A Stochastic Model of Oligopolistic Market Equilibrium Problems

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1 Introduction

We provide a stochastic formulation of the classical deterministic oligopolistic market equilibrium, *à la Cournot* [2] in this short note. Equilibria of this kind are particular cases of Nash equilibria, and it is well known (see, e.g., [1] for the general Hilbert space case, and [4] for a finite-dimensional framework close to operations research problems) that under standard hypotheses solutions can be obtained by solving a variational inequality. Thus, we can apply the theory of random (or stochastic) variational inequalities in Lebesgue spaces to our model. This approach has been proposed quite recently to study many stochastic equilibrium problems arising from applied sciences and operations research [5–8, 10]. Other approaches to stochastic variational inequalities have been proposed by other authors. Here we cite only the very recent paper [13] which also contains applications to Nash equilibrium problems.

The paper is structured in four sections. In the remainder of this introduction we briefly recall the connection between Nash equilibrium problems and variational inequalities in the deterministic, finite-dimensional setting; in Sect. 2 we introduce random data in the deterministic oligopolistic market model; in Sect. 3 we present the Lebesgue-space formulation of the stochastic model; in Sect. 4 we study a particular class of utility functions, and use them to illustrate our model by means of a numerical example.

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Consider m players each acting in a selfish manner in order to maximize their individual welfare. Each player i has a strategy vector $q_i = (q_{i1}, \ldots, q_{in}) \in X_i$, where $X_i \subset \mathbb{R}^n$ is a convex and closed set, and a utility (or welfare) function $w_i : X_1 \times X_2 \times \cdots \times X_m \to \mathbb{R}$. He/she chooses his/her strategy vector q_i so as to maximize w_i, given the moves $(q_j)_{j \neq i}$ of the other players. We will use the notation $q_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m)$ and $q = (q_i, q_{-i})$.

Definition 1. A Nash equilibrium is a vector $q^* = (q_1^*, \ldots, q_m^*) \in X$, such that:

$$w_i(q_i^*, q_{-i}^*) \geq w_i(q_i, q_{-i}^*),\ \forall q_i \in X_i, \forall i \in \{1, \ldots, m\}.$$

The following theorem (see e.g. [12, Chap. 6]) relates Nash equilibrium problems and variational inequalities.

Theorem 1. *Let $w_i \in C^1(X), \forall i$, and concave with respect to q_i. Let $F : \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ be the mapping built with the partial gradients of the utility functions as follows:*

$$F(q) = (-D_{q_1} w_1(q), \ldots, -D_{q_m} w_m(q)).$$

Then, $q^* \in X$ is a Nash equilibrium if and only if it satisfies the variational inequality:

$$\sum_{r=1}^{mn} F_r(q^*) \cdot (q_r - q_r^*) \geq 0, \forall q \in X$$

2 The Stochastic Oligopoly Model

We consider here the model in which m players are the producers of the same commodity. The quantity produced by firm i is denoted by q_i so that $q \in \mathbb{R}^m$ denotes the global production vector. Let (Ω, P) be a probability space and for every $i \in \{1, \ldots, m\}$ consider functions $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ and $p : \Omega \times \mathbb{R}^m \to \mathbb{R}$.

More precisely, for almost every $\omega \in \Omega$, (i.e. P-almost surely in probabilistic language), $f_i(\omega, q_i)$ represents the cost of producing the commodity by firm i, and is assumed to be nonnegative, increasing, concave, and C^1, while $p(\omega, q_1 + \cdots + q_m)$ represents the demand price associated with the commodity. For almost every $\omega \in \Omega$, p is assumed nonnegative, increasing, convex w.r.t. q_i, and C^1. We also assume that all these functions are random variables w.r.t. ω, i.e. they are measurable with respect to the probability measure P on Ω. In this way, we have introduced the possibility that both the production cost and the demand price are affected by a certain degree of uncertainty or randomness.

Thus, the welfare (or utility) function of player i is given by:

$$w_i(\omega, q_1, \ldots, q_m) = p(\omega, q_1 + \cdots + q_m) q_i - f_i(\omega, q_i). \tag{1}$$

Although many authors assume no bounds on the production, in a more realistic model the production capability is bounded from above and we allow also for the upper bound being a random variable: $0 \le q_i \le \overline{q}_i(\omega)$.

Thus, the specific Nash equilibrium problem associated with this model takes the following form. For a.e. $\omega \in \Omega$, find $q^*(\omega) = (q_1^*(\omega), \dots, q_m^*(\omega))$:

$$w_i(q^*(\omega)) = \max_{0 \le q_i \le \overline{q}_i(\omega)} \left\{ p(\omega, q_i + \sum_{j \ne i} q_j^*(\omega)) q_i - f_i(\omega, q_i) \right\}, \forall i. \quad (2)$$

In order to write the equivalent variational inequality, consider the closed and convex subset of $\mathbb{R}^m$:

$$K(\omega) = \{(q_1, \dots, q_m) : 0 \le q_i \le \overline{q}_i(\omega), \forall i\}$$

for each ω and define the functions

$$F_i(\omega, q) := \frac{\partial f_i(\omega, q_i)}{\partial q_i} - \frac{\partial p(\omega, \sum_{j=1}^m q_j)}{\partial q_i} q_i - p\left(\omega, \sum_{j=1}^m q_j\right). \quad (3)$$

The Nash problem is then equivalent to the following variational inequality: for a.e. $\omega \in \Omega$, find $q^*(\omega) \in K(\omega)$ such that

$$\sum_{j=1}^m F_j[\omega, q^*(\omega)](q_j - q_j^*(\omega)) \ge 0, \ \forall q \in K(\omega). \quad (4)$$

Since $F(\omega, \cdot)$ is continuous, and $K(\omega)$ is convex and compact, problem (4) is solvable for almost every $\omega \in \Omega$, due to the Stampacchia's theorem. Moreover, we assume that $F(\omega, \cdot)$ is monotone, i.e.:

$$\sum_{i=1}^m (F_i(\omega, q) - F_i(\omega, q'))(q_i - q_i') \ge 0 \ \ \forall \omega \in \Omega, \forall q, q' \in \mathbb{R}^m.$$

F is said to be strictly monotone if the equality holds only for $q = q'$ and in this case (4) has a unique solution. In the sequel the following uniform strong monotonicity property will be useful:

$$\exists \alpha > 0 : \sum_{i=1}^m (F_i(\omega, q) - F_i(\omega, q'))(q_i - q_i') \ge \alpha \|q - q'\|^2 \ \ \forall \omega \in \Omega, \forall q, q' \in \mathbb{R}^m. \quad (5)$$

Although the uniform strong monotonicity property is quite demanding, nonetheless it is verified by some classes of utility functions frequently used in the literature (see, e.g., Sect. 4).

3 The Lebesgue Space Formulation

Now we are interested in computing statistical quantities associated with the solution $q^*(\omega)$, in particular its mean value. For this purpose we introduce a Lebesgue space formulation of problems (2) and (4). Moreover, in view of the numerical approximation of the solution, from now on, we assume that the random and the deterministic part of the operator can be separated. Thus, let:

$$w_i(\omega,q) = p\left(\sum_{j=1}^{m} q_j\right) + \beta(\omega) - \alpha(\omega) f_i(q_i) - g_i(q_i)$$

where α, β are real random variables, with $0 < \underline{\alpha} \leq \alpha(\omega) \leq \overline{\alpha}$, and the part of the cost which is affected by uncertainty is denoted now by f_i (with an abuse of notation). As a consequence, the operator F takes the form:

$$F_i(\omega,q) = \alpha(\omega)\frac{\partial f_i(q_i)}{\partial q_i} + \frac{\partial g_i(q_i)}{\partial q_i} - p\left(\sum_{j=1}^{m} q_j\right) - \beta(\omega) - \frac{\partial p\left(\sum_{j=1}^{m} q_j\right)}{\partial q_i} q_i.$$

The separation of variables allows us to use the approximation procedure developed in [6]. Furthermore, we assume that F is uniformly strongly monotone according to (5) and satisfies the following growth condition:

$$|F_i(\omega,q)| \leq c(1+|q|), \forall q \in \mathbb{R}^m, \forall \omega \in \Omega, \forall i \tag{6}$$

and $w_i(\omega,0) \in L^1(\Omega)$. Moreover, we shall assume that $\alpha \in L^\infty(\Omega)$, while $\beta, \overline{q}_i \in L^2(\Omega)$. Under these assumptions the following Nash equilibrium problem can be derived (see [9] or [3] for a similar derivation which can be easily extended to our functional setting):

Find $u^* \in L^2(\Omega, P, \mathbb{R}^m)$ such that, $\forall i$

$$\int_\Omega w_i(\omega, u^*(\omega)) dP_\omega = \max_{0 \leq u_i \leq \overline{q}_i} \int_\Omega w_i(\omega, (u_i(\omega), u^*_{-i}(\omega)) dP_\omega, \tag{7}$$

where we used the notation: $(u_i, u^*_{-i}) := (u^*_1, \ldots, u^*_{i-1}, u_i, u^*_{i+1}, \ldots, u^*_m)$. Then, we define a closed and convex set K_P by

$$K_P = \{u \in L^2(\Omega, P, \mathbb{R}^m) : 0 \leq u_i(\omega) \leq \overline{q}_i(\omega), P\text{-a.s.}, \forall i\}$$

and consider the variational inequality formulation of (7): Find $u^* \in K_P$ such that

$$\int_\Omega \sum_{j=1}^{m} F_j(\omega, u^*(\omega))(u_j(\omega) - u^*(\omega)) \geq 0, \forall u \in K_P. \tag{8}$$

The relation between problems (7) and (8) is clarified by the following theorem.

Theorem 2. *u^* is a solution of (7) if and only if it is a solution of (8).*

Proof. The proof can be obtained along the same lines as in [3], with minor modifications. □

Since the stochastic oligopolistic market problem will be studied through (8), we ensure its solvability by the following:

Theorem 3. *Let $f_i(\cdot, q_i), p(\cdot, \sum_{j=1}^m q_j)$ be measurable, and $f_i(\omega, \cdot), d_i(\omega, \cdot)$ are of class C^1. Let F be uniformly strongly monotone and satisfy the growth condition (6). Then (8) admits a unique solution.*

Proof. Under our assumption $F : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function and it is well known that for each measurable function $u(\omega)$, the function $F(\omega, u(\omega))$ is also measurable. Under the growth condition (6) the superposition operator $N_F : u(\omega) \to F(\omega, u(\omega))$ maps $L^2(\Omega, P, \mathbb{R}^m)$ in $L^2(\Omega, P, \mathbb{R}^m)$ and is continuous, being P a probability measure. Moreover the uniform strong monotonicity of F implies the strong monotonicity of N_F. The set K_P is convex, closed, and (norm) bounded, hence weakly compact. Then, monotone operator theory applies (see, e.g., [11] for a recent survey on existence theorems) and (8) admits a unique solution. □

Remark 1. The Lebesgue formulation is the natural one for our stochastic problem, in that the solution of (8) is a function which, by definition, admits finite mean value and variance. If the unique solution of (4) is square integrable, then it also satisfies (8) (see also Proposition 1 in [8]).

Let us note that we worked with the abstract probability space (Ω, P) up to this point, and this was sufficient in providing the general formulation of our problem in Lebesgue spaces in a concise manner. However, in concrete applications the sample space Ω is not known. On the other hand, one can measure the distributions of the real valued random variables that are involved in the model. Hence, it is natural to work with the probability distributions induced on the images of the functions: $A = \alpha(\omega), B = \beta(\omega), Q_i = \overline{q}_i(\omega)$. Thus, let $y = (A, B, Q)$ and consider the probability space $(\mathbb{R}^d, \mathbb{P})$ with $d = 2 + m$. In order to formulate the problem (8) in the image space we introduce the closed convex set $K_{\mathbb{P}}$ by:

$$K_{\mathbb{P}} = \{u \in L^2(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^m) : 0 \le u_i(A, B, Q) \le Q_i, \forall i, \mathbb{P}\text{-a.s.}\}$$

and consider the following problem: Find $u^* \in K_{\mathbb{P}}$ such that $\forall u \in K_{\mathbb{P}}$

$$\int_{\mathbb{R}^d} \sum_{i=1}^m \left[A \frac{\partial f_i(u_i^*(y))}{\partial q_i} + \frac{\partial g_i(u_i^*(y))}{\partial q_i} - p\left(\sum_{j=1}^m u_j^*(y)\right) - B \right.$$
$$\left. - \frac{\partial p\left(\sum_{j=1}^m u_j^*(y)\right)}{\partial q_i} u_i^* \right] (u_i(y) - u_i^*(y)) dP(y) \ge 0. \quad (9)$$

We assume that all the random variables are independent. Moreover, as it is verified in most applications, we assume that each probability distribution is characterized by its density φ. Thus, we have $\mathbb{P} = \mathbb{P}_A \otimes \mathbb{P}_B \otimes \mathbb{P}_Q$, $dP_\alpha(A) = \varphi_\alpha(A)dA$, $dP_\beta(B) = \varphi_\beta(B)dB$, $dP_{\overline{q}}(Q) = \varphi_{\overline{q}}(Q)dQ$, where we used the compact notation $\varphi_x(X) = \prod_{i=1}^{n} \varphi_{x_i}(X_i)$. Hence, we can write (8) using the Lebesgue measure:

$$\int_{\underline{\alpha}}^{\overline{\alpha}} \int_{\mathbb{R}} \int_{\mathbb{R}^n_+} \sum_{i=1}^{m} \left[A \frac{\partial f_i(u_i^*(A,B,Q))}{\partial q_i} + \frac{\partial g_i(u_i^*(A,B,Q))}{\partial q_i} - p\left(\sum_{j=1}^{m} u_j^*(A,B,Q) \right) - B \right.$$
$$\left. - \frac{\partial p\left(\sum_{j=1}^{m} u_j^*(A,B,Q) \right) u_i^*}{\partial q_i} \right] (u_i(y) - u_i^*(y)) \varphi_\alpha(A) \varphi_\beta(B) \varphi_{\overline{q}}(Q) dA\, dB\, dQ \geq 0 \tag{10}$$

for all $u \in K_{\mathbb{P}}$. The advantage of this formulation is that it is suitable for an approximation procedure based on discretization and truncation. The approximation method is applied to the example presented in Sect. 4.1, for the details of the method we refer the interested reader to [6, 8]. The outcome of the above mentioned procedure is a sequence of simple functions $(u_k^*)_k$ which converges in L^2 to the exact solution u^* when $k \to \infty$ (see [8, Theorem 4.2]). We can then use this sequence to approximate the mean value of the solution, which is defined in the standard way as

$$\langle u^* \rangle := \int_{\mathbb{R}^d} u^*(y) dP(y).$$

4 A Class of Utility Functions

In this section we consider a random version of a class of utility functions widely used in the literature (see, e.g., [12, Chap. 6]) and show that these functions satisfy the theoretical requirements stated in the preceding section.
Thus, let

$$f_i(\omega, q_i) = a(\omega)\, a_i q_i^2 + b_i q_i + c_i$$
$$p\left(\omega, \sum_{i=1}^{m} q_i \right) = -d \sum_{i=1}^{m} q_i + e(\omega)$$

where $0 < \underline{a} \leq a(\omega) \leq \overline{a}$, $a \in L^\infty(\Omega)$, $e \in L^2(\Omega)$, and a_i, b_i, d, c_i are positive real numbers. Thus, $w_i(\omega, q) = -[a(\omega)\, a_i q_i^2 + b_i q_i + c_i] + (-d \sum_{i=1}^{m} q_i + e(\omega)) q_i$, and

$$F_i(\omega, q) = [2a(\omega)\, a_i + 2d] q_i + d \sum_{j \neq i} q_j + b_i - e(\omega) \tag{11}$$

For each ω the operator F consists of a linear part and a constant vector. The following theorem shows that $F(\omega, q)$ satisfies the monotonicity requirement mentioned in the previous section.

Theorem 4. *Let $F : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ defined as in (11). Then F is strongly monotone, uniformly with respect to ω.*

Proof. Let T be the matrix associated to the linear part of F. A straightforward computation gives that the diagonal elements of T are $2a(\omega)\, a_i + 2d$ while its off diagonal elements are all equal to d. Now let us decompose T as the sum of three matrices:

$$T = 2a(\omega)\, diag(a_1, a_2, \ldots, a_m) + d\, I_m + \mathbf{d} \tag{12}$$

The first matrix is a diagonal matrix which as $a(\omega)\min_i\{a_i\}$ as its minimum eigenvalue. Given that $0 < \underline{a} \leq a(\omega)$ this matrix is positive definite, uniformly with respect to ω. The second matrix is a scalar matrix, and because d is strictly positive this matrix is positive definite. The third matrix, $\mathbf{d}$, has each entry equal to d, hence it is positive semidefinite. Hence, T is positive definite, uniformly with respect to ω, and as a consequence, F is strongly monotone, uniformly with respect to ω. □

4.1 Numerical Example

We consider the random version of a classical oligopoly problem presented in [12] where three producers are involved in the production of a homogeneous commodity. The cost f_i of producing the commodity by firm i and the demand function p are given by

$$\begin{aligned}
f_1(\omega, q_1) &= a(\omega) q_1^2 + q_1 + 1 \\
f_2(\omega, q_2) &= 0.5 a(\omega) q_2^2 + 4 q_2 + 2 \\
f_3(\omega, q_3) &= a(\omega) q_3^2 + 0.5 q_3 + 5 \\
p\left(\omega, \sum_{i=1}^{3}\right) &= -\sum_{i=1}^{3} q_i + e(\omega)
\end{aligned}$$

where $a(\omega)$ and $e(\omega)$ are random parameters that follow truncated normal distributions:

$$\begin{aligned}
a &\sim 0.5 \leq N(1, 0.25) \leq 1.5 \\
e &\sim 4.5 \leq N(5, 0.25) \leq 5.5
\end{aligned}$$

Although we do not put upper bounds on the production capabilities, the existence of the solution is ensured because of the coercivity of the operator generated by f and p. Solution of the nonrandom problem $(q_1, q_2, q_3) = (23/30, 0, 14/15)$ where $a(\omega) \equiv 1$, $e(\omega) \equiv 5$ is given in [12]. We use the following approximation procedure to evaluate mean value of q (see [6] for a detailed description of the method). First, we choose a discretization of the parameter domain $[0.5, 1.5] \times [4.5, 5.5]$ using

$N_1 \times N_2$ points and solve the problem for each pair $(a(i), e(j))$ using an extragradient method. Then we evaluate the mean value of q by using appropriate probability distribution functions. Approximate mean values of q_1, q_2, and q_3 are shown in Table 1.

Table 1 Mean value of $q = (q_1, q_2, q_3)$

	$N_1 = 100, N_2 = 100$	$N_1 = 200, N_2 = 200$	$N_1 = 400, N_2 = 400$
$\langle q_1 \rangle$	0.76935	0.77154	0.77262
$\langle q_2 \rangle$	$2.903E{-}08$	$2.9109E{-}08$	$2.9185E{-}08$
$\langle q_3 \rangle$	0.94103	0.9436	0.94487

5 Conclusions and Future Developments

We used the theory of random variational inequalities to incorporate uncertain data in an oligopolistic market model. The model presented makes use of quadratic cost functions and a linear demand price, which yields to a linear random variational inequality. In future work we plan to treat other classes of functions which yield to nonlinear variational inequalities and to perform more extended numerical experiments.

References

1. Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities: Applications to Free Boundary Problems. Wiley, Chichester (1984)
2. Cournot, A.A.: Researches into the Mathematical Principles of the Theory of Wealth, 1838 (English Translation). MacMillan, London (1897)
3. Faraci, F., Raciti, F.: On generalized Nash equilibrium in infinite dimension: the Lagrange multipliers approach. Optimization (published online first, December 2012). doi:10.1080/02331934.2012.747090
4. Gabay, D., Moulin, H.: On the uniqueness and stability of Nash Equilibria in noncooperative games. In: Bensoussan, A., Kleindorfer, P., Tapiero, C.S. (eds.) Applied Stochastic Control in Econometrics and Management Sciences, pp. 271–294. North Holland, Amsterdam (1980)
5. Gwinner, J., Raciti, F.: Random equilibrium problems on networks. Math. Comput. Model. **43**, 880–891 (2006)
6. Gwinner, J., Raciti, F.: On a class of random variational inequalities on random sets. Numer. Funct. Anal. Optim. **27**(5–6), 619–636 (2006)
7. Gwinner, J., Raciti, F.: On monotone variational inequalities with random data. J. Math. Inequalities **3**(3), 443–453 (2009)
8. Gwinner, J., Raciti, F.: Some equilibrium problems under uncertainty and random variational inequalities. Ann. Oper. Res. **200**, 299–319 (2012). doi:10.1007/s10479-012-1109-2
9. Jadamba, B., Raciti, F.: On the modelling of some environmental games with uncertain data. J. Optim. Theory Appl. (published online first). doi:10.1007/s10957-013-0389-2

10. Jadamba, B., Khan, A.A., Raciti, F.: Regularization of stochastic variational inequalities and a comparison of an L_p and a sample-path approach. Nonlinear Anal. **94**, 65–83 (2014). http://dx.doi.org/10.1016/j.na.2013.08.009
11. Maugeri. A., Raciti, F.: On existence theorems for monotone and nonmonotone variational inequalities. J. Convex Anal. **16**(3 and 4), 899–911 (2009)
12. Nagurney, A.: Network Economics: A Variational Inequality Approach, 2nd and revised edn. Kluwer, Dordrecht (1999)
13. Ravat, U., Shanbhag, U.V.: On the existence of solutions to stochastic variational inequality and complementarity problems. arXiv:1306.0586v1 [math.OC] (3 Jun 2013)