (3).

(2) ⇒ (1) If $\mathrm{End}_{hb}(E) \cong_{am} A \times \mathrm{End}(\Bbbk^n)$, then

$$E \cong_{vb} \big(A \times \mathrm{GL}_n(\Bbbk)\big) *_{\mathrm{GL}_n(\Bbbk)} \Bbbk^n \cong_{vb} A \times \Bbbk^n,$$

since $E \cong_{vb} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0$.

(3) ⇒ (1). Let now $\sigma : A \to \mathrm{Aut}_{hb}(E)$, $\sigma(a) = \big(\sigma_1(a), t_a\big)$, be a schematic section, and let $\varphi : A \times E_0 \to \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0 \cong_{vb} E$ be the morphism given by $\varphi(a, v) = \big[(\sigma_1(a), t_a), v\big]$. Clearly, φ is a homomorphism of homogeneous vector bundles. It is enough to prove that φ is injective, since both vector bundles have the same rank.

Suppose that $(a, v), (a', v') \in A \times E_0$ are such that $\varphi(a, v) = \varphi(a, v')$. Then

$$\big[(\sigma_1(a), t_a), v\big] = \big[(\sigma_1(a'), t_{a'}), v'\big],$$

and it follows that $a = a'$. Therefore $v = v'$ and hence φ is an isomorphism. □

Corollary 5. *Let E, E' be two homogeneous vector bundle over A. Then the following statements are equivalent:*

(i) $E \cong_{vb} E'$;
(ii) $\mathrm{Aut}_{hb}(E) \cong_{am} \mathrm{Aut}_{hb}(E')$, *and* $E_0 \cong E_0'$ *as rational* $\mathrm{Aut}_A(E)$*-modules;*
(iii) $\mathrm{Aut}_{hb}(E) \cong_{am} \mathrm{Aut}_{hb}(E')$, *and* $E_0 \cong E_0'$ *as rational* $\mathrm{Z}_A^0(E)$*-modules.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are clear.

Assume that (iii) holds. Let $\psi : \mathrm{Aut}_{hb}(E) \to \mathrm{Aut}_{hb}(E')$ be an isomorphism of algebraic groups and let $\Phi : E_0 \to E'_0$ be a morphism of $\mathrm{Z}^0_A(E)$-modules. Then the morphism $\varphi : \mathrm{Z}^0(E) \times E_0 \to E'$, defined as $\varphi(g, v) = \psi(g)\big(\Phi(v)\big)$ induces the required isomorphism $E \to E'$. □

Remark 7. It is well known that there exist homogeneous vector bundles E, E' such that $E \ncong E'$ whereas $\mathrm{Aut}_A(E) \cong_{am} \mathrm{Aut}_A(E')$. Even the stronger condition $\mathrm{End}_A(E) \cong_{am} \mathrm{End}_A(E')$ is not sufficient in order to guarantee that $E \cong E'$. However, Corollary 4 shows that the trivial bundle is characterized by its endomorphisms monoid. One can see that $\mathrm{Z}^0_A(I_r) = \Bbbk^* \,\mathrm{Id}$ acts by homotheties in the fiber. In general, the group $\mathrm{Z}^0_A(E)$ could be larger and there could exist two different irreducible representations of the same dimension. This is the main problem we encounter when trying to generalize Corollary 4. Thus, it raises the following question.

Question 1. Let $E, E' \to A$ be two indecomposable homogeneous vector bundles over A. Does the existence of an isomorphism $\mathrm{Aut}_{hb}(E) \cong_{am} \mathrm{Aut}_{hb}(E')$ (or $\mathrm{End}_{hb}(E) \cong_{am} \mathrm{End}_{hb}(E')$) imply that $E \cong E'$?

The following lemma is a straightforward generalization of [13, Lemma 4.3] and may be proved in much the same way (see also [18, Theorem 2]).

Lemma 1. *Let $\rho : L \to A$ be a homogeneous line bundle. Then there exists a structure of a commutative algebraic monoid on L such that ρ is a morphism of algebraic monoids. The fiber $\rho^{-1}(0) = L_0 \cong \Bbbk$ is central in L. Moreover, the unit group is $G(L) = L \setminus \Theta(L)$, where $\Theta(L)$ is the image of the zero section of L.*

Corollary 6. *If $L \to A$ is a homogeneous line bundle, then* $\mathrm{End}_{hb}(L) \cong_{vb} L$.

Proof. By Lemma 1 L is an algebraic monoid. For any $x \in L$ let $l_x : L \to L$ be the endomorphisms defined as $l_x(y) = xy$ (the product on the algebraic monoid L). Hence, L is a sub-bundle of $\mathrm{End}_{hb}(L)$. But $\mathrm{End}_A(L) \cong \Bbbk$; hence $\mathrm{End}_{hb}(L)$ is a line bundle, and $L = \mathrm{End}_{hb}(L)$. □

4.2 *The Vector Bundle Structure of* $\mathbf{End}_{hb}(E)$

For $i = 1, 2$ let E_i be a homogeneous vector bundle over A. In order to study the structure of $\mathrm{Hom}_{hb}(E_1, E_2)$ as a vector bundle it suffices to assume that E_i is indecomposable (see Remark 6). In this case, we have that $E_i \cong L_i \otimes F_i$, where $L_i \in \mathrm{Pic}^0(A)$ and F_i is a unipotent homogeneous vector bundle.

Proposition 3. *Let $E_i \cong L_i \otimes F_i$ be indecomposable homogeneous vector bundles of rank $r_i = \mathrm{rk}(E_i)$, for $i = 1, 2$.*

(1) *If* $L_1 \ncong_{hb} L_2$*, then*

$$\mathrm{Hom}_{hb}(E_1, E_2) = \{\theta_a : E_1 \to E_2 : a \in A\} \cong_{vb} A \times \{0\},$$

where if $v \in (E_1)_x$*, then* $\theta_a(v) = 0_{a+x} \in E_2$*.*
(2) *If* $L_1 \cong_{vb} L_2$*, then*

$$\mathrm{Hom}_{hb}(E_1, E_2) \cong_{vb} L_1 \otimes \mathrm{Hom}_{hb}(F_1, F_2).$$

Moreover, $\mathrm{End}_{hb}(E_1) \cong_{vb} L_1 \otimes \mathrm{End}_{hb}(F_1)$.

Proof. By Proposition 2,

$$\mathrm{Hom}_{hb}(E_1, E_2) \cong \mathrm{Aut}_{hb}(E_1) *_{\mathrm{Aut}_A(E_1)} \mathrm{Hom}_A(E_1, E_2).$$

We claim that if $L_1 \ncong L_2$, then $\mathrm{Hom}_A(E_1, E_2) = 0$, and hence,

$$\mathrm{Hom}_{hb}(E_1, E_2) \cong \mathrm{Aut}_{hb}(E_1) *_{\mathrm{Aut}_A(E_1)} \{0\} \cong A \times \{0\}.$$

Indeed, suppose that there exists $0 \neq \varphi \in \mathrm{Hom}_A(E_1, E_2)$ and let

$$L = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_{r_1-1} \subset H_{r_1} = E_1$$

be the filtration associated with E_1. Let $k \in \{0, \ldots, r_1 - 1\}$ be such that $H_k \subset \mathrm{Ker}(\varphi)$ but $H_{k+1} \not\subset \mathrm{Ker}(\varphi)$. Let $j \in \{0, \ldots, r_2 - 1\}$ be such that $\mathrm{Im}(\varphi) \subset K'_{j+1}$, $\mathrm{Im}(\varphi) \not\subset K'_j$ where

$$0 = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{r_2-1} \subset K_{r_2} = E_2$$

is the filtration associated with E_2.

Then φ induces a non zero morphism $\tilde{\varphi} : L_1 \cong H_{k+1}/H_k \to K_{j+1}/K_j \cong L_2$. Since both are algebraically equivalent to zero, $\tilde{\varphi}$ is an isomorphism, and $L_1 \cong L_2$ as claimed.

Suppose now that $L_1 \cong_{vb} L_2 = L$. Then

$$\mathrm{Hom}_A(E_1, E_2) \cong (L \otimes F_1)^\vee \otimes (L \otimes F_2) \cong F_1^\vee \otimes F_2 \cong \mathrm{Hom}_A(F_1, F_2).$$

It follows that $\mathrm{Hom}_{hb}(E_1, E_2)$ and $L \otimes \mathrm{Hom}_{hb}(F_1, F_2)$ are homogeneous vector bundles of the same rank. Consider now the homomorphism of vector bundles $\varphi : L \otimes \mathrm{Hom}_{hb}(F_1, F_2) \to \mathrm{Hom}_{hb}(E_1, E_2)$ given by $\varphi\big((l, t_a) \otimes (h, t_a)\big) = (l \otimes h, t_a)$, where we use the identification $L \cong_{vb} \mathrm{End}_{hb}(L)$, and $(l \otimes h)(v \otimes w) = l(v) \otimes h(w)$, for $v \otimes w \in L \otimes F$. Since φ is injective, it is an isomorphism of vector bundles, and the proof is completed. □

Theorem 4 and Proposition 3 give the following explicit description of $\mathrm{End}_{hb}(E)$.

Theorem 6. *Let* $E = \bigoplus_{i,j} L_i \otimes F_{i,j}$ *and* $E' = \bigoplus_{i,j} L_i \otimes F'_{i,j}$ *be two homogeneous vector bundles, where* L_i *are homogeneous line bundles,* F_{ij} *and* F'_{ij} *unipotent homogeneous vector bundles and* $L_i \not\cong L_j$ *if* $i \neq j$*. Then*

$$\mathrm{Hom}_{hb}(E, E') \cong_{vb} \bigoplus_i L_i \otimes \big(\oplus_{j,k} \mathrm{Hom}_{hb}(F_{i,j}, F'_{i,k})\big).$$

Moreover, $\mathrm{End}_{hb}(E) \cong_{vb} \bigoplus_i L_i \otimes \big(\oplus_{j,k} \mathrm{Hom}_{hb}(F_{i,j}, F_{i,k})\big)$.

4.3 The Algebraic Monoid Structure of $\mathrm{End}_{hb}(E)$

Let us start with an important consequence of Corollary 1.

Proposition 4. *The Kernel of* $\mathrm{End}_{hb}(E)$ *of a homogeneous vector bundle* $\rho : E \to A$ *is given by*

$$\mathrm{Ker}\big(\mathrm{End}_{hb}(E)\big) = \Theta\big(\mathrm{End}_{hb}(E)\big) = \big\{\theta_a : E \to E : \theta_a(v) = 0_{\rho(v)+a}\big\},$$

where Θ *is the zero section. Moreover,* $\mathrm{Ker}\big(\mathrm{End}_{hb}(E)\big)$ *is an algebraic group and isomorphic to* A.

Recall that an endomorphism $f \in \mathrm{End}_A(E)$ is called *nilpotent of index* n if $f^n(v) \in \Theta(E)$ for all $v \in E$ and there exists $v_0 \in E$ such that $f^{n-1}(v_0) \notin \Theta(E)$. In other words, $f^n = 0 \in \mathrm{End}_A(E)$ whereas $f^{n-1} \neq 0$. The set $N_A(E)$ of nilpotent endomorphisms is an ideal of $\mathrm{End}_A(E)$, see Sect. 2.2 and [2].

Definition 3. Let $E \to A$ be a homogeneous vector bundle. An endomorphism $f \in \mathrm{End}_{hb}(E)$ is called *pseudo-nilpotent of index* n if $f^n \in \Theta\big(\mathrm{End}_{hb}(E)\big) = \mathrm{Ker}\big(\mathrm{End}_{hb}(E)\big)$ whereas $f^{n-1} \notin \Theta\big(\mathrm{End}_{hb}(E)\big)$. We denote by $\mathscr{N}_{hb}(E)$ the set of pseudo-nilpotent endomorphisms. It is clear that $N_A(E) = \mathscr{N}_{hb}(E) \cap \mathrm{End}_A(E)$.

If L is a homogeneous line bundle, then $\mathrm{End}_{hb}(L) = L$ and $\mathrm{Aut}_{hb}(L) = L \setminus \Theta(L)$. Hence, $\mathrm{End}_{hb}(L) = \mathrm{Aut}_{hb}(L) \sqcup \Theta(L)$. In particular, $\mathscr{N}_{hb}(L) = \Theta(L) = \mathrm{Ker}(L)$. For indecomposable vector bundles of higher rank we have an analogue of Atiyah's results (see [2]).

Theorem 7. *Let* $E \to A$ *be an indecomposable homogeneous vector bundle. Then:*

(1) The algebraic monoid $\mathrm{End}_{hb}(E)$ *decomposes as*

$$\mathrm{End}_{hb}(E) = \mathrm{Aut}_{hb}(E) \sqcup \mathscr{N}_{hb}(E).$$

Moreover, $\mathscr{N}_{hb}(E)$ *is an ideal of* $\mathrm{End}_{hb}(E)$.

(2) *The set* $\mathcal{N}_{hb}(E)$ *of pseudo-nilpotent elements is a homogeneous vector bundle over* A *of* $\operatorname{rk}\mathcal{N}_{hb}(E) = \operatorname{rk}\operatorname{End}_{hb}(E) - 1$. *Moreover,*

$$\mathcal{N}_{hb}(E) = \mathrm{Z}^0_{hb}(E) \cdot N_A(E) \cong \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} N_A(E).$$

where $\cdot$ *denotes the product in* $\operatorname{End}_{hb}(E)$. *The fiber of* $\pi : \mathcal{N}_{hb}(E) \to A$ *is isomorphic to* $N_A(E)$, *and* π *is a morphism of algebraic semigroups.*

Proof. From Remark 2 we have that

$$\operatorname{End}_{hb}(E) = \mathrm{Z}^0_{hb}(E) \cdot \operatorname{End}_A(E) \cong \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} \operatorname{End}_A(E).$$

Let $f \in \operatorname{End}_A(E)$ and $z \in \mathrm{Z}^0_{hb}(E)$. Since $\operatorname{End}_A(E) = \Bbbk\operatorname{Id} \oplus N_A(E)$ it follows that either f is in $\operatorname{Aut}_A(E)$ or in $N_A(E)$. In the first case $z \cdot f \in \operatorname{Aut}_{hb}(E)$ and in the second if $f^n = 0$, then $(z \cdot f)^n = z^n \cdot f^n = \theta_{\pi(z^n)}$, where $\pi : \mathrm{Z}^0_{hb}(E) \to A$ is the canonical projection. Therefore,

$$\operatorname{End}_{hb}(E) = \operatorname{Aut}_{hb}(E) \sqcup \mathcal{N}_{hb}(E).$$

Note that in particular we have proved that

$$\mathcal{N}_{hb}(E) = \mathrm{Z}^0_{hb}(E) \cdot N_A(E) \cong \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} N_A(E).$$

Since $\mathcal{N}_{hb}(E) = \operatorname{End}_{hb}(E) \setminus \operatorname{Aut}_{hb}(E)$, it follows that $\mathcal{N}_{hb}(E)$ is an ideal. In particular, $\mathcal{N}_{hb}(E)$ is $\operatorname{Aut}_{hb}(E)$-stable, and hence a homogeneous vector bundle. Finally, the equality $\operatorname{rk}\mathcal{N}_{hb}(E) = \operatorname{rk}\operatorname{End}_{hb}(E) - 1$ follows again from the fact that $\operatorname{End}_A(E) = \Bbbk\operatorname{Id} \oplus N_A(E)$ and $N_A(E) \neq 0$. ◻

The next theorem yields information about the geometric structure of $\operatorname{End}_{hb}(E)$ when the rank is ≥ 2.

Theorem 8. *If* $E \to A$ *is an indecomposable vector bundle of rank* $r \geq 2$ *obtained by successive extensions of the homogeneous line bundle* L, *then* $\operatorname{End}_{hb}(E)$ *and* $\mathcal{N}_{hb}(E)$ *are also obtained by successive extensions of* L. *Moreover, if* $r \geq 2$, *then* $\operatorname{rk}\mathcal{N}_{hb}(E) \geq 1$.

Proof. Let $L' \subset \operatorname{End}_{hb}(E)$ be a homogeneous, rank-one sub-bundle and let $f \in L' \cap \operatorname{End}_A(E) = L'_0 \setminus \{\theta_0\}$ be a non zero nilpotent element. Let $e \in E_0$ be such that $f(e) \neq 0 \in E_0$. Since $\operatorname{End}_{hb}(E) \cong_{vb} \operatorname{Aut}_{hb}(E) *_{\operatorname{Aut}_A(E)} \operatorname{End}_A(E)$, for every $a \in A$, there exists $(h_a, t_a) \in \operatorname{Aut}_{hb}(E)$ such that $L'_a = \Bbbk(h_a \circ f)$. Hence, $\varphi : L' \to E$, $\varphi(l) = l(e)$ is an injective morphism of homogeneous vector bundles, and since E is obtained by successive extensions of L, it follows that $L' \cong L$. Thus, $\operatorname{End}_{hb}(E)$ is also obtained by successive extensions of L.

From Theorems 7 and 8, $\mathcal{N}_{hb}(E)$ is a homogeneous vector bundle, obtained by successive extensions of L. It remains to show that $\operatorname{rk}\mathcal{N}_{hb}(E) \geq 1$. But by Proposition 1, there exists $0 \neq \varphi \in \operatorname{End}_A(E)$ such that $\varphi^2 = 0$, which is the desired conclusion. □

We are thus led to the following generalization of Miyanishi's Structure Theorem (see Remark 4).

Theorem 9. *Let $E \cong L \otimes F \to A$ be an indecomposable homogeneous vector bundle. Then there exists an exact sequence of vector bundles over A*

$$0 \longrightarrow \mathcal{N}_{hb}(E) \longrightarrow \operatorname{End}_{hb}(E) \stackrel{\rho}{\longrightarrow} \operatorname{End}_{hb}(L) \cong L \longrightarrow 0.$$

Moreover, the morphisms in the above sequence are compatible with the algebraic semigroup structures. Furthermore, if $E \not\cong L$, then the sequence is non-split.

Proof. By Remark 4, $L \subset E$ is $\operatorname{Aut}_{hb}(E)$-stable and $\operatorname{End}_{hb}(E)$-stable. It is easy to see that the restriction $\rho : \operatorname{End}_{hb}(E) \to \operatorname{End}_{hb}(L)$ is a morphism of algebraic monoids. In particular, it is compatible with the underlying vector bundle structures.

By Theorem 7, $\operatorname{End}_{hb}(E) = \operatorname{Aut}_{hb}(E) \sqcup \mathcal{N}_{hb}(E)$. It is clear that if $g \in \operatorname{Aut}_{hb}(E)$, then $\rho(g) \in \operatorname{Aut}_{hb}(L) = L \setminus \Theta(L)$ and if $(f, t_a) \in \mathcal{N}_{hb}(E)$, there exists $n \in \mathbb{N}$ such that $f^n = \theta_{na}$. It follows that the restriction $f|_L$ belongs to $\mathcal{N}_{hb}(L) = \Theta(L)$. Therefore, $\mathcal{N}_{hb}(E) = \operatorname{Ker}(\rho)$.

Assume now that the exact sequence splits. Then there exists an immersion of homogeneous vector bundles $\iota : L \hookrightarrow \operatorname{End}_{hb}(E)$, such that $\rho \circ \iota = \operatorname{Id}_L$. In particular, $\iota\big(L \setminus \Theta(L)\big) \subset \operatorname{Aut}_{hb}(E)$. Let E_0 be the fiber of E over $0 \in A$ and consider the morphism of vector bundles

$$\varphi : L \otimes E_0 \cong \operatorname{End}_{hb}(L) \otimes E_0 \to E, \qquad \varphi(f \otimes v) = f(v).$$

Let $e \in E$ be such that $\pi(e) = a$ and $f \in L \setminus \Theta(L)$ be such that $\alpha(f) = a$. Then $\varphi\big(\iota(f) \otimes \iota(f)^{-1}(e)\big) = e$, and it follows that φ is a surjective morphism of homogeneous vector bundles of the same rank. Thus, φ is an isomorphism. But $L \otimes E_0$ is decomposable unless $\dim E_0 = 1$. Therefore, $E \cong_{vb} L$, and the proof is completed. □

5 Explicit Calculations for Small Rank

The algebra of endomorphisms of successive extensions of line bundles over a curve, of small rank, has been studied in [5–9]. We use a fairly straightforward generalization of such results to give an explicit description of the endomorphisms monoid of indecomposable homogeneous vector bundles of rank 2 and 3 over abelian varieties.

5.1 Homomorphisms Between a Homogeneous Line Bundle and a Homogeneous Vector Bundle

As we saw in Sect. 4, any homogeneous line bundle is an algebraic monoid, and is isomorphic to its endomorphisms monoid. In this section we give a description of $\mathrm{Hom}_{hb}(E, E')$ when one of the homogeneous vector bundles is a line bundle and the other is an indecomposable homogeneous vector bundle.

Proposition 5. *Let $E = L \otimes F$ be an indecomposable homogeneous vector bundle of rank $\mathrm{rk}\, E = n \geq 2$, and L' a homogeneous line bundle. Then,*

1. *If $L = L'$, then*
   a. $\mathrm{Hom}_{hb}(L, E) \cong_{hb} \oplus^r L$ *where* $r = \dim H^0(A, F)$.
   b. $\mathrm{Hom}_{hb}(E, L) \cong_{hb} \oplus^s L$ *where* $s = \dim H^0(A, F^\vee)$.
2. *If $L \neq L'$, then* $\mathrm{Hom}_{hb}(E, L') \cong_{hb} \mathrm{Hom}_{hb}(L', E) \cong_{hb} A \times \{0\}$.

Proof. From what has been already proven, we have that

$$\begin{aligned}\mathrm{Hom}_{hb}(\mathscr{O}_A, F) &\cong_{vb} \mathrm{Aut}_{hb}(\mathscr{O}_A) *_{\mathrm{Aut}_A(\mathscr{O}_A)} \mathrm{Hom}_A(\mathscr{O}_A, F)\\ &=_{vb} (A \times \Bbbk^*) *_{\Bbbk^*\,\mathrm{Id}} \mathrm{Hom}_A(\mathscr{O}_A, F) \cong_{vb} A \times \mathrm{Hom}_A(\mathscr{O}_A, F).\end{aligned}$$

It follows that $\mathrm{Hom}_{hb}(\mathscr{O}_A, F)$ is a trivial bundle, with fiber isomorphic to $\mathrm{Hom}_A(\mathscr{O}_A, F) = H^0(A, F)$, i.e. $\mathrm{Hom}_{hb}(\mathscr{O}_A, F) \cong_{vb} A \times H^0(A, F)$, which by Proposition 3 proves (1).

The proof for (2) is similar. In this case we have that $\mathrm{Hom}_{hb}(F, \mathscr{O}_A) \cong_{vb} A \times H^0(A, F^\vee)$. □

5.2 Homomorphisms Between Indecomposable Homogeneous Vector Bundles of Rank 2

Let E and E' be two non-isomorphic indecomposable homogeneous vector bundles of rank 2. Let

$$\rho_E : 0 \to L \xrightarrow{j} E \xrightarrow{\pi} L \to 0$$

and

$$\rho_{E'} : 0 \to L' \xrightarrow{i_1} E' \xrightarrow{p_1} L' \to 0$$

be the extensions associated with E and E', respectively. By Proposition 3, if $L \not\cong_{vb} L'$, then $\mathrm{Hom}_A(E, E') = 0$. If $L \cong_{vb} L'$, then $\mathrm{Hom}_A(E, E') \neq 0$, since $0 \neq \phi = i_1 \circ \pi \in \mathrm{Hom}_A(E, E')$.

We are thus led to the following strengthening of Theorem 8.

Proposition 6. *Let E, E' be as above. If $L \cong_{vb} L'$, then* $\mathrm{Hom}_{hb}(E, E') \cong_{vb} L$.

Proof. It is sufficient to prove that $\mathrm{Hom}_A(E, E') = \Bbbk\phi$ (see Theorem 8).

Let $0 \neq \varphi \in \mathrm{Hom}_A(E, E')$. Since E and E' are non-isomorphic, the image $\varphi(E)$ is a line sub-bundle L_0 of E'. Moreover, $\psi : p_1 \circ \varphi : E \to L$ is a non zero homomorphism of homogeneous vector bundles. If $\varphi(E) = L_0 \neq L$ we get a contradiction, since, by Proposition 5, $\mathrm{Hom}_{hb}(E, L) \cong_{hb} A \times \{0\}$. Thus, $L_0 \cong_{hb} L$.

If $\varphi \neq \lambda\phi$ with $\lambda \in \Bbbk$, $0 \neq \varphi \circ j : L \to L_0$ is an isomorphism, and then $\varphi \circ (\varphi \circ j)^{-1} : E \to L$ is a spitting. This contradicts our assumption. Hence, $\varphi = \lambda\phi$, which completes the proof. □

When $E = E'$ we have that $\dim \mathrm{End}_A(E) \leq 2$ (see Theorem 1). To describe $\mathrm{End}_{hb}(E)$ we consider the associated exact sequence

$$0 \to L \xrightarrow{i} E \xrightarrow{p} L \to 0$$

where L is a homogeneous line bundle.

Note that $\mathrm{rk}\,\mathcal{N}_{hb}(E) \geq 1$, since $0 \neq \varphi = i \circ p : E \to E$ satisfies $\varphi^2 = 0$ and $\mathrm{rk}\,\mathrm{End}_{hb}(E) = 2$. Hence, $\mathrm{End}_A(E) \cong \Bbbk\,\mathrm{Id} \oplus \Bbbk\varphi$ (see [2]). Therefore,

Proposition 7. *Let E be a homogeneous vector bundle of rank 2. Then* $\mathrm{End}_{hb}(E)$ *is a commutative algebraic monoid, and* $\mathrm{End}_A(E) \cong \Bbbk[t]/(t^2)$. *Moreover,* $\mathrm{End}_{hb}(E) \cong_{vb} E$.

Proof. The only assertion that still needs proof here is the last one. For this, observe that

$$\mathrm{End}_A(E) \cong_{am} \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) : a, b \in \Bbbk \right\},$$

with action on the fiber E_0 given as follows: consider an isomorphism $E_0 \cong \Bbbk^2$ such that $(1, 0) \in \mathrm{Ker}(p)_0$. In other words, $(1, 0)$ belongs to the fiber L_0, of the line bundle $L \subset E$ which is $\mathrm{Aut}(E)$-stable. Under this identification, the action $\mathrm{End}_A(E) \times E_0 \to E_0$ is given by $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \cdot (x, y) = (ax + by, ay)$.

On the other hand, the action of $\mathrm{Aut}_A(E)$ on $\mathrm{End}_A(E)$ is given by $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax & ay+bx \\ 0 & ax \end{smallmatrix}\right)$. Thus, there exists an isomorphism of $\mathrm{Aut}_A(E)$-modules $\varphi : E_0 \to \mathrm{End}_A(E)$, which implies that the morphism

$$\psi : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0 \to \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{End}_A(E)$$
$$\psi\big([(f, t_a), e_0]\big) = [(f, t_a), \varphi(e_0)]$$

is an isomorphism of vector bundles, and $E \cong_{vb} \mathrm{End}_{hb}(E)$ as claimed. □

5.3 The Endomorphisms Monoid of an Indecomposable, Homogeneous, Rank 3 Vector Bundle

For indecomposable homogeneous vector bundles E of rank 3, $\dim \operatorname{End}_A(E) \leq 4$ (see Theorem 1). As in [5, 6], $\operatorname{End}_A(E)$ is a commutative algebra of dimension $2 \leq \dim \operatorname{End}_A(E) \leq 3$ and the possibilities are

$$\operatorname{End}_A(E) = \begin{cases} \Bbbk[t]/(t^2) & \text{or} \\ \Bbbk[t]/(t^3) & \text{or} \\ \Bbbk[r,s]/(r,s)^2 \end{cases}$$

The structure of $\operatorname{End}_A(E)$ depends on the extensions associated with E and their relations. For higher rank there will be more possibilities for $\operatorname{End}_A(E)$. The study of these cases should help us find the answer to Question 1. However, this topic exceeds the scope of this paper.

The remainder of this section is devoted to the study of the case $\operatorname{End}_A(E) \cong \Bbbk[t]/(t^r)$.

Assume that $E \to A$ has rank $r \geq 3$ and $\operatorname{End}_A(E) \cong \Bbbk[t]/(t^r)$. As in the rank 2 case, we claim that there exists an isomorphism $\varphi : E_0 \to \operatorname{End}_A(E)$ of $\operatorname{Aut}_A(E)$-modules which induces an isomorphism

$$\psi : \operatorname{Aut}_{hb}(E) *_{\operatorname{Aut}_A(E)} E_0 \to \operatorname{Aut}_{hb}(E) *_{\operatorname{Aut}_A(E)} \operatorname{End}_A(E)$$
$$\psi\big(\big[(f,t_a),e_0\big]\big) = \big[(f,t_a),\varphi(e_0)\big]$$

of vector bundles and hence, $E \cong_{vb} \operatorname{End}_{hb}(E)$. We give the proof of the claim only for the case $r = 3$. The proof of the general case is similar, and is left to the reader.

Proposition 8. *If* $\operatorname{rk}(E) = 3$ *and* $\operatorname{End}_A(E) \cong \Bbbk[t]/(t^3)$*, then* $\operatorname{End}_{hb}(E)$ *is a commutative algebraic monoid, and* $\operatorname{End}_{hb}(E) \cong_{vb} E$.

Proof. It suffices to prove that the representations $\operatorname{Aut}_A(E) \times \operatorname{End}_A(E) \to \operatorname{End}_A(E)$ and $\operatorname{Aut}_A(E) \times E_0 \to E_0$ are isomorphic. In this case,

$$\operatorname{End}_A(E) \cong_{am} \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} a, b \in \Bbbk \right\},$$

and the action over the fiber E_0 is given as follows: consider an isomorphism $E_0 \cong \Bbbk^3$, such that $(1,0,0) \in (E_1)_0$, where $L = E_1 \subset E_2 \subset E$ is a $\operatorname{Aut}(E)$-stable filtration. Under this identification the action $\operatorname{End}_A(E) \times E_0 \to E_0$ is given by

$$\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \cdot (x,y,z) = (ax+by+cz, ay+bz, az).$$

On the other hand, the action of $\mathrm{Aut}_A(E)$ on $\mathrm{End}_A(E)$ is given by

$$\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \cdot \begin{pmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} = \begin{pmatrix} ax & bx+ay & ax+by+cz \\ 0 & ax & bz+ay \\ 0 & 0 & ax \end{pmatrix}.$$

Therefore, there exists an isomorphism $\varphi : E_0 \to \mathrm{End}_A(E)$ of $\mathrm{Aut}_A(E)$-modules and hence the homomorphism

$$\psi : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0 \to \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{End}_A(E)$$
$$\psi\big(\big[(f, t_a), e_0\big]\big) = \big[(f, t_a), \varphi(e_0)\big]$$

is an isomorphism of vector bundles and $E \cong_{vb} \mathrm{End}_{hb}(E)$ as claimed. □

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On Certain Semigroups Derived from Associative Algebras

Jan Okniński

To Mohan and Lex on the occasion of their anniversaries

Abstract This paper is based on a general project concerning semigroup theoretical methods in the study of associative rings. Let A be an associative algebra over a field K. The main idea is to introduce semigroup constructions of certain types that strongly reflect the properties of A. Then the aim is to study the structure of these semigroups and to derive certain invariants of the algebra. Some of the classical constructions that motivate our approach include: the lattice of left ideals of A and the set of orbits on A under the action of certain groups derived from the unit group $\mathrm{U}(A)$ of A. The focus is on the case of finite dimensional algebras over an algebraically closed field.

Keywords Finite dimensional algebra • Left ideals • Semigroup • Unit group • Orbit semigroup

Subject Classifications: 16D80, 16G99, 20M25, 20M99

1 Introduction

Let A be an associative unital algebra over a field K. The motivating idea is to study semigroups S_A of certain specific types that are naturally associated to A. Then to describe the semigroup structure of S_A as well as the ring theoretical properties of

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the semigroup algebra $K[S_A]$. Finally, to derive semigroup theoretical invariants and numerical invariants of A determined by S_A. Hence, the starting general simple-minded problem is to find and study possible constructions of S_A that might be useful in this context, at least for some important classes of algebras.

Clearly, several classes of semigroups show up naturally in ring theory and representation theory. We mention just a few such examples: path semigroups (and algebras) [2], semigroups coming from multiplicative bases [3], monoids arising from representations of quivers [13], or monoids arising in the context of cancellation property for projective modules over von Neumann regular rings [1]. However, the starting point of this project comes from the following two important constructions, that provide us with some ideas in the search for possible semigroups S_A.

1. The set $S(A)$ of all K-subspaces of A equipped with the operation

$$X \cdot Y = \mathrm{lin}_K(XY),$$

called the subspace semigroup of A, [8]. Such a semigroup carries a lot of information on A and might be quite useful, but it is very big and seems difficult to handle. So, the idea is to derive a smaller semigroup from $S(A)$.
2. The set $O_A(G)$ of double cosets GaG of elements $a \in A$, called G-orbits, where $G = \mathrm{U}(A)$, the unit group of A, or some important subgroup of $\mathrm{U}(A)$. A natural problem that arises here is to find a natural semigroup structure on the set of orbits.

A model case is $A = \mathrm{M}_n(K)$, $G = B$, a Borel subgroup of $\mathrm{U}(A) = \mathrm{GL}_n(K)$, (for example, the group of upper triangular matrices in $\mathrm{GL}_n(K)$). Then

$$\mathrm{GL}_n(K) = \bigcup_{x \in W} BxB$$

(Bruhat decomposition) and

$$\mathrm{M}_n(K) = \bigcup_{x \in R} BxB$$

(Renner decomposition), where W is the corresponding Weyl group (in this case, the group of permutation matrices), and R is the Renner monoid corresponding to $\mathrm{M}_n(K)$ (consisting of all $\{0, 1\}$-matrices with at most one nonzero entry in each row and in each column).

Such decompositions also hold in the more general case of reductive algebraic groups and reductive algebraic monoids. The latter are connected monoids M in the sense of [11] (meaning that M is an affine algebraic semigroup whose underlying variety is irreducible), whose unit group is a reductive group. The same applies to the abstractly defined class of finite monoids of Lie type [14]. Then $M = \bigcup_{e \in \Lambda} \mathrm{U}(M)e\mathrm{U}(M)$ for a finite semilattice Λ and $M = \bigcup_{x \in R} BxB$ for a finite inverse monoid R, see [14], Theorems 4.5 and 8.8.

Notice that if A is a finite dimensional algebra over an algebraically closed field, then the multiplicative semigroup $(A, \cdot)$ can be considered as a connected algebraic monoid. However, A is a reductive monoid exactly when A is a semisimple algebra. So the theory of reductive monoids can provide us with some intuition and methods, but it only applies directly to a very restricted class of algebras.

It is worth mentioning that the classical Hecke algebras, as well as the Hecke algebras corresponding to certain special classes of monoids, can be interpreted as algebras with bases indexed by orbits. However, the set of orbits is not closed under the corresponding multiplication. We refer to [12] and [15] for basic results on Hecke algebras in the context of monoids and their representations.

One of the natural tools that are useful in the study of $S(A)$ is the so called Zariski closed subsets semigroup $Cl(A)$, considered in [8]. So, we start with a discussion of some of the properties of $S(A)$ and $Cl(A)$ and give some examples. Using the semigroup of left ideals $L(A) \subseteq S(A)$, we introduce the semigroup $C(A)$ of conjugacy classes of left ideals of A, which is much smaller than $S(A)$ and potentially easier to handle. Moreover, $C(A)$ is related to the finite representation type property of the algebra A, and the finiteness of $C(A)$ becomes crucial in this context. Next, a reduction to nilpotent left ideals is presented. Finally, properties of A recognizable in terms of $C(A)$ and the isomorphism problem are defined and discussed. The last section is devoted to another approach towards a construction of a semigroup associated to certain finite dimensional algebras A, namely the so called orbit semigroup O_A and the simple orbit semigroup O_A^s. The origin of this project comes from [8, 9], while some of the recent results come from [4]. Several aspects and open problems of the presented program form a part of a joint project with A. Mecel.

2 Conjugacy Classes of Left Ideals and Related Semigroups

Let A be a unital algebra over a field K. Let $C(A)$ be the set of conjugacy classes $[I]$ of left ideals I in A. Then $C(A)$ is a semigroup for the operation $[I] \cdot [J] = [IJ]$. This definition was introduced in [9].

Example 1. For any division algebra D and every $n \geq 1$ the semigroup $C(\mathrm{M}_n(D))$ is isomorphic to the semigroup with zero $\{f_1, \ldots, f_n\} \cup \{0\}$, where the operation is defined by: $f_i f_j = f_j$, for $i, j = 1, \ldots, n$. This is clear because every left ideal of $\mathrm{M}_n(D)$ is conjugate to an ideal of the form $\mathrm{M}_n(D)e_i$, where $e_i = e_{11} + \cdots + e_{ii}$, a diagonal rank i idempotent.

Our motivation is to look for invariants of A that can be expressed in terms of $C(A)$. Some of the main problems can be formulated as follows:

Problem 1. Determine necessary and sufficient conditions under which $C(A)$ is finite. Determine the structure of $C(A)$.

Problem 2. Determine properties of the algebra A that can be recognized by the semigroup $C(A)$.

Problem 3. Assume that $C(A)$ is finite. Is A artinian?

In particular, one might even ask whether (or when) $C(A) \cong C(B)$ for two algebras A, B implies that $A \cong B$. In view of Example 1, in this context even if we restrict to finite dimensional algebras over K, the field should be algebraically closed. As the latter hypothesis is also standard in representation theory of finite dimensional algebras, in this paper we shall assume that K is algebraically closed and A is finite dimensional over K.

The idea to study $C(A)$ is motivated by two observations that will be explained later:

1. The finiteness of $C(A)$ is strongly related to the finite representation type property of finite dimensional algebras.
2. The latter is related also to the set of double cosets $\mathrm{U}(A)a\mathrm{U}(A)$, $a \in A$, where $\mathrm{U}(A)$ is the group of units of A.

We start with a discussion of some related useful semigroups. Let $L(A)$ be the semigroup of left ideals of A with operation $I \cdot J = IJ$. Clearly, $L(A)$ is a right ideal in $S(A)$. Moreover, $C(A) = L(A)/\rho$, where ρ is the congruence defined by

$$(I, J) \in \rho \Leftrightarrow I = Ju \quad \text{for some} \quad u \in \mathrm{U}(A).$$

Equivalently, $I = u^{-1}Ju$ for some $u \in \mathrm{U}(A)$, or $[I] = [J]$ in $C(A)$.

One also uses the semigroup $Cl(M)$ of Zariski closed subsets of an irreducible algebraic monoid M (with $M = A$) subject to $X \cdot Y = \overline{XY}$, the Zariski closure, and the natural embedding

$$S(A) \longrightarrow Cl(A).$$

In order to present the main results on the structure of $S(A)$ and $Cl(A)$ it is convenient to first recall the classical facts about the structure of the multiplicative monoid $\mathrm{M}_n(K)$ of $n \times n$ matrices over K, [5]. This is a regular semigroup with finitely many ideals

$$\{0\} = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = \mathrm{M}_n(K) \tag{1}$$

with completely 0-simple factors I_k/I_{k-1}. A Rees presentation of these factors can be explicitly given, but it is extremely complicated and intriguing, in terms of the properties of the corresponding sandwich matrix.

The next step is to look at the structure of the subspace semigroup $S(\mathrm{M}_n(K))$. This is done by first considering the semigroup $Cl(A)$, in which some basic tools of algebraic geometry can be used and which reveals a lot of information in terms of the associated algebraic groups.

Theorem 1 ([8]).

1. *Every regular $\mathscr{J}$-class of $Cl(M)$ is completely 0-simple.*
2. *$Cl(M)$ has a finite ideal chain with every factor nilpotent or a regular semigroup.*
3. *Every regular factor of this chain is a 0-disjoint union of completely 0-simple semigroups.*
4. *Regular $\mathscr{J}$-classes correspond to conjugacy classes of connected subgroups of certain linear algebraic groups, and the structure of every such $\mathscr{J}$-class can be described in group theoretical terms.*

Using this, one can prove the following result.

Theorem 2 ([8]). *There exists an ideal chain*

$$\{0\} = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_t = S(\mathrm{M}_n(K)) \tag{2}$$

where every factor J_k/J_{k-1} is either nilpotent or a 0-disjoint union of (infinitely many in general) completely 0-simple semigroups.

More generally, the same is true for $S(A)$, where A is any finite dimensional algebra.

We list some further results on $S(\mathrm{M}_n(K))$ obtained in [6–8]. $J(A)$ stands for the radical of an algebra A.

Proposition 1. *Let J be a regular $\mathscr{J}$-class of $S(\mathrm{M}_n(K))$. Then*

1. *J contains a basic algebra D; the latter means that $D/J(D) \cong K \times \cdots \times K$.*
2. *Every two basic algebras in J are conjugate in $\mathrm{M}_n(K)$.*
3. *If $N_{\mathrm{GL}_n(K)}(D)$ is the normalizer of D in $\mathrm{GL}_n(K)$ then the $\mathscr{H}$-class of $S(\mathrm{M}_n(K))$ containing D is of the form $\mathscr{H}_D = \{Dx \mid x \in N_{\mathrm{GL}_n(K)}(D)\}$.*

We continue with some basic properties of $C(A)$ for an arbitrary algebra A.

Proposition 2. *$C(A)$ has the following properties:*

1. *$C(A)$ is a periodic semigroup (of bounded index),*
2. *Every $\mathscr{L}$-class of $C(A)$ is a singleton,*
3. *The number of regular $\mathscr{J}$-classes is equal to the number of ideals in the algebra $A/J(A)$, hence it is finite.*

In particular every regular $\mathscr{J}$-class of $C(A)$ is a right zero semigroup.

As a consequence one can prove an analogue of the structural flavor of $S(\mathrm{M}_n(K))$.

Theorem 3. *$C(A)$ has a finite ideal chain with every factor nilpotent or a right zero semigroup.*

We continue with some simple examples.

Example 2. Let A be a semisimple algebra. Then $A \cong \mathrm{M}_{n_1}(K) \oplus \cdots \oplus \mathrm{M}_{n_k}(K)$ and $C(A) = C_1 \times \cdots \times C_k$, where every $C_j = \{f_{j1}, \ldots, f_{jn_j}, 0\}$, $f_{jm} f_{jn} = f_{jn}$ (right zero semigroup with zero adjoined).

Example 3. Let A be a principal left ideal algebra (every left ideal is a principal ideal). Then $A \cong \mathrm{M}_{n_1}(B_1) \oplus \cdots \oplus \mathrm{M}_{n_k}(B_k)$, where every B_j is a local algebra whose radical is a principal (left) ideal. Then

$$B_j \cong K_0[S_j], \quad S_j = \{0, x_j, x_j^2, \ldots, x_j^{r_j}\}, \quad \text{where} \quad x^{r_j+1} = 0.$$

Moreover, $C(A) = C_1 \times \cdots \times C_k$, where C_j can be identified with the set of all sequences $(x_j^{i_1}, \ldots, x_j^{i_k}, 0, \ldots, 0)$ of length n_j such that $0 \le i_1 \le i_2 \le \cdots \le i_k \le r_j$, with respect to the product induced from S_j.

One might expect that some of the general properties of $S(A)$ can be understood by looking at the class of subspace semigroups of algebras over finite fields $\mathbf{F}_q$, which seems to be of independent interest.

Example 4. The subspace semigroup $S(\mathrm{M}_2(\mathbf{F}_q))$ has 8 $\mathscr{J}$-classes; only one of them is not regular. Every regular $\mathscr{J}$-class J is 'square' (the numbers of $\mathscr{R}$- and $\mathscr{L}$-classes are equal) and has an invertible (over $\mathbf{C}[G]$ for the maximal subgroup G of J) sandwich matrix.

In particular, this leads to the following perhaps naive question.

Question. Does there exist a natural bijection between the set of $\mathscr{R}$- and $\mathscr{L}$-classes of every regular $\mathscr{J}$-class J of $C(A)$? In particular, if $K = \mathbf{F}_q$, is such a J a 'square'?

2.1 Connection with Finite Representation Type

A special attention for the semigroup $C(A)$ is due to the relation with the following classical family of well-behaved algebras.

Definition 1. A is of finite representation type if there are finitely many isomorphism classes of finitely generated indecomposable left A-modules. Equivalently: finitely many isomorphism classes of finitely generated indecomposable right A-modules.

Theorem 4 ([3]). *Assume that A is of finite representation type. Then A has a multiplicative basis. This means there exists a basis B such that $ab \in B \cup \{0\}$ for all $a, b \in B$.*

In particular, for every $n \ge 1$ there are finitely many algebras of finite type and of dimension n. The following observations are used to connect the finite type property with the semigroup $C(A)$.

Lemma 1. *Assume that $Ax = Ay$ for some $x, y \in A$. Then there exists $u \in \mathrm{U}(A)$ such that $x = uy$.*

Lemma 2. *Let I, J be left ideals of A. Then the left A-modules $A/I, A/J$ are isomorphic iff $J = Iu$ for some $u \in \mathrm{U}(A)$.*

If there are finitely many isomorphism classes of left A-modules of the form A/I, for $I \in L(A)$, then one gets the assertion of the following corollary.

Corollary 1. *If A is of finite representation type then $C(A)$ is a finite semigroup.*

Let $\mathscr{I}(A)$ denote the lattice of (two-sided) ideals of A.

Theorem 5 ([9]). *Consider the following conditions:*

1. *A is of finite representation type,*
2. *$C(A)$ is finite,*
3. *A has finitely many $\mathrm{U}(A)$-orbits,*
4. *$\mathscr{I}(A)$ is a distributive lattice.*

The following implications hold: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

We notice that condition (3) is equivalent to the finiteness of the set of conjugacy classes of principal left ideals. On the other hand, condition (4) is equivalent to the finiteness of $\mathscr{I}(A)$, see [10], Ex.4 in § 2.2 and Ex.3 in § 2.6. It is easy to see that in this case every ideal of A must also be principal. So the map $\mathrm{U}(A)x\mathrm{U}(A) \mapsto AxA$ is onto (from the set of orbits to $\mathscr{I}(A)$).

A partial converse to the implication 1. $\Longrightarrow$ 2. can also be proved.

Theorem 6 ([9]). *A is of finite representation type if and only if $C(\mathrm{M}_n(A))$ is finite for every $n \geq 1$.*

Clearly, if A is of finite type, then so is $\mathrm{M}_n(A)$ and $C(\mathrm{M}_n(A))$ is finite by the previous result. For the converse, one uses the fact that if I, J is a submodules of $_AA^n$ and I', J' are the left ideals of $\mathrm{M}_n(A)$ consisting of matrices with every row in I (respectively in J), then

$$J'g = I' \text{ for some } g \in \mathrm{U}(\mathrm{M}_n(A)) \quad \Leftrightarrow \quad Jg = I, \quad \text{which implies} \quad A^n/I \cong A^n/J.$$

This yields that for every $n \geq 1$ there are finitely many indecomposable A-modules with n generators. Then A is of finite type, by the Brauer-Thrall conjecture, proved by Nazarova and Roiter, see [10], Chapter 7.

Next, we turn to a preliminary discussion of the finiteness of $C(A)$.

Theorem 7 ([4]). *The following conditions are equivalent:*

(1) $C(A)$ is finite,
(2) The number of conjugacy classes of nilpotent left ideals in A is finite.

Actually, the following is a direct consequence of the proof.

Corollary 2. *Assume that A has n conjugacy classes of idempotents and that the number of conjugacy classes of nilpotent left ideals in A is finite and equal to m. Then the semigroup $C(A)$ is finite and $|C(A)| \leq nm$.*

2.2 Basic Algebras

We start with an important notion of the representation theory of finite dimensional algebras.

Definition 2. Assume that $\{e_1, e_2, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents. By a basic algebra associated to A we mean the algebra

$$A^b = e_A A e_A$$

where $e_A = e_{j_1} + \ldots + e_{j_t}$, and $e_{j_1}, \ldots, e_{j_t}$ are chosen so that $Ae_{j_i} \ncong Ae_{j_k}$ for $i \neq k$ and each A-module Ae_j is isomorphic to one of the modules $Ae_{j_1}, \ldots, Ae_{j_t}$.

A^b does not depend on the choice of the sets $e_1, e_2, \ldots, e_n$ and $e_{j_1}, \ldots, e_{j_t}$, up to a K-algebra isomorphism. The algebra A^b is basic ($A^b/J(A^b) \cong K^t$) and there is an equivalence of categories (of left modules) $\mathrm{mod}(A) \cong \mathrm{mod}(A^b)$; see [2].

We write $[L]_e$ for the conjugacy class of a left ideal L in eAe, where $e = e^2 \in A$.

Proposition 3. *Assume that $e = e^2$ is an idempotent of A. Let*

$$\phi([L]_e) = [AL] \quad \textit{for} \quad L \in L(eAe).$$

Then $\phi : C(eAe) \to C(A)$ is well defined and it is a semigroup monomorphism. So: if $C(A)$ is finite then $C(eAe)$ is finite.

The following construction plays an important role in representation theory of algebras and also in the approach towards a classification of algebras with finite semigroups $C(A)$. We follow [10], which differs from the general definition used for example in [2].

Definition 3. Let $\{e_1, \ldots, e_n\}$ be a complete set of orthogonal primitive idempotents of A. A directed graph $\Gamma = (V, E)$ is called the quiver of A, denoted by $\Gamma(A)$, if the vertex set of Γ is of the form $V = \{1, 2, \ldots, n\}$ and the edge set E is equal to $\{(i, j) \mid e_i J(A) e_j \neq 0\}$. Associated with Γ is the quiver $\Gamma^s = (V^s, E^s)$, where $V^s = V \times \{0, 1\}$ and $E^s = \{((i, 0), (j, 1)) \mid (i, j) \in E\}$. It is called the separated quiver of A and it is denoted by $\Gamma^s(A)$.

Theorem 8 ([9]). *Let A be a finite dimensional basic algebra with a distributive lattice of ideals over an algebraically closed field K and such that $J(A)^2 = 0$. Then the following conditions are equivalent*

(1) C(A) is finite,
(2) The separated quiver $\Gamma^s(A)$ of A has no cycles (with orientation ignored) and $\dim(eJ(A)) \leq 3$ for every primitive idempotent $e \in A$.

One can easily give examples of non-basic algebras with a 2-nilpotent Jacobson radical which do not satisfy the above bound on the dimension of $eJ(A)$.

Example 5. Consider $A = \mathrm{M}_n(K[x]/(x^2))$. This is an algebra of finite representation type, so $C(A)$ is finite. (Another proof: it can be verified that $\mathrm{M}_n(K[x]/(x^2))$ has exactly $n + 1$ conjugacy classes of left nilpotent ideals. Thus, by a previous result, $C(A)$ is finite.)

The dimension of $eJ(A)$ could be arbitrarily large in this example. This shows that a direct generalization of the above theorem to the non-basic case is not possible.

Example 6. Consider the subalgebra B of the matrix algebra $\mathrm{M}_7(K)$ of the form $B = \sum_{i=1}^{7} Ke_{ii} + Ke_{12} + Ke_{13} + Ke_{15} + Ke_{45} + Ke_{46} + Ke_{47}$, where e_{ij} denote the matrix units of $\mathrm{M}_7(K)$. The algebra B is basic, and $C(B)$ is finite. Indeed, it is easy to see that the separated quiver $\Gamma^s(B)$ has no cycles. Moreover $\dim(eJ(B)) \leq 3$ for every primitive idempotent e of B. From Theorem 8 it follows that the semigroup $C(B)$ is finite. However, $\Gamma^s(B)$ is not a disjoint union of Dynkin graphs. It is known that the latter implies that B is not of finite representation type, see [10], Theorem 11.8.

2.3 Recognizable Properties and the Isomorphism Problem

One of the aims is to determine properties of A which can be derived from its semigroup $C(A)$. Such properties might be called recognizable. A simple example follows.

Definition 4. The radical of $C(A)$ is defined as the largest semigroup ideal of $C(A)$ consisting of nilpotent elements. The radical of $C(A)$ will be denoted by $\mathcal{N}(C(A))$.

Observe that

$$\mathcal{N}(C(A)) = \{[L] \in C(A) \mid L \subseteq J(A)\}. \tag{3}$$

Therefore $\mathcal{N}(C(A))$ is the set of all nilpotent elements of $C(A)$ and it is the largest nilpotent deal of $C(A)$.

Notice that $C(A)$ is a regular semigroup if and only if A is a semisimple algebra, and a description of $C(A)$ follows from Example 1 in this case.

Consider the case where $J(A)^2 = 0$. We have

$$A = A' \oplus J(A) \quad \text{as spaces,}$$

where $A' \cong \overline{A}$, $A' = A_1 \oplus \cdots \oplus A_n$, $A_i \cong \mathrm{M}_{s_i}(K)$, for $1 \leq i \leq n$.

Let $J_{ij} = A_i J(A) A_j$. Then J_{ij} are $A_i - A_j$-bimodules. Hence J_{ij} are right modules over the algebra $A_i^{op} \otimes_K A_j \cong \mathrm{M}_{s_i s_j}(K)$. So they are of the form

$$J_{ij} = \underbrace{N_{ij} \oplus N_{ij} \oplus \ldots \oplus N_{ij}}_{m_{ij}},$$

where m_{ij} are positive integers and N_{ij} is isomorphic to a minimal right ideal of $\mathrm{M}_{s_i s_j}(K)$.

One shows that the properties of $C(A)$ allow us to determine the set $\{s_1, s_2, \ldots, s_n\}$, as well as the matrix $[m_{ij}]$. For example:

Proposition 4. *Assume that $J(A)^2 = 0$ and $C(A)$ is finite. There exists a collection of recognizable sets $\mathscr{J}_{ij} \subseteq C(A)$, $1 \le i, j \le n$, such that if $[m_{ij}]$ is the matrix corresponding to the algebra A, then*

$$m_{ij} = \begin{cases} 0 \text{ if } \mathscr{J}_{ij} = \{[0]\} \\ 1 \text{ if } \mathscr{J}_{ij} \neq \{[0]\}. \end{cases} \tag{4}$$

This leads to the following conclusion.

Theorem 9 ([4]). *Let A, B be finite dimensional algebras over an algebraically closed field K. Assume that $J(A)^2 = 0$ and $C(A)$ is finite. If the semigroups $C(A)$ and $C(B)$ are isomorphic then the algebras A and B are isomorphic.*

3 Orbit Semigroups

In this section we introduce and discuss certain semigroups arising from the double coset decomposition of the multiplicative monoid $(A, \cdot)$ with respect to important subgroups of the unit group $\mathrm{U}(A)$ of A.

Recall that a finite dimensional algebra over an algebraically closed field may be considered as a connected algebraic monoid in the sense of [11]. If $A = J(A) + \overline{A}$ (a direct sum of subspaces), where $\overline{A} \cong A/J(A)$ then

$$\mathrm{U}(A) = J(A) + \mathrm{U}(\overline{A}).$$

Let B be a Borel subgroup of the algebraic group $\mathrm{U}(A)$. So, B is a maximal closed solvable normal subgroup of $\mathrm{U}(A)$. Such B is unique up to conjugation. Since $1 + J(A)$ is a nilpotent normal subgroup of $\mathrm{U}(A)$, it is easy to see that B can be identified with $J(A) + T(\overline{A})$, where

$$T(\overline{A}) = T_1 \times \cdots \times T_n \tag{5}$$

with $\overline{A} = A_1 \times \cdots \times A_n$ and $A_j = \mathrm{M}_{s_j}(K)$ for some s_j and T_j is the group of upper triangular matrices in $\mathrm{GL}_{s_j}(K)$. In particular, if A is a basic algebra then $B = \mathrm{U}(A)$.

Let E be a complete set of primitive orthogonal idempotents of A compatible with B. The latter means that $E = \{e_{i,j} \mid i = 1, \ldots, n,\ j = 1, \ldots, s_i\}$ in the notation of (5) with $e_{i,j}$ denoting the j-th diagonal rank one idempotent in $A_i = \mathrm{M}_{s_i}(K)$.

Assume first that the set $O_A(B)$ of B-orbits BxB, $x \in A$, is finite. Then for every $x, y \in A$ there exists exactly one orbit BzB that is a dense subset of $BxBByB$. We define

$$BxB \cdot ByB = BzB. \tag{6}$$

One verifies that in this way $O_A(B)$ becomes a monoid with unity B. Since every two Borel subgroups B, B' of $\mathrm{U}(A)$ are conjugate, the corresponding monoids $O_A(B)$, $O_A(B')$ are isomorphic. Hence, we can write $O_A = O_A(B)$ for simplicity.

Assume now that the lattice $\mathscr{I}(A)$ of ideals of A is distributive (which holds if there are finitely many B-orbits on A, by Theorem 5). This implies that every $eAe - fAf$-bimodule eAf is uniserial, [3]. So, if $x \in eAe'$, $y \in fAf'$ then there exists $w_{xy} \in eAf'$ such that $eAew_{xy}f'Af'$ is the unique maximal submodule of the $eAe - f'Af'$ – bimodule eAf' among all those generated by elements of xBy. Let

$$A_0 = \bigcup_{e,f \in E} eAf.$$

The elements BxB, for $x \in A_0$, are referred to as the simple B-orbits in A. The set of all simple B-orbits is denoted by O_A^s. One verifies that it becomes a semigroup under the operation

$$BxB \diamond ByB = Bw_{xy}B. \tag{7}$$

Lemma 3. *Assume the lattice $\mathscr{I}(A)$ of ideals of A is distributive. Then $BaB \mapsto (eAe)a(fAf)$, for $a \in eAf, e, f \in E$, defines a bijection between O_A^s and the set of submodules of all $eAe - fAf$-bimodules eAf. In particular, O_A^s is a finite semigroup.*

Lemma 4. *If A has finitely many B-orbits then the product defined in O_A by (6) for $x, y \in O_A^s$ satisfies*

$$BxB \cdot ByB = Bw_{xy}B.$$

The above means that the semigroup O_A (if defined) contains O_A^s as a subsemigroup, which might seem more suitable for our purpose. In other words, the geometric and the algebraic definitions of the product coincide on O_A^s.

Example 7. Let $A = \mathrm{M}_n(K)$. Then

$$O_A^s = \{Be_{ij}B \mid i, j = 1, \dots, n\} \cup \{0\}$$

where e_{ij} are matrix units and $B =$ the group of upper triangular matrices. Thus

$$Be_{ij}B \cdot B_{kl}B = Be_{il} \text{ if } j \leq k \text{ and } 0 \text{ otherwise.}$$

Let

$$T = \{f_{ij} = e_{ij} + e_{ij+1} + \cdots + e_{in} \mid i, j = 1, \ldots, n\} \cup \{0\}.$$

Then $T \cong O_A^s$ and $T \setminus \{0\}$ is a basis of $\mathrm{M}_n(K)$. Hence

$$\mathrm{M}_n(K) \cong K_0[T] \cong K_0[O_A^s],$$

the contracted semigroup algebras. Notice that T is a completely 0-simple semigroup isomorphic to $\mathscr{M}^0(\{1\}, n, n, P)$ where the sandwich matrix $P = (p_{jk})$ is defined by $p_{jk} = 0$ if $j > k$ and $p_{jk} = 1$ if $j \leq k$.

The first natural question is to determine the structural properties of O_A and of O_A^s, when they are defined. Notice that, in view of Example 7 we get that O_A^s is regular if and only if A is semisimple, and a description of O_A^s follows in this case.

The next step is to look for properties of A that can be derived from O_A^s. We conclude with some sample results in this direction.

Theorem 10. *Assume that the lattice $\mathscr{I}(A)$ is distributive. Then*

1. $|O_A^s| = \dim_K(A) + 1$.
2. *A and $K_0[O_A^s]$ are isomorphic modulo the squares of their radicals. Moreover $J(K_0[O_A^s]) = K_0[N]$, where $N = \{BxB \mid x \in J(A) \cap A_0\}$.*
3. *If A is hereditary and basic then $A \cong K_0[O_A^s]$.*

A sample natural problem that arises here is to determine for which algebras A one has $A \cong K_0[O_A^s]$.

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The Betti Numbers of Simple Embeddings

Lex E. Renner

Abstract Let G be a simple algebraic group with Weyl group (W, S) and let $w \in W$. We consider the *descent set* $D(w) = \{s \in S \mid l(ws) < l(w)\}$. This has been generalized to the situation of the Bruhat poset W^J, where $J \subset S$. To do this one identifies a certain subset $S^J \subset W^J$ that plays the role of $S \subset W$ in the well known case $J = \emptyset$. One ends up with the *descent system* (W^J, S^J). On the other hand, each subset $J \subset S$ determines a projective, simple $G \times G$-embedding $\mathbb{P}(J)$ of G. The case where $J = \emptyset$ is closely related to the wonderful embedding. One obtains a complete list of all subsets $J \subset S$ such that $\mathbb{P}(J)$ is a *rationally smooth* algebraic variety. In such cases we determine the Betti numbers of $\mathbb{P}(J)$ in terms of (W^J, S^J). It turns out that $\mathbb{P}(J)$ can be decomposed into a union of "rational" cells. The descent system is used here to help record the dimension of each cell.

Keywords Betti numbers • Descent systems • H-polynomials • Rationally smooth

Subject Classifications: Primary 14M27; Secondary 20M32, 52B15, 51F15, 14L30

1 Introduction

Let G_0 be a semisimple algebraic group and let $\rho : G_0 \to End(V)$ be a representation of G_0. Define Y_ρ to be the Zariski closure of $G = [\rho(G_0)] \subseteq \mathbb{P}(End(V))$, the projective space associated with $End(V)$. Finally, let X_ρ be the normalization of Y_ρ. X_ρ is a projective, normal $G \times G$-embedding of G. That is,

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there is an open embedding $G \subseteq X_\rho$ such that the action $G \times G \times G \to G$, $(g, h, x) \rightsquigarrow gxh^{-1}$, extends over X_ρ. Furthermore $B \times B$ acts on X_ρ with a finite number of orbits. The problem here is to find a homologically useful description of how X_ρ fits together from these $B \times B$-orbits. This has been accomplished by several authors in case X is the wonderful embedding of an adjoint semisimple group. See [4, 7, 12] for an assortment of approaches. The purpose of this survey is to describe what can be done here for any rationally smooth embedding of the form X_ρ when ρ is irreducible.

The main problem here is easy to describe. Suppose we have a rationally smooth, embedding X_ρ. We wish to calculate the Betti numbers of X_ρ in terms of the $B \times B$-orbit structure of X_ρ. The challenge here is to first organize the $B \times B$-orbits into homologically meaningful "cells", and then to quantify these cells in terms combinatorial data coming from the Weyl group.

1.1 Motivation

We consider the following two examples as motivation for the present discussion. The first example is $\mathbb{P}^{(n+1)^2-1}(K)$ which arises as a simple embedding of $PGl_n(K)$. The second example is related to the well-known "wonderful" compactification which is associated with any semisimple group.

Example 1. Let $M = M_{n+1}(K)$. Then the two-sided action of $Gl_n(K)$ on M results in the structure of a simple embedding on $X = \mathbb{P}^{(n+1)^2-1}(K)$. It is well-known that the Poincaré polynomial P_X of X is given by the formula

$$P_X(t^{1/2}) = \sum_{i=0}^{(n+1)^2-1} t^i = \left(\sum_{i=0}^{n} t^{(n-i)(n+1)}\right)\left(\sum_{i=0}^{n} t^i\right)$$

Example 2. A *canonical monoid* M is a $\mathcal{J}$-irreducible monoid of type $J = \phi$. This is just a monoid theoretic way of saying that $\mathbb{P}(M) := (M \setminus \{0\})/K^*$ is closely related to the wonderful compactification of $G/Z(G)$. Let M be a canonical monoid with unit group G, and let G_0 be the commutator subgroup of G. In fact, if G_0 is a group of adjoint type, then $X = \mathbb{P}(M)$ is the canonical compactification of G_0. According to [7], the Poincaré polynomial P_X of X is given by

$$P_X(t^{1/2}) = \left(\sum_{w \in W} t^{l(w_0)-l(w)+|I_w|}\right)\left(\sum_{v \in W} t^{l(v)}\right).$$

where $I_w = \{s \in S \mid w < ws\}$ and $w_0 \in W$ is the longest element.

The whole point of this survey is to reveal a general method that explains these two extreme cases. Furthermore we also explain the factorization of $P_X(t^{1/2})$.

The Poincaré polynomial of the wonderful compactification was originally obtained by DeConcini and Procesi in [7]. It was that calculation that motivated many of the results contained in this survey. See [12, 14–18].

1.2 Betti Numbers and Rational Cells

Let X be a complex, algebraic variety of dimension n. We say that X is **rationally smooth** at $x \in X$ if there is a neighbourhood U of x in the complex topology such that, for any $y \in U$,

$$H^m(X, X \setminus \{y\}) = (0)$$

for $m \neq 2n$ and

$$H^{2n}(X, X \setminus \{y\}) = \mathbb{Q}.$$

Here $H^*(X)$ denotes the cohomology of X with rational coefficients. Danilov [6] has characterized the rationally smooth toric varieties in combinatorial terms.

If X is a rationally smooth affine variety with $\mathbb{C}^*$-action, and attractive fixed point $x \in X$, we refer to (X, x) as a **rational cell**.

Proposition 1. *Let suppose that X is a complex variety with a filtration*

$$\{x_1\} = X_1 \subset X_2 \ldots \subset X_n = X$$

such that each $X_i \subset X$ is closed, and each $C_i = X_i \setminus X_{i-1}$ is a rational cell. Then

$$(dim_{\mathbb{Q}} H^{2k}(X; \mathbb{Q})) = \textit{the number of cells of dimension } k,$$

and the rational cohomology of X is zero in odd dimensions.

Proof. Inductively, apply the long exact sequence of rational cohomology to the pair (X_i, X_{i-1}), using the fact that, if (C, x) is a $2k$-dimensional rational cell, then $C \setminus \{x\}$ is a rational cohomology $2k - 1$-sphere.

Any rationally smooth, projective embedding X has a filtration of the type indicated in Proposition 1. See [14].

1.3 H-Polynomials

The H-polynomial is the obvious synthesis of two extremes, the h-polynomial of a torus embedding, and the length polynomial of a Weyl group. In the former case one collects summands of the form $(t-1)^a$ (coming from an a-dimensional orbit of a torus group) while in the latter case one collects summands of the form t^b (coming from a b-dimensional orbit of a unipotent group). But in each case the corresponding polynomial yields the desired coefficients. The common theme here is that, in both cases, we are summing over a finite number of K-orbits for the appropriate solvable group K. In more general cases, like $G \times G$-embeddings of G with the $B \times B$-action, there are a finite number of $B \times B$-orbits, and each one is composed of a unipotent part and a diagonalizable part. In this situation, we need to collect summands of the form $(t-1)^a t^b$ for the appropriate integers a and b. Indeed, for each $B \times B$-orbit BxB, define

$$a(x) = rank(B \times B) - rank(B \times B)_x$$

and

$$b(x) = dim(UxU).$$

Here $(B \times B)_x = \{(g,h) \in B \times B \mid gxh^{-1} = x\}$. The summand associated with this orbit is $(t-1)^{a(x)} t^{b(x)}$. Thus we make the following fundamental definition.

Definition 1. Let $\rho : G \to End(V)$ be an irreducible representation and let

$$X = X_\rho$$

be as above. The H**-polynomial** of X is defined to be

$$H_X(t) = \sum_{x \in \mathcal{R}} (t-1)^{a(x)} t^{b(x)}.$$

where $\mathcal{R}$ is a set of representatives for the $B \times B$-orbits of X.

Remark 1. If X is rationally smooth then we have the relation

$$H_X(t) = P_X(t^{1/2})$$

where $P_X(t^{1/2})$ is the Poincaré polynomial of X. This H-polynomial is not the correct tool for investigating varieties with singularities that are not rationally smooth. In the case of Schubert varieties, and Kazhdan-Lusztig theory, the correct formulation incorporates a "correction factor" (*aka* the *KL-polynomial*) that takes into account local intersection cohomology groups. See Theorem 6.2.10 of [2]. The authors of [5] calculate the Poincaré polynomial, for intersection cohomology, of a large class of $G \times G$-embeddings using the stratification by $G \times G$-orbits.

Example 3. Let $G_0 = PGL_3(\mathbb{C})$, and let $\rho : G_0 \to End(V)$ be any irreducible representation whose highest weight is in general position. Then the H-polynomial of X_ρ is given by

$$H(t) = \left[1 + 2t^2 + 2t^3 + t^5\right]\left[1 + 2t + 2t^2 + t^3\right]$$

See Example 7 below for more information related to this example.

2 Descent Systems and Simple Embeddings

2.1 Simple Embeddings

Let $X = X_\rho$ be a projective, normal embedding, as discussed in the introduction. If the representation ρ is irreducible then one can check that X is a **simple** embedding. Namely there is a unique $G\times G$-orbit $Y \subset X$ such that, for any $G\times G$-orbit $V \subset X$, $Y \subset \overline{V}$. Furthermore one can check that any projective, simple embedding Z is of the form $Z = X_\rho$ for some irreducible representation ρ of G. Another way to look as this is using $\mathcal{J}$-irreducible monoids.

Let M be a reductive monoid [13] with unit group G and Borel subgroup $B \subseteq G$. Let $T \subseteq B$ be a maximal torus. We let

$$E(\overline{T}) = \{e \in \overline{T} \mid e = e^2\}$$

be the set of **idempotents** of $\overline{T}$ and $E_1(\overline{T}) = \{e \in E(\overline{T}) \mid dim(eT) = 1\}$ the set of **rank-one** idempotents of $\overline{T}$. The corresponding **cross section lattice** of M, relative to T and B, is

$$\Lambda = \{e \in E(\overline{T}) \mid eB = eBe\} = \{e \in E(\overline{T}) \mid eb = ebe \text{ for all } b \in B\}.$$

It turns out that, for any cross section lattice Λ,

$$M = \bigsqcup_{e\in\Lambda} GeG.$$

Notice that Λ is a multiplicatively closed. Furthermore, if Λ' is another cross section lattice of M then there exists $g \in G$ such that $\Lambda' = g\Lambda g^{-1}$. There is a one-to-one correspondence between the set of cross section lattices of M and the set of pairs $\{(T, B)\}$ where T is a maximal torus contained in the Borel subgroup B.

Lemma 1. *Let M be a reductive monoid with zero element $0 \in M$. Let $\Lambda \subseteq E(\overline{T})$ be the cross section lattice relative to T and B. The following are equivalent.*

(a) $\Lambda\backslash\{0\}$ has a unique minimal element e_0 (so that $e_0 f = e_0$ for all $f \in \Lambda\backslash\{0\}$);
(b) there exists a rational representation $\rho : M \longrightarrow End(V)$ such that

  (i) V is irreducible over M.
  (ii) ρ is a finite morphism.

See Lemma 7.8 of [13] for the proof. A reductive monoid M is called *$\mathcal{J}$-irreducible* if it satisfies the conditions of Lemma 1. Any $\mathcal{J}$-irreducible monoid is also semisimple. Lemma 1 establishes a fundamental link between the orbit structure of a $\mathcal{J}$-irreducible and its representation-theoretic structure. The orbit structure of a $\mathcal{J}$-irreducible monoid has been described explicitly in [11]. Let S denote the set of simple involutions of G relative to T and B. It turns out that for each proper subset $J \subseteq S$ there is essentially one $\mathcal{J}$-irreducible monoid. The main result of [11] provides an algorithm for computing Λ in terms of J. If M is $\mathcal{J}$-irreducible then the center of G is one-dimensional $X = (M\setminus\{0\})/k^*$ is a simple embedding.

2.2 Rationally Smooth Embeddings

We first state a general theorem from [14] that quantifies the Betti numbers of a rationally smooth embedding in terms of combinatorial invariants (e.g. length functions, G/Bs, etc.) To state this theorem we first recall that if (W, S) is a Weyl group and $J \subset S$ then W^J is the set of minimal length representatives for the cosets of W_J in W. In particular, the canonical composition

$$W^J \to W \to W/W_J$$

is bijective.

Theorem 1 ([14]). *Let M be a reductive monoid such that $X = \mathbb{P}_\epsilon(M)$ is rationally smooth. Write G/P_e for $(eG)/\mathbb{C}^*$. Let $e_1 \in \Lambda_1$ and let $we_1w^{-1} = e$, where $w \in W^J$. $w_0 \in W^J$ is the unique element of maximal length, and $J = \{s \in S \mid se_1 = e_1s\}$. Also, let $H(G/P_e) = \sum_{w\in W^J} t^{l(w)}$.*

1. If we let $w(e) = w$ then

$$P_X(t) = \sum_{e\in E_1} \left[t^{l(w_0)-l(w(e))+m(e)} H(G/P_e)\right].$$

2. In case P_e and $P_{e'}$ are conjugate for all $e, e' \in E_1$ the sum can be rewritten as

$$P_X(t) = \left[\sum_{e\in E_1} t^{l(w_0)-l(w(e))+m(e)}\right] H(G/P_e).$$

Furthermore, there is a certain idempotent $f_e \in E(\overline{T})$ such that $m(e) = dim(f_eM_e)$, where $M_e = \{g \in G(M) \mid ge = eg = e\}$.

See Theorem 5.5 of [14] for more details about Theorem 1 and Definition 5.4 of [14] for more details about $m(e)$.

It is possible to choose a 1-parameter subgroup $\lambda : k^* \to G$ such that the BB-decomposition

$$X = \bigsqcup_{e\in E_1} X(e)$$

is indexed by $E_1(\overline{T})$. Furthermore, the irreducible component $X(e)^\lambda$, $e \in E_1(\overline{T})$, of the fixed point set of λ, is given by $X(e)^\lambda = (eG)/K^* \cong G/P_e$.

Each summand $t^m H(G/P_e)$ (with $m = l(w_0) - l(w(e)) + m(e)$) in the formula for $H_{\mathbb{P}(M)}(t)$ in Theorem 1 mirrors the BB-projection

$$\pi_e : X(e) \to G/P_e \cong (eG)/K^*.$$

which is given by $\pi_e([x]) = [ex]$. A careful inspection of this projection yields that the fibre of π_e has dimension $l(w_0) - l(w(e)) + m(e)$.

The most intriguing problem here is to calculate the quantity $m(e)$. It contains a subtle contribution from the induced BB-decomposition of the associated maximal torus. See Sections 4.2 and 5.1 of [14] for more details. It turns out that, if $\mathbb{P}_\epsilon(M)$ is a simple embedding (where M is $\mathcal{J}$-irreducible of type J) and $e \in E_1(\overline{T})$, then

$$X(e) = \{[y] \in X \mid eBy = eBey \subseteq eG\}.$$

In this case the calculation of $m(e)$ involves the *descent system* (W^J, S^J) of (W, S) for the appropriate $J \subset S$ [15]. See Sect. 2.3 below for a discussion of these descent systems. The descent system serves as an effective combinatorial substitute for the infinitesimal part of Bialynicki-Birula's method [1]. See Theorem 6 and the examples that follow it for more illustration of how $m(e)$ is quantified in the case of a simple embedding of the form X_ρ. We are particularly interested in the situation where the embedding X_ρ is obtained from an irreducible representation $\rho = \rho_\lambda$, of type $J = \{s \in S \mid s(\lambda) = \lambda\}$, of G. We refer to this embedding as $\mathbb{P}(J)$. This is well-defined since $\mathbb{P}(J)$ depends only on J and not on λ.

It turns out that $m(e) = dim(f_e M_e)$ where f_e is a certain idempotent of $\overline{T}_e$. Thus it remains for us to calculate $dim(f_e M_e)$ for each $e \in E_1 \cong W^J$. This requires some calculation related to the fact that M_e is essentially a product of matrix monoids

$$M_e \sim \Pi_{s \in S \setminus J} M_{\delta(s)}(K),$$

so that f_e breaks up into a "sum" of idempotents $\{f_{e,s}\}$, one for each $e \in E_1$ and each $s \in S \setminus J$. We obtain that

$$\nu(e) = dim(fM_e) = \sum_{s \in S \setminus J} dim(f_{e,s} M_e) = \sum_{s \in S \setminus J} \delta(s)(rank(f_{e,s}) - 1),$$

where the rank here is in M (not in M_e). The purpose of the descent system (W^J, S^J) is to identify the "oriented edges" $(u, v) \in W^J \times W^J$ and use them to calculate the numbers $\{rank(f_{e,s}) - 1\}$.

2.3 Descent Systems

Let $r : W \to GL(V)$ be the usual reflection representation of the Weyl group W, where V is a rational vector space. Along with this goes the **Weyl chamber** $\mathcal{C} \subseteq V$ and the corresponding set of **simple reflections** $S \subseteq W$. The Weyl group W is generated by S, and $\mathcal{C}$ is a fundamental domain for the action of W on V.

Let $\lambda \in \mathcal{C}$. Consider the face lattice $\mathcal{F}_\lambda$ of the polytope

$$\mathcal{P}_\lambda = Conv(W \cdot \lambda),$$

the convex hull of $W \cdot \lambda$ in V. This lattice $\mathcal{F}_\lambda$ depends only on $W_\lambda = \{w \in W \mid w(\lambda) = \lambda\} = W_J = \langle s \mid s \in J \rangle$, where $J = \{s \in S \mid s(\lambda) = \lambda\}$. One can describe $\mathcal{F}_\lambda = \mathcal{F}_J$ explicitly in terms of $J \subseteq S$. See [11].

Definition 2. We refer to J as **combinatorially smooth** if $\mathcal{P}_\lambda$ is a simple polytope.

It is important to characterize the very interesting condition of Definition 2. If $J \subseteq S$ we let $\pi_0(J)$ denote the set of connected components of J. To be more precise, let $s, t \in J$. Then s and t are in the same connected component of J if there exist $s_1, \ldots, s_k \in J$ such that $ss_1 \neq s_1 s$, $s_1 s_2 \neq s_2 s_1, \ldots, s_{k-1} s_k \neq s_k s_{k-1}$, and $s_k t \neq t s_k$.

The following theorem indicates exactly how to detect, combinatorially, the condition of Definition 2.

Theorem 2 ([15]). *Let $\lambda \in \mathcal{C}$. The following are equivalent.*

1. *$\mathcal{P}_\lambda$ is a simple polytope.*
2. *There are exactly $|S|$ edges of $\mathcal{P}_\lambda$ meeting at λ.*
3. *$J = \{s \in S \mid s(\lambda) = \lambda\}$ has the properties*
   (a) *If $s \in S \backslash J$, and $J \nsubseteq C_W(s)$, then there is a unique $t \in J$ such that $st \neq ts$. If $C \in \pi_0(J)$ is the unique connected component of J with $t \in C$ then $C \backslash \{t\} \subseteq C$ is a setup of type $A_{l-1} \subseteq A_l$.*
   (b) *For each $C \in \pi_0(J)$ there is a unique $s \in S \backslash J$ such that $st \neq ts$ for some $t \in C$.*
4. *$\mathbb{P}(J)$ is rationally smooth.*

Definition 3. Let (W, S) be a Weyl group and let $J \subseteq S$ be a proper subset. Define

$$S^J = (W_J (S \setminus J) W_J) \cap W^J.$$

We refer to (W^J, S^J) as the **descent system** associated with $J \subseteq S$.

Proposition 2. *Let $u, v \in W^J$ be such that $u^{-1} v \in S^J W_J$. In particular, $u \neq v$. Then either $u < v$ or $v < u$ in the Bruhat order $<$ on W^J.*

For a proof see [16]. These pairs (u, v) give us the edges of the associated polytope $\mathcal{P}_\lambda$. See Sect. 2.4 below for more interpretation relating descent systems and polytopes.

For $s \in S \setminus J$ we let

$$S_s^J = W_J s W_J \cap W^J.$$

Definition 4. Let $w \in W^J$. Define

1. $D_s^J(w) = \{r \in S_s^J \mid wrc < w \text{ for some } c \in W_J\}$, and
2. $A_s^J(w) = \{r \in S_s^J \mid w < wr\}$.

We refer to $D^J(w) = \bigsqcup_{s \in S \setminus J} D_s^J(w)$ as the **descent set** of w relative to J, and $A^J(w) = \sqcup_{s \in S \setminus J} A_s^J(w)$ as the **ascent set** of w relative to J.

By Proposition 2, for any $w \in W^J$, $S^J = D^J(w) \sqcup A^J(w)$. One should think of w as a vertex on the polytope $\mathcal{P}_\lambda$, and S^J as a labelling of the edges at w. The elements of $A^J(w)$ correspond to edges going "up" from w and the elements of $D^J(w)$ correspond to edges going "down" from w. See Sect. 2.4 below for a more elaborate discussion of the geometric underpinnings of the descent system.

Remark 2. Notice that $wrc < w$ for some $c \in W_J$ if and only if $(wr)_0 < w$, where $(wr)_0 \in wrW_J$ is the element of minimal length in wrW_J. It is useful to illustrate the fact that $S^J = D^J(w) \sqcup A^J(w)$, for each $w \in W^J$, by doing some specific calculations.

Definition 5. For each $w \in W^J$ and each $s \in S \setminus J$ define

$$\nu_s(w) = |A_s^J(w)|.$$

We refer to $(W^J, \leq, \{\nu_s\})$ as the **augmented poset** of J. For convenience we let

$$\nu(w) = \sum_{s \in S \setminus J} \nu_s(w).$$

The point here is this. We use $(W^J, \leq, \{\nu_s\})$ to quantify how the underlying torus embedding of $\mathbb{P}(J)$ is involved in calculating the H-polynomial of $\mathbb{P}(J)$.

The following Theorem gives us a clear picture of how these S^J work.

Theorem 3. *Assume that $J \subset S$ is combinatorially smooth. Then*

1. $S^J = \bigsqcup_{s \in S \setminus J} S_s^J$.
2. *Let $s \in S \setminus J$. In case $st = ts$ for all $t \in J$, $S_s^J = \{s\}$. Otherwise,*

$$S_s^J = \{s, t_1 s, t_2 t_1 s, \ldots, t_m \cdots t_2 t_1 s\}$$

where $C = C(s) = \{t_1, t_2, \ldots, t_m\}$, $st_1 \neq t_1 s$ and $t_i t_{i+1} \neq t_{i+1} t_i$ for $i = 1, \ldots m - 1$.

3. $S_s^J \cong \{g \in E_2 \mid ge_1 = e_1 \text{ and } cgc^{-1} = g_s \text{ for some } c \in W_J\}$.

Example 4. Let

$$W = < s_1, \ldots s_n >$$

be the Weyl group of type A_n (so that $W \cong S_{n+1}$), and let

$$J = \{s_3, \ldots s_n\} \subseteq S.$$

If $w \in W^J$ then $w = a_p$, $w = b_q$, or else $w = a_p b_q$. Here $a_p = s_p \cdots s_1$ $(1 \le p \le n)$ and $b_q = s_q \cdots s_2$ $(2 \le q \le n)$. If we adopt the useful convention $a_0 = 1$ and $b_1 = 1$, then we can write

$$W^J = \{a_p b_q \mid 0 \le p \le n \text{ and } 1 \le q \le n\}$$

with uniqueness of decomposition. Let $w = a_p b_q \in W^J$. After some tedious calculation with braid relations and reflections, we obtain that

(a) $A^J_{s_1}(a_p b_q) = \{s_1\}$ if $p < q$.
$A^J_{s_1}(a_p b_q) = \emptyset$ if $q \le p$.
Thus $\nu_{s_1}(a_p b_q) = 1$ if $p < q$ and $\nu_{s_1}(a_p b_q) = 0$ if $q \le p$.

(b) $A^J_{s_2}(a_p b_q) = \{s_m \cdots s_n \mid m > q\}$ if $q < n$.
$A^J_{s_2}(a_p b_q) = \emptyset$ if $q = n$.
Thus $\nu_{s_2}(a_p b_q) = n - q$.

2.4 The Story Told by the Descent System

The following table provides the reader with a check list of the important quantities associated with a descent system. We include also a translation into the corresponding quantities from reductive monoids. Although these monoids are not strictly needed here they were a very enthusiastic participant in the birth of the descent system. See [15] for more details.

Recall that $E = E(\overline{T})$ is the set of idempotents of $\overline{T}$ and $E_i = \{f \in E \mid dim(fT) = i\} \subset E$. As always, $e_1 \in E_1 = E_1(\overline{T})$ is the unique element such that $e_1 B = e_1 B e_1$. For $e \in E_1$ let $v \in W^J$ be the unique element such that $e = v e_1 v^{-1}$. We write $e = e_v$. For $e, f \in E$ we write $e \sim f$ if there exists $w \in W$ such that $wew^{-1} = f$. If $s \in S \setminus J$ let $g_s \in E_2$ be the unique idempotent

Reductive monoid jargon	Bruhat order jargon
$e_1 \in \Lambda_1 = \{e_1\}$	$1 \in W^J$
$e = e_v \in E_1$	The $v \in W^J$ with $e = v e_1 v^{-1}$
$e_v \le e_w$ in E_1, i.e. $e_v B e_w \ne 0$	$w \le v$ in W^J
$E_2 = \{g \in E \mid dim(gT) = 2\}$	$\{(u, v) \in W^J \times W^J \mid$ $u < v$ and $u^{-1}v \in S^J W_J\}$
$\{g \in E_2 \mid gB = gBg\}$	$S \setminus J$
$\{g \in E_2 \mid ge_1 = e_1\}$	$S^J = (W_J(S \setminus J)W_J) \cap W^J$
$\{g \in E_2 \mid ge_1 = e_1, g \sim g_s\}$	$S^J_s = (W_J s W_J) \cap W^J$
$E_2(e_w) = \{g \in E_2 \mid ge_w = e_w\}$	$\{v \in W^J \mid w^{-1}v \in S^J W_J\}$
$\Gamma(e_w) = \{g \in E_2(e_w) \mid ge' = e'$ for some $e' < e_w\}$	$A^J(w) = \{r \in S^J \mid w < wr\}$
$\Gamma_s(e_w) = \Gamma(e_w) \cap \{g \in E_2 \mid g \sim g_s\}$	$A^J_s(w) = \{r \in S^J_s \mid w < wr\}$
$E(\overline{T}) \setminus \{0\}$	$\{(w, I) \mid I \in \Lambda^\times,\ w < ws$ if $s \in I^*\}$

such that $g_s s = s g_s \neq g_s$ and $g_s B = g_s B g_s$ (or what is the same, $g_s B \subseteq B g_s$). Let $\Lambda^\times = \{I \subset S \mid$ no component of I is contained in $J\}$ and for $I \in \Lambda^\times$ let $I^* = I \cup \{t \in J \mid ts = st$ for all $s \in I\}$.

The "picture" here is this. W^J is canonically identified with the set of vertices of the rational polytope $\mathcal{P}_\lambda = Conv(W \cdot \lambda)$, where λ is any highest weight with $\{w \in W \mid w(\lambda) = \lambda\} = W_J$. On the other hand there is a canonical ordering on $E_1 = E_1(\overline{T})$ coming from the associated reductive monoid. Evidently $(E_1, \leq)$ and $(W^J, \leq)$ are anti-isomorphic as posets. Furthermore the set of edges $Edg(\mathcal{P}_\lambda)$ of $\mathcal{P}_\lambda$ is canonically identified with $E_2 = E_2(\overline{T})$. If $g(v,w) = g(w,v) \in Edg(\mathcal{P}_\lambda)$ is the edge of $\mathcal{P}_\lambda$ joining the distinct vertices $v, w \in W^J$ then either $v < w$ or else $w < v$. Given $v \in W^J$, with edges $Edg(v) = \{g \in E_2 \mid g = g(v,w)$ for some $w \in W^J\}$, the question of whether $v < w$ or $w < v$ (for a given $g(v,w) \in Edg(v)$) is coded in the descent system (W^J, S^J).

We can illustrate this with the following "descent picture" of type A_3 with $J = \emptyset$. Then $S^J = S = \{(2134), (1324), (1243)\}$. Consider $\sigma = (1324) \in W = S_4$. The theory tells us that, for $s \in S$, either $s\sigma < \sigma$ or $s\sigma > \sigma$. Simple calculation yields that $(2134)(1324) = (2314)$, $(1324)(1324) = (1234)$ and $(1243)(1324) = (1423)$. Thus, $A(\sigma) = \{(2134), (1243)\}$ and $D(\sigma) = \{(1324)\}$.

The ascents $A(\sigma)$ correspond to edges going "up" from σ and the descents $D(\sigma)$ correspond to edges going "down" from σ. The beauty of this story is that the descent system makes it all work for any proper subset $J \subset S$ such that $\mathcal{P}_\lambda = Conv(W \cdot \lambda)$ is a simple polytope.

These ascent sets give us a combinatorial picture of the cell structure of the associated toric variety $X(J)$. Each T-fixed point $x \in X(J)$ corresponds to a rank-one idempotent $e \in E_1(\overline{T})$; as well as to a certain element σ of W^J. The set of ascents at σ correspond to the one-dimensional T-orbits of the BB-cell C_x that is defined by

$$C_x = \{y \in X(J) \mid lim_{t \to 0} \lambda(t) y = x\}.$$

Evidently, the dimension of C_x is equal to $|A(\sigma)|$, the number of ascents at σ. It is useful to think of $A(\sigma)$ as a combinatorial replacement for the infinitesimal part of the BB-method [1].

2.4.1 The Betti Numbers of $X(J)$ and Descent Systems

Associated with the polytope $\mathcal{P}_\lambda = Conv(W \cdot \lambda)$ is a certain toric variety $X(J)$. This toric variety can also be obtained as the closure of a maximal torus in the simple embedding $\mathbb{P}(J)$ of G. See [16] for more details. Here we explain how the descent system (W^J, S^J) encodes the Betti numbers of $X(J)$.

Definition 6. Let X be a complex algebraic variety. The *Poincaré polynomial* of X is the polynomial $P(X,t)$ with the signed Betti numbers of X as coefficients.

$$P(X,t) = \sum_{i\geq 0}(-1)^i dim_{\mathbb{Q}}[H^i(X;\mathbb{Q})]t^i.$$

Assume that $J \subseteq S$ is combinatorially smooth. In this section we describe the Poincaré polynomial of $X(J)$ in terms of the augmented poset $(W^J, \leq, \{\nu_s\})$.

By assumption J is combinatorially smooth. Thus, by the results of [6], the Betti numbers of $X(J)$ can be calculated by calculating the h-polynomial. Let

$$f_i = \text{the number of codimension } (i+1) - \text{orbits of } X(J)$$

where $i = -1, 0, \dots, n-1$. The **h-polynomial** is defined by insisting that

$$\sum_{i=0}^{n} h_i t^{n-i} = \sum_{i=-1}^{n-1} f_i (t-1)^{n-i-1}.$$

Notice, in particular, that $f_{-1} = 1$. A simple calculation yields that

$$h_k = \sum_{i=0}^{k}(-1)^{k-i}\binom{n-i}{k-i} f_{i-1}.$$

By Theorem 10.8, Remark 10.9 and Proposition 12.11 of [6], the Poincaré polynomial of $X(J)$ is given by

$$P(X(J),t) = \sum_k h_k t^{2k}.$$

On the other hand we can describe the h-polynomial of $X(J)$ in terms of the augmented poset $(W^J, \leq, \{\nu_s\})$. This is the main point of the entire discussion.

Theorem 4 ([16]). *Assume that $X(J)$ is rationally smooth. Then the Poincaré polynomial of $X(J)$ is*

$$P(X(J),t) = \sum_{w\in W^J} t^{2\nu(w)}.$$

Example 5. In this example we consider the root system of type B_l. Let E be a real vector space with orthonormal basis $\{\epsilon_1, \dots, \epsilon_l\}$. Then

$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\} \cup \{\epsilon_i + \epsilon_j \mid i \neq j\} \cup \{\epsilon_i\}$, and
$\Delta = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{l-1} - \epsilon_l, \epsilon_l\} = \{\alpha_1, \dots, \alpha_l\}$.

Let $S = \{s_1, s_2, \dots, s_{l-1}, s_l\}$ be the corresponding set of simple reflections. Here we consider the case

$$J = \{s_1, \dots, s_{l-1}\}.$$

We first calculate $W^J = \{w \in W \mid w(\alpha_i) \in \Phi^+ \text{ for all } 1 \leq i \leq l-1\}$. This leads to a simple calculation and we obtain

$$W^J \cong \bigsqcup_{k=0}^{n} \{(i_1, i_2, \ldots, i_k) \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq l\},$$

via the correspondence

$$(i_1, i_2, \ldots, i_k) \mapsto w$$

where

$$w(\epsilon_v) = \epsilon_{i_v} \text{ for } 1 \leq v \leq k,$$

and

$$w(\epsilon_{k+v}) = -\epsilon_{j_v} \text{ for } 1 \leq v \leq l-k,$$

where $l \geq j_1 > j_2 > \ldots > j_{l-k} \geq 1$ (so that $\{1, \ldots, l\} = \{i_1, i_2, \ldots, i_k\} \sqcup \{j_1, j_2, \ldots, j_{l-k}\}$).

Let

$$\lambda = 2\lambda_l = \epsilon_1 + \ldots + \epsilon_l = \alpha_1 + 2\alpha_2 \ldots + l\alpha_l.$$

By Proposition 4.1 of [9], for $v, w \in W^J$, $w \leq v$ if and only if $w(\lambda) - v(\lambda)$ is a sum of positive roots. A simple calculation yields that

$$w \leq v \text{ if and only if } m_w(i) \leq m_v(i) \text{ for all } i = 1, \ldots, l,$$

where

$$m_w(i) = |\{j \leq i \mid w(\epsilon_v) = -\epsilon_j \text{ for some } v = 1, \ldots, l\}|.$$

Let $M(w) = \{j \mid w(\epsilon_v) = -\epsilon_j \text{ for some } v = 1, \ldots, l\}$. If $M(w) \subset M(v)$ then also $M(v)^c \subset M(w)^c$ (complement of sets) and we obtain that

$$w(\lambda) - v(\lambda) = A + B$$

where

$A = \sum_{j \in M(w)^c} (\alpha_j + \alpha_{j+1} + \ldots + \alpha_l) - \sum_{j \in M(v)^c} (\alpha_j + \alpha_{j+1} + \ldots + \alpha_l)$, and
$B = \sum_{j \in M(v)} (\alpha_j + \alpha_{j+1} + \ldots + \alpha_l) - \sum_{j \in M(w)} (\alpha_j + \alpha_{j+1} + \ldots + \alpha_l)$.

Thus $M(w) \subset M(v)$ implies that $w \leq v$, at least for elements of W^J.

We now wish to calculate $A^J(w)$ for each $w \in W^J$. Recall that

$$A^J(w) = \{r \in S^J \mid w < wr\}$$

and

$$S^J = \{s_l, s_{l-1}s_l, \ldots, s_i s_{i-1} \cdots s_{l-1}s_l, \ldots, s_1 \cdots s_l\}.$$

Let $w \in W^J$ correspond, as above, to $i_1 < \ldots < i_k$ and $j_1 > \ldots > j_{l-k}$. Let $r_i = s_i \cdots s_l \in S^J$. One checks that

$$M(wr) = M(w) \cup \{j\} \text{ if } i \leq k,$$

and

$$M(wr) = M(w) \setminus \{j\} \text{ if } i > k.$$

Hence by our previous calculations $w < wr_i$ if and only if $i \leq k$. Thus we obtain

$$A^J(w) = \{s_k \cdots s_l, \ldots, s_1 \cdots s_l\} = \{r \in S^J \mid w < wr\}.$$

Thus if $w \in W^J$ we obtain

$$\nu(w) = |\{j \mid w(\epsilon_v) = \epsilon_j \text{ for some } v\}|.$$

We can use this information to calculate the Poincaré polynomial of $X(J)$. An easy calculation yields

$$P(X(J), t) = \sum_{w \in W^J} t^{2\nu(w)} = \sum_{A \subset \{1,\ldots,l\}} t^{2|A|} = (1 + t^2)^l.$$

2.4.2 The Betti Numbers of $X(J)$ and Eulerian Polynomials

Consider the root system of type A_n and let $J(k,n) = \{s_{n-k+1}, \ldots, s_n\}$ for $1 \leq k \leq n$. In [8] Golubitsky found a formula for $P_{X(J(k,n))}(t^{1/2})$ in terms of Eulerian polynomials. Let $h_k(t)$ denote the h-polynomial of $X(J(k,n))$ (so that $P_{X(J(k,n))}(t^{1/2}) = h_k(t)$). Finally let

$$E_{n+1}(t) = \sum_{I \subseteq S} \frac{(n+1)!}{|W_I|}(t-1)^{|I|}$$

be the $(n+1)$-*Eulerian polynomial*. Notice that $E_{n+1}(t)$ is the h-polynomial of $X(\emptyset)$. In Theorem 5 of [8] the author determines the following recursion formula.

Theorem 5. *Let h and E be as above. Then*

$$h_k(t) = h_{k-1}(t) - \binom{n+1}{k+1}(t^k + \ldots + t)E_{n-k}(t).$$

It would be interesting to find the extent to which the Poincaré polynomial of $X(J)$ can be expressed in terms of generalized Eulerian Polynomials [3].

2.5 The List

One can list all possible subsets $J \subseteq S$ that are combinatorially smooth. We do this according to the type of the underlying simple group. The numbering of the elements of S is as follows. For types A_n, B_n, C_n, F_4, and G_2 it is the usual numbering. In these cases the end nodes are s_1 and s_n. For type E_6 the end nodes are s_1, s_5 and s_6 with $s_3s_6 \neq s_6s_3$. For type E_7 the end nodes are s_1, s_6 and s_7 with $s_4s_7 \neq s_7s_4$. For type E_8 the end nodes are s_1, s_7 and s_8 with $s_5s_8 \neq s_8s_5$. In each case of type E_n, the nodes corresponding to $s_1, s_2, \ldots, s_{n-1}$ determine the unique subdiagram of type A_{n-1}. For type D_n the end nodes are s_1, s_{n-1} and s_n. The two subdiagrams of D_n, of type A_{n-1}, correspond to the subsets $\{s_1, s_2, \ldots, s_{n-2}, s_{n-1}\}$ and $\{s_1, s_2, \ldots, s_{n-2}, s_n\}$ of S.

The reader is referred to [15] for the details. The key ingredient here is Theorem 2.

1. A_1.

   (a) $J = \phi$.

   $A_n, n \geq 2$. Let $S = \{s_1, \ldots, s_n\}$.

   (a) $J = \phi$.
   (b) $J = \{s_1, \ldots, s_i\}$, $1 \leq i < n$.
   (c) $J = \{s_j, \ldots, s_n\}$, $1 < j \leq n$.
   (d) $J = \{s_1, \ldots, s_i, s_j, \ldots s_n\}$, $1 \leq i, i \leq j - 3$ and $j \leq n$.

2. B_2.

   (a) $J = \phi$.
   (b) $J = \{s_1\}$.
   (c) $J = \{s_2\}$.

   $B_n, n \geq 3$. Let $S = \{s_1, \ldots, s_n\}$, α_n short.

   (a) $J = \phi$.
   (b) $J = \{s_1, \ldots, s_i\}$, $1 \leq i < n$.
   (c) $J = \{s_n\}$.
   (d) $J = \{s_1, \ldots, s_i, s_n\}$, $1 \leq i$ and $i \leq n - 3$.

3. $C_n, n \geq 3$. Let $S = \{s_1, \ldots, s_n\}$, α_n long.
   (a) $J = \phi$.
   (b) $J = \{s_1, \ldots, s_i\}$, $1 \leq i < n$.
   (c) $J = \{s_n\}$.
   (d) $J = \{s_1, \ldots, s_i, s_n\}$, $1 \leq i$ and $i \leq n-3$.
4. $D_n, n \geq 4$. Let $S = \{s_1, \ldots s_{n-2}, s_{n-1}, s_n\}$.
   (a) $J = \phi$.
   (b) $J = \{s_1, \ldots, s_i\}$, $1 \leq i \leq n-3$.
   (c) $J = \{s_{n-1}\}$.
   (d) $J = \{s_n\}$.
   (e) $J = \{s_1, \ldots, s_i, s_{n-1}\}$, $1 \leq i \leq n-4$.
   (f) $J = \{s_1, \ldots, s_i, s_n\}$, $1 \leq i \leq n-4$.
5. E_6. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$.
   (a) $J = \phi$.
   (b) $J = \{s_1\}$ or $\{s_1, s_2\}$.
   (c) $J = \{s_5\}$ or $\{s_4, s_5\}$.
   (d) $J = \{s_6\}$.
   (e) $J = \{s_1, s_5\}, \{s_1, s_2, s_5\}$ or $\{s_1, s_4, s_5\}$.
   (f) $J = \{s_1, s_6\}$.
   (g) $J = \{s_5, s_6\}$
   (h) $J = \{s_1, s_5, s_6\}$.
6. E_7. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$.
   (a) $J = \phi$.
   (b) $J = \{s_1\}, \{s_1, s_2\}$ or $\{s_1, s_2, s_3\}$.
   (c) $J = \{s_6\}$ or $\{s_5, s_6\}$.
   (d) $J = \{s_7\}$.
   (e) $J = \{s_1, s_6\}, \{s_1, s_2, s_6\}, \{s_1, s_2, s_3, s_6\}, \{s_1, s_5, s_6\}$, or $\{s_1, s_2, s_5, s_6\}$.
   (f) $J = \{s_6, s_7\}$.
   (g) $J = \{s_1, s_7\}$ or $\{s_1, s_2, s_7\}$.
   (h) $J = \{s_1, s_6, s_7\}, \{s_1, s_2, s_6, s_7\}$.
7. E_8. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$.
   (a) $J = \phi$.
   (b) $J = \{s_1\}, \{s_1, s_2\}, \{s_1, s_2, s_3\}$ or $\{s_1, s_2, s_3, s_4\}$.
   (c) $J = \{s_7\}$ or $\{s_6, s_7\}$.
   (d) $J = \{s_8\}$.
   (e) $J = \{s_1, s_7\}, \{s_1, s_2, s_7\}, \{s_1, s_2, s_3, s_7\}, \{s_1, s_2, s_3, s_4, s_7\}$,
   $\{s_1, s_6, s_7\}, \{s_1, s_2, s_6, s_7\}, \{s_1, s_2, s_3, s_6, s_7\}$ or $\{s_1, s_2, s_5, s_6\}$.
   (f) $J = \{s_7, s_8\}$.
   (g) $J = \{s_1, s_8\}, \{s_1, s_2, s_8\}$ or $\{s_1, s_2, s_3, s_8\}$.
   (h) $J = \{s_1, s_7, s_8\}, \{s_1, s_2, s_7, s_8\}$.

8. F_4. Let $S = \{s_1, s_2, s_3, s_4\}$.
   (a) $J = \phi$.
   (b) $J = \{s_1\}$ or $\{s_1, s_2\}$.
   (c) $J = \{s_4\}$ or $\{s_3, s_4\}$.
   (d) $J = \{s_1, s_4\}$.
9. G_2. Let $S = \{s_1, s_2\}$.
   (a) $J = \phi$.
   (b) $J = \{s_1\}$.
   (c) $J = \{s_2\}$.

3 The Poincaré Polynomial of a Simple Embedding

Recall from Theorem 1 that there is a certain quantity $m(w)$ (the "je ne sais quoi") that is missing in the formula for the Poincaré polynomial of a simple embedding. In this section we put it all together in terms of the descent system.

3.1 That Certain "Je ne sais quoi"

We are finally in position to state the main result. Let M be a $\mathcal{J}$-irreducible monoid such that $M \setminus \{0\}$ is rationally smooth. In this section we obtain the sought-after H-polynomial of M in terms of the augmented poset $(E_1, \leq, \{\nu_s\})$.

Let $J \subset S$ be combinatorially smooth and let $s \in S \setminus J$, $w \in W^J$. We define

(a) $\delta(s) = |C(s)| + 1$, and
(b) $\nu_s(w) = |A_s(w)|$ where $e = we_1w^{-1}$.

The subset $C(s) \subseteq J$ is the unique connected component of J containing the unique element $t \in J$ such that $st \neq ts$ (otherwise $C(s) = \emptyset$ if there is no such t). Hence $\delta(s) = 1$ if and only if $st = ts$ for all $t \in J$. Let $w_0 \in W^J$ be the longest element (so that $l(w_0) = dim(U_e)$).

Theorem 6. *The H-polynomial $H(M)$ of M is given by*

$$H(M) = \left(\sum_{w \in W^J} t^{l(w_0) - l(w) + m(w)} \right) H(J)$$

where $m(w) = \sum_{s \in S \setminus J} \delta(s)\nu_s(w)$, and $H(J) = \sum_{v \in W^J} t^{l(v)}$ the H-polynomial of G/P_J. Thus $P_X(t^{1/2}) = H(M)$ where $X = (M \setminus \{0\})/K^*$.

Proof. Now $X = \bigsqcup_{e \in E_1} X(e)$ where

$$X(e) = \{[y] \in X \mid eBy = eBey \subseteq eG\}.$$

We write $X = \bigsqcup_{w \in W^J} X(w)$, where we denote $X(w) = X(e)$ if $we_1w^{-1} = e$. Thus we have, for each $w \in W^J$,

$$\pi : X(w) \to (eG)/K^*,$$

and $\pi^{-1}(K^*e) = Bf_eM_ek^*$. Here f_e is a certain idempotent. Notice how this fibration "mirrors" the factorization of $H(M)$. Since eG is isomorphic to a cone on the projective variety $P_J\backslash G$, it has the usual Bruhat decomposition. It turns out that $X(w)/K^*$ has a cell decomposition

$$X(w) = \bigsqcup_{v \in W^J} E_v$$

where, for each $v \in W^J$, $dim(E_v) = l(v) + dim(BC_e^*) - 1$. But from [14], $Bf_eM_ek^* \cong (U_e \cap B_u) \times f_eM_ek^*$. Hence, we obtain that

$$dim(Bf_eM_ek^*) - 1 = dim(U_e \cap B_u) + dim(f_eM_e) = l(w_0) - l(w) + dim(f_eM_e).$$

By counting up all these cells our preliminary calculation of $H(M)$, using the formula for $H(M)$ from Theorem 1, is

$$H(M) = \sum_{w \in W^J} \left(t^{l(w_0) - l(w) + dim(f_eM_e)} \sum_{v \in W^J} t^{l(v)} \right)$$

where f_e is a certain idempotent of $\overline{T}_e$. Thus it remains for us to calculate $dim(f_eM_e)$ for each $e \in E_1$. This requires some calculation related to the fact that M_e is essentially a product of matrix monoids of size $\delta(s) \times \delta(s)$ (for each $s \in S \setminus J$) and $f = f_e$ breaks up into a "sum" of idempotents $\{f_s\}$ such that $rank(f_s) - 1 = \nu_s(e)$ (where the rank is in M not in M_e). In any case we obtain

$$\nu(e) = dim(fM_e) = \sum_{s \in S \setminus J} dim(fM_e) = \sum_{s \in S \setminus J} \delta(s)\nu_s(e).$$

By substituting this expression (for the value of $dim(f_eM_e)$) into our preliminary formula for $H(M)$ (and collecting terms appropriately) we obtain the desired result. See Corollary 6.5 and Theorem 6.6 of [18] for more details.

Remark 3. We already know from [14] that

$$\nu(e) = dim(f_eM_e)$$

is the "obscure object of desire" in the pursuit of the Betti numbers of rationally smooth group embeddings. See Theorem 1 above for the basic idea, or Definition 5.4 and Theorems 5.1 and 5.5 of [14] for all the details.

Most of the results about descent systems [15, 16] were developed for the purpose of calculating $dim(f_e M_e)$. But one should not overlook the fact that descent systems are a natural combinatorial part of any Coxeter group (W, S).

3.2 Examples

In this section we use Theorem 6 to calculate the H-polynomial of four classes of examples. In each case this boils down to finding W^J and calculating $l(w)$ and $m(w)$ for each $w \in W^J$. Complete details of these examples can be found in [15, 16, 18].

Example 6. Let $M = M_{n+1}(K)$. Then M is $\mathcal{J}$-irreducible of type $J \subset S$, where $J = \{s_2, s_3, \ldots, s_n\}$ and $S = \{s_1, s_2, \ldots, s_n\} \subset W$ is of type A_n $(n \geq 1)$. In this example

$S^J = \{s_1, s_2 s_1, s_3 s_2 s_1, \ldots, s_n \cdots s_1\}$, and
$W^J = S^J \sqcup \{1\}$.

Write $a_i = s_i \cdots s_1$ if $i > 1$, and $a_0 = 1$. An elementary calculation yields

$S \setminus J = \{s_1\}$,
$l(a_i) = i$,
$w_0 = s_n \cdots s_1$,
$\delta(s_1) = n$,
$\nu_{s_1}(a_i) = n - i$,
$P(J) = \sum_{i=0}^{n} t^{2i}$, and
$X = \mathbb{P}(M) = \mathbb{P}^{(n+1)^2-1}(K)$, projective space.

Another elementary calculation (using Theorem 6) then yields

$$P_X(t^{1/2}) = \left(\sum_{i=0}^{n} t^{(n-i)(n+1)}\right)\left(\sum_{i=0}^{n} t^i\right) = \sum_{i=0}^{(n+1)^2-1} t^i.$$

Example 7. A *canonical monoid* M is a $\mathcal{J}$-irreducible monoid of type $J = \phi$. It follows from Theorem 2 that if M is a canonical monoid then $M \setminus \{0\}$ is rationally smooth. Let M be a canonical monoid with unit group G, and let G_0 be the commutator subgroup of G. If G_0 is a group of of adjoint type, then $\mathbb{P}(M)$ is the canonical compactification of G_0. The augmented poset in this example is $(W, \leq, \{\nu_s\}_{s \in S})$. $(W, \leq)$ is the Weyl group with the usual Bruhat order and

(i) $\nu_s(w) = 1$ if $w < ws$,
(ii) $\nu_s(w) = 0$ if $w > ws$,
(iii) $\delta(s) = 1$ for all $s \in S = S \setminus J$, and (therefore)
(iv) $m(w) = \sum_{s \in S} \delta(s)\nu_s(w) = |\{s \in S \mid w < ws\}|$.

Thus

$$P_X(t^{1/2}) = \left(\sum_{w\in W} t^{l(w_0)-l(w)+|I_w|}\right)\left(\sum_{v\in W} t^{l(v)}\right).$$

where $I_w = \{s \in S \mid w < ws\}$. Observe how $H(M)$ is determined by $(W, \leq, \{\nu_s\}_{s\in S})$.

The H-polynomial of the M is related to the Poincaré polynomial of $X = \mathbb{P}(M)$ by the rule $H(M)(t) = P_X(t^{1/2})$. The Poincaré polynomial of the canonical compactification was originally obtained by DeConcini and Procesi in [7]. It was that calculation that motivated many of the results of this paper.

Example 8. In this example we illustrate Theorem 6 by calculating the Poincaré polynomial of $\mathbb{P}(M)$ where M is $\mathcal{J}$-irreducible of type $J \subset S$, where $S = \{s_1, s_2, \ldots, s_n\} \subset W$ is of type A_n $(n \geq 2)$ and $J = J_n = \{s_3, s_4, \ldots, s_n\}$. We shall refer the reader to Example 4.6 of [16] for some of the details.

If $w \in W_n^J$ we can write $w = a_p b_q$ where $a_p = s_p \cdots s_1$ $(1 \leq p \leq n)$ and $b_q = s_q \cdots s_2$ $(2 \leq q \leq n)$. We also adopt the peculiar but useful convention $a_0 = 1$ and $b_1 = 1$. Thus

$$W_n^J = \{a_p b_q \mid 0 \leq p \leq n \text{ and } 1 \leq q \leq n\}$$

with uniqueness of decomposition.

Now $S \setminus J = \{s_1, s_2\}$ so that $C(s_1) = \phi$ and $C(s_2) = \{s_3, \ldots, s_n\}$. Thus,

(i) $\delta(s_1) = 1$, and
(ii) $\delta(s_2) = (n-2) + 1 = n - 1$.

From Example 4 above,

(i) $\nu_{s_1}(a_p b_q) = 1$ if $p < q$ and
$\nu_{s_1}(a_p b_q) = 0$ if $p \geq q$.
(ii) $\nu_{s_2}(a_p b_q) = n - q$.

Thus, by definition,

(i) $m(a_p b_q) = (n-1)(n-q) + 1$ if $p < q$ and
(ii) $m(a_p b_q) = (n-1)(n-q)$ if $p \geq q$.

Finally,

(i) $l(a_p b_q) = p + q - 1$, and
(ii) $a_n b_n \in W^J$ is the longest element.

Thus, for $w = a_p b_q \in W^J$, we obtain by elementary calculation that

$$l(w_0) - l(w) + m(w) = n - p + n(n-q) + \epsilon$$

where $\epsilon = 1$ if $0 \le p < q \le n$, and $\epsilon = 0$ if $n \ge p \ge q \ge 1$. Thus

$$\sum_{w\in W^J} t^{l(w_0)-l(w)+m(w)} = \sum_{0\le p<q\le n} t^{n-p+n(n-q)+1} + \sum_{n\ge p\ge q\ge 1} t^{n-p+n(n-q)}$$

The other factor here is

$$H(J) = \sum_{w\in W^J} t^{l(w)} = \sum_{i=1}^{n} i(t^{i-1} + t^{2n-i}).$$

Finally we obtain

$$P_X(t^{1/2}) = \left(\sum_{0\le p<q\le n} t^{n-p+n(n-q)+1} + \sum_{n\ge p\ge q\ge 1} t^{n-p+n(n-q)}\right)\left(\sum_{i=1}^{n} i(t^{i-1} + t^{2n-i})\right).$$

Example 9. In this example we consider the root system of type B_l. Let E be a real vector space with orthonormal basis $\{\epsilon_1, \dots, \epsilon_l\}$. Then

$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\} \cup \{\epsilon_i + \epsilon_j \mid i \ne j\} \cup \{\epsilon_i\}$, and
$\Delta = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{l-1} - \epsilon_l, \epsilon_l\} = \{\alpha_1, \dots, \alpha_l\}$.

Let $S = \{s_1, s_2, \dots, s_{l-1}, s_l\}$ be the corresponding set of simple reflections. Here we consider the case the $\mathcal{J}$-irreducible monoid M of type

$$J = \{s_1, \dots, s_{l-1}\} \subset S.$$

We make the following identification.

$$W^J \cong \{1 \le i_1 < i_2 < \dots < i_k \le l\}$$

as follows. Given such a sequence, $1 \le i_1 < i_2 < \dots < i_k \le l$, we define

$$w(\epsilon_v) = \epsilon_{i_v} \text{ for } 1 \le v \le k,$$

and

$$w(\epsilon_{k+v}) = -\epsilon_{j_v} \text{ for } 1 \le v \le l-k,$$

where $l \ge j_1 > j_2 > \dots > j_{l-k} \ge 1$ (so that $\{1, \dots, l\} = \{i_1, i_2, \dots, i_k\} \sqcup \{j_1, j_2, \dots, j_{l-k}\}$). One can check that $w \in W^J$ and that, conversely, any element of W^J is of this form.

With these identifications we let $w \in W^J$. We now recall that

$$A^J(w) = \{r \in S^J \mid w < wr\}$$

and that

$$S^J = \{s_1 \cdots s_l, s_2 \cdots s_l, \ldots, s_i \cdots s_l, \ldots, s_{l-1}s_l, s_l\}.$$

Let $w \in W^J$ correspond, as above, to $i_1 < \ldots < i_k$ and $j_1 > \ldots > j_{l-k}$. Let $r_i = s_i \cdots s_l \in S^J$. By the calculations of [16], $w < wr_i$ if and only if $i \leq k$. Thus we obtain

$$A^J(w) = \{s_1 \cdots s_l, \ldots, s_k \cdots s_l\} = \{r \in S^J \mid w < wr\}.$$

Now we can use Theorem 6 above to obtain the H-polynomial of M. Let us first assemble the relevant information.

1. $S \setminus J = \{s_l\}$.
2. $\delta(s_l) = |C(s_l)| + 1 = |\{s_1, \ldots, s_{l-1}\}| + 1 = l$.
3. $\nu(w) = \nu_{s_l}(w)$.
4. If $w \in W^J$ then $\nu(w) = k$ where

$$w \longleftrightarrow \{1 \leq i_1 < i_2 < \ldots < i_k \leq l\}$$

   as above.
5. $m(w) = l\nu(w) = kl$.
6. $l(w_0) - l(w) = \sum_{i \in M'(w)} i$ where $M'(w) = \{i \mid w(\epsilon_j) = \epsilon_i \text{ for some } j\} = \{i_1, i_2, \ldots, i_k\}$, and where $w_0 \in W^J$ is the longest element (notice that $l(w_0) = l(l+1)/2$).

Collecting terms we obtain that, for $w \in W^J$,

$$l(w_0) - l(w) + m(w) = \left(\sum_{i \in M'(w)} i\right) + l|M'(w)| = \sum_{i \in M'(w)} (i + l).$$

After recalling some elementary generating functions, and applying Theorem 6, we obtain that

$$P_X(t^{1/2}) = \left(\Pi_{k=1}^l (1 + t^{k+l})\right) \left(\Pi_{k=1}^l (1 + t^k)\right).$$

The $\Pi_{k=1}^l (1 + t^k)$ factor here is $H(G/P_J) = \sum_{v \in W^J} t^{l(v)}$ and the $\Pi_{k=1}^l (1 + t^{k+l})$ factor is $\sum_{w \in W^J} t^{l(w_0) - l(w) + m(w)}$.

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SL_2-Regular Subvarieties of Complete Quadrics

Mahir Bilen Can and Michael Joyce

We dedicate our work with admiration to Professors Mohan Putcha and Lex Renner on the occasion of their 60th birthday.

Abstract We determine SL_n-stable, SL_2-regular subvarieties of the variety of complete quadrics. We use the methods of Akyıldız and Carrell given in Proc Natl Acad Sci USA 86(11):3934–3937, 1989 to give a factorization of Poincaré polynomials of these regular subvarieties.

Keywords Complete quadrics • SL_2-regular varieties • Kostant-Macdonald identity

Subject Classifications: 14L30, 05E05, 05E10

1 Introduction

The study of the variety $\mathscr{X} := \mathscr{X}_n$ of $(n-2)$-dimensional complete quadrics, a completion of the variety of smooth quadric hypersurfaces in $\mathbb{P}^{n-1}(\mathbb{C})$, dates back to the nineteenth century, where it was used to answer fundamental questions in enumerative geometry. Complete quadrics received renewed attention in the second half of the twentieth century for two primary reasons: (1) the toolkit of modern algebraic geometry made it possible to develop Schubert calculus rigorously, thereby addressing Hilbert's 15-th problem [12, 14]; and (2) the interpretation of $\mathscr{X}$ by De Concini and Procesi as an example of a wonderful embedding [9].

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Identifying quadrics with the symmetric matrices defining them (up to scaling), the change of variables action of $\mathrm{SL}_n := \mathrm{SL}_n(\mathbb{C})$ on smooth quadrics corresponds to the action on symmetric matrices given by $g \cdot A = \theta(g)Ag^{-1}$, for $g \in \mathrm{SL}_n$, A a non-degenerate symmetric matrix, and θ the involution $\theta(g) = (g^\top)^{-1}$. Modulo the center of SL_n, the variety of smooth quadric hypersurfaces can be identified as $\mathrm{SL}_n/\mathrm{SO}_n$ since $\mathrm{SO}_n \subset \mathrm{SL}_n$ is the stabilizer of the smooth quadric defined by the identity matrix.

Any semi-simple, simply connected complex algebraic group G equipped with an involution σ has a canonical wonderful embedding X. Letting H denote the normalizer of G^σ, X is a smooth projective G-variety containing an open G-orbit isomorphic to G/H and whose boundary $X-(G/H)$ is a union of smooth G-stable divisors with smooth transversal intersections. Boundary divisors are canonically indexed by the elements of a certain subset Δ of a root system associated to (G,σ). Each G-orbit in X corresponds to a subset $I \subseteq \Delta$. The Zariski closure of the orbit is smooth and is equal to the transverse intersection of the boundary divisors corresponding to the elements of I.

The wonderful embedding of the symmetric pair (SL_n,θ) above is $\mathscr{X}$, where $\sigma(A) = (A^\top)^{-1}$. In this case, Δ is the set of simple roots associated to SL_n relative to its maximal torus of diagonal matrices contained in the Borel subgroup $B \subseteq \mathrm{SL}_n$ of upper triangular matrices, and is canonically identified with the set $[n-1] = \{1,2,\dots n-1\}$.

The study of cohomology theories of wonderful embeddings, initiated in [9], has been carried out through several different approaches. Poincaré polynomials have been computed in [10, 18, 23], while the structure of the (equivariant) cohomology rings have been described in [5, 7, 11, 15, 24].

We study the cohomology of SL_n-stable subvarieties of $\mathscr{X}$ that are SL_2-regular. An SL_2-regular variety is one which admits an action of SL_2 such that any one-dimensional unipotent subgroup of SL_2 fixes a single point. Akyıldız and Carrell developed a remarkable approach for studying the cohomology algebra $H^*(X;\mathbb{C})$ of such varieties [1–3]. Their method, when applied to flag varieties, has important representation theoretic consequences.

Let us briefly describe the contents of this paper. Section 2 sets some notation and recalls some basic facts about SL_2-regular varieties, wonderful embeddings, and complete quadrics. In Sect. 3, we precisely identify which SL_n-stable subvarieties of $\mathscr{X}$ are SL_2-regular. When combined with an earlier result of Strickland [23], our result takes an especially nice form: an SL_n-stable subvariety of $\mathscr{X}$ is SL_2-regular if and only if the dense orbit of the subvariety contains a fixed point of the maximal torus of SL_n.

In Sect. 4, we apply the machinery developed by Akyıldız and Carrell to compute the Poincaré polynomial

$$P_X(t) := \sum_{i=0}^{2\dim X} \dim_{\mathbb{C}} H^i(X;\mathbb{C})t^i$$

of a SL_2-regular, SL_n-stable subvariety X of $\mathscr{X}$. If $I \subset [n-1]$ is the corresponding set of simple roots, then

$$P_X(t) = \left(\frac{1-t^6}{1-t^4}\right)^{|I|} \prod_{k=1}^{n} \frac{1-t^{2k}}{1-t^2}.$$

This factorization of the Poincaré polynomial should be viewed as an extension of the results of [2] on the famous identity of Kostant and Macdonald [13, 16]

$$\sum_{\pi \in S_n} q^{\ell(\pi)} = \prod_{k=1}^{n} \frac{1-q^k}{1-q},$$

recognizing the left-hand side as the Poincaré polynomial of the complete flag variety, evaluated at $q = t^2$.

2 Preliminaries

2.1 Notation and Conventions

All varieties are defined over $\mathbb{C}$ and all algebraic groups are complex algebraic groups. Throughout, n is a fixed integer, and $\mathscr{X} := \mathscr{X}_n$ denotes the SL_n-variety of $(n-2)$-dimensional complete quadrics, which is reviewed in Sect. 2.4. The set $\{1, 2, \ldots, m\}$ is denoted $[m]$ and if $I \subset [m]$, then its complement is denoted I^c. If I and K are sets, then $I - K$ denotes the set complement $\{a \in I : a \notin K\}$. The transpose of a matrix A is denoted $A^\top$.

We denote by $B' \subset SL_2$ the subgroup of upper triangular matrices, with its usual decomposition $B' = T'U'$ into a semidirect product of a maximal torus T' consisting of the diagonal matrices and the unipotent radical U' of B'. Let $\mathfrak{b}', \mathfrak{t}', \mathfrak{u}'$ denote their Lie algebras.

Finally, the symmetric group of permutations of $[n]$ is denoted by S_n, and for $w \in S_n$, $\ell(w)$ denotes $\ell(w) = |\{(i, j) : 1 \leq i < j \leq n,\ w(i) > w(j)\}|$.

2.2 SL_2-Regular Varieties

Let X be a smooth projective variety over $\mathbb{C}$ on which an algebraic torus T acts with finitely many fixed points. Let T' be a one-parameter subgroup with $X^{T'} = X^T$. For $p \in X^{T'}$ define the sets $C_p^+ = \{y \in X : \lim_{t\to 0} t \cdot y = p, t \in T'\}$ and $C_p^- = \{y \in X : \lim_{t\to\infty} t \cdot y = p, t \in T'\}$, called the *plus cell* and *minus cell* of p, respectively.

Theorem 1 ([4]). *Let X, T and T' be as above. Then*

1. *C_p^+ and C_p^- are locally closed subvarieties isomorphic to affine space;*
2. *If T_pX is the tangent space of X at p, then C_p^+ (resp., C_p^-) is T'-equivariantly isomorphic to the subspace T_p^+X (resp., T_p^-X) of T_pX spanned by the positive (resp., negative) weight spaces of the action of T' on T_pX.*

As a consequence of Theorem 1, there exists a filtration

$$X^{T'} = V_0 \subset V_1 \subset \cdots \subset V_n = X, \qquad n = \dim X,$$

of closed subsets such that for each $i = 1, \ldots, n$, $V_i - V_{i-1}$ is the disjoint union of the plus (resp., minus) cells in X of (complex) dimension i. It follows that the odd-dimensional integral cohomology groups of X vanish, the even-dimensional integral cohomology groups of X are free, and the Poincaré polynomial $P_X(t) := \sum_{i=0}^{2n} \dim_{\mathbb{C}} H^i(X;\mathbb{C})t^i$ of X is given by

$$P_X(t) = \sum_{p \in X^{T'}} t^{2\dim C_p^+} = \sum_{p \in X^{T'}} t^{2\dim C_p^-}.$$

Because the odd-dimensional cohomology vanishes, we will prefer to study the q-Poincaré polynomial, $P_X(q) = P_X(t^2)$.

Now suppose that X has an action of SL_2. The action of U' gives rise to a vector field V on X. Note that $p \in X$ is fixed by U' if and only if $V(p) \in T_pX$ is zero. The SL_2-variety X is said to be *SL_2-regular* if there is a unique U'-fixed point on X. An SL_2-regular variety has only finitely many T'-fixed points [2].

A smooth projective G-variety X is *SL_2-regular* if there exists an injective homomorphism $\phi : \mathrm{SL}_2 \hookrightarrow G$ such that the induced action makes X into an SL_2-regular SL_2-variety. Recall that the Jacobson-Morozov Theorem [8, Section 5.3] implies that when G is simply-connected (the only case we consider) such ϕ are determined by specifying $h \in \mathfrak{t}'$ and $e \in \mathfrak{u}'$ satisfying $[h, e] = 2e$. As an abuse of notation, we will often identify B', T', U' (resp., $\mathfrak{b}', \mathfrak{t}', \mathfrak{u}', h, e$) with their images under ϕ (resp., $d\phi$).

Let p be the unique U'-fixed point of the SL_2-regular variety X. The minus cell C_p^- is open in X [1], and hence, the weights of T' on T_pX are all negative. Let $x_1, \ldots, x_n$ be a T'-equivariant basis for the cotangent space T_p^*X of X. Then the coordinate ring $\mathbb{C}[C_p^-] = \mathbb{C}[x_1, \ldots, x_n]$ is a graded algebra with $\deg x_i > 0$. Viewing the vector field V associated to the U' action as a derivation of $\mathbb{C}[x_1, \ldots, x_n]$, $V(x_i)$ is homogeneous of degree $\deg x_i + 2$ and $V(x_1), V(x_2), \ldots V(x_n)$ is a regular sequence in $\mathbb{C}[x_1, \ldots, x_n]$ [2].

Theorem 2 ([1, Proposition 1.1]). *Let Z be the zero scheme of the vector field V, supported at the point $p \in X$, and let $I(Z) = (V(x_1), \ldots, V(x_n)) \subset \mathbb{C}[x_1, \ldots, x_n]$ be the ideal of Z, graded as above. Then there exists a degree-doubling isomorphism of graded algebras $\mathbb{C}[C_p^-]/I(Z) \cong H^*(X;\mathbb{C})$.*

Consequently, the q-Poincaré polynomial of X is given by

$$P_X(q) = \prod_{i=1}^{n} \frac{1 - q^{\deg x_i + 1}}{1 - q^{\deg x_i}}.$$

2.3 Wonderful Embeddings

We briefly review the theory of wonderful embeddings, referring the reader to [9] and [17] for more details.

Definition 1. Let X be a smooth, complete G-variety containing a dense open homogeneous subvariety X_0. Then X is a *wonderful embedding* of X_0 if

1. $X - X_0$ is the union of finitely many G-stable smooth codimension one subvarieties X_i for $i = 1, 2, \ldots, r$;
2. For any $I \subset [r]$, the intersection $X^I := \cap_{i \notin I} X_i$ is smooth and transverse;
3. Every irreducible G-stable subvariety has the form X^I for some $I \subset [r]$.

If a wonderful embedding of X_0 exists, it is unique up to G-equivariant isomorphism.

The G-orbits of X are also parameterized by the sets $I \subset [r]$. We denote by $\mathscr{O}^I$ the unique dense G-orbit in X^I. There is a fundamental decomposition

$$X^I = \bigsqcup_{K \subset I} \mathscr{O}^K. \tag{1}$$

Note that X contains a unique closed orbit Z, corresponding to $I = \emptyset$.

Remark 1. Fix a Borel subgroup $B \subset G$ and let B^- denote the opposite Borel subgroup of B. Fix a maximal torus $T \subset B$ and let $p \in Z$ be the unique B^--fixed point. The *spherical roots* of X are the T-weights of T_pX/T_pZ and the set $[r]$ in Definition 1 can be intrinsically identified with the set of spherical roots of X.

2.4 Complete Quadrics

There is a vast literature on the variety $\mathscr{X}$ of complete quadrics. See [14] for a survey, as well as [9] and [11] for recent work on the cohomology ring of $\mathscr{X}$. We briefly recall the relevant definitions.

Let X_0 denote the open set of the projectivization of Sym_n, the space of symmetric n-by-n matrices, consisting of matrices with non-zero determinant. Elements of X_0 should be interpreted as (the defining equations of) smooth quadric hypersurfaces in $\mathbb{P}^{n-1}$. The group SL_n acts on X_0 by change of variables defining the quadric hypersurfaces, which translates to the action

$$g \cdot A = (g^\top)^{-1} A g^{-1} \tag{2}$$

on Sym_n. X_0 is a homogeneous space under this SL_n action and the stabilizer of the quadric $x_1^2 + x_2^2 + \ldots x_n^2 = 0$ (equivalently, the identity matrix) is SO_n.

The classical definition of $\mathscr{X}$ (see [19, 20, 25]) is as the closure of the image of the map

$$[A] \mapsto ([A], [\Lambda^2(A)], \ldots, [\Lambda^{n-1}(A)]) \in \prod_{i=1}^{n-1} \mathbb{P}(\Lambda^i(\mathrm{Sym}_n)).$$

Renewed interest in the variety of complete quadrics can be attributed in large part to the following theorem, which gives two alternative descriptions of $\mathscr{X}$.

Theorem 3. *1. Vainsencher [26] $\mathscr{X}$ can be obtained by the following sequence of blow-ups: in the naive compactification $\mathbb{P}^{n-1}$ of X_0, first blow up the locus of rank* 1 *quadrics; then blow up the strict transform of the rank* 2 *quadrics; …; then blow up the strict transform of the rank $n-1$ quadrics.*
2. De Concini and Procesi [9] $\mathscr{X}$ is the wonderful embedding of X_0 and the spherical roots of $\mathscr{X}$ are the simple positive roots of the $\mathrm{A_n}$ *root system.*

From Theorem 3(1), a point $\mathscr{P} \in \mathscr{X}$ is described by the data of a flag

$$\mathscr{F} : V_0 = 0 \subset V_1 \subset \cdots \subset V_{s-1} \subset V_s = \mathbb{C}^n \tag{3}$$

and a collection $\mathscr{Q} = (Q_1, \ldots Q_s)$ of quadrics, where Q_i is a quadric in $\mathbb{P}(V_i)$ whose singular locus is $\mathbb{P}(V_{i-1})$. It is clear from Theorem 3(2) that r of Definition 1 is equal to $n-1$, and moreover, $i \in [n-1]$ corresponds to the simple root $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ in the $\mathrm{A_n}$ root system (see Remark 1).

Additionally, for each $K \subset [n-1]$, the map $(\mathscr{F}, \mathscr{Q}) \mapsto \mathscr{F}$ is an SL_n-equivariant projection

$$\pi_K : \mathscr{X}^K \to \mathrm{SL}_n/P_K, \tag{4}$$

where P_K is the standard parabolic subgroup associated with the roots corresponding to K. The fiber over $\mathscr{F} \in \mathrm{SL}_n/P_K$ is isomorphic to a product of varieties of complete quadrics of smaller dimension.

$\mathscr{O}^K$ consists of complete quadrics whose flag $\mathscr{F}$ satisfies $\{\dim V_i : i = 1, 2, \ldots, s-1\} = K^c$. $\mathscr{X}^K$ is a wonderful embedding of $\mathscr{O}_K$, the variety of complete quadrics whose flag satisfies $\{\dim V_i : i = 1, 2, \ldots, s-1\} \subset K^c$ [9].

In Fig. 1 we depict the cell decomposition of $\mathscr{X}_3$, the variety of complete conics in $\mathbb{P}^2$. Each colored disk represents a B-orbit and edges stand for the covering relations between closures of B-orbits. A cell is a union of all B-orbits of the same color. We include the label $I \subseteq \{1, 2\}$, which indicates the SL_3-orbit containing the given B-orbit. We use the label T to indicate the presence of a fixed point under the maximal torus T of SL_3.

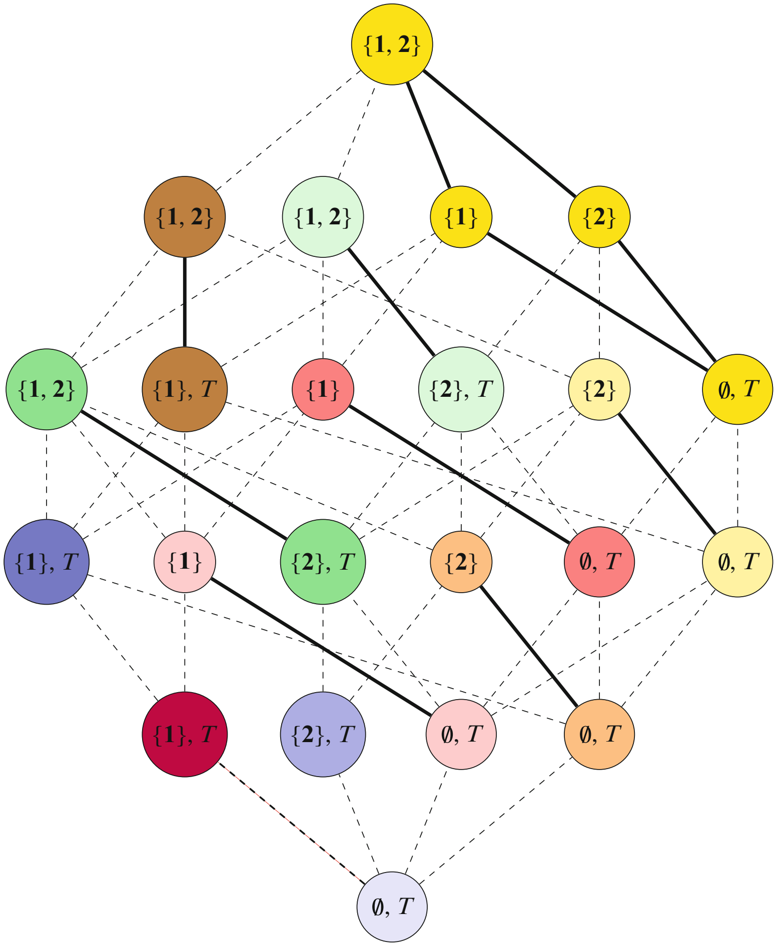

Fig. 1 Cell decomposition of the complete quadrics for $n = 3$

2.5 Unipotent Fixed Flags

We need the following elementary theorem on the fixed point loci of any partial flag variety SL_n/P under the action of a one dimensional unipotent subgroup $U' \hookrightarrow \mathrm{SL}_n$. Such loci are completely classified by Spaltenstein [22] and Shimomura [21].

Theorem 4. *Fix a one-dimensional unipotent subgroup $U' \hookrightarrow SL_n$ with the Lie algebra $\mathfrak{u}' = Lie(U')$.*

1. *The fixed point locus* $(SL_n/P)^{U'}$ *is non-empty.*
2. *If a non-zero element* $e \in \mathfrak{u}'$ *is regular, i.e. has a single Jordan block, then* $(SL_n/P)^{U'}$ *consists of a single point.*
3. *If* $(SL_n/B)^{U'}$ *is a single point, then any non-zero* $e \in \mathfrak{u}'$ *is regular.*

3 SL_2-Regular Subvarieties of Complete Quadrics

Definition 2. A subset $I \subset [n-1]$ is *special* if it does not contain any consecutive numbers. Equivalently, $I = \{i_1 < i_2 < \cdots < i_s\} \subset [n-1]$ is special if $i_{j+1} - i_j \geq 2$ for $j = 1, 2, \ldots, s-1$.

Remark 2. Given a special subset $I \subset [n-1]$, any subset $K \subset I$ is also special.

Theorem 5. *Let* $I \subset [n-1]$. *The following are equivalent:*

1. I *is special;*
2. $\mathscr{X}^I$ *is* SL_2*-regular;*
3. $\mathscr{O}^I$ *contains a* T*-fixed point.*

Remark 3. The equivalence (1) $\Leftrightarrow$ (3) in Theorem 5 is due to Strickland [23, Proposition 2.1].

Proof of (1) $\Rightarrow$ *(2).* Let I be a special subset of $[n-1]$. Let

$$e = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{5}$$

and

$$h = \begin{pmatrix} 2n & 0 & 0 & \ldots & 0 \\ 0 & 2n-2 & 0 & \ldots & 0 \\ 0 & 0 & 2n-4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2 \end{pmatrix}.$$

A routine calculation shows that $[h, e] = 2e$, so let $\phi : \mathrm{SL}_2 \to \mathrm{SL}_n$ be the associated embedding.

Next we show that $\mathscr{X}^I$ is SL_2-regular by proving that the unique U'-fixed point of $\mathscr{X}^I$ is the standard flag in $\mathbb{C}^n$, viewed as a point in $\mathscr{O}^{[n-1]} \cong \mathrm{SL}_n/B$. Since $\pi_K : \mathscr{O}^K \to \mathrm{SL}_n/P_K$ is SL_n-equivariant (4), any U'-fixed point $\mathscr{P} = (\mathscr{F}, \mathscr{Q} = (Q_1, \ldots, Q_s)) \in \mathscr{O}^K \subset \mathscr{X}^I$ maps to a U'-fixed partial flag $\mathscr{F} = \pi_K(\mathscr{P})$.

By Theorem 4, there is a unique U'-fixed partial flag $\mathscr{F}_K$ in each SL_n/P_K. Moreover, writing $K^c = \{k_1 < k_2 < \cdots < k_t\}$, $\mathscr{F}_K$ is the flag whose i-th vector space is spanned by the first k_i standard basis vectors.

For each $K \subset I$, we determine the U'-fixed locus of the fiber of π_K over $\mathscr{F}_K$. Since the flag

$$\mathscr{F}_K := (0) = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{t-1} \subset V_t = \mathbb{C}^n$$

is U'-fixed, the action of $u \in U'$ on a quadric Q_i defined by the symmetric matrix A_i in V_i/V_{i-1} is given by restricting u to a linear transformation on V_i/V_{i-1}. Because K is special, $\dim V_i/V_{i-1} \leq 2$. Moreover, the matrix of u with respect to the basis of standard basis vectors in $V_i - V_{i-1}$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if $\dim(V_i/V_{i-1}) = 2$.

If

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

defines such a quadric on a two-dimensional vector space, then

$$u \cdot A = \begin{pmatrix} a & b-a \\ b-a & c-2b+a \end{pmatrix}.$$

The only fixed quadric is degenerate and defined by $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Therefore, if the point $(\mathscr{F}_K, \mathscr{Q})$ is U'-fixed, then each V_i/V_{i-1} is one-dimensional. In other words, $K = \emptyset$ and $(\mathscr{F}_K, \mathscr{Q})$ is the standard flag in SL_n/B.

Proof of (2) ⇒ (1). Assume that I is not special. Let $\phi : \mathrm{SL}_2 \to \mathrm{SL}_n$ be any homomorphism, giving rise to $B' = T'U' \subset \mathrm{SL}_n$. To show that $\mathscr{X}^I$ is not regular, we must show that U' does not have a unique fixed point. First, consider the action of U' on SL_n/B. By Theorem 4, there are always U'-fixed flags and there is a unique U'-fixed flag if and only if any non-zero $e \in \mathfrak{u}'$ is regular. Thus, we assume that the Jordan form of e is

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Since I is not special, there exists a, $1 \leq a < n-1$, such that $a, a+1 \in I$. Let $K = \{a, a+1\}$. Since $K \subset I$, $\mathscr{O}^K \subset \mathscr{X}^I$. Moreover, U'-fixed points in $\mathscr{O}^K$ are in canonical bijection with the quadrics, defined on the three-dimensional vector space spanned by the a-th, $(a+1)$-st, and $(a+2)$-nd standard basis vectors, that are fixed by the restricted action of U'. Without loss of generality, we can assume that $U' \subset \mathrm{SL}_3$ and

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{u}'.$$

A standard Lie theory calculation using (2) shows that a quadric defined by

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

is fixed by U' if and only if $e^\top A + A^\top e^\top = 0$ if and only if

$$A = \begin{pmatrix} 0 & 0 & c \\ 0 & -c & 0 \\ c & 0 & f \end{pmatrix}.$$

Thus, $\mathscr{O}^K$ contains a positive dimensional family of U'-fixed quadrics, and consequently $\mathscr{X}^I$ is not regular.

Remark 4. The proof that (2) $\Leftrightarrow$ (3) in Theorem 5 is achieved by using the explicit combinatorics at hand to show that each of the two statements is equivalent to (1). It is natural to wonder whether either direction of the implication holds in a more general setting.

4 Poincaré Polynomial of $\mathscr{X}^I$

We apply Theorem 2 to compute the cohomology of $\mathscr{X}^I$ when I is a special subset of $[n-1]$.

Proposition 1. *If $I \subset [n-1]$ is special, then the q-Poincaré polynomial of $\mathscr{X}^I$ is equal to*

$$P_{\mathscr{X}^I}(q) = \left(\frac{1-q^3}{1-q^2}\right)^{|I|} \prod_{k=1}^{n} \frac{1-q^k}{1-q}. \tag{6}$$

Proof. Fix a regular SL_2-action on $\mathscr{X}^I$ corresponding to

$$e = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathfrak{u} \text{ and } h = \begin{pmatrix} n-1 & 0 & \dots & 0 \\ 0 & n-3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -(n-1) \end{pmatrix} \in \mathfrak{t}$$

so that T' is included in T via

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto \begin{pmatrix} t^{n-1} & 0 & \dots & 0 \\ 0 & t^{n-3} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{-(n-1)} \end{pmatrix}.$$

From Theorem 2 and the discussion preceding it, to compute the Poincaré polynomial of $\mathscr{X}^I$, we must understand the T'-weight decomposition of the tangent space $T_{\mathscr{P}}(\mathscr{X}^I)$ of the unique U'-fixed point $\mathscr{P}$, corresponding to the standard flag in the complete flag variety $\mathrm{SL}_n/B \subset \mathscr{X}^I$. Since T' is a subtorus of the maximal torus T of SL_n, consider the T-equivariant decomposition

$$T_{\mathscr{P}}(\mathscr{X}^I) = T_{\mathscr{P}}(\mathrm{SL}_n/B) \oplus N_{\mathscr{P}}(\mathrm{SL}_n/B, \mathscr{X}^I).$$

Here, $N_{\mathscr{P}}(\mathrm{SL}_n/B, \mathscr{X}^I)$ denotes the fiber of the normal bundle of SL_n/B in $\mathscr{X}^I$ at the point $\mathscr{P}$.

As a T-module, $T_{\mathscr{P}}(\mathrm{SL}_n/B) \cong \mathfrak{u}^- = \oplus_{\alpha>0}\mathfrak{u}_{-\alpha}$. Since T' has weight 2 acting on any simple positive root space, the weight of T' on $\mathfrak{u}_{-\alpha}$ is $-2\mathrm{ht}(\alpha)$, where $\mathrm{ht}(\alpha)$ is the height of α (cf. [6]). The height of $\alpha = \varepsilon_i - \varepsilon_j$, $i < j$ is $j - i$.

Since $\mathscr{X}^I$ is a wonderful embedding of $\mathscr{O}^I$, SL_n/B is a transverse intersection of the T-stable subvarieties $\mathscr{X}^K$ where $K \subset I$ has cardinality $|I| - 1$. Thus, as a T-module,

$$N_{\mathscr{P}}(\mathrm{SL}_n/B, \mathscr{X}^I) \cong \bigoplus_{j\in I} T_{\mathscr{P}}(\mathscr{X}^I)/T_{\mathscr{P}}(\mathscr{X}^{I-\{j\}}).$$

Since the T-weight of $T_{\mathscr{P}}(\mathscr{X}^I)/T_{\mathscr{P}}(\mathscr{X}^{I-\{j\}})$ is $-2\alpha_j$ [9], its T'-weight is -4.

Combining these calculations with Theorem 2 gives (6), using the elementary identity

$$\prod_{\alpha>0} \frac{1-q^{ht(\alpha)+1}}{1-q^{ht(\alpha)}} = \prod_{1\le i<j\le n} \frac{1-q^{j-i+1}}{1-q^{j-i}} = \prod_{k=1}^{n} \frac{1-q^k}{1-q}.$$

We interpret Theorem 1 as a generalization of the classical Kostant-Macdonald identity [13, 16] for the complete flag variety:

$$\sum_{\pi \in S_n} q^{\ell(\pi)} = [n]_q! := \prod_{k=1}^{n} \frac{1-q^k}{1-q}. \tag{7}$$

Akyıldız and Carrell recovered (7) as a corollary of Theorem 2 applied to the variety $X = \mathrm{SL}_n/B$.

In order to derive a similar "sum = product" identity in the case of the varieties $\mathscr{X}^I$, I special, we compute $P_{\mathscr{X}^I}(q)$ by describing a decomposition into cells and applying Theorem 1. To do so, we make use of a result of De Concini and Springer [10] to reduce the calculation to that of a cell decomposition for $\mathscr{X}$, which was first computed by Strickland in [23].

Let $K \subset [n-1]$ be special and let $W = S_n$ be the symmetric group on $[n]$. Let W_K be the parabolic subgroup of W generated by transpositions $(i, i+1)$ for $i \in K$, let W^K be the set of minimal coset representatives of W/W_K, and let $w_{0,K} = \prod_{i \in K} (i, i+1)$ denote the longest element of W_K. The T-fixed points in $\mathscr{O}^K$ are indexed by W^K [23, Proposition 2.3].

W acts on the free abelian group generated by $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ by $w \cdot \varepsilon_i = \varepsilon_{w(i)}$ and the simple roots $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ lie in this group. We write $v = \sum_{i=1}^{n} c_i \varepsilon_i > 0$ (resp., < 0) if the first non-zero coefficient c_i which appears in the decomposition is positive (resp., negative). Interpreting the ε_i as characters of SL_n, $v > 0$ is equivalent to the corresponding character being positive along a suitable one-dimensional torus $T' \subset \mathrm{SL}_n$. Define the set

$$R_K(w) := \{i \in K^{\mathrm{c}} : w(\alpha_i + w_{0,K}(\alpha_i)) < 0\}.$$

Proposition 2 ([23], Theorem 2.7 and Proposition 2.6). *Let $p \in \mathscr{O}^K$ be a T-fixed point of $\mathscr{X}$ corresponding to $w \in W^K$. Let C_p^+ denote the plus cell of p in $\mathscr{X}$ associated to the action of T'. Then*

$$\dim C_p^+ = \ell(w) + |K| + |R_K(w)|.$$

Proposition 3 ([10], Lemma 4.1). *Retaining the notation of Proposition 2, an orbit $\mathscr{O}^K$ intersects C_p^+ if and only if $K \subset I \subset R_K(w) \cup K$. If $K \subset I$, then $\mathscr{X}^I \cap C_p^+$ is the plus cell of $\mathscr{X}^I$ containing p and has dimension* $\dim C_p^+ - |(I^c \cap R_K(w))|$.

Theorem 6. *Let $I \subset [n-1]$ be a special subset. Then*

$$P_{\mathscr{X}^I}(q) = \sum_{K \subset I} \sum_{w \in W^K} q^{\ell(w)+|K|+s_{K,I}(w)},$$

where $s_{K,I}(w) = |\{i \in I - K : w(\alpha_i + w_{0,K}(\alpha_i)) < 0\}|$.

Corollary 1. *Let I be a special subset of $[n-1]$. Then*

$$\sum_{K\subset I}\sum_{w\in W^K} q^{\ell(w)+|K|+s_{K,I}(w)} = \left(\frac{1-q^3}{1-q^2}\right)^{|I|}\prod_{k=1}^{n}\frac{1-q^k}{1-q}. \tag{8}$$

Example 1. We illustrate Corollary 1 in the case $n = 3$. If $I = \emptyset$, then we recover the classical Kostant-Macdonald identity for SL_3/B (cf. [2]):

$$1 + 2q + 2q^2 + q^3 = \frac{(1-q^2)(1-q^3)}{(1-q)^2} = (1+q)(1+q+q^2).$$

If $I = \{1\}$, we obtain a new identity:

$$(1+q^2)(1+q+q^2)+q(1+q+q^2) = \frac{(1-q^3)}{(1-q^2)}\cdot\frac{(1-q^2)(1-q^3)}{(1-q)^2} = (1+q+q^2)^2.$$

The decomposition of the left-hand side reflects the sums over individual subsets $K \subset I$. The identity for $I = \{2\}$ yields the same identity as $I = \{1\}$.

Remark 5. If I is any special subset of $[n-1]$ of cardinality l and $K \subset I$ has cardinality k, then one can show directly that

$$\sum_{w\in W^K} q^{\ell(w)+|K|+s_{K,I}(w)} = \left(\frac{q}{1+q^2}\right)^k\left(\frac{1+q^2}{1+q}\right)^l\prod_{i=1}^{n}\left(1+q+\cdots+q^{i-1}\right)$$

by verifying $s_{K,I}(w) = |\{i \in I \setminus K : \ell(ws_i) < \ell(w)\}|$ (cf. [23, proof of Proposition 2.6]). Then (8) is obtained by summing over all $K \subset I$ and applying the Binomial Theorem.

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Markov Chains for Promotion Operators

Arvind Ayyer, Steven Klee, and Anne Schilling

Dedicated to Mohan Putcha and Lex Renner on the occasion of their 60th birthdays

Abstract We consider generalizations of Schützenberger's promotion operator on the set $\mathscr{L}$ of linear extensions of a finite poset. This gives rise to a strongly connected graph on $\mathscr{L}$. In earlier work (Ayyer et al., J. Algebraic Combinatorics 39(4), 853–881 (2014)), we studied promotion-based Markov chains on these linear extensions which generalizes results on the Tsetlin library. We used the theory of $\mathscr{R}$-trivial monoids in an essential way to obtain explicitly the eigenvalues of the transition matrix in general when the poset is a rooted forest. We first survey these results and then present explicit bounds on the mixing time and conjecture eigenvalue formulas for more general posets. We also present a generalization of promotion to arbitrary subsets of the symmetric group.

Keywords Posets • Linear extensions • Promotion • Markov chains • Tsetlin library • $\mathscr{R}$-trivial monoids

Subject Classifications: Primary 06A07, 20M32, 20M30, 60J27; Secondary 47D03

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1 Introduction

Schützenberger [25] introduced the notion of evacuation and promotion on the set of linear extensions of a finite poset P of size n. This generalizes promotion on standard Young tableaux defined in terms of jeu-de-taquin moves. Haiman [16] as well as Malvenuto and Reutenauer [22] simplified Schützenberger's approach by expressing the promotion operator ∂ in terms of more fundamental operators τ_i ($1 \leq i < n$), which either act as the identity or as a simple transposition. A beautiful survey on this subject was written by Stanley [28].

In earlier work, we considered a slight generalization of the promotion operator [2] defined as $\partial_i = \tau_i \tau_{i+1} \cdots \tau_{n-1}$ for $1 \leq i \leq n$ with $\partial_1 = \partial$ being the original promotion operator. In Sect. 2 we define the extended promotion operator, give examples and state some of its properties. We survey our results on Markov chains based on the operators ∂_i, which act on the set of all linear extensions of P (denoted $\mathscr{L}(P)$) in Sect. 3.

Our results [2] can be viewed as a natural generalization of the results of Hendricks [17, 18] on the Tsetlin library [33], which is a model for the way an arrangement of books in a library shelf evolves over time. It is a Markov chain on permutations, where the entry in the ith position is moved to the front with probability p_i. From our viewpoint, Hendricks' results correspond to the case when P is an anti-chain and hence $\mathscr{L}(P) = S_n$ is the full symmetric group. Many variants of the Tsetlin library have been studied and there is a wealth of literature on the subject. We refer the interested reader to the monographs by Letac [20] and by Dies [11], as well as the comprehensive bibliographies in [14] and [5].

One of the most interesting properties of the Tsetlin library Markov chain is that the eigenvalues of the transition matrix can be computed exactly. The exact form of the eigenvalues was independently investigated by several groups. Notably Donnelly [12], Kapoor and Reingold [19], and Phatarfod [23] studied the approach to stationarity in great detail. There has been some interest in finding exact formulas for the eigenvalues for generalizations of the Tsetlin library. The first major achievement in this direction was to interpret these results in the context of hyperplane arrangements [4, 5, 10]. This was further generalized to a class of monoids called left regular bands [8] and subsequently to all bands [9] by Brown. This theory has been used effectively by Björner [6,7] to extend eigenvalue formulas on the Tsetlin library from a single shelf to hierarchies of libraries.

We present without proof our explicit combinatorial formulas [2] for the eigenvalues and multiplicities for the transition matrix of the promotion Markov chain when the underlying poset is a rooted forest in Sect. 4 (see Theorem 4). The proof of eigenvalues and their multiplicities follows from the $\mathscr{R}$-triviality of the underlying monoid using results by Steinberg [30, 31]. Intuition on why the promotion monoid is $\mathscr{R}$-trivial is stated in Sect. 5.

The remainder of the paper contains new results and is outlined as follows. In Sect. 6, we prove a formula for the mixing time of the promotion Markov chain. This improves the result stated without proof in the Outlook section of [2]. In Sect. 7, we present a partial conjecture for the eigenvalues of the transition matrix of posets which are not rooted forests. We give supporting data for our conjectures with formulas for all posets of size 4. Lastly, Sect. 8 defines a generalization of promotion on arbitrary subsets of S_n and gives a formula for its stationary distribution.

2 Extended Promotion on Linear Extensions

Let P be an arbitrary poset of size n, with partial order denoted by $\preceq$. We assume that the elements of P are labeled by integers in $[n] := \{1, 2, \ldots, n\}$. In addition, we assume that the poset is naturally labeled, that is if $i, j \in P$ with $i \preceq j$ in P then $i \leq j$ as integers. Let $\mathscr{L} := \mathscr{L}(P)$ be the set of its **linear extensions**,

$$\mathscr{L}(P) = \{\pi \in S_n \mid i \prec j \text{ in } P \implies \pi_i^{-1} < \pi_j^{-1} \text{ as integers}\}, \tag{1}$$

which is naturally interpreted as a subset of the symmetric group S_n. Note that the identity permutation e always belongs to $\mathscr{L}$. Let P_j be the natural (induced) subposet of P consisting of elements k such that $j \preceq k$ [27].

We now briefly recall the idea of **promotion** of a linear extension of a poset P. Start with a linear extension $\pi \in \mathscr{L}(P)$ and imagine placing the label π_i^{-1} in P at the location i. By the definition of the linear extension, the labels will be well-ordered. The action of promotion of π will give another linear extension of P as follows:

1. The process starts with a seed, the label 1. First remove it and replace it by the minimum of all the labels covering it, i, say.
2. Now look for the minimum of all labels covering i in the original poset, and replace it, and continue in this way.
3. This process ends when a label is a "local maximum." Place the label $n + 1$ at that point.
4. Decrease all the labels by 1.

This new linear extension is denoted $\pi\partial$ [28].

Example 1. Figure 1 shows a poset (left) to which we assign the identity linear extension $\pi = 123456789$. The linear extension $\pi' = \pi\partial = 214537869$ obtained by applying the promotion operator is depicted on the right. Note that indeed we place $\pi_i'^{-1}$ in position i, namely 3 is in position 5 in π', so that 5 in $\pi\partial$ is where 3 was originally. Figure 2 illustrates the steps used to construct the linear extension $\pi\partial$ from the linear extension π from Fig. 1.

The definition of promotion was originally motivated by the following construction. The Young diagram of a partition λ (with English notation) can naturally be viewed as a poset on the boxes of the diagram ordered according to the rule that a box is covered by any boxes immediately below it or to its right. The linear extensions of this poset are standard Young tableaux of shape λ. In this context,

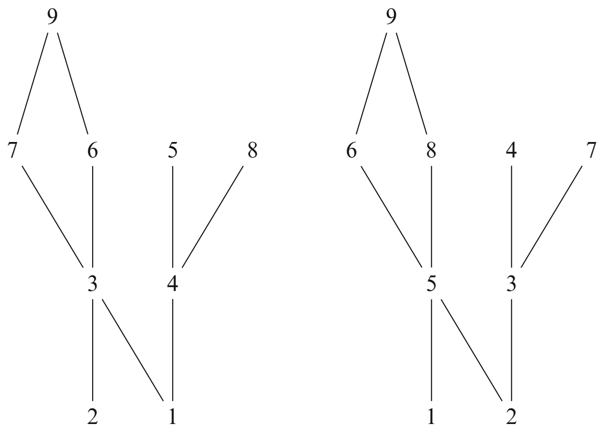


Fig. 1 A linear extension π (*left*) and $\pi\partial$ (*right*)

the definition of promotion is a natural generalization of the standard promotion operator used in the RSK algorithm. On semistandard tableaux, promotion is also used to define affine crystal structures in type A [26] and it has applications to the cyclic sieving phenomenon [24]. The above definition of promotion for arbitrary posets is originally due to Schützenberger [25].

We now generalize the above construction to **extended promotion**, whose seed is any of the numbers $1, 2, \ldots, n$. The algorithm is similar to the original one, and we describe it for seed j. Start with the subposet P_j and perform steps 1–3 in a completely analogous fashion. Now decrease all the labels strictly larger than j by 1 in P (not only P_j). Clearly this gives a new linear extension, which we denote $\pi\partial_j$. Note that ∂_n is always the identity.

The extended promotion operator can be expressed in terms of more elementary operators τ_i $(1 \le i < n)$ as shown in [16, 22, 28] and has explicitly been used to count linear extensions in [13]. Let $\pi = \pi_1 \cdots \pi_n \in \mathscr{L}(P)$ be a linear extension of a finite poset P in one-line notation. Then

$$\pi\tau_i = \begin{cases} \pi_1 \cdots \pi_{i-1}\pi_{i+1}\pi_i \cdots \pi_n & \text{if } \pi_i \text{ and } \pi_{i+1} \text{ are not} \\ & \text{comparable in } P, \\ \pi_1 \cdots \pi_n & \text{otherwise.} \end{cases} \tag{2}$$

Alternatively, τ_i acts non-trivially on a linear extension if interchanging entries π_i and π_{i+1} yields another linear extension. Then as an operator on $\mathscr{L}(P)$,

$$\partial_j = \tau_j \tau_{j+1} \cdots \tau_{n-1}. \tag{3}$$

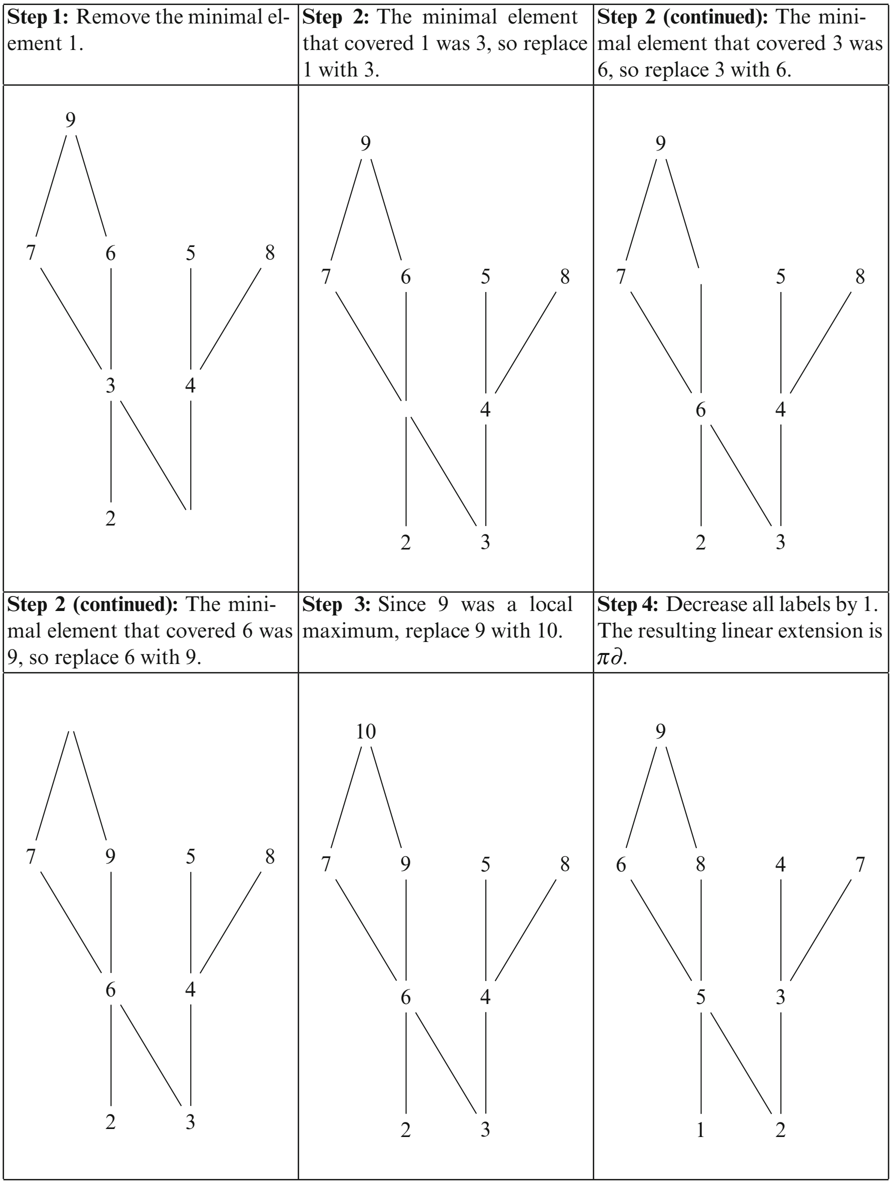


Fig. 2 Constructing $\pi\partial$ from π

3 Promotion Markov Chains

We now consider two discrete-time Markov chains related to the extended promotion operator. For completeness, we briefly review the part of the theory relevant to us.

Fix a finite poset P of size n. The operators $\{\partial_i \mid 1 \leq i \leq n\}$ define a directed graph on the set of linear extensions $\mathscr{L}(P)$. The vertices of the graph are the elements in $\mathscr{L}(P)$ and there is an edge from π to π' if $\pi' = \pi\partial_i$. We can now consider random walks on this graph with probabilities given formally by $x_1, \ldots, x_n$ which sum to 1. We give two ways to assign the edge weights, see Sects. 3.1 and 3.2. An edge with weight x_i is traversed with that rate. A priori, the x_i's must be positive real numbers for this to make sense according to the standard techniques of Markov chains. However, the ideas work in much greater generality and one can think of this as an "analytic continuation."

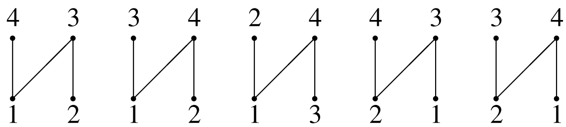
A discrete-time Markov chain is defined by the **transition matrix** M, whose entries are indexed by elements of the state space. In our case, they are labeled by elements of $\mathscr{L}(P)$. We take the convention that the (π', π) entry gives the probability of going from $\pi \to \pi'$. The special case of the diagonal entry at (π, π) gives the probability of a loop at the π. This ensures that column sums of M are one and consequently, one is an eigenvalue with row (left-) eigenvector being the all-ones vector. A Markov chain is said to be **irreducible** if the associated digraph is strongly connected. In addition, it is said to be **aperiodic** if the greatest common divisor of the lengths of all possible loops from any state to itself is one. For irreducible aperiodic chains, the Perron-Frobenius theorem guarantees that there is a unique **stationary distribution**. This is given by the entries of the column (right-) eigenvector of M with eigenvalue 1. Equivalently, the stationary distribution $w(\pi)$ is the solution of the **master equation**, given by

$$\sum_{\pi' \in \mathscr{L}(P)} M_{\pi,\pi'}\, w(\pi') = \sum_{\pi' \in \mathscr{L}(P)} M_{\pi',\pi}\, w(\pi). \tag{4}$$

Edges which are loops contribute to both sides equally and thus cancel out. For more on the theory of finite state Markov chains, see [21].

We set up a running example that will be used for each case.

Example 2. Define P by its covering relations $\{(1,3), (1,4), (2,3)\}$, so that its Hasse diagram is the first diagram in the list below:



The remaining graphs correspond to the linear extensions

$$\mathscr{L}(P) = \{1234, 1243, 1423, 2134, 2143\}.$$

3.1 Uniform Promotion Graph

The vertices of the **uniform promotion graph** are labeled by elements of $\mathscr{L}(P)$ and there is an edge between π and π' if and only if $\pi' = \pi \partial_j$ for some $j \in [n]$. In this case, the edge is given the symbolic weight x_j.

Example 3. The uniform promotion graph for the poset in Example 2 is illustrated in Fig. 3. The transition matrix, with the lexicographically ordered basis, is given by

$$\begin{pmatrix} x_4 & x_3 & x_1 + x_2 & 0 & 0 \\ x_2 + x_3 & x_4 & 0 & x_1 & 0 \\ 0 & x_2 & x_3 + x_4 & 0 & x_1 \\ 0 & x_1 & 0 & x_4 & x_2 + x_3 \\ x_1 & 0 & 0 & x_2 + x_3 & x_4 \end{pmatrix}.$$

Note that the row sums are one although the matrix is not symmetric, so that the stationary state of this Markov chain is uniform. We state this for general finite posets in Theorem 1.

The variable x_4 occurs only on the diagonal in the above transition matrix. This is because the action of ∂_4 (or in general ∂_n) maps every linear extension to itself resulting in a loop.

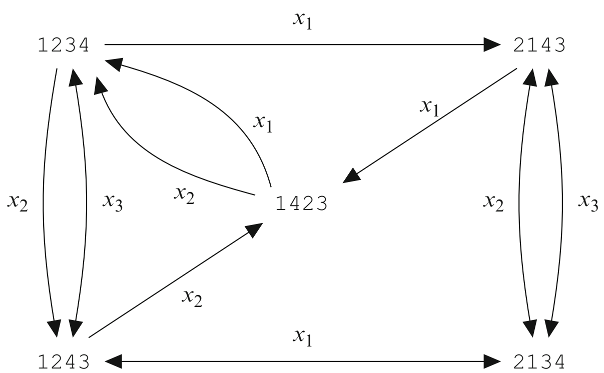


Fig. 3 Uniform promotion graph for Example 2. Every vertex has four outgoing edges labeled x_1 to x_4 and self-loops are not drawn

3.2 Promotion Graph

The **promotion graph** is defined in the same fashion as the uniform promotion graph with the exception that the edge between π and π' when $\pi' = \pi\partial_j$ is given the weight x_{π_j}.

Example 4. The promotion graph for the poset of Example 2 is illustrated in Fig. 4. Although it might appear that there are many more edges here than in Fig. 3, this is not the case. The transition matrix this time is given by

$$\begin{pmatrix} x_4 & x_4 & x_1+x_4 & 0 & 0 \\ x_2+x_3 & x_3 & 0 & x_2 & 0 \\ 0 & x_2 & x_2+x_3 & 0 & x_2 \\ 0 & x_1 & 0 & x_4 & x_1+x_4 \\ x_1 & 0 & 0 & x_1+x_3 & x_3 \end{pmatrix}.$$

Notice that row sums are no longer one. The stationary distribution, as a vector written in row notation is

$$\left(1, \frac{x_1+x_2+x_3}{x_1+x_2+x_4}, \frac{(x_1+x_2)(x_1+x_2+x_3)}{(x_1+x_2)(x_1+x_2+x_4)}, \frac{x_1}{x_2}, \frac{x_1(x_1+x_2+x_3)}{x_2(x_1+x_2+x_4)}\right)^T.$$

Again, we will give a general such result in Theorem 2.

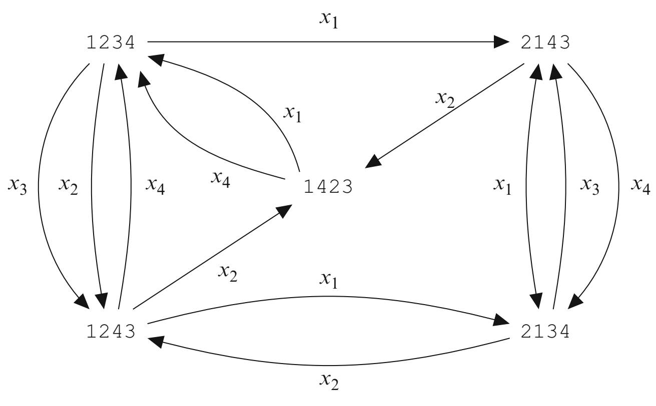


Fig. 4 Promotion graph for Example 2. Every vertex has four outgoing edges labeled x_1 to x_4 and self-loops are not drawn

3.3 Irreducibility and Stationary States

In this section we summarize some properties of the promotion Markov chains of Sects. 3.1 and 3.2 and state their stationary distributions. Proofs of these statements can be found in [2].

Proposition 1. *Consider the digraph G whose vertices are labeled by elements of $\mathscr{L}$ and whose edges are given as follows: for $\pi, \pi' \in \mathscr{L}$, there is an edge between π and π' in G if and only if $\pi' = \pi \partial_j$ for some $j \in [n]$. Then G is strongly connected.*

Corollary 1. *Assuming that the edge weights are strictly positive, the two Markov chains of Sects. 3.1 and 3.2 are irreducible and ergodic. Hence their stationary states are unique.*

Next we state properties of the stationary state of the two discrete time Markov chains, assuming that all x_i's are strictly positive.

Theorem 1. *The discrete time Markov chain according to the uniform promotion graph has the uniform stationary distribution, that is, each linear extension is equally likely to occur.*

We now turn to the promotion graphs. In this case we find nice product formulas for the stationary weights.

Theorem 2. *The stationary state weight $w(\pi)$ of the linear extension $\pi \in \mathscr{L}(P)$ for the discrete time Markov chain for the promotion graph is given by*

$$w(\pi) = \prod_{i=1}^{n} \frac{x_1 + \cdots + x_i}{x_{\pi_1} + \cdots + x_{\pi_i}} \,, \tag{5}$$

assuming $w(e) = 1$.

Remark 1. The entries of w do not, in general, sum to one. Therefore this is not a true probability distribution, but this is easily remedied by a multiplicative constant Z_P depending only on the poset P.

When P is the n-antichain, then $\mathscr{L} = S_n$. In this case, the probability distribution of Theorem 2 has been studied in the past by Hendricks [17, 18] and is known as the **Tsetlin library** [33], which we now describe. Suppose that a library consists of n books $b_1, \ldots, b_n$ on a single shelf. Assume that only one book is picked at a time and is returned before the next book is picked up. The book b_i is picked with probability x_i and placed at the end of the shelf.

We now explain why promotion on the n-antichain is the Tsetlin library. A given ordering of the books can be identified with a permutation π. The action of ∂_k on π gives $\pi \tau_k \cdots \tau_{n-1}$ by (3), where now all the τ_i's satisfy the braid relation since none of the π_j's are comparable. Thus the k-th element in π is moved all the way to the end. The probability with which this happens is x_{π_k}, which makes this process identical to the action of the Tsetlin library.

The stationary distribution of the Tsetlin library is a special case of Theorem 2. In this case, Z_P of Remark 1 also has a nice product formula, leading to the probability distribution,

$$w(\pi) = \prod_{i=1}^{n} \frac{x_{\pi_i}}{x_{\pi_1} + \cdots + x_{\pi_i}}. \tag{6}$$

Letac [20] considered generalizations of the Tsetlin library to rooted trees (meaning that each element in P besides the root has precisely one successor). Our results hold for any finite poset P.

4 Partition Functions and Eigenvalues for Rooted Forests

For a certain class of posets, we are able to give an explicit formula for the probability distribution for the promotion graph. Note that this involves computing the partition function Z_P (see Remark 1). We can also specify all eigenvalues and their multiplicities of the transition matrix explicitly. Proofs of these statements can be found in [2].

Before we can state the main theorems of this section, we need to make a couple of definitions. A **rooted tree** is a connected poset, where each node has at most one successor. Note that a rooted tree has a unique largest element. A **rooted forest** is a union of rooted trees. A **lower set** (resp. **upper set**) S in a poset is a subset of the nodes such that if $x \in S$ and $y \preceq x$ (resp. $y \succeq x$), then also $y \in S$. We first give the formula for the partition function.

Theorem 3. *Let P be a rooted forest of size n and let $x_{\preceq i} = \sum_{j \preceq i} x_j$. The partition function for the promotion graph is given by*

$$Z_P = \prod_{i=1}^{n} \frac{x_{\preceq i}}{x_1 + \cdots + x_i}. \tag{7}$$

Let L be a finite poset with smallest element $\hat{0}$ and largest element $\hat{1}$. Following [8, Appendix C], one may associate to each element $x \in L$ a **derangement number** d_x defined as

$$d_x = \sum_{y \succeq x} \mu(x, y) f([y, \hat{1}]) , \tag{8}$$

where $\mu(x, y)$ is the Möbius function for the interval $[x, y] := \{z \in L \mid x \preceq z \preceq y\}$ [27, Section 3.7] and $f([y, \hat{1}])$ is the number of maximal chains in the interval $[y, \hat{1}]$.

A permutation is a **derangement** if it does not have any fixed points. A linear extension π is called a **poset derangement** if it is a derangement when considered as a permutation. Let $\mathfrak{d}_P$ be the number of poset derangements of the poset P.

A **lattice** L is a poset in which any two elements have a unique supremum (also called join) and a unique infimum (also called meet). For $x, y \in L$ the join is denoted by $x \vee y$, whereas the meet is $x \wedge y$. For an **upper semi-lattice** we only require the existence of a unique supremum of any two elements.

Theorem 4. *Let P be a rooted forest of size n and M the transition matrix of the promotion graph of Sect. 3.2. Then*

$$\det(M - \lambda \mathbb{1}) = \prod_{\substack{S \subseteq [n] \\ S \text{ upper set in } P}} (\lambda - x_S)^{d_S},$$

where $x_S = \sum_{i \in S} x_i$ and d_S is the derangement number in the lattice L (by inclusion) of upper sets in P. In other words, for each subset $S \subseteq [n]$, which is an upper set in P, there is an eigenvalue x_S with multiplicity d_S.

The proof of Theorem 4 follows from the fact that the monoid corresponding to the transition matrix M is $\mathscr{R}$-trivial. When P is a union of chains, which is a special case of rooted forests, we can express the eigenvalue multiplicities directly in terms of the number of poset derangements.

Theorem 5. *Let $P = [n_1] + [n_2] + \cdots + [n_k]$ be a union of chains of size n whose elements are labeled consecutively within chains. Then*

$$\det(M - \lambda \mathbb{1}) = \prod_{\substack{S \subseteq [n] \\ S \text{ upper set in } P}} (\lambda - x_S)^{\mathfrak{d}_{P \setminus S}},$$

where $\mathfrak{d}_\emptyset = 1$.

Note that the antichain is a special case of a rooted forest and in particular a union of chains. In this case the Markov chain is the Tsetlin library and all subsets of $[n]$ are upper (and lower) sets. Hence Theorem 4 specializes to the results of Donnelly [12], Kapoor and Reingold [19], and Phatarford [23] for the Tsetlin library.

The case of unions of chains, which are consecutively labeled, can be interpreted as looking at a parabolic subgroup of S_n. If there are k chains of lengths n_i for $1 \leq i \leq k$, then the parabolic subgroup is $S_{n_1} \times \cdots \times S_{n_k}$. In the realm of the Tsetlin library, there are n_i books of the same color. The Markov chain consists of taking a book at random and placing it at the end of the stack.

5 $\mathscr{R}$-Trivial Monoids

In this section we briefly outline the proof of Theorem 4. More details can be found in [2].

A finite **monoid** $\mathscr{M}$ is a finite set with an associative multiplication and an identity element. Green [15] defined several preorders on $\mathscr{M}$. In particular for $x, y \in \mathscr{M}$ the $\mathscr{R}$- and $\mathscr{L}$-order is defined as

$$\begin{aligned} x \geq_{\mathscr{R}} y \quad & \text{if } y = xu \text{ for some } u \in \mathscr{M}, \\ x \geq_{\mathscr{L}} y \quad & \text{if } y = ux \text{ for some } u \in \mathscr{M}. \end{aligned} \tag{9}$$

This ordering gives rise to equivalence classes ($\mathscr{R}$-classes or $\mathscr{L}$-classes)

$$\begin{aligned} x \,\mathscr{R}\, y \quad & \text{if and only if } x\mathscr{M} = y\mathscr{M}, \\ x \,\mathscr{L}\, y \quad & \text{if and only if } \mathscr{M}x = \mathscr{M}y. \end{aligned}$$

The monoid $\mathscr{M}$ is said to be $\mathscr{R}$**-trivial** (resp. $\mathscr{L}$**-trivial**) if all $\mathscr{R}$-classes (resp. $\mathscr{L}$-classes) have cardinality one.

Now let P be a rooted forest of size n and $\hat{\partial}_i$ for $1 \leq i \leq n$ the operators on $\mathscr{L}(P)$ defined by the promotion graph of Sect. 3.2. That is, for $\pi, \pi' \in \mathscr{L}(P)$, the operator $\hat{\partial}_i$ maps π to π' if $\pi' = \pi \partial_{\pi_i^{-1}}$. We are interested in the monoid $\mathscr{M}^{\hat{\partial}}$ generated by $\{\hat{\partial}_i \mid 1 \leq i \leq n\}$. The next lemma shows that the action of the generators $\hat{\partial}_i$ for rooted forests is very similar to the action of the operators of the Tsetlin library by moving the letter i to the end; the difference in this case is that letters above i need to be reordered according to the poset.

Lemma 1. *Let P and $\hat{\partial}_i$ be as above, and $\pi \in \mathscr{L}(P)$. Then $\pi\hat{\partial}_i$ is the linear extension in $\mathscr{L}(P)$ obtained from π by moving the letter i to position n and reordering all letters $j \succeq i$.*

Example 5. Let P be the union of a chain of length 3 and a chain of length 2, where the first chain is labeled by the elements $\{1, 2, 3\}$ and the second chain by $\{4, 5\}$. Then $41235\ \hat{\partial}_1 = 41253$, which is obtained by moving the letter 1 to the end of the word and then reordering the letters $\{1, 2, 3\}$, so that the result is again a linear extension of P.

Let M be the transition matrix of the promotion graph of Sect. 3.2. Define $\mathscr{M}$ to be the monoid generated by $\{G_i \mid 1 \leq i \leq n\}$, where G_i is the matrix M evaluated at $x_i = 1$ and all other $x_j = 0$. We are now ready to state the main result of this section.

Theorem 6. *$\mathscr{M}$ is $\mathscr{R}$-trivial.*

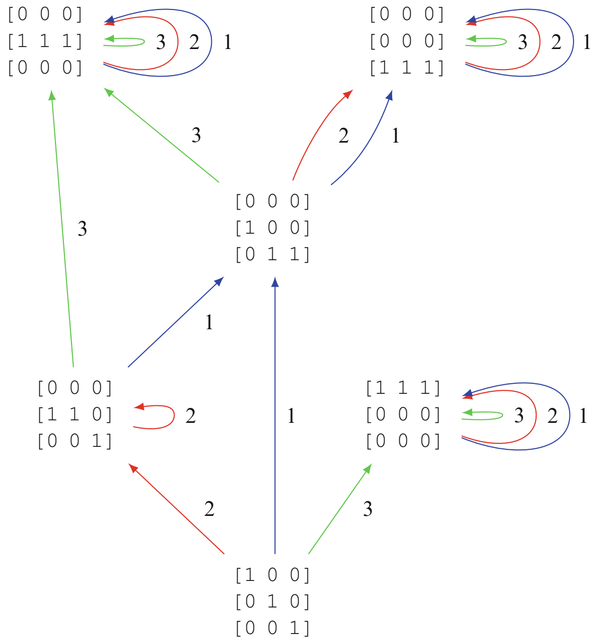


Fig. 5 Monoid $\mathscr{M}$ in right order for the poset of Example 6. With the conventions in (9), the identity is the biggest element in $\mathscr{R}$-order

Remark 2. Considering the matrix monoid $\mathscr{M}$ is equivalent to considering the abstract monoid $\mathscr{M}^{\hat{\partial}}$ generated by $\{\hat{\partial}_i \mid 1 \leq i \leq n\}$. Since the operators $\hat{\partial}_i$ act on the right on linear extensions, the monoid $\mathscr{M}^{\hat{\partial}}$ is $\mathscr{L}$-trivial instead of $\mathscr{R}$-trivial.

The proof of Theorem 6 exploits Lemma 1 by proving that there is an order on idempotents using right factors. For $x \in \mathscr{M}^{\hat{\partial}}$, let rfactor$(x)$ be the maximal common right factor of all elements in the image of x, that is, all elements $\pi \in \text{im}(x)$ can be written as $\pi = \pi_1 \cdots \pi_m\, \text{rfactor}(x)$ and there is no bigger right factor for which this is true. Let us also define the set of entries in the right factor $\text{Rfactor}(x) = \{i \mid i \in \text{rfactor}(x)\}$. Note that since all elements in the image set of x are linear extensions of P, Rfactor(x) is an upper set of P. Theorem 6 is then established by showing that for idempotents x, the set Rfactor(x) is the same as the left stabilizer $\{i \mid \hat{\partial}_i x = x\}$ which imposes a partial order.

Example 6. Let P be the poset on three elements $\{1, 2, 3\}$, where 2 covers 1 and there are no further relations. The linear extensions of P are $\{123, 132, 312\}$. The monoid $\mathscr{M}$ with $\mathscr{R}$-order, where an edge labeled i means right multiplication by G_i, is depicted in Fig. 5. From the picture it is clear that the elements in the monoid are partially ordered.

This confirms Theorem 6 that the monoid is $\mathscr{R}$-trivial. The proof of Theorem 4 now follows from [30, Theorems 6.3 and 6.4] and some further considerations regarding the lattice L. For more details see [2, Section 6].

6 Mixing Times

For random walks on hyperplane arrangements, Brown and Diaconis [10] (see also [1]) give explicit bounds for the rates of convergence to stationarity. These bounds still hold for Markov chains related to left-regular bands [8]. Here we present analogous results for the Markov chains corresponding to the $\mathscr{R}$-trivial monoids of Sect. 4. The methods are very similar to the ones we used for Markov chains related to nonabelian sandpile models [3], which also turn out to yield $\mathscr{R}$-trivial monoids.

The **rate of convergence** is the total variation distance from stationarity after k steps, that is,

$$||\mathbb{P}^k - w|| = \frac{1}{2} \sum_{\pi \in \mathscr{L}(P)} |\mathbb{P}^k(t) - w(\pi)|,$$

where $\mathbb{P}^k$ is the distribution after k steps and w is the stationary distribution.

Theorem 7. *Let P be a rooted forest with $n := |P|$ and $p_x := \min\{x_i \mid 1 \leq i \leq n\}$. Then, as soon as $k \geq (n^2-1)/p_x$, the distance to stationarity of the promotion Markov chain satisfies*

$$||\mathbb{P}^k - w|| \leq \exp\left(-\frac{(kp_x - (n^2-1))^2}{2kp_x}\right).$$

The **mixing time** [21] is the number of steps k until $||\mathbb{P}^k - w|| \leq e^{-c}$ (where different authors use different conventions for the value of c). Using Theorem 7 we require

$$(kp_x - (n^2-1))^2 \geq 2kp_x c \, ,$$

which shows that the mixing time is at most $\frac{2(n^2+c-1)}{p_x}$. If the probability distribution $\{x_i \mid 1 \leq i \leq n\}$ is uniform, then p_x is of order $1/n$ and the mixing time is of order at most n^3.

The proof of Theorem 7 follows the same outline as the proof in [3, Section 5.3]. We need to define a statistic $u(x)$ for $x \in \mathscr{M}$ such that $u(x)$ is minimal if and only if x is the constant map and furthermore

1. u **decreases along $\mathscr{R}$-order:** $u(xx') \leq u(x)$ for any $x, x' \in \mathscr{M}$.
2. **Existence of generator with strict decrease:** There exists a generator G_i such that $u(xG_i) < u(x)$.

Unlike in [3], we take $u(x) \in \mathbb{Z}^2_{\geq 0}$ with lexicographic ordering on $\mathbb{Z}^2_{\geq 0}$, that is $(x, y) < (x', y')$ if either $x < x'$, or $x = x'$ and $y < y'$. Set $u(x) := (n - |\operatorname{Rfactor}(x)|, |\operatorname{des}(x)|)$, where $\operatorname{des}(x) = \{i \mid xG_i = x\}$. It is clear that $u(x) = (0, n)$ if and only if x is a constant map, which is the minimal value u can achieve. The maximal value of u is achieved by the identity $u(e) = (n, 0)$. The two conditions follow from [2, Section 6]: either the right factor rfactor(x) increases by right multiplication by a generator G_i; if not, then $\{i\} \cup \operatorname{Rfactor}(x)$ must be an upper set again and $\operatorname{des}(xG_i) = \operatorname{des}(x) \setminus \{j \mid j \text{ covers } i \text{ in } P\}$.

Therefore, the probability that $(n, 0) \geq u(x) > (0, n)$ after k steps of the right random walk on $\mathscr{M}$ is bounded above by the probability of having at most $(n + 1)(n - 1) = n^2 - 1$ successes in k Bernoulli trials with success probability p_x. A successful step is one that decreases the statistic u. Using Chernoff's inequality for the cumulative distribution function of a binomial random variable as in [3] we obtain Theorem 7.

7 Other Posets

So far [2], we have characterized posets where the Markov chains for the promotion graph yield certain simple formulas for their eigenvalues and multiplicities. The eigenvalues have explicit expressions for rooted forests and there is an explicit combinatorial interpretation for the multiplicities as derangement numbers of permutations for unions of chains by Theorem 5.

However, we have not classified all possible posets whose promotion graphs have nice properties. For example, the eigenvalues (other than 1) of the transition matrix of the promotion graph of the poset in Example 2 are given by

$$x_3 + x_4, \quad x_3, \quad 0 \quad \text{and} \quad -x_1,$$

even though the corresponding monoid is not $\mathscr{R}$-trivial (in fact, it is not even aperiodic). The egg-box picture of the monoid is given in Fig. 6. Notice that one of the eigenvalues is negative.

On the other hand, not all posets have this property. In particular, the poset with covering relations $1 < 2, 1 < 3$ and $1 < 4$ has six linear extensions, but the characteristic polynomial of its transition matrix does not factorize at all. It would be interesting to classify all posets with the property that all the eigenvalues of

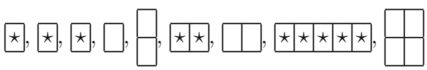
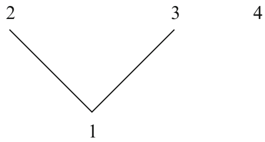
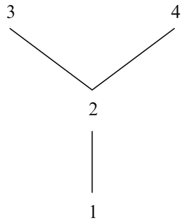
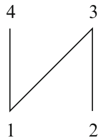
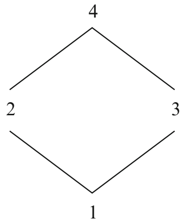
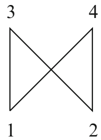


Fig. 6 Egg-box picture for the monoid associated to the promotion Markov chain for the poset in Example 2

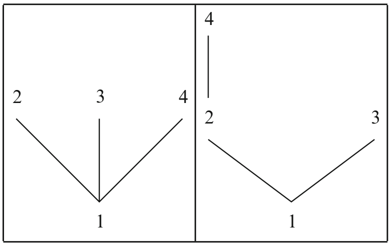
Table 1 All inequivalent posets of size 4 whose promotion transition matrices have simple expressions for their eigenvalues

Poset	Eigenvalues (other than 1)
	$0, 0, 0, x_2, x_3, x_2 + x_3, x_4 - x_1$
	$-x_1 - x_2$
	$0, x_3, -x_1, x_3 + x_4$
	$x_4 - x_1$
	$0, x_3 + x_4, -x_1 - x_2$

the transition matrices of the promotion Markov chain are linear in the probability distribution x_i. In such cases, one would also like an explicit formula for the multiplicity of these eigenvalues.

We list all posets of size 4, which are not down forests and which nonetheless have simple linear expressions for their eigenvalues in Table 1 along with the eigenvalues. For all such posets, there is at least one eigenvalue which contains a negative term. The posets, which are not down forests and the eigenvalues of whose promotion transition matrices have nonlinear expressions, are given in Table 2. Comparing the two tables, it is not obvious how to characterize those posets where

Table 2 All inequivalent posets of size 4 whose promotion transition matrices do not have simple expressions for their eigenvalues



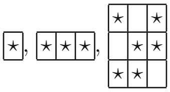


Fig. 7 Egg-box picture for the monoid associated to the promotion Markov chain for the second poset in Table 2

the eigenvalues are simple. It would be interesting to classify posets where all eigenvalues are linear in the parameters and understand the eigenvalues and their multiplicities completely. For comparison, the egg-box picture of the second poset in Table 2 is presented in Fig. 7.

Using data from all posets which are not down forests of sizes up to 7, we have the following necessary (but not sufficient) conjecture.

Conjecture 1. *Let P be a poset of size n which is not a down forest and M be its promotion transition matrix. If M has eigenvalues which are linear in the parameters $x_1, \ldots, x_n$, then the following hold*

1. *The coefficients of the parameters in the eigenvalues are only one of* ± 1*,*
2. *Each element of P has at most two successors,*
3. *The only parameters whose coefficients in the eigenvalues are* -1 *are those which either have two successors or one of whose successors have two successors.*

8 Subsets of S_n

We define a generalization of the action of promotion on an arbitrary nonempty subset of S_n inspired by the ideas in [16,22,28]. Let A be such a subset and suppose $\pi = \pi_1 \cdots \pi_n \in A$ in one-line notation. Then we define the operator σ_i for $i \in \{1, \ldots, n\}$ as

$$\pi \sigma_i = \begin{cases} \pi_1 \cdots \pi_{i-1} \pi_{i+1} \pi_i \cdots \pi_n & \text{if } \pi_1 \cdots \pi_{i-1} \pi_{i+1} \pi_i \cdots \pi_n \in A \\ \pi & \text{otherwise.} \end{cases} \tag{10}$$

In other words, σ_i acts non-trivially on a permutation in A if interchanging entries π_i and π_{i+1} yields another permutation in A, and otherwise acts as the identity. Then the generalized promotion operator, also denoted ∂_j, is an operator on A defined by

$$\partial_j = \sigma_j \sigma_{j+1} \cdots \sigma_{n-1}. \tag{11}$$

As in Sects. 3.1 and 3.2, we can define a promotion graph whose vertices are the elements of the set A and where there is an edge between permutations π and π' if and only if $\pi' = \pi \partial_j$. In the uniform promotion case, such an edge has weight x_j and in the promotion case, the edge has weight x_{π_j}. In both cases, we have analogous Markov chains. We describe the stationary distribution of these chains below.

Theorem 8. *Assuming the promotion graph for A is strongly connected, the unique stationary state weight $w(\pi)$ of the permutation $\pi \in A$ for the corresponding discrete time Markov chain is*

1. *In the uniform promotion case*

$$w(\pi) = \frac{1}{|A|}, \tag{12}$$

2. *In the promotion case*

$$w(\pi) = \prod_{i=1}^{n} \frac{x_1 + \cdots + x_i}{x_{\pi_1} + \cdots + x_{\pi_i}} \ . \tag{13}$$

The proofs are essentially identical to the proofs of Theorems 1 and 2 given in [2] and are skipped.

Remark 3. 1. The entries of w do not, in general, sum to one. Therefore this is not a true probability distribution, but this is easily remedied by a multiplicative constant Z_A depending only on the subset A.
2. Even if the set A is such that the promotion graph is not strongly connected, (12) and (13) hold. However, the formula need not be unique. The proofs of Theorem 8 still go through because all we need to do is to verify that the master equation (4) holds.

There is a natural way to build subsets A which cannot be the set of linear extensions $\mathscr{L}(P)$ for any poset P, and whose promotion graphs are yet strongly connected. The idea is to consider a union of sorting networks. A **sorting network** from the identity permutation e to any permutation π is a shortest path from one to the other by a series of nearest-neighbor transpositions. In other words, these are maximal chains in right weak order starting at the identity. For example, one sorting network to the permutation 24153 is

$$12345 \to 12435 \to 21435 \to 24135 \to 24153.$$

Proposition 2. *Let $A \subset S_n$ be a union of sorting networks. Consider the digraph G_A whose vertices are labeled by the elements of A and whose edges are given as follows: for $\pi, \pi' \in A$, there is an edge between π and π' in G_A if and only if $\pi' = \pi \partial_j$ for some $j \in [n]$. Then G_A is strongly connected.*

Proof. The operators ∂_i are each invertible, which means that each vertex of G_A has exactly one edge pointing in and one pointing out for each i. Therefore, it suffices to show that there is a directed path from e to π for every π in A.

By definition of a sorting network, π can be written as $e\sigma_{i_k} \ldots \sigma_{i_1}$. Although the action of each σ_{i_j} depends crucially on the set A, they satisfy $\sigma_{i_j}^2 = 1$. Using the fact that $\sigma_{n-1} = \partial_{n-1}$ and (11), one can recursively express each σ_{i_j} as a product of ∂_ℓ's analogous to the proof of Lemma 2.3 in [2].

As a consequence of Proposition 2, the unique stationary distribution of a subset which is a union of sorting networks is given by (13). One is naturally led to ask whether the eigenvalues of these transition matrices are also linear in the parameters. This does not seem to be true in any general sense.

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The Markov chains presented in this paper are implemented in a `Maple` package by the first author (AA) available from his home page and in `Sage` [29, 32] by the third author (AS). Many of the pictures presented here were created with `Sage`.

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Fomin-Greene Monoids and Pieri Operations

Carolina Benedetti and Nantel Bergeron

Abstract We explore monoids generated by operators on certain infinite partial orders. Our starting point is the work of Fomin and Greene on monoids satisfying the relations $(\mathbf{u}_r + \mathbf{u}_{r+1})\mathbf{u}_{r+1}\mathbf{u}_r = \mathbf{u}_{r+1}\mathbf{u}_r(\mathbf{u}_r + \mathbf{u}_{r+1})$ and $\mathbf{u}_r\mathbf{u}_t = \mathbf{u}_t\mathbf{u}_r$ if $|r - t| > 1$. Given such a monoid, the non-commutative functions in the variables $\mathbf{u}$ are shown to commute. Symmetric functions in these operators often encode interesting structure constants. Our aim is to introduce similar results for more general monoids not satisfying the relations of Fomin and Greene. This paper is an extension of a talk by the second author at the workshop on algebraic monoids, group embeddings and algebraic combinatorics at The Fields Institute in 2012.

Keywords Monoids • Pieri operators • Partial orders • Symmetric functions • Quasisymmetric functions • Structure constants • Combinatorial Hopf algebra

Subject Classifications: 05E05, 16S99, 20M25

1 Introduction

In their work on the plactic monoid, Lascoux and Schützenberger [22] constructed the Schur functions in terms of noncommutative variables satisfying only Knuth relations. It was subsequently discovered that symmetric functions can be

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constructed using different monoid algebras, for example the nil-plactic monoid, the nil-coxeter monoid or the $H_n(0)$ algebra. A uniform understanding of these constructions can be found in the seminal work of Fomin and Greene [15].

One of the main advantages of the work of Fomin and Green is that it shows the Schur positivity of certain generating functions defined on those monoid algebras. This is a central problem in algebraic combinatorics and we still have several open problems of this kind. The theory in [15] works very well for the problems it is set to solve, but it also has its limitations.

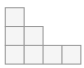
Here we want to show that this quest of understanding symmetric functions inside a monoid algebra is very alive and new results are still underway and needed. In this presentation, very close to the approach of Fomin and Greene, we look at monoids generated by operators acting on an infinite poset. We show that a certain space of functions on the monoid algebra of operators is isomorphic to symmetric functions (or a subspace of symmetric functions). These subspaces are obtained via Pieri operators as defined in [10]. The posets we consider are very often produced from a combinatorial Hopf algebra as defined in [1, 11]. Unlike the theory in [15], we are not guaranteed to have Schur positivity. Even when the object in question is Schur positive the rule of Fomin and Greene is not applicable. One has to develop new techniques to deal with this. It has been done in some cases, but it is still open in others.

We keep this paper as a talk, like a story. We introduce the results as they come from the examples. In the first part, Sect. 2, we look at a classical example. Next, in Sect. 3, we look at less known examples and constructions which are unrelated to [15]. We then look at what can be done in the future in Sect. 4.

2 A Classical Example

2.1 *Operators on the Young Lattice*

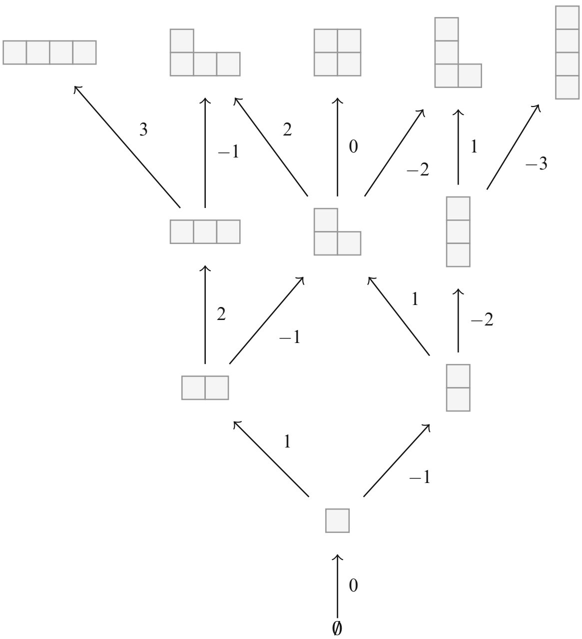
We start by a classical construction of Schur functions inspired by [14]. A *partition* of an integer n is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. When λ is a partition of n we denote it by $\lambda \vdash n$, the number of parts of λ will be denoted by $\ell(\lambda) = \ell$ and its size by $|\lambda| = n$. The diagram of a partition λ, denoted λ as well, is the subset of $\mathbb{Z}\times\mathbb{Z}$ given by $\lambda = \big\{(i, j) : 1 \leq j \leq \ell,\ 1 \leq i \leq \lambda_j\big\}$. We draw this by putting a unit box with coordinates (i, j) in the bottom left corner. For example the partition $\lambda = (4, 2, 1)$ is depicted by

 .

The Young lattice $\mathscr{Y}$ consists of all partitions $\lambda \vdash n \geq 0$, ordered by inclusion of diagrams. The empty partition is the unique partition for $n = 0$. An inclusion $\mu \subset \lambda$ is a cover if and only if $\mu \cup \{(i, j)\} = \lambda$ for a unique cell (i, j). We will label such a cover by an edge labeled by $c_{i,j} = j - i$:

$$\mu \xrightarrow{c_{i,j}} \lambda$$

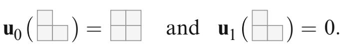
We can draw the lower part of this poset as



Consider the free $\mathbb{Z}$-module $\mathbb{Z}\mathscr{Y}$ spanned by all partitions of $n \geq 0$. We define linear operators $\mathbf{u}_r$ for each $r \in \mathbb{Z}$ as follows

$$\begin{aligned} \mathbf{u}_r \colon \mathbb{Z}\mathscr{Y} &\longrightarrow \quad \mathbb{Z}\mathscr{Y}, \\ \mu &\longmapsto \begin{cases} \lambda & \text{if } \mu \xrightarrow{r} \lambda \text{ in } \mathscr{Y} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

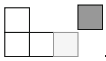
For example



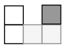
We are interested in the monoid $\mathcal{M}\langle \mathbf{u}_r \rangle$ generated by the operators $\mathbf{u}_r$ for $r \in \mathbb{Z}$ and the zero operator $\mathbf{0}$. By the nature of these operators, it is not very hard to see that they satisfy the following relations:

$$\begin{array}{lrll} (1) & \mathbf{u}_r^2 & = \mathbf{0} & \\ (2) & \mathbf{u}_r\mathbf{u}_{r+1}\mathbf{u}_r = \mathbf{u}_{r+1}\mathbf{u}_r\mathbf{u}_{r+1} & = \mathbf{0} & \\ (3) & \mathbf{u}_r\mathbf{u}_t & = \mathbf{u}_t\mathbf{u}_r & \text{if } |r-t| > 1. \end{array} \tag{2}$$

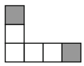
These relations can be understood graphically. The first relation states that once we add a cell in a given diagonal, if we try to add a second cell in the same diagonal we will not get a partition:


.

The second relation states that if we add two consecutive cells in a row (or column) and if we try to add a third cell in the same diagonal as the first added cell we will not get a partition:


.

The third relation states that we can add two cells independently in diagonals that are far from each other:


.

Proposition 1. *$\mathcal{M}\langle \mathbf{u}_r \rangle$ is the monoid freely generated by the $\mathbf{u}_r$ for $r \in \mathbb{Z}$ and $\mathbf{0}$ modulo the relations (2).*

This is a consequence of a more general theorem and it can be shown using some very well known facts about the symmetric group and the combinatorics of partitions. However to our knowledge this statement is not mentioned as such in the literature. To see that the relations (2) generate all the relations of the monoid $\mathcal{M}\langle \mathbf{u}_r \rangle$ requires a deeper understanding of the relations. We will sketch a proof here. Recall that the symmetric group is generated by simple reflections s_r satisfying the braid relations:

$$\begin{array}{lrll} (1) & s_r^2 & = Id & \\ (2) & s_r s_{r+1} s_r & = s_{r+1} s_r s_{r+1} & \\ (3) & s_r s_t & = s_t s_r & \text{if } |r-t| > 1. \end{array} \tag{3}$$

For a permutation w, the length $\ell(w)$ is the minimal number of generators s_r necessary to express w as a product of generators. If $w = s_{i_1} s_{i_2} \cdots s_{\ell(w)}$, then we say that the word $s_{i_1}s_{i_2}\cdots s_{\ell(w)}$ is a reduced word for w. There is a small abuse of notation here: a reduced word is an element of the free monoid generated by the s_r's. Here, we are studying the equivalence classes of words modulo the relations (3). It is a well known fact that any two reduced words for a given permutation w are connected together using only (2) and (3) of the relations (3). Moreover, if a word $s_{i_1}s_{i_2}\cdots s_k$ is not reduced, then at least one instance of the relation (1) of (3) will be used to reduce it (see [16]). The set of equivalence classes of words that do not have any occurrence of $s_rs_{r+1}s_r$ are in bijection with 321-avoiding permutation(s). These are permutations w with no $i<j<k$ such that $w(i)>w(j)>w(k)$ (see [29]).

Consider now the infinite group $S_{\mathbb{Z}}$ of permutations of $\mathbb{Z}$ with only finitely many non-fixed points. This is the group generated by the simple reflections s_r for $r\in\mathbb{Z}$ subject to the relations in (3). For $w\in S_{\mathbb{Z}}$ we define the operator

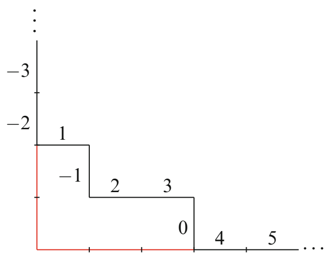
$$\mathbf{u}_w=\mathbf{u}_{i_1}\mathbf{u}_{i_2}\cdots\mathbf{u}_{i_{\ell(w)}}$$

where $s_{i_1}s_{i_2}\cdots s_{\ell(w)}$ is any reduced word for w. Comparing the relations (3) with the relations (2) we see that this is a well defined operator. Moreover, if w is not 321-avoiding, then relation (2) of (2) gives $\mathbf{u}_w=0$ and if $s_{i_1}s_{i_2}\cdots s_k$ is not a reduced word, then $\mathbf{u}_{i_1}\mathbf{u}_{i_2}\cdots\mathbf{u}_{i_k}=0$. In order to show that the relations (2) generate all the relations of $\mathscr{M}\langle\mathbf{u}_r\rangle$ it is enough to prove that

Lemma 1. *(a) For each $w\in S_{\mathbb{Z}}$ 321-avoiding, we have $\mathbf{u}_w\neq 0$,*
(b) For $w,w'\in S_{\mathbb{Z}}$ 321-avoiding, we have that $w\neq w'$ implies $\mathbf{u}_w\neq\mathbf{u}_{w'}$.

This will indeed show that the map from the free monoid generated by the $\mathbf{u}_r$'s modulo the relations (2) to $\mathscr{M}\langle\mathbf{u}_r\rangle$ has no kernel and is surjective. These results are known in some different form (see [12, 30]) and are not trivial. We will provide a proof here in this context for completeness.

Let us start with the lattice $\mathscr{Y}$ and its labelled covers. It is possible to encode this lattice and its covers with a subset of the 321-avoiding permutation(s) in $S_{\mathbb{Z}}$. Given a partition λ, add the two positive x-y axis. We put the numbers $\ldots,-3,-2,-1,0$ for every vertical step from infinity on the y-axis following the border of the partition. We put the numbers $1,2,3,\ldots$ one on each horizontal step from left to right. The example below describes this procedure better for $\lambda=(3,1)$,



When we read the entries on the y-axis, then the outer boundary of λ followed by the x-axis, we obtain a 321-avoiding permutation $v(\lambda) \in S_{\mathbb{Z}}$ (the entries on the axis are fixed points). In the example above we get

$$v(\lambda) = (\cdots, -3, -2, 1, -1, 2, 3, 0, 4, 5, \cdots).$$

If we have a cover $\mu \xrightarrow{r} \lambda$, then the entry $v(\mu)(r) \leq 0 < v(\mu)(r+1)$. Adding a box on the diagonal of content r has the effect of interchanging these two entries in $v(\mu)$. We have shown the following:

Lemma 2.

$$\mu \xrightarrow{r} \lambda \qquad \Longrightarrow \qquad v(\lambda) = v(\mu)s_r \quad \textit{and} \quad \ell(v(\lambda)) = \ell(v(\mu)) + 1.$$

This lemma allows us to show Lemma 1 (b) if we know that $\mathbf{u}_w \neq 0$. Indeed, if $\mathbf{u}_w(\mu) = \lambda$, then the above lemma gives us that $v(\lambda) = v(\mu)w$. Hence if $w \neq w'$, then $v(\mu)w \neq v(\mu)w'$ and $\mathbf{u}_w \neq \mathbf{u}_{w'}$.

Now, in order to prove Lemma 1 (a) we need to construct a partition μ such that $\mathbf{u}_w(\mu) = \lambda \neq 0$ for each 321-avoiding $w \in S_{\mathbb{Z}}$. When $\mu \subseteq \lambda$, we say that the diagram λ/μ obtained by removing the cells of μ from λ is a skew diagram. For $w \in S_{\mathbb{Z}}$ that is 321-avoiding, we construct recursively on the length $\ell(w)$ a skew diagram λ/μ such that $\mathbf{u}_w(\mu) = \lambda$. Moreover, if we read the content of the cells of λ/μ, row by row, from left to right, starting at the bottom, then we get a sequence of integers $(j_1, j_2, \ldots, j_k)$ such that $s_{j_1}s_{j_2}\cdots s_{j_k}$ is a reduced word for w. Finally, if $(i, j) \in \mu$ and $(i+1, j) \notin \mu$ and $(i, j+1) \notin \mu$, then either $(i+1, j) \in \lambda$ or $(i, j+1) \in \lambda$ (see Example 1 below).

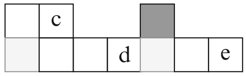
If $\ell(w) = 0$, then the result is immediate as $\lambda/\mu = \emptyset/\emptyset$ does the trick. We assume that for all 321-avoiding permutations such that $\ell(w) < \ell$ we can construct λ/μ as above. Let $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ be a reduced expression for a 321-avoiding permutation of length $\ell(w) = \ell$. By induction hypothesis we assume we have constructed λ/μ for $w' = s_{i_1}s_{i_2}\cdots s_{i_{\ell-1}}$. We can moreover assume that $(i_1, i_2, \ldots, i_{\ell-1})$ is the sequence of contents we read from λ/μ. We consider a cell on the diagonal of content $d = i_\ell$ sliding from infinity downward and stop at $(i, j) = (i, i+d)$ the first contact of either μ, λ/μ or one of the x-y-axes. We claim that

$$\text{if } (i-1, j-1) \in \lambda/\mu, \text{ then both } (i-1, j) \in \lambda/\mu \text{ and } (i, j-1) \in \lambda/\mu.$$

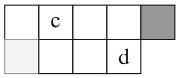
In the sequence $(i_1, i_2, \ldots, i_{\ell-1})$, let k be such that $(i_k, i_{k+1}, \ldots, i_\ell)$ are the contents of the cells in rows $i+1$ and up in λ/μ. Since no cell of λ/μ is in column j and up in row i and up, we have that $i_{k'} < j - i - 1 = d - 1$ for all $k \leq k' \leq \ell - 1$. This means that $s_{i_{k'}}$ and s_d commute for all $k \leq k' \leq \ell - 1$. We have that

$$s_{i_1}s_{i_2}\cdots s_{i_\ell} = s_{i_1}s_{i_2}\cdots s_d s_{i_k}\cdots s_{i_{\ell-1}}. \tag{4}$$

Now suppose $(i, j-1) \notin \lambda/\mu$. This means that s_d commutes with all s_c where c is the content of cells in row i of λ/μ and all cells of content e in row $i-1$ and column $j' > j+1$. We depict this as follows

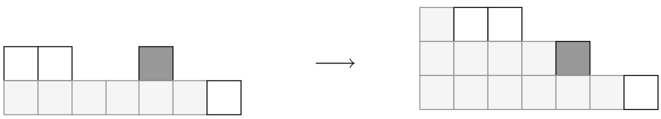


where the dark cell corresponds to the added cell in position (i, j) of content d. Since the cell $(i, j-1) \notin \lambda/\mu$ all cells in row i have content $c < d-1$. The cells in row $i-1$ and column $j' > j$ have content $e > d+1$. If $(i, j-1) \notin \lambda/\mu$, then we get $s_d s_d$ in the reduced expression of w, a contradiction. If in addition $(i-1, j) \in \lambda/\mu$, then we get $s_d s_{d+1} s_d$ which contradicts the fact that w is 321-avoiding. Hence we must have that $(i, j-1) \in \lambda/\mu$. Now if we assume that $(i-1, j) \notin \lambda/\mu$ and $(i, j-1) \in \lambda/\mu$ the picture is now



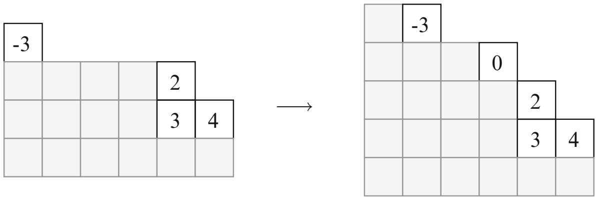
The cell in position $(i-1, j-1)$ has content d. All cells in row i and column $j' < j-1$ have content $c < d-1$. This time we can move the reflection s_d corresponding to the cell in position $(i-1, j-1)$ to pass the s_c in row i up to $s_{d-1}s_d$. Again we get a contradiction as $s_d s_{d-1} s_d$ cannot occur in the reduced word of a 321-avoiding permutation w. This concludes the case when $(i-1, j-1) \in \lambda/\mu$. In this case we simply add the cell (i, j) to λ and not to μ. The diagram $(\lambda \cup (i, j))/\mu$ is a skew shape with all the desired properties and the right hand side of (4) is the reduced word of w that we read from this diagram.

We now consider the case where $(i-1, j-1) \in \mu$ or falls outside the first quadrant. If both $(i-1, j) \in \lambda/\mu$ and $(i, j-1) \in \lambda/\mu$, then again the diagram $(\lambda \cup (i, j))/\mu$ is a skew shape with all the desired properties and the right hand side of (4) is the reduced word of w that we read from this diagram. By induction hypothesis, it is not the case that both $(i-1, j) \notin \lambda/\mu$ and $(i, j-1) \notin \lambda/\mu$. If $(i-1, j) \in \lambda/\mu$ and $(i, j-1) \notin \lambda/\mu$, then we move all the boxes of λ/μ in row $r \geq i$ up each diagonal by 1 unit. This increases the size of λ and μ proportionally but keeps the relative shape of λ/μ invariant along the diagonal lines. We then add the box (i, j) to lambda and add all the boxes (i', j) for $i' < i$ to both λ and μ. Graphically we have



The case where $(i-1, j) \notin \lambda/\mu$ and $(i, j-1) \in \lambda/\mu$ is exactly transposed, interchanging the roles of row and column. In any case we obtain the desired skew shape λ'/μ' such that $\mathbf{u}_w(\mu') = \lambda' \neq 0$.

Example 1. Let us illustrate the induction procedure involved in the proof of Lemma 1 (a). Start with $w = s_3 s_{-3} s_4 s_2$ and its skew shape as illustrated on the left hand side of the figure below. The induction step tells us that the operator $\mathbf{u}_{ws_0}$ is not zero since $\mathbf{u}_{ws_0}(\mu) = \lambda$ where $\mu = (6, 4, 4, 3, 1)$ and $\lambda = (6, 6, 5, 4, 2)$:



2.2 Pieri Operators on Young Lattice and Symmetric Functions

In the previous section, we obtained a very good understanding of the noncommutative monoid $\mathscr{M}\langle \mathbf{u}_r \rangle$. We now introduce a commutative algebra $\mathbf{B}\langle H_k \rangle$ that is isomorphic to the (Hopf) algebra of symmetric functions Sym. The algebra $\mathbf{B}\langle H_k \rangle$ is generated by certain homogeneous series H_k in the elements of $\mathscr{M}\langle \mathbf{u}_r \rangle$. This is using the Pieri operators theory as developed in [10] related to the multiplication of symmetric functions (see [25]).

There are several combinatorial Hopf algebras of interest for our study. As it turns out, $\mathscr{Y}$ is intimately related to Sym. The space of symmetric functions is well known to have different bases indexed by partitions. We refer the reader to [25, 27] for more details about our presentation of Sym. We use the standard notation for the common bases of Sym: h_λ for complete homogeneous; e_λ for elementary; m_λ for monomial; and s_λ for Schur functions. For simplicity, we let h_i and e_i denote the corresponding generators indexed by the partition (i).

There is a correspondence between the representation theory of all symmetric groups and symmetric functions. The multiplication and comultiplication in Sym corresponds to some induction and restriction of representations. In this identification, Schur functions encode irreducible representations. In particular we must have that the coefficients $C^{\nu}_{\lambda,\mu}$ in

$$s_\lambda s_\mu = \sum_\nu C^{\nu}_{\lambda,\mu} s_\nu \tag{5}$$

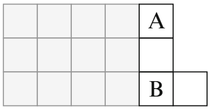
are non-negative integers. They count the multiplicity of an irreducible in certain induced representations. This shows the nonnegativity of the constants $C^{\nu}_{\lambda,\mu}$ but does not give us a combinatorial formula for them. One is interested in a positive combinatorial rule to describe these numbers. This combinatorial rule is classically known as the Littlewood-Richardson rule. A particular case of this rule is Pieri rule that describes the multiplication by h_k:

$$s_\lambda h_k = \sum_{\nu/\lambda \text{ a } k\text{-row strip}} s_\nu$$

where a k-row strip is a diagram with k cells in distinct columns. In terms of the lattice $\mathscr{Y}$ we have the following characterization of k-row strip.

Lemma 3. *ν/λ is a k-row strip if and only if there is a strictly increasing path of length k in $\mathscr{Y}$ from λ to ν. Moreover, if such a path exists from λ to ν, then it is unique.*

Proof. If ν/λ is a k-row strip, then we can add the cells of ν/λ to λ one by one from left to right. Since the cells are in distinct columns, they are in distinct diagonals as well. Adding them from left to right will give us the desired strictly increasing path from λ to ν. Conversely, if we have a strictly increasing path from λ to ν, then the cells of ν/λ are in distinct diagonals. Assume two cells of ν/λ are in the same column as pictured bellow



The cell A has content strictly smaller than the content of the cell B. In the path from λ to ν the cell A would be added before the cell B. But this is a contradiction since when the cell A is added without cell B this would not be a partition. Hence, ν/λ is a k-row strip. □

This allows us to reconstruct the multiplication by h_k using operators on $\mathscr{Y}$. Let

$$H_k = \sum_{i_1 < i_2 < \cdots < i_k} \mathbf{u}_{i_k} \cdots \mathbf{u}_{i_2} \mathbf{u}_{i_1}.$$

This is an infinite series of operators of degree k in $\mathscr{M}\langle \mathbf{u}_r \rangle$. In view of Lemma 1, no term in the series H_k vanishes. If one fixes λ, there are only finitely many paths of length k from λ in $\mathscr{Y}$. This means that $H_k : \mathbb{Z}\mathscr{Y} \to \mathbb{Z}\mathscr{Y}$ is a well defined operator.

Proposition 2.

$$H_k = \sum_{\ell(\zeta)=k} \mathbf{u}_\zeta,$$

where ζ runs over all permutations such that its disjoint cycle decomposition $\zeta = C_1 C_2 \cdots C_s$ has only cycles of the form $C_i = (a+b, \ldots, a+1, a)$ for some $a, b \in \mathbb{Z}$ and $b > 0$.

Proof. It suffices to show that

$$\mathbf{u}_{i_k} \cdots \mathbf{u}_{i_2} \mathbf{u}_{i_1} = \mathbf{u}_\zeta$$

with $i_1 < i_2 < \ldots < i_k$ if and only if ζ decomposes into disjoint cycles of the form $(a+b, \ldots, a+1, a)$. The disjoint cycles of $\zeta = s_{i_k} \cdots s_{i_2} s_{i_1}$ for $i_1 < i_2 < \ldots < i_k$ correspond to the consecutive segments $s_{a+b} \cdots s_{a+1} s_a = (a+b, \ldots, a+1, a)$.

Using Lemma 3

$$H_k(\lambda) = \sum_{\nu/\lambda \text{ a } k\text{-row strip}} \nu \qquad \Longleftrightarrow \qquad s_\lambda h_k = \sum_{\nu/\lambda \text{ a } k\text{-row strip}} s_\nu \, .$$

This implies that

$$H_b H_a(\lambda) = \sum_\nu d^\nu_{\lambda,(a,b)} \nu \qquad \Longleftrightarrow \qquad s_\lambda h_a h_b = \sum_\nu d^\nu_{\lambda,(a,b)} s_\nu \, .$$

In particular, for all λ we have $H_b H_a(\lambda) = H_a H_b(\lambda)$ since $h_a h_b = h_b h_a$. Again the result below is derived from very classical results.

Theorem 1. *The algebra* $\mathbf{B}\langle H_k \rangle$ *spanned by* $\{H_1, H_2, H_3, \ldots\}$ *is isomorphic to* Sym.

Proof. We have seen above that $H_b H_a(\lambda) = H_a H_b(\lambda)$, but to see that the product of series $H_b H_a = H_a H_b$ requires a little bit more argument. As we multiply $H_a H_b$ and $H_b H_a$, some terms will go to zero and others will survive. The terms that survive in $H_a H_b$ are of the form

$$\mathbf{u}_w = \mathbf{u}_{i_1} \mathbf{u}_{i_2} \cdots \mathbf{u}_{i_a} \mathbf{u}_{j_1} \mathbf{u}_{j_2} \cdots \mathbf{u}_{j_b}$$

where w is 321-avoiding, $i_1 < i_2 < \cdots < i_a$ and $j_1 < j_2 < \cdots < j_b$. Showing that $H_a H_b = H_b H_a$ requires the construction of a bijection between the possible reduced expressions of $w = s_{i_1} \cdots s_{i_a} s_{j_1} \cdots s_{j_b}$ and the ones of the form $w = s_{j'_1} \cdots s_{j'_b} s_{i'_1} \cdots s_{i'_b}$ where $i'_1 < i'_2 < \cdots < i'_a$ and $j'_1 < j'_2 < \cdots < j'_b$. This is done in [29] and in [9] using jeu-de-taquin. We then have that $\{H_\mu = H_{\mu_1} H_{\mu_2} \cdots H_{\mu_1} : \mu \text{ partition}\}$ spans $\mathbf{B}\langle H_k \rangle$. To see that the H_μ are linearly independent, it suffices to remark that

$$H_\mu(\emptyset) = \mu$$

hence they have distinct values on $\emptyset$. □

Remark 1. Theorem 1 follows easily from the more general Theorem 1.1 of [15]. The approach of Fomin and Greene has the advantage that one does not need to have all the relations of the $\mathbf{u}_r$. It is enough to show that they satisfy the relations:

$$\begin{array}{llll} (1) & \mathbf{u}_r\mathbf{u}_t &= \mathbf{u}_t\mathbf{u}_r & \text{if } |r-t|>1, \\ (2) & (\mathbf{u}_r+\mathbf{u}_{r+1})\mathbf{u}_{r+1}\mathbf{u}_r &= \mathbf{u}_{r+1}\mathbf{u}_r(\mathbf{u}_r+\mathbf{u}_{r+1}) & \end{array} \qquad (6)$$

It is clear that our operators $\mathbf{u}_r$ satisfy the relations (6). In later sections we will give examples where Fomin and Greene theory is not applicable.

2.3 NSym and QSym

For the theory of Pieri operators as developed in [10] we need to introduce two graded dual Hopf algebras. First, the algebra of non-commutative symmetric functions $Nsym$ is a non-commutative analogue of Sym that arises by considering an algebra with one non-commutative generator at each positive degree. We define $Nsym$ as the algebra with generators $\{\mathbf{h}_1, \mathbf{h}_2, \dots\}$ and no relations. Each generator $\mathbf{h}_i$ is defined to be of degree i, giving $Nsym$ the structure of a graded algebra. We let $Nsym_n$ denote the graded component of $Nsym$ of degree n. A basis for $Nsym_n$ is given by the set of *complete homogeneous functions* $\{\mathbf{h}_\alpha := \mathbf{h}_{\alpha_1}\mathbf{h}_{\alpha_2}\cdots\mathbf{h}_{\alpha_m}\}_{\alpha\vDash n}$ indexed by compositions α of n.

We have the projection morphism $\chi\colon Nsym \to Sym$ defined by sending the basis element $\mathbf{h}_\alpha$ to the complete homogeneous symmetric function

$$\chi(\mathbf{h}_\alpha) := h_{\alpha_1}h_{\alpha_2}\cdots h_{\alpha_{\ell(\alpha)}}$$

and extended linearly to all of $Nsym$. A second basis of $NSym$ is given by the R_α, usually called the *ribbon basis*. For this, given a composition $\alpha = (\alpha_1, \cdots, \alpha_m) \vDash n$ we denote its length m by $\ell(\alpha)$. The ribbon basis R_α are defined by

$$R_\alpha = \sum_{\beta\geq\alpha}(-1)^{\ell(\alpha)-\ell(\beta)}\mathbf{h}_\beta, \qquad \text{or equivalently} \qquad \mathbf{h}_\alpha = \sum_{\beta\geq\alpha} R_\beta \qquad (7)$$

where $\alpha \leq \beta$ if α is finer than β.

The product expansion follows easily from the non-commutative product on the generators

$$\mathbf{h}_\alpha\mathbf{h}_\beta = \mathbf{h}_{\alpha_1,\dots\alpha_{\ell(\alpha)},\beta_1,\dots\beta_{\ell(\beta)}} \ .$$

$Nsym$ has a coalgebra structure, which is defined on the generators by

$$\Delta(\mathbf{h}_j) = \sum_{i=0}^{j}\mathbf{h}_i \otimes \mathbf{h}_{j-i} \ .$$

This determines the action of the coproduct on the basis $\mathbf{h}_\alpha$ since the coproduct is an algebra morphism with respect to the product.

Second, the Hopf algebra of quasi-symmetric functions, $Qsym$ is dual to $Nsym$ and contains Sym as a subalgebra. The graded component $Qsym_n$ is indexed by compositions of n. This algebra is most readily realized within the ring of power series of bounded degree $\mathbb{Q}[\![x_1, x_2, \dots]\!]$. The monomial quasi-symmetric function indexed by a composition α is defined as

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_m}^{\alpha_m}. \tag{8}$$

The algebra of quasi-symmetric functions, $Qsym$, can then be defined as the algebra with the monomial quasi-symmetric functions as a basis, whose multiplication is inherited as a subalgebra of $\mathbb{Q}[\![x_1, x_2, \dots]\!]$. We define the coproduct on this basis as:

$$\Delta(M_\alpha) = \sum_{S \subset \{1,2,\dots,\ell(\alpha)\}} M_{\alpha_S} \otimes M_{\alpha_{S^c}},$$

where if $S = \{i_1 < i_2 < \cdots < i_{|S|}\}$, then $\alpha_S = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{|S|}})$.

We view Sym as a subalgebra of $Qsym$. In fact, the usual monomial symmetric functions $m_\lambda \in Sym$ expand positively in the quasi-symmetric monomial functions:

$$m_\lambda = \sum_{sort(\alpha) = \lambda} M_\alpha,$$

where $sort(\alpha)$ is the partition obtained by organizing the parts of α from the largest to the smallest.

The *fundamental quasi-symmetric functions*, denoted by F_α form another basis of $Qsym_n$ and are defined by their expansion in the monomial quasi-symmetric basis:

$$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta.$$

The algebras $Qsym$ and $Nsym$ form graded dual Hopf algebras. The monomial basis of $Qsym$ is dual in this context to the complete homogeneous basis of $Nsym$, and the fundamental basis of $Qsym$ is dual to the ribbon basis of $Nsym$. $Nsym$ and $Qsym$ have a pairing $\langle \cdot, \cdot \rangle : Nsym \times Qsym \to \mathbb{Q}$, defined under this duality as either $\langle \mathbf{h}_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$, or $\langle R_\alpha, F_\beta \rangle = \delta_{\alpha,\beta}$.

2.4 Skew Function $K_{[\lambda,\nu]}$

Associated to any $\lambda \subseteq \nu$ in $\mathscr{Y}$, we construct a quasisymmetric function $K_{[\lambda,\nu]}$ following the notion of Pieri operators as developed in [10]. Let $\langle \lambda, \mu \rangle = \delta_{\lambda,\mu}$ define a scalar product on $\mathbb{Z}\mathscr{Y}$. Using the operators H_k on $\mathbb{Z}\mathscr{Y}$ we can define

$$K_{[\lambda,\nu]} = \sum_{\alpha} \langle H_\alpha(\lambda), \nu \rangle M_\alpha.$$

In view of the commutation relation $H_a H_b = H_b H_a$, the function $K_{[\lambda,\nu]}$ is not only quasisymmetric but symmetric as well. Indeed since $H_\alpha = H_{sort(\alpha)}$ and since $m_\lambda = \sum_{sort(\alpha)=\lambda} M_\alpha$, we have that

$$K_{[\lambda,\nu]} = \sum_{\mu} \langle H_\mu(\lambda), \nu \rangle m_\mu$$

is symmetric. We are interested in knowing the coefficients of $K_{[\lambda,\nu]}$ when expanded in different bases. We remark that we have an action of $NSym$ on $\mathbb{Z}\mathscr{Y}$ given by $\mathbf{h}_\alpha.\lambda = H_\alpha(\lambda)$. In this case the action factors through the projection $\chi \colon NSym \to Sym$. As observed earlier, the basis $\mathbf{h}_\alpha$ of $NSym$ is dual to the basis M_α of $QSym$. A straightforward computation shows that

$$K_{[\lambda,\nu]} = \sum_{\alpha} \langle \mathbf{h}_\alpha.\lambda, \nu \rangle M_\alpha = \sum_{\alpha} \langle \mathbf{x}_\alpha.\lambda, \nu \rangle Y_\alpha,$$

for any dual bases $\mathbf{x}_\alpha$ and Y_α of $NSym$ and $QSym$ respectively. We thus have that

Theorem 2.

$$K_{[\lambda,\nu]} = \sum_{\alpha} \langle R_\alpha.\lambda, \nu \rangle F_\alpha = \sum_{\mu} C^{\nu}_{\lambda,\mu} s_\mu$$

where $C^{\nu}_{\lambda,\mu}$ is given in (5). Moreover for $\alpha = (\alpha_1, \ldots, \alpha_k)$ a composition of n, we have that $\langle R_\alpha.\lambda, \nu \rangle$ counts the number of paths in $\mathscr{Y}$ from λ to ν with labels $i_1, i_2, \ldots, i_n$ such that $i_r > i_{r+1}$ if and only if $r \in D(\alpha) = \{\alpha_1, \alpha_1+\alpha_2, \ldots, n-\alpha_k\}$.

Proof. The first equality follows from duality between the R_α and the F_α. For the second equality, from the definition of H_k we remark that for

$$K_{[\lambda,\nu]} = \sum_{\mu} \langle H_\mu(\lambda), \nu \rangle m_\mu$$

the coefficient $\langle H_\mu(\lambda), \nu \rangle = d^{\nu}_{\lambda,\mu}$ is the coefficient of s_ν in the product $s_\lambda h_\mu$. In Sym, the bases h_μ and m_μ are dual and the basis s_μ is self dual. Hence the coefficient of s_μ in $K_{[\lambda,\nu]}$ is the same as the coefficient of s_ν in $s_\lambda s_\mu$.

The fact that $\langle R_\alpha.\lambda, \nu \rangle$ counts the paths as described follows from a simple inclusion-exclusion argument and the fact that by definition $\langle \mathbf{h}_\alpha.\lambda, \nu \rangle$ counts the paths in $\mathscr{Y}$ from λ to ν with labels $i_1, i_2, \ldots, i_n$ such that $i_r > i_{r+1}$ only if $r \in D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, n - \alpha_k\}$. ☐

Remark 2. The function $K_{[\lambda,\nu]}$ in Theorem 2 is the well known skew-Schur function $s_{\nu/\lambda}$. It is denoted $F_{\nu/\lambda}$ by Fomin and Greene in [15]. Theorem 1.2 of [15] shows that the coefficients $C^{\nu}_{\lambda,\mu}$ are positive and count paths in $\mathscr{Y}$ satisfying a precise rule.

This is a very powerful method that works for any monoid of operators $\mathbf{u}_r$ satisfying the relations (6). Several classical examples are solved by this theory which gives a method to understand the coefficients we are interested in. This includes the weak order of the symmetric group and the Stanley symmetric function $F_{w/u}$ originally defined in [29]. There are many new situations where Fomin and Greene theory cannot be applied and we will give some examples of this in the next sections.

3 Schubert vs Schur

We present an example of a monoid that does not satisfy Fomin and Greene's conditions, yet it is interesting and still yields some symmetry and positivity. In this example, which is taken from the theory of Schubert polynomials (see [7, 23, 24]), positivity results are highly non-trivial. We consider operators on the infinite symmetric group defined from Monk's rule. From these operators one defines Pieri operators that mimic the multiplication of Schubert polynomials by symmetric functions. Symmetry follows from the commutativity of multiplication and positivity follows from geometry. A combinatorial proof of positivity is much harder to obtain and was only recently achieved in [4] using the techniques of [2].

Let $u \in \mathscr{S}_\infty := \bigcup_{n\geq 0} \mathscr{S}_n$ be an infinite permutation where all but a finite number of positive integers are fixed. Schubert polynomials $\mathfrak{S}_u$ are indexed by such permutations [23, 24]. These polynomials form a homogeneous basis of the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$ in countably many variables. The coefficients $c^w_{u,v}$ in

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c^w_{u,v} \mathfrak{S}_w, \tag{9}$$

are known to be positive from geometry.

3.1 Operators on the r-Bruhat Order

We now define operators on the r-Bruhat order on $\mathscr{S}_\infty$. Let $\ell(w)$ be the length of a permutation $w \in \mathscr{S}_\infty$. We define the r-Bruhat order $<_r$ by its covers. Given permutations $u, w \in \mathscr{S}_\infty$, we say that $u \lessdot_r w$ if $\ell(u) + 1 = \ell(w)$ and $u^{-1}w = (i, j)$, where (i, j) is a reflection with $i \leq r < j$.

For $0 < a < b$, let $\mathbf{u}_{ab}$ denote the operator on $\mathbb{Z}S_\infty$ defined by

$$\begin{aligned} \mathbf{u}_{ab} \colon \mathbb{Z}S_\infty &\longrightarrow \quad \mathbb{Z}S_\infty, \\ u &\longmapsto \begin{cases} (a, b)u & \text{if } u \lessdot_r (a, b)u, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{10}$$

We have shown in [8] that these operators satisfy the following relations:

$$\begin{array}{llll}
(1) & \mathbf{u}_{bc}\mathbf{u}_{cd}\mathbf{u}_{ac} \equiv \mathbf{u}_{bd}\mathbf{u}_{ab}\mathbf{u}_{bc}, & & \text{if } a < b < c < d, \\
(2) & \mathbf{u}_{ac}\mathbf{u}_{cd}\mathbf{u}_{bc} \equiv \mathbf{u}_{bc}\mathbf{u}_{ab}\mathbf{u}_{bd}, & & \text{if } a < b < c < d, \\
(3) & \mathbf{u}_{ab}\mathbf{u}_{cd} \equiv \mathbf{u}_{cd}\mathbf{u}_{ab}, & & \text{if } b < c \text{ or } a < c < d < b, \\
(4) & \mathbf{u}_{ac}\mathbf{u}_{bd} \equiv \mathbf{u}_{bd}\mathbf{u}_{ac} & \equiv \mathbf{0}, & \text{if } a \le b < c \le d, \\
(5) & \mathbf{u}_{bc}\mathbf{u}_{ab}\mathbf{u}_{bc} \equiv \mathbf{u}_{ab}\mathbf{u}_{bc}\mathbf{u}_{ab} & \equiv \mathbf{0}, & \text{if } a < b < c.
\end{array} \qquad (11)$$

The **0** in relations (4) and (5) mean(s) that no chain in any r-Bruhat order can contain such a sequence of transpositions. On the other hand, relations (1), (2) and (3) are complete and transitively connect any two chains in a given interval $[u, w]_r$. It is interesting to notice that the relations are independent of r. This is a fact noticed in [7]: a nonempty interval $[u, w]_r$ in the r-Bruhat order is isomorphic to a nonempty interval $[x, y]_{r'}$ in an r'-Bruhat order as long as $wu^{-1} = yx^{-1}$. It is important to remark that if one fixes r, there are in fact more relations than (11). We will clarify this after Proposition 3. For the moment we assume that the operator $\mathbf{u}_{ab}$ acts on the disjoint union of all r-Bruhat orders for $r > 0$. Let $\mathscr{M}\langle \mathbf{u}_{ab}\rangle$ be the monoid generated by the **0** operator and all operators $\mathbf{u}_{ab}$ for $a < b$. A consequence of [8] is the following proposition.

Proposition 3. *$\mathscr{M}\langle \mathbf{u}_{ab}\rangle$ is the monoid freely generated by the $\mathbf{u}_{ab}$ for $0 < a < b \in \mathbb{Z}$ and **0** modulo the relations (11).*

Remark 3. When we specify a chain $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$ in the interval $[u, w]_r$, it is understood that this is the actual sequence $(\mathbf{u}_{a_nb_n}, \ldots, \mathbf{u}_{a_2b_2}, \mathbf{u}_{a_1b_1})$ of operators we are referring to. This is a slight abuse of notation but it simplifies notation and the context will make it clear. □

In fact we can say much more about the monoid $\mathscr{M}\langle \mathbf{u}_{ab}\rangle$. Given any $\zeta \in S_\infty$ we produce a chain in a nonempty interval $[u, w]_r$ for some r as follows. Let $up(\zeta) = \{a : \zeta^{-1}(a) < a\}$. This is a finite set and we can set $r = |up(\zeta)|$. To construct w, we sort the elements in $up(\zeta) = \{i_1 < i_2 < \cdots < i_r\}$ and its complement $up^c(\zeta) = \mathbb{Z}_{>0} \setminus up(\zeta) = \{j_1 < j_2 < \ldots\}$. Next, we put $w = [i_1, i_2, \ldots, i_r, j_1, j_2, \ldots] \in S_\infty$ and then we let $u = \zeta^{-1}w$. Notice that u, w and r constructed this way depend on ζ. From [7, 8], we have that $[u, w]_r$ is non-empty and now we want to construct a chain in $[u, w]_r$. This is done recursively as follows: let

$$a_1 = u(i_1) \text{ where } i_1 = \max\{i \le r : u(i) < w(i)\} \quad \text{ and}$$
$$b_1 = u(j_1) \text{ where } j_1 = \min\{j > r : u(j) > u(i_1) \ge w(j)\}$$

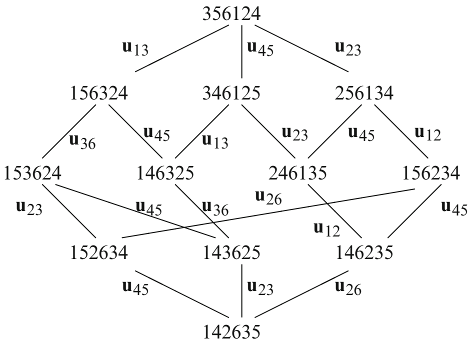
then $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$ is a chain in $[u, w]_r$ for any chain $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}$ in $[(a_1, b_1)u, w]_r$. Here we have that all the other possible chains in the interval $[u, w]_r$ are obtained from the chain $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$ by sequences of transformations given in Eq. (11). This means that the operator $\mathbf{u}_\zeta = \mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$ is well defined, non zero for any $r' \ge r$ and if $\zeta \ne \zeta'$ then $\mathbf{u}_\zeta \ne \mathbf{u}_{\zeta'}$. For a fix r,

$$\mathbf{u}_\zeta(u) = \begin{cases} \zeta u & \text{if } u <_r \zeta u, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2. Consider $\zeta = [3, 6, 2, 5, 4, 1, \ldots]$ where all other values are fixed. We have that $up(\zeta) = \{3, 5, 6\}$ and $up^c(\zeta) = \{1, 2, 4, \ldots\}$. In this case, $r = 3$, $w = [3, 5, 6, 1, 2, 4, \ldots]$ and $u = [1, 4, 2, 6, 3, 5, \ldots]$. The recursive procedure above produces the chain $\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{45}\mathbf{u}_{26}$ in $[u, v]_3$. We get all other chains by using the relations (11):

$$\begin{array}{c} \mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{45}\mathbf{u}_{26},\ \mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{26}\mathbf{u}_{45},\ \mathbf{u}_{23}\mathbf{u}_{45}\mathbf{u}_{12}\mathbf{u}_{26},\ \mathbf{u}_{45}\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{26}, \\ \mathbf{u}_{45}\mathbf{u}_{13}\mathbf{u}_{36}\mathbf{u}_{23},\ \mathbf{u}_{13}\mathbf{u}_{45}\mathbf{u}_{36}\mathbf{u}_{23},\ \mathbf{u}_{13}\mathbf{u}_{36}\mathbf{u}_{45}\mathbf{u}_{23},\ \mathbf{u}_{13}\mathbf{u}_{36}\mathbf{u}_{23}\mathbf{u}_{45}. \end{array} \tag{12}$$

The interval obtained in this case is



Since $u <_r \zeta u$ in this case, we have $\mathbf{u}_\zeta(u) = \zeta u \neq 0$ for this r. Now for any $r' \geq r$, we can build $w' = [3, 5, 6, 7, 8, \ldots, 7+r'-r, 1, 2, 4, \ldots]$, $u' = [1, 4, 2, 7, 8, \ldots, 7+r'-r, 6, 3, 5, \ldots]$ by adding fixed points of $\zeta = wu^{-1}$ before the position r'. In this way we construct a permutation u' such that $u' <_{r'} \zeta u'$ and $\mathbf{u}_\zeta(u') = \zeta u' \neq 0$ for any $r' \geq r$.

The above discussion shows the following corollary:

Corollary 1. *The monoid $\mathscr{M}\langle \mathbf{u}_{ab} \rangle$ is precisely*

$$\mathscr{M}\langle \mathbf{u}_{ab} \rangle = \{\mathbf{u}_\zeta : \zeta \in \mathscr{S}_\infty\} \cup \{\mathbf{0}\}.$$

Moreover, if we let $\mathscr{M}_r\langle \mathbf{u}_{ab} \rangle$ be the monoid generated by the operator $\mathbf{u}_{ab}$ acting on r-Bruhat order for a fixed r, we have

$$\mathscr{M}_r\langle \mathbf{u}_{ab} \rangle = \{\mathbf{u}_\zeta : \zeta \in \mathscr{S}_\infty,\ |up(\zeta)| \leq r\} \cup \{\mathbf{0}\}.$$

Here the multiplication in $\mathscr{M}\langle \mathbf{u}_{ab} \rangle$ is given by $\mathbf{u}_\zeta \mathbf{u}_\eta = \mathbf{u}_{\zeta\eta}$ if $\eta u <_r \zeta\eta u$ for some u and r, and is $\mathbf{0}$ otherwise.

3.2 Pieri Operators on r-Bruhat Order

We now introduce some Pieri operators related to the operators $\mathbf{u}_{ab}$. These Pieri operators are defined in such a way that they mimic the multiplication of a Schubert polynomial by the homogeneous symmetric polynomial $h_k(x_1, \ldots, x_r)$.

A permutation $v \in \mathscr{S}_\infty$ such that $v(1) < v(2) < \cdots < v(r)$ and $v(r+1) < v(r+2) < \cdots$ is called *r-grassmannian*. Any partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ with at most r non-zero parts defines a unique r-grassmannian permutation

$$v(\lambda, r) = [\lambda_r + 1, \lambda_{r-1} + 2, \ldots, \lambda_1 + r, v(r+1), \ldots],$$

where $v(r+1) < v(r+2) < \cdots$ are the positive integers not in $\{\lambda_r + 1, \lambda_{r-1} + 2, \ldots, \lambda_1 + r\}$. As seen in [23, 24], for any such partition λ we have that the Schur polynomial $S_\lambda(x_1, x_2, \ldots, x_r)$ is equal to the Schubert polynomial

$$\mathfrak{S}_{v(\lambda,r)} = S_\lambda(x_1, x_2, \ldots, x_r).$$

In particular, the homogeneous polynomial $h_k(x_1, \ldots, x_r)$ is the Schubert polynomial $\mathfrak{S}_{v((k),r)}$. The multiplication of an arbitrary Schubert polynomial by $h_k(x_1, \ldots, x_r)$ is known as the Pieri formula for Schubert polynomials. It was originally stated as a theorem by Lascoux and Schützenberger [23] with a very brief outline of a proof. Sottile later proved this formula geometrically and clarified the history for us [28]. Using the operators $\mathbf{u}_{ab}$ on the r-Bruhat order, this can be stated as follows.

$$\mathfrak{S}_u h_k(x_1, \ldots, x_r) = \mathfrak{S}_u \mathfrak{S}_{v((k),r)} = \sum_w \mathfrak{S}_w, \tag{13}$$

where the sum is over all $w >_r u$ such that $\mathbf{u}_{a_k b_k} \cdots \mathbf{u}_{a_2 b_2} \mathbf{u}_{a_1 b_1}(u) = w$ for some $b_1 < b_2 < \cdots < b_k$. It is known (see [7, 8]) that in such interval $[u, w]_r$, there must be a chain from u to w that is increasing in the sense that $\mathbf{u}_{a_k b_k} \cdots \mathbf{u}_{a_2 b_2} \mathbf{u}_{a_1 b_1}(u) = w$ with $b_1 < b_2 < \cdots < b_k$. Such a chain, when it exists, is unique among all saturated chains in $[u, w]_r$.

We now introduce series H_k similar to Sect. 2.2 that will commute with each other and encode the Pieri formula for Schubert polynomials. Let

$$H_k = \sum_{\substack{b_1 < b_2 < \cdots < b_k \\ a_i < b_i}} \mathbf{u}_{a_k b_k} \cdots \mathbf{u}_{a_2 b_2} \mathbf{u}_{a_1 b_1}. \tag{14}$$

Many of the terms in this sum are zero, the non-zero terms have a very special form. In [23], we see that it is important to look at the disjoint decomposition of ζ into disjoint cycles. In the next proposition we describe the $\mathbf{u}_\zeta$ appearing in H_k and the structure of the disjoint cycles. For $\zeta \in \mathscr{S}_\infty$, let $\zeta = C_1 C_2 \cdots C_s$ be the decomposition of ζ in disjoint non-trivial cycles. There are only finitely many non-fixed points, so only finitely many non-trivial cycles. Given a cycle $C = (c_1, c_2, \ldots, c_m)$, we say that C is *increasing* if $c_m < c_{m-1} < \cdots < c_1$. Given two disjoint increasing cycles $C = (c_1, c_2, \ldots, c_m)$ and $C' = (c'_1, c'_2, \ldots, c'_n)$ we say that they are *totally disjoint* if any of the following happens

1. $[c_m, c_1] \cap [c'_n, c'_1] = \emptyset$, or
2. $[c_m, c_1] \cap \{c'_1, c'_2, \ldots, c'_n\} = \emptyset$, or
3. $[c'_n, c'_1] \cap \{c_1, c_2, \ldots, c_m\} = \emptyset$.

In case (1), the two cycles have support in disjoint intervals. In cases (2) and (3), If the intervals intersect, their intersection must fall between two successive elements in the support of the other cycles. For $C = (c_1, c_2, \ldots, c_m)$ let $\|C\| = m - 1$. For $\zeta = C_1 C_2 \cdots C_s$ a product of totally disjoint increasing cycles such that $k = \sum_{i=1}^{s} \|C_i\|$, we say that ζ is *k-increasing*.

Proposition 4.

$$H_k = \sum_{\zeta} \mathbf{u}_\zeta,$$

where ζ runs over k-increasing permutations.

Proof. We proceed by induction on k. The result is clear for $k = 1$. Assume the result is true for any non-zero product $\mathbf{u}_{\zeta'} = \mathbf{u}_{a_{k-1}b_{k-1}} \cdots \mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$ such that $b_1 < b_2 < \ldots < b_{k-1}$. We assume that $\mathbf{u}_{\zeta'} = \mathbf{u}_{C_1}\mathbf{u}_{C_2} \cdots \mathbf{u}_{C_s}$ where $\zeta = C_1 C_2 \cdots C_s$ are totally disjoint increasing cycles. For $C = (c_1, c_2, \ldots, c_m)$, an increasing cycle, we have $\mathbf{u}_C = \mathbf{u}_{c_2c_1}\mathbf{u}_{c_3c_2} \cdots \mathbf{u}_{c_mc_{m-1}}$. A careful analysis of the relation (11) shows that for totally disjoint increasing cycles $C_1 C_2 \cdots C_s$, the operators $\mathbf{u}_{C_i}$ and $\mathbf{u}_{C_j}$ commute for $i \neq j$. We will assume that a_{k-1} and b_{k-1} belong to the cycle C_1.

We investigate what happens when we perform a non-zero product $\mathbf{u}_{a_kb_k}\mathbf{u}_{\zeta'}$ where $b_k > b_{k-1}$. If $a_k > b_{k-1}$, then (b_k, a_k) is a new increasing cycle totally disjoint from any cycle of ζ'. If $a_k = b_{k-1}$, then a_k, b_k increases the cycle C_1 of ζ' and is still totally disjoint from the other cycles of ζ'.

If $a_k < b_{k-1}$, then from (11)–(4) we must have $a_k < a_{k-1}$ and the operators $\mathbf{u}_{a_kb_k}\mathbf{u}_{a_{k-1}b_{k-1}} \neq \mathbf{0}$ commute. Let $C_1 = (c_1, c_2, \ldots, c_m)$ and recall that we have $b_{k-1} = c_1$ and $a_{k-1} = c_2$. We have $\mathbf{u}_{C_1} = \mathbf{u}_{c_2c_1}\mathbf{u}_{c_3c_2} \cdots \mathbf{u}_{c_mc_{m-1}}$ and $b_k > b_{k-1} = c_1 > c_i$ for all i. Since $a_k < a_{k-1} = c_2$, then $\mathbf{u}_{a_kb_k}\mathbf{u}_{c_3c_2} \neq \mathbf{0}$ implies $a_k < c_3$ and $\mathbf{u}_{a_kb_k}\mathbf{u}_{c_3c_2}$ commutes. Continuing this process, we find that $\mathbf{u}_{a_kb_k}\mathbf{u}_{C_1} = \mathbf{u}_{C_1}\mathbf{u}_{a_kb_k} \neq \mathbf{0}$ and $a_k < c_m < c_1 < b_k$. This means C_1 and (b_k, a_k) are totally disjoint increasing cycles. We have

$$\mathbf{u}_{a_kb_k}\mathbf{u}_{\zeta'} = \mathbf{u}_{C_1}\mathbf{u}_{a_kb_k}\mathbf{u}_{C_2} \cdots \mathbf{u}_{C_s}.$$

From the induction hypothesis, the result holds for $\mathbf{u}_{a_kb_k}\mathbf{u}_{C_2} \cdots \mathbf{u}_{C_s}$ and decomposes into totally disjoint increasing cycles. Moreover C_1 will be totally disjoint from the cycles of $(b_ka_k)C_2 \cdots C_s$.

As in Corollary 1, the expression in Proposition 4 is valid as long as we consider all possible r-Bruhat orders for $r > 1$. If we fix r, then most of the $\mathbf{u}_\zeta$ in H_k will act as zero on the r-Bruhat order. For a fixed r, we see that $H_k : \mathbb{Z}\mathscr{S}_\infty \to \mathbb{Z}\mathscr{S}_\infty$ is a well defined operator on the r-Bruhat order. From Corollary 1, for a fixed r,

$$H_k = \sum_{\substack{\zeta \text{ is } k\text{-increasing} \\ |up(\zeta)| \le r}} \mathbf{u}_\zeta .$$

By definition of H_k and Eq. (13), we have

$$H_k(w) = \sum_{wu^{-1} \ k\text{-increasing}} u \qquad \Longleftrightarrow \qquad \mathfrak{S}_w h_k(x_1, \ldots, x_r) = \sum_{wu^{-1} \ k\text{-increasing}} \mathfrak{S}_u .$$

This implies that

$$H_b H_a(w) = \textstyle\sum_v d^u_{w,(a,b)} u \\ \Longleftrightarrow \qquad \mathfrak{S}_w h_a(x_1, \ldots, x_r) h_b(x_1, \ldots, x_r) = \textstyle\sum_u d^u_{w,(a,b)} \mathfrak{S}_u . \tag{15}$$

In particular, for all w we have $H_b H_a(w) = H_a H_b(w)$ since $h_a h_b = h_b h_a$. The result below is not as well known as Theorem 1.

Theorem 3. *The algebra* $\mathbf{B}\langle H_k \rangle$ *spanned by* $\{H_1, H_2, H_3, \ldots\}$ *as operators on the* r*-Bruhat order for* $r > 0$ *is isomorphic to* Sym.

Proof. As we multiply $H_a H_b$ and $H_b H_a$, some terms will go to zero and others will survive. The terms that survive in $H_a H_b$ are of the form

$$\mathbf{u}_w = \mathbf{u}_\zeta \mathbf{u}_\eta$$

where ζ is a-increasing and η is b-increasing. Let $d^w_{(a,b)}$ be the coefficient of $\mathbf{u}_w$ in $H_a H_b$. From Corollary 1, for any $w \in \mathscr{S}_\infty$ we can find u and an $r > 0$ such that $\mathbf{u}_w(u) = v \ne 0$. So $d^w_{(a,b)}$ is the coefficient of v in $H_a H_b(u)$. From (15), for all w, we have

$$d^w_{(a,b)} = \text{Coeff of } v \text{ in } H_a H_b(u) = \text{Coeff of } v \text{ in } H_b H_a(u) = d^w_{(b,a)} .$$

Hence $H_a H_b = H_b H_a$.

The algebra $\mathbf{B}\langle H_k \rangle$ is clearly spanned by $H_\lambda = H_{\lambda_1} \cdots H_{\lambda_\ell}$ where λ runs over all partitions. To show the isomorphism with Sym, we only need to show that the H_λ are linearly independent. Let $r \ge \ell(\lambda)$. Using (15), we have that $H_\lambda(Id) = \sum_\mu d^\mu_\lambda v_\mu$ where v_μ is the unique grassmannian permutation defined by $v(v_\mu, r) = \mu$ and the d^μ_λ satisfy

$$h_\lambda(x_1, \ldots, x_r) = \sum_\mu d^\mu_\lambda s_\mu(x_1, \ldots, x_r).$$

If we have a finite linear combination $\Phi = \sum_\lambda c_\lambda H_\lambda$, then for $r \geq \max\{\ell(\lambda) : c_\lambda \neq 0\}$ we have that $\Phi(Id)$ corresponds to the symmetric function $\sum_\lambda c_\lambda h_\lambda$. This is zero if and only if all $c_\lambda = 0$. □

As in Sect. 2.4, let $\langle v, w\rangle = \delta_{v,w}$ define a scalar product on $\mathbb{Z}\mathscr{S}_\infty$. For a fixed $r > 0$ and $u <_r w$, we define the quasisymmetric function

$$K_{[u,w]_r} = \sum_\alpha \langle H_\alpha(u), w\rangle M_\alpha . \tag{16}$$

As before, since $H_a H_b = H_b H_a$, the function $K_{[u,w]_r}$ is in fact a symmetric function. As shown in [9, 10], we have the following theorem:

Theorem 4.

$$K_{[u,w]_r} = \sum_\alpha \langle R_\alpha(u), w\rangle F_\alpha = \sum_\mu c^w_{u,v(\mu,r)} s_\mu ,$$

where $c^w_{u,v(\mu,r)}$ are defined in (9). Moreover for α a composition of n we have that $\langle R_\alpha(u), w\rangle$ counts the number of paths in the r-Bruhat order $\mathscr{S}_\infty$ from u to w of the form $\mathbf{u}_{a_n b_n} \cdots \mathbf{u}_{a_2 b_2}\mathbf{u}_{a_1 b_1}$ where $b_i > b_{i+1}$ if and only if $i \in D(\alpha)$.

Example 3. Using the chains in (12) and Theorem 4 we can compute the quasisymmetric function associated to this interval and we get

$$\begin{aligned} K_{[142635,356124]_3} &= F_{13} + F_{121} + F_{22} + F_{112} + F_{121} + F_{31} + F_{211} + F_{22} \\ &= S_{31} + S_{22} + S_{211} . \end{aligned}$$

Remark 4. The monoid generated by the operators $\mathbf{u}_{ab}$ does not satisfy relations that resemble (6), hence we cannot use the work of Fomin and Greene to conclude that $K_{[u,w]_r}$ is symmetric nor deduce a combinatorial rule for constructing the coefficient $c^w_{u,v(\mu,r)}$ in $K_{[u,w]_r}$. In fact all known attempts to give such a rule so far have failed. In the next section we outline how it is shown combinatorially in [1] that the coefficients are positive (without giving an explicit rule in all cases) using techniques developed by [2].

3.3 Combinatorial Proof of Positivity of $c^w_{u,v(\mu,r)}$

Let $Comp_n$ denote the set of compositions of n. Given a finite family of objects $\mathscr{C}$ and a function $\alpha : \mathscr{C} \to Comp_n$ we can define a quasisymmetric function as follows

$$K_{\mathscr{C}} = \sum_{x \in \mathscr{C}} F_{\alpha(x)} .$$

The function $K_{[u,w]_r}$ of Theorem 4 is clearly of this form. In that case $\mathscr{C}$ is the set of saturated chains $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$ in the interval $[u,w]_r$ and $\alpha=\alpha(\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1})$ is the unique composition where $b_i > b_{i+1}$ if and only if $i \in D(\alpha)$.

Assaf [2] develops new combinatorial techniques to show that quasisymmetric functions of the form $K_{\mathscr{C}}$ are symmetric with a positive expansion in terms of Schur functions. To this end one must construct partially commuting involutions $\phi_i\colon\mathscr{C}\to\mathscr{C}$ for $1<i<n$ satisfying a set of axioms. When $\mathscr{C}$ consists of words (or saturated chains), the involutions ϕ_i can be viewed as an analogue of the dual Knuth relations. In [1] we have defined such involution ϕ_i on the set of chains of $[u,w]_r$. Given a chain $x=\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$, the involution ϕ_i will only affect the three entries $\mathbf{u}_{a_{i+1}b_{i+1}}\mathbf{u}_{a_ib_i}\mathbf{u}_{a_{i-1}b_{i-1}}$. We set $\phi_i(x)=x$ if and only if $\left|D(\alpha(x))\cap\{i-1,i\}\right|\neq 1$. When $\left|D(\alpha(x))\cap\{i-1,i\}\right|=1$, the entries $\mathbf{u}_{a_{i+1}b_{i+1}}\mathbf{u}_{a_ib_i}\mathbf{u}_{a_{i-1}b_{i-1}}$ of x can be one of twelve cases. To define ϕ_i, we match the twelves cases as follows:

$$
\begin{array}{lll}
\textbf{(A)} & \mathbf{u}_{\gamma c}\mathbf{u}_{\alpha a}\mathbf{u}_{\beta b} \leftrightarrow \mathbf{u}_{\alpha a}\mathbf{u}_{\gamma c}\mathbf{u}_{\beta b}, & \\
 & \mathbf{u}_{\beta b}\mathbf{u}_{\alpha a}\mathbf{u}_{\gamma c} \leftrightarrow \mathbf{u}_{\beta b}\mathbf{u}_{\gamma c}\mathbf{u}_{\alpha a}, & \text{if } \{a,\alpha\}\cap\{c,\gamma\}=\emptyset \text{ and } a<b<c,\\
\textbf{(B)} & \mathbf{u}_{bc}\mathbf{u}_{ab}\mathbf{u}_{bd} \leftrightarrow \mathbf{u}_{ac}\mathbf{u}_{cd}\mathbf{u}_{bc}, & \\
 & \mathbf{u}_{bd}\mathbf{u}_{ab}\mathbf{u}_{bc} \leftrightarrow \mathbf{u}_{bc}\mathbf{u}_{cd}\mathbf{u}_{ac}, & \text{if } a<b<c<d,\\
\textbf{(C)} & \mathbf{u}_{\beta b}\mathbf{u}_{\alpha a}\mathbf{u}_{ac} \leftrightarrow \mathbf{u}_{\alpha a}\mathbf{u}_{ac}\mathbf{u}_{\beta b}, & \\
 & \mathbf{u}_{ac}\mathbf{u}_{\alpha a}\mathbf{u}_{\beta b} \leftrightarrow \mathbf{u}_{\beta b}\mathbf{u}_{ac}\mathbf{u}_{\alpha a}, & \text{if } \{\alpha,a,c\}\cap\{b,\beta\}=\emptyset \text{ and } a<b<c.
\end{array}
$$

This matching is completely determined by the relations in (11). We see them as the analogue of the dual Knuth relations for this problem. Instead of using the relation (11) one can investigate the free monoid spanned by the $\mathbf{u}_{ab}$ modulo the dual Knuth relations above. Under certain axioms described in [1,2], the component of the equivalent classes of these relations will be combinatorially symmetric and Schur positive. To our knowledge this is the best we can do so far, and is the best generalization of the work of Fomin and Greene.

4 k-Schur Functions

In this section we present a monoid of operators for which much less is known but that is expected to behave as in Sect. 3. This monoid is related to the so-called k-Schur functions [18, 21]. This time we will define operators on the Bruhat order of the k-affine symmetric group. The operators we define will be related to the multiplication of dual k-Schur functions. There are still many open questions in this case, but we will present our program and we believe that it can be solved in the same spirit as in Sect. 3. There is another order one may consider on the k-affine symmetric group, namely the *weak* order. The operators corresponding to the weak order are related to the multiplication of k-Schur functions, but we will discuss only briefly the difficulties which arise in this situation.

The k-Schur functions were originally defined combinatorially in terms of k-atoms, and conjecturally provide a positive decomposition of the Macdonald polynomials [21]. These functions have several definitions and it is conjectural that they are equivalent (see [18]). In this paper we will adopt the definition given by the k-Pieri rule and k-tableaux (see [18, 20]) since this gives us a relation with the homology and cohomology of the affine grassmannians and we therefore get positivity in their structure constants.

Different objects index k-Schur functions: 0-grassmannian in k-affine permutations, $k+1$-cores, k-bounded partitions. Originally (as in [21]), k-Schur functions were indexed by k-bounded partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where $\lambda_1 \leq k$. These partitions are in bijection with $k+1$-cores (see [19]). By definition, $k+1$-cores are integer partitions $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ with no hook of length $k+1$. To close the loop, in [13] it is shown that $k+1$-cores are in bijection with $0-$grassmannian permutations in the k-affine symmetric group (see [6, 18]).

4.1 Affine Symmetric Group

The k-affine symmetric group $W = \tilde{A}_k$ is generated by reflections s_i for $i \in \{0, 1, \ldots, k\}$, subject to the relations:

$$s_i^2 = 1; \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}; \qquad s_i s_j = s_j s_i \text{ if } i - j \neq \pm 1,$$

where $i - j$ and $i+1$ are understood to be taken modulo $k+1$. Let $w \in W$ and denote its length by $\ell(w)$, given by the minimal number of generators needed to write a reduced expression for w. We let W_0 denote the parabolic subgroup obtained from W by removing the generator s_0. This is naturally isomorphic to the symmetric group S_{k+1}. For more details on the affine symmetric group see [13].

Let $u \in W$ be an affine permutation. This permutation can be represented using window notation. That is, u can be seen as a bijection from $\mathbb{Z}$ to $\mathbb{Z}$, so that if u_i is the image of the integer i under u, then it can be seen as a sequence:

$$u = \cdots | u_{-k} \ \cdots \ u_{-1} \ u_0 \underbrace{| u_1 \ u_2 \ \cdots \ u_{k+1} |}_{\text{main window}} u_{k+2} \ u_{k+3} \ \cdots \ u_{2k+2} | \cdots$$

Moreover, u satisfies the property that $u_{i+k+1} = u_i + k + 1$ for all i, and the sum of the entries in the main window $u_1 + u_2 + \cdots + u_{k+1} = \binom{k+2}{2}$. Notice that in view of the first property, u is completely determined by the entries in the main window. In this notation, the generator $u = s_i$ is the permutation such that $u_{i+m(k+1)} = i+1+m(k+1)$ and $u_{i+1+m(k+1)} = i+m(k+1)$ for all m, and $u_j = j$ for all other values. The multiplication uw of permutations u, w in W is the usual composition given by $(uw)_i = u_{w_i}$. In view of this, the parabolic subgroup W_0 corresponds to the $u \in W$ such that the numbers $\{1, 2, \ldots, k+1\}$ appear in the main window.

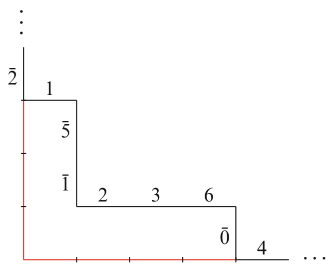
Now, let W^0 denote the set of minimal length coset representatives of W/W_0. In this paper we take right coset representatives, although left coset representatives could be taken as well. The set of permutations in W^0 are the *affine grassmannian permutations* of W, or 0-grassmannians for short.

Definition 1. The *affine* 0-*grassmannians* W^0 are the permutations $u \in W$ such that the numbers $1, 2, \ldots, k+1$ appear from left to right in the sequence u.

Example 4. Let $k = 4$ and

$$u = \cdots|\bar{3}\ \bar{2}\ 1\ \bar{5}\ \bar{1}\ \underbrace{|2\ 3\ 6\ \bar{0}\ 4|}_{\text{main window}}\ 7\ 8\ 11\ 5\ 9|\cdots$$

where $\bar{i}$ stands for $-i$. By convention we say that 0 is negative. This permutation u is 0-grassmannian and it corresponds to the 5-core $\mu = (4, 1, 1)$. The correspondence is easy to see from the window notation. We just need to read the sequence of entries of u, drawing a vertical step down for each negative entry, and an horizontal step right for each positive entry. The result is the diagram of μ:



4.2 *k-Schur Functions and Weak Order*

As previously mentioned, 0-grassmannian permutations index k-Schur functions, which we denote by $S_u^{(k)}$ for some $u \in W^0$.

Given $u \in W$, we say that $u \lessdot_w us_i$ is a cover for the weak order if $\ell(us_i) = \ell(u) + 1$. The weak order on W is the transitive closure of these covers. We can define operators

$$\begin{aligned} \mathbf{s}_i \colon \mathbb{Z}W^0 &\longrightarrow \quad \mathbb{Z}W^0, \\ u &\longmapsto \begin{cases} us_i & \text{if } u \lessdot_w us_i \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{17}$$

on the weak order of W restricted to W^0. The definition and multiplication of k-Schur functions is based on the operators $\mathbf{s}_i$ so it is worthwhile to study the monoid they generate. As we will see in Example 5 there are difficulties with the behavior of this case which make it very difficult at this point to understand its combinatorics. For this reason, we will quickly turn our attention to the dual k-Schur after Example 5.

The Pieri rule for k-Schur functions is described by certain chains in the weak order of W restricted to W^0. This result is given in [17, 18, 20]. A saturated chain $w = \mathbf{s}_{i_m} \cdots \mathbf{s}_{i_2}\mathbf{s}_{i_1}(u)$ in an interval $[u,w]_w$ of the weak order restricted to W^0 gives us a sequence of labels $(i_1, i_2, \dots, i_m)$. We say that the sequence $(i_1, i_2, \dots, i_m)$ is cyclically increasing if $i_1, i_2, \dots, i_m$ lies clockwise on a clock with hours $0, 1, \dots, k$ and if the min $\{j \ : \ 0 \le j \le k; \ j \notin \{i_1, i_2, \dots, i_m\}\}$ lies between i_m and i_1. In particular we must have $1 \le m \le k$. Now, to express the Pieri rule, we first remark that for $1 \le m \le k$, the homogeneous symmetric function h_m corresponds to the k-Schur function $S^{(k)}_{v(m)}$ where $v(m)$ is a 0-grassmannian whose main window is given by $|2 \ \cdots \ m \ \bar{0} \ m+1 \ \cdots \ k \ k+2|$. Then, the multiplication of a k-Schur function $S^{(k)}_u$ by a homogeneous symmetric function h_m is given by

$$S^{(k)}_u h_m := \sum_{\substack{(i_1,i_2,\dots,i_m) \text{ cyclically increasing} \\ \mathbf{s}_{i_m}\cdots\mathbf{s}_{i_2}\mathbf{s}_{i_1}(u)\neq 0}} S^{(k)}_{\mathbf{s}_{i_m}\cdots\mathbf{s}_{i_2}\mathbf{s}_{i_1}(u)}. \tag{18}$$

Iterating Eq. (18) one can easily see that

$$h_\lambda = \sum_u \mathrm{K}_{\lambda,u} S^{(k)}_u \tag{19}$$

is a triangular relation [20]. One way to define k-Schur functions is to start with Eq. (18) as a rule, and define them as follows.

Definition 2. The *k-Schur functions* are the unique symmetric functions $S^{(k)}_u$ obtained by inverting the matrix $[\mathrm{K}_{\lambda,u}]$ obtained from (19) above.

It is clear that we can define a Pieri operator

$$H_m = \sum_{(i_1,i_2,\dots,i_m) \text{ cyclically increasing}} \mathbf{s}_{i_m} \cdots \mathbf{s}_{i_2}\mathbf{s}_{i_1},$$

for $1 \le m \le k$. Again we can show that $H_a H_b = H_b H_a$ and define $K_{[u,w]_w}$ using the original definition. The example below shows the main problems we have with this function.

Example 5. Let $k = 2$ and $u = |\bar{0} \ 2 \ 4|$. We consider the interval $[u,w]_w$ in the weak order restricted to W^0, where $w = |\bar{3} \ 4 \ 5|$. This interval is a single chain $w = \mathbf{s}_0\mathbf{s}_2\mathbf{s}_1(u)$. In this case, we remark that

$$\langle H_1H_1H_1(u), w\rangle = \langle H_1H_2(u), w\rangle = \langle H_2H_1(u), w\rangle = 1$$

are the only nonzero entries in $K_{[u,w]_w}$ and we get

$$\begin{aligned} K_{[u,w]_w} &= M_{111} + M_{21} + M_{12} \\ &= F_{12} + F_{21} - F_{111} \\ &= S_{21} - S_{111}. \end{aligned}$$

This small example shows some of the behavior of the (quasi)symmetric function $K_{[u,w]_w}$ for the weak order of W. In general, it is neither F-positive nor Schur positive. Although, these functions contain some information about the structure constants, it is not enough to fully understand them combinatorially. In particular, these functions lack some of the properties needed to use the theory developed in [2]. The functions $K_{[u,w]_w}$ were first defined in [10, 26] but the combinatorial expansion in terms of Schur functions is still open.

4.3 Dual *k*-Schur Functions

Recall that $Sym = \mathbb{Z}[h_1, h_2, \dots]$ is the Hopf algebra of symmetric functions. The space of k-Schur functions $Sym_{(k)}$ can be seen as a subalgebra of Sym spanned by $\mathbb{Z}[h_1, h_2, \dots, h_k]$. In fact, it is a Hopf subalgebra whose comultiplication defined in the homogeneous basis is given by

$$\Delta(h_m) = \sum_{i=0}^{m} h_i \otimes h_{m-i}$$

and extended algebraically. The degree map is given by $\deg(h_m) = m$. The space Sym is a self dual Hopf algebra where the Schur functions S_λ form a self dual basis under the pairing $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$.

The map dual to the inclusion $Sym_{(k)} \hookrightarrow Sym$, is a projection $Sym \twoheadrightarrow Sym^{(k)}$, where $Sym^{(k)} = Sym^*_{(k)}$ is the graded dual of $Sym_{(k)}$. It can be checked that the kernel of this projection is the linear span of $\{m_\lambda : \lambda_1 > k\}$, hence

$$Sym^{(k)} \;\cong\; Sym\big/\langle m_\lambda : \lambda_1 > k \rangle \, .$$

The graded dual basis to $S_u^{(k)}$ will be denoted here by $\mathfrak{S}_u^{(k)} = S_u^{(k)*}$ which are also known as the affine Stanley symmetric functions. The multiplication of the dual k-Schur $\mathfrak{S}_u^{(k)}$ is described in terms of operator on the affine Bruhat order, as we will see in the next section.

4.4 Affine Bruhat Order

For $b - a \le k$, let $t_{a,b}$ be the transposition in W such that for all $m \in \mathbb{Z}$, it transposes $a + m(k+1)$ and $b + m(k+1)$. The *affine Bruhat order* is given by its covering relation. Namely, for $u \in W$, we have $u \lessdot ut_{a,b}$ is a cover in the affine Bruhat order if $\ell(ut_{a,b}) = \ell(u) + 1$.

Proposition 5 (see [13]). *For $u \in W$ and $b - a \le k$, we have that $u \lessdot ut_{a,b}$ is a cover in the* Bruhat order *if and only if $u(a) < u(b)$ and for all $a < i < b$ we have $u(i) < u(a)$ or $u(i) > u(b)$.*

Notice that if $a' = a + m(k+1)$ and $b' = b + m(k+1)$ then $t_{a',b'} = t_{a,b}$, therefore, many different choices of a and b give the same covering as long as they satisfy the conditions of the proposition. The affine 0-Bruhat order arises as a suborder of the Bruhat order. We define it by its covers. For $u \in W$, we get a covering $u \lessdot_0 ut_{a,b}$ if there exists a transposition $t_{a,b}$ satisfying Proposition 5 and also $u(a) \le 0 < u(b)$. As previously noted, a transposition $t_{a',b'}$ satisfying the same conditions as $t_{a,b}$ gives the same affine Bruhat covering relation as long as $a' \equiv a$, $b' \equiv b$ modulo $k+1$. In view of this, we introduce operators on the affine 0-Bruhat order restricted to W^0. To keep track of the distinct a, b such that $u \lessdot_0 ut_{a,b}$ is an affine 0-Bruhat covering for a given u. For any $b - a \le k$, let

$$\begin{aligned} \mathbf{t}_{ab}\colon \mathbb{Z}W^0 &\longrightarrow \quad \mathbb{Z}W^0, \\ u &\longmapsto \begin{cases} ut_{a,b} & \text{if } u \lessdot ut_{a,b} \text{ and } u(a) \le 0 < u(b) \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{20}$$

We write these operators as acting on the right: $u\mathbf{t}_{ab}$. Remark now that if $u\mathbf{t}_{ab} \neq 0$, then $u\mathbf{t}_{ab} = u\mathbf{t}_{a',b'} \neq 0$ for only finitely many values of m with $a' = a + m(k+1)$ and $b' = b + m(k+1)$. To see this, it is enough to notice that there exists m such that $u(a + m(k+1)) \ge 0$ and m' such that $u(b + m'(k+1)) < 0$.

Example 6. In Fig. 1 below, we have the interval $[|\bar{6}\,8\,3\,\bar{1}\,4\,13|, |8\,\bar{6}\,\bar{2}\,9\,13\,\bar{1}|]$ in the affine 0-Bruhat graph: In this example we see that there are three operators from $u = |\bar{6}\,8\,3\,\bar{1}\,4\,13|$ to $w = |8\,\bar{6}\,3\,\bar{1}\,13\,4|$. We have $u\mathbf{t}_{\bar{5}\bar{4}} = u\mathbf{t}_{12} = u\mathbf{t}_{78} = w$ labeled by $\bar{4}, 2, 8$, respectively. All other operators evaluate to 0. For example $u\mathbf{t}_{\overline{11}\,\overline{10}} = 0$.

When restricted to 0-grassmannian permutations, the affine 0-Bruhat order behaves well, as shown in the next lemma whose proof (for left coset) can be consulted in [18, Prop. 2.6]. Therefore, our operators $\mathbf{t}_{ab}$ are well defined.

Lemma 4. *If $u\mathbf{t}_{ab} = w$ and $u \in W^0$, then we have that $w \in W^0$.*

At this point, there are a few questions we would like to answer regarding the monoid $\mathscr{M}\langle \mathbf{t}_{ab} \rangle$ generated by the operators $\mathbf{t}_{ab}$. The main questions are:

(I) Can we describe all the relations satisfied by the operators $\mathbf{t}_{ab}$ (as in Proposition 3)?
(II) Is there a combinatorial object that characterizes all the elements of $\mathscr{M}\langle \mathbf{t}_{ab} \rangle$ (as in Corollary 1)?
(III) Can we define Pieri operators H_k related to the multiplication $\mathfrak{S}_u h_m$?
(IV) Can we find a good expression for H_k as in Proposition 4?
(V) Is the algebra spanned by the H_k isomorphic $Sym_{(k)}$ (as in Theorem 1)?

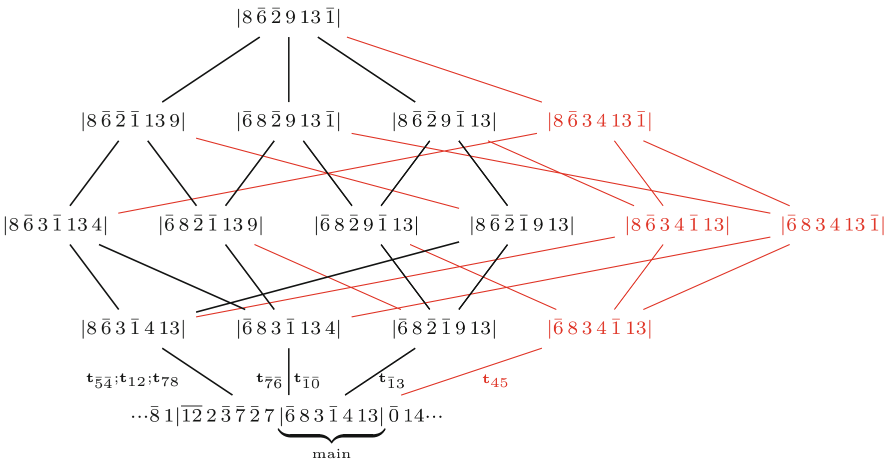


Fig. 1 The interval $[|\bar{6}\,8\,3\,\bar{1}\,4\,13|, |8\,\bar{6}\,\bar{2}\,9\,13\,\bar{1}|]$

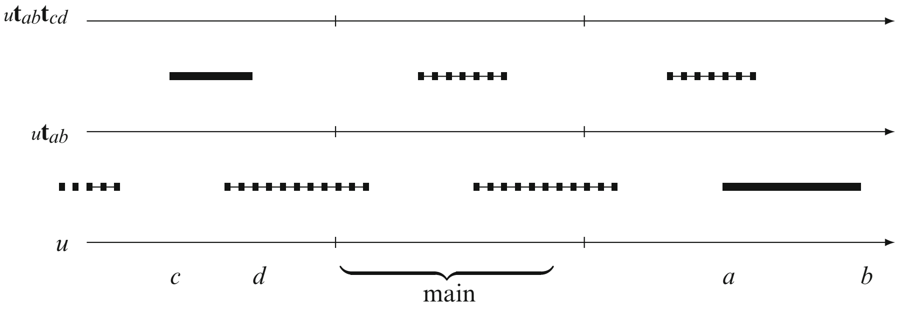
(VI) What is the analogue of Theorem 4?

(VII) Can we show combinatorially the positivity of the structure constants in the product $\mathfrak{S}_u h_m$ as done in Sect. 3.3?

We have some partial answers to question (I) that we will discuss next. Questions (II) and (IV) seem very difficult at this point and are still open. Questions (III), (V) and (VI) are done in the literature (see [5, 17]), although (V) is not stated as it is here. We are in the process of solving question (VII); this involves analyzing 3, 4, 5, and 6-tuples of the operators $\mathbf{t}_{ab}$. The number of possibilities are much greater than the situation in Sect. 3.3 and will be available in subsequent work.

4.5 *Relations of the Operators* $\mathbf{t}_{ab}$

The purpose of this section is to understand some of the relations satisfied by the $\mathbf{t}_{ab}$ operators restricted to W^0. Our main goal at this point is not to understand all the defining relations, but to find enough that will allow us to answer question (VII). Answering question (II) is a very worthwhile project for future work. Most of the relations we present here were given and proven in [5]. The relations depend on the following data: for $\mathbf{t}_{ab}$ we need to consider $a, b, \overline{a}, \overline{b}$ where $\overline{a}$ and $\overline{b}$ are the residue modulo $k+1$ of a and b respectively. Remark that $\overline{a} \neq \overline{b}$ since $b-a < k+1$. Let $u \in W^0$. Lemma 4 implies that, if non-zero, $u\mathbf{t}_{ab}$ and $u\mathbf{t}_{ab}\mathbf{t}_{cd}$ are both in W^0. The different relations satisfied by the operators $\mathbf{t}_{ab}$ and $\mathbf{t}_{cd}$ depend on the relation among $\overline{a}, \overline{b}, \overline{c}, \overline{d}$. For this reason it is useful to visualize these operators as follows.



Above the permutation u, the operator $\mathbf{t}_{ab}$ is represented by drawing a bold line connecting positions a, b and repeating this pattern to the left and to the right in all positions congruent to a, b modulo $k + 1$. Next we apply $\mathbf{t}_{cd}$ to the resulting permutation, drawing a bold line connecting positions c, d and repeating that pattern modulo $k + 1$. The importance of visualizing not only the bold line but the dotted ones as well, relies on the fact that even if in the diagram, the line representing $\mathbf{t}_{ab}$ does not intersect the line representing $\mathbf{t}_{cd}$, their "virtual" copies (or dotted copies) might intersect and this will determine the commutation relation satisfied by these operators. Therefore, it will be important to consider the pattern produced by these two operators in the main window.

With these definitions in mind we present some of the relations satisfied by the $\mathbf{t}$ operators restricted to W^0 (there are less relations if we consider all of W).

(A) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv \mathbf{t}_{cd}\mathbf{t}_{ab}$ if $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ are distinct.
(B1) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv \mathbf{t}_{cd}\mathbf{t}_{ab} \equiv 0$ if $(a \leq c < b \leq d)$ or $(b = c$ and $d - a > k + 1)$.
(B2) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv 0$ if $(\overline{a} = \overline{c}$ and $b \leq d)$ or $(\overline{b} = \overline{d}$ and $c \leq a)$.
(B3) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv 0$ if $(\overline{b} = \overline{c}$ or $\overline{a} = \overline{d})$ and $(d - c + b - a > k + 1)$.

There are more possible zeros than what we present in (B). If the numbers a, b, c, d are not distinct, then we must have $b = c$ or $d = a$. If $b = c$, then $d - a \leq k + 1$ in view of (B). Similarly if $d = a$ then $b - c \leq k + 1$.

(C) $\mathbf{t}_{ab}\mathbf{t}_{bd} = \mathbf{t}_{ab}\mathbf{t}_{b-k-1,a}$ if $d - a = k + 1$,

if $d - a < k + 1$ then there is no relation between $\mathbf{t}_{ab}\mathbf{t}_{bd}$ and $\mathbf{t}_{bd}\mathbf{t}_{ab}$. Now we look at the cases $\mathbf{t}_{ab}\mathbf{t}_{cd}$ where a, b, c, d are distinct but some equalities exists between $\overline{a}, \overline{b}$ and $\overline{c}, \overline{d}$. By symmetry of the relation we will assume that $b < d$, which (excluding (B)) implies that $a < b < c < d$.

(D) $\mathbf{t}_{ab}\mathbf{t}_{cd} = \mathbf{t}_{d-k-1,c}\mathbf{t}_{b-k-1,a}$ if $\overline{b} = \overline{c}$, $\overline{d} = \overline{a}$ and $(b-a)+(d-c) = k+1$.

All the relations above are *local*. This means that if $\mathbf{t}_{ab}\mathbf{t}_{cd} = \mathbf{t}_{c'd'}\mathbf{t}_{a'b'}$, then $|a'-a|$, $|b'-b|$, $|c'-c|$ and $|d'-d|$ are strictly less than $k+1$. For example, in (D) we have $|b-k-1-a|$, $|a-b|$, $|d-k-1-c|$ and $|c-d|$ which are strictly less than $k+1$.

Remark 5. The relations we care about in this paper and its sequel are all local. There are some relations that are not local:

$$\mathbf{t}_{ab}\mathbf{t}_{cd} = \mathbf{t}_{a-k-1,b-k-1}\mathbf{t}_{cd} = \mathbf{t}_{a+k+1,b+k+1}\mathbf{t}_{cd},$$

if $c < a < b < d$. The full description of the relations of the operators $\mathbf{t}$ is rather complicated. It would take too much space here and are not all understood.

We now consider some more relations of length three:

(E1) $\mathbf{t}_{bc}\mathbf{t}_{cd}\mathbf{t}_{ac} \equiv \mathbf{t}_{bd}\mathbf{t}_{ab}\mathbf{t}_{bc}$ if $a < b < c < d$,
(E2) $\mathbf{t}_{ac}\mathbf{t}_{cd}\mathbf{t}_{bc} \equiv \mathbf{t}_{bc}\mathbf{t}_{ab}\mathbf{t}_{bd}$ if $a < b < c < d$.

additionally we have

(F) $\mathbf{t}_{bc}\mathbf{t}_{ab}\mathbf{t}_{bc} \equiv \mathbf{t}_{ab}\mathbf{t}_{bc}\mathbf{t}_{ab} \equiv \mathbf{0}$ if $a < b < c$ and $c-a < k+1$.

Remark 6. If we fix a permutation u we can derive more relations of length 2. Let $r = |b-a| + |d-c|$:

(X1) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}$ if $r < k+1$, $\overline{d} = \overline{a}$, $u(c) \le 0$ and $u(d) \le 0$,
(X2) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{cd}\mathbf{t}_{b-r,b}$ if $r < k+1$, $\overline{d} = \overline{a}$ and $u(d) > 0$,
(X3) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{d-r,d}\mathbf{t}_{ab}$ if $r < k+1$, $\overline{b} = \overline{c}$ and $u(a+r) \le 0$,
(X4) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{d-r,c}\mathbf{t}_{b,a+r}$ if $r < k+1$, $\overline{b} = \overline{c}$, $u(b) > 0$ and $u(a+r) > 0$,
(X5) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{cd}\mathbf{t}_{a,b+c-d}$ if $\overline{b} = \overline{d}$, $b-a > d-c$ and $u(d-b+a) > 0$,
(X6) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{c,d-b+a}\mathbf{t}_{a,b}$ if $\overline{b} = \overline{d}$, $b-a < d-c$ and $u(a) \le 0$.

In the (X) relations, the conditions we impose on u are minimal to assure that both sides of the equality are non-zero. These conditions are not given by the definition of the operators $\mathbf{t}_{ab}$. For example in (X1), the left hand side is non-zero regardless of the value of $u(d)$ but to guarantee that the right hand side is non-zero, we must have $u(d) \le 0$. This shows that as operators $\mathbf{t}_{ab}\mathbf{t}_{cd} \ne \mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}$.

4.6 Multiplication of Dual k-Schur

For dual k-Schur functions $\mathfrak{S}_u^{(k)}$, the analogue of the Pieri formula (18) is given by

$$\mathfrak{S}_u^{(k)} h_m := \sum_{\substack{u\mathbf{t}_{a_1b_1}\cdots\mathbf{t}_{a_mb_m}\neq 0 \\ b_1<b_2<\cdots<b_m}} \mathfrak{S}^{(k)}_{u\mathbf{t}_{a_1b_1}\cdots\mathbf{t}_{a_mb_m}}, \tag{21}$$

where the sum is over all increasing paths $b_1 < b_2 < \cdots < b_m$ starting at u [18]. Since the Pieri formula is encoded by increasing composition of operators in the affine 0-Bruhat order restricted to W^0, we can define Pieri operators similar to Eq. (14) using increasing composition of operators $\mathbf{t}_{ab}$. We can then define a Pieri operator

$$H_m = \sum_{\substack{b_1<b_2<\cdots<b_k \\ a_i<b_i}} \mathbf{t}_{a_1b_1}\mathbf{t}_{a_2b_2}\cdots\mathbf{t}_{a_mb_m}. \tag{22}$$

Many terms in this sum may be zero. At this point we do not have a good description of the terms that survive or how to express the non-zero terms as in Proposition 4. The definition of the operator H_m in this case allows us to see that

By definition of H_k and Eq. (21), we have

$$wH_bH_a = \textstyle\sum_v d^u_{w,(a,b)}u \iff \mathfrak{S}_w h_a h_b = \textstyle\sum_u d^u_{w,(a,b)}\mathfrak{S}_u\,. \tag{23}$$

In particular, for all w we have $H_bH_a(w) = H_aH_b(w)$ since $h_ah_b = h_bh_a$.

Theorem 5. *The algebra $\mathbf{B}\langle H_k\rangle$ spanned by $\{H_1, H_2, \ldots, H_k\}$ as operators on the k-affine Bruhat order restricted to W^0 is isomorphic to $Sym_{(k)}$.*

Proof. As we multiply H_mH_n and H_nH_m, some terms go to zero and others survive. The terms that survive in H_mH_m are of the form

$$\omega = \mathbf{t}_{a_1b_1}\mathbf{t}_{a_2b_2}\cdots\mathbf{t}_{a_mb_m}\mathbf{t}_{c_1d_1}\mathbf{t}_{c_2c_2}\cdots\mathbf{t}_{a_nb_n}.$$

where $b_1 < b_2 < \ldots < b_k$ and $d_1 < d_2 < \ldots < d_n$. Let $d^{\omega}_{(a,b)}$ be the coefficient of ω in H_mH_n. Since $\omega \neq \mathbf{0}$, there is a $u \in W^0$ such that $u\omega = v \neq 0$. As before, for all ω, we have

$$d^{\omega}_{(a,b)} = \text{Coeff of } v \text{ in } H_aH_b(u) = \text{Coeff of } v \text{ in } H_bH_a(u) = d^{\omega}_{(b,a)}.$$

Hence $H_aH_b = H_bH_a$.

The algebra $\mathbf{B}\langle H_k\rangle$ is clearly spanned by $H_\lambda = H_{\lambda_1}\cdots H_{\lambda_\ell}$ where λ runs over all partitions. Again, we only need to show that the H_λ's are linearly independent. Using the definition of the H_m, we have that $\mathrm{Id}H_\lambda = \sum_\mu d^\mu_\lambda v_\mu$ where v_μ is the unique 0-grassmannian permutation with shape μ and the d^μ_λ satisfy

$$h_\lambda(x_1,\ldots,x_r) = \sum_\mu d^\mu_\lambda s_\mu(x_1,\ldots,x_r).$$

As we have seen in the proof of Theorem 3 this implies the linear independence of the H_λ. □

As in Sect. 2.4, let $\langle v, w\rangle = \delta_{v,w}$ define a scalar product on $\mathbb{Z}W^0$. For a $u < w$ in the 0-Bruhat order, we define the quasisymmetric function

$$K_{[u,w]} = \sum_{\alpha} \langle uH_\alpha, w\rangle M_\alpha . \tag{24}$$

Again, since $H_a H_b = H_b H_a$, the function $K_{[u,w]}$ is in fact a symmetric function. As shown in [5, 10]

Theorem 6.

$$K_{[u,w]} = \sum_{\alpha} \langle uR_\alpha, w\rangle F_\alpha = \sum_{\mu} c^w_{u,\mu} s_\mu ,$$

where $c^w_{u,\mu}$ are defined by

$$\mathfrak{S}^{(k)}_u s_\mu = \sum_{w} c^w_{u,\mu} \mathfrak{S}^{(k)}_w .$$

Moreover for α a composition of n, we have that $\langle uR_\alpha, w\rangle$ count the number of compositions $\omega = \mathbf{t}_{a_1 b_2}\mathbf{t}_{a_2 b_2}\cdots\mathbf{t}_{a_m b_m}$ such that $u\omega = w$ and $b_i > b_{i+1}$ if and only if $i \in D(\alpha)$.

Example 7. Considering the interval $[u, w] = [|\bar{6}\,8\,3\,\bar{1}\,4\,13|, |8\,\bar{6}\,\bar{2}\,9\,13\,\bar{1}|]$ from Example 6. The total number of composition of operators is 240. In this case

$$K_{[u,w]} = 9F_{1111} + 30F_{112} + 51F_{121} + 30F_{13} + 30F_{211} + 51F_{22} + 30F_{31} + 9F_4$$

is symmetric and the expansion in term of Schur functions is positive

$$K_{[u,w]} = 9S_4 + 30S_{31} + 21S_{22} + 30S_{211} + 9S_{1111} .$$

4.7 Comments on the Combinatorial Proof of the Positivity of $c^w_{u,\mu}$

If one considers an interval $[u, w]$ of rank 3 and computes $K_{[u,w]}$, then by Theorem 24 the coefficient of F_{21} and F_{12} must be the same in $K_{[u,w]}$. This means that every time we have a descent followed by an ascent in a chain, we must have another chain with an ascent followed by a descent. This should be reflected in relations like (X) and could depend on u. To achieve a result similar to [1] for $K_{[u,w]}$, one needs first to build a full set of relations of length 3 that pairs every ascent-descent type to a descent-ascent. This cannot be done independently from u. The purpose of this will be to define Dual-Knuth operations on the maximal chains in intervals $[u, w]$ in order to construct dual graphs as in [2].

We give here a partial list of the relations of length 3 that would be the analogue for dual k-Schur of (A)–(B)–(C) in Sect. 3.3. The complete full list of 3-relations needed is too long for this survey. In future work, we will need to show that the corresponding ϕ_i defined by those relations satisfy the axioms of [2]. This is a long analysis that will appear in subsequent work. This will show that the monoid defined by the $\mathbf{t}_{ab}$ behaves like the monoid of Sect. 3, even if it does not satisfy the Fomin and Greene's hypothesis. This shows that these monoids are worthwhile to investigate.

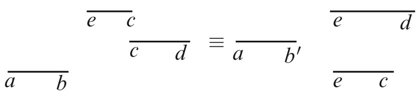
We have already listed some of the relations satisfied by triplets of operators $\mathbf{t}_{ab}$. Relations **(A),(E1),(E2),(F)** resemble the relations listed in (11). However, as noted before in the case of the operators $\mathbf{t}_{ab}$, more relations can be derived making the analysis of relations much more complex than the $\mathbf{u}_{ab}$ operators.

$$
\begin{array}{lll}
(1a) & \mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ec} \equiv \mathbf{t}_{ec}\mathbf{t}_{a,b-|c-e|}\mathbf{t}_{ed}, & \text{if } a<b<e<c<d \text{ and } \bar a=\bar d<\bar e<\bar b=\bar c\\
(1b) & \mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ec} \equiv \mathbf{t}_{\bar d c}\mathbf{t}_{\bar b a}\mathbf{t}_{ec}, & \text{if } a<b<e<c<d \text{ and } \bar a=\bar d=\bar e,\ \bar b=\bar c\\
(1c) & \mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv \mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd}, & \text{if } a<b<c<e<f<d \text{ and } \bar a=\bar d,\ \bar b=\bar c\\
(1d) & \mathbf{t}_{ab}\mathbf{t}_{bc}\mathbf{t}_{db} \equiv \mathbf{t}_{db}\mathbf{t}_{ad}\mathbf{t}_{dc}, & \text{if } a<d<b<c \text{ and } \bar a=\bar c\\
(1e) & \mathbf{t}_{ab}\mathbf{t}_{bc}\mathbf{t}_{db} \equiv \mathbf{t}_{ab}\mathbf{t}_{b-m,c-m}\mathbf{t}_{db}, & \text{if } a=d<b<c \text{ and } \bar a=\bar c,\ m=k+1
\end{array}
$$

In analogy with relations **(X1)–(X6)**, let us list more relations that depend on the permutation u we apply them to. Let $r=|d-c|+|b-a|<k+1$

$$
\begin{array}{lll}
(2a) & u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{d,c+s}\mathbf{t}_{b-s,a}\mathbf{t}_{ef}, & \text{if } a<b<e<f\le c<d,\bar a=\bar d,\ u(c)\le 0,\\
 & & \quad u(d)\le 0\\
(2b) & u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{cd}\mathbf{t}_{b-r,b}\mathbf{t}_{ef}, & \text{if } a<b<e<f\le c<d,\bar a=\bar d,\ u(d)>0\\
(3a) & u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{d-r,c}\mathbf{t}_{b,a+r}\mathbf{t}_{ef}, & \text{if } a<b<e<f\le c<d,\bar b=\bar c,\bar e\ge\bar d,\\
 & & \quad u(a+r)>u(b)>0\\
(3b) & u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{d-r,d}\mathbf{t}_{ab}\mathbf{t}_{ef}, & \text{if } a<b<e<f\le c<d,\bar b=\bar c,\ \bar e\ge\bar d,\\
 & & \quad u(a+r)\le 0\\
(4a) & u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{eb} \equiv u\mathbf{t}_{eb}\mathbf{t}_{ae}\mathbf{t}_{c-|b-e|,d}, & \text{if } a<e<b<c<d,\bar b=\bar c,\bar e>\bar a,\\
 & & \quad u(a+r)>u(b)>0\\
(4b) & u\mathbf{t}_{eb}\mathbf{t}_{ae}\mathbf{t}_{c-|b-e|,d} \equiv u\mathbf{t}_{eb}\mathbf{t}_{d-r,d}\mathbf{t}_{ab}, & \text{if } a<e<b<c<d,\bar b=\bar c,\bar e>\bar a,\\
 & & \quad u(a+r)\le 0
\end{array}
$$

If the reader represents these relations as a system of bars, they can be interpreted as exchanging an ascent-descent by a descent-ascent. As an example, putting $b'=b-|c-e|$ in relation $(1a)$ we can represent it graphically as



Next we list more ascent-descent relations equivalent to descent-ascent. This is not an exhaustive list but it gives a good sense of the behaviour of these operators.

$$
\begin{array}{llll}
(6a) & u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{eb}\mathbf{t}_{c,d-|a-e|}\mathbf{t}_{ea}, & \text{if } c < d < e < a < b, \bar{c} < \bar{e}, \bar{a} = \bar{d}, \\
& & u(b-r) \leq 0 \\
(6b) & u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{eb}\mathbf{t}_{c,c+r}\mathbf{t}_{ab}, & \text{if } c < d < e < a < b, \bar{c} \leq e, \bar{a} = \bar{d}, \\
& & u(b-r) > 0 \\
(6c) & u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{eb}\mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}, & \text{if } c < d < e < a < b, \bar{c} > \bar{e}, \bar{a} = \bar{d}, \\
& & u(b-r) \leq 0 \\
(6d) & u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{eb}\mathbf{t}_{d-r,c}\mathbf{t}_{b,a+r}, & \text{if } c < d < e < a < b, \bar{c} \neq \bar{e} \leq \bar{d}, \bar{b} = \bar{c}, \\
& & u(c) > 0 \\
(6e) & u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{eb}\mathbf{t}_{cd}\mathbf{t}_{a,a+r}, & \text{if } c < d < e < a < b, \bar{c} \neq \bar{e} \leq \bar{d}, \bar{b} = \bar{c}, \\
& & u(c) \leq 0
\end{array}
$$

We encourage the reader to draw the corresponding diagrams of the given relations together with their *virtual* copies in order to realize what these relations look like and understand better the interaction of these triplets. A full understanding of the relations satisfied by tuples of the operators $\mathbf{t}_{ab}$ will lead us to describe connected components of these relations. This is work in progress that we aim to use, for instance, to solve question (VII) as stated before.

Remark 7. In a recent paper, Assaf and Billey [3] have constructed involutions ϕ_i on the so called star-tableaux. Such involutions preserve the spin statistic. Star-tableaux are equivalent to non-zero sequences of operators $\mathbf{t}_{ab}$ acting on the identity 0-grassmannian permutation Id. These transformations ϕ_i are strongly related to the relations we study satisfied by triplets $\mathbf{t}_{ab}$. Showing that these triplets satisfy the spin statistic as well will in fact give us a much stronger positive result. We expect to include this as well in future work.

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Affine Permutations and an Affine Catalan Monoid

Tom Denton

Abstract We describe results on pattern avoidance arising from the affine Catalan monoid. The schema of affine codes as canonical decompositions in conjunction with two-row moves is detailed, and then applied in studying the Catalan quotient of the 0-Hecke monoid. We prove a conjecture of Hanusa and Jones concerning periodicity in the number of fully-commutative affine permutations. We then re-frame prior results on fully commutative elements using the affine codes.

Keywords Affine permutations • Rank enumeration • Pattern avoidance

Subject Classifications: 05A05, 05A16, 05A30

1 Introduction

The Hecke Algebra of a Weyl group is a deformation by a parameter $q = \frac{q_1}{q_2}$ which for generic values yields an algebra with representation theory equivalent to that of the group algebra of the original Weyl group. Our interest is ultimately in the affine symmetric group, $\tilde{S}_n$, but we will describe the basic constructions of algebras and monoids in full generality, and then specialize where necessary.

At $q_1 = 0$, however, we obtain the 0-*Hecke algebra*, which can be interpreted as a monoid algebra of the 0-*Hecke monoid*. If the original Weyl group is generated by simple reflections s_i for i in the index set I, then the 0-Hecke monoid is generated by π_i for $i \in I$. The commutation and braid relations between the π_i match the relations on the s_i, but we have $\pi_i^2 = \pi_i$ instead of $s_i^2 = 1$. Thus,

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the 0-Hecke monoid is generated by projections, instead of reflections. There is a bijection between elements of the original group and elements of the monoid, but the representation theory changes considerably. This representation theory was initially studied by Norton for the 0-Hecke algebra of the symmetric group [24], and expanded to arbitrary finite Weyl type by Carter [5]. Later, the 0-Hecke algebra was shown to have characters determined by the pairing of noncommutative and quasi-symmetric functions [17], and was explored as a special case of the representation theory of $\mathscr{J}$-trivial monoids [10].

At $q_1 = q_2 = 0$, one obtains the *nilHecke algebra*, which, like the 0-Hecke algebra, can be considered as a monoid algebra. This *NilHecke monoid* is generated by a_i for $i \in I$, again with the same commutation and braid relations as the original group, but now with $a_i^2 = 0$. The nilHecke algebra has proven useful in studying reduced word combinatorics, since any non-reduced word in the a_i evaluates to 0 [1]. It has also proven important in the categorification of quantum algebras [16, 26], and provides a very useful model for the study of k-Schur functions [19–21].

In studying the product formula on the k-Schur functions, the present author developed a new combinatorial model for the affine symmetric group [8], combining aspects of the inversion vector or affine code and RC-graphs, originally developed to study reduced word combinatorics [1]. The model provides an interpretation of the inversion vector as a unique maximal decomposition of the affine permutation into cyclically decreasing elements, originally introduced in [19]. This combinatorial model is equally functional in the affine symmetric group, as well as its 0-Hecke monoid and nilHecke monoid. This model was immediately used for calculating a special case of the k-Littlewood-Richardson rule.

In the author's dissertation, a certain quotient of the 0-Hecke algebra for the symmetric group was studied in relation to pattern avoidance. This led to an algebraic interpretation of certain kinds of pattern avoidance via the fibers of the quotient map. The quotient is known as the Catalan monoid, since it has Catalan-many elements, and each fiber of the quotient contains a unique maximal-length element avoiding the pattern $[2, 3, 1]$, and a unique minimal-length element avoiding the pattern $[3, 2, 1]$. The author also extended this result to the affine setting, defining an affine Catalan monoid, and generalizing the $[3, 2, 1]$ avoidance result to that setting. This provides a bijection between $[2, 3, 1]$ and $[3, 2, 1]$-avoiding permutations. The bijection is equivalent to the bijection of Simion and Schmidt [27], a fundamental early result in pattern avoidance, but we place the bijection in a new algebraic setting. Mazorchuk and Steinberg [23] identified a 'double Catalan monoid' with a natural morphism from the 0-Hecke monoid. The fibers of this monoid each contain a unique minimal-length $[4, 3, 2, 1]$-avoiding element, and possibly several maximal-length $[4, 2, 3, 1]$-avoiding elements. These results are similar to results in the author's thesis [7], but Mazorchuk and Steinberg go on to give a presentation of the monoid and an interesting generalization involving quotients by parabolic submonoids. Finally, it's worth noting that Grensing and Mazorchuck have recently posted a categorification of the Catalan monoid [14].

In the present article, we will review the connection between cyclic decompositions of affine permutations and the affine code, the connection between the 0-Hecke algebra and pattern avoidance, and link the two topics via 'shadow diagrams.' Along the way, we will confirm a conjecture of Hanusa and Jones [15] concerning periodicity in the length generating function of the fully commutative elements of the affine symmetric group. This is the content of Theorem 2. While in review, an extended abstract submitted to FPSAC 2013 also solved this conjecture [2], and describes many interesting results in other Lie types.

2 Background and Notation

2.1 Affine Permutations, Hecke Algebras, Specializations

An n-affine permutation (we usually omit the n when no ambiguity will arise) is a permutation $x : \mathbb{Z} \to \mathbb{Z}$ satisfying the following properties:

- $\sum_{i=1}^{n} x(i) = \binom{n+1}{2}$, and
- $x(i + n) = x(i) + n$.

One can show easily that such permutations form a group, which is called the *affine symmetric group* $\tilde{S}_n$. Many of the basic facts about the affine permutation group are collected in [4].

The second condition implies that one may completely specify an affine permutation simply by specifying $x(i)$ for i in the set $\{1, 2, \ldots, n\}$. The list $[x(1), \ldots, x(n)]$ is called the *window notation* for x. We call any list $[x_1, x_2, \ldots, x_n]$ a *valid window notation* if it is the window notation of an affine permutation. One can observe that a list of n integers is a valid window notation if and only if the sum is $\binom{n+1}{2}$, and when each entry of the list is reduced mod n one obtains a permutation of $\mathbb{Z}_n$. (If any residues were repeated, then the second condition could be used to show that the presumed permutation has repeated entries, and is thus not a permutation.) For example, $[5, -2, 3]$ is a valid window notation for an affine permutation (the sum is 6, and the list reduces to $[2, 1, 0]$ modulo 3). On the other hand, $[6, -3, 3]$ is not a window for an affine permutation σ. The sum is 6, but the permutation reduces to $[0, 0, 0]$ modulo 3. As a result, we have:

$$\sigma(8) = \sigma(2 + 3 + 3) = \sigma(2) + 3 + 3 = -3 + 3 + 3 = 3 = \sigma(3),$$

so that σ is not a bijection. We will usually identify an affine permutation with its window notation.

Generators and relations for $\tilde{S}_n$ are given as follows. There is one simple reflection s_i for each $i \in \mathbb{Z}_n$, being the permutation which exchanges $mn + i$ and $mn + i + 1$ for every $m \in \mathbb{Z}$, while fixing all other numbers. The action of the simple reflection is either on the left or the right: The left action exchanges the values, while the right action exchanges the numbers in the given positions. For example, if we take $x = [1, 3, 2]$, then $s_1 x = [2, 3, 1]$, exchanging the values 1 and 2. On the other hand, $xs_1 = [3, 1, 2]$, exchanging the first and second positions.

These generators satisfy the following relations:

- $s_i^2 = 1$,
- Commutation relations: $s_i s_j = s_j s_i$ for all $j > i$ with $j - i > 1$, and
- Braid relations: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all i.

Note that by omitting the generator s_0, we recover a group isomorphic to the usual symmetric group.

The Iwahori-Hecke algebra $H_q(\tilde{S}_n)$ is a q-deformation of the group algebra $\mathbb{C}\tilde{S}_n$, generated by elements T_i for $i \in \mathbb{Z}_n$ with relations:

- $T_i^2 = (q - 1)T_i + q$,
- Commutation relations: $T_i T_j = T_j T_i$ for all $j > i$ with $j - i > 1$, and
- Braid relations: $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for all i.

In short, the quadratic relation is deformed while the relations between different generators are preserved. At $q = 1$, we recover $\mathbb{C}\tilde{S}_n$. At $q = 0$, we have $T_i^2 = -T_i$. For convenience, we set $\pi_i := -T_i$, so that $\pi_i^2 = \pi_i$. It is easy to see that the π_i satisfy the commutation and braid relations. The monoid generated by the π_i is then called the 0-Hecke monoid $\tilde{H}_n$. There is a bijection between elements of this monoid and the set of affine permutations: In particular, reduced words in the affine symmetric group are also reduced in $\tilde{H}_n$. (This is a standard fact in the study of the Iwahori-Hecke algebra of a general Weyl group W. See, for example, [4].)

In the finite case, Margolis and Steinberg [22] noticed that the 0-Hecke monoid arises as a submonoid of the power-set of S_n, with a proof appearing in [23]. In fact, the proof given there applies verbatim to the affine case, though we do not make use of this realization in this paper.

The π_i may be considered as *anti-sorting* operators on the collection of affine permutations, transposing values if they are in order, and leaving them fixed if not. Thus, if $x = [3, 1, 2]$, then $\pi_1 x = [3, 2, 1]$, anti-sorting the values 1 and 2. On the other hand, $x\pi_1 = [3, 1, 2] = x$, since the first and second positions are already anti-sorted.

For expedience, we will often write words in the generators as subscripts on the generator. Furthermore, we will avoid examples with $n \geq 10$ in this paper, and will thus may omit commas in the writing of lists of indices. Thus, $\pi_1\pi_2\pi_1$ may be written as π_{121} unambiguously.

2.2 Pattern Avoidance

A permutation may be thought of as a sequence of numbers, and an affine permutation may be thought of as a doubly-infinite sequence of numbers. Let $\sigma = [\sigma_1, \ldots, \sigma_k]$ be a permutation, and x a permutation or affine permutation.

We say that x *contains* σ if there exist $i_1 < i_2 < \cdots < i_k$ such that $x_{i_1}, x_{i_1}, \ldots, x_{i_k}$ are in the same relative order as σ. If x does not contain σ, then we say that x **avoids** σ, or that x is σ**-avoiding.**

For example, the pattern $[1,2]$ appears in any x such that there exists a $x_i < x_j$ for some $i < j$. The only $[1,2]$-avoiding permutation in S_N, then, is the longest element, which is strictly decreasing in one-line notation. As a larger example, the permutation $[\mathbf{3}, \mathbf{4}, 5, \mathbf{2}, 1, 6]$ contains the pattern $[2,3,1]$ at the bold positions. In fact, this permutation contains six distinct instances of the pattern $[2,3,1]$.

Of particular interest is the pattern $[3,2,1]$. A permutation (or affine permutation) which avoids $[3,2,1]$ is called *fully commutative*, or FC for short. The name arises because one can show that if x is FC, then any reduced word for x can be obtained from any other via a sequence of commutation relations [3, 13, 19].

3 Canonical Decompositions of Affine Permutations

Given a subset $A \subsetneq \mathbb{Z}_n$ with $|A| = m < n$ we define the *cyclically decreasing element* d_A (d for 'decreasing') to be the product $d_A := T_{i_1} \cdots T_{i_m}$ for $i_l \in A$, where if $j, j-1 \in A$ then j appears to the left of $j-1$ in any reduced word for d_A. (One may similarly define cyclically increasing elements, where j appears to the right of $j-1$.) Note that such products may be specialized to either the affine symmetric group or the 0-Hecke monoid. The cyclically decreasing elements are also known as *Pieri factors* in the literature, and play an important role in the theory of k-Schur functions [19].

Heuristically, we want to write the product of the generators in a decreasing list, but then the index set $\mathbb{Z}_n$ has no greatest element. Thus, we rely on the 'local' relations of $j > j-1$ in $\mathbb{Z}_n$. Since $A \subsetneq \mathbb{Z}_n$, we can choose an element not in A to act as the 'top' element, and write the elements of A in decreasing order with respect to this element. For example, if we take $n = 9$ and $A = \{0,1,3,4,5,8\} \subset \mathbb{Z}_9$ at $q = 0$, we can take (for example) $2 \notin A$ to be the 'top' element, and write have $d_A = \pi_{108543}$. We could also have taken 6 as the 'top' element, which would produce the reduced word π_{543108}, which is related to π_{108543} by a sequence of commutation relations. In fact, the words obtained by different choice of 'top' element are always related by commutation relations: There is a bijection between cyclically decreasing elements d_A and proper subsets of $\mathbb{Z}_n$.

Almost all affine permutations can be written in many different ways as a product of cyclically decreasing elements, $x = d_{A_l} d_{A_{l-1}} \cdots d_{A_1}$. (For example, given a reduced word $w = w_1 w_2 \cdots w_k$ for x, we may write $A_i = \{w_i\}$ and write x as a product of k cyclically decreasing elements.) Such a decomposition is called an *α-decomposition of x*. We write $\mathbf{A} = (A_1, A_2, \ldots, A_l)$, and may then form a composition $\mathrm{sh}((A)) = (|A_1|, \ldots, |A_n|)$. As mentioned, a given x may have many different α-decompositions. (In fact, the affine Stanley symmetric functions are defined as a sum over all such decompositions [19].) We may order the

α-decompositions lexicographically (or reverse lexicographically) by comparison of $\mathrm{sh}((A))$. Under this ordering, we have the following theorem:

Theorem 1 (Canonical Cyclically Decreasing Decomposition [8, 19]). *Every affine permutation x admits a unique maximal decomposition under the anti-lexicographic ordering as a product of cyclically decreasing elements $x = d_{\mathbf{A}}$. This decomposition has $A_{i+1} \subset \{j-1 \mid j \in A_i\}$ for each i, and thus $\mathrm{sh}(\mathbf{A})$ is a partition.*

For an easy example, consider the permutation $[3,1,2] = s_2 s_1$. This can be written as $d_{\{2\}} d_{\{1\}}$ or as $d_{\{1,2\}}$. The first decomposition is associated to the composition $(1,1)$ and the second decomposition has composition (2). Thus, the second decomposition is the maximal decomposition.

At this point, we note that there is also a notion of cyclically *increasing* elements u_A, formed in the obvious way: Choose a 'bottom' element not in A, then take an increasing product of the generators indexed by elements of A. For our example above, we would have $u_A = \pi_{801345}$. There is a corresponding notion of cyclically increasing decompositions and every affine permutation admits a unique maximal cyclically increasing decomposition.

We've made two choices so far: a choice between cyclically increasing and decreasing elements in building the decomposition of x, and the choice of whether to find the maximal decomposition according to lexicographic or anti-lexicographic ordering on $\mathrm{sh}(\mathbf{A})$. We'll henceforth call the maximal lexicographic decomposition the *left* decomposition, and the anti-lexicographic decomposition the *right* decomposition. Thus, the theorem is stated for the right decreasing decomposition, and may be modified for any of the other three choices. (In particular, the containment property for the A_i must be modified for other cases, though an analogous statement holds.)

The maximal decomposition is closely related to the *affine code* of the permutation, which is also known as the inversion vector or affine Lehmer code. And there are actually four different affine codes one can associate to any affine permutation, corresponding to the four possible choices. The right decreasing code $\mathrm{RD}(x)$ is the list $[c_1, c_2, \ldots, c_n]$ where c_i is the number of $j < i$ with $x(j) > x(i)$. One can show that the numbers c_i are always finite, and that one of the c_i must be equal to 0.

One may recover the maximal decomposition for an affine permutation x from the affine code by carrying out the following steps:

- Make a Ferrer's diagram of the code of x. This diagram has n columns, and column i has c_i boxes.
- Fill each box with a *residue*, by filling the jth box from the bottom in the c_i column with the number $i - j \mod n$. (We start counting j from 1, so the bottom-most residue in the c_i column is just $i-1$; we then count backwards up the column modulo n.)
- Now each *row* of the resulting diagram corresponds to a cyclically decreasing element, obtained by reading cyclically right-to-left starting from any empty column. (One should imagine the diagram drawn on a cylinder, just as the cyclically decreasing elements come from the 'circular' index set $\mathbb{Z}_n$.)

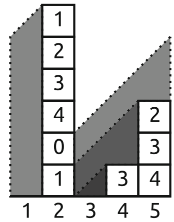

Fig. 1 Diagram of the maximal decomposition of x with window notation $[4, -4, 7, 5, 3]$

Example 1. Let $n = 5$, and x be the affine permutation with window notation $[4, -4, 7, 5, 3]$. Then the affine code of x is $[0, 6, 0, 1, 3]$. For example, there are six elements to the left of the entry $x(2) = -4$ which are larger than -4. Specifically, these are $x(-7) = -3, x(-4) = -1, x(-2) = 2, x(-1) = 0$, and $x(1) = 4$. The diagram of the permutation is pictured in Fig. 1. (The grey areas are the 'shadows' of the columns, described later.) Starting with one of the empty columns, we read right-to-left, top-to-bottom, considering the diagram as though it were on a cylinder. The maximal decomposition of x is then:

$$x = d_{\{1\}}d_{\{2\}}d_{\{3\}}d_{\{2,4\}}d_{\{3,0\}}d_{\{4,3,1\}}.$$

By using a very small modification of the 'moves' on RC-diagrams considered by Bergeron and Billey [1], one can move between different α-decompositions of x. There are two kinds of 'moves' available, by which one can move boxes in the diagram into other rows. A *commutation move* moves a box with no neighbours above or to the right up into the next row directly, using only the commutation relations. A *chute move* moves a box past a two-by-l block of boxes, changing the residue of the given box, using a sequence of braid relations. (The precise rule for this is obtained by repeated application of the basic braid relations in $\tilde{S}_n$.) This is illustrated in Fig. 2, with an example in Fig. 3.

How can we apply this schema to affine pattern avoidance? It is known that an affine permutation is $[3, 2, 1]$-avoiding if and only if it has no reduced words in which one may apply a braid relation of the form $xyx = yxy$. This was shown independently by Lam [19] and Green [13]. Given a diagram of a maximal decomposition, we can convert it to a *shadow diagram*. This simply involves drawing the shadows of each column, extending down-to-the-left at 45°. The shadows also wrap-around the diagram. (Refer back to Fig. 1 for an example.) One can then observe the following:

Proposition 1. *An affine permutation x is* $[3, 2, 1]$*-avoiding if and only if no column of the diagram of x is completely in the shadow of another column. Equivalently, $c_{i-j} >= c_i - j$ for all $i, j \in \mathbb{Z}_n$, considering the indices as elements of $\mathbb{Z}_n$.*

Thus, the example permutation with window $[4, -4, 7, 5, 3]$ contains a $[3, 2, 1]$ pattern, since $c_5 = c_{2-2} = 3 < c_2 - 2 = 4$. This allows for a very fast check of whether a given affine permutation is $[3, 2, 1]$ avoiding. It takes linear time

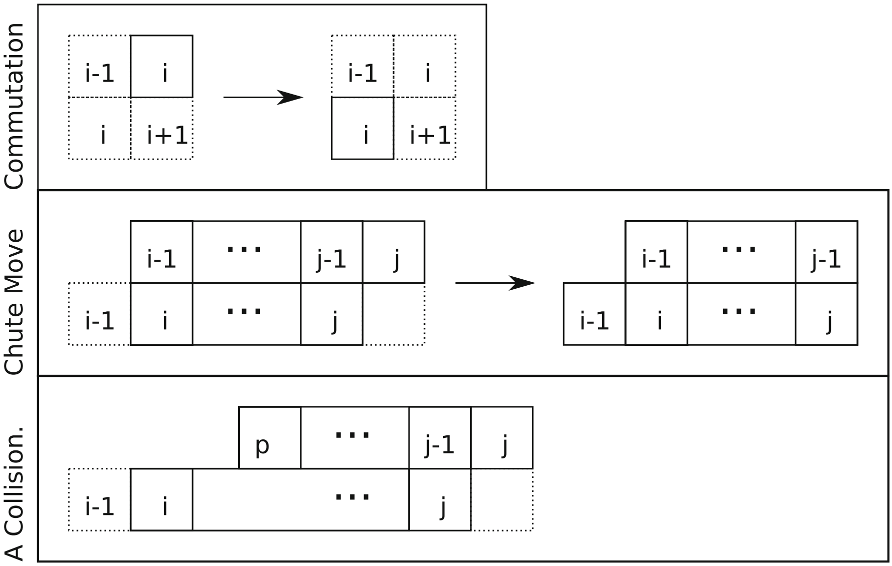

Fig. 2 The two-row moves on an α-decomposition. The collision shows a situation where a chute move will lead to a situation involving squaring a generator. This doesn't happen in reduced decompositions, and the result depends on whether one is working in the affine symmetric group (in which case the two colliding boxes annihilate one another), the nilHecke monoid (in which case the whole permutation is equal to 0), or the 0-Hecke monoid (where one of the boxes is removed while the other remains), according to the various specializations of T_i^2 at different values of q

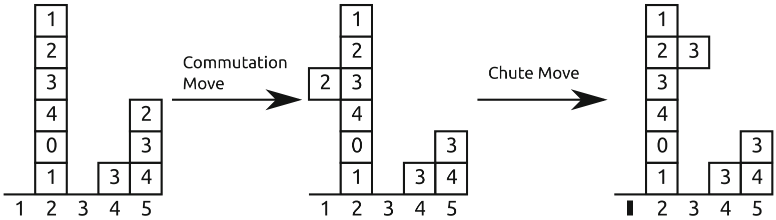

Fig. 3 Examples of other α-decompositions of x with window notation $[4, -4, 7, 5, 3]$. The leftmost picture is the maximal decomposition of x. The middle is obtained from the maximal α-decomposition by a commutation move. And the rightmost is obtained from the middle by a chute move (in this case, a simple braid relation)

to construct the affine code (indeed, only one pass through the main window is required), and only one pass through the code to determine whether the permutation is FC.

Proof. First we show that if any non-empty column is completely shadowed, then the element is not fully commutative. The strategy is to show that one can apply a chute move to a diagram with a fully-shadowed column, either directly or after applying a few commutation moves. (This direction is fairly intuitive to work through on paper: draw some diagrams corresponding to the cases and try to produce chute moves. What follows is a concise but rigorous 'proof by regular expressions.')

- *Case 1: There exists a shadowed column adjacent to the shadowing column.* Suppose column i is completely shadowed by column $i+1$. We consider only the two *rows* c_i and c_i+1. We can consider the diagram for these two rows in isolation; this two-row diagram will still have a 1-box column i completely shadowed by a 2-box column $i+1$. Consider the code d for just these two rows, which contains only 0's, 1's, and 2's. This code must contain at least one zero, and any instance of the consecutive subsequence $[1,2]$ indicates a column of the original diagram completely shadowed by a neighbouring column. Choose any $[1,2]$-instance where reading to the right one finds a 0 before finding another $[1,2]$ instance. Now consider the subword of d which begins with the chosen $[1,2]$ instance and ends with the first 0 one meets when reading to the right. The subword must then be of the form $[1, 2^p, 1^q, 0]$ (meaning a 1, followed by p 2's and q 1's, then a 0), possibly with $q=0$. Consider the diagram of this subword. One can apply a sequence of commutation moves to move the 1^q trailing boxes into the upper row, and then apply a chute move on the last box in the lower row.
- *Case 2: A shadowed column not adjacent to the shadowing column.* Choose a completely shadowed column a. Then find the first column right of a which completely shadows a: suppose this is the jth column. Now find the first column to the left of j which j fully shadows, and suppose this is the ith column. If $j=i+1$, we're in the first case. Otherwise, $c_{i+1}<c_i$, so the top box of c_i can be moved up-and-to-the-right using a commutation move. Since c_j is the nearest column to the right of c_i which shadows c_i, the box can be brought adjacent to column c_j. Because of full shadowing of c_i, the box will be below the level of the top of c_j once adjacent. We can then recycle the arguments of Case 1 to create a chute move.

For the converse, we show that if x is not fully commutative, then x has a shadowed column. Every α-decomposition of x can be obtained from the maximal decomposition by a sequence of two-row moves [8], which includes all reduced words for x. Let $w=[w_1, w_2, \dots, w_l]$ be a reduced word for x with a consecutive subword $s_i s_{i+1} s_i$ occurring farthest to the left amongst all reduced words for x; in particular, let the braid $s_i s_{i+1} s_i$ occur in positions w_j, w_{j+1}, w_{j+2}, with j minimal amongst all reduced words for x. One may then find the maximal decomposition of the permutation with word $[w_{j+3}, \dots, w_l]$. Then inserting the three letters w_j, w_{j+1}, w_{j+2} will ensure that the column with top box w_j is fully shadowed by the column with top box w_{j+2}. The column which now has top box w_j cannot grow any larger through the addition of further boxes from $[w_1, \dots, w_{j-1}]$ without the application of a braid move, which is impossible by the minimality of j. Therefore, there is a completely shadowed column in the maximal decomposition of x, as desired. □

Corollary 1. *Each FC affine permutation has diagram given by a union of partitions, separated by empty columns, such that shadow condition is satisfied.*

Proof. If $0 \neq c_i < c_{i+1}$, then the shadow condition is violated. Thus, for every i, we have $c_i > c_{i+1}$ or $c_i = 0$. □

3.1 Enumeration of Fully Commutative Affine Permutations

We can also use this structure to study the length-enumeration of the fully commutative elements. It was shown by Crites that there are infinitely many $[3, 2, 1]$-avoiding elements [6], but one can still ask how many affine permutations there are of length l. Let $F_n(l)$ be the number of fully commutative affine permutations of length l in $\tilde{S}_n$. This was studied by Jones and Hanusa via generating functions [15]. In particular, they were able to show that the number of FC elements of length l is eventually periodic in l. Observationally, this periodic behaviour began at $l = 1 + \lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor$. We now show that this bound is sharp.

Suppose that x has code c. Then let $U(x)$ be the affine permutation with code obtained by adding one to each non-zero entry of c. Likewise, let $D(x)$ be the permutation with code obtained by reducing each non-zero entry of c by one. For our running example with window $[4, -4, 7, 5, 3]$, and affine code $[0, 6, 0, 1, 3]$, then $U(x)$ has code $[0, 7, 0, 2, 3]$. Thus, $U(x)$ is the permutation with window $[6, -5, 8, 4, 2]$. Likewise, $D(x)$ has code $[0, 5, 0, 0, 2]$, which gives the permutation $[3, -3, 5, 6, 4]$.

It is clear that if x is fully commutative then so is $U(x)$ and $D(x)$; reducing or increasing the size of each column does not affect the shadow condition of Proposition 1. It is also clear that $U(D(x)) = x$ unless the code of x has some entry c_i equal to 1. We will call these *shift-minimal* elements.

Definition 1. A fully commutative affine permutation x is *shift-minimal* if the code c of x has at least one entry equal to 1. For any fully commutative affine permutation, we set $\text{col}(x)$ to be the number of non-zero entries of c, and $r(x)$ to be the length of x reduced modulo $\text{col}(x)$. Set $M_{i,j}$ to be the number of shift-minimal fully commutative affine permutations x with $\text{col}(x) = i$ and $r(x) = j$.

For example, let y be the affine permutation with window $[-3, -1, 8, 1, 10]$. Then the code of y is $[4, 3, 0, 3, 0]$, and y is FC. We have $\text{col}(y) = 3$, and $r(y) = 10 \mod 3 = 1$.

Lemma 1. *There are only finitely many shift-minimal FC elements. The maximal length of a shift-minimal element is* $\leq 1 + \lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor$, *and there exist elements attaining this length.*

Proof. Since each shift-minimal element contains a 1 in its code c, there can be (by the shadow condition) no $c_i > n$. This establishes an upper bound of $1 + (n - 2)(n - 1)$ on the length of a shift-minimal element, which in turn implies that there are finitely many such elements.

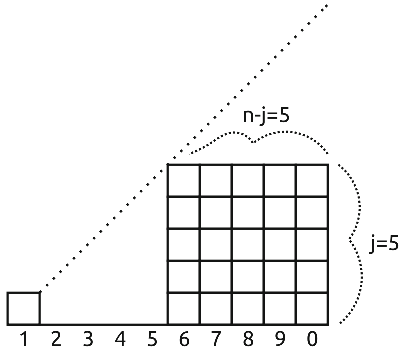

Fig. 4 A shift-minimal element of maximal length for $n = 10$. The single-box column ensures the element is shift-minimal. Then building a large rectangle allows us to maximize the total number of boxes. We should then maximize the size of rectangles over the possible heights j and widths $n - j$

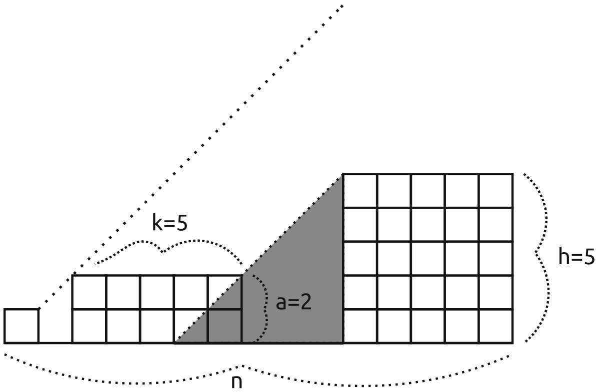

Fig. 5 Ruling out the two-rectangle case. The rectangle on the right is freely chosen, so as not to shadow the single box int he first column. The middle rectangle, in order to maximize the number of boxes, should have its upper-right point on the shadow cast by the right-most rectangle. Maximizing the width gives the constant k. Then the rectangle has area ak. Maximizing a then 'merges' the rectangle with the original rectangle

To construct an element of maximal length, suppose without loss of generality that $c_1 = 1$. Then we claim an optimal strategy to construct a long element is to make a $j \times (n - j)$ rectangle to the right of the first column. (See Fig. 4.)

By Corollary 1, the diagram must be a union of partitions, $P_1, P_2, \ldots, P_l$, placed so that they satisfy the shadow rule. Since we are maximizing the total number of boxes, we may take each partition P_i and complete it to a rectangle of height equal to the first column of P_i. This operation will never violate the shadow rule. Thus, we consider a sequence of rectangles. The last rectangle may as well be pushed as far to the right as possible, so that it is adjacent (on the cylinder) to the single box in the first column.

Now consider the second-to-last rectangle. On inspection, we see that it has a maximal width k determined by the height of the last rectangle and its placement relative to the first column. (See Fig. 5.) If it has less than this maximal width, then we can add more columns to the rectangle until the maximal width is reached.

Then this rectangle has width k, and some height a which is less than or equal to the height of the last rectangle. But to maximize the size of the rectangle, we should then maximize the height a. Then a is equal to the height of the last rectangle, which shows that they are, in fact, a single big rectangle. Applying this argument to all of the partitions, we see that we only need to consider a single rectangle!

Now we should then maximize the area of our single rectangle: Thus, we maximize the product $j(n-j)$ over j. The maximum product of two integers that sum to n is $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$. Including the initial column of size 1, we obtain a maximal length shift-minimal element of length $1 + \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$, as desired. □

Now we consider our function $F_n(l)$. It is clear that for every FC affine permutation x of length l, there exists some minimal k such that $D^k(x)$ is shift-minimal. This involves subtracting some multiple of $\mathrm{col}(x)$ until we are left with a shift-minimal element. Let $M_{i,j}$ be the number of shift-minimal FC affine permutations with $\mathrm{col}(x) = i$ and $r(x) = j$. Then we observe that for l large:

$$F_n(l) = \sum_{i=1}^{n-1} M_{i,l\%i},$$

where $l\%i$ denotes $l \mod i$. Then every $F_n(l)$ is a sum of $M_{i,j}$'s; therefore, once all of the shift-minimal elements are accounted for, the function $F_n(l)$ must begin being periodic. And this occurs exactly at $1 + \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$. Thus we have shown:

Theorem 2. *The function $F_n(l)$ is periodic in l, with periodic behaviour beginning at $l = 1 + \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$.*

Ostensibly, from this construction, the period of $F_n(l)$ would be $\leq n!$. In fact, it was shown by Hanusa and Jones that the period divides n. This implies that there should be some interesting relations between the numbers $M_{i,j}$ which remain to be fully explored.

4 Affine Catalan Monoid and Pattern Avoidance

In this section, we mainly summarize results from [9] and the author's thesis, connecting the prior results to the point of view of affine codes.

An interesting quotient of the 0-Hecke monoid may be obtained by introducing the relation, for every $i \in I$:

$$\pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}.$$

In the finite case, this gives a monoid isomorphic to the *Catalan monoid* of functions $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ satisfying:

- $f(p) \leq p$, and
- $p \leq\Rightarrow f(p) \leq f(q)$.

These functions form a monoid under composition, called the *non-decreasing parking functions*. (See e.g. [28]; it also is described under the notation $\mathscr{C}_N$ in e.g. [25, Chapter XI.4] and, together with many variants, in [11, Chapter 14].) Similarly, it is a natural quotient of Kiselman's monoid [12, 18]. In [10], this monoid was studied as an instance of the larger class of order-preserving regressive functions on monoids, and a set of explicit orthogonal idempotents in the algebra was described.

The fibers of this quotient have a nice property:

Theorem 3. *Each fiber of the quotient map* $H_0(S_N) \to \mathrm{NDPF}_N$ *contains a unique* $[3, 2, 1]$*-avoiding element for minimal length and a unique* $[2, 3, 1]$*-avoiding element of maximal length.*

As mentioned in the introduction, one can check that this directly implements the Simone-Schmidt bijection, which is one of the essential early results in the study of pattern avoidance. The bijection's proof was originally combinatorial, but this theorem shows that there is actually an algebraic reason for the bijection.

We also note that by taking the quotient of the 0-Hecke monoid by the relation $\pi_{i+1}\pi_i\pi_{i+1} = \pi_{i+1}\pi_i$, one obtains another monoid isomorphic to the Catalan monoid, though the fibers contain a unique $[3, 1, 2]$-avoiding permutation instead of a $[2, 3, 1]$-avoiding permutation.

We can use the affine codes of the finite permutations to directly construct the fibers. The finite permutation group is a parabolic subgroup of the affine permutation group, so all of the technology for working with affine codes descends to the finite case. Here, the code becomes a familiar Lehmer code or inversion vector, and the two-row moves become very close to moves on RC-graphs.

For finite permutations, the code is calculated in exactly the same way, but there are no π_0 generators and no 0-residues in the diagram of a finite permutation. Thus, we have $c_i \leq i - 1$ for each i. Given the shadow diagram of a finite permutation, we obtain the $[3, 2, 1]$-avoiding permutation by deleting all columns that are completely shadowed. Likewise, to obtain the maximal, $[3, 1, 2]$-avoiding element, we insert fully shadowed columns. Such a fiber is pictured in Fig. 6.

4.1 Affine Pattern Avoidance

In the affine case, the introduction of the extra relation gives a monoid isomorphic to the affine non-decreasing parking functions.

Definition 2. The extended affine non-decreasing parking functions are the functions $f : \mathbb{Z} \to \mathbb{Z}$ which are:

- Regressive: $f(i) \leq i$,
- Order Preserving: $i \leq j \Rightarrow f(i) \leq f(j)$, and
- Skew Periodic: $f(i + N) = f(i) + N$.

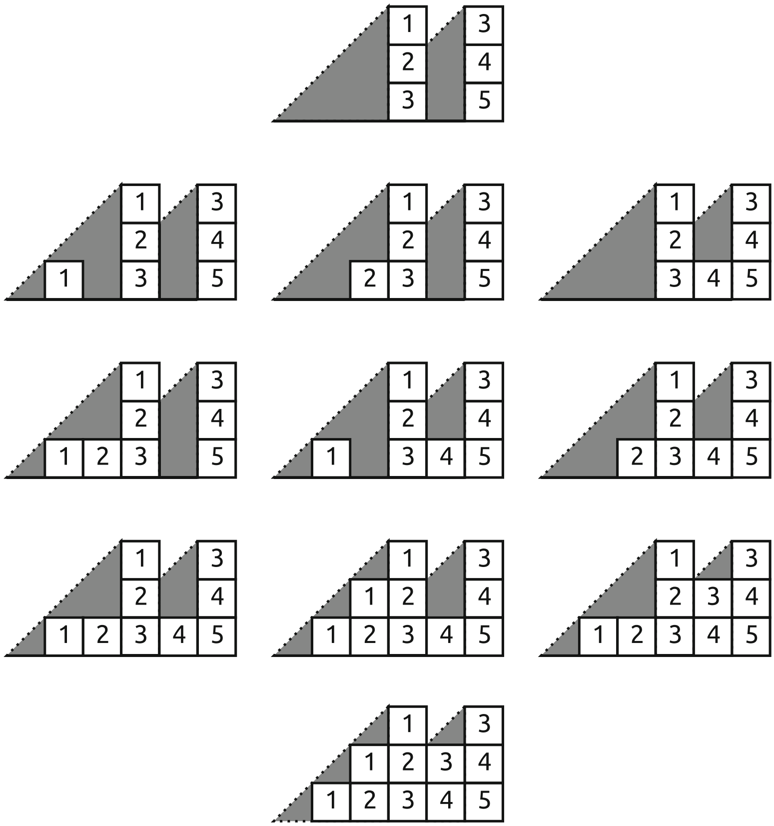

Fig. 6 A fiber of the finite quotient from $H_0(S_N)$ to the Catalan monoid, NDPF_N. At the top is the diagram of the $[3,2,1]$-avoiding permutation $[2,4,5,1,6,3]$. At the bottom is the $[2,3,1]$-avoiding permutation $[6,5,2,1,4,3]$. All other elements of the fiber lie between. One may notice that the fiber has an implicit poset structure arising from containment of diagrams; since this is obtained by deletion of individual letters from reduced expressions, this is simply the Bruhat order on the associated permutations

Define the **shift functions** sh_t as the functions sending $i \to i - t$ for every i.

The **affine non-decreasing parking functions** ${\mathrm{NDPF}^{(1)}}_N$ are obtained from the extended affine non-decreasing parking functions by removing the shift functions for all $t \neq 0$.

It is worth noting that the quotient seems to work well for the 0-Hecke monoid, but isn't useful in the case of general q.

We can summarize the correspondence in the following theorem:

Theorem 4 ([7, 9]). *The affine non-decreasing parking functions* ${\mathrm{NDPF}^{(1)}}_N$ *are a* $\mathscr{J}$*-trivial monoid which can be obtained as a quotient of the* 0*-Hecke monoid of the affine symmetric group by the relations* $\pi_j\pi_{j+1}\pi_j = \pi_j\pi_{j+1}$*, where the subscripts are interpreted modulo* N*. Each fiber of this quotient contains a unique* $[3,2,1]$*-avoiding affine permutation.*

By a result of Crites, there are infinitely many affine permutations that avoid a pattern σ if and only if σ contains the pattern $[3, 2, 1]$ [6]. Thus, there are infinitely many $[3, 2, 1]$-avoiding affine permutations, but only finitely many $[2, 3, 1]$-avoiding affine permutations. As in the finite case, the $[3, 2, 1]$-avoiding element is obtained by deleting all fully-shadowed columns. But filling in shadowed columns may be ambiguous, since there must be at least one empty column; thus we cannot expect that there is a unique element of maximal length in the fibers of the affine Catalan quotient.

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