Prime Ideals in Polynomial and Power Series Rings over Noetherian Domains

Ela Celikbas, Christina Eubanks-Turner, and Sylvia Wiegand

Abstract In this article we survey recent results concerning the set of prime ideals in two-dimensional Noetherian integral domains of polynomials and power series. We include a new result that is related to current work of the authors [Celikbas et al., *Prime Ideals in Quotients of Mixed Polynomial-Power Series Rings*; see http://www.math.unl.edu/$\sim$swiegand1 (preprint)]: Theorem 5.4 gives a general description of the prime spectra of the rings $R[\![x,y]\!]/P$, $R[\![x]\!][y]/Q$ and $R[y][\![x]\!]/Q'$, where x and y are indeterminates over a one-dimensional Noetherian integral domain R and P, Q, and Q' are height-one prime ideals of $R[\![x,y]\!]$, $R[\![x]\!][y]$, and $R[y][\![x]\!]$, respectively. We also include in this survey recent results of Eubanks-Turner, Luckas, and Saydam describing prime spectra of simple birational extensions $R[\![x]\!][f(x)/g(x)]$ of $R[\![x]\!]$, where $f(x)$ and $g(x)$ are power series in $R[\![x]\!]$ such that $f(x) \neq 0$ and is a prime ideal of $R[\![x]\!][y]$—this is a special case of Theorem 5.4. We give some examples of prime spectra of homomorphic images of mixed power series rings when the coefficient ring R is the ring of integers $\mathbb{Z}$ or a Henselian domain.

Keywords Commutative ring • Noetherian ring • Integral domain • Polynomial ring • Power series ring • Prime ideals • Prime spectrum

1991 *Mathematics Subject Classification.* Primary 13B35, 13J10, 13A15

E. Celikbas
Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
e-mail: celikbase@missouri.edu

C. Eubanks-Turner
Department of Mathematics, Loyola Marymount University, Los Angeles, CA 90045, USA
e-mail: ceturner@lmu.edu

S. Wiegand (✉)
Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588, USA
e-mail: swiegand1@math.unl.edu

M. Fontana et al. (eds.), *Commutative Algebra: Recent Advances in Commutative Rings, Integer-Valued Polynomials, and Polynomial Functions*, DOI 10.1007/978-1-4939-0925-4_4,

1 Introduction

Prime ideals play a fundamental role in commutative ring theory, especially in the theory of ideals and modules. By the primary decomposition theorem, every nonzero ideal of a Noetherian ring has a unique set of associated prime ideals. Often if a property can be demonstrated for prime ideals, then it holds for all ideals, for example, finite generation, by Cohen's theorem [15, Theorem 3.4]. Murthy uses prime ideals to show that a regular local ring is a UFD in [18]. For a Noetherian ring R, the Grothendieck group of all finitely generated R-modules is generated by the modules of the form R/P, where P is a prime ideal of R (see [2]). The Wiegands demonstrate many connections between the set of prime ideals of a ring R and the set of indecomposable R-modules in [28].

For R a commutative ring, we denote by Spec(R) the *prime spectrum* of R, that is, the set of prime ideals of R, considered as a partially ordered set, or *poset*, under inclusion. In 1950, Irving Kaplansky asked:

Question 1.1. Which partially ordered sets occur as Spec(R), for some Noetherian ring R?

This problem remains open, although there have been many and varied results related to Question 1.1:

(1) Hochster's characterization of the prime spectrum of a commutative ring as a topological space [9],
(2) Lewis' result that every finite poset is the prime spectrum of a commutative ring [12],
(3) Some properties of prime spectra of Noetherian rings [16, 27, 29],
(4) Examples of Noetherian rings such as those of Nagata, McAdam, and Heitmann that do not have other properties that might be expected of Noetherian rings [8, 17, 19, 21], and
(5) Characterizations of prime spectra of other specific classes of Noetherian rings or of particular Noetherian rings (see, for example, [6, 13, 24, 26]).

Many of these results are discussed in more detail in [29], along with other results.

In this article we focus on results over the past decade concerning prime spectra for two-dimensional Noetherian integral domains of polynomials and power series. We include background information related to this focus. In particular, our results are related to S. Wiegand's theorem from the 1980s, Theorem 2.3, proved using techniques developed by Heitmann and others; see [25]. Theorem 2.3 shows that any finite amount of "misbehavior" is possible for prime ideals of a Noetherian ring. Other results such as McAdam's Theorem 2.4 suggest that the converse is also true: Perhaps, in some sense, the amount of such misbehavior is finite. Our current and recent investigations of prime spectra show that certain finite subsets of these spectra determine the partially ordered sets that are prime spectra for our rings; see Theorem 5.4 and Definition 5.5.

The *characterization* of the prime spectrum of a particular ring requires (1) a list of axioms that describe the prime spectrum as a poset, and (2) a proof that any two posets satisfying these axioms are order-isomorphic. In order to characterize prime spectra for a class of Noetherian rings, the axioms of (1) may contain "genetic codes" that allow for some variety in the spectra of rings of the class; they depend upon cardinalities associated to the ring. In this case we require (2′). Each "genetic code" should determine a unique partially ordered set up to order-isomorphism and (3) examples to show that every poset fitting the axioms can be realized as a prime spectrum for some ring in the class.

For the remainder of this article, let x and y be indeterminates over a one-dimensional Noetherian domain R. Wiegand characterizes $\text{Spec}(\mathbb{Z}[y])$, where $\mathbb{Z}$ is the ring of integers, in [26]; see Theorem 2.9. If R is a countable, semilocal one-dimensional Noetherian domain, Wiegand and Heinzer characterize $\text{Spec}(R[y])$ in [6]; this characterization of course depends upon the number of maximal ideals of R. Shah and Wiegand extend this result to $\text{Spec}(R[y])$, for R a semilocal one-dimensional Noetherian domain of any cardinality in [23,29]—this characterization depends upon the number of maximal ideals of R, the cardinality of R, and the cardinality of $R/\mathbf{m}$ for each maximal ideal $\mathbf{m}$ of R; see Theorem 2.13.

Several recent articles describe prime spectra for power series rings. In [7], Heinzer, Rotthaus, and Wiegand describe $\text{Spec}(R[\![x]\!])$; see Theorem 2.15. In [5], Eubanks-Turner, Luckas, and Saydam describe prime spectra of simple birational extensions of $R[\![x]\!]$, that is, $\text{Spec}(R[\![x]\!][g/f])$, where $g, f \in R[\![x]\!]$, $f \neq 0$, and either g, f is an $R[\![x]\!]$-sequence or $(g, f) = R[\![x]\!]$; see Theorem 6.1. In current work, the present authors describe prime spectra of rings of the form $R[\![x]\!][y]/Q$ or $R[y][\![x]\!]/Q$, where Q is a height-one prime ideal of the appropriate ring and $x \notin Q$; see Sect. 5 and [4].

In Sect. 2 we give notation and background results on prime spectra, and we mention some related items, such as the intriguing Conjecture 2.12 of Roger Wiegand. We give some general properties of mixed power series in Sect. 3. In Sect. 4 we characterize $\text{Spec}(R[\![x]\!][\![y]\!]/Q)$, where R is a one-dimensional Noetherian domain and Q is a height-one prime ideal of $R[\![x, y]\!]$; see Theorem 4.1. In Sect. 5 we give new results related to the characterization of $\text{Spec}(R[\![x]\!][y]/Q)$ and $\text{Spec}(R[y][\![x]\!]/Q)$ from [4]; see Theorems 5.2 and 5.4. In Sect. 6 we give results from [5] concerning prime spectra of simple birational extensions of $R[\![x]\!]$; this yields a characterization in the case R is a countable Dedekind domain. In Sect. 7, we show two prime spectra examples of dimension two: $\text{Spec}(\mathbb{Z}[y][\![x]\!]/Q)$, where Q is a specified prime ideal of $\mathbb{Z}[y][\![x]\!]$, and $\text{Spec}(R[\![x]\!][y]/Q)$, where Q is a specified prime ideal of $R[\![x]\!][y]$ and R is a Henselian domain.

All rings are commutative with identity throughout the paper. Let $\mathbb{N}$ denote the natural numbers, let $\mathbb{Z}$ denote the integers, and let $\mathbb{R}$ denote the real numbers. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\aleph_0 := |\mathbb{N}|$.

2 Background

In this section, we give background information related to our focus on recent work concerning prime spectra of two-dimensional Noetherian integral domains of polynomials and power series. We refer the reader to [28, 29] for more general information concerning prime spectra in Noetherian rings.

We first introduce some notation.

Notation 2.1. Let U be a partially ordered set, sometimes abbreviated *poset*; let S be a subset of U and let $u, v \in U$. We define

$$u^{\uparrow(U)} = u^{\uparrow} := \{w \in U \mid u < w\}, \quad u^{\downarrow} := \{w \in U \mid w < u\}, \quad \mathrm{L_e}(S) := \{u \in U \mid u^{\uparrow} = S\};$$

$$\max(S) := \{\text{maximal elements of } S\}, \quad \text{and} \quad \min(S) := \{\text{minimal elements of } S\}.$$

For $u \in U$, the *height* of u, $\mathrm{ht}(u)$, is the length $t \in \mathbb{N}_0$ of a maximal length chain in U of form

$$u_0 < u_1 < u_2 < \cdots < u_t = u.$$

Set $\mathcal{H}_i(U) := \{u \in U \mid \mathrm{ht}(u) = i\}$, for each $i \in \mathbb{N}_0$. The *dimension* of U, $\dim(U)$, is the maximum of the heights of all elements of U.

We say v *covers* u and write $u \ll v$ if $u < v$ and there are no elements of U strictly between u and v. The *minimal upper bound set* of u and v, if $u \not\leq v$ and $v \not\leq u$, is the set $\mathrm{mub}(u, v) := \min(u^{\uparrow} \cap v^{\uparrow})$ and their *maximal lower bound set* is $\mathrm{Mlb}(u, v) := \max(u^{\downarrow} \cap v^{\downarrow})$.

Let R be a commutative ring. We use notation similar to that for the partially ordered set $U = \mathrm{Spec}(R)$. For example, if $P \in \mathrm{Spec}(R)$, $P^{\uparrow} = \{Q \in \mathrm{Spec}(R) \mid P \subsetneq Q\}$; $\min(R)$ is the set of minimal prime ideals of R; $\max(R)$ is the set of maximal ideals of R; and $\dim(R)$ is the supremum of the heights that occur for maximal ideals of R. We also use $V(S) := V_R(S) := \{\mathbf{q} \in \mathrm{Spec}(R) \mid S \subseteq \mathbf{q}\}$, for a subset S of R; for $a \in R$, put $V_R(a) := V_R(\{a\})$. For each $i \in \mathbb{N}_0$, we set $\mathcal{H}_i(R) := \{\mathbf{q} \in \mathrm{Spec}(R) \mid \mathrm{ht}(\mathbf{q}) = i\}$.

In Remarks 2.2 we establish that the rings we study are well behaved.

Remarks 2.2. (1) If a ring A is Cohen–Macaulay, $n, m \in \mathbb{N}_0$, and x_i and y_j are indeterminates over A, for $1 \leq i \leq n$, $1 \leq j \leq m$, then the *mixed polynomial-power series rings*, $A[\![\{x_i\}_{i=1}^n]\!][\{y_j\}_{j=1}^m]$ and $A[\{y_j\}_{j=1}^m][\![\{x_i\}_{i=1}^n]\!]$, are Cohen–Macaulay; see [15, Theorem 17.7]. Thus they are *catenary*: If $P \subseteq Q$ in $\mathrm{Spec}(R)$, then any two maximal chains of prime ideals from P to Q have the same length [15, Theorem 17.9].

(2) If R is a Noetherian integral domain of dimension one, then R is Cohen–Macaulay; see [15, Exercise 17.1, p. 139]. Thus every mixed polynomial-power series ring over a one-dimensional Noetherian domain R that involves a finite number of variables is catenary by item (1).

Theorem 2.3 was inspired by many examples of Noetherian prime spectra with finite amounts of "misbehavior" that were produced by Nagata, McAdam, Heitmann, and others; they show, for example, that Noetherian rings can be noncatenary and that there exist Noetherian rings containing two height-two prime ideals whose intersection contains no height-one prime ideal; see [8, 17, 20]. The proof of Theorem 2.3 uses techniques of these and other researchers. The statement of Theorem 2.3 summarizes the situation: All sorts of finite noncatenary or prescribed intersecting behavior is possible in the prime spectrum of some Noetherian rings. This idea is related to the later sections of this article where we show that the prime spectra that occur for our rings have a similar finite amount of "prescribed" discrepancy within a general form of the spectra; see Sects. 5 and 6. The difference here is that our rings are catenary by Remark 2.2.

Theorem 2.3 ([25, Theorem 1]). *Let F be an arbitrary finite poset. There exist a Noetherian ring A and a saturated order-embedding $\varphi : F \to \mathrm{Spec}(A)$ such that φ preserves minimal upper bound sets and maximal lower bound sets. In detail, for $u, v \in F$, we have*

(i) $u < v$ *if and only if* $\varphi(u) < \varphi(v)$;
(ii) v *covers* u *if and only if* $\varphi(v)$ *covers* $\varphi(u)$;
(iii) $\varphi(\mathrm{mub}_F(u, v)) = \mathrm{mub}(\varphi(u), \varphi(v))$; *and*
(iv) $\varphi(\mathrm{Mlb}_F(u, v)) = \mathrm{Mlb}(\varphi(u), \varphi(v))$.

A related theorem of Steve McAdam, Theorem 2.4, guarantees that noncatenary misbehavior cannot be too widespread in the prime spectrum of a Noetherian ring:

Theorem 2.4 ([16]). *Let P be a prime ideal of height n in a Noetherian ring. Then all but finitely many covers of P have height $n + 1$.*

Perhaps one might conjecture from Theorem 2.4 that in general prime spectra of Noetherian rings behave well, like the spectra of excellent rings, if a finite "bad" subset is removed.[1]

Corollary 2.5, which follows from Theorem 2.3, relates to our focus for this article because it describes exactly the countable posets that arise as prime spectra of two-dimensional semilocal Noetherian domains.

Corollary 2.5 ([25, Theorem 2]). *Let U be a countable poset of dimension two. Assume that U has a unique minimal element and $\max(U)$ is finite. Then $U \cong \mathrm{Spec}(R)$ for some countable Noetherian domain R if and only if $\mathrm{L_e}(u)$ is infinite for each element u with $\mathrm{ht}(u) = 2$.*

Lemma 2.6 is useful for counting prime ideals in our rings.

[1]For the definition of "excellent ring" see [15, p. 260]. Basically "excellence" means the ring is catenary and has other nice properties that polynomial rings over a field possess.

Lemma 2.6 ([29, Lemma 4.2] and [5, Lemma 3.6, Remarks 3.7]). *Let T be a Noetherian domain, let y be an indeterminate, and let I be a proper ideal of T. Let $\beta = |T|$ and $\rho = |T/I|$. Then:*

(1) $|(T/I)[y]| = \rho \cdot \aleph_0 \leq \beta \cdot \aleph_0 = |T[y]|$.
(2) $|T[\![y]\!]| = \beta^{\aleph_0} = \rho^{\aleph_0}$.
(3) If $\beta \leq \aleph_0$*, then* $|(T/I)[y]| = \aleph_0 = |T[y]|$.
(4) If $\beta = \aleph_0$ *and* $\max(T)$ *is infinite, then* $\beta = |\max(T)| = |T/I| \cdot \aleph_0$.
(5) If k is a field, y, y' are indeterminates, and c is an irreducible element of $k[y]$, then

$$|(k[y]/ck[y])[y']| = |k| \cdot \aleph_0 = |\max(k[y])|.$$

2.1 Prime Ideals in Polynomial Rings

This subsection includes basic facts, previous results, and technical lemmas concerning $\mathrm{Spec}(A[y])$, where A is a Noetherian domain and y is an indeterminate over A.

In Remarks 2.7, we give some basic facts.

Remarks 2.7. Let A be a Noetherian domain of dimension d and let y be an indeterminate over A.

(1) If P is a prime ideal of $A[y]$, then $\mathrm{ht}(P \cap A) \leq \mathrm{ht}(P) \leq \mathrm{ht}(P \cap A) + 1$; see [15, Theorem 15.1].
(2) If M is a prime ideal of $A[y]$ of height $d + 1$, then M is a maximal ideal of $A[y]$, the prime ideal $\mathbf{m} = M \cap A$ is a maximal ideal of A of height d, and $M = (\mathbf{m}, h(y))A[y]$, where $\overline{h(y)}$ is irreducible in $\overline{A[y]} = A[y]/(\mathbf{m}[y]) \cong (A/\mathbf{m})[y]$. This follows from item (1) and [10, Theorem 28, p. 17].
(3) If I is a nonzero ideal of $A[y]$ such that $I \cap A = (0)$, then $I = h(y)K[y] \cap A[y]$, where K is the field of fractions of A and $h(y) \in A[y]$ with $\deg(h(y)) \geq 1$. This follows since $K[y] = (A \setminus \{0\})^{-1}A[y]$ is a principal ideal domain (PID). If P is a prime ideal of $A[y]$ such that $P \cap A = (0)$, then $\mathrm{ht}(P) = 1$. The set of prime ideals P of $A[y]$ such that $P \cap A = (0)$ is in one-to-one correspondence with the set of height-one prime ideals of $K[y]$, via $P \mapsto PK[y] \mapsto PK[y] \cap A[y]$.

The proof of Lemma 2.8 is straightforward and follows from material in [10] on G-domains; see [4]. A *G-domain* is an integral domain A such that $A[y]$ contains a maximal ideal that intersects A in (0).

Lemma 2.8 ([4, 10]). *Let A be a Noetherian domain. If Q is a maximal ideal of $A[y]$ of height one, then*

(1) $Q \cap A = (0)$;
(2) $\dim(A) \leq 1$ *and* $|\max(A)| < \infty$; *say* $\max(A) = \{\mathbf{m}_1, \ldots, \mathbf{m}_t\}$; *and*
(3) Q contains an element of form $h(y) = yg(y) + 1$, *where* $0 \neq g(y) \in (\cap_{i=1}^t \mathbf{m}_i)[y]$.

Moreover, if A is one-dimensional and semilocal with maximal ideals $\mathbf{m}_1, \ldots, \mathbf{m}_t$, and Q is a prime ideal of $A[y]$ that is minimal over an element of form $h(y) = yg(y) + 1$, where $g(y) \in (\cap_{i=1}^t \mathbf{m}_i)[y]$, then Q is a height-one maximal ideal of $A[y]$.

Theorem 2.9, due to Roger Wiegand, characterizes $\mathrm{Spec}(\mathbb{Z}[y])$, the spectrum of the ring of polynomials in the variable y over the integers $\mathbb{Z}$. The most important distinguishing feature of $\mathrm{Spec}(\mathbb{Z}[y])$ is Axiom RW.

Theorem 2.9 ([26, Theorem 2]). *Let $U = \mathrm{Spec}(\mathbb{Z}[y])$, the partially ordered set of prime ideals of the ring of polynomials in one variable over the integers. Then U is characterized by the following axioms:*

(P1) U is countable and has a unique minimal element.
(P2) U has dimension two.
(P3) For each element u of height-one, $u^{\uparrow}$ is infinite.
(P4) For each pair u, v of distinct elements of height-one, $u^{\uparrow} \cap v^{\uparrow}$ is finite.
(RW) Every pair (S, T) of finite subsets S and T of U such that $\emptyset \neq S \subseteq \mathcal{H}_1(U)$ and $T \subseteq \mathcal{H}_2(U)$ has a "radical element" in U. A "radical element" for such a pair (S, T) is a height-one element $w \in U$ such that $s^{\uparrow} \cap w^{\uparrow} \subseteq T \subseteq w^{\uparrow}$, for every $s \in S$.

A partially ordered set U satisfies the axioms of Theorem 2.9 if and only if U is order-isomorphic to $\mathrm{Spec}(\mathbb{Z}[y])$.

Theorem 2.9 leads to Question 2.10:

Question 2.10. For which two-dimensional Noetherian domains A is $\mathrm{Spec}(A) \cong \mathrm{Spec}(\mathbb{Z}[y])$?

Remarks 2.11. (1) The following is known about rings that fit Question 2.10:

(a) Let k be a field and let z be another indeterminate. Then $\mathrm{Spec}(k[z, y])$ is order-isomorphic to $\mathrm{Spec}(\mathbb{Z}[y]) \iff k$ is an algebraic extension of a finite field. The ($\Leftarrow$) direction is due to Wiegand in [26, Theorem 2]; for the ($\Rightarrow$) direction, see [29].
(b) Let D be an order in an algebraic number field; that is, D is the ring of algebraic integers in a field K that is a finite extension of the rational numbers. Roger Wiegand shows $\mathrm{Spec}(D[y])$ is order-isomorphic to $\mathrm{Spec}(\mathbb{Z}[y])$ in [26, Theorem 1].
(c) In their 1998 article Li and Wiegand prove that if $B := \mathbb{Z}[y][\frac{g_1}{f}, \ldots, \frac{g_m}{f}]$, where f is nonzero and $f, g_1, \ldots, g_m \in \mathbb{Z}[y]$, then $\mathrm{Spec}(B)$ is order-isomorphic to $\mathrm{Spec}(\mathbb{Z}[y])$; see [14].
(d) Saydam and Wiegand extend the result of Li and Wiegand in 2001 to show, for D an order in an algebraic number field and for B a finitely generated extension of $D[y]$ contained in the field of fractions of $D[y]$, that $\mathrm{Spec}(B) \cong \mathrm{Spec}(\mathbb{Z}[y])$ in [22].

(2) The prime spectrum of $R[y]$ is not known in general, for y an indeterminate over a one-dimensional Noetherian domain R with infinitely many maximal ideals. In fact $\mathrm{Spec}(R[y])$ is barely known beyond the examples of item (1) above and the rings of Theorem 2.13 below; see [29].

(3) The prime spectrum of $\mathbb{Q}[z, y]$, where $\mathbb{Q}$ is the field of rational numbers, is unknown, but Wiegand shows that it is *not* order-isomorphic to $\mathrm{Spec}(\mathbb{Z}[y])$ in [26]; see also [29, Remark 2.11.3].

In relation to Question 2.10, Wiegand's 1986 conjecture is still open:

Conjecture 2.12 (Wiegand [26]). For every two-dimensional Noetherian integral domain D that is finitely generated as a $\mathbb{Z}$-algebra, $\mathrm{Spec}(D) \cong \mathrm{Spec}(\mathbb{Z}[y])$.

The next theorem, Theorem 2.13, was first proved by Heinzer and Wiegand in case R is countable. Later Shah, Wiegand, and Wiegand proved it for cardinalities; see also [11]. By Theorem 2.13, the prime spectrum of a polynomial ring over a semilocal one-dimensional Noetherian domain is dependent upon whether or not the coefficient ring is Henselian.[2] For example, complete local rings, such as power series rings over a field, are Henselian.

Theorem 2.13 ([6, Theorem 2.7], [23, Theorem 2.4], and [29, Theorem 3.1]). *Let R be a semilocal one-dimensional Noetherian domain, let $\mathbf{m}_1, \ldots, \mathbf{m}_n$ be the maximal ideals of R where $n \in \mathbb{N}$, let y be an indeterminate, let $\beta = |R[y]|$, and let $\gamma_i = |(R/\mathbf{m}_i)[y]|$, for each i with $1 \le i \le n$. Then there exist exactly two possibilities for $U = \mathrm{Spec}(R[y])$ up to cardinality, depending upon whether or not R is Henselian and, if R is not Henselian, depending upon the number n of maximal ideals of R.*

- *In case R is **not Henselian**, U satisfies these axioms:*

  (I_β) $|U| = \beta$ *and U has a unique minimal element* $u_0 = (0)$.
  (II_β) $|\mathcal{H}_1(U) \cap \max(U)| = \beta$.
  (III_γ) $\dim(U) = 2$.
  (IV_n) *There exist exactly n height-one elements $u_1, \ldots, u_n \in U$ such that $u_i^\uparrow$ is infinite. Also:*
    (i) $u_1^\uparrow \cup \cdots \cup u_n^\uparrow = \mathcal{H}_2(U)$.
    (ii) $u_i^\uparrow \cap u_j^\uparrow = \emptyset$ *if* $i \neq j$.
    (iii) $|u_i^\uparrow| = \gamma_i$, *for* $1 \le i \le n$.
  (V_n) *If $v \in U$, v is not maximal,* $\mathrm{ht}(v) = 1$ *and* $v \notin \{u_1, \ldots, u_n\}$, *then $v^\uparrow$ is finite.*
  (VI_β) *For every nonempty finite subset T of $\mathcal{H}_2(U)$, we have* $|L_e(T)| = \beta$.

[2] Essentially a "Henselian" ring is one that satisfies Hensel's Lemma; see the definition in [20].

- *If R is* ***Henselian****, then $n = 1$ and U satisfies Axioms $I_\beta, II_\beta, III_\gamma, IV_1$ and the adjusted axioms V_1^h and VI_β^h below:*

(V_1^h) *If $v \in U$, v is not maximal,* $\text{ht}(v) = 1$ *and $v \neq u_1$, then* $|v^\uparrow| = 1$.
(VI_β^h) *For every nonempty finite subset T of $\mathcal{H}_2(U)$, we have $|L_e(T)| = \beta \iff |T| = 1$, and $L_e(T) = \emptyset \iff |T| > 1$.*

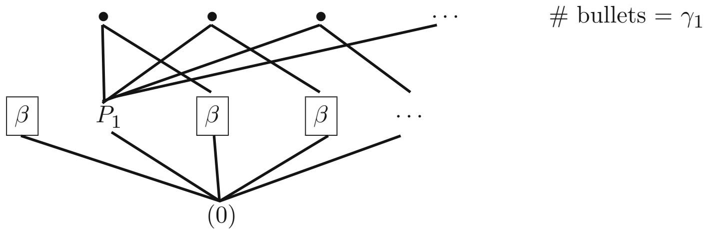


Diagram 2.13.h: Spec($R[y]$), R Henselian

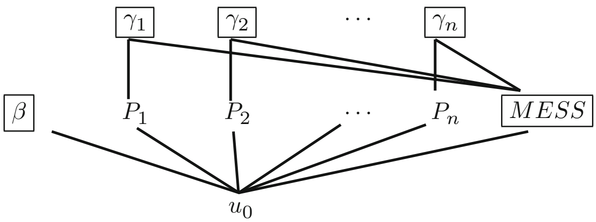


Diagram 2.13.nh: Spec($R[y]$), R non-Henselian

These diagrams show Spec($R[y]$) for the two cases of the theorem, where P_i is the prime ideal of $R[y]$ corresponding to u_i, for each i with $1 \leq i \leq n$, and each block $\boxed{\beta}$ represents β primes in that position.
The relations satisfied by the MESS box in Diagram 2.13 are too complicated to show. They are described in Axiom VI_β.

2.2 Prime Ideals in Power Series Rings

In this subsection we describe prime ideals in power series rings over a Noetherian domain. In the remainder of the paper we use the following straightforward remarks, particularly Remark 2.14(1).

Remarks 2.14. Let x be an indeterminate over a Noetherian domain A. Then

(1) Every maximal ideal of $A[\![x]\!]$ has the form $(\mathbf{m}, x)A[\![x]\!]$, where $\mathbf{m}$ is a maximal ideal of A; see [20, Theorem 15.1] (Nagata). Thus x is in every maximal ideal of $A[\![x]\!]$.

(2) If $\mathbf{p}$ is a prime ideal of A, then $\mathbf{p}A[\![x]\!] \in \operatorname{Spec}(A[\![x]\!])$ and $\operatorname{ht}(\mathbf{p}A[\![x]\!]) = \operatorname{ht}(\mathbf{p})$; see [3, Theorem 4] or [1, Theorem 4].
(3) Thus every maximal ideal of $A[\![x]\!]$ of maximal possible height in a Noetherian catenary domain has the form $(\mathbf{m}, x)A[\![x]\!]$, where $\mathbf{m}$ is a maximal ideal of A with $\operatorname{ht}(\mathbf{m}) = \dim(A)$.

Heinzer, Rotthaus, and Wiegand almost characterized $\operatorname{Spec}(R[\![x]\!])$ for R a one-dimensional Noetherian domain, except for specifying the cardinalities of the $\mathrm{L_e}(\{M\})$ sets of height-two maximal ideals M of $R[\![x]\!]$. Later Wiegand and Wiegand showed that $|\mathrm{L_e}(\{M\})| = |R[\![x]\!]|$ for each M.

Theorem 2.15 ([7, Theorem 3.4] and [29, Theorem 4.3]). *Let R be a one-dimensional Noetherian domain and let x be an indeterminate. Set $\beta := |R[\![x]\!]|$ and set $\alpha := |\max(R)|$. Then the partially ordered set $U := \operatorname{Spec}(R[\![x]\!])$ is determined by axioms similar to those of the Henselian version of Theorem 2.13:*

(I_β) *$|U| = \beta$ and U has a unique minimal element $u_0 = (0)$.*
(II_0) $\mathcal{H}_1(U) \cap \max(U) = \emptyset$.
(III_α) $\dim(U) = 2$, $|\mathcal{H}_2(U)| = \alpha$.
(IV_1) *There exists a height-one element $u_1 \in U$ such that $u_1^{\uparrow} = H_2(U)$ namely, $u_1 = xR[\![x]\!]$.*
(V_1^h) *If $v \in U$, v is not maximal, $\operatorname{ht}(v) = 1$, and $v \neq u_1$, then $|v^{\uparrow}| = 1$.*[3]
(VI_β^h) *For every nonempty finite subset T of $\mathcal{H}_2(U)$, we have $|L_e(T)| = \beta$ if and only if $|T| = 1$, and $L_e(T) = \emptyset$ if and only if $|T| > 1$.*

Thus $\operatorname{Spec}(R[\![x]\!])$ is as shown in the following diagram:

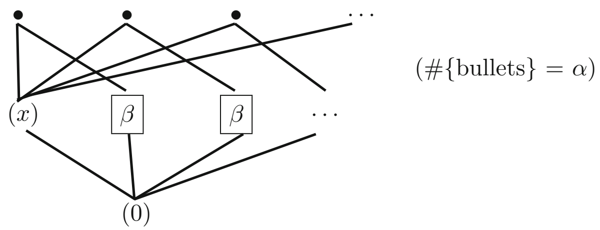


Diagram 2.15.0: $\operatorname{Spec}(R[\![x]\!])$

In Diagram 2.15.0, the cardinality of the set of bullets equals the cardinality of $\max(R)$ since the set of height-two maximal ideals of $R[\![x]\!]$ is in one-to-one correspondence with the set of maximal ideals of the coefficient ring R by Remark 2.14(1). The boxed β beneath each maximal ideal of $R[\![x]\!]$ means that there are exactly β prime ideals in that position (beneath that maximal ideal and

[3] Since axiom II_0 holds, this axiom could be stated here without saying "v is not maximal."

no other). For each value of α and β, any two posets described by Diagram 2.15.0 are order-isomorphic.

Remark 2.16. From Diagrams 2.15.0 and 2.13.h, we see that $\mathrm{Spec}(\mathbb{Q}[y][\![x]\!]) \not\cong \mathrm{Spec}(\mathbb{Q}([\![x]\!][y])$, where $\mathbb{Q}$ is the field of rational numbers. Moreover the difference between the prime spectrum of a power series ring over a one-dimensional Noetherian domain, such as $\mathbb{Q}[y][\![x]\!]$, and that of a polynomial ring over a Henselian ring, such as $\mathbb{Q}[\![x]\!][y]$, is that the partially ordered set described in the Henselian case of Theorem 2.13 has β height-one maximal elements, whereas the other partially ordered set has no height-one maximal elements.

In our characterizations of prime spectra, we identify those prime ideals that are an intersection of maximal ideals, such as the prime ideal (x) in Diagram 2.15.0 and the prime ideal P_1 in Diagram 2.13.h. These are called *j-prime ideals*.

Definitions 2.17. (1) Let A be a commutative ring.

- A *j-prime (ideal)* of A is a prime ideal of A that is an intersection of maximal ideals of A.
- The *j-spectrum* of A is $j\text{-}\mathrm{Spec}(A) := \{j\text{-primes} \in \mathrm{Spec}(A)\}$.

(2) For U a partially ordered set, we say that $u \in U$ is a *j-element* if u is a maximal element of U or if $\min(u^{\uparrow})$ is infinite. Then $j\text{-}set(U) := \{j\text{-elements of } U\}$.

Thus, if A is a two-dimensional integral domain, $\{j\text{-elements of } U = \mathrm{Spec}(A)\} = \{j\text{-prime ideals of } A\}$.

Examples 2.18. We show $j\text{-}\mathrm{Spec}(\mathbb{Q}[y][\![x]\!])$ and $j\text{-}\mathrm{Spec}(\mathbb{Q}[\![x]\!][y])$, respectively, in Diagram 2.18.0; they are parts of Diagrams 2.15.0 and 2.13.h.

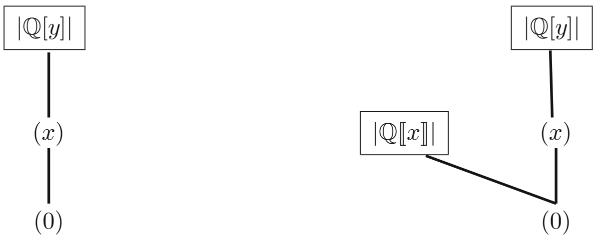


Diagram 2.18.0: $j\text{-}\mathrm{Spec}(\mathbb{Q}[y][\![x]\!])$ and $j\text{-}\mathrm{Spec}(\mathbb{Q}[\![x]\!][y])$

3 Properties of Mixed Polynomial-Power Series Rings

In this section we give some properties of prime spectra of three-dimensional Noetherian mixed polynomial-power series rings. We use the following setting:

Setting 3.1. Let x and y be indeterminates over a one-dimensional Noetherian domain R. Let A be either $R[y][\![x]\!]$, $R[\![x]\!][y]$, or $R[\![x, y]\!]$. Let $A_1 = R[y]$ if

$A = R[y][\![x]\!]$ or $R[\![x]\!][y]$, and let $A_1 = R[\![y]\!]$ if $A = R[\![x, y]\!]$. Then $A/xA \cong A_1$ and, depending on which A_1 we have, for every $\mathbf{m} \in \max(R)$.

$$A/(\mathbf{m}, x)A \cong A_1/\mathbf{m}A_1 \cong (R/\mathbf{m})[y] \quad \text{or} \quad A/(\mathbf{m}, x)A \cong (R/\mathbf{m})[\![y]\!].$$

Proposition 3.2 gives a description of the maximal ideals of A having maximal height, that is, height three.

Proposition 3.2 ([4]). *Assume Setting 3.1 and let $\mathcal{M}$ be a height-three maximal ideal of A. Then*

(1) $\mathcal{M} = (\mathbf{m}, x, h(y))A$, for some $\mathbf{m} \in \max(R)$ and some $h(y) \in A_1$ with $\overline{h(y)}$ irreducible in $\overline{A_1} = A_1/(\mathbf{m}A_1) \cong (R/\mathbf{m})[y]$ or $\overline{h(y)}$ irreducible in $\overline{A_1} \cong (R/\mathbf{m})[\![y]\!]$.
(2) Conversely, the ideals $(\mathbf{m}, x, h(y))A$ are maximal and have height three, for every $\mathbf{m} \in \max(R)$ and for every $h(y) \in A_1$ such that $\overline{h(y)}$ is irreducible in $\overline{A_1}$.
(3) If $A = R[\![x, y]\!]$, then every maximal ideal of $R[\![x, y]\!]$ has height 3; there are $|\max(R)|$ maximal ideals in $R[\![x, y]\!]$; and $\max(R[\![x, y]\!]) = \{(\mathbf{m}, x, y)R[\![x, y]\!]$, where $\mathbf{m} \in \max(R)\}$.
(4) For $A = R[\![x]\!][y]$ or $A = R[y][\![x]\!]$, there are $|(R/\mathbf{m})| \cdot \aleph_0$ height-three maximal ideals that contain $\mathbf{m}$, for each fixed $\mathbf{m} \in \max(R)$.

Proof. Item (3) follows from Remark 2.14(1). For the remaining items, see [4, Proposition 4.2]. □

Proposition 3.3 is also straightforward to prove using Remark 2.14(1) and Lemma 2.8; see [4].

Proposition 3.3 ([4, Propostion 4.3]). *There are no height-one maximal ideals in $R[\![x, y]\!]$, $R[y][\![x]\!]$, or in $R[\![x]\!][y]$.*

Proposition 3.4 is the reason that the prime spectra of $R[\![x]\!][y]$ and $R[y][\![x]\!]$ is much simpler than $\operatorname{Spec}(R[x, y])$.

Proposition 3.4 ([5, Proposition 3.11] and [4, Proposition 3.3]). *Assume Setting 3.1. Let P be a height-two prime ideal of A such that $x \notin P$. Then P is contained in a unique maximal ideal of A.*

In Proposition 3.5, with Setting 3.1, we observe that certain obvious conditions on a height-one prime ideal Q of A are equivalent to saying that Q is not contained in any height-three maximal ideal of A. For $A = R[\![x, y]\!]$, these conditions never occur; see Proposition 3.2(3) or Theorem 4.1.

Proposition 3.5 ([4, Proposition 3.8]). *Assume Setting 3.1, so that R is a one-dimensional Noetherian domain and A is $R[\![x]\!][y]$, $R[y][\![x]\!]$, or $R[\![x, y]\!]$. Let Q be a height-one prime ideal of A. Then statements 1–4 are equivalent:*

(1) Every prime ideal of A containing $(Q, x)A$ is a maximal ideal.
(2) For every $\mathbf{m} \in \max(R)$, every prime ideal of A containing $(Q, \mathbf{m})A$ is maximal.

(3) Q is contained in no height-three maximal ideal of A.
(4) $\dim(A/Q) = 1$.

Moreover,

- *If $(Q, x)A = A$, then item (1) holds.*
- *If $\mathbf{m} \in \max(R)$ and $(Q, \mathbf{m})A = A$, then every prime ideal containing $(Q, \mathbf{m})A$ is maximal.*
- *Thus either of the conditions*
  (i) $(Q, x)A = A$ *or*
  (ii) $(Q, \mathbf{m})A = A$, *for every* $\mathbf{m} \in \max(R)$,

  implies (4) $\dim(A/Q) = 1$.

Proposition 3.6 holds for higher-dimensional rings and more variables (one variable must be a power series variable), but to fit our focus in this article, we consider prime ideals of A, where $A = R[\![x, y]\!]$, $R[y][\![x]\!]$, or $R[\![x]\!][y]$ has dimension three. One case of Proposition 3.6 is given in [5, Proposition 3.8].

Proposition 3.6 ([4, Proposition 2.18]). *Assume Setting 3.1 and let Q and $\mathcal{M}$ be prime ideals of A with $x \notin Q$,* $\mathrm{ht}(Q) = 1$, *and* $\mathrm{ht}(\mathcal{M}) = 3$. *Then $Q^{\uparrow} \cap \mathcal{M}^{\downarrow}$ contains exactly $|R[\![x]\!]|$ height-two prime ideals.*

3.1 *j-Spectra of Quotients of Mixed Polynomial-Power Series Rings*

We use Setting and Notation 3.7 in the remainder of this section.

Setting and Notation 3.7. Let R be a one-dimensional Noetherian domain and let x and y be indeterminates. Let A be $R[\![x]\!][y]$, $R[y][\![x]\!]$, or $R[\![x, y]\!]$ and let Q be a height-one prime ideal of A such that $x \notin Q$ and $(Q, x)A \neq A$. Set $B := A/Q$. By Remarks 2.2, A is catenary and has dimension three, and so B is a Noetherian integral domain with $\dim(B) \leq 2$. Let I be a nonzero ideal of $R[y]$ such that $(I, x)A = (Q, x)A$; that is, $I = \{$ all constant terms in $R[y]$ of power series in $Q\}$.

Note 3.8. If $I \neq R[y]$, then the ideal I from Setting and Notation 3.7 is a nonzero height-one ideal of $R[y]$; that is, every prime ideal P of $R[y]$ minimal over I has height one.

Proof. Let P be a prime ideal of $R[y]$ minimal over I. If $I = (0)$, then $(I, x) = (x) \neq (Q, x)$, since $Q \neq (0)$ and $x \notin Q$. Thus $I \neq (0)$, and so $\mathrm{ht}(P) \geq 1$. Now $(Q, x)A \neq A$ by assumption and $1 = \mathrm{ht}(Q) < \mathrm{ht}(Q, x)$ since $x \notin Q$ and A is catenary by Remarks 2.2. Also $\mathrm{ht}(Q, x) \leq 2$ by Krull's principal ideal theorem. Thus $\mathrm{ht}(Q, x) = 2$. Now $(P, x) \neq A$ since $P \in \mathrm{Spec}(R[y])$. Also (P, x) is a minimal prime ideal of $(I, x) = (Q, x)$. Thus $\mathrm{ht}(P, x) = 2$, and so $P \in \mathrm{Spec}(R[y])$ implies $\mathrm{ht}(P) = 1$. $\square$

We show in this subsection that the j-primes of A that contain Q also contain x. It follows that each j-prime of A corresponds to a minimal prime ideal of $R[y]/I$. We begin to demonstrate this correspondence with the following remarks.

Remarks 3.9. With Setting and Notation 3.7, consider the following canonical surjections:

$$\pi : A \longrightarrow B = A/Q \text{ with } \ker(\pi) = Q,$$

$$\pi_x : A \longrightarrow R[y] = A/xA \text{ with } \ker(\pi_x) = (x).$$

(i) The maps π and π_x yield isomorphisms:

$$\operatorname{Spec}(B) \cong \operatorname{Spec}\left(\frac{A}{Q}\right); \quad \text{and} \quad \operatorname{Spec}(R[y]) \cong \operatorname{Spec}\left(\frac{A}{xA}\right);$$
$$\operatorname{Spec}\left(\frac{B}{xB}\right) \cong \operatorname{Spec}\left(\frac{A}{(x,Q)A}\right) = \operatorname{Spec}\left(\frac{A}{(x,I)A}\right) \cong \operatorname{Spec}\left(\frac{R[y]}{I}\right).$$

(ii) Since A is catenary, the correspondences in Remark 3.9(i) above imply that for each $n \leq 2$, the ht-n prime ideals of B can be identified with the ht-$(n+1)$ prime ideals of A containing Q; the ht-n prime ideals of $R[y]$ can be identified with the ht-$(n+1)$ prime ideals of A containing x; and the ht-n primes of B containing x can be identified with the ht-n prime ideals of $R[y]$ containing I.

Proposition 3.10 ([4, Proposition 3.20]). *Assume Setting 3.7.*

(1) $\operatorname{Spec}(B/xB) \cong \operatorname{Spec}(R[y]) \cap (V_{R[y]}(I)) \cong \operatorname{Spec}(R[y]/I)$.
(2) The height-one prime ideals of B that contain x correspond to the height-one prime ideals of $R[y]$ that contain I.
(3) Every nonmaximal j-prime ideal of B contains x and thus corresponds to a j-prime ideal of $R[y]$ containing I.
(4) j-$\operatorname{Spec}(B) \setminus \{(0)\} \setminus \{$*height-one maximal elements*$\} \cong j$-$\operatorname{Spec}(B/xB) \cong j$-$\operatorname{Spec}(R[y]/I)$.
(5) If $\max(R)$ *is infinite, then* j-$\operatorname{Spec}(B/xB) = \operatorname{Spec}(B/xB)$*; that is, every prime ideal of B containing x is a j-prime, and every prime ideal of $R[y]$ containing I is a j-prime.*

Proof. Items (1)–(4) follow from Remarks 3.9; see [4, Proposition 3.22]. For item (5), if $P \in \operatorname{Spec}(B)$ has height one and contains x, then P corresponds to a height-one prime ideal P' of $R[y]$ that contains I. Therefore it suffices to show that every prime ideal P' of $R[y]$ containing I is contained in an infinite number of height-two maximal ideals. If $P' = \mathbf{m}R[y]$, for some $\mathbf{m} \in \max(R)$, then

$$|(P')^{\uparrow(R[y])}| = |(\mathbf{m}R[y])^{\uparrow(R[y])}| = |R[y]/(\mathbf{m}R[y])| = |(R/\mathbf{m})[y]| = |R/\mathbf{m}| \cdot \aleph_0,$$

using Lemma 2.6; thus P'/I is a j-prime of $R[y]/I$. On the other hand, if $P' \cap R = (0)$, then every element of P' has positive degree and $P' = h(y)K[y] \cap R[y]$, where K is the field of fractions of R and $h(y) \in R[y]$, by Remarks 2.7(3). The leading coefficient h_n of $h(y)$ is contained in at most finitely many maximal ideals of R.

Claim 3.11. $P' \subsetneq (\mathbf{m}, P')R[y] \neq R[y]$, for every maximal ideal $\mathbf{m}$ such that $h_n \notin \mathbf{m}$.

Proof of Claim 3.11. Since $\mathbf{m}R[y]$ contains elements of degree 0, $(\mathbf{m}, P')$ is properly bigger than P'. For the inequality, write $h(y) = h_n y^n + h_{n-1}y^{n-1} + \cdots + h_0$, where $n \geq 1$, each $h_i \in R$ and $h_n \neq 0$. If $h_n \notin \mathbf{m}$, then $h_i/h_n \in R_{\mathbf{m}}$, for each i with $0 \leq i \leq n$. Thus

$$\begin{aligned} P'R_{\mathbf{m}}[y] &= h(y)K[y] \cap R_{\mathbf{m}}[y] = \left(y^n + \frac{h_{n-1}}{h_n}y^{n-1} + \cdots + \frac{h_0}{h_n}\right) R_{\mathbf{m}}[y] \\ &\Longrightarrow \frac{P'R_{\mathbf{m}}[y] + \mathbf{m}R_{\mathbf{m}}[y]}{\mathbf{m}R_{\mathbf{m}}[y]} \subsetneq \frac{R_{\mathbf{m}}[y]}{\mathbf{m}R_{\mathbf{m}}[y]}, \end{aligned}$$

and so Claim 3.11 is proved. □

For item (5), since $\max(R)$ is infinite, there are infinitely many $\mathbf{m} \in \max(R)$ such that $h_n \notin \mathbf{m}$. Each pair $(\mathbf{m}, P')$ with $\mathbf{m} \in \max(R)$ is in a distinct maximal ideal of $R[y]$; that is, a maximal ideal containing $(\mathbf{m}, P')$ cannot contain $(\mathbf{m}', P')$ if $\mathbf{m} \neq \mathbf{m}'$ and $\mathbf{m}, \mathbf{m}' \in \max(R)$. Thus $|(P')^{\uparrow(R[y])}| = |\max(R)|$, since removing finitely many $\mathbf{m} \in \max(R)$ such that $h_n \in \mathbf{m}$ from the infinite set $\max(R)$ leaves the same number. This completes the proof of item (5) and thus Proposition 3.10 is proved. □

Remark 3.12. With Setting 3.7, assume that R is semilocal and $I \subseteq P' \in \operatorname{Spec}(R[y])$. If P' is $\mathbf{m}R[y]$, for some $\mathbf{m} \in \max(R)$, or P' is a maximal ideal of $R[y]$, then P' is a j-prime ideal. However not every prime ideal containing I is necessarily a j-prime ideal. See Example 3.13 and Theorem 7.2.

Example 3.13. Let $R = \mathbb{Z}_{(2)}$ and $I = 2y(2y-1)(y+2)$. Then $\operatorname{Spec}(R/I)$ is shown below:

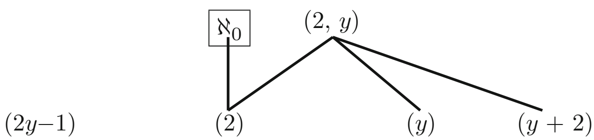


Diagram 3.13.0: $\operatorname{Spec}(\mathbb{Z}_{(2)}[y]/(2y(2y-1)(y+2))$

The structure of $\operatorname{Spec}(\mathbb{Z}_{(2)}[y]/(2y(2y-1)(y+2)))$ is determined by the finite partially ordered subset

$$F = \{(2y-1), (2), (y), (y+2), (2, y)\}.$$

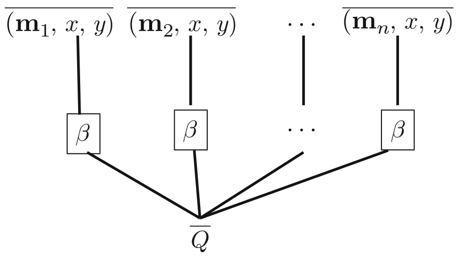
We see that $\text{Spec}(\mathbb{Z}_{(2)}[y]/(2y(2y-1)(y+2)))$ and its j-spec are not the same, since (y) and $(y+2)$ are prime ideals that are not j-prime ideals.

4 Two-Dimensional Prime Spectra of Form $R[\![x, y]\!]/Q$

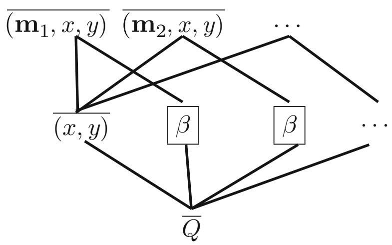
In this short section we discuss the prime spectra of homomorphic images by a height-one prime ideal of the ring of power series in two variables over a one-dimensional Noetherian domain. These prime spectra are similar to those for images of mixed polynomial-power series rings. If both variables x and y are power series variables, however, the analysis is simplified.

Theorem 4.1. *Let R be a one-dimensional Noetherian domain, let x and y be indeterminates, and let Q be a height-one prime ideal of $R[\![x, y]\!]$. Set $B = R[\![x, y]\!]/Q$ and $\beta = |R[\![x]\!]|$. Then:*

(1) If $Q \not\subseteq (x, y)R[\![x, y]\!]$, then there exist $n \in \mathbb{N}$ and $\mathbf{m}_1, \ldots, \mathbf{m}_n \in \max(R)$ such that $\text{Spec}(B)$ *has the form shown below:*



(2) If $Q \subseteq (x, y)R[\![x, y]\!]$, then $\text{Spec}(B)$ *is order-isomorphic to* $\text{Spec}(R[\![x]\!])$*; that is,* $\text{Spec}(B)$ *has the form shown below:*



where the $\mathbf{m}_i$ range over all the elements of $\max(R)$ *and $\beta = |R[\![x]\!]|$.*

As the diagrams show, $\text{Spec}(B)$ *is characterized by the description for each case.*

Proof. For both items we use that every maximal ideal of $R[\![x, y]\!]$ has the form $(\mathbf{m}, x, y)$ where $\mathbf{m} \in \max(R)$, by Proposition 3.2(3). For item (1), let Q_0 be the ideal of R generated by all the constant terms of elements of Q. Then $1 \notin Q_0$, since an element with constant term 1 is a unit of $R[\![x, y]\!]$—every such element is outside every maximal ideal. Also $Q_0 \neq (0)$, since $Q \not\subseteq (x, y)$. Therefore Q_0 is contained in finitely many maximal ideals of R, say $\mathbf{m}_1, \dots, \mathbf{m}_n$. It follows that Q is contained in n maximal ideals of $R[\![x, y]\!]$, namely $(\mathbf{m}_1, x, y), \dots, (\mathbf{m}_n, x, y)$. By Proposition 3.6, we have $|R[\![x]\!]|$ height-two prime ideals between Q and $(\mathbf{m}_i, x, y)$, for each i. Every height-two prime ideal P between Q and a maximal ideal $(\mathbf{m}, x, y)$ is missing either x or y, since $Q \not\subseteq (x, y)R[\![x, y]\!]$. Therefore by Proposition 3.4 each such prime ideal P is contained in a unique maximal ideal, namely $(\mathbf{m}, x, y)$. Thus $\text{Spec}(R[\![x, y]\!]/Q)$ has the form given in the first diagram.

For item (2), since $Q \subseteq (x, y)$, we have $Q \subseteq (\mathbf{m}, x, y)$, for every $\mathbf{m} \in \max(R)$. By Proposition 3.6 again the number of primes between Q and every maximal ideal $(\mathbf{m}, x, y)$ is $|R[\![x]\!]|$, and so we have $|R[\![x]\!]|$ height-two prime ideals between Q and each $(\mathbf{m}_i, x, y)$, for each i. As for item (1), every height-two prime ideal P other than (x, y) that is between Q and a maximal ideal $(\mathbf{m}, x, y)$ is missing either x or y, since $Q \not\subseteq (x, y)R[\![x, y]\!]$. Therefore every such prime ideal P is in a unique maximal ideal of $R[\![x, y]\!]$, and so we get the form of the second diagram in this case. □

5 Spectra of Quotients of Mixed Polynomial-Power Series Rings

Let x and y be indeterminates over a one-dimensional Noetherian domain R, let $A = R[\![x]\!][y]$ or $R[y][\![x]\!]$, and let Q be a height-one prime ideal of A. In this section we describe $\text{Spec}(A/Q)$; this is work in progress from [4]. In some cases, we determine the spectra precisely. We need not consider $A = R[\![x, y]\!]$, since Theorem 4.1 contains a complete description of $\text{Spec}(R[\![x, y]\!]/Q)$, for Q a height-one prime ideal of $R[\![x, y]\!]$.

First we consider the exceptional cases where the dimension of A/Q is 1. We need a definition:

Definition 5.1. A *fan* is a one-dimensional poset with a unique minimal element.

Theorem 5.2 ([4, Theorem 5.2]). *Let R be a one-dimensional Noetherian domain and let x and y be indeterminates over R. Let $A = R[\![x]\!][y]$ or $R[y][\![x]\!]$, let Q be a height-one prime ideal of A, and let $B = A/Q$. Then* $\text{Spec}(B)$ *is a fan if one of the following two cases occur:*

(i) Every height-two prime ideal of A containing $(Q, x)A$ is maximal.
(ii) For every $\mathbf{m} \in \max(R)$, *every height-two prime ideal of A containing $(Q, \mathbf{m})A$ is maximal.*

Moreover, if $A = R[y][\![x]\!]$, then $\text{Spec}(B)$ *is a fan with a finite number of elements, but at least two. If $A = R[\![x]\!][y]$, then* $\text{Spec}(B)$ *is a fan with $|R[\![x]\!]|$ elements.*

Proof. By Proposition 3.5, either of these conditions implies that $\operatorname{Spec}(B)$ is a fan.

For the "Moreover" statement, every maximal ideal of B is the image of a height-two maximal ideal of A that contains Q. In case $A = R[y][\![x]\!]$, every height-two maximal ideal has the form (M, x), where M is a height-one maximal ideal of $R[y]$, by Remarks 2.14. There are just finitely many such height-two maximal ideals that contain $(Q, x)A$. For both of the rings $A = R[y][\![x]\!]$ and $A = R[\![x]\!][y]$, since A has no height-one maximal ideals by Proposition 3.3, there must be a maximal ideal containing Q that is bigger than Q and so the cardinality of the fan is at least two. For $A = R[\![x]\!][y]$, $|\operatorname{Spec}(A/Q)| = |R[\![x]\!]|$; see [4, Theorem 5.2]. □

Except for the special cases of Theorem 5.2, prime spectra of homomorphic images of mixed polynomial-power series rings $R[\![x]\!][y]$ and $R[y][\![x]\!]$ by height-one prime ideals are two dimensional. In order to describe the partially ordered sets that arise, we need a kind of *genetic code*. Definition 5.5 of this section contains such a code and a general set of axioms involving the code that are satisfied by two-dimensional images $B = R[\![x]\!][y]/Q$ and $B' = R[y][\![x]\!]/Q'$, where Q and Q' are height-one prime ideals of $R[\![x]\!][y]$ and $R[y][\![x]\!]$, respectively. Basically the code tells us, for each two-dimensional partially ordered set, how many elements are at each level and what relationships hold between elements.

We use the following setting and notation for the rest of this section.

Setting and Notation 5.3. Let x and y be indeterminates over a one-dimensional Noetherian domain R, let $A = R[\![x]\!][y]$ or $R[y][\![x]\!]$, and set $\beta := \big|R[\![x]\!]\big|$. Let Q be a height-one prime ideal of A such that $x \notin Q$ and the domain $B := A/Q$ has dimension two. Let I be the height-one ideal of $R[y]$ such that $(I, x)A = (Q, x)A$, and let $\{q_1, \dots, q_\ell\}$, for $\ell \in \mathbb{N}$, be the minimal primes of I in $R[y]$. Define:

- $F := \{q_1, \dots, q_\ell\} \cup \{q_i^{\uparrow} \cap q_j^{\uparrow}\}_{1 \le i < j \le \ell}$, a subset of $V_{R[y]}(I)$;
- $\gamma_i := |q_i^{\uparrow} \setminus (\bigcup_{j \ne i} q_j^{\uparrow})|$, for each i with $1 \le i \le \ell$; that is, each γ_i is the number of height-two maximal ideals of $R[y]$ that contain q_i but none of the other q_js; and
- $\varepsilon := |\{\text{ht } 1 \text{ maximal ideals of } B\}|$.

The main theorem of [4] describes $\operatorname{Spec}(B)$ in Setting and Notation 5.3. We remark that, as might be expected from Proposition 3.10, these prime spectra are largely determined by $\operatorname{Spec}(R[y]/I)$.

Main Theorem 5.4 ([4, Theorem 6.5]). *Assume Setting and Notation 5.3. Then there exists an order-monomorphism $\varphi : F \to U$ such that U and φ have the following properties:*

(1) $|U| = \beta$, $\dim(U) = 2$, *and U has a unique minimal element u_0.*
(2) $|\{\mathcal{H}_1(U) \cap \max(U)\}| = \varepsilon$; $\quad \{\varphi(q_1), \dots, \varphi(q_\ell)\} \subseteq \mathcal{H}_1(U)$.
(3) $\mathcal{H}_2(U) = \bigcup \varphi(q_i)^{\uparrow} = (\varphi(F) \setminus \{\varphi(q_1), \dots, \varphi(q_\ell)\}) \cup \bigcup_{i=1}^{\ell} T_i$, *where each* $T_i = \varphi(q_i)^{\uparrow} \setminus (\cup_{j \ne i} \varphi(q_j)^{\uparrow})$ *and* $|T_i| = \gamma_i$.
(4) $\{\varphi(q_1), \dots, \varphi(q_\ell)\}$ *contains the set* $\{u \in U \mid |u^{\uparrow}| = \infty, \ \operatorname{ht}(u) = 1\}$ *of nonmaximal nonzero j-elements of U.*

(5) For every $u \in \mathcal{H}_1(U) \setminus \varphi(F)$*, there exists a unique maximal element in* U *that is greater than or equal to* u*.*

(6) For every $1 \leq i < j \leq \ell$, $\varphi(q_i)^{\uparrow} \cap \varphi(q_j)^{\uparrow} = \varphi(q_i^{\uparrow} \cap q_j^{\uparrow}) \subseteq \varphi(F)$*.*

(7) For every finite nonempty subset $T \subseteq \mathcal{H}_2(U) \setminus F$, $L_e(T) = \emptyset$ *if* $|T| > 1$ *and* $|L_e(T)| = \beta$ *if* $|T| = 1$.[4]

These properties determine U *as a partially ordered set. Moreover* $\varepsilon \leq \beta$*. If* $A = R[y][\![x]\!]$*, then* ε *is finite; if* $A = R[y][\![x]\!]$ *and* $\max(R)$ *is infinite, then* $\varepsilon = 0$*.*

Proof. We give some notes about the proof: The map $\varphi : F \hookrightarrow U$ is given by $\varphi(P) = P/I$, for every $P \in F$, so that $\mathrm{ht}(\varphi(P)) = 1 + \mathrm{ht}(P)$. Then item (4) follows from Note 3.8 and Proposition 3.10, and items (5) and (7) follow from Proposition 3.4. The "Moreover" statement holds since every ideal of A is finitely generated, and thus the total number of prime ideals of A and of B is at most β. The remaining statements follow from Remark 2.14(1) and Lemma 2.8; if $\max(R)$ is finite, every height-one maximal ideal of B corresponds to a height-two maximal ideal of A such that $N = (M, x)$, where M is a height-one maximal ideal of $R[y]$ and $(Q, x) \subseteq N$. There are just finitely many of these. For more details, see [4]. □

Definition 5.5. Let $\ell \in \mathbb{N}_0$ and let $\varepsilon, \beta, \gamma_1, \ldots, \gamma_\ell$ be cardinal numbers with $\varepsilon, \gamma_i \leq \beta$, for each γ_i. We say that a partially ordered set U is *image polynomial-power series of type* $(\varepsilon; \beta; F; \ell; (\gamma_1, \ldots, \gamma_\ell))$, if there exist a finite partially ordered set F of dimension at most one with ℓ minimal elements such that every non-minimal maximal element of F is greater than at least two minimal elements of F and an order-monomorphism φ such that U satisfies properties (1)–(7) of Theorem 5.4.

For examples of these prime spectra, see Sects. 6 and 7.

6 Prime Spectra of Simple Birational Extensions of Power Series Rings

By a *simple birational extension* of an integral domain A with field of fractions K, we mean a ring of form $A[g/f]$ between A and K, where $f, g \in A$ with $f \neq 0$, and either f, g is an A-sequence or $(f, g)A = A$. As noted in Remarks 2.11(c), the prime spectra of simple birational extensions of $\mathbb{Z}[y]$ are order-isomorphic to $\mathrm{Spec}(\mathbb{Z}[y])$; see [14]. In this section, for R a one-dimensional Noetherian domain and x an indeterminate, we present some recent work of Eubanks-Turner, Luckas, and Saydam on prime spectra of simple birational extensions of $R[\![x]\!]$; see [5]. Generally the prime spectrum of a simple birational extension of $R[\![x]\!]$ is rather more complicated than that of $R[\![x]\!]$.

[4] The term "$\mathrm{L_e}(T)$" is defined in Notation 2.1.

Theorem 6.1 summarizes the possible prime spectra of simple birational extensions of a power series ring $R[\![x]\!]$ if R is a one-dimensional Noetherian domain with infinitely many maximal ideals. The original statement of this theorem is given incorrectly in [5]. We do not necessarily know that all the γ_i are the same, as was assumed there.

Theorem 6.1 ([5, Theorem 4.1]). *Let R be a one-dimensional Noetherian domain such that $\alpha = |\max(R)|$ is infinite, let x and y be indeterminates, and let f and g be elements of $R[\![x]\!]$ with $f \neq 0$. Let a and b be the constant terms of f and g respectively. Set $\beta = \left|R[\![x]\!]\right|$, and $v = |V_R(a,b)|$. Let $V_R(a,b) = \{\mathbf{m}_1, \ldots, \mathbf{m}_v\}$ be a numbering of the maximal ideals of R that contain a and b. For each i with $1 \leq i \leq v$, let $\gamma_i := |R/\mathbf{m}_i| \cdot \aleph_0$. Let $B = R[\![x]\!][g/f]$.*

(1) Suppose that $(f,g)R[\![x]\!] = R[\![x]\!]$ and x divides f; equivalently $B = R[\![x]\!][1/f]$, $a = 0$, and b is a unit. Then $\operatorname{Spec}(B)$ is a fan of cardinality β.

(2) Suppose that $g = 0$ or $(f,g)R[\![x]\!] = R[\![x]\!]$ and x does not divide f, so that $B = R[\![x]\!]$ or $B = R[\![x]\!][1/f]$, $a \neq 0$, and $(a,b)R = R$. Then $\operatorname{Spec}(B)$ is either order-isomorphic to $\operatorname{Spec}(R[\![x]\!])$ or to $\operatorname{Spec}(R[\![x]\!])$ with $|R[\![x]\!]|$ height-one maximal elements adjoined.

(3) (a) If f, g is an $R[\![x]\!]$-sequence and x divides f, then $a = 0$ and b is a nonzero nonunit.

(b) If $a = 0$ and b is a nonzero nonunit, then $\operatorname{Spec}(B)$ is D-birational of type $(\beta; \beta; (v,0); \gamma_1, \cdots, \gamma_v)$, as defined in Definition 6.3.

(4) (a) If f, g is an $R[\![x]\!]$-sequence and x does not divide f, then $a \neq 0$ and $(a,b)R \neq R$.

(b) If $a \neq 0$ and $(a,b)R \neq R$, then $\operatorname{Spec}(B)$ is N-birational of type $(\beta; \beta; v+1; \gamma_1, \cdots, \gamma_v, \alpha; t_1, \cdots, t_v)$, for some $t_1, \cdots, t_v$ in $\mathbb{N}_0$; the list $\mathbf{m}_1, \ldots, \mathbf{m}_v$ is to be reordered so that the corresponding list $t_1, \cdots, t_v$ is in increasing order. See Definition 6.2.

Proof. See [5, Theorem 4.1]; it is an easy adjustment to put in the γ_i instead of γ. □

The "type" referred to in Definitions 6.2 and 6.3 below is like a *genetic code* that describes the numbers of prime ideals in various positions of $\operatorname{Spec}(R[\![x]\!][g/f])$ in general. These definitions are related to Definition 5.5; here we give more details and restrictions on the partially ordered set F than in that definition.

Definition 6.2. Let $\ell, t_1, \ldots, t_{\ell-1} \in \mathbb{N}_0$ be such that $t_1 \leq t_2 \leq \cdots \leq t_{\ell-1}$, and let $\beta, \gamma_1, \ldots, \gamma_\ell$ be infinite cardinal numbers with each $\gamma_i \leq \beta$. Let $\varepsilon = 0$ or β; if $\ell = 0$, there are no t_i or γ_i and we require $\varepsilon = \beta \neq 0$. Then a partially ordered set U is *N-birational of type* $(\varepsilon; \beta; \ell; \gamma_1, \ldots, \gamma_\ell; t_1, \ldots, t_{\ell-1})$ if axioms 1–6 hold:

(1) $|U| = \beta$, and U has a unique minimal element u_0.

(2) $|\{\text{height-one maximal elements of } U\}| = \varepsilon$.

(3) If $\ell \neq 0$, then $\dim(U) = 2$. If $\ell = 0$ (and so $\varepsilon = \beta \neq 0$), then $\dim(U) = 1$ and U is a fan; see Definition 5.1.

(4) U has exactly ℓ height-one elements $u \in U$ such that $|u^{\uparrow}| = \infty$. Moreover, if we list these elements as $P_1, P_2, \ldots, P_\ell$, then they satisfy:

- $|P_i^{\uparrow}| = \gamma_i$, for each i with $1 \le i \le \ell$;
- $\bigcup_{i=1}^{\ell} P_i^{\uparrow} = \{$height-two maximal elements of $U\}$; and
- $1 \le i, j < \ell$, and $i \ne j \implies |P_i^{\uparrow} \cap P_\ell^{\uparrow}| = t_i$ and $|P_i^{\uparrow} \cap P_j^{\uparrow}| = 0$.

(5) For every height-one element $u \in U \setminus \{P_1, P_2, \ldots, P_\ell\}$, there exists a unique maximal element in U that is greater than or equal to u.
(6) For every height-two element $t \in U$, $|L_e(t)| = \beta$.

If each $t_i = 0$, then every pair (P_i, P_j) with $i \ne j$ is comaximal by the condition of axiom 4; otherwise P_ℓ is usually distinguishable from the other P_i because there are t_i maximal elements bigger than both P_ℓ and P_i, for each i with $1 \le i < \ell$. Schematically, the condition of axiom 4 yields the following j-sets, where the unique minimal element below all other elements has been removed:

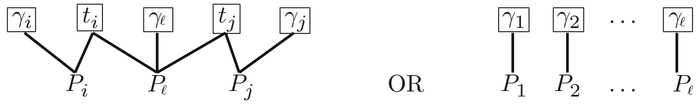
(4) 

Diagram 6.2.0: Parts of N-birational posets; on the right each $t_i = 0$

Abbreviations 6.2.a. If $\gamma_1 = \cdots = \gamma_\ell = \gamma$ (as in all of our examples), then we write the type as $(\varepsilon; \beta; \ell; \gamma; t_1, \ldots, t_{\ell-1})$.

Definition 6.3 for *D-birational* is the case where every $t_i \le 1$ of the N-birational condition. The D-birational posets correspond to prime spectra for simple birational extensions of $D[\![x]\!]$, for some Dedekind domain D; see Theorem 6.5. In this case, we group the nonmaximal j-primes into a comaximal subset and a non-comaximal subset.

Definition 6.3. Let $m, n \in \mathbb{N}_0$ and let $\beta, \gamma_1, \ldots, \gamma_m, \delta_1, \ldots, \delta_n$ be infinite cardinal numbers with each $\gamma_i, \delta_j \le \beta$. Let $\varepsilon = 0$ or β; if $m = n = 0$, require $\varepsilon = \beta \ne 0$. Then a partially ordered set U is *D-birational of type* $(\varepsilon; \beta; (m, n); \gamma_1, \ldots, \gamma_m; \delta_1, \ldots, \delta_n)$ if axioms (1), (2), (5), (6) of Definition 6.2 hold as well as axioms (3′) and (4′) below:

(3′) If $m \ne 0$ or $n \ne 0$, then $\dim(U) = 2$. If $m = n = 0 = \beta$, then $\varepsilon = \beta \ne 0$, $\dim(U) = 1$, and U is a fan.
(4′) U has exactly $m + n$ height-one elements u such that $u^{\uparrow}$ is infinite: $P_1, P_2, \ldots, P_m, Q_1, \cdots, Q_n$, where for $i, j, r, i', j' \in \mathbb{N}$ with $1 \le i, j \le m, i \ne j$, $1 \le r \le n$, $1 \le i', j' < n$, and $i' \ne j'$, we have:

- $|P_i^{\uparrow}| = \gamma_i$, $|Q_r^{\uparrow}| = \delta_r$;

- $|P_i^{\uparrow} \cap P_j^{\uparrow}| = 0 = |P_i^{\uparrow} \cap Q_r^{\uparrow}| = |Q_{i'}^{\uparrow} \cap Q_{j'}^{\uparrow}|$ and $|Q_{i'}^{\uparrow} \cap Q_n^{\uparrow}| = 1$;
- $\bigcup_{i=1}^{m} P_i^{\uparrow} \cup \bigcup_{r=1}^{n} Q_r^{\uparrow} = \{$height-two maximal elements of $U\}$.

Abbreviations 6.3.a. If $\gamma_i = \gamma_j = \delta_s = \delta_t = \gamma$, for every i, j, s, t with $1 \le i < j \le m$, $1 \le s < t \le n$, we write the type as $(\varepsilon; \beta; (m, n); \gamma)$.

Diagram 6.3.0 shows the j-set of a D-birational poset of type $(\varepsilon; \beta; (m, n); \gamma_1, \ldots, \gamma_m; \delta_1, \ldots, \delta_n)$, where $n > 1$. (The complete poset U would have clumps of size β beneath each height-two maximal element by axiom 6.)

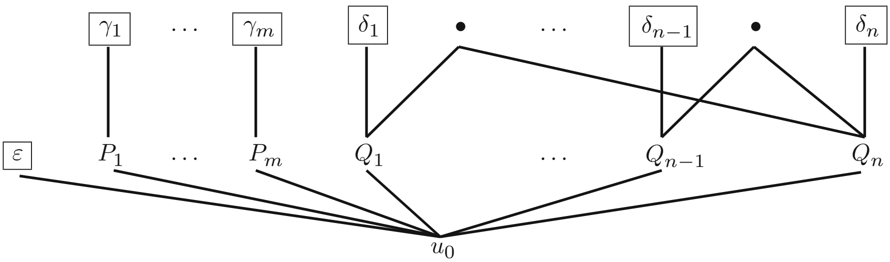


Diagram 6.3.0: j-set of a D-birational poset of type $(\varepsilon; \beta; (m, n); \gamma_1, \ldots, \gamma_m; \delta_1, \ldots, \delta_n)$, where $n > 1$

Remark 6.4. Ambiguity: If U is an N-birational poset of type $(\varepsilon; \beta; \ell; \alpha; 0, \cdots, 0)$ where $\ell = 1$, or where $\ell > 1$ and each $t_i = 0$, then U is D-birational, but there is some ambiguity about the type. We could take either $(m, n) = (\ell, 0)$ or $(m, n) = (\ell - 1, 1)$. The picture for the spectra is the same in either case. A D-birational partially ordered set of type $(\varepsilon; \beta; (1, 0); \alpha)$ is order-isomorphic to one of type $(\varepsilon; \beta; (0, 1); \alpha)$ and a D-birational partially ordered set of type $(\varepsilon; \beta; (m, 0); \alpha)$, for $m > 1$, is order-isomorphic to one of type $(\varepsilon; \beta; (m - 1, 1); \alpha)$, but *not* to one of type $(\varepsilon; \beta; (m - 2, 2); \alpha)$. We keep this ambiguity because the different types arise in different circumstances when the notation is applied to $\mathrm{Spec}(R[\![x]\!][g/f])$.

When R is a countable Dedekind domain, the cardinalities in Theorem 6.1 can be given more explicitly, yielding a true characterization. For R countable, all the γ_i and δ_j are equal, by Lemma 2.6, and so we use the abbreviated form of the code in Abbreviations 6.3.a. Recall that $(aR :_R b) = \{c \in R \mid bc \in aR\}$, if $a, b \in R$.

Theorem 6.5 ([5, Theorem 4.3]). *Let R be a countable Dedekind domain with quotient field K such that* $\max(R)$ *is infinite, let x be an indeterminate, and let B be a simple birational extension of $R[\![x]\!]$, as described below for $f, g \in R[\![x]\!]$ an $R[\![x]\!]$-sequence such that f, g have constant terms a, b respectively. Set $v = |V_R(a, b)|$ and $w = |\{q \in V_R(a, b) \mid (aR : b) \not\subseteq q\}|$. Then:*

(1) If $a = 0$ and $B := R[\![x]\!][1/f]$, then $\mathrm{Spec}(B)$ *is a fan.*

(2) If $a \neq 0$ *and* $B := R[\![x]\!][1/f]$*, then* $\operatorname{Spec}(B)$ *is order-isomorphic to* $\operatorname{Spec}(\mathbb{Q}[y][\![x]\!])$ *or* $\operatorname{Spec}(\mathbb{Q}[\![x]\!][y])$.
(3) If $B = R[\![x]\!][g/f]$*,* $a = 0$*,* $b \neq 0$*, and* $bR \neq R$*, then* $\operatorname{Spec}(B)$ *is* D*-birational of type* $(|\mathbb{R}|; |\mathbb{R}|; (v, 0); \aleph_0)$.
(4) If $B = R[\![x]\!][g/f]$*,* $a \neq 0$*, and* $(a, b)R \neq R$*, then* $\operatorname{Spec}(B)$ *is* D*-birational of type*

$$(|\mathbb{R}|; |\mathbb{R}|; (v - w, w + 1); \aleph_0).$$

In order to show that Theorem 6.5 is a characterization, we show every D-birational poset occurs, for some Dedekind domain D. In fact this is true with $D = \mathbb{Z}$, as we see in Theorem 6.6.

Theorem 6.6 ([5, Theorem 4.8]). *Let* R *be a PID with* α *maximal ideals, where* α *is infinite, let* x *and* y *be indeterminates, set* $\beta = |R[\![x]\!]|$*, and suppose that* $\alpha = |R| \cdot \aleph_0 = |R/\mathbf{m}| \cdot \aleph_0$ *is constant, for each* $\mathbf{m} \in \max(R)$*. Then, for each* $m, n \in \mathbb{N}_0$*, there exists a simple birational extension of* $R[\![x]\!]$ *that is* D*-birational of type* $(\beta; \beta; (m, n); \alpha)$*. In particular, if* R *is a PID with* $|R| = |\max(R)| = \aleph_0$ *and* $m, n \in \mathbb{N}_0$*, then, for every* D*-birational partially ordered set* U *of type* $(|\mathbb{R}|; |\mathbb{R}|; (m, n); \aleph_0)$ *from Definition 6.3, there is a simple birational extension* $B := R[\![x]\!][g/f]$ *of* $R[\![x]\!]$ *so that the prime spectrum of* B *is order-isomorphic to* U.

We give an example adjusted from [5] to illustrate Theorems 6.5 and 6.6.

Example 6.7. Let B be the simple birational extension $B := \mathbb{Z}[\![x]\!][g/f]$ of $\mathbb{Z}[\![x]\!]$, where $f = x + 2079$ and $g = x + 4851$. Then, in the notation of Theorem 6.5, $a = 2079 = 3^3 \cdot 7 \cdot 11, b = 4851 = 3^2 \cdot 7^2 \cdot 11$, $(a\mathbb{Z} : b) = 3\mathbb{Z}$, and so $v = |\{3, 7, 11\}| = 3$, $w = 2$, and $(f, g)\mathbb{Z}[\![x]\!] \neq \mathbb{Z}[\![x]\!]$, since $(f, g) \subseteq (x, 3)$. Since $7f - 3g = 4x \in (f, g)$, we have $(f, x) \subseteq P$ or $(f, 2) \subseteq P$, for every prime ideal P minimal over (f, g). Therefore the ideal (f, g) has height two, and so, by [15, Theorem 17.4], f, g is a $\mathbb{Z}[\![x]\!]$-sequence. Also, in $\mathbb{Z}[y]$, the ideal

$$(fy - g)\mathbb{Z}[y] = (2079y - 4851)\mathbb{Z}[y] = (3^2 \cdot 7 \cdot 11(3y - 7))\mathbb{Z}[y].$$

The height-one prime ideals in $\mathbb{Z}[y]$ containing this ideal are (3), (7), (11), and $(3y - 7)$. By Theorem 6.5, $\operatorname{Spec}(B)$ is D-birational of type $(|\mathbb{R}|; |\mathbb{R}|; (1, 3); \aleph_0)$, since the cardinality of $\mathbb{Z}[\![x]\!]$ is $|\mathbb{R}|$ and the cardinality of $\max(\mathbb{Z})$ is $|\mathbb{Z}| \cdot \aleph_0 = \aleph_0$. Thus Diagram 6.7.1 shows the partially ordered set $\operatorname{Spec}(B)$, except that we cannot show the clumps of size $|\mathbb{R}|$ beneath every height-two maximal ideal. Here π is the canonical map from $\mathbb{Z}[\![x]\!][y] \to B := \mathbb{Z}[\![x]\!][g/f]$, and $\pi(x, 3)$ denotes the image in B under π of the ideal $(x, 3)\mathbb{Z}[\![x]\!]$. There is one j-prime ideal, namely $\pi(x, 3)$, that is unrelated to the others; the other three are connected by height-two maximal ideals that contain the last j-prime ideal, $\pi(x, 3y - 7)$.

Diagram 6.7.2 is a close-up picture showing relations for elements of the set labeled $C_{\aleph_0}$, to show, for every $M \in C_{\aleph_0}$, that $|\mathrm{L_e}(M)| = |\mathbb{R}|$.

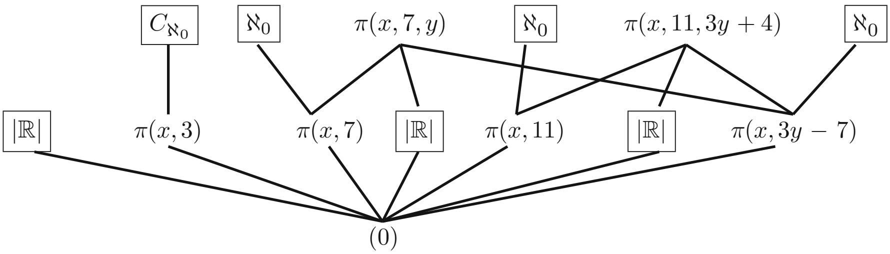


Diagram 6.7.1: Spec(B) for Example 6.7

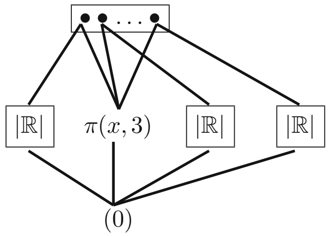


Diagram 6.7.2: Relations in $C_{\aleph_0}$ from Diagram 6.7.1

Remark 6.8. If R is a countable one-dimensional Noetherian domain R such that Spec$(R[y]/(ay-b))$ is know then one can also find Spec(B); see [5]. When R is not Dedekind, however, the relations among the minimal elements of Spec$(R[y]/(ay-b))$ may be more complex than they are for Dedekind domains, and we do not know what posets are realizable as Spec$(R[y]/(ay-b))$. It is not clear that every form of the axioms for that situation can be realized.

The following example from [5] gives a simple birational extension B of $\mathbb{Z}[5i][\![x]\!]$ that has two distinct maximal ideals containing two distinct nonmaximal height-one j-primes. Thus Spec(B) is not D-birational.

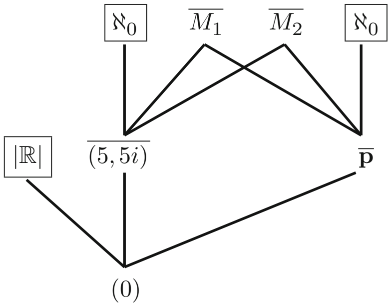
Example 6.9. For $R = \mathbb{Z}[5i]$, a non-Dedekind ring, let $B = \mathbb{Z}[5i][\![x]\!][g/f]$, $f = x+5$, $g = 5i$. Then the two nonmaximal height-one j-primes of B correspond in $\mathbb{Z}[5i][y]$ to $(5,5i)\mathbb{Z}[5i][y]$ and to

$$\mathbf{p} := (y-i)\mathbb{Z}[i][y] \cap \mathbb{Z}[5i][y] = (y^2-1, 5y-5i, 5iy+5)\mathbb{Z}[5i][y].$$

Therefore $(5,5i)+\mathbf{p} = (y^2-1,5,5i) = (y^2-4,5,5i) \subseteq (y-2,5,5i) \cap (y+2,5,5i)$. If we let $M_1 = (y-2,5,5i)\mathbb{Z}[5i][y]$, $M_2 = (y+2,5,5i)\mathbb{Z}[5i][y]$, and $\overline{}$ denotes the image in B of the map Spec$(\mathbb{Z}[5i][y]/(5y-5i)) \to V_B(x)$ from Remarks 3.9.1, we have j-Spec(B) in Diagram 6.9.1:

To make Diagram 6.9.1 show all of Spec(B), we would add clumps of size $|\mathbb{R}|$ beneath every height-two prime ideal but beneath no other height-two prime ideal. This partially ordered set is N-birational of type $(|\mathbb{R}|; |\mathbb{R}|; 2; \aleph_0; 2)$, since the number of height-one maximal ideals is $|\mathbb{R}|$ and $|\mathrm{L_e}(P)| = |\mathbb{R}|$ for every height-two

Diagram 6.9.1: j-Spec(B), for $B := \mathbb{Z}[5i][\![x]\!][g/f]$, $f = x + 5, g = 5i$



$P \in \text{Spec}(B)$; also ℓ, the number of nonmaximal j-prime ideals, is 2. Since the number t_1 of maximal ideals containing both of them is 2, we have that j-Spec(B) is not D-birational.

The following question raised in [5] is still unknown:

Question 6.10. Does every N-birational poset of type $(|\mathbb{R}|; |\mathbb{R}|; \ell; \aleph_0; t_1, \cdots, t_{\ell-1})$, for all values of ℓ, and $t_1 \leq \cdots \leq t_\ell$, occur as Spec$(R[\![x]\!][g/f])$, where g, f are as described in Theorem 6.1?

7 Examples of Two-Dimensional Polynomial-Power Series Prime Spectra

To illustrate Theorem 5.4, we give an example with $R = \mathbb{Z}$; this partially ordered set is the prime spectrum of $\mathbb{Z}[y][\![x]\!]/Q$, for an appropriately chosen height-one prime ideal Q of $\mathbb{Z}[y][\![x]\!]$:

Example 7.1. For $\alpha = (2y-1) \cdot 3 \cdot (y+1) \cdot y \cdot (y(y+1)+6) \cdot 2 \cdot (3y+1)$, we describe Spec$(\mathbb{Z}[y][\![x]\!]/(x-\alpha))$, by displaying j-Spec$(\mathbb{Z}[y][\![x]\!]/(x-\alpha))$ in this diagram:

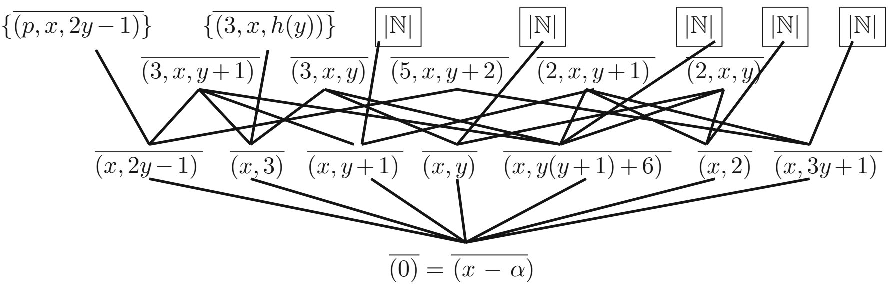


Diagram 7.1.1: j-Spec$(\mathbb{Z}[y][\![x]\!]/(x-\alpha))$

The partially ordered set in Diagram 7.1.1 is image polynomial-power series of type $(0; |\mathbb{R}|; 7; \aleph_0; F)$ from Definition 5.5, where F is the part of the poset shown that includes only the ideals at the second and third levels and their connections; that is, F corresponds to the twelve ideals $\overline{(x, 2y-1)}, \overline{(x, 3)}, \ldots$ and $\overline{(3, x, y+1)}, \overline{(3, x, y)}, \ldots$ and with the relations (lines) connecting them.

We close with a description of the two-dimensional partially ordered sets that arise as $\operatorname{Spec}(A/Q)$, where $A = R[\![x]\!][y]$ or $R[y][\![x]\!]$ and R is a one-dimensional Henselian integral domain with unique maximal ideal $\mathbf{m}$. As with Example 7.1, $\operatorname{Spec}(A/Q)$ is largely determined by $\operatorname{Spec}(R[y]/I)$, where I is a height-one prime ideal of $R[y]$ such that $(I, x)A = (Q, x)A$.

Theorem 7.2 ([4, Theorem 7.2]). *Let $(R, \mathbf{m})$ be a Henselian integral domain. Let x and y be indeterminates; let $A = R[\![x]\!][y]$ or $R[y][\![x]\!]$. Let Q be a height-one prime ideal of A and let $B = A/Q$. Then* $\operatorname{Spec}(B) \setminus \{$*the set of height-one maximal ideals*$\}$ *is determined by* $\operatorname{Spec}(R[y]/I)$*, where I is a height-one prime ideal of $R[y]$ such that $(I, x)A = (Q, x)A$. If I is contained in $\mathbf{m}R[y]$, then* $\operatorname{Spec}(B)$ *is given in Diagram 7.2.0.*

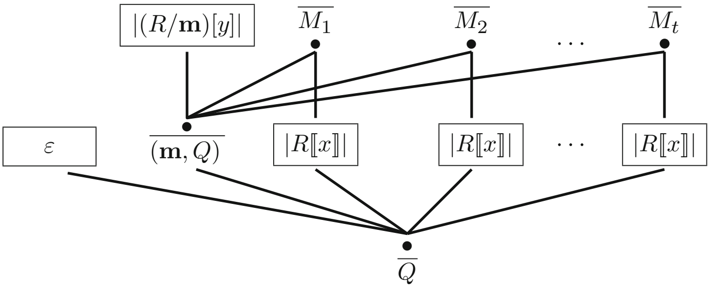


Diagram 7.2.0: $\operatorname{Spec}(B)$ if $I \subseteq \mathbf{m}R[y]$

Notes

- *$M_1, M_2, \cdots, M_t$ are the height-two ideals of $R[y]$ that contain $\mathbf{m}$ and another height-one prime ideal of $R[y]$ that contains I.*
- *$\overline{M_1}, \cdots, \overline{M_t}$ denote the image of $(M_i, x)A$ in A/Q.*
- *If $A = R[y][[x]]$, ε is 0; for $A = R[[x]][y]$, ε is sometimes finite and sometimes $|R[\![x]\!]|$.*
- *The sets $\mathrm{L_e}(M)$, for elements M of the block of size $|(R/\mathbf{m})[y]|$, are not shown.*
- *The partially ordered set in Diagram 7.2.0 is image polynomial-power series of type $(\varepsilon; |R[\![x]\!]|; F; 1; |R/\mathbf{m}| \cdot \aleph_0)$, where F corresponds to $\{\overline{(\mathbf{m}, Q)}\}$ in* $\operatorname{Spec}(B)$.

References

1. J.T. Arnold, Prime ideals in power series rings. In *Conference on Commutative Algebra (University of Kansas, Lawrence, Kansas 1972)*. Lecture Notes in Mathematics, vol. 311 (Springer, Berlin, 1973), pp. 17–25
2. H. Bass, *Algebraic K-Theory*. Mathematics Lecture Note Series (W.A. Benjamin Inc, New York/Amsterdam, 1968), pp. 762
3. J.W. Brewer, *Power Series Over Commutative Rings*. Lecture Notes in Pure and Applied Mathematics, vol. 64 (Marcel Dekker Inc, New York, 1981)
4. E. Celikbas, C. Eubanks-Turner, S. Wiegand, *Prime Ideals in Quotients of Mixed Polynomial-Power Series Rings*. See http://www.math.unl.edu/~swiegand1 (preprint)
5. C. Eubanks-Turner, M. Luckas, S. Saydam, Prime ideal in birational extensions of two-dimensional power series rings. Comm. Algebra **41**(2), 703–735 (2013)
6. W. Heinzer, S. Wiegand, Prime ideals in two-dimensional polynomial rings. Proc. Am. Math. Soc. **107**(3), 577–586 (1989)
7. W. Heinzer, C. Rotthaus, S. Wiegand, Mixed polynomial-power series rings and relations among their spectra. In *Multiplicative Ideal Theory in Commutative Algebra* (Springer, New York, 2006), pp. 227–242
8. R.C. Heitmann, Prime ideal posets in Noetherian rings. Rocky Mountain J. Math. **7**(4), 667–673 (1977)
9. M. Hochster, Prime ideal structure in commutative rings. Trans. Am. Math. Soc. **142**, 43–60 (1969)
10. I. Kaplansky, *Commutative Rings* (Allyn and Bacon Inc, Boston, 1970)
11. K. Kearnes, G. Oman, Cardinalities of residue fields of Noetherian integral domains. Comm. Algebra, **38**, 3580–3588 (2010)
12. W. Lewis, The spectrum of a ring as a partially ordered set. J. Algebra **25**, 419–434 (1973)
13. W. Lewis, J. Ohm, The ordering of Spec R. Can. J. Math. **28**, 820–835 (1976)
14. A. Li, S. Wiegand, Prime ideals in two-dimensional domains over the integers. J. Pure Appl. Algebra **130**, 313–324 (1998)
15. H. Matsumura, *Commutative Ring Theory*, 2nd edn. Cambridge Studies in Advanced Mathematics, vol. 8 (Cambridge University Press, Cambridge, 1989). Translated from the Japanese by M. Reid
16. S. McAdam, Saturated chains in Noetherian rings. Indiana Univ. Math. J. **23**, 719–728 (1973/1974)
17. S. McAdam, Intersections of height 2 primes. J. Algebra **49**(2), 315–321 (1977)
18. M.P. Murthy, A note on factorial rings. Archiv der Mathematik **15**(1), 418–420 (1964)
19. M. Nagata, On the chain problem of prime ideals. Nagoya Math. J. **10**, 51–64 (1956)
20. M. Nagata, *Local Rings*. Interscience Tracts in Pure and Applied Mathematics, vol. 13 (Interscience Publishers a division of John Wiley and Sons, New York-London, 1962)
21. L.J. Ratliff, *Chain Conjectures in Ring Theory: An Exposition of Conjectures on Catenary Chains*. Lecture Notes in Mathematics, vol. 647 (Springer, Berlin, 1978)
22. A.S. Saydam, S. Wiegand, Noetherian domains with the same prime ideal structure as $\mathbb{Z}_{(2)}[x]$. Arab. J. Sci. Eng. Sect. C Theme Issues (Comm. Algebra) **26**(1), 187–198, (2001)
23. C. Shah, Affine and projective lines over one-dimensional semilocal domains. Proc. Am. Math. Soc. **124**(3), 697–705 (1996)
24. S. Wiegand, Locally maximal Bezout domains. Am. Math. Soc. **47**, 10–14 (1975)
25. S. Wiegand, Intersections of prime ideals in Noetherian rings. Comm. Algebra **11**, 1853–1876 (1983)
26. R. Wiegand, The prime spectrum of a two-dimensional affine domain. J. Pure Appl. Algebra **40**(2), 209–214 (1986)
27. S. Wiegand, R. Wiegand, The maximal ideal space of a Noetherian ring. J. Pure Appl. Algebra **08**, 129–141 (1976)

28. R. Wiegand, S. Wiegand, Prime ideals and decompositions of modules. In *Non-Noetherian Commutative Ring Theory*, ed. by S. Chapman, S. Glaz. Mathematics and its Applications, vol. 520 (Kluwer Academic Publishers, Dordrecht, 2000), pp. 403–428
29. R. Wiegand, S. Wiegand, Prime ideals in Noetherian rings: a survey. In *Ring and Module Theory*, ed. by T. Albu, G.F. Birkenmeier, A. Erdo **u** gan, A. Tercan (Birkhäuser, Boston, 2010), pp. 175–193