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The Fields Institute for Research in Mathematical Sciences

Robert Connelly
Asia Ivić Weiss
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Editors

Rigidity and Symmetry

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The Fields Institute for Research in Mathematical Sciences

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Rigidity and Symmetry

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Preface

The thematic program on Discrete Geometry and Applications took place at the Fields Institute for Research in Mathematical Sciences in Toronto between July 1 and December 31, 2011. The papers included in this book are based on some research conducted during the semester and on some of the lectures there, in particular those related to the part of the program under the heading "Rigidity and Symmetry".

This includes the study of the theory of rigidity as applied to discrete objects such as bar and joint frameworks, tensegrities, body and bar frameworks especially including such symmetric objects, periodic frameworks, and the combinatorics when the objects are symmetric. When the configuration of points that define the object is generic, the rigidity properties reduce to combinatorial properties usually of some underlying graph. When the object is symmetric, it automatically becomes non-generic, but nevertheless it is possible to consider the case when the configuration is generic modulo the symmetry group. This leads to a lot of interesting and intricate theory. It is useful to keep in mind that there are two approaches to a symmetric rigid object. Incidental rigidity is when the object is rigid and symmetric, but it is not constrained to stay rigid under a flex. Forced rigidity is when the object is rigid and symmetric, and the symmetry is part of constraints. Both situations occur here.

Another part concerns symmetry as applied to abstract as well as geometric objects. Central to this theme are polytopes, the generalizations of polygons and polyhedra to higher rank (the abstract analogue of dimension). Several articles are devoted to regular maps on surfaces, which are just polyhedra in a general sense. These usually permit operations – replacing faces by different edge-circuits – that change their combinatorial type, an important idea relating different maps. Such operations can be applied in higher rank as well. Regular and chiral polytopes (the latter roughly speaking half-regular) often correspond to interesting groups, particularly simple ones; such connexions are explored in several papers. Variants of regularity, further weakening the condition, also lead to interesting questions. Closely related to polytopes are graphs and complexes; these are the subject of other articles. More metrical in scope are papers on volume in non-euclidean spaces, symmetric configurations in the plane, and a concept of rigidity of polytopes that provides a bridge to the previous part.

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Contents

Volumes of Polytopes in Spaces of Constant Curvature

Nikolay Abrosimov and Alexander Mednykh

Abstract We overview volume calculations for polyhedra in Euclidean, spherical and hyperbolic spaces. We prove the Sforza formula for the volume of an arbitrary tetrahedron in H^3 and S^3. We also present some results, which provide a solution for the Seidel problem on the volume of non-Euclidean tetrahedron. Finally, we consider a convex hyperbolic quadrilateral inscribed in a circle, horocycle or one branch of equidistant curve. This is a natural hyperbolic analog of the cyclic quadrilateral in the Euclidean plane. We find several versions of the Brahmagupta formula for the area of such quadrilateral. We also present a formula for the area of a hyperbolic trapezoid.

Keywords Volumes of polyhedra • Constant curvature spaces • Sforza formula • Seidel problem • Brahmagupta formula

Subject Classifications: Primary 51M20; Secondary 51M25, 51M09, 52B15

1 Volumes of Euclidean Polyhedra

Calculating volumes of polyhedra is a classical problem, that has been well known since Euclid and remains relevant nowadays. This is partly due to the fact that the volume of a fundamental polyhedron is one of the invariants for a three-dimensional manifold.

One of the first results in this direction was obtained by Tartaglia (1499–1557), who had described an algorithm for calculating the height of a tetrahedron with

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some specified lengths of its edges. The formula which expresses the volume of an Euclidean tetrahedron in terms of its edge lengths was given by Euler (see [53], p. 256). The multidimensional analogue of this result is known as the Cayley–Menger formula (see [50], p. 124).

Theorem 1 (Tartaglia, XVI AD). *Let T be an Euclidean tetrahedron with edge lengths d_{ij}, $1 \le i < j \le 4$. Then the volume $V = V(T)$ is given by the formula*

$$288\,V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

In the above relation the volume is evaluated as a root of the quadratic equation whose coefficients are polynomials with integer coefficients. Surprisingly, but this result can be generalized to an arbitrary Euclidean polyhedron. About 15 years ago, I. Kh. Sabitov [42, 43] proved the corresponding theorem for any polyhedron in R^3. Then Robert Connelly, Idzhad Sabitov and Anke Walz gave the second proof for general orientable 2-dimensional polyhedral surfaces using the theory of places instead of resultants [13].

Theorem 2 (Sabitov [42]; Connelly, Sabitov, and Walz [13]). *Let P be Euclidean polyhedron with triangular faces and edge lengths d_{ij}. Then the volume $V(P)$ is a root of a polynomial whose coefficients depend on d_{ij}^2 and combinatorial type of P only.*

Note that the explicit form of the above mentioned polynomial is known only in some special cases, in particular for octahedra with symmetries [21]. On the other hand, Theorem 2 can be used to prove the well-known Bellows conjecture stated by R. Connelly, N. Kuiper and D. Sullivan [11].

Bellows conjecture (Connelly, Kuiper, Sullivan, late 1970s). *The generalized volume of a flexible polyhedron does not change when it is bending.*

Recalling [42] we note that the *generalized volume* of an oriented geometrical polyhedron is equal to the sum of volumes of consistently oriented tetrahedra with a common vertex and with bases on the faces of the polyhedron. *Bending* of a polyhedron is a continuous isometric deformation, provided the rigidity of the faces.

Cauchy's rigidity theorem [8] states that a convex polyhedron with rigid faces is itself rigid. For non-convex polyhedra this is not true; there are examples of flexible polyhedra among them. The first example of a flexible polyhedron was constructed by Bricard [7]. It was a self-intersecting octahedron. The first example of a flexible polyhedron embedded into Euclidean 3-space was presented by Connely [10]. The smallest example of such polyhedron is given by Steffen. It has 14 triangular faces and 9 vertices (see Fig. 1).

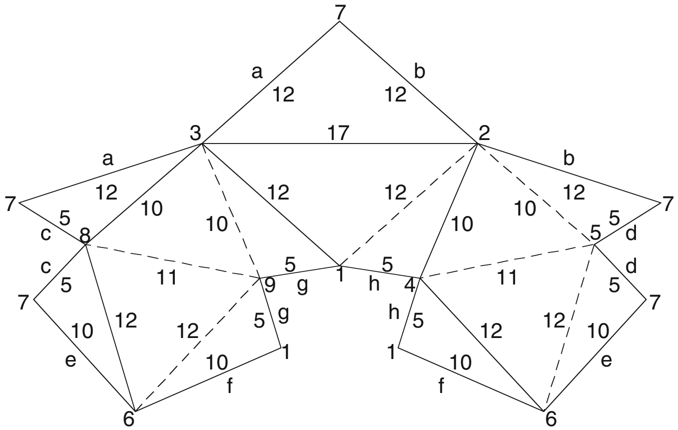


Fig. 1 Development of Steffen's flexible polyhedron

A flexible polyhedron keeps the same combinatorial type and the same set of edge lengths under bending. Then, by Theorem 2 the volume of this polyhedron can take only a finite number of values corresponding to the roots of a polynomial. Since the bending of a polyhedron is a continuous isometric deformation, the volume is constant.

A few months ago a new result by A. A. Gaifullin [20] was published in arXiv. He proved an analog of Theorem 2 for the generalized volume of a n-dimensional polyhedron. In the spherical space the analog of Theorem 2 is not true (see [4]) while in the hyperbolic space the question is still open. One will see below that in many cases the volume of a non-Euclidean polyhedron is not expressed in terms of elementary functions.

2 Volumes of Non-Euclidean Tetrahedra

In the hyperbolic and spherical cases, the situation becomes more complicated. Gauss, who was one of the creators of hyperbolic geometry, referred to the calculation of volumes in non-Euclidean geometry as *die Dschungel*.

It is well known that a tetrahedron in S^3 or H^3 is determined by an ordered set of its dihedral angles up to isometry of the space (see, e.g., [3]). Recall that in the Euclidean case this is true up to simility.

2.1 *Orthoschemes in S^3 and H^3*

Volume formulas for non-Euclidean tetrahedra in some special cases has been known since Lobachevsky, Bolyai and Schläfli. For example, Schläfli [46] found the

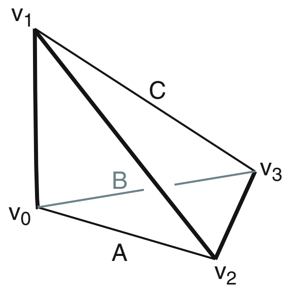


Fig. 2 Orthoscheme $T = T(A, B, C)$ with essential dihedral angles A, B, C, all other angles are $\frac{\pi}{2}$

volume of an orthoscheme in S^3. Recall that an *orthoscheme* is an n-dimensional simplex defined by a sequence of edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ that are mutually orthogonal. In three dimensions, an orthoscheme is also called a *birectangular tetrahedron* (see Fig. 2).

Theorem 3 (Schläfli [45]). *Let T be a spherical orthoscheme with essential dihedral angles A, B, C. Then the volume $V = V(T)$ is given by the formula*

$$V = \frac{1}{4} S(A, B, C), \quad \textit{where} \quad S\left(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z\right) = \hat{S}(x, y, z)$$
$$= \sum_{m=1}^{\infty} \left(\frac{D - \sin x \sin z}{D + \sin x \sin z}\right)^m \frac{\cos 2mx - \cos 2my + \cos 2mz - 1}{m^2} - x^2 + y^2 - z^2,$$

and $D \equiv \sqrt{\cos^2 x \cos^2 z - \cos^2 y}$.

The function $S(x, y, z)$ appeared in Theorem 3 is called the *Schläfli function*. The volume of a hyperbolic orthoscheme was obtained independently by Janos Bolyai [6] and Nikolai Lobachevsky [26]. The following theorem represents a result of Lobachevsky as a quite simple formula. In such form it was given by H. S. M. Coxeter [14].

Theorem 4 (Lobachevsky [26]; Coxeter [14]). *Let T be a hyperbolic orthoscheme with dihedral angles A, B, C. Then the volume $V = V(T)$ is given by the formula $V = \frac{i}{4} S(A, B, C)$, where $S(A, B, C)$ is the Schläfli function.*

J. Bolyai (see, e.g., [6]) found the volume of a hyperbolic orthoscheme in terms of planar angles and an edge length. Consider a hyperbolic orthoscheme T with essential dihedral angles along AC, CB, BD, and all other dihedral angles equal to $\frac{\pi}{2}$ (Fig. 3).

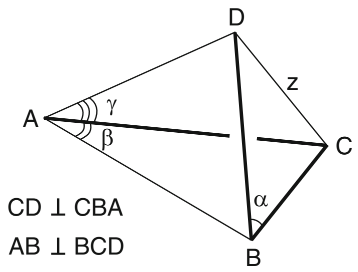


Fig. 3 Orthoscheme T given by planar angles α, β, γ and edge length z

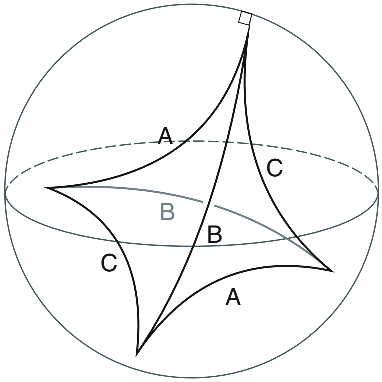


Fig. 4 Ideal tetrahedron $T = T(A, B, C)$ with dihedral angles A, B, C

Theorem 5 (Bolyai [6]). *Let T be a hyperbolic orthoscheme with planar angles α, β, γ and edge length z. Then the volume $V = V(T)$ is given by the formula*

$$V = \frac{\tan\gamma}{2\tan\beta} \int_0^z \frac{u \sinh u \, du}{\left(\frac{\cosh^2 u}{\cos^2\alpha} - 1\right)\sqrt{\frac{\cosh^2 u}{\cos^2\gamma} - 1}}.$$

2.2 Ideal Tetrahedra

An *ideal tetrahedron* is a tetrahedron with all vertices at infinity. Opposite dihedral angles of an ideal tetrahedron are pairwise equal and the sum of dihedral angles at the edges adjacent to one vertex is $A + B + C = \pi$ (see Fig. 4).

The volume of an ideal tetrahedron has been known since Lobachevsky [26]. J. Milnor presented it in very elegant form [31].

Theorem 6 (Lobachevsky [26]; Milnor [31]). *Let T be an ideal hyperbolic tetrahedron with dihedral angles A, B and C. Then the volume $V = V(T)$ is given by the formula*

$$V = \Lambda(A) + \Lambda(B) + \Lambda(C),$$

where $\Lambda(x) = -\int\limits_0^x \log|2\sin t|\,dt$ *is the Lobachevsky function.*

More general case of tetrahedron with at least one vertex at infinity was investigated by E. B. Vinberg [3].

2.3 General Hyperbolic Tetrahedron

Despite the fact that partial results on the volume of non-Euclidean tetrahedra were known a volume formula for hyperbolic tetrahedra of general type remained unknown until recently. A general algorithm for obtaining such a formula was indicated by W.–Yi. Hsiang in [23]. A complete solution was obtained more than 10 years later. In paper by Korean mathematicians Y. Cho and H. Kim [9] proposed a general formula. However, it was asymmetric with respect to permutation of angles. The next advance was achieved by Japanese mathematicians. First, J. Murakami and Y. Yano [36] proposed a formula expressing the volume by dihedral angles in symmetric way. A year later, A. Ushijima [52] presented a simple proof of the Murakami–Yano's formula. He also investigated the case of a truncated hyperbolic tetrahedron.

It should be noted that in all these studies the volume is expressed as a linear combination of 16 dilogarithms or Lobachevsky functions. The arguments of these functions depend on the dihedral angles of the tetrahedron, and an additional parameter, which is the root of a quadratic equation with complex coefficients in a sophisticated form.

The geometric meaning of this formula was explained by P. Doyle and G. Leibon [18] in terms of Regge symmetry. A clear description of these ideas and a complete geometric proof of this formula was given by Yana Mohanty [34]. In particular, she was able to prove the equivalence of Regge symmetry and homogeneity (scissors congruence) in the hyperbolic space [35].

In 2005, D. A. Derevnin and A. D. Mednykh [16] proposed the following integral formula for the volume of a hyperbolic tetrahedron.

Theorem 7 (Derevnin and Mednykh [16]). *Let* $T(A, B, C, D, E, F)$ *be a compact hyperbolic tetrahedron with dihedral angles* A, B, C, D, E, F. *Then the volume* $V = V(T)$ *is given by the formula*

$$V = -\frac{1}{4}\int\limits_{z_1}^{z_2} \log \frac{\cos\frac{A+B+C+z}{2}\cos\frac{A+E+F+z}{2}\cos\frac{B+D+F+z}{2}\cos\frac{C+D+E+z}{2}}{\sin\frac{A+B+D+E+z}{2}\sin\frac{A+C+D+F+z}{2}\sin\frac{B+C+E+F+z}{2}\sin\frac{z}{2}}\,dz,$$

where z_1 *and* z_2 *are the roots of the integrand satisfied the condition* $0 < z_2 - z_1 < \pi$. *More precisely,*

$$
\begin{aligned}
z_1 &= \arctan\frac{k_3}{k_4} - \arctan\frac{k_1}{k_2}\,, \quad z_2 = \pi - \arctan\frac{k_3}{k_4} - \arctan\frac{k_1}{k_2}\,, \quad \textit{where}\\
k_1 &= -\cos S - \cos(A+D) - \cos(B+E) - \cos(C+F) - \cos(D+E+F) -\\
&\quad - \cos(D+B+C) - \cos(A+E+C) - \cos(A+B+F)\,,\\
k_2 &= \sin S + \sin(A+D) + \sin(B+E) + \sin(C+F) + \sin(D+E+F) +\\
&\quad + \sin(D+B+C) + \sin(A+E+C) + \sin(A+B+F)\,,\\
k_3 &= 2\,(\sin A \sin D + \sin B \sin E + \sin C\, \sin F)\,,\\
k_4 &= \sqrt{k_1^2 + k_2^2 - k_3^2}\,,\\
S &= A + B + C + D + E + F\,.
\end{aligned}
$$

2.4 Sforza Formula for Non-Euclidean Tetrahedron

Surprisingly, more than 100 years ago, in 1907, Italian mathematician Gaetano Sforza (or Scorza in some of his papers) found a fairly simple formula for the volume of a non-Euclidean tetrahedron. This fact has became widely known following the discussion between the second author and J. M. Montesinos at the conference in El Burgo de Osma (Spain) in August, 2006. Unfortunately, the outstanding work of Sforza [48] written in Italian has been forgotten.

The original arguments by Sforza are based on some identity given by H. W. Richmond [40] and the Schläfli formula. He also used the Pascal's equation [37] for minors of Gram matrix and some routine calculations. In this section we provide a new proof of Sforza formula for the volume of an arbitrary tetrahedron in H^3 or S^3. The idea of this proof belongs to authors, it was never published before.

Consider a hyperbolic (or a spherical) tetrahedron T with dihedral angles A, B, C, D, E, F. Assume that A, B, C are dihedral angles at the edges adjacent to one vertex and the respectively opposite dihedral angles are D, E, F (see Fig. 5). Then the Gram matrix $G(T)$ is defined as follows

$$
G = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}.
$$

Denote by $\mathscr{C} = \langle c_{ij} \rangle_{i,j=1,2,3,4}$ a matrix of cofactors $c_{ij} = (-1)^{i+j} G_{ij}$, where G_{ij} is (i, j)-th minor of G.

In the following proposition we collect some known results about hyperbolic tetrahedron (see, e.g., [52]).

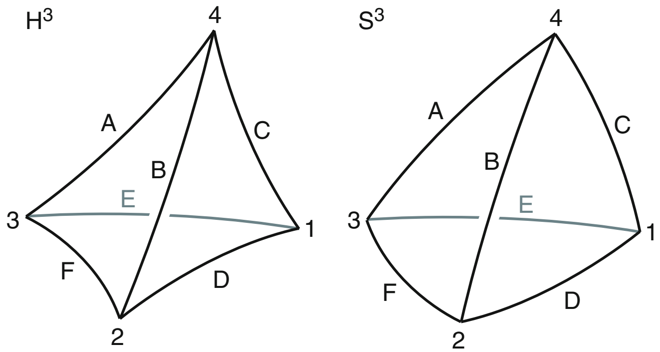


Fig. 5 Tetrahedron $T = T(A, B, C, D, E, F)$ in H^3 or S^3

Proposition 1. *Let T be a compact hyperbolic tetrahedron. Then*

(i) $\det G < 0$;

(ii) $c_{ii} > 0,\ i \in \{1, 2, 3, 4\}$;

(iii) $c_{ij} > 0,\ i \neq j;\ i, j \in \{1, 2, 3, 4\}$;

(iv) $\cosh \ell_{ij} = \dfrac{c_{ij}}{\sqrt{c_{ii} c_{jj}}}$,

where ℓ_{ij} is a hyperbolic length of the edge joining vertices i and j.

Further, we need the following assertion due to Jacobi ([39], Theorem 2.5.1, p. 12).

Proposition 2 (Jacobi theorem). *Let $M = \langle m_{ij} \rangle_{i,j=1,\dots,n}$ be a matrix and $\Delta = \det M$ is determinant of M. Denote by $\mathscr{C} = \langle c_{ij} \rangle_{i,j=1,\dots,n}$ the matrix of cofactors $c_{ij} = (-1)^{i+j} \det M_{ij}$, where M_{ij} is $(n-1) \times (n-1)$ minor obtained by removing i-th row and j-th column of the matrix M. Then for any k, $1 \leq k \leq n-1$ we have*

$$\det \langle c_{ij} \rangle_{i,j=1,\dots,k} = \Delta^{k-1} \det \langle m_{ij} \rangle_{i,j=k+1,\dots,n} .$$

One of the main tools for volume calculations in H^3 and S^3 is the classical Schläfli formula.

Proposition 3 (Schläfli formula). *Let X^n be a space of constant curvature K. Consider a family of convex polyhedra P in X^n depending on one or more parameters in a differential manner and keeping the same combinatorial type. Then the differential of the volume $V = V(P)$ satisfies the equation*

$$(n-1) K \, dV = \sum_F V_{n-2}(F) \, d\theta(F),$$

where the sum is taken over all $(n-2)$*-facets of* P, $V_{n-2}(F)$ *is* $(n-2)$*-dimensional volume of* F, *and* $\theta(F)$ *is the interior angle along* F.

In the classical paper by Schläfli [45] this formula was proved for the case of a spherical n-simplex. For the hyperbolic case, it was obtained by H. Kneser [25] (for more details, see also [3] and [32]). In the Euclidean case this formula reduces to the identity $0 = 0$.

Now we are able to prove the following theorem.

Theorem 8 (Sforza formula in H^3). *Let* T *be a compact hyperbolic tetrahedron with Gram matrix* G. *Consider* $G = G(A)$ *as a function of dihedral angle* A. *Then the volume* $V = V(T)$ *is given by the formula*

$$V = \frac{1}{4} \int_{A_0}^{A} \log \frac{c_{34}(A) - \sqrt{-\det G(A)} \sin A}{c_{34}(A) + \sqrt{-\det G(A)} \sin A} \, dA,$$

where A_0 *is a suitable root of the equation* $\det G(A) = 0$ *and* $c_{34} = c_{34}(A)$ *is* $(3,4)$*-cofactor of the matrix* $G(A)$.

Proof (Abrosimov, Mednykh). Let $\Delta = \det G$. By the Jacobi theorem (Proposition 2) applied to Gram matrix G for $n = 4$ and $k = 2$ we obtain

$$c_{33}c_{44} - c_{34}^2 = \Delta(1 - \cos^2 A).$$

By the Cosine rule (Proposition 1, (iv)) we get $\cosh \ell_A = \dfrac{c_{34}}{\sqrt{c_{33}c_{44}}}$. Hence,

$$\sinh \ell_A = \sqrt{\cosh^2 \ell_A - 1} = \sqrt{\frac{c_{34}^2 - c_{33}c_{44}}{c_{33}c_{44}}} = \frac{\sqrt{-\Delta}\,\sin A}{\sqrt{c_{33}c_{44}}}.$$

Since $\exp(\pm\ell_A) = \cosh \ell_A \pm \sinh \ell_A$ we have

$$\exp(\pm\ell_A) = \frac{c_{34} \pm \sqrt{-\Delta}\,\sin A}{\sqrt{c_{33}c_{44}}}.$$

Hence,

$$\exp(2\,\ell_A) = \frac{\exp(\ell_A)}{\exp(-\ell_A)} = \frac{c_{34} + \sqrt{-\Delta}\,\sin A}{c_{34} - \sqrt{-\Delta}\,\sin A}$$

and

$$\ell_A = \frac{1}{2} \log \frac{c_{34} + \sqrt{-\Delta}\,\sin A}{c_{34} - \sqrt{-\Delta}\,\sin A}.$$

By the Schläfli formula (Proposition 3) for $V = V(T)$ we have

$$-dV = \frac{1}{2} \sum_{\alpha} \ell_\alpha \, d\alpha, \quad \alpha \in \{A, B, C, D, E, F\}.$$

By assumption that angle A is variable and all other angles are constant, we get

$$-dV = \frac{1}{2} \ell_A \, dA.$$

Note that $\det G \to 0$ as $A \to A_0$. Thus, $V \to 0$ as $A \to A_0$. Then integrating both sides of the equation we obtain

$$V = \int\limits_{A_0}^{A} \left(-\frac{\ell_A}{2}\right) dA = \frac{1}{4} \int\limits_{A_0}^{A} \log \frac{c_{34} - \sqrt{-\det G}\ \sin A}{c_{34} + \sqrt{-\det G}\ \sin A} \, dA,$$

where the lower limit A_0 is a suitable root of the equation $\det G(A) = 0$. □

In the following proposition we collect some known results about spherical tetrahedron (see, e.g., [27]).

Proposition 4. *Let T be a spherical tetrahedron. Then*

(i) $\det G > 0$;

(ii) $c_{ii} > 0, i = 1, 2, 3, 4$;

(iii) $\cos \ell_{ij} = \dfrac{c_{ij}}{\sqrt{c_{ii} c_{jj}}}$,

where ℓ_{ij} is a length in S^3 of the edge joining vertices i and j.

The next theorem presents a spherical version of the Sforza formula.

Theorem 9 (Sforza formula in S^3). *Let T be a spherical tetrahedron with Gram matrix G. Consider $G = G(A)$ as a function of dihedral angle A. Then the volume $V = V(T)$ is given by the formula*

$$V = \frac{1}{4i} \int\limits_{A_0}^{A} \log \frac{c_{34}(A) + i \sqrt{\det\ G(A)} \sin A}{c_{34}(A) - i \sqrt{\det\ G(A)} \sin A} \, dA,$$

where A_0 is a suitable root of the equation $\det G(A) = 0$ *and* $c_{34} = c_{34}(A)$ *is* $(3, 4)$*-cofactor of the matrix* $G(A)$.

The proof is similar to one given in the hyperbolic case.

3 Seidel Conjecture

In 1986, J. J. Seidel [47] conjectured that the volume of an ideal hyperbolic tetrahedron can be expressed as a function of the determinant and the permanent of its Gram matrix. Recall that the formula expressing the volume of such tetrahedron in terms of dihedral angles has been known since Lobachevsky and Bolyai (Theorem 6). In spite of this, the Seidel problem had not been solved for a long time. Ten years later, a strengthened version of Seidel conjecture was suggested by Igor Rivin and Feng Luo. They supposed that the volume of a non-Euclidean tetrahedron (hyperbolic or spherical one) depends only on the determinant of its Gram matrix. It was shown in [1] and [2] that the strengthened conjecture is false, while Seidel conjecture is true within certain conditions.

Consider a non-Euclidean tetrahedron T with dihedral angles A, B, C, D, E, F in S^3 or H^3 (Fig. 5). Denote vertices by numbers $1, 2, 3, 4$. Let A_{ij} denote a dihedral angle at the edge joining vertices i and j. For convenience, we set $A_{ii} = \pi$ for $i = 1, 2, 3, 4$. It is well known [4] that, in the hyperbolic and spherical spaces, the tetrahedron T is uniquely (up to isometry) determined by its Gram matrix

$$G = \langle -\cos A_{ij} \rangle_{i,j=1,2,3,4} = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}.$$

Recall that the permanent of a matrix $M = \langle m_{ij} \rangle_{ij=1,2,\ldots,n}$ is defined by

$$\operatorname{per} M = \sum_{i=1}^{n} m_{ij} \operatorname{per} M_{ij}, \quad \operatorname{per}(m_{ij}) = m_{ij},$$

where M_{ij} is the matrix obtained from M by removing i-th row and the j-th column. Conditions for the existence of spherical and hyperbolic tetrahedra in terms of Gram matrices are given in [27] and [52], respectively.

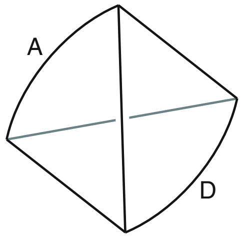
3.1 *Strengthened Conjecture*

In this section we provide a counterexample to the Seidel's strengthened problem. In the spherical case, the answer is given by the following theorem (see [1]).

Theorem 10 (Abrosimov [1]). *There exists a one-parameter family of spherical tetrahedra with unequal volumes and the same determinant of Gram matrix.*

We prove this theorem by constructing such a family. Consider a tetrahedron $T(A, D)$ with two opposite dihedral angles equaled to A and D, and the remaining dihedral angles equaled to $\frac{\pi}{2}$ (Fig. 6).

Fig. 6 Tetrahedron $T = T(A, D)$



It is easy to show that the volume of such a tetrahedron equals $\dfrac{A\,D}{2}$ and the determinant of its Gram matrix is $\det G = \sin^2 A \sin^2 D$. Among all tetrahedra $T(A, D)$ with $0 < A, D < \pi$, we choose a family of tetrahedra

$$T_c(A, D) = T\left(A, \arcsin\frac{c}{\sin A}\right)$$

whose Gram matrices have the same determinant $\det G = c^2$, where c is a constant satisfying the inequalities $0 < c < \min\{\sin A, \sin D\}$.

The volume of such tetrahedra is given by equation

$$V(T_c) = \frac{A}{2}\arcsin\frac{c}{\sin A}.$$

Thus, it depends not only on the constant c, but also on the value of the free parameter A.

We have constructed the required family of tetrahedra. In the hyperbolic case, we have failed to construct an elementary counterexample to the Seidel's strengthened conjecture. Nevertheless, a similar theorem is also valid (see [2]).

Theorem 11 (Abrosimov [1]). *There exists a one-parameter family of hyperbolic tetrahedra with unequal volumes and the same determinant of Gram matrix.*

The proof of this theorem is based on the following considerations. Consider any hyperbolic tetrahedron T with dihedral angles A, B, C, D, E, F. Assume that the angles A, B, C are along the edges adjacent to one vertex, and D, E, F are opposite to them. We fix all dihedral angles except two opposite ones, say A and D. Since the set of hyperbolic tetrahedra is open (see [27, 52]), it follows that, varying A and D within sufficiently small limits, we obtain hyperbolic tetrahedra. In the set $T(A, D)$ of such tetrahedra, we choose a family of tetrahedra $T_c(A, D)$ with the same Gram determinant $\det G = c^2 < 0$. The latter condition means that the differential of the function $\det G$ is zero. Since the angles A and D vary and the remaining angles are fixed, it follows that

$$-d\det G = 2c_{12}\sin A\,dA + 2c_{34}\sin D\,dD = 0,$$

where c_{ij} is (i, j)-th cofactor of the matrix G. Due to this relation, we can treat the angle D as a function of the angle A. We have

$$\frac{dD}{dA} = -\frac{c_{12} \sin A}{c_{34} \sin D}.$$

The derivative of the volume as a composite function of the angle A equals

$$\frac{dV}{dA} = \frac{\partial V}{\partial A} + \frac{\partial V}{\partial D} \frac{dD}{dA}.$$

According to classical Schläfli formula (Proposition 3), we have

$$\frac{\partial V}{\partial A} = -\frac{\ell_A}{2}, \quad \frac{\partial V}{\partial D} = -\frac{\ell_D}{2},$$

where ℓ_A and ℓ_D are the lengths of the corresponding edges of the tetrahedron.

In turn, the lengths of the edges can be expressed in terms of dihedral angles (see [30, 52])

$$\ell_A = \operatorname{arccoth} \frac{\sqrt{-\det G}\ \sin A}{c_{34}},$$
$$\ell_D = \operatorname{arccoth} \frac{\sqrt{-\det G}\ \sin D}{c_{12}}.$$

Comparing these expressions and performing simple calculations, we obtain

$$\frac{dV}{dA} = -\frac{\tanh \ell_A}{2} \left(\frac{\ell_A}{\tanh \ell_A} - \frac{\ell_D}{\tanh \ell_D} \right).$$

It is required in the theorem that the volume is varying according to parameter A, i.e., $\dfrac{dV}{dA} \neq 0$, which is equivalent to condition $\ell_A \neq \ell_D$.

Thus, two sufficiently closed tetrahedra from the family $T_c(A, D)$ have the same Gram determinant and unequal volume provided that $\ell_A \neq \ell_D$. It is not hard to construct an infinite family of tetrahedra satisfying the last condition for $A \neq D$. For example, this condition is satisfied by "almost symmetric" tetrahedra with angles $A \neq D, B = E$, and $C = F$. Recall that, for fixed c, the family $T_c(A, D)$ still depends on one free parameter.

3.2 Solution of the Seidel Conjecture

The solution of the Seidel problem, which was posed in [47], is given by the following theorem in [2].

Theorem 12 (Abrosimov [2]). *In each of classes of acute or obtuse tetrahedra the volumes of ideal tetrahedra are determined uniquely by the determinant and permanent of Gram matrix.*[1]

It is known (see, e.g., [31]), that the opposite dihedral angles of an ideal tetrahedron are equal and the sum of dihedral angles at the edges adjacent to one vertex is $A + B + C = \pi$ (Fig. 4).

Thus, we can take $C = \pi - A - B$. The Gram matrix has the form

$$G = \begin{pmatrix} 1 & -\cos A & -\cos B & \cos(A+B) \\ -\cos A & 1 & \cos(A+B) & -\cos B \\ -\cos B & \cos(A+B) & 1 & -\cos A \\ \cos(A+B) & -\cos B & -\cos A & 1 \end{pmatrix}.$$

We have

$$\det G = -4\sin^2 A\,\sin^2 B\,\sin^2(A+B),$$

$$\operatorname{per} G = 4 + 4\cos^2 A\,\cos^2 B\,\cos^2(A+B).$$

To prove Theorem 12, we show that the dihedral angles of an ideal tetrahedron are uniquely (up to a permutation) determined by $\det G$ and $\operatorname{per} G$ in each of the cases: for acute angled tetrahedron and for obtuse angled tetrahedron.

Without loss of generality, we can assume that $0 < A \leq B \leq C = \pi - A - B$. Then the dihedral angles A, B are a priori acute, and the angle C is either acute or obtuse. In the former case, the tetrahedron under consideration is acute-angled, and in the latter case, it is obtuse-angled.

Let us introduce the new variables

$$x = \sin A\,\sin B, \quad y = \cos A\,\cos B$$

and show that, for a fixed left-hand side, the solutions of the system of equations

$$-\frac{1}{4}\det G = x^2(1-(y-x)^2),$$

$$\frac{1}{4}\operatorname{per} G = y^2(y-x)^2 + 1$$

correspond to one tetrahedron (up to isometry) in each of the two cases mentioned above.

Suppose that the system has a pair of different solutions (a, b) and (x, y). Then we have

[1] By an *obtuse tetrahedron* we mean a tetrahedron with at least one dihedral angle $> \frac{\pi}{2}$.

$$a^2(1-(b-a)^2) = x^2(1-(y-x)^2),$$
$$b^2(b-a)^2 = y^2(y-x)^2.$$

The angle C being acute means that $\cos A \cos B - \sin A \sin B = \cos C < 0$, i.e., both solutions satisfy the inequalities $b(b-a) < 0$ and $x(x-y) < 0$. If the angle C is obtuse, then the reverse inequality. This allows us to take a square root in the second equation without loosing solutions:

$$a^2(1-(b-a)^2) = x^2(1-(y-x)^2),$$
$$b\,(b-a) = y\,(y-x).$$

Expressing x from the second equation and substituting the resulting expression into the first one, we obtain a sixth-degree polynomial equation in y. Fortunately, it decomposes into the linear factors and the biquadratic polynomial

$$(b-y)\cdot(b+y)\cdot\big(y^4-(a^2+a^4+2\,ab-2\,a^3b-b^2+2\,ab^3-b^4)\,y^2+$$
$$a^4b^2-4\,a^3b^3+6\,a^2b^4-4\,ab^5+b^6\big)=0.$$

Thus, all solutions can be we found in radicals. Substituting the expressions for x and y in terms of dihedral angles, we see that different solutions of the system correspond to the same ideal tetrahedron $T(A, B, C)$ up to reordering of the dihedral angles.

Note that, in Theorem 12, the assumption that the tetrahedron is either acute-angled or obtuse-angled cannot be dispensed with. This is demonstrated by the following example.

Example 1. Consider a pair of ideal tetrahedra, the obtuse-angled tetrahedron $T_1(s, s, \pi-2s)$ and the acute-angled tetrahedron $T_2\left(t, \dfrac{\pi-t}{2}, \dfrac{\pi-t}{2}\right)$, where

$$s = \arccos\frac{\sqrt{2+\sqrt{4+\sqrt{170\sqrt{17}-698}}}}{2\sqrt{2}},$$
$$t = \arccos\frac{-1+\sqrt{17}+\sqrt{-26+10\sqrt{17}}}{8}.$$

The determinants and the permanents of the Gram matrices of these tetrahedra coincide; they are

$$\det G(T_1) = \det G(T_2) = \frac{107-51\sqrt{17}}{128},$$
$$\mathrm{per} G(T_1) = \mathrm{per} G(T_2) = \frac{163+85\sqrt{17}}{128}.$$

At the same time, the volumes of the tetrahedra T_1 and T_2 are different and equal 0.847365 and 1.01483, respectively.

4 Heron and Brahmagupta Formulas

Heron of Alexandria (c. 60 BC) is credited with the following formula that relates the area S of a triangle to its side lengths a, b, c

$$S^2 = (s-a)(s-b)(s-c)s,$$

where $s = (a + b + c)/2$ is the semiperimeter. For polygons with more than three sides, the side lengths do not in general determine the area, but they do if the polygon is convex and cyclic (inscribed in a circle). Brahmagupta, in the seventh century, gave the analogous formula for a convex cyclic quadrilateral with side lengths a, b, c, d

$$S^2 = (s-a)(s-b)(s-c)(s-d),$$

where $s = (a + b + c + d)/2$. See [15] for an elementary proof. An interesting consideration of the problem can be found in the Möbius paper [33]. Independently, D. P. Robbins [41] and V. V. Varfolomev [55] found a way to generalize these formulas. The main idea of both papers was to determine the squared area S^2 as a root of an algebraic equation whose coefficients are integer polynomials in the squares of the side lengths. See also [12, 19] and [44] for more detailed consideration.

In the present section of our work we deal with the hyperbolic plane instead of the Euclidean one. The hyperbolic plane under consideration is equipped with a Riemannian metric of constant curvature $k = -1$. All necessary definitions from hyperbolic geometry can be found in the book [3].

By definition, a *cyclic polygon* in the hyperbolic plane is a convex polygon inscribed in a circle, horocycle or one branch of equidistant curve. Useful information about cyclic polygons can be found in [56] and [57]. In particular, it is shown in [56] that any cyclic polygon in the hyperbolic plane is uniquely determined (up to isometry) by the ordered sequence of its side lengths. In addition, among all hyperbolic polygons with fixed positive side lengths there exist polygons of maximal area. Every such a maximal polygon is cyclic (see [57]).

The following four non-Euclidean versions of the Heron formula in the hyperbolic plane have been known for a long time.

Theorem 13. *Let T be a hyperbolic triangle with side lengths a, b, c. Then the area $S = S(T)$ is given by each of the following formulas*

(i) Sine of 1/2 Area Formula

$$\sin^2 \frac{S}{2} = \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s)}{4\cosh^2\left(\frac{a}{2}\right)\cosh^2\left(\frac{b}{2}\right)\cosh^2\left(\frac{c}{2}\right)};$$

(ii) Tangent of 1/4 Area Formula

$$\tan^2 \frac{S}{4} = \tanh\left(\frac{s-a}{2}\right)\tanh\left(\frac{s-b}{2}\right)\tanh\left(\frac{s-c}{2}\right)\tanh\left(\frac{s}{2}\right);$$

(iii) Sine of 1/4 Area Formula

$$\sin^2 \frac{S}{4} = \frac{\sinh(\frac{s-a}{2})\sinh(\frac{s-b}{2})\sinh(\frac{s-c}{2})\sinh(\frac{s}{2})}{\cosh\left(\frac{a}{2}\right)\cosh\left(\frac{b}{2}\right)\cosh\left(\frac{c}{2}\right)};$$

(iv) Bilinski Formula

$$\cos \frac{S}{2} = \frac{\cosh a + \cosh b + \cosh c + 1}{4\cosh\left(\frac{a}{2}\right)\cosh\left(\frac{b}{2}\right)\cosh\left(\frac{c}{2}\right)}.$$

The first two formulas are contained in the book [3] (p. 66). The third formula can be obtained by the squaring of the product of the first two. The forth one was derived by Stanko Bilinski in [5] (see also [57]).

It should be noted that the analogous formulas in spherical space are also known. For example, the spherical version of (i) is called the Cagnoli's Theorem (see [51], sec. 100), (ii) is called the Lhuilier's Theorem (see [51], sec. 102), (iii) and (iv) are proven in [51] (sec. 103).

4.1 Preliminary Results for Cyclic Quadrilaterals

We recall a few well known facts about cyclic quadrilaterals. A convex Euclidean quadrilateral with interior angles A, B, C, D is cyclic if and only if $A + C = B + D = \pi$. A similar result for hyperbolic quadrilateral was obtained by V. F. Petrov [38] and L. Wimmer [58]. They proved the following proposition.

Proposition 5. *A convex hyperbolic quadrilateral with interior angles A, B, C, D is cyclic if and only if $A + C = B + D$.*

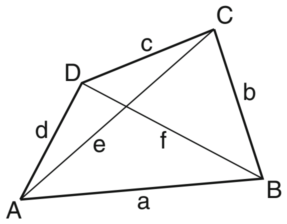


Fig. 7 Cyclic quadrilateral Q with angles A, B, C, D, where $A + C = B + D$

Note that the sum of angles of a hyperbolic quadrilateral is less then 2π. Hence, for any cyclic hyperbolic quadrilateral we have $A + C = B + D < \pi$.

It was shown in [56] that a cyclic n-gon is uniquely up to isometry determined by the lengths of its sides. Denote side and diagonal lengths of a quadrilateral as indicated on Fig. 7. Then, in the Euclidean case, a quadrilateral is cyclic if and only if $ef = a\,c + b\,d$. This is the Ptolemy's theorem. A similar result for hyperbolic quadrilateral is contained in the paper by J. E. Valentine [54].

Proposition 6. *A convex hyperbolic quadrilateral with side lengths a, b, c, d and diagonal lengths e, f is cyclic if and only if*

$$\sinh\frac{e}{2}\sinh\frac{f}{2} = \sinh\frac{a}{2}\sinh\frac{c}{2} + \sinh\frac{b}{2}\sinh\frac{d}{2}.$$

An important supplement to the Ptolemy's theorem is the following property relating lengths of the sides and diagonals

$$\frac{e}{f} = \frac{a\,d + b\,c}{a\,b + c\,d}. \tag{1}$$

Together with the Ptolemy's theorem this equation allows us to express the diagonal lengths of a cyclic quadrilateral by its side lengths.

It was noted in [22] that the above mentioned relationships between sides and diagonals of a cyclic quadrilateral are also valid in the hyperbolic geometry when (the length side) a is replaced by $s(a) = \sinh\frac{a}{2}$. In particular, formula (1) can be rewritten in the following way.

Proposition 7. *The side lengths a, b, c, d and diagonal lengths e, f of a cyclic hyperbolic quadrilateral are related by the following equation*

$$\frac{s(e)}{s(f)} = \frac{s(a)s(d) + s(b)s(c)}{s(a)s(b) + s(c)s(d)}.$$

By making use of Propositions 6 and 7 we derive the following formulas for the diagonal lengths e, f of a cyclic hyperbolic quadrilateral. Then we have

$$s^2(e) = \frac{s(a)s(d) + s(b)s(c)}{s(a)s(b) + s(c)s(d)}(s(a)s(c) + s(b)s(d)), \tag{2}$$

$$s^2(f) = \frac{s(a)s(b) + s(c)s(d)}{s(a)s(d) + s(b)s(c)}(s(a)s(c) + s(b)s(d)). \tag{3}$$

Note that formulas (2) and (3) are valid also in the Euclidean and spherical geometries. In these cases, instead of function $s(a)$ one should take the functions $s(a) = a$ and $s(a) = \sin \frac{a}{2}$ respectively. See [22] and [17] for the arguments in the spherical case.

All the above propositions will be used in the next section to obtain a few versions of the Brahmagupta formula for a cyclic hyperbolic quadrilateral.

4.2 Area of Cyclic Hyperbolic Quadrilateral

In this section we consider the four versions of the Brahmagupta formula for a cyclic hyperbolic quadrilateral given by second author in [29]. They are generalizations of the respective statements (i)–(iv) of Theorem 13.

In particular, the first statement (i) has the following analog.

Theorem 14 (Sine of $1/2$ area formula). *Let Q be a cyclic hyperbolic quadrilateral with side lengths a, b, c, d. Then the area $S = S(Q)$ is given by the formula*

$$\sin^2 \frac{S}{2} = \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s-d)}{4\cosh^2\frac{a}{2}\cosh^2\frac{b}{2}\cosh^2\frac{c}{2}\cosh^2\frac{d}{2}}(1-\varepsilon),$$

where $\varepsilon = \dfrac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\sinh\frac{d}{2}}{\cosh\frac{s-a}{2}\cosh\frac{s-b}{2}\cosh\frac{s-c}{2}\cosh\frac{s-d}{2}}$ *and* $s = \dfrac{a+b+c+d}{2}.$

Note that the number ε vanishes if $d = 0$. In this case, we get formula (i) again.

The second statement (ii) for the case of a hyperbolic quadrilateral has the following form.

Theorem 15 (Tangent of $1/4$ area formula). *Let Q be a cyclic hyperbolic quadrilateral with side lengths a, b, c, d. Then the area $S = S(Q)$ is given by the formula*

$$\tan^2 \frac{S}{4} = \frac{1}{1-\varepsilon}\tanh\frac{s-a}{2}\tanh\frac{s-b}{2}\tanh\frac{s-c}{2}\tanh\frac{s-d}{2},$$

where s and ε are the same as in Theorem 14.

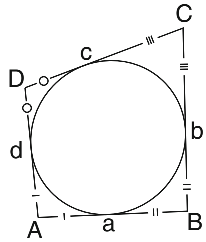


Fig. 8 Circumscribed cyclic quadrilateral Q with side lengths a, b, c, d

It follows from Theorem 14 that for any $a, b, c, d \neq 0$ we have $1 - \varepsilon > 0$ and $\varepsilon > 0$. Hence, $0 < \varepsilon < 1$. Taking into account these inequalities as an immediate consequence of Theorems 14 and 15 we obtain the following corollary.

Corollary 1. *For any cyclic hyperbolic quadrilateral the following inequalities take a place*

$$\sin^2 \frac{S}{2} < \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s-d)}{4\cosh^2(\frac{a}{2})\cosh^2(\frac{b}{2})\cosh^2(\frac{c}{2})\cosh^2(\frac{d}{2})}$$

and

$$\tan^2 \frac{S}{4} > \tanh\left(\frac{s-a}{2}\right)\tanh\left(\frac{s-b}{2}\right)\tanh\left(\frac{s-c}{2}\right)\tanh\left(\frac{s-d}{2}\right).$$

By squaring the product of formulas in Theorems 14 and 15 we obtain the following result. It can be considered as a direct generalization of the statement (iii) in Theorem 13.

Theorem 16 (Sine of $1/4$ area formula). *Let Q be a cyclic hyperbolic quadrilateral with side lengths a, b, c, d. Then the area $S = S(Q)$ is given by the formula*

$$\sin^2 \frac{S}{4} = \frac{\sinh\frac{s-a}{2}\sinh\frac{s-b}{2}\sinh\frac{s-c}{2}\sinh\frac{s-d}{2}}{\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh\frac{d}{2}},$$

where $s = \dfrac{a+b+c+d}{2}$.

The analogous formula in spherical space can be obtained by replacing sinh and cosh with sin and cos correspondingly (see [28], p. 182, proposition 5).

Consider a circumscribed quadrilateral Q with side lengths a, b, c, d (Fig. 8). In this case we have $s - a = c$, $s - b = d$, $s - c = a$, $s - d = b$. As a result we obtain the following assertion.

Corollary 2 (Brahmagupta formula for bicentric quadrilateral). *Let Q be a bicentric (i.e. inscribed and circumscribed) hyperbolic quadrilateral with side lengths a, b, c, d. Then the area $S = S(Q)$ is given by the formula*

$$\sin^2 \frac{S}{4} = \tanh\left(\frac{a}{2}\right) \tanh\left(\frac{b}{2}\right) \tanh\left(\frac{c}{2}\right) \tanh\left(\frac{d}{2}\right).$$

The analogous formula in spherical space can be given by replacing tanh with tan (see [28], p. 46). An Euclidean version of this result is well known (see, e.g., [24]). In this case

$$S^2 = a\,b\,c\,d.$$

The next theorem [29] presents a version of the Bilinski formula for a cyclic quadrilateral.

Theorem 17 (Bilinski formula). *Let Q be a cyclic hyperbolic quadrilateral with side lengths a, b, c, d. Then the area $S = S(Q)$ is given by the formula*

$$\cos \frac{S}{2} = \frac{\cosh a + \cosh b + \cosh c + \cosh d - 4 \sinh\left(\frac{a}{2}\right) \sinh\left(\frac{b}{2}\right) \sinh\left(\frac{c}{2}\right) \sinh\left(\frac{d}{2}\right)}{4 \cosh\left(\frac{a}{2}\right) \cosh\left(\frac{b}{2}\right) \cosh\left(\frac{c}{2}\right) \cosh\left(\frac{d}{2}\right)}.$$

4.3 Proof of Brahmagupta Formulas

Consider a cyclic hyperbolic quadrilateral Q with side lengths a, b, c, d and interior angles A, B, C, D shown on Fig. 7. By the Gauss–Bonnet formula we get the area

$$S = S(Q) = 2\pi - A - B - C - D.$$

To prove Theorem 14 let us find the quantities $\sin^2 \frac{S}{4}$ and $\cos^2 \frac{S}{4}$ in terms of a, b, c, d. Since $A + C = B + D$ (see Proposition 5) we have

$$2\sin^2 \frac{S}{4} = 1 - \cos \frac{S}{2} = 1 - \cos(\pi - (A + C)) = 1 + \cos(A + C).$$

Hence,

$$\sin^2 \frac{S}{4} = \frac{1 + \cos A \, \cos C - \sin A \, \sin C}{2}. \tag{4}$$

Now we show that $\cos A, \cos C$ and the product $\sin A \cdot \sin C$ can be expressed in terms of elementary functions in a, b, c, d. To find $\cos A$ we use the Cosine rule for hyperbolic triangle ABD.

$$\cos A = \frac{\cosh a\ \cosh d - \cosh f}{\sinh a\ \sinh d}. \tag{5}$$

We note that $\cosh f = 2\,s^2(f) + 1, \cosh a = 2\,s^2(a) + 1$ and $\cosh d = 2\,s^2(d) + 1$. Putting these identities into Eqs. (3) and (5) we express $\cos A$ in terms of a, b, c, d. After straightforward calculations we obtain

$$\cos A = \frac{s^2(a) - s^2(b) - s^2(c) + s^2(d) + 2s(a)\,s(b)\,s(c)\,s(d) + 2s^2(a)\,s^2(d)}{2\,(s(a)\,s(d) + s(b)\,s(c))\cosh\frac{a}{2}\cosh\frac{d}{2}}. \tag{6}$$

In a similar way we get the formula

$$\cos C = \frac{-s^2(a) + s^2(b) + s^2(c) - s^2(d) + 2s(a)\,s(b)\,s(c)\,s(d) + 2s^2(b)\,s^2(c)}{2\,(s(a)\,s(d) + s(b)\,s(c))\cosh\frac{b}{2}\cosh\frac{c}{2}}. \tag{7}$$

Note that $\sin A\ \sin C > 0$ and $\sin^2 A\ \sin^2 C = (1 - \cos^2 A)(1 - \cos^2 C)$. Then we take a square root in the latter equation, where $\cos A$ and $\cos C$ are given by the formulas (6) and (7). Thus we express a product $\sin A \cdot \sin C$ in terms of a, b, c, d. Substituting $\cos A, \cos C$ and $\sin A \cdot \sin C$ into (4) and simplifying we get

$$\sin^2\frac{S}{4} = \frac{\sinh\frac{-a+b+c+d}{4}\sinh\frac{a-b+c+d}{4}\sinh\frac{a+b-c+d}{4}\sinh\frac{a+b+c-d}{4}}{\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh\frac{d}{2}}. \tag{8}$$

This proves Theorem 16.

In a similar way, from identity $2\cos^2\frac{S}{4} = 1 + \cos\frac{S}{2} = 1 - \cos(A + C)$ we have

$$\cos^2\frac{S}{4} = \frac{\cosh\frac{a+b-c-d}{4}\cosh\frac{a-b+c-d}{4}\cosh\frac{a-b-c+d}{4}\cosh\frac{a+b+c+d}{4}}{\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh\frac{d}{2}}. \tag{9}$$

The following lemma can be easy proved by straightforward calculations.

Lemma 1. *The expression*

$$H = \frac{\cosh\frac{a+b-c-d}{4}\cosh\frac{a-b+c-d}{4}\cosh\frac{a-b-c+d}{4}\cosh\frac{a+b+c+d}{4}}{\cosh\frac{-a+b+c+d}{4}\cosh\frac{a-b+c+d}{4}\cosh\frac{a+b-c+d}{4}\cosh\frac{a+b+c-d}{4}}$$

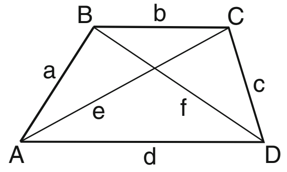


Fig. 9 Trapezoid $T = T(a, b, c, d)$

can be rewritten in the form $H = 1 - \varepsilon$, *where*

$$\varepsilon = \frac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\sinh\frac{d}{2}}{\cosh\frac{s-a}{2}\cosh\frac{s-b}{2}\cosh\frac{s-c}{2}\cosh\frac{s-d}{2}} \text{ and } s = \frac{a+b+c+d}{2}.$$

Taking four times product of Eqs. (8) and (9) we have

$$\sin^2\frac{S}{2} = \frac{\sinh\frac{-a+b+c+d}{2}\sinh\frac{a-b+c+d}{2}\sinh\frac{a+b-c+d}{2}\sinh\frac{a+b+c-d}{2}}{4\cosh^2\frac{a}{2}\cosh^2\frac{b}{2}\cosh^2\frac{c}{2}\cosh^2\frac{d}{2}} \cdot H, \quad (10)$$

where H is the same as in Lemma 1.

Then the statement of Theorem 14 follows from Eq. (10), Lemma 1 and the evident identity $s - a = \dfrac{-a+b+c+d}{2}$.

To prove Theorem 15 we divide (8) by (9). As a result we have

$$\tan^2\frac{S}{4} = \frac{\sinh\frac{-a+b+c+d}{4}\sinh\frac{a-b+c+d}{4}\sinh\frac{a+b-c+d}{4}\sinh\frac{a+b+c-d}{4}}{\cosh\frac{a+b-c-d}{4}\cosh\frac{a-b+c-d}{4}\cosh\frac{a-b-c+d}{4}\cosh\frac{a+b+c+d}{4}}. \quad (11)$$

Hence, applying Lemma 1 we obtain the statement of Theorem 15.

Finally, the Bilinski formula (Theorem 17) follows from the identity $\cos\dfrac{S}{2} = \cos^2\dfrac{S}{4} - \sin^2\dfrac{S}{4}$ and the above mentioned Eqs. (8) and (9).

4.4 Area of Hyperbolic Trapezoid

In this section we give a formula for the area of a hyperbolic trapezoid in terms of its side lengths.

A convex hyperbolic quadrilateral with interior angles A, B, C, D is called a *trapezoid* if $A + B = C + D$ (see Fig. 9). This definition is also valid for Euclidean case.

Denote the lengths of sides and diagonals as shown on Fig. 9. We assume that $b \neq d$. Otherwise, in the case $b = d$, the area of trapezoid T is not determined by lengths of its sides a, b, c, d. The next formula was obtained by Dasha Sokolova in [49].

Theorem 18 (Sokolova [49]). *Let T be a hyperbolic trapezoid with side lengths a, b, c, d. Then the area $S = S(T)$ is given by the formula*

$$\tan^2\frac{S}{4} = \frac{\sinh^2\frac{b+d}{2}\sinh\frac{a+b-c-d}{4}\sinh\frac{a+b+c-d}{4}\sinh\frac{-a+b+c-d}{4}\sinh\frac{a-b+c+d}{4}}{\sinh^2\frac{b-d}{2}\cosh\frac{a-b-c-d}{4}\cosh\frac{a-b+c-d}{4}\cosh\frac{a+b-c+d}{4}\cosh\frac{a+b+c+d}{4}}.$$

Remark 1. The following formula gives the area of an Euclidean trapezoid in terms of its side lengths.

$$S_E{}^2 = \frac{(b+d)^2(a+b-c-d)(a+b+c-d)(-a+b+c-d)(a-b+c+d)}{16\,(b-d)^2}.$$

Note that $\tan^2\frac{S}{4} \sim \left(\frac{S_E}{4}\right)^2$ for sufficiently small values of a, b, c, d.

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Cubic Cayley Graphs and Snarks

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Abstract The well-known conjecture that there are no snarks amongst Cayley graphs is considered. Combining the theory of Cayley maps with the existence of certain kinds of independent sets of vertices in arc-transitive graphs, some new partial results are obtained suggesting promising future research directions in regards to this conjecture.

Keywords Snark • Cubic graph • Cayley graph • Coloring • Arc-transitive • Independent set

Subject Classifications: 05C25, 05C15, 20E32

1 Introductory Remarks

A *snark* is a connected, cyclically 4-edge-connected cubic graph which is not 3-edge-colorable, that is, a connected, cyclically 4-edge-connected cubic graph whose edges cannot be colored by three colors in such a way that adjacent edges receive distinct colors. While examples of snarks were initially scarce – the Petersen

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graph being the first known example of a snark – infinite families of snarks are now known to exist. The first and second Blanuša snarks, the second and third snarks discovered [5], are actually the smallest members of two infinite families of snarks [31].

A *Cayley snark* is a cubic Cayley graph which is a snark. Although most known examples of snarks exhibit a lot of symmetry, none of them is a Cayley graph. In fact it was conjectured in [3] that no such graphs exist.

Conjecture 1 ([3]). There are no Cayley snarks.

The proof of this conjecture would contribute significantly to various open problems regarding Cayley graphs. One of such problems is the well-known conjecture that every connected Cayley graph contains a Hamilton cycle. Namely, every hamiltonian cubic graph is easily seen to be 3-edge-colorable. It is perhaps also worth mentioning that Conjecture 1 is in fact a special case of the conjecture that all Cayley graphs on groups of even order are 1-factorizable (see [32]). (A graph is 1*-factorizable* if its edge set can be partitioned into edge-disjoint 1-factors (perfect matchings).)

A large number of articles, directly or indirectly related to this problem, have appeared in the literature affirming the non-existence of Cayley snarks. For example, in [28] it is proved that the smallest example of a Cayley snark, if it exists, comes either from a non-abelian simple group or from a group which has a single non-trivial proper normal subgroup. The subgroup must have index two and must be either non-abelian simple or the direct product of two isomorphic non-abelian simple groups. In 2004, Potočnik [30], motivated by the fact that there are only two known examples of connected cubic vertex-transitive graphs which are not 3-edge-colorable, namely, the Petersen graph and its truncation, asked whether every connected cubic vertex-transitive graph, other than these two graphs, is 3-edge-colorable, and gave the answer for graphs admitting transitive solvable groups of automorphisms. In particular, it is proved in [30] that every connected cubic graph (different from the Petersen graph) whose automorphism group contains a solvable subgroup acting transitively on the set of vertices is 3-edge-colorable.

In this paper we will present an innovative approach to finding a possible solution to Conjecture 1 combining the theory of Cayley maps with the existence of a certain kind of independent set of vertices in arc-transitive graphs (see Theorem 1).

The paper is organized as follows. In Sect. 2 we gather various concepts that are needed in the subsequent sections. In Sect. 3 we discuss the structure of Cayley snarks. In Sect. 4 we describe the above mentioned approach and then use it to prove Conjecture 1 for a certain class of graphs in Sect. 5. Finally, in Sect. 6 we give possible future directions in regards to the Cayley snark problem.

2 Terminology and Notation

Throughout this paper graphs are simple and, unless otherwise specified, undirected and connected. Furthermore, all graphs and groups are assumed to be finite. For group-theoretic terms not defined here we refer the reader to [34].

Let X be a graph. Then for adjacent vertices u and v in X, we write $u \sim v$ and denote the corresponding edge by uv. We let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}X$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. If $u \in V(X)$ then $N_X(u)$ denotes the set of neighbors of u. The *girth* of X is the length of a shortest cycle in X. For a non-negative integer k, a *k-arc* in X is a sequence of $k+1$ vertices $(u_1, u_2, \ldots, u_{k+1})$, not necessarily all distinct, such that any two consecutive terms are adjacent and any three consecutive terms are distinct. For a subset U of $V(X)$ the subgraph of X induced by U is denoted by $X[U]$. If $X[U]$ is an empty graph then U is called an *independent set of vertices*.

A *k-factor* of a graph is a spanning k-regular subgraph of the graph. Therefore a 2-factor is a collection of cycles spanning all vertices of the graph. A 2-factor is said to be *even* if all of these cycles are of even length. A *Hamilton path* of a graph is a simple path going through all vertices of the graph. A *Hamilton cycle* of a graph is a cycle going through all vertices of the graph; in other words, it is a connected 2-factor of a graph. A graph possessing a Hamilton cycle is said to be *hamiltonian*.

A subgroup $G \leq \mathrm{Aut}X$ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* provided it acts transitively on the set of vertices, edges and arcs of X, respectively. A graph is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if its automorphism group is vertex-transitive, edge-transitive and arc-transitive, respectively. A subgroup $G \leq \mathrm{Aut}X$ is said to be *k-regular* if it acts transitively on the set of k-arcs and the stabilizer of a k-arc in G is trivial. A graph X is said to be *(G,k)-regular* if $G \leq \mathrm{Aut}X$ is k-regular. In particular, a subgroup $G \leq \mathrm{Aut}X$ is said to be *1-regular* if it acts transitively on the set of arcs and the stabilizer of an arc in G is trivial.

Given a group G and a subset $S \subseteq G$ such that $S = S^{-1}$ and $1 \notin S$, the *Cayley graph* $\mathrm{Cay}(G,S)$ on G relative to S has vertex set G and edge set $\{g \sim gs \mid g \in G, s \in S\}$. If G is cyclic then $\mathrm{Cay}(G,S)$ is said to be a *circulant*. Note that $\mathrm{Cay}(G,S)$ is connected if and only if S is a generating set of the group G. Denote by $\mathrm{Aut}(G,S)$ the set of all automorphisms of a group G which fix the set $S \subseteq G$ setwise, that is,

$$\mathrm{Aut}(G,S) = \{\sigma \in \mathrm{Aut}(G) | S^\sigma = S\}.$$

It is easy to check that $\mathrm{Aut}(G,S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G,S))$ contained in the stabilizer of the identity element $1 \in G$. It follows from the definition of Cayley graphs that the left regular representation G_L of G induces a regular subgroup of $\mathrm{Aut}(\mathrm{Cay}(G,S))$, implying that $\mathrm{Cay}(G,S)$ is a vertex-transitive graph. Following [38], $\mathrm{Cay}(G,S)$ is called a *normal Cayley graph* if G_L is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$, that is, if $\mathrm{Aut}(G,S)$ coincides with the vertex stabilizer $1 \in G$. Moreover, if $\mathrm{Cay}(G,S)$ is a normal Cayley graph, then $\mathrm{Aut}(\mathrm{Cay}(G,S)) = G_L \rtimes \mathrm{Aut}(G,S)$.

In order to state the classification of connected arc-transitive circulants, which has been obtained independently by Kovács [19] and Li [23], we need to recall certain graph products. The *wreath (lexicographic) product* $X[Y]$ of a graph X by a graph Y is the graph with vertex set $V(X) \times V(Y)$ such that $\{(u_1, u_2), (v_1, v_2)\}$ is an edge if and only if either $\{u_1, v_1\} \in E(X)$, or $u_1 = v_1$ and $\{u_2, v_2\} \in E(Y)$. For a positive integer b and a graph X, denote by bX the graph consisting of b vertex-disjoint copies of the graph X. Then the graph $X[\overline{K_b}] - bX$ is called the *deleted wreath (deleted lexicographic) product* of X and $\overline{K_b}$, where $\overline{K_b} = bK_1$.

Proposition 1 ([19, 23]). *If X is a connected arc-transitive circulant of order n, then one of the following holds:*

(i) $X \cong K_n$;
(ii) $X = \Sigma[\overline{K}_d]$, *where* $n = md$, $m, d > 1$ *and* Σ *is a connected arc-transitive circulant of order* m;
(iii) $X = \Sigma[\overline{K}_d] - d\Sigma$, *where* $n = md$, $d > 3$, $gcd(d, m) = 1$ *and* Σ *is a connected arc-transitive circulant of order* m;
(iv) X *is a normal circulant.*

Near-bipartite graphs are a natural generalization of bipartite graphs. A *near-bipartite* graph is a graph X in which there exists an independent set $I \subset V(X)$ of vertices, such that the induced graph $X[V(X) \setminus I]$ is bipartite. A *chromatic number* $\chi(X)$ of a graph X is the minimum number of colors needed to color the vertices of X in such a way that adjacent vertices have different colors. If the graph is near-bipartite, than we can color the vertices of this graph with three colors, since we can color the vertices in the independent set with one color, and for the remaining vertices two colors suffice. Therefore, near-bipartite graphs have chromatic number at most 3. Conversely, if a graph has chromatic number 3, then it is near-bipartite, since we can choose the vertices of one color to be the required independent set, and the remaining vertices are colored with two colors, which means that the remaining graph is bipartite.

The following result about chromatic numbers of tetravalent circulants will be needed in Sect. 4.

Proposition 2 ([16, Theorem 3.2.]). *Let $X = C_n(a, b) = Cay(\mathbb{Z}_n, \{\pm a, \pm b\})$ be a tetravalent circulant of order n. Then*

$$\chi(X) = \begin{cases} 2 & \text{if } a \text{ and } b \text{ are odd and } n \text{ is even} \\ 4 & \text{if } 3 \nmid n,\ n \neq 5, \text{ and } (b \equiv \pm 2a \pmod{n}) \text{ or } a \equiv \pm 2b \pmod{n} \\ 4 & \text{if } n = 13 \text{ and } (b \equiv \pm 5a \pmod{13} \text{ or } a \equiv \pm 5b \pmod{13}) \\ 5 & \text{if } n = 5 \\ 3 & \text{otherwise} \end{cases}.$$

Following [29] we say that, given a graph (or more generally a loopless multigraph) X, a subset S of $V(X)$ is *cyclically stable* if the induced subgraph $X[S]$ is acyclic, that is, a forest. The size $|S|$ of a maximum cyclically stable subset S of $V(X)$ is called the *cyclic stability number* of X.

Given a connected graph X, a subset $F \subseteq E(X)$ of edges of X is said to be *cycle-separating* if $X - F$ is disconnected and at least two of its components contain cycles. We say that X is *cyclically k-edge-connected* if no set of fewer than k edges is cycle-separating in X. Furthermore, the *edge-cyclic connectivity* $\zeta(X)$ of X is the largest integer k not exceeding the Betti number $|E(X)| - |V(X)| + 1$ of X for which X is cyclically k-edge-connected. (This distinction is indeed necessary as, for example, the theta graph Θ_2, K_4 and $K_{3,3}$ possess no cycle-separating sets of edges and are thus cyclically k-edge-connected for all k, however their edge-cyclic connectivities are 2, 3 and 4, respectively.)

Regarding the cyclic stability number, the following result may be deduced from [29, Théorème 5].

Proposition 3 ([29]). *Let X be a cyclically 4-edge-connected cubic graph of order n, and let S be a maximum cyclically stable subset of $V(X)$. Then $|S| = \lfloor (3n - 2)/4 \rfloor$ and more precisely, the following hold.*

(i) If $n \equiv 2 \pmod 4$, then $|S| = (3n-2)/4$, and $X[S]$ is a tree and $V(X) \setminus S$ is an independent set of vertices.
(ii) If $n \equiv 0 \pmod 4$, then $|S| = (3n-4)/4$, and either $X[S]$ is a tree and $V(X) \setminus S$ induces a graph with a single edge, or $X[S]$ has two components and $V(X) \setminus S$ is an independent set of vertices.

The following result concerning cyclic edge connectivity of cubic vertex-transitive graphs is proved in [27, Theorem 17].

Proposition 4 ([27]). *The cyclic edge connectivity $\zeta(X)$ of a cubic connected vertex-transitive graph X equals its girth $g(X)$.*

3 On Cayley (Non)Snarks

Any cubic Cayley graph is bridgeless, and so, by Petersen's theorem [33], it has a 1-factor and a 2-factor. In [20] the existence of 2-factors with long cycles in cubic graphs is investigated, whereas, in this paper we are after even 2-factors. Namely, a cubic graph containing such a 2-factor is not a snark for we can color the edges of an even 2-factor with two colors and the remaining edges with the third color.

For a cubic Cayley graph $\mathrm{Cay}(G, S)$ the generating set S is of two forms: either it consists of three involutions or it consists of an involution, a non-involution and its inverse. In the first case, $\mathrm{Cay}(G, S)$ is clearly 3-edge-colorable, and we may therefore restrict ourselves to the study of cubic Cayley graphs with respect to generating sets with a single involution. More precisely, we consider Cayley graphs X arising from groups having a $(2, s, t)$-generation, that is, from groups $G = \langle a, x \mid a^2 = x^s = (ax)^t = 1, \dots \rangle$ generated by an involution a and an element x of order $s \geq 3$ such that their product ax has order t. Here "…" denotes the extra relations needed in the presentation of the group. Such graphs are called *$(2, s, t)$-Cayley graphs*. If s is even, then a $(2, s, t)$-Cayley graph is not a snark, since in this case the set of edges $\{\{g, gx\} \mid g \in G\}$ obviously forms an even 2-factor in X. The following proposition thus holds.

Proposition 5. *A Cayley snark is a* $(2, s, t)$*-Cayley graph, where* s *is odd, not possessing an even* 2*-factor.*

As cubic graphs are of even order, by Proposition 5, the existence of a Hamilton cycle in a cubic Cayley graph implies that the graph is 3-edge-colorable, and thus a non-snark. In particular, Conjecture 1 is essentially a weaker form of the folklore conjecture that every connected Cayley graph with order greater than 2 possesses a Hamilton cycle, which is in fact a Cayley variation of Lovász's conjecture [24] that every connected vertex-transitive graph possesses a Hamilton path. These hamiltonicity questions have been challenging mathematicians for more than 40 years, but only partial results have been obtained thus far. Most results proved thus far depend on various restrictions made either on the class or the order of the group or the structure of the corresponding generating sets. For example, one may easily see that connected Cayley graphs on abelian groups have a Hamilton cycle. Further, it is known that the same holds for hamiltonian groups (see [2]), for metacyclic groups with respect to standard generating sets (see [1]), and for groups with a cyclic commutator subgroup of prime-power order (see [9,18,25,35]). This last result was generalized to connected vertex-transitive graphs admitting a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order, where the Petersen graph is the only exception (see [8]). In addition, every connected Cayley digraph on any p-group has a directed Hamilton cycle (see [36]). On the other hand, it is still not known whether Cayley graphs on dihedral groups have a Hamilton cycle. The best result in this respect is due to Alspach, Chen and Dean [4] who solved the problem for generalized dihedral groups of order divisible by 4. Furthermore, combining results from [10, 11, 22, 26] it follows that every connected Cayley graph on a group G has a Hamilton cycle if $|G| = kp$, where p is prime, $1 \leq k < 32$, and $k \neq 24$. These results show, among other, that every connected Cayley graph on a group of order $n < 120$, $n \neq 72$, is hamiltonian. Moreover, since every group of order 72 is solvable, these results and [30, Theorem 1.5.] combined together imply the following proposition.

Proposition 6 ([10,11,22,26,30]). *There are no Cayley snarks of order* $n < 120$.

For further results not explicitly mentioned or referred to here see the survey articles [7,21,37].

4 Constructing Even 2-Factors in $(2, s, t)$-Cayley Graphs

The methods used in this paper to construct even 2-factors in cubic Cayley graphs are a generalization of the methods introduced by Glover and Yang in [13]. These methods were later used in [12, 14, 15] where the hamiltonian problem for $(2, s, 3)$-Cayley graphs was considered.

Let $s \geq 3$ and $t \geq 3$ be positive integers and let $X = \mathrm{Cay}(G, \{a, x, x^{-1}\})$ be a $(2, s, t)$-Cayley graph on a group $G = \langle a, x \mid a^2 = x^s = (ax)^t = 1, \ldots \rangle$. The graph X is cubic and has a canonical Cayley map $\mathscr{M}(X)$ given by an embedding in the closed orientable surface of genus

$$g = 1 + |G| \cdot \left(\frac{1}{4} - \frac{1}{2s} - \frac{1}{2t} \right) \tag{1}$$

with $|G|/s$ disjoint s-gons and $|G|/t$ $2t$-gons as the corresponding faces. (For a detailed description of Cayley maps we refer the reader to [6, 17].) This map is given using the same rotation of the x, a, x^{-1} edges at every vertex and results in one s-gon and two $2t$-gons adjacent to each vertex. Generalizing the approach of [12] where the so called hexagon graph was associated with each $(2, s, 3)$-Cayley graph, a $2t$*-gonal graph* $O(X)$ is associated with our $(2, s, t)$-Cayley graph X in the following way. Its vertex set consists of all $2t$-gons in $\mathscr{M}(X)$ arising from the relation $(ax)^t = 1$, where two such $2t$-gons are adjacent if they share an edge in X. Observe that $O(X)$ may also be seen as the *orbital graph* of the left action of G on the set $\mathscr{C}$ of left cosets of the subgroup $C = \langle ax \rangle$, arising from the suborbit $\{aC, caC, c^2aC, c^3aC, \ldots c^{t-1}aC\}$ of length t, where $c = ax$. More precisely, $O(X)$ has vertex set $\mathscr{C}$, with adjacency defined as follows: a coset yC is adjacent to the t cosets yaC, $ycaC$, yc^2aC, ..., $yc^{t-2}aC$ and $yc^{t-1}aC$. Clearly, G acts 1-regularly on $O(X)$. Conversely, let Y be a connected arc-transitive graph of valency t admitting a 1-regular action of a subgroup G of $\mathrm{Aut}Y$. Let $v \in V(Y)$ and let h be a generator of $C = G_v \cong \mathbb{Z}_t$. Then there must exist an element $a \in G$ such that $G = \langle a, h \rangle$ and such that Y is isomorphic to the orbital graph of G relative to the suborbit $\{aC, haC, h^2aC, \ldots, h^{t-1}aC\}$. Moreover, a short computation shows that a may be chosen to be an involution, and letting $x = ah$ we get the desired generation for G. There is therefore a well-defined correspondence between these two classes of objects. This gives us the following result.

Proposition 7. *Let X be a $(G, 1)$-regular graph of valency t, $t \geq 3$, with the vertex stabilizer G_v isomorphic to $\mathbb{Z}_t$. Then X can be constructed via a Cayley graph on the group G with respect to its $(2, s, t)$-generation. In particular, X is isomorphic to a $2t$-gonal graph of a $(2, s, t)$-Cayley graph on the group G.*

The method for constructing an even 2-factor in X consists in identifying a subset T of vertices $V = V(O(X))$ inducing a bipartite subgraph in $O(X)$, the complement $V \setminus T$ of which is an independent set of vertices. (Of course the existence of such an even 2-factor is obvious in the case that s is even.) The bipartite subgraph $O(X)[T]$ gives rise to a "bipartite graph of faces" in X such that every vertex in the Cayley graph X lies on the boundary of at least one of the faces in this subgraph. (The concept of a "bipartite graph of faces" is defined in the obvious way, see the examples below.) Since all faces in this graph of faces are of even length (they are $2t$-gonal faces), its boundary is the desired even 2-factor in the $(2, s, t)$-Cayley graph X. In particular, this method works in case $O(X)$ is a near-bipartite graph (see Theorem 1).

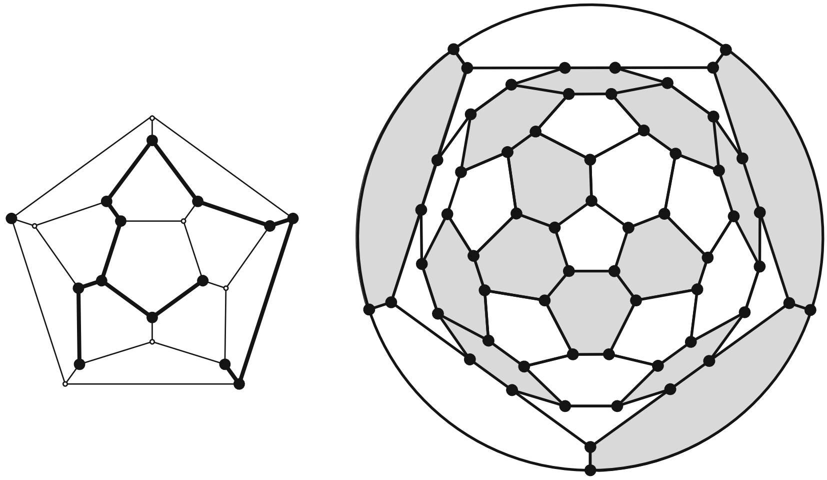

Fig. 1 A forest of faces in the spherical Cayley map $\mathscr{M}(X)$ of the $(2,5,3)$-Cayley graph X on A_5 giving rise to an even 2-factor in X, and the corresponding induced forest in the hexagon graph $Hex(X)$

Example 1. On the right-hand picture of Fig. 1 we show a forest of faces whose boundary is a 2-factor consisting of even cycles in the spherical Cayley map $\mathscr{M}(X)$ of the Cayley graph X on the $(2,5,3)$-generated group $A_5 = \langle a, x \mid a^2 = x^5 = (ax)^3 = 1\rangle$. The corresponding forest in $O(X)$ is shown on the left-hand picture of Fig. 1.

Example 2. On the left-hand picture of Fig. 2 we show a graph of faces whose boundary is an even 2-factor in the spherical Cayley map of the Cayley graph on the symmetric group S_4 with respect to its $(2,3,4)$-generation $\langle a, x \mid a^2 = x^3 = (ax)^4 = 1, \dots\rangle$, where $a = (13)$ and $x = (234)$. The corresponding bipartite induced subgraph of $O(X)$ is shown on the right-hand picture of Fig. 2.

Example 3. Let $G = \mathbb{Z}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3) = \langle x, y, f \mid x^3 = y^3 = f^4 = 1, x^y = x, x^f = y, y^f = x^{-1}\rangle$. Then $a := f^2xy$ and $b := fy$ give rise to a $(2,4,4)$-generation of G. Let X be a $(2,4,4)$-Cayley graph on G with respect to this $(2,4,4)$-generation. Then using (1) it can be seen that X has a toroidal Cayley map given by an embedding in the torus with 18 faces, 9 disjoint squares and 9 octagons. By Proposition 7 the corresponding 8-gonal graph $O(X)$ is an arc-transitive graph of order 9 admitting a 1-regular action of G with a vertex stabilizer isomorphic to $\mathbb{Z}_4$. Figure 3 shows that $O(X) \cong K_3[3K_1] - 3K_3$ is a near-bipartite graph.

Theorem 1. *Let $X = Cay(G, \{a, x, x^{-1}\})$ be a $(2,s,t)$-Cayley graph on a group $G = \langle a, x \mid a^2 = x^s = (ax)^t = 1, \dots\rangle$, $s, t \geq 3$ and let $O(X)$ be the corresponding $2t$-gonal graph of X. If $O(X)$ is a near-bipartite graph, then X*

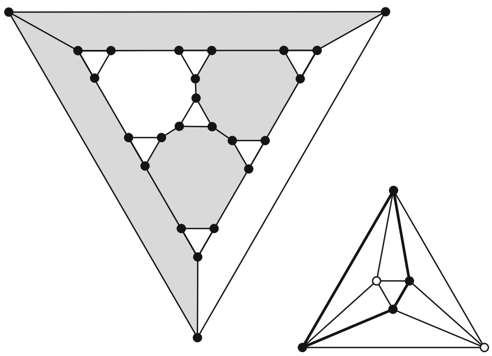

Fig. 2 The *left-hand picture* shows an even 2-factor in a $(2,3,4)$-Cayley graph $X = \mathrm{Cay}(S_4, \{a, x, x^{-1}\})$ on $S_4 = \langle a, x \mid a^2 = x^3 = (ax)^4 = 1, \ldots \rangle$, where $a = (13)$ and $x = (234)$. The *right-hand picture* shows the independent set of vertices in $O(X)$ whose complement is a bipartite graph giving the even 2-factor in X

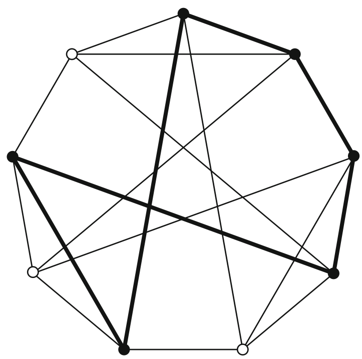

Fig. 3 An induced bipartite subgraph in $K_3[3K_1] - 3K_3$ whose complement is an independent set of three vertices. This graph is in fact the 8-gonal graph $O(X)$ of the $(2,4,4)$-Cayley graph X on $\mathbb{Z}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$

is not a snark. Moreover, if the vertex set V of $O(X)$ decomposes into $\{T, V \setminus T\}$ where T induces a tree and $V - T$ is an independent set of vertices, then X contains a Hamilton cycle.

Proof. Consider the canonical Cayley map $\mathscr{M}(X)$ of $X = \mathrm{Cay}(G, \{a, x, x^{-1}\})$ embedded with s-gonal and $2t$-gonal faces in a closed orientable surface with genus as given in (1). Suppose that the corresponding $2t$-gonal graph $O(X)$ is near-bipartite. Then the vertex set V of $O(X)$ decomposes into $\{T, V \setminus T\}$ where T induces a bipartite graph and $V \setminus T$ is an independent set of vertices. Each vertex of $O(X)$ corresponds to a $2t$-gonal face of $\mathscr{M}(X)$ as illustrated in the beginning of this section. Since every vertex of X belongs to two $2t$-gonal faces in $\mathscr{M}(X)$ and $V \setminus T$ is an independent set of vertices, we can conclude that every vertex of X belongs to at least one $2t$-gonal faces whose corresponding vertex of $O(X)$ is in T. The bipartite graph $O(X)[T]$ then translates into a graph of faces in $\mathscr{M}(X)$ whose boundary contains all the vertices of X. Since all faces in this graph of faces are of even length (they are $2t$-gonal faces), its boundary is an even 2-factor in X. By Proposition 5 it now follows that X is not a snark. Furthermore, if $O(X)[T]$ is a tree, then it translates into a tree of faces in $\mathscr{M}(X)$ containing all of the vertices of X and as a subspace of the Cayley map it is a topological disk. The boundary of this topological disk is a (simple) cycle passing through all vertices of the Cayley graph X, and so X is hamiltonian. $\square$

Theorem 2. *Let $X = Cay(G, \{a, x, x^{-1}\})$ be a $(2, s, 4)$-Cayley graph on a group $G = \langle a, x \mid a^2 = x^s = (ax)^4 = 1, \ldots \rangle$, $s \geq 3$, such that the corresponding 8-gonal graph $O(X)$ is a circulant. Then X is not a snark.*

Proof. By Proposition 7, the 8-gonal graph $O(X)$ is a tetravalent $(G, 1)$- regular circulant, and so Proposition 1 implies that $O(X)$ is either isomorphic to the complete graph K_5, or to $K_2[5K_1] - 5K_2 \cong K_{5,5} - 5K_2$, or to $K_2[4K_1] \cong K_{4,4}$, or to $C_n[2K_1]$, or it is a normal circulant. Since the order of $O(X)$ is equal to $|G|/4 = |V(X)|/4$, in the first three possibilities Proposition 6 implies that X is not a snark. In what follows we may therefore assume that $O(X)$ is of order ≥ 30 and we only need to consider two cases.

Case 1. $O(X) \cong C_n[2K_1]$.

We can clearly color the vertices of C_n with three colors (namely, C_n is near-bipartite). With the use of such a vertex coloring of C_n we can now color the graph $C_n[2K_1]$ in the following way. If a vertex v of C_n is colored with color i, then we color the two vertices corresponding to this vertex (the two vertices in the same $2K_1$ corresponding to v) with color i. This gives us a good 3-vertex coloring of $C_n[2K_1]$, implying that $O(X)$ is near bipartite, and therefore, by Theorem 1, X is not a snark.

Case 2. $O(X)$ is a normal circulant.

In this case $O(X) \cong \mathrm{Cay}(\mathbb{Z}_n, S)$ and the stabilizer of the vertex $0 \in \mathrm{Cay}(\mathbb{Z}_n, S)$ in the full automorphism group of $O(X)$ is isomorphic to $\mathrm{Aut}(\mathbb{Z}_n, S) \leq \mathrm{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. Since $O(X)$ is connected and $\langle S \rangle = \mathbb{Z}_n$ we may, without loss of generality, assume that $S = \{\pm 1, \pm 1^{\alpha}\}$ for some $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$. Since $O(X)$ is of order ≥ 30, Proposition 2 implies that $\chi(O(X)) \leq 3$. Therefore $O(X)$ is a near-bipartite graph and thus, by Theorem 1, X is not a snark. $\square$

5 Existence of Even 2-Factors in $(2, s, 3)$-Cayley Graphs

It was proved in [12] that a $(2, s, 3)$-Cayley graph on a group G has a Hamilton path when $|G|$ is congruent to 0 modulo 4, and has a Hamilton cycle when $|G|$ is congruent to 2 modulo 4. The Hamilton cycle was constructed, combining the theory of Cayley maps with classical results on cyclic stability in cubic graphs, as the contractible boundary of a "tree of faces" in the corresponding Cayley map.

Further, with a generalization of these methods, the existence of a Hamilton cycle in a $(2, s, 3)$-Cayley graph was proved in [14] when apart from $|G|$ also s is congruent to 0 modulo 4. More recently, with a further extension of the above "tree of faces" approach, a Hamilton cycle was shown to exist whenever $|G|$ is congruent to 0 modulo 4 and s is odd (see [15]). This leaves $|G|$ congruent to 0 modulo 4 with s congruent to 2 modulo 4 as the only remaining case in which the existence of a Hamilton cycle in $(2, s, 3)$-Cayley graphs has not yet been proven. In this last case, however, the "tree of faces" approach cannot be applied, and so entirely different techniques will have to be introduced if one is to complete the proof of the existence of Hamilton cycles in $(2, s, 3)$-Cayley graphs. These results combined together with Proposition 5 imply that there are no $(2, s, 3)$-Cayley snarks. For the sake of completeness, however, a self-contained proof of this fact is provided below. These demonstrate that it is somewhat easier to deal with the snark problem than the hamiltonian problem, bearing in mind of course that both problems are among the hardest problems in graph theory. The following proposition, which combined together with Proposition 3 shows that cyclically 4-edge-connected cubic graphs are near-bipartite, will be needed in this respect. In fact, the statement of this proposition really says that in part (ii) of Proposition 3, a particular one of the two possibilities for the set S may be chosen.

Proposition 8. *Let X be a cyclically 4-edge-connected cubic graph of order $n \equiv 0 \pmod 4$. Then there exists a cyclically stable subset S of $V(X)$ such that $X[S]$ is a forest and $V(X) \setminus S$ is an independent set of vertices.*

Proof. Let X be a cyclically 4-edge-connected cubic graph of order $n \equiv 0 \pmod 4$. By Proposition 3(ii) we may assume that there exists a maximum cyclically stable subset S of $V(X)$ such that $X[S]$ is a tree and $V(X) \setminus S$ induces a graph with a single edge, say $uv \in E(X)$. Consider the neighbors of the vertex $u \in V(X)$ in $X[S]$. Since $v \in N_X(u) \cap V(X) \setminus S$ is the only neighbor of u in $V(X) \setminus S$ it follows that $|N_{X[S]}(u)| = 2$. Let $N_{X[S]}(u) = \{u_i \mid i \in \{1, 2\}\}$. Then $|N_{X[S]}(u_i)| \leq 2$, $i \in \{1, 2\}$. Furthermore, if for some $i \in \{1, 2\}$ we have $|N_{X[S]}(u_i)| = 2$ then the set $\{u\} \cup S \setminus \{u_i\}$ induces a forest whose complement is an independent set of vertices. We may therefore assume that $|N_{X[S]}(u_i)| = 1$ for every $i \in \{1, 2\}$. Also, since $X[S]$ is connected there exists a path $P = u_1 w_1 w_2 \ldots w_k u_2$ between vertices u_1 and u_2 in $X[S]$. If there exists $j \in \{1, \ldots, k\}$ such that $|N_{X[S]}(w_j)| = 3$, then the set $\{u\} \cup S \setminus \{w_j\}$ induces a forest whose complement is an independent set of vertices. Hence, we may assume that $X[S] = P$. Now, repeating the argument for the neighbors $N_{X[S]}(v) = \{v_i \mid i \in \{1, 2\}\}$ of the vertex v in $X[S]$ it follows that we can restrict ourselves to the case that $|N_{X[S]}(v_i)| = 1$ for every $i \in \{1, 2\}$. But u_1 and u_2 are the only vertices of valency 1 in $X[S]$, and so $\{v_i \mid i \in \{1, 2\}\} = \{u_i \mid i \in \{1, 2\}\}$. It follows that uvu_iu, $i \in \{1, 2\}$, is a 3-cycle in X, a contradiction (since X is cyclically 4-edge-connected). □

Theorem 3. *There are no $(2, s, 3)$-Cayley snarks.*

Proof. Let X be a $(2, s, 3)$-Cayley graph on a group $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots \rangle$, $s \geq 3$, and let $O(X)$ be the corresponding 6-gonal graph of X.

By Proposition 7, $O(X)$ is a $(G, 1)$-regular graph. By Proposition 6 we may assume that $O(X)$ is of order $|G|/3 \geq 60$, and therefore [12, Proposition 3.4.] implies that $O(X)$ is of girth strictly bigger than 5. Proposition 4 implies that $O(X)$ is a cyclically 4-edge-connected graph, and so Propositions 3 and 8 combined together imply that $O(X)$ is a near-bipartite graph. Thus, by Theorem 1, X is not a snark. □

6 Further Research Directions

By Proposition 5 and Theorem 3, a Cayley snark, if it exists, is a $(2, s, t)$-Cayley graph on a group $G = \langle a, x \mid a^2 = x^s = (ax)^t = 1, \ldots \rangle$, where $s \geq 3$ is odd and $t > 3$, which does not have an even 2-factor, and, by Theorem 1, its corresponding $2t$-gonal graph, arising from the canonical Cayley map $\mathscr{M}(X)$ given by an embedding of it in the closed orientable surface with s-gonal and $2t$-gonal faces, is not near-bipartite. The converse is not true. In particular, there exist $(2, s, t)$-Cayley graphs which are not snarks but their $2t$-gonal graphs are not near-bipartite. For example, the $2t$-gonal graph arising from the spherical Cayley map of a $(2, 3, 3)$-Cayley graph X on the alternating group $A_4 = \langle a, x \mid a^2 = x^3 = (ax)^3 = 1 \rangle$, where $a = (12)(34)$ and $x = (123)$, is isomorphic to the complete graph K_4, and it is therefore not near-bipartite. By Proposition 6, however, X is not a snark.

In view of Proposition 7 the $2t$-gonal graph $O(X)$ of a $(2, s, t)$-Cayley graph X on a group $G = \langle a, x \mid a^2 = x^s = (ax)^t = 1, \ldots \rangle$, $s, t \geq 3$, is a $(G, 1)$-regular graph of valency t with the vertex stabilizer G_v isomorphic to $\mathbb{Z}_t$. It therefore seems that the thoughtful study of the structure of such graphs is in order if one is to make a progress in regards to Conjecture 1. We pose the following problem.

Problem 1. Let G be a finite group. Characterize non-near-bipartite t-valent $(G, 1)$-regular graphs with the vertex stabilizer G_v isomorphic to $\mathbb{Z}_t$.

Since a graph is near-bipartite if and only if its vertices can be colored with less than four colors, Problem 1 can be reformulated as follows.

Problem 2. Let G be a finite group. Characterize t-valent $(G, 1)$-regular graphs X with the vertex stabilizer G_v isomorphic to $\mathbb{Z}_t$ such that $\chi(X) \geq 4$.

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Local, Dimensional and Universal Rigidities: A Unified Gram Matrix Approach

A.Y. Alfakih

Abstract This chapter is a unified treatment, based on projected Gram matrices (PGMs), of the problems of local, dimensional, and universal rigidities of bar frameworks. This PGM-based approach makes these problems amenable to semidefinite programming methodology; and naturally gives rise to results expressed in terms of Gale matrices. We survey known results emphasizing numerical examples and proofs which highlight the salient aspects of this approach.

Keywords Bar-and-joint frameworks • Infinitesimal rigidity • Rigidity matrix • Dual rigidity matrix • Dimensional rigidity • Universal rigidity • Gram matrix and Gale transform

Subject Classifications: 52C25, 05C50, 15A57

1 Introduction

A *configuration* p in $\mathbb{R}^r$ is a finite collection of n labelled points $p^1, \ldots, p^n$. A *bar framework* (or simply a *framework*) in $\mathbb{R}^r$, denoted by $G(p)$, is a configuration p in $\mathbb{R}^r$ together with a simple graph G on the vertices $1, \ldots, n$. A framework $G(p)$ in $\mathbb{R}^r$ is said to be r*-dimensional* if the dimension of the affine span of $p^1, \ldots, p^n$ is equal to r. To avoid trivialities, we assume that graph G is connected and not complete. Figure 1 depicts three 2-dimensional frameworks in $\mathbb{R}^2$.

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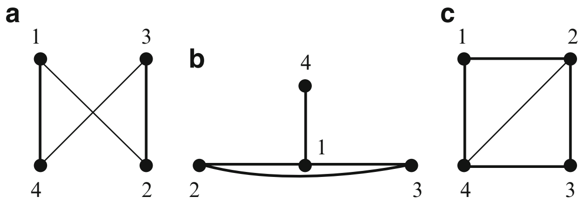

Fig. 1 An example of three 2-dimensional frameworks in $\mathbb{R}^2$. In framework (**b**), the edge $(2, 3)$ is shown as an arc to make the edges $(2, 1)$ and $(1, 3)$ visible

Two frameworks $G(p)$ and $G(q)$ in $\mathbb{R}^r$ are said to be *congruent* if $||q^i - q^j|| = ||p^i - p^j||$ for all $i, j = 1, \ldots, n$, where $||.||$ denotes the Euclidean norm. Further, two frameworks $G(p)$ in $\mathbb{R}^r$ and $G(q)$ in $\mathbb{R}^s$ are said to be *equivalent* if $||q^i - q^j|| = ||p^i - p^j||$ for all $(i, j) \in E(G)$, where $E(G)$ denotes the edge set of graph G.

Each configuration p defines an $n \times n$ matrix $D_p = (d_{ij}) = (||p^i - p^j||^2)$. D_p is called the *Euclidean distance matrix* generated by p. Let H denote the adjacency matrix of graph G, then two frameworks $G(p)$ and $G(q)$ are congruent iff $D_p = D_q$; and they are equivalent iff $H \circ D_p = H \circ D_q$, where $\circ$ denotes the *Hadamard product*, i.e., the element-wise product.

An r-dimensional framework $G(p)$ in $\mathbb{R}^r$ is said to be *locally rigid* if for some $\epsilon > 0$, there does not exist any r-dimensional framework $G(q)$ in $\mathbb{R}^r$ such that: (i) $||q^i - p^i|| \le \epsilon$ for all $i = 1, \ldots, n$; and (ii) $G(q)$ is equivalent, but not congruent, to $G(p)$. For instance, in Fig. 1, only framework (c) is locally rigid.

We say that an r-dimensional framework $G(p)$ in $\mathbb{R}^r$ is *dimensionally rigid* if there does not exist an r'-dimensional framework $G(q)$ in $\mathbb{R}^{r'}$ that is equivalent to $G(p)$, for any $r' \ge r + 1$. For example, in Fig. 1, framework (b) is dimensionally rigid since it has no equivalent frameworks in $\mathbb{R}^3$. On the other hand, frameworks (a) and (c) are not dimensionally rigid since each has an infinite number of equivalent frameworks in $\mathbb{R}^3$.

An r-dimensional framework $G(p)$ in $\mathbb{R}^r$ is said to be *universally rigid* if every r'-dimensional framework $G(q)$ in $\mathbb{R}^{r'}$, for any r', that is equivalent to $G(p)$, is in fact congruent to $G(p)$. None of the frameworks in Fig. 1 is universally rigid. If a framework is not (locally, dimensionally, universally) rigid, we say that it is (locally, dimensionally, universally) *flexible*.

Finally, a framework $G(p)$ is *generic* if the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the integers. That is, if there does not exist a non-zero polynomial f with integer coefficients such that $f(p^1, \ldots, p^n) = 0$.

The remainder of this chapter is organized as follows. Section 2 is devoted to mathematical preliminaries. Section 3 discusses local and infinitesimal rigidities and introduces the dual rigidity matrix. Dimensional rigidity is the subject of Sect. 4, while universal rigidity is discussed in Sect. 5.

The related notion of global rigidity [14,18] will not be considered here as it falls outside the scope of this chapter.

1.1 Notation

We denote the edge set and the adjacency matrix of a simple graph G by E(G) and H respectively. $||.||$ denotes the Euclidean norm. $\mathscr{S}_n$ denotes the space of $n \times n$ real symmetric matrices. Positive semi-definiteness (positive definiteness) of a symmetric matrix A is denoted by $A \succeq \mathbf{0}$ ($A \succ \mathbf{0}$). We denote by $\text{diag}(A)$ the vector formed from the diagonal entries of a matrix A. The vector of all 1's in $\mathbb{R}^n$ is denoted by e. The Hadamard product of two matrices A and B is denoted by $A \circ B$. The $n \times n$ identity matrix is denoted by I_n; and $\mathbf{0}$ denotes the zero matrix or the zero vector of the appropriate dimension. E^{ij} denotes the $n \times n$ symmetric matrix with 1's in the ijth and jith entries and 0's elsewhere. Finally, throughout this chapter, $G(p)$ is a bar framework in $\mathbb{R}^r$ with n nodes and m edges. The number of missing edges of G is denoted by $\bar{m} = n(n-1)/2 - m$.

2 Preliminaries

An $n \times n$ real matrix A is said to be *positive semidefinite* if $x^T Ax \geq 0$ for all x in $\mathbb{R}^n$; and it is said to be *positive definite* if $x^T Ax > 0$ for all x in $\mathbb{R}^n$, $x \neq \mathbf{0}$. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be $n \times n$ matrices. Then $C = A \circ B$, i.e., C is the Hadamard product of A and B, iff $c_{ij} = a_{ij} b_{ij}$ for all $i, j = 1, \ldots, n$. The left null space of a matrix A is the null space of A^T.

2.1 Projected Gram Matrices and Euclidean Distance Matrices

In this section we present a one-to-one correspondence between the set of Euclidean distance matrices of order n and the set of projected Gram matrices of order $n-1$. This correspondence enables us to obtain a characterization of equivalent frameworks in terms of projected Gram matrices. To this end, let $G(p)$ be a given r-dimensional framework in $\mathbb{R}^r$, then the $n \times r$ matrix

$$P = \begin{bmatrix} (p^1)^T \\ (p^2)^T \\ \vdots \\ (p^n)^T \end{bmatrix} \tag{1}$$

is called the *configuration matrix* of $G(p)$. Thus the Gram matrix of configuration p is given by $B = PP^T$. Note that B is an $n \times n$ symmetric positive semidefinite matrix with rank r. Also note that B is invariant under orthogonal transformations. In order to make B invariant under translations, we impose the condition

$$Be = \mathbf{0}, \tag{2}$$

where e is the vector of all 1's. This condition is equivalent to setting the origin at the centroid of the points $p^1, \dots, p^n$ since $Be = \mathbf{0}$ iff $P^T e = \mathbf{0}$. Let V be an $n \times (n-1)$ matrix such that

$$V^T e = \mathbf{0}\,, \quad V^T V = I_{n-1}\,. \tag{3}$$

Then Condition (2) is also equivalent to

$$B = VXV^T, \tag{4}$$

where X is $(n-1) \times (n-1)$, $X \succeq \mathbf{0}$ and rank $X = r$. Furthermore,

$$X = V^T B V = V^T P P^T V. \tag{5}$$

Accordingly, X is called the *projected Gram matrix (PGM)* of framework $G(p)$ since VV^T is the orthogonal projection on $e^{\perp}$, the orthogonal complement of e in $\mathbb{R}^n$.

Since PGMs are invariant under rigid motions, all congruent frameworks have the same PGM. As a result, we will identify congruent frameworks. Given an r-dimensional framework $G(p)$ in $\mathbb{R}^r$, its PGM X is given by Eq. (5). On the other hand, given a PGM X of rank r, then its configuration matrix P can be recovered by factorizing $B = VXV^T = PP^T$. Moreover, P is unique since we don't distinguish between congruent frameworks. Our approach is based on representing a configuration p by its corresponding PGM X.

Let $D = (d_{ij})$ be the Euclidean distance matrix generated by $G(p)$. Then

$$\begin{aligned} d_{ij} &= ||p^i - p^j||^2, \\ &= (p^i)^T p^i + (p^j)^T p^j - 2\,(p^i)^T p^j, \\ &= (PP^T)_{ii} + (PP^T)_{jj} - 2\,(PP^T)_{ij}. \end{aligned}$$

Therefore,

$$D = \text{diag}(B)e^T + e(\text{diag}(B))^T - 2B,$$

where B is the Gram matrix of $G(p)$.

Let $\mathscr{S}_H = \{A \in \mathscr{S}_n : \text{diag}(A) = \mathbf{0}\}$ and define the linear transformations $\mathscr{K}_V : \mathscr{S}_{n-1} \to \mathscr{S}_H$ and $\mathscr{T}_V : \mathscr{S}_H \to \mathscr{S}_{n-1}$ such that

$$\mathscr{K}_V(A) := \text{diag}(VAV^T)e^T + e(\text{diag}(VAV^T))^T - 2VAV^T, \tag{6}$$

and

$$\mathscr{T}_V(A) := -\frac{1}{2} V^T A V. \tag{7}$$

The transformations $\mathscr{K}_V$ and $\mathscr{T}_V$ are mutually inverse [5]. Thus, for an r-dimensional framework $G(p)$ in $\mathbb{R}^r$ with PGM X_p we have

$$D_p = \mathscr{K}_V(X_p) \text{ and } X_p = \mathscr{T}_V(D_p) \text{ with rank } X_p = r. \tag{8}$$

Now let $G(q)$ in $\mathbb{R}^s$ be an s-dimensional framework equivalent to $G(p)$ and let D_q be the Euclidean distance matrix generated by $G(q)$. Then $H \circ D_q = H \circ D_p$ where H is the adjacency matrix of graph G. Thus,

$$H \circ (D_q - D_p) = H \circ \mathscr{K}_V(X_q - X_p) = \mathbf{0}, \tag{9}$$

where X_q is the PGM of $G(q)$.

Let E^{ij} be the $n \times n$ symmetric matrix with 1's in the ijth and jith entries and zeros elsewhere. Further, let

$$M^{ij} := \mathscr{T}_V(E^{ij}) = -\frac{1}{2} V^T E^{ij} V. \tag{10}$$

Then one can easily show that the set $\{M^{ij} : i \neq j,\ (i,j) \notin E(G)\}$ forms a basis for the kernel of $H \circ \mathscr{K}_V$. Hence, it follows from (9) that

$$X_q - X_p = \sum_{ij : i \neq j, (i,j) \notin E(G)} \hat{y}_{ij} M^{ij},$$

for some scalars $\hat{y}_{ij}$. Thus,

$$X_q = X_p + \sum_{ij : i \neq j, (i,j) \notin E(G)} \hat{y}_{ij} M^{ij} \succeq \mathbf{0}; \tag{11}$$

and rank $X_q = s$, where s is the dimension of framework $G(q)$. Equation (11) expresses the fact that $||q^i - q^j|| \neq ||p^i - p^j||$ only if $i \neq j, (i,j) \notin E(G)$. More precisely, it follows from (8) and (10) that

$$||q^i - q^j||^2 = \hat{y}_{ij} + ||p^i - p^j||^2 \text{ for all } ij : i \neq j, (i,j) \notin E(G).$$

Let us define

$$X(y) := X_p + \mathscr{M}(y), \text{ where } \mathscr{M}(y) := \sum_{ij : i \neq j, (i,j) \notin E(G)} y_{ij} M^{ij}. \tag{12}$$

Then

$$\{X(y) : X(y) \succeq \mathbf{0}\} \tag{13}$$

is the set of PGMs of all s-dimensional frameworks in $\mathbb{R}^s$ that are equivalent to $G(p)$, where $s = \text{rank } X(y)$. The next theorem is an immediate consequence of (12).

Theorem 1. *Let $G(p)$ be an $(n-1)$-dimensional bar framework on n vertices in $\mathbb{R}^{n-1}$ and assume that G is not the complete graph. Then $G(p)$ is locally flexible.*

Proof. Since $G(p)$ is $(n-1)$-dimensional we have $X_p \succ \mathbf{0}$. The result follows since for a sufficiently small $\epsilon > 0$, $X(y) = X_p + \mathscr{M}(y) \succ \mathbf{0}$ for all y such that $||y|| \leq \epsilon$.

Let

$$X_p = [W \quad U] \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} W^T \\ U^T \end{bmatrix} = Q \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q^T \tag{14}$$

be the spectral decomposition of X_p, where $Q = [W \quad U]$ is orthogonal and Λ is the $r \times r$ diagonal matrix consisting of the positive eigenvalues of X_p. Here, W is $(n-1) \times r$ and U is $(n-1) \times (n-1-r)$ where the columns of U form an orthonormal basis for the null space of X_p.

The following lemma plays a major role in our approach. It will be used in subsequent sections as the starting point for our discussion of local, dimensional and universal rigidities.

Lemma 1. *Let $G(p)$ be a given r-dimensional framework in $\mathbb{R}^r$ and let $X(y)$ and $Q = [W \quad U]$ be as defined in (12) and (14) respectively. Then*

$$\left\{ X(y) : Q^T X(y)\, Q = \begin{bmatrix} \Lambda + W^T \mathscr{M}(y)\, W & W^T \mathscr{M}(y)\, U \\ U^T \mathscr{M}(y)\, W & U^T \mathscr{M}(y)\, U \end{bmatrix} \succeq \mathbf{0} \right\}, \tag{15}$$

is the set of PGMs of all s-dimensional frameworks in $\mathbb{R}^s$ that are equivalent to $G(p)$, where

$$s = \; rank \left(\begin{bmatrix} \Lambda + W^T \mathscr{M}(y)\, W & W^T \mathscr{M}(y)\, U \\ U^T \mathscr{M}(y)\, W & U^T \mathscr{M}(y)\, U \end{bmatrix} \right).$$

Lemma 1 follows since $X(y) \succeq \mathbf{0}$ if and only if $Q^T X(y)\, Q \succeq \mathbf{0}$, and since rank $Q^T X(y)\, Q = \text{rank } X(y)$.

2.2 Gale Matrices and Stress Matrices

In this section we show how Gale matrices arise naturally when configurations are represented by their PGMs. We also establish the connection between Gale matrices and stress matrices.

Let $G(p)$ be an r-dimensional framework on n vertices in $\mathbb{R}^r$, $r \leq n-2$, and let P be the configuration matrix of $G(p)$. Then the following $(r+1) \times n$ matrix

$$\mathscr{P} := \begin{bmatrix} P^T \\ e^T \end{bmatrix} = \begin{bmatrix} p^1 & \dots & p^n \\ 1 & \dots & 1 \end{bmatrix} \tag{16}$$

has full row rank since $p^1, \dots, p^n$ affinely span $\mathbb{R}^r$.

Definition 1. Let $\bar{r}$ denote the dimension of the null space of matrix $\mathscr{P}$ defined in (16), that is,

$$\bar{r} = n - 1 - r. \tag{17}$$

Definition 2. Assume that the null space of $\mathscr{P}$ is nontrivial, i.e., $\bar{r} \geq 1$. Then any $n \times \bar{r}$ matrix Z whose columns form a basis of the null space of $\mathscr{P}$ is called a *Gale matrix* of configuration p (or framework $G(p)$). Furthermore, the ith row of Z, considered as a vector in $\mathbb{R}^{\bar{r}}$, is called a *Gale transform* of p^i [16].

Gale transform is widely used in the theory of polytopes [21]. Note that if Z and Z' are two Gale matrices of $G(p)$, then $Z' = ZQ'$ for some $\bar{r} \times \bar{r}$ nonsingular matrix Q'. Hence, without loss of generality we can assume that the top $\bar{r} \times \bar{r}$ submatrix of Z is the identity matrix $I_{\bar{r}}$. As a result, Z is sparse.

An *equilibrium stress* of a framework $G(p)$ is a real-valued function ω on $E(G)$ such that:

$$\sum_{j:(i,j)\in E(G)} \omega_{ij}(p^i - p^j) = \mathbf{0} \text{ for all } i = 1, \dots, n. \tag{18}$$

Let $\omega = (\omega_{ij})$ be an equilibrium stress for $G(p)$. Define the following $n \times n$ symmetric matrix $S = (s_{ij})$ where

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } i \neq j, (i,j) \in E(G), \\ 0 & \text{if } i \neq j, (i,j) \notin E(G), \\ \sum_{k:(i,k)\in E(G)} \omega_{ik} & \text{if } i = j. \end{cases} \tag{19}$$

S is called a *stress matrix* of $G(p)$. Note that (16) and (19) imply that S is a stress matrix of $G(p)$ if and only if

$$\mathscr{P} S = \mathbf{0}, \text{ and } s_{ij} = 0 \text{ for all } ij : i \neq j, (i,j) \notin E(G). \tag{20}$$

As the following lemma shows, Gale matrices and stress matrices are closely related.

Lemma 2 (Alfakih [3]). *Let Z and S be, respectively, a Gale matrix and a stress matrix of an r-dimensional framework $G(p)$ in $\mathbb{R}^r$. Then there exists an $\bar{r} \times \bar{r}$ symmetric matrix Ψ such that*

$$S = Z \Psi Z^T. \tag{21}$$

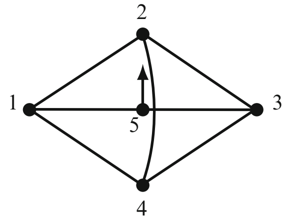

Fig. 2 A framework in $\mathbb{R}^2$ which is infinitesimally flexible and locally rigid. The edge (2,4) is shown as an arc in order to make it visible that nodes 5 and 2, and nodes 5 and 4 are nonadjacent. A non-trivial infinitesimal flex is $\delta^1 = \delta^2 = \delta^3 = \delta^4 = (0\ ,\ 0)^T$, $\delta^5 = (0\ ,\ 1)^T$

On the other hand, let Ψ' be any $\bar{r} \times \bar{r}$ symmetric matrix such that $(z^i)^T \Psi' z^j = 0$ for all $ij : i \neq j, (i,j) \notin E(G)$, where $(z^i)^T$ denotes the ith row of Z. Then $S' = Z\Psi' Z^T$ is a stress matrix of $G(p)$.

For later use, observe that $S = Z\Psi Z^T \succeq \mathbf{0}$, rank $S = \bar{r}$ if and only if $\Psi \succ \mathbf{0}$.

Example 1. Consider the framework $G(p)$ in Fig. 2. $G(p)$ has configuration matrix

$$P = \begin{bmatrix} -1.5 & 0 \\ 0 & 1 \\ 1.5 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \text{ and a Gale matrix } Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{bmatrix}.$$

Moreover, the missing edges of G are $(1,3)$, $(2,5)$ and $(4,5)$. It is easy to show that

$$\omega = (\omega_{12} = 1,\ \omega_{14} = 1,\ \omega_{15} = -2,\ \omega_{23} = 1,\ \omega_{24} = -1,\ \omega_{34} = 1,\ \omega_{35} = -2)$$

is an equilibrium stress of $G(p)$, and that the corresponding stress matrix S of $G(p)$ can be written as

$$S = \begin{bmatrix} 0 & -1 & 0 & -1 & 2 \\ -1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 2 \\ -1 & 1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & -4 \end{bmatrix} = Z \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} Z^T.$$

The following lemma shows that Gale matrices are related to the null space of projected Gram matrices. As a result, Gale matrices arise naturally in our approach.

Lemma 3. *Let P be the configuration matrix of an r-dimensional framework $G(p)$ in $\mathbb{R}^r$ and let the matrices V, U and W be as defined in (3) and (14). Then*

1. VU *is a Gale matrix of* $G(p)$*,*
2. $VW = PQ'$ *for some* $r \times r$ *non-singular matrix* Q'*.*

See [4] for a proof of Lemma 3.

3 Local and Infinitesimal Rigidities

Our approach leads to a new rigidity matrix $\bar{R}$, called the *dual rigidity matrix*, which is different from, but carries the same information as, the "usual" rigidity matrix R. Our presentation here follows closely that in [2]. We begin by recalling how R is derived.

Let $G(p)$ be a given r-dimensional framework in $\mathbb{R}^r$ and let $G(q)$ be an r-dimensional framework in $\mathbb{R}^r$ that is equivalent to $G(p)$, where $q^i = p^i + \delta^i$ for $i = 1, \ldots, n$. Then

$$||p^i + \delta^i - p^j - \delta^j||^2 - ||p^i - p^j||^2 = 0 \text{ for all } (i, j) \in E(G). \qquad (22)$$

Hence, the linear (in δ^i's) term of Eq. (22) is given by

$$2(p^i - p^j)^T \delta^i + 2(p^j - p^i)^T \delta^j = 0 \text{ for all } (i, j) \in E(G). \qquad (23)$$

Any $\delta = [(\delta^1)^T \ (\delta^2)^T \ \ldots \ (\delta^n)^T]^T \in \mathbb{R}^{nr}$ that satisfies Eq. (23) is called an *infinitesimal flex* of $G(p)$. The *rigidity matrix* R of an r-dimensional framework $G(p)$ in $\mathbb{R}^r$, with n vertices and m edges, is the $m \times nr$ matrix whose rows and columns are indexed, respectively, by the edges and the vertices of G, where the (i, j)th row is given by

$$[\ 0 \ldots 0 \ \overbrace{(p^i - p^j)^T}^{\text{vertex } i} \ 0 \ldots 0 \ \overbrace{(p^j - p^i)^T}^{\text{vertex } j} \ 0 \ldots 0 \ ]. \qquad (24)$$

Obviously, δ is an infinitesimal flex of framework $G(p)$ if and only if δ is in the null space of its rigidity matrix R. Also, ω is an equilibrium stress of $G(p)$ if and only if ω is in the left null space of R.

An infinitesimal flex is said to be *trivial* if it results from a rigid motion. The dimension of the space of trivial flexes is $r(r+1)/2$. A framework $G(p)$ is said to be *infinitesimally rigid* if it has only trivial infinitesimal flexes. Otherwise, $G(p)$ is said to be *infinitesimally flexible* [11, 12, 15, 20, 22]. The notion of infinitesimal rigidity is stronger than that of local rigidity.

Theorem 2 (Gluck [17]). *If a framework* $G(p)$ *is infinitesimally rigid then it is locally rigid.*

The converse of Theorem 2 is false. Figure 2 depicts a framework which is infinitesimally flexible and locally rigid.

It is well known [10, 17] that local rigidity is a generic property, i.e., if an r-dimensional generic framework $G(p)$ in $\mathbb{R}^r$ is locally rigid, then all r-dimensional generic frameworks $G(q)$ in $\mathbb{R}^r$ are also locally rigid. Furthermore, the notions of local rigidity and infinitesimal rigidity coincide for generic frameworks.

Theorem 3 (Asimow and Roth [9]). *Let $G(p)$ be a generic r-dimensional framework on n vertices in $\mathbb{R}^r$ and let R be its rigidity matrix. Then $G(p)$ is locally rigid if and only if*

$$rank\ R = nr - \frac{r(r+1)}{2}. \tag{25}$$

3.1 The Dual Rigidity Matrix

The following well-known lemma [23], which follows from Schur complement, will be needed later.

Lemma 4. *Let*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

be a partitioned real symmetric matrix, where A is an $r \times r$ positive definite matrix. Then M is positive semi-definite with rank r if and only if $C - B^T A^{-1} B = \mathbf{0}$.

Our starting point in deriving the dual rigidity matrix is Lemma 1. Let X_p be the PGM of an r-dimensional framework $G(p)$ in $\mathbb{R}^r$. Then $X(y) = X_p + \mathscr{M}(y)$ is the PGM of a framework in $\mathbb{R}^r$ that is equivalent to $G(p)$, if and only if the matrix

$$\begin{bmatrix} \Lambda + W^T \mathscr{M}(y) W & W^T \mathscr{M}(y) U \\ U^T \mathscr{M}(y) W & U^T \mathscr{M}(y) U \end{bmatrix}, \tag{26}$$

defined in Lemma 1, is positive semidefinite with rank r.

Now, for a sufficiently small $\epsilon > 0$, the matrix $\Lambda + W^T \mathscr{M}(y) W$ is positive definite for all y such that $||y|| \leq \epsilon$. Therefore, it follows from Lemma 4 that for all y such that $||y|| \leq \epsilon$, the matrix in (26) is positive semidefinite with rank r, if and only if

$$\Phi(y) := U^T \mathscr{M}(y) U - \ U^T \mathscr{M}(y) W \ (\Lambda + W^T \mathscr{M}(y) W)^{-1} \ W^T \mathscr{M}(y) U = \mathbf{0}. \tag{27}$$

Thus

$$\{X(y) = X_p + \mathscr{M}(y) : \Phi(y) = \mathbf{0}\}$$

is the set of PGM of all r-dimensional frameworks in $\mathbb{R}^r$ that are: (i) equivalent, but not congruent, to $G(p)$, (ii) arbitrarily close to $G(p)$.

The linear (in y) term of $\Phi(y)$ is given by

$$U^T \mathcal{M}(y) U = \mathbf{0}. \tag{28}$$

Therefore, framework $G(p)$ is infinitesimally flexible if and only if there exists a non-zero y satisfying (28). Next we express Eq. (28) in terms of a Gale matrix of $G(p)$.

Theorem 4 (Alfakih [2]). *Let Z be a Gale matrix of an r-dimensional framework $G(p)$ in $\mathbb{R}^r$. Then $G(p)$ is infinitesimally flexible if and only if there exists a non-zero $y = (y_{ij})$ in $\mathbb{R}^{\bar{m}}$ such that*

$$Z^T \mathcal{E}(y) Z = \mathbf{0}, \text{ where } \mathcal{E}(y) = \sum_{ij: i \neq j, (i,j) \notin E(G)} y_{ij} E^{ij}, \tag{29}$$

and where E^{ij} is the symmetric matrix of order n with 1's in the ijth and the jith entries and zeros elsewhere.

Proof. This follows from Lemma 3 and Eqs. (10) and (12).

Example 2. Let $G(p)$ be the framework in Fig. 2 considered in Example 1. The missing edges of G are $(1,3)$, $(2,5)$ and $(4,5)$ and a Gale matrix Z of $G(p)$ was given in Example 1. Then

$$\begin{aligned} Z^T \mathcal{E}(y) Z &= y_{13}\, Z^T E^{13} Z + y_{25}\, Z^T E^{25} Z + y_{45}\, Z^T E^{45} Z \\ &= \begin{bmatrix} 2y_{13} & -2y_{25} - 2y_{45} \\ -2y_{25} - 2y_{45} & -4y_{25} - 4y_{45} \end{bmatrix} = \mathbf{0} \end{aligned}$$

has the non-zero solution $y_{13} = 0$, $y_{25} = -1$ and $y_{45} = 1$. Thus the framework $G(p)$ in Fig. 2 is infinitesimally flexible.

Equation (29) can be written as the system of linear equations $\bar{R} y = \mathbf{0}$. In the spirit of (23), $\bar{R}$ is called the *dual rigidity matrix*. Next, we show how to construct $\bar{R}$ in terms of Gale matrix Z using the symmetric Kronecker product. We start, first, with some necessary definitions.

Given an $n \times n$ symmetric matrix A, let svec(A) denote the $\frac{n(n+1)}{2}$-vector formed by stacking the columns of A from the main diagonal downwards after having multiplied the off-diagonal entries of A by $\sqrt{2}$. For example, if A is a 3×3 matrix, then

$$\text{svec}(A) = \begin{bmatrix} a_{11} \\ \sqrt{2}\, a_{21} \\ \sqrt{2}\, a_{31} \\ a_{22} \\ \sqrt{2}\, a_{32} \\ a_{33} \end{bmatrix}. \tag{30}$$

Let B be an $m \times n$ matrix and let A be an $n \times n$ symmetric matrix. Then the *symmetric Kronecker product* between B and itself, denoted by $B \otimes_s B$, is defined such that

$$(B \otimes_s B)\ \mathrm{svec}(A) = \mathrm{svec}(BAB^T). \tag{31}$$

For more details on the symmetric Kronecker product see [8].

Definition 3. Let Z be a Gale matrix of an r-dimensional framework $G(p)$ in $\mathbb{R}^r$. Then $\bar{R}^T$, the transpose of the dual rigidity matrix of $G(p)$, is the sub-matrix of $Z \otimes_s Z$ obtained by keeping only rows that correspond to missing edges of G.

Recall that $\bar{m}$ denotes the number of missing edges of G and $\bar{r} = n - 1 - r$. Also recall that $z^{i\,T}$ denotes the ith row of Z. Then the dual rigidity matrix $\bar{R}$ of $G(p)$ is the $\frac{\bar{r}(\bar{r}+1)}{2} \times \bar{m}$ matrix whose columns are indexed by the missing edges of G, where the (i,j)th column is equal to $\frac{1}{\sqrt{2}}\ \mathrm{svec}(z^i z^{j\,T} + z^j z^{i\,T})$. For example, if the missing edges of G are $(i_1, j_1), (i_2, j_2), \ldots, (i_{\bar{m}}, j_{\bar{m}})$, then

$$\bar{R} = \frac{1}{\sqrt{2}} \left[\mathrm{svec}\left(z^{i_1} z^{j_1\,T} + z^{j_1} z^{i_1\,T}\right)\ \ldots\ \mathrm{svec}\left(z^{i_{\bar{m}}} z^{j_{\bar{m}}\,T} + z^{j_{\bar{m}}} z^{i_{\bar{m}}\,T}\right)\right]. \tag{32}$$

Note that the dual rigidity matrix $\bar{R}$ is usually sparse since, wlog, the top $\bar{r} \times \bar{r}$ sub-matrix of Z can be chosen to be the identity matrix $I_{\bar{r}}$.

The following theorem, which is the analogue of Theorem 3, characterizes infinitesimal rigidity in terms of $\bar{R}$.

Theorem 5 (Alfakih [2]). *Let $\bar{R}$ be the dual rigidity matrix of an r-dimensional framework $G(p)$ in $\mathbb{R}^r$. Then $G(p)$ is infinitesimally rigid if and only if $\bar{R}$ has a trivial null space, i.e., if and only if*

$$\mathit{rank}\ \bar{R} = \bar{m}. \tag{33}$$

It is worth remarking that the rank of $\bar{R}$ would not change if the factors of $\sqrt{2}$ are dropped from the definition of $\bar{R}$ in (32). These factors are kept in order to make the definition of $\bar{R}$ in terms of the symmetric Kronecker product simple.

Example 3. The framework $G(p)$ in Fig. 2 considered earlier in Examples 1 and 2 has missing edges $(1,3)$, $(2,5)$ and $(4,5)$; and a Gale matrix

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{bmatrix}.$$

Thus $\bar{m} = 3$ and $\bar{r} = 2$. Therefore, the dual rigidity matrix of $G(p)$ is

$$\bar{R} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2\sqrt{2} & -2\sqrt{2} \end{bmatrix}.$$

Note that the rigidity matrix R of $G(p)$ is 7×10. Also note that $y = (0, -1, 1)^T$ is a basis of the null space of $\bar{R}$, and $x = (0, -\sqrt{2}, 1)^T$ is a basis of the left null space of $\bar{R}$.

3.2 Similarities and Differences Between R and $\bar{R}$

There are two main differences between the dual rigidity matrix $\bar{R}$ and the rigidity matrix R. First, $\bar{R}$ is $\bar{r}(\bar{r}+1)/2$-by-$\bar{m}$ while R is m-by-nr. Second, $\bar{R}$ is invariant under rigid motions while R is not. Hence, in Eq. (33), there is no need to account for the trivial flexes as was the case in (25).

R and $\bar{R}$ are similar in the sense that the null space and the left null space of $\bar{R}$ are closely related to those of R. Recall that the infinitesimal flexes of $G(p)$ belong to the null space of R, and the equilibrium stresses of $G(p)$ belong to the left null space of R.

Theorem 6 (Alfakih [2]). *Let R and $\bar{R}$ be, respectively, the rigidity and the dual rigidity matrices of an r-dimensional framework $G(p)$ in $\mathbb{R}^r$. Then:*

1. *The left null space of $\bar{R}$ is isomorphic to the left null space of R.*
2. *The dimension of the null space of $\bar{R}$ = the dimension of the null space of $R - \dfrac{r(r+1)}{2}$.*

At the heart of the isomorphism between the left null spaces of R and $\bar{R}$ is the fact that $S = Z\Psi Z^T$ is a stress matrix of $G(p)$ if and only if svec(Ψ) is in the left null space of $\bar{R}$. This fact is a simple consequence of Lemma 2 and the definition of $\bar{R}$.

Example 4. It is easy to verify that in Examples 1 and 3

$$\text{svec}(\Psi) = \text{svec}\left(\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

is in the left null space of $\bar{R}$.

4 Dimensional Rigidity

We will need the following well-known Farkas lemma on the cone of positive semidefinite matrices in the proof of Theorem 7 below.

Lemma 5. *Let $A^1, \ldots, A^k$ be given $n \times n$ symmetric real matrices. Then exactly one of the following two statements holds:*

1. *There exists $Y \succ \mathbf{0}$ such that trace $(A^i Y) = 0$ for all $i = 1, \ldots, k$.*
2. *There exists $x = (x_i) \in \mathbb{R}^k$ such that $x_1 A^1 + \cdots + x_k A^k \succeq \mathbf{0}, \neq \mathbf{0}$.*

For a proof of this lemma, see for example [4]. A sufficient condition for dimensional rigidity is given in the following theorem.

Theorem 7 (Alfakih [1]). *Let $G(p)$ be an r-dimensional bar framework on n vertices in $\mathbb{R}^r$ for some $r \leq n-2$; and let Z be a Gale matrix of $G(p)$. If there exists a matrix $\Psi \succ \mathbf{0}$ of order $\bar{r}$ such that*

$$(z^i)^T \Psi z^j = 0 \text{ for all } ij : i \neq j, (i,j) \notin E(G). \tag{34}$$

Then $G(p)$ is dimensionally rigid.

Proof. Suppose that there exists $\Psi \succ \mathbf{0}$ such that $(z^i)^T \Psi z^j = 0$ for all $ij : i \neq j, (i,j) \notin E(G)$. Then, by Lemma 5, there does not exist $y = (y_{ij}) \in \mathbb{R}^{\bar{m}}$ such that the matrix $\sum_{ij:i\neq j,(i,j)\notin E(G)} y_{ij} (z^i (z^j)^T + z^j (z^i)^T)$ is non-zero positive semidefinite. But $z^i (z^j)^T + z^j (z^i)^T = Z^T E^{ij} Z$. Therefore, there does not exist $y = (y_{ij}) \in \mathbb{R}^{\bar{m}}$ such that $Z^T \mathscr{E}(y) Z \succeq \mathbf{0}, \neq \mathbf{0}$. Hence, by Lemma 3 and (10), there does not exist $y = (y_{ij}) \in \mathbb{R}^{\bar{m}}$ such that $U^T \mathscr{M}(y) U \succeq \mathbf{0}, \neq \mathbf{0}$.

Now assume that $G(p)$ is not dimensionally rigid. Then it follows from (13) that there exists a nonzero $\hat{y}$ such that $X(\hat{y}) = X_p + \mathscr{M}(\hat{y}) \succeq \mathbf{0}$, where rank $X(\hat{y}) \geq r+1$. Thus it follows from Lemma 1 that the matrix

$$\begin{bmatrix} \Lambda + W^T \mathscr{M}(\hat{y}) W & W^T \mathscr{M}(\hat{y}) U \\ U^T \mathscr{M}(\hat{y}) W & U^T \mathscr{M}(\hat{y}) U \end{bmatrix},$$

defined in Lemma 1, is positive semidefinite with rank $\geq r+1$. But $\Lambda + W^T \mathscr{M}(\hat{y}) W$ is $r \times r$, therefore, $U^T \mathscr{M}(\hat{y}) U \succeq \mathbf{0}, \neq \mathbf{0}$, a contradiction. Hence, $G(p)$ is dimensionally rigid. $\square$

In light of Lemma 2, Theorem 7 can also be stated in terms of stress matrices.

Theorem 8 (Alfakih [1]). *Let $G(p)$ be an r-dimensional bar framework on n vertices in $\mathbb{R}^r$ for some $r \leq n-2$. If $G(p)$ admits a positive semidefinite stress matrix S of rank $\bar{r} = n - r - 1$. Then $G(p)$ is dimensionally rigid.*

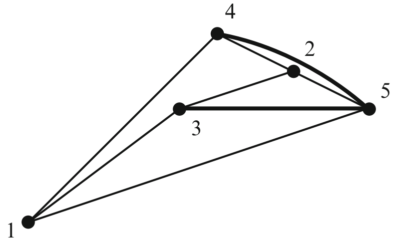

Fig. 3 A dimensionally rigid framework $G(p)$ in $\mathbb{R}^2$ that does not admit a positive semidefinite stress matrix of rank 2. Note that the points p^2, p^4, and p^5 are collinear. The edge $(4, 5)$ is drawn as an arc to make the edges $(2, 4)$ and $(2, 5)$ visible

We remark that the converse of Theorem 7 is not true. Consider [1] the framework $G(p)$ on 5 vertices in $\mathbb{R}^2$, where the configuration matrix P and a Gale matrix Z of $G(p)$ are given by

$$P = \begin{bmatrix} -5 & -3 \\ 2 & 1 \\ -1 & 0 \\ 0 & 2 \\ 4 & 0 \end{bmatrix} \text{ and } Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 0 \\ 3/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix},$$

and where the missing edges of G are $(1, 2)$ and $(3, 4)$ (see Fig. 3). It is clear that $G(p)$ is dimensionally rigid (in fact, $G(p)$ is also universally rigid). Moreover, one can easily verify that there does not exist a 2×2 symmetric positive definite matrix Ψ such that $(z^1)^T \Psi z^2 = (z^3)^T \Psi z^4 = 0$. In other words, $G(p)$ has no positive semidefinite stress matrix of rank 2.

5 Universal Rigidity

Affine motions play a pivotal role in the problem of universal rigidity.

5.1 Affine Motion

An affine motion in $\mathbb{R}^r$ is a map $f : \mathbb{R}^r \to \mathbb{R}^r$ of the form

$$f(p^i) = Ap^i + b,$$

for all p^i in $\mathbb{R}^r$, where A is an $r \times r$ matrix and b is an r-vector. A rigid motion is an affine motion where matrix A is orthogonal.

We say that framework $G(q)$ in $\mathbb{R}^r$ is *affinely-equivalent* to a given r-dimensional framework $G(p)$ in $\mathbb{R}^r$ if: (i) $G(q)$ is equivalent to $G(p)$ and (ii) configuration q is obtained from configuration p by an affine motion; i.e., $q^i = Ap^i + b$, for all $i = 1, \ldots, n$, for some $r \times r$ matrix A and an r-vector b.

Affine-equivalence can be characterized in terms of points $p^1, \ldots, p^n$ and the edges of G.

Lemma 6 (Connelly [14]). *Let $G(p)$ be an r-dimensional bar framework on n vertices in $\mathbb{R}^r$. Then the following two conditions are equivalent:*

1. *There exists a bar framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$,*
2. *There exists a non-zero symmetric $r \times r$ matrix Φ such that*

$$(p^i - p^j)^T \Phi (p^i - p^j) = 0, \text{ for all } (i, j) \in E(G). \tag{35}$$

Affine-equivalence can also be characterized in terms of Gale matrix and the missing edges of G. Recall from (29) that

$$\mathscr{E}(y)_{ij} = \begin{cases} y_{ij} & \text{if } i \neq j \text{ and } (i, j) \notin E(G), \\ 0 & \text{Otherwise}. \end{cases} \tag{36}$$

Lemma 7 (Alfakih [3]). *Let $G(p)$ be an r-dimensional bar framework on n vertices in $\mathbb{R}^r$ and let Z be any Gale matrix of $G(p)$. Then the following two conditions are equivalent:*

1. *There exists a bar framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$,*
2. *There exists a non-zero $y = (y_{ij}) \in \mathbb{R}^{\bar{m}}$ such that:*

$$V^T \mathscr{E}(y) Z = \mathbf{0}, \tag{37}$$

where V is defined in (3).

The following theorem characterizes universal rigidity in terms of dimensional rigidity and affine-equivalence.

Theorem 9 (Alfakih [1]). *Let $G(p)$ be an r-dimensional bar framework on n vertices in $\mathbb{R}^r$, $r \leq n-2$. Then $G(p)$ is universally rigid if and only if the following two conditions hold:*

1. *$G(p)$ is dimensionally rigid.*
2. *There does not exist a bar framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$.*

Proof. Let $G(p)$ be a given r-dimensional framework on n vertices in $\mathbb{R}^r$ for some $r \leq n-2$. Clearly, if $G(p)$ is universally rigid then $G(p)$ is dimensionally rigid and there does not exist a framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$.

To prove the other direction, assume that $G(p)$ is not universally rigid. Then there exists an s-dimensional framework $G(q)$ in $\mathbb{R}^s$, that is equivalent, but not congruent, to $G(p)$, for some s: $1 \leq s \leq n-1$. Therefore, it follows from (13) that there exists a non-zero $\hat{y}$ in $\mathbb{R}^{\bar{m}}$ such that $X(\hat{y}) = X_p + \mathscr{M}(\hat{y}) \succeq \mathbf{0}$. Now for a sufficiently small $\epsilon > 0$, it follows that[1]

$$X(t\hat{y}) = X_p + \mathscr{M}(t\hat{y}) \succeq \mathbf{0}, \text{ and rank } X(t\hat{y}) \geq r \tag{38}$$

for all $t : 0 \leq t \leq \epsilon$. Thus it follows from Lemma 1 that for all $t : 0 \leq t \leq \epsilon$, the matrix

$$Y(t) = \begin{bmatrix} \Lambda + tW^T \mathscr{M}(\hat{y})\, W & tW^T \mathscr{M}(\hat{y})\, U \\ tU^T \mathscr{M}(\hat{y})\, W & tU^T \mathscr{M}(\hat{y})\, U \end{bmatrix},$$

defined in Lemma 1 is positive semidefinite with rank $\geq r$. Consequently, $U^T \mathscr{M}(\hat{y})\, U \succeq \mathbf{0}$ and the null space of $U^T \mathscr{M}(\hat{y})\, U$ is a subset of the null space of $W^T \mathscr{M}(\hat{y})\, U$.

Therefore, if rank $Y(t_0) \geq r + 1$ for some $t_0 : 0 < t_0 \leq \epsilon$, then we have a contradiction since $G(p)$ is dimensionally rigid. Hence, rank $Y(t) = r$ for all $t : 0 \leq t \leq \epsilon$. Thus, both matrices $U^T \mathscr{M}(\hat{y})\, U$ and $W^T \mathscr{M}(\hat{y})\, U$ must be zero. This implies that $\mathscr{M}(\hat{y})\, U = \mathbf{0}$, i.e., $V^T \mathscr{E}(\hat{y}) Z = \mathbf{0}$, which is also a contradiction by Lemma 7. Therefore, $G(p)$ is universally rigid.

By combining Theorems 9 and 7, we obtain the following sufficient condition for universal rigidity.

Theorem 10 (Connelly [11, 13], Alfakih [1]). *Let $G(p)$ be an r-dimensional bar framework on n vertices in $\mathbb{R}^r$, for some $r \leq n-2$. If the following two conditions hold:*

1. *There exists $\Psi \succ \mathbf{0}$ such that $(z^i)^T \Psi z^j = 0$ for all $ij, i \neq j, (i,j) \notin E(G)$.*
2. *There does not exist a bar framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$.*

Then $G(p)$ is universally rigid.

5.2 Universal Rigidity for Generic Frameworks

Theorem 10 becomes simpler if the framework $G(p)$ is assumed to be generic.

Lemma 8 (Connelly [14]). *Let $G(p)$ be a generic r-dimensional bar framework on n nodes in $\mathbb{R}^r$. Assume that the degree of each node of G is $\geq r$. Then there does not exist a non-zero symmetric matrix Φ such that $(p^i - p^j)^T \Phi (p^i - p^j) = 0$ for all $(i,j) \in E(G)$.*

---

[1] The rank function is lower semi-continuous on the set of matrices of order $n-1$.

Therefore, if $G(p)$ is generic, and if the degree of each node of G is at least r. Then it follows from Lemmas 8 and 6 that Condition 2 of Theorem 10 holds. As a result, Theorem 10 reduces to the following theorem.

Theorem 11 (Connelly [11, 13], Alfakih [1]). *Let $G(p)$ be an r-dimensional generic bar framework on n vertices in $\mathbb{R}^r$, for some $r \le n-2$. Suppose that there exists $\Psi \succ \mathbf{0}$ such that $(z^i)^T \Psi z^j = 0$ for all $ij, i \neq j, (i,j) \notin E(G)$; or equivalently, suppose that $G(p)$ admits a positive semidefinite stress matrix S of rank $\bar{r} = n-r-1$. Then $G(p)$ is universally rigid.*

The converse of Theorem 11 is also true.

Theorem 12 (Gortler and Thurston [19]). *Let $G(p)$ be an r-dimensional generic bar framework on n nodes in $\mathbb{R}^r$, for some $r \le n-2$. If $G(p)$ is universally rigid, then there exists a positive semidefinite stress matrix S of $G(p)$ of rank $\bar{r} = n-r-1$.*

5.3 *Universal Rigidity for Frameworks in General Position*

The genericity assumption is a strong one. A weaker assumption is that of points in general position. A configuration p (or a framework $G(p)$) in $\mathbb{R}^r$ is said to be in *general position* if no $r+1$ points in $p^1, \ldots, p^n$ are affinely dependent. For example, a set of points in the plane are in general position if no three of them are collinear.

The following theorem shows that Theorem 11 still holds under the general position assumption.

Theorem 13 (Alfakih and Ye [7]). *Let $G(p)$ be an r-dimensional bar framework on n nodes in general position in $\mathbb{R}^r$, for some $r \le n-2$. Suppose that there exists $\Psi \succ \mathbf{0}$ such that $(z^i)^T \Psi z^j = 0$ for all $ij, i \neq j, (i,j) \notin E(G)$; or equivalently, suppose that $G(p)$ admits a positive semidefinite stress matrix S of rank $\bar{r} = n - r - 1$. Then $G(p)$ is universally rigid.*

The proof of Theorem 13 [7] relays on the following useful property of Gale matrices under the general position assumption.

Lemma 9. *Let $G(p)$ be an r-dimensional bar framework on n nodes in general position in $\mathbb{R}^r$ and let Z be any Gale matrix of $G(p)$. Then every $\bar{r} \times \bar{r}$ sub-matrix of Z is nonsingular.*

In particular, using Lemmas 7 and 9, it is proved in [7] that if a framework $G(p)$ in general position in $\mathbb{R}^r$ admits a positive semidefinite stress matrix S of rank $n-r-1$, then there does not exist a framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$.

On the other hand, a constructive proof was given in [6] that the converse of Theorem 13 holds for r-dimensional frameworks $G(p)$ in $\mathbb{R}^r$, where G is an $(r+1)$-lateration graph. Such frameworks were shown to be universally rigid in [24].

A graph G on n vertices is called an $(r+1)$-*lateration graph* if there exists a permutation π of the vertices of G, such that:

1. The first $(r+1)$ vertices, $\pi(1), \ldots, \pi(r+1)$, induce a clique in G.
2. Each remaining vertex $\pi(j)$, for $j = (r+2), \ldots, n$, is adjacent to exactly $(r+1)$ vertices in the set $\{\pi(1), \pi(2), \ldots, \pi(j-1)\}$.

Theorem 14 (Alfakih et al. [6]). *Let $G(p)$ be an r-dimensional bar framework on n nodes in general position in $\mathbb{R}^r$, for some $n \geq r+2$, where G is an $(r+1)$-lateration graph. Then there exists a positive semidefinite stress matrix S of $G(p)$ of rank $n-r-1$.*

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Geometric Constructions for Symmetric 6-Configurations

Leah Wrenn Berman

Abstract A *geometric k-configuration* is a collection of points and lines, typically in the Euclidean plane, with k points on each line, k lines passing through each point, and non-trivial geometric symmetry; that is, it is a (n_k). configuration for some number n of points and lines. We say a k-configuration is *symmetric* if it has non-trivial geometric symmetry. While 3-configurations have been studied since the mid-1800s, and 4-configurations have been studied since 1990, little is known about more highly incident configurations, such as 5- or 6-configurations. This article surveys several known geometric construction techniques that produce highly symmetric 6-configurations.

Keywords Configurations • Incidence geometry

Subject Classifications: 51E30, 05E30

A *geometric k-configuration* is a collection of points and straight lines, typically in the Euclidean or projective plane, so that every point lies on k lines and every line passes through k points. Since the number of point-line incidences must equal the number of line-point incidences, the number of points and lines in a k-configuration must be equal, and such configurations are also referred to as (n_k) configurations, indicating that there are n points and n lines in the configuration. Other related objects of study are *topological* k-configurations, where the straight lines are replaced with topological lines, i.e., simple closed curves that pairwise transversally intersect exactly once in the projective plane, and *combinatorial* k-configurations, where the straight lines are replaced by "combinatorial lines", i.e., sets of points whose intersection has size at most 1. In this paper, unless otherwise indicated, by

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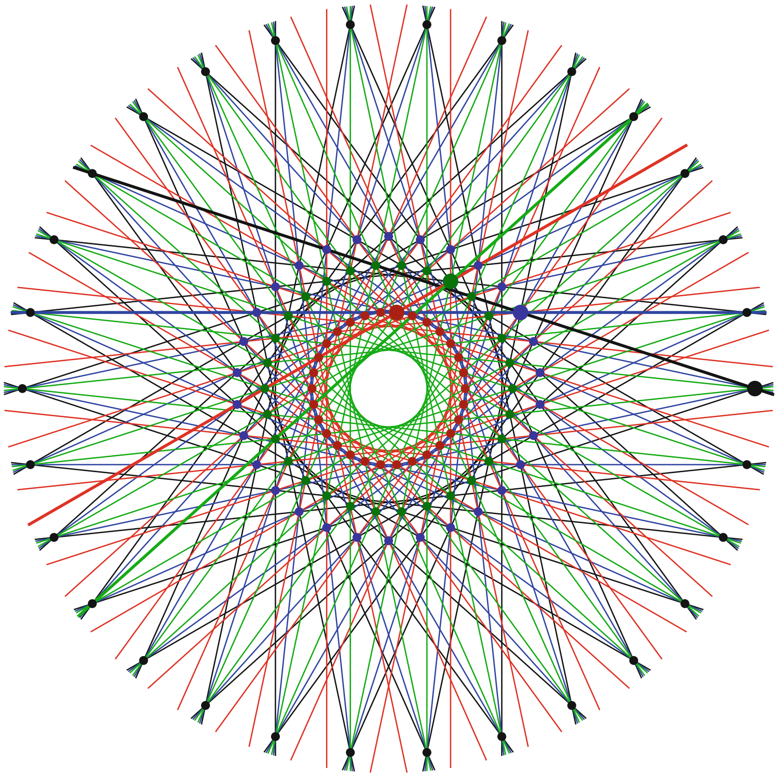

Fig. 1 A (120_6) configuration, with four symmetry classes of points and lines (indicated by color)

'configuration' we mean a geometric k-configuration for some positive integer k. Figure 1 shows a connected 6-configuration with 120 points and lines, although we do not require that k-configurations in general are connected.

The study of k-configurations began in the mid-1800s with various questions about 3-configurations and certain specific combinatorial 4-configurations, but it had mostly died out by the 1940s. The modern study of configurations began in 1990, with the first intelligible geometric representation of a combinatorial 4-configuration, shown and discussed in the paper "The real (21_4) configuration" by Branko Grünbaum and John Rigby [19]; most papers have focused on 3- and 4-configurations. Various mathematical disciplines have been used to investigate questions about configurations, including graph theory, the theory of oriented matroids, projective geometry, combinatorial designs, and Euclidean geometry. Of particular utility has been the use of various computer software, including purpose-built programs written in various programming languages; programs to investigate various group- and graph-theoretic questions, such as NAUTY and GAP; general-purpose computer algebra software such as *Mathematica*; and drawing software such as *Geometer's Sketchpad*. (For example, the configurations shown in this paper were produced using a combination of *Geometer's Sketchpad* and *Mathematica*.)

Despite some significant recent interest in the study of various types of configurations, there has been relatively little study of k-configurations with $k > 4$. A few papers (of the author and coauthors) discussed 5-configurations [8, 13] and a few 8-configurations were described in [9]. However, until recently, basically no general methods for producing intelligible 6-configurations were known. Grünbaum, in his monograph *Configurations of points and lines* [18, Section 1.1, 4.2] described two known ways of constructing two specific geometric 6-configurations, but the configurations that are produced are geometrically unintelligible. The focus of the current paper is to describe a number of methods for producing geometric 6-configurations which have a relatively small number of points and lines and large amounts of geometric symmetry; in addition, two of the methods produce infinite families of 6-configurations.

1 Symmetric Configurations; Levi and Reduced Levi Graphs

A geometric configuration is *symmetric* if there exist nontrivial isometries of the Euclidean or extended Euclidean plane that map the configuration to itself; the orbits under the isometries partition the points and lines of the configuration into various *symmetry classes*, which in this paper are typically indicated by color. For example, the (120_6) configuration shown in Fig. 1 has four symmetry classes of points, colored black, red, green and blue, and four symmetry classes of lines, also colored black, red, green and blue. This definition of symmetric configuration is more general than the notion of *polycyclic* configurations [15], which have the added constraint that every symmetry class must contain the same number of elements under the largest cyclic subgroup of the symmetry group of the configuration; the more general definition of symmetric used here allows for the possibility that, for example, lines of the configuration might pass through the center of the configuration. (All configurations discussed in the remainder of this paper are polycyclic, however.) Note that in other papers in the literature (e.g., [14]), following the use of the term in combinatorial designs, the term "symmetric" has been used to refer to configurations which have the same number of points on each line as lines passing through each point—that is, to k-configurations. However, since the geometry of the configuration plays such an important role in this discussion, we follow Grünbaum [18, p. 16] and say that k-configurations are *balanced*, and we reserve the use of *symmetric* to refer to configurations with nontrivial geometric symmetry.

The same combinatorial configuration can have embeddings with very different symmetry properties. For example, the (9_3) Pappus configuration has embeddings with no symmetry, with one line of mirror symmetry, with two lines of mirror symmetry, with 3-fold rotational symmetry, and with 3-fold dihedral symmetry. In this paper, we will not be concerned with whether or not two geometric configurations are combinatorially isomorphic; indeed, there is no known general answer to the question, even for well-understood classes of geometric configurations, such as celestial 4-configurations (discussed in the next section).

Often, we are interested in classifying configurations according to the number of symmetry classes of points and lines. Following Grünbaum [18, p. 34], we say that k-configurations with h symmetry classes of points and of lines are *h-astral* configurations. For example, the configuration shown in Fig. 1 is a 4-astral 6-configuration, because there are 4 symmetry classes of points and 4 symmetry classes of lines. If there are h_1 symmetry classes of lines and h_2 symmetry classes of points, we say the configuration is (h_1, h_2)-astral. In the case where $h = \lfloor \frac{k+1}{2} \rfloor$, which is smallest possible (because in a configuration at most two points on a given line can be in the same symmetry class, and there are k points/line), we say the configuration is *astral*. Of particular interest are astral 4-configurations, with 2 symmetry classes of points and lines; note that in [3, 5] it was shown that there are no astral k-configurations for $k \geq 6$, and it is still open (but unlikely, see [8]) as to whether there exist astral 5-configurations.

Given an (n_k) configuration, the *Levi graph* is a bipartite graph with $2n$ nodes, one corresponding to each point and line of the configuration, and nk arcs, where a line-node and a point-node are joined by an arc if and only if the line and point are incident in the configuration. If the configuration has cyclic symmetry of order m, then the Levi graph can be quotiented by that cyclic group to form the *reduced Levi graph*, also known as a *voltage* graph, a bipartite multigraph with one node for each symmetry class of points and lines, and labelled arcs connecting the point-nodes and line-nodes indicating the incidence. In particular, if one symmetry class of lines is labelled L, with elements of the symmetry class labelled cyclically as $L_0, L_1, \ldots, L_{m-1}$, and similarly a vertex class v is labelled cyclically as $v_0, \ldots, v_{m-1}$, then an arc with label a connects node L and node v precisely when line L_0 passes through vertex v_a. Note that this means that arcs are labelled "from the point of view" of the vertices, which is opposite to the labelling scheme in [15]. In particular, given a drawing of a configuration in which the 0-th elements of each symmetry class of points and lines are highlighted, and the rest of the symmetry classes of points and lines are labelled cyclically, to label the corresponding reduced Levi graph, look at line L_0 and then write down the subscripts of the vertices it intersects as the labels of the arcs adjacent to L in the reduced Levi graph.

In what follows, we often need to describe or construct the intersection of two lines or the line that passes between two points. If L and M are lines, the intersection of L and M is denoted $L \wedge M$; if p and q are points, the line that passes between p and q is denoted $p \vee q$.

2 Celestial 4-Configurations

Several of the constructions of 6-configurations depend on using a certain type of symmetric 4-configuration, called a *celestial* 4-configuration (which will be described below), as building blocks. The configuration shown in Fig. 2a is the smallest celestial 4-configuration. Celestial configurations are reasonably

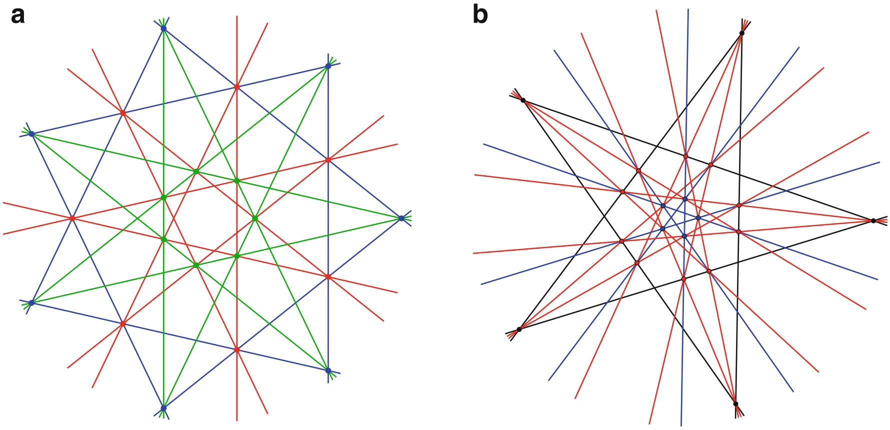

Fig. 2 Symmetric 4-configurations. (**a**) The (21_4) celestial configuration $7\#(2,1;3,2;1,3)$, which is 3-astral and 3-celestial: there are three symmetry classes of points and lines, and each line (resp. point) is incident with two elements from each of two symmetry classes of points (reps. lines). (**b**) A dihedrally symmetric 3-astral (20_4) configuration, first discussed in [17], that is non-celestial; the *red lines* are incident with points from three symmetry classes

well-understood, although they are not completely classified. Every celestial 4-configuration has a corresponding *celestial symbol* whose parameters must satisfy four axioms, and given a valid symbol, there is a straightforward, iterative construction technique, which uses nothing more than drawing straight lines between points (given an initial starting convex m-gon of vertices) and constructing the intersection of lines to determine the vertices and lines of the configuration.

For more details on celestial 4-configurations, the interested reader is referred to Branko Grünbaum's book [18, Section 3.5–3.8]; the following discussion is mostly adapted from that reference. However, in that reference, celestial 4-configurations are unfortunately referred to as k*-astral* configurations: all the configurations discussed in those sections are celestial k-astral 4-configurations, but there are 4-configurations with k symmetry classes of points and lines (which are therefore k-astral) that are not k-celestial; e.g., see Fig. 2b. We reserve the term k-astral to indicate the number of symmetry classes of points and lines, and use the term *celestial* to refer to a more constrained class. For other discussions of celestial 4-configurations, see [4, 6, 10, 12], where they are referred to as *celestial*; referred to as *polycyclic* see [15], and some early images appeared in [20].

A celestial 4-configuration has mh points and lines, for some $m \geq 7$ and $h \geq 3$, and the dihedral symmetry of an m-gon, with the same number of elements in each symmetry class. A defining characteristic is that every point has two lines from each of two symmetry classes of lines incident with it, and every line has two points from each of two symmetry classes incident with it. A celestial 4-configuration with h symmetry classes of points and lines is called an h*-celestial 4-configuration* (or often, simply an h-celestial configuration).

The notation

$$m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h)$$

is called a *configuration symbol*; a symbol is *valid* if it obeys the following four constraints on the sequence $(s_1, t_1, \ldots; s_h, t_h)$:

(A1): No adjacent entries in the sequence are equal (taken cyclically: $t_h \neq s_1$ also);
(A2): The sum $(s_1 - t_1) + \ldots + (s_h - t_h)$ is even;
(A3): No substring $(s_j, t_j; \ldots; s_k)$ or $(t_j; s_{j+1}, t_{j+1}; \ldots; s_k, t_k)$ can be completed to a valid configuration symbol using the same m;
(A4): (The cosine condition)

$$\prod_{j=1}^{h} \cos\left(\frac{s_j \pi}{m}\right) = \prod_{j=1}^{h} \cos\left(\frac{t_j \pi}{m}\right).$$

(Axiom (A3) is quite technical and prevents extra unintended incidences; for details on these axioms see [18, Section 3.5–3.8].)

Every celestial 4-configuration can be represented by a symbol

$$m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h),$$

although many symbols can correspond to the same unlabeled geometric configuration, and any valid symbol corresponds to an iterative construction method for the configuration.

The iterative construction method for a configuration given a valid symbol is as follows.

Initial steps:

- Construct vertices v_0 to be the vertices of a regular convex m-gon:

$$(v_0)_i = \left(\cos\left(\frac{2\pi i}{m}\right), \sin\left(\frac{2\pi i}{m}\right)\right).$$

- Construct lines $(L_0)_i$ of span s_1:

$$(L_0)_i = (v_0)_i \vee (v_0)_{i+s_1}$$

Iterative step: For $1 < j < h$, construct vertices $(v_j)_i$ and lines $(L_j)_i$ as follows:

- Vertex $(v_j)_i := (L_{j-1})_i \wedge (L_{j-1})_{i-t_j}$; this is sometimes referred to as a "t_j-th intersection point" of the $(L_{j-1})_i$.

- Having constructed the vertices v_j, the lines L_j are span s_j with respect to the v_j; that is,

$$(L_j)_i := (v_j)_i \vee (v_j)_{i+s_j}.$$

At the $(h-1)$-st step, we construct the vertices v_{h-1} to be the $(t-1)$-st intersection of the lines L_{h-2}, and we construct the lines L_{h-1} which are of span s_h with respect to the vertices v_{h-1}. The configuration symbol is valid if the set of vertices v_h constructed as

$$(v_h)_i := (L_{h-1})_i \wedge (L_{h-1})_{i-t_h}$$

coincide *as sets* with the set of vertices v_0. Then the configuration closes up.

The configuration shown in Fig. 2a may be described as the 3-celestial 4-configuration $7\#(2,1;3,2;1,3)$; in the diagram, the points and lines v_0 and L_0 are blue, the points and lines v_1 and L_1 are red, and the points and lines v_2 and L_2 are green.

Given a valid sequence $(s_1, t_1; \ldots; s_h, t_h)$, shifting the sequence cyclically by an even number of steps corresponds to choosing a different ring of points to label as v_0 and hence to a geometrically congruent configuration, while cyclically shifting the sequence an odd number of steps, or reversing the sequence, results in a configuration that is the geometric polar of the configuration. If any two entries s_i, s_j (or t_i, t_j) in the sequence $(s_1, t_1; \ldots; s_h, t_h)$ are switched, then the resulting sequence satisfies axioms (A2) and (A4), which we call the *cohort axioms*. We can represent a whole collection of valid configuration symbols using a more compact notation, called *cohort notation*: $m\#\{S; T\}$ represents a collection of configurations, using any valid assignment of s_i's and t_i's chosen alternately from S and T, respectively, to form the symbol of a celestial configuration. In cohort notation, the configuration in Fig. 2a is represented as $7\#\{\{1,2,3\};\{1,2,3\}\}$.

If $S = T$, then the corresponding cohort necessarily satisfies the cohort axioms. Configuration cohorts of the form $m\#\{S, S\}$ are called *trivial*, as are the corresponding configurations. The configuration in Fig. 2a is trivial, with $S = T = \{1,2,3\}$. Some cohorts/configurations fall into an infinite family and are called *systematic*; cohorts/configurations which are *provably* neither systematic nor trivial are called *sporadic*.

Let $\delta = \frac{1}{2}((s_1 + \ldots + s_h) - (t_1 + \ldots + t_h))$, and note that by axiom (A4), δ must be an integer; Boben and Pisanski [15] called δ the "twist" of the configuration. The reduced Levi graph for $m\#(s_1, t_1; \ldots; s_h, t_h)$ is a $2h$-cycle with all arcs doubled, with one of the parallel arcs labelled 0 and the other s_i or t_i, alternately around the cycle, with the final double-arc labelled with t_h and δ. For a more detailed discussion about the relationship between celestial 4-configurations and reduced Levi graphs, see [15] (using the name voltage graphs), where the celestial 4-configuration $m\#(s_1, t_1; \ldots; s_h, t_h)$ in their notation would be denoted $\mathscr{C}_4(m, (s_1, \ldots, s_h), (t_1, \ldots, t_h), \delta)$.

A 2-celestial 4-configuration is also an *astral* 4-configuration, and these have been completely classified; the first conjecture of a complete classification appeared in [16] and a complete, albeit very technical, proof of the classification appeared in [2]. Subsequently, Grünbaum presented a significantly more streamlined proof in [18, Section 3.6].

The conclusion of the proof is that

1. 2-celestial configurations exist only if $6 \mid m$ and $m \geq 12$;
2. For each $m = 6k$, there is a single one-parameter family of cohorts: $6k\#\{3k - j, j\}, \{3k - 2j, 2k\}$.
3. If $m = 30, 42, 60$ there are a number of *sporadic* 4-configurations (see, e.g., Table 3.6.1 in [18, p. 207]).

The situation for $h \geq 3$ is considerably less well-understood. In distinct contrast to the 2-celestial case, h-celestial configurations with $h > 2$ can be constructed for any "large-enough" m: in particular, it is possible to construct trivial h-celestial configurations for any $h > 2$. (There are no trivial 2-celestial configurations, because although the cohort axioms are satisfied, it is impossible to find a valid ordering for the sequence if S contains only two numbers.) In addition, a number of systematic families have been identified for $h \geq 3$, including 19 infinite families for $h = 3$, for m divisible by 2, 3, 6, 10, 12, and 30 (these do not exhaust the known data, however), 4 infinite families for $h = 4$, an infinite family when $m = 2q$ and $h = 2^{k+1}$ for any k, and an infinite family for $m = 10q$ and $h = 4j$ for any j (see [1] for details). However, no complete classification exists, even for $h = 3$.

3 Celestial 6-Configurations

If we extend the definition of "celestial" configurations to mean $2k$-configurations with the dihedral symmetry of an m-gon in which two lines from each of h symmetry classes pass through each point and two points from each of h symmetry classes of points lie on each line, we can discuss more highly incident celestial configurations. In particular, we can try to construct celestial 6-configurations, which have two lines from each of three symmetry classes of lines passing through each point and two points from each of three symmetry classes of points lying on each line.

The construction is technical, but it was motivated by the following idea: is it possible to simultaneously construct two celestial 4-configurations in such a way as to get extra incidences? If so, can that construction be extended to construct a 6-configuration?

Suppose you begin with the vertices of a regular convex m-gon, say, $m = 11$, and then simultaneously construct two trivial 3-celestial configurations of the form $m\#(a, b; c, a; b, c)$ and $m\#(a, d; c, a; d, c)$, say $11\#(3, 2; 4, 3; 2, 4)$ (Fig. 3a) and $11\#(3, 1; 4, 3; 1, 4)$ (Fig. 3b). Since $a = 3$ in both configurations, the first set of points (dark blue) and lines (black) constructed will be the same (in this case, the

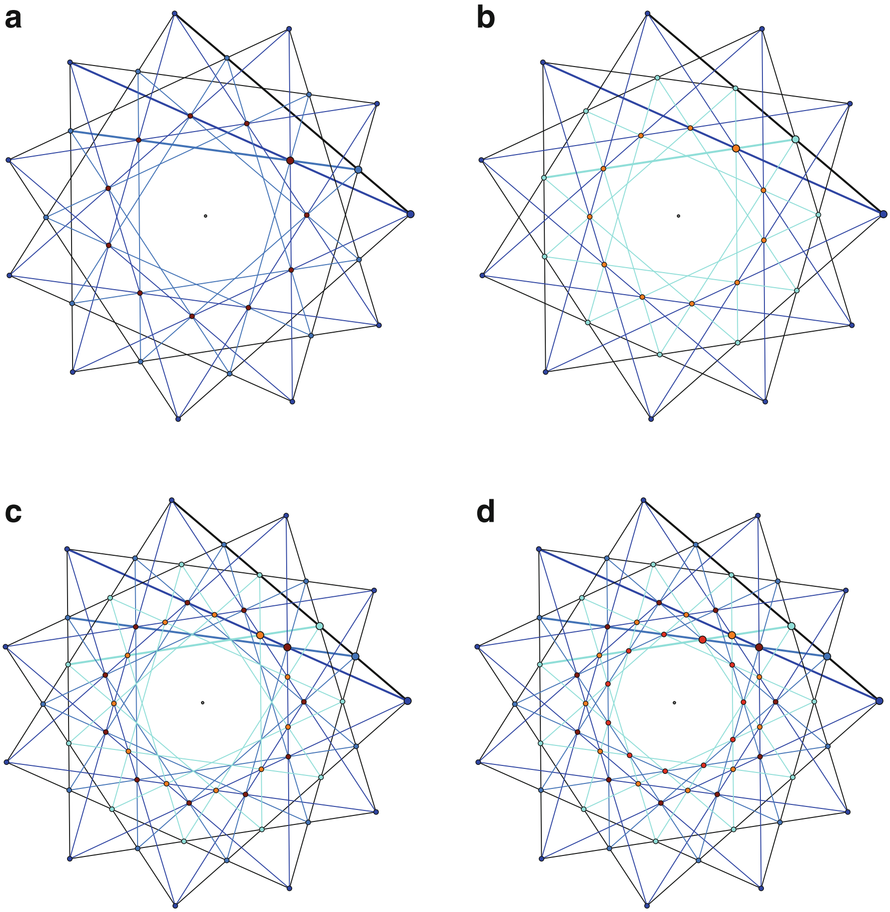

Fig. 3 Simultaneously constructing the trivial celestial configurations $11\#(3, 2; 4, 3; 2, 4)$ and $11\#(3, 1; 4, 3; 1, 4)$ results in additional lines intersecting four at a time, and a $(4, 6)$-configuration. (**a**) $11\#(3, 2; 4, 3; 2, 4)$. (**b**) $11\#(3, 1; 4, 3; 1, 4)$. (**c**) Constructing $11\#(3, 2; 4, 3; 2, 4)$ and $11\#(3, 1; 4, 3; 1, 4)$ simultaneously. (**d**) Adding new points forms a (4, 6)-configuration

lines are span 3), but in Fig. 3a we chose the $t_1 = b = 2$ point of intersection (medium blue) and in Fig. 3b we chose the $t_1 = d = 1$ point of intersection (cyan); in both configurations we constructed lines of span 4 with respect to these sets of points (also colored medium blue and cyan, respectively). In Fig. 3a, we next chose the 3rd points of intersection of the medium blue lines to construct the third set of points, since $t_2 = a = 3$, and colored these burgundy, and then constructed lines (colored dark blue) of span 2 with respect to the burgundy points; note these lines are span 4 with respect to the original dark blue points. In Fig. 3b, we instead constructed the 3rd (since $a = t_2 = 3$) points of intersection of the cyan lines, which we colored orange, and finally constructed lines of span 1 (dark blue) with respect

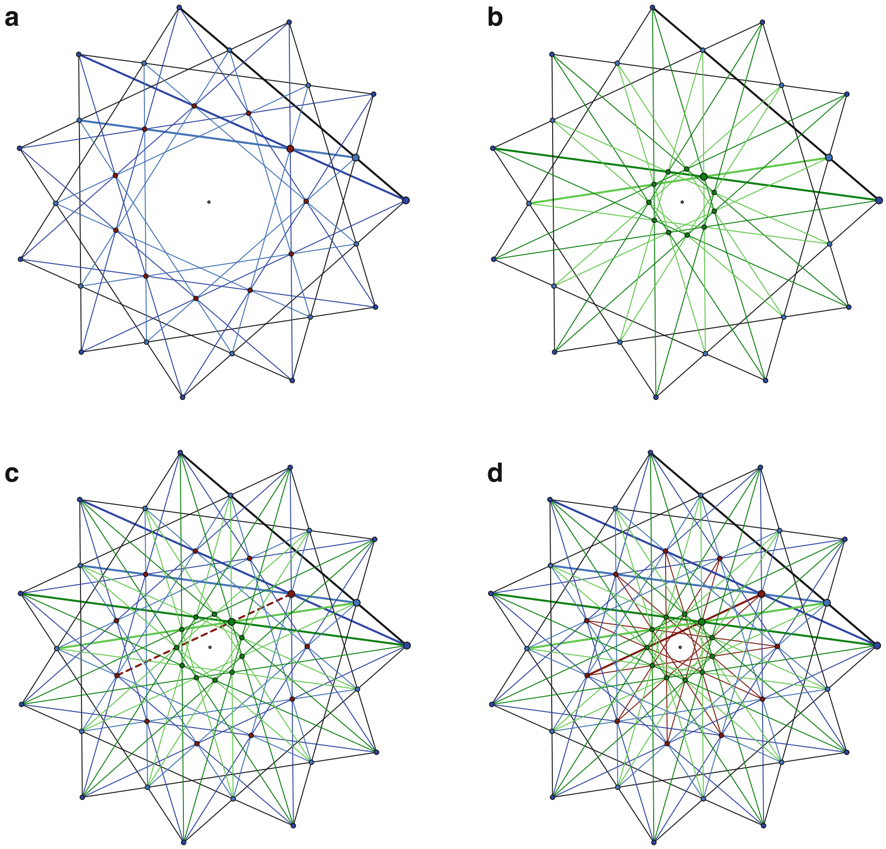

Fig. 4 Simultaneously constructing the trivial celestial configurations 11#(3, 2; 4, 3; 2, 4) and 11#(3, 2; 5, 3; 2, 5) results in additional points being collinear four at a time, and a (6, 4)-configuration. (**a**) 11#(3, 2; 4, 3; 2, 4). (**b**) 11#(3, 2; 5, 3; 2, 5). (**c**) Constructing 11#(3, 2; 4, 3; 2, 4) and 11#(3, 2; 5, 3; 2, 5) simultaneously. (**d**) Adding new lines forms a (6,4)-configuration

to these orange points. These lines are also of span 4 with respect to the original dark blue points! If we construct all these points and lines simultaneously, by first constructing the black points and lines, then constructing both the medium blue and cyan points and lines at the same time, then the burgundy and orange points and lines, and finally the dark blue lines (which were span 4 with respect to the black vertices in both Fig. 3a, b), as in Fig. 3c, then the medium blue and cyan lines intersect 4-at-a-time. If we place a new set of points (colored red) on this four-fold intersection, then we have constructed a (4, 6)-configuration, shown in Fig. 3d.

On the other hand, if we begin by constructing the configurations $m\#(a, b; c, a; b, c)$ and $m\#(a, b; e, a; b, e)$, say 11#(3, 2; 4, 3; 2, 4) (repeated as Fig. 4a) and 11#(3, 2; 5, 3; 2, 5) (Fig. 4b, using dark and light green lines of span 5 with respect to the dark blue and medium blue points), then when we combine the two configurations (Fig. 4c), we notice (somewhat less obviously) that the burgundy points and the dark green points are actually collinear; when we add in all the new burgundy lines, we get a $(6, 4)$-configuration (Fig. 4d).

These coincidences and collinearities are not just happenstance; rather, they are a consequence of the Crossing Spans Lemma [4, 6, 13], which says the following (this lemma has been phrased in various ways in the literature): Begin with a regular convex m-gon of points u_i and construct lines L_i and R_i of spans a and b, respectively, with respect to the u_i. Place new points v_i arbitrarily on the lines L_i and construct new lines N_i of span b with respect to the points v_i. Then the three lines N_i, N_{i-b} and R_{i-a} all are coincident at a new point w_i. If the points v_i happen to lie on an intersection of two of the lines L_i, say $v_i = L_i \wedge L_{i-d}$, then the points w_i lie on N_i, N_{i-b}, R_{i-a}, and R_{i-a-d}. (See [4] for a proof using elementary Euclidean geometry results about circles and cyclic quadrilaterals; the Crossing Spans Lemma is listed as Theorem 3.) The Dual Crossing Spans Lemma results if the roles of points and lines are interchanged; see [6] for a careful statement.

The fact that trivial celestial configurations exist at all may be interpreted as a consequence of the Crossing Spans Lemma or Dual Crossing Spans Lemma. In addition, in Fig. 3, the fact that the additional burgundy points lie on the common intersection of the medium blue lines and the cyan lines is an immediate consequence of the extension of the Crossing Spans Lemma, using the assignments that the burgundy points are the u_i, the dark blue lines are the L_i, the medium blue lines are the R_i, the orange points are the v_i, and the cyan lines are the N_i; we conclude that two cyan and two medium blue lines do, in fact, intersect in a single point. Similarly, in Fig. 4, the collinearity of the burgundy and dark green points is a consequence of the Dual Crossing Spans Lemma.

The next natural question is to ask if it is possible to extend the simultaneous construction of these celestial configurations $m\#(a, b; c, a; b, c)$, $m\#(a, d; c, a; d, c)$ and $m\#(a, b; e, a; b, e)$ to construct a 6-configuration. Obviously, there are other trivial celestial configurations associated with the five discrete parameters a, b, c, d, e, such as $m\#(a, d; e, a; d, e)$. But there are also other implicit trivial celestial configurations hidden away inside the configurations we've already constructed: just considering the burgundy, orange and red points and the various blue lines in Fig. 3d, we see an $11\#(2, 1; 3, 2; 1, 3)$ configuration, which in general would correspond to an $m\#(b, c; d, b; c, d)$ configuration.

Appropriately generalizing this notion of simultaneously constructing certain trivial celestial configurations with certain discrete parameters and adding new points and lines that satisfy certain incidences requires some careful bookkeeping. (Proofs that the various objects have the incidences stated below, and a more technical description of the general construction method that produces a $(2s, 2t)$-configuration for any $s, t \geq 2$, can be found in [6, 7].) The iterative construction for producing a 6-configuration is as follows; an example, built upon the configurations shown in Figs. 3 and 4, is shown in Fig. 5. The initial data for the construction is a value of m that determines the symmetry of the configuration, with $m \geq 11$, and two pairs of sets of discrete parameters t_0, t_1, t_2 with index set $T = \{0, 1, 2\}$ and s_0, s_1 with index set $S = \{0, 1\}$, where the s_i and t_i are distinct positive integers strictly less than $\frac{m}{2}$. In our examples, $t_0 = a = 3$, $t_1 = b = 2$, $t_2 = d = 1$ and $s_0 = c = 4$, $s_1 = e = 5$.

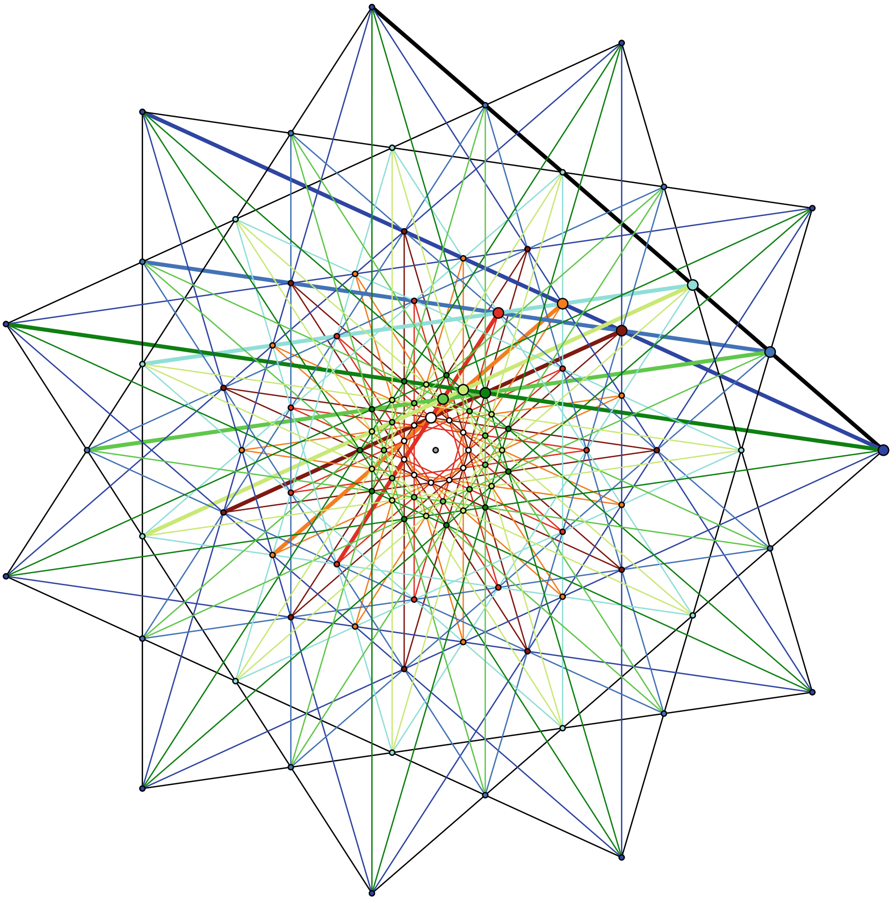

Fig. 5 A 6-configuration, with $m = 11$, $t_0 = 3$, $t_1 = 2$, $t_2 = 1$ and $s_0 = 4$, $s_1 = 5$, with 10 symmetry classes of points and 10 symmetry classes of lines. The 0th element of each symmetry class is shown larger

(Note that the discrete parameters t_0, t_1, t_2 and s_0, s_1 listed here are *not* the same parameters as those that are listed in a general celestial configuration $m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h)$.)

Step 1. Begin with a single class of lines, labelled $(L^{\emptyset}_{\emptyset})_i$ (the black lines in the previous examples and in Fig. 5), that form the extended sides of a regular convex m-gon.
In this step we have constructed a single class of lines, corresponding to the fact that there is one way to choose a subset of T of size 0 for the lower index and one way to choose a subset of S of size 0 for the upper index.

Step 2. Construct three sets of vertices, indexed $(v_0^{\emptyset})_i$, $(v_1^{\emptyset})_i$, $(v_2^{\emptyset})_i$, where $v_j^{\emptyset}$ is the t_j-th intersection of the lines $L_{\emptyset}^{\emptyset}$: that is,

$$(v_j^{\emptyset})_i = (L_{\emptyset}^{\emptyset})_i \wedge (L_{\emptyset}^{\emptyset})_{i-t_j}.$$

These correspond to the dark blue, medium blue, and cyan points, respectively, in the previous examples and in Fig. 5; the upper subscript $\emptyset$ refers to the fact that the new points lie on the lines $L_{\emptyset}^{\emptyset}$. In this step, we constructed three classes of vertices, corresponding to the way to choose a subset of S of size 0 for the upper index and a subset of T of size 1 for the lower index. (In practice, it is easier to construct the vertices $(v_0^{\emptyset})_i$ first, and then construct the lines $(L_{\emptyset}^{\emptyset})_i = (v_0^{\emptyset})_i \vee (v_0^{\emptyset})_{i+t_0}$.)

Step 3. Now, construct 6 new sets of lines

$$(L_j^p)_i := (v_j^{\emptyset})_i \vee (v_j^{\emptyset})_{i-s_p},$$

where $p \in S$ and $j \in T$. The downstairs index j in L_j^p, which is 0, 1 or 2, indicates that each $(L_j^p)_i$ is incident with vertex $(v_j^{\emptyset})_i$, and the upstairs index p in L_j^p, which is either 0 or 1, indicates whether the lines are span s_0 or span s_1 with respect to $v_j^{\emptyset}$. For example, the lines L_1^0 are of span $s_0 = 4$ with respect to the points $v_1^{\emptyset}$; these were the lines colored medium blue in Fig. 5. In that figure, the other span $s_0 = 4$ lines are L_0^0 colored dark blue and L_2^0 colored cyan, and the span $s_1 = 5$ lines are L_0^1 colored dark green, L_1^1 colored light green, and L_2^1 colored yellow-green. Note that each of the 6 classes of lines is indexed by one of the $\binom{3}{1}$ subsets of T that form the lower index and one of the $\binom{2}{1}$ subsets of S that form the upper index.

Step 4. Construct six new classes of vertices

$$(v_{jk}^p)_i := (L_j^p)_i \wedge (L_k^p)_i.$$

For example, the vertex $(v_{12}^0)_i$ is the intersection of $(L_1^0)_i$ (medium blue) and $(L_2^0)_i$ (cyan). That is, the vertex class (v_{12}^0) is precisely the set of red vertices in Fig. 3a. It is not obvious from the definition, but it turns out that vertex $(v_{jk}^p)_i$ also lies on $(L_j^p)_{i-t_k}$ and $(L_k^p)_{i-t_j}$; this follows from the Crossing Spans Lemma.

As an example, in Fig. 5 (and also in Fig. 3a), where $p = 0$ and $jk = 01$ the burgundy vertex $(v_{01}^0)_0$ lies on the span $s_0 = 4$ lines $(L_0^0)_0$ (thick dark blue), $(L_0^0)_{-2}$ (thinner dark blue), $(L_1^0)_0$ (thick medium blue) and $(L_1^0)_{-3}$ (thinner medium blue); notice that $t_0 = 3$ and $t_1 = 2$. Similarly, the vertex $(v_{01}^1)_0$ should lie on $(L_1^1)_0$ and $(L_1^1)_{-3}$ and $(L_0^1)_0$ and $(L_0^1)_{-2}$. In Fig. 4b, the lines (L_1^1) are shown in light green, where they are span $s_1 = 5$ with respect to the medium blue vertices ($v_1^{\emptyset}$), and the lines (L_0^1) are shown in dark green, where they are span $s_1 = 5$ with respect to the dark blue vertices ($v_0^{\emptyset}$). Thus, the vertices (v_{01}^1) are those shown in Fig. 4b as dark green. In Fig. 5, the vertices v_{01}^0 are burgundy, v_{02}^0 are orange and v_{12}^0 are red, while the vertices v_{01}^0 are dark green, v_{02}^0 are yellow-green and v_{12}^0 are light green. At this step, we have constructed one vertex class for each of the $\binom{3}{2}$ choices of subsets of T that index the vertex class downstairs and each of the $\binom{2}{1}$ choices of subsets of S (that is, choices of spans) that index the vertex class upstairs, for a total of 6 vertex classes constructed at this step.

Step 5. Construct three new line classes

$$(L_{jk}^{01})_i := (v_{jk}^0)_i \vee (v_{jk}^1)_i.$$

For example, line $(L_{01}^{01})_0$ connects vertex $(v_{01}^0)_0$ and vertex $(v_{01}^1)_0$. In Fig. 4c, the vertices $(v_{01}^0)_0$ are burgundy, and $(v_{01}^1)_0$ are dark green, and the line $(L_{01}^{01})_0$ is shown dashed burgundy. In that figure, we observed that the dashed burgundy line also passes through two other vertices; in particular, it passes through the burgundy point $(v_{01}^0)_5$ (recall $s_1 = 5$) and the dark green point $(v_{01}^1)_4$ (and $s_0 = 4$). This generalizes; as a consequence of the Dual Crossing Spans Lemma, line $(L_{jk}^{01})_i$ passes through the four points $(v_{jk}^0)_i, (v_{jk}^1)_i, (v_{jk}^0)_{s_1}$, and $(v_{jk}^1)_{s_0}$. In Fig. 5, the lines (L_{01}^{01}) are burgundy, (L_{02}^{01}) are orange, and (L_{12}^{01}) are red. Note that each line class corresponds to using a subset of size 2 from S for the upstairs index (of which there is only one, S itself), and a subset of size 2 from T for the downstairs index, so we have constructed a total of three new line classes at this step.

Step 6. Finally, we construct a single new class of vertices (v_{012}^{01}) defined by

$$(v_{012}^{01})_i = (L_{01}^{01})_i \wedge (L_{02}^{01})_i \wedge (L_{12}^{01})_i;$$

it is true but not obvious that these three lines intersect, and moreover, the three lines $(L_{01}^{01})_{i-s_2}$, $(L_{02}^{01})_{i-s_1}$ and $(L_{12}^{01})_{i-s_0}$ also intersect at this point. (Proving that this vertex class and its generalizations is well-defined is one of the main results from [6, 7].) These are the white vertices in Fig. 5.

Counting everything up, we see that each vertex has six lines, two from each of two symmetry classes, passing through it, and each line has six vertices, two from each of two symmetry classes, so the resulting incidence structure is a 6-configuration, with 10 symmetry classes of points and lines.

The reduced Levi graph of this configuration has a very beautiful structure. The underlying unlabeled graph is the Desargues graph with all arcs doubled; Fig. 6 shows the labelled general reduced Levi graph for the celestial 6-configuration with m, (t_0, t_1, t_2), (s_0, s_1) with like-labelled arcs colored the same. Unlabelled arcs have label 0.

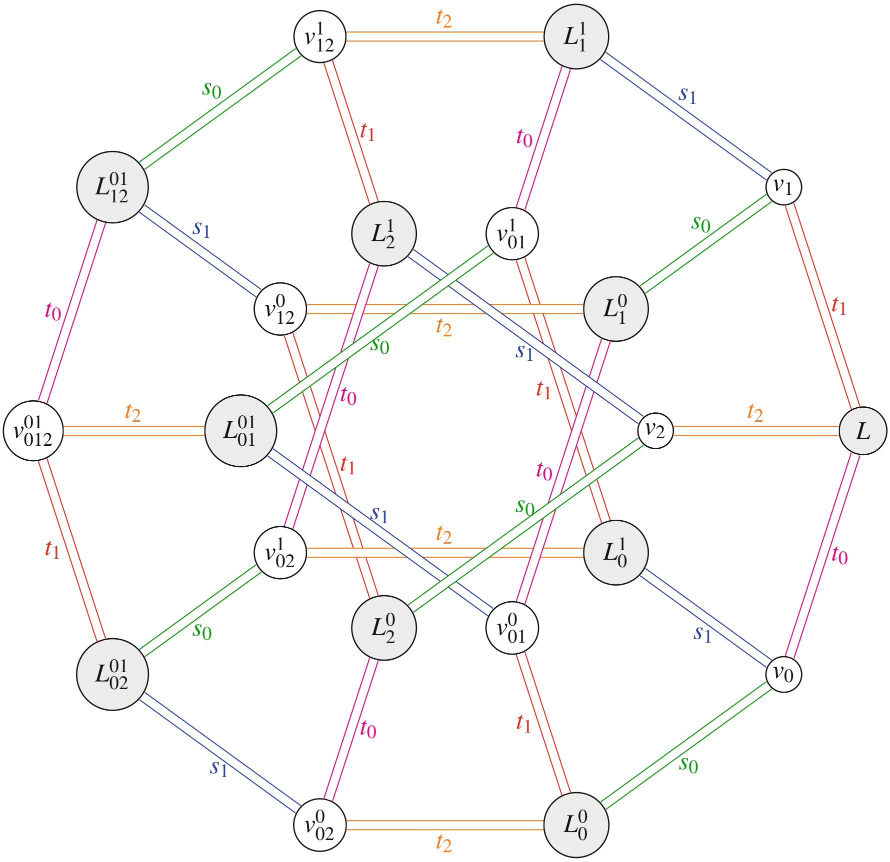

Fig. 6 The reduced Levi graph, with group $\mathbb{Z}_m$, for the 6-configuration with parameters $m, (t_0, t_1, t_2), (s_0, s_1)$ whose construction is described in Sect. 3. Unlabelled edges are considered to have label 0 (Note pairs of edges with the same color have the same pair of labels)

4 Sporadic 6-Configurations

There are other 6-configurations that are formed by simultaneously constructing certain celestial configurations, in this case 2-celestial (a.k.a. astral) 4-configurations; the resulting configurations have four symmetry classes of points and lines, which is the best known, since it was shown in [3] that no astral (that is, 3-astral) 6-configurations exist. At the moment, only five such configurations are known. They are conjectured to be sporadic.

The motivating idea behind the discovery of these configurations was the following: when considering the list of all astral configurations with $m = 30$, which is the smallest case in which there are both sporadic and systematic astral 4-configurations, there are a number of parameters that appear in multiple

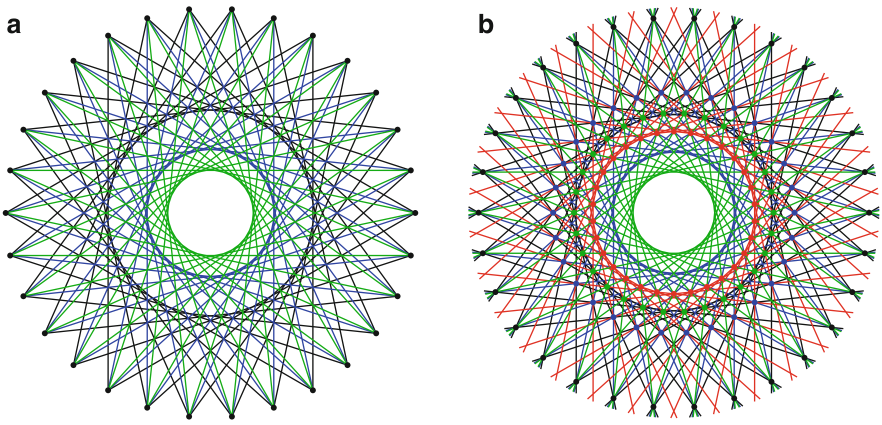

Fig. 7 A 4-astral (120_6) configuration, whose outside lines (*black*, *green*, *blue*) are span 10, 12, and 13 with respect to the outside (*black*) vertices, is formed by selecting certain 4-fold intersections of the outside lines (the vertices colored *blue*, *green*, and *red*) and then adding a particular *red line* that intersects two *blue*, two *green* and two *red* vertices. (**a**) If $m = 30$, lines of span 10, 12, 13 have many 4-fold intersection points. (**b**) Choosing the right selection and adding an extra set of lines generates a 6-configuration

configurations; in particular, several configurations use the spans 10, 11, 12, 13, 14 as "outside" spans (i.e., as spans with respect to the outermost symmetry class of points). If all lines of, say, spans 10, 12, and 13 (drawn in black, blue and green, respectively, in Fig. 7a) are constructed beginning with the vertices of a convex 30-gon, there are a number of points that can be constructed that lie on four lines at a time, because they form part of some astral 4-configuration. If three appropriate symmetry classes of intersections are chosen, one for each pair of colors (there may be more than one choice), then the spans will have three points from each of three symmetry classes of points lying on them. It is sometimes also possible to construct a new set of lines that is incident with all three of the new sets of points; if so, then we have constructed a 6-configuration. Figure 7 shows an initial set of lines, and then a particular choice of new point classes which are collinear, leading to the construction of a 6-configuration with 4 symmetry classes of points and lines.

There are five 6-configurations that have been found by this ad hoc technique; they correspond to starting spans of (10, 11, 13), (10, 12, 13), (10,12,14), (11, 12, 14) and (12, 13, 14) (which was shown in Fig. 1). Of the remaining 3-subsets of $\{10, 11, 12, 13, 14\}$, the starting spans $(10, 11, 14)$ and $(11, 12, 13)$ already form astral $(6, 4)$-configurations (all three types of lines intersect in a single point), and the rest cannot be completed to a 6-configuration in this way. The other choices of three spans from $\{10, 11, 12, 13, 14\}$ for $m = 30$ do not yield enough intersections to complete to a 6-configuration with four symmetry classes, and beginning with other possible spans for 4-configurations with m = 30 does not appear to yield other configurations either.

These "sporadic" 6-configurations have very nice reduced Levi graphs, shown in Figs. 8 and 9. The underlying graph for all the reduced Levi graphs is the

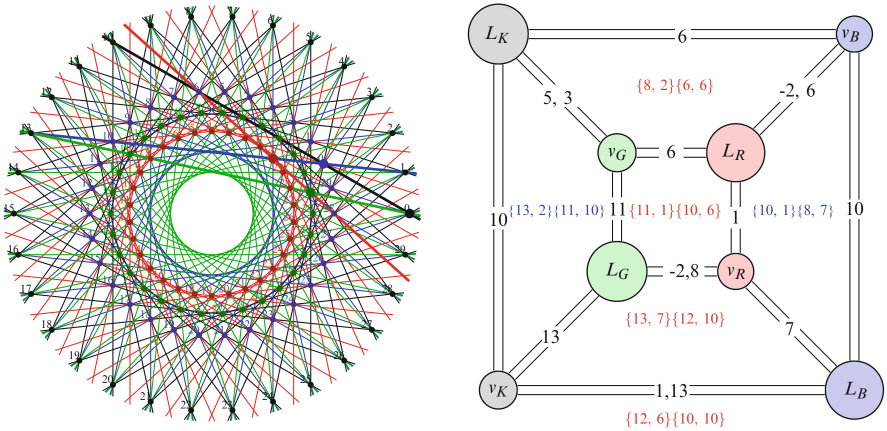

Fig. 8 A 4-astral 6-configuration and its reduced Levi graph, with outside spans 10, 12, 13. Arcs in the reduced Levi graph with a single label have the other label 0. The span of a line corresponding to a double-arc labelled a, b is $b - a$. The annotations inside the cells (including the outside cell) of the reduced Levi graph correspond to the component cohorts for the astral (2-celestial) 4-configurations; *red cohorts* are sporadic and *blue cohorts* are systematic

cubical graph with group $\mathbb{Z}_{30}$; each doubled four-cycle in the reduced Levi graph corresponds to a certain astral 4-configuration, indicated in the center of the cell. The red annotations list the corresponding cohort for a sporadic astral 4-configurations, while the blue cohorts are systematic; the cohorts are listed in the order corresponding to their list in Grünbaum's monograph [18, Tables 3.61, 3.62].

It is likely that the configuration shown in Fig. 7b and the configuration described in Fig. 9d are polars, as are the configurations described in Fig. 9a, b, while the configuration described in Fig. 9c, which is completely made of sporadic astral 4-configurations, is probably self-polar. However, this has not yet been proved.

By brute-force methods, it is possible to show that it is impossible to label the edges of the cubical reduced Levi graph using only labels from the single systematic astral cohort $6k\#\{3k - j, j\}, \{3k - 2j, 2k\}$. No other ways to label the cubical reduced Levi graph using other combinations of sporadic and systematic astral 4-configurations have been found, but there is no complete proof known that there are no others.

5 Constructing 6-Configurations with Chiral Symmetry

Recently, Jill Faudree and the author discovered a method of constructing highly incident configurations that does not use celestial 4-configurations as building blocks; in fact, this new method produces highly symmetric but non-celestial (and potentially chiral) 4-configurations, in addition to 5- and 6-configurations (relevant

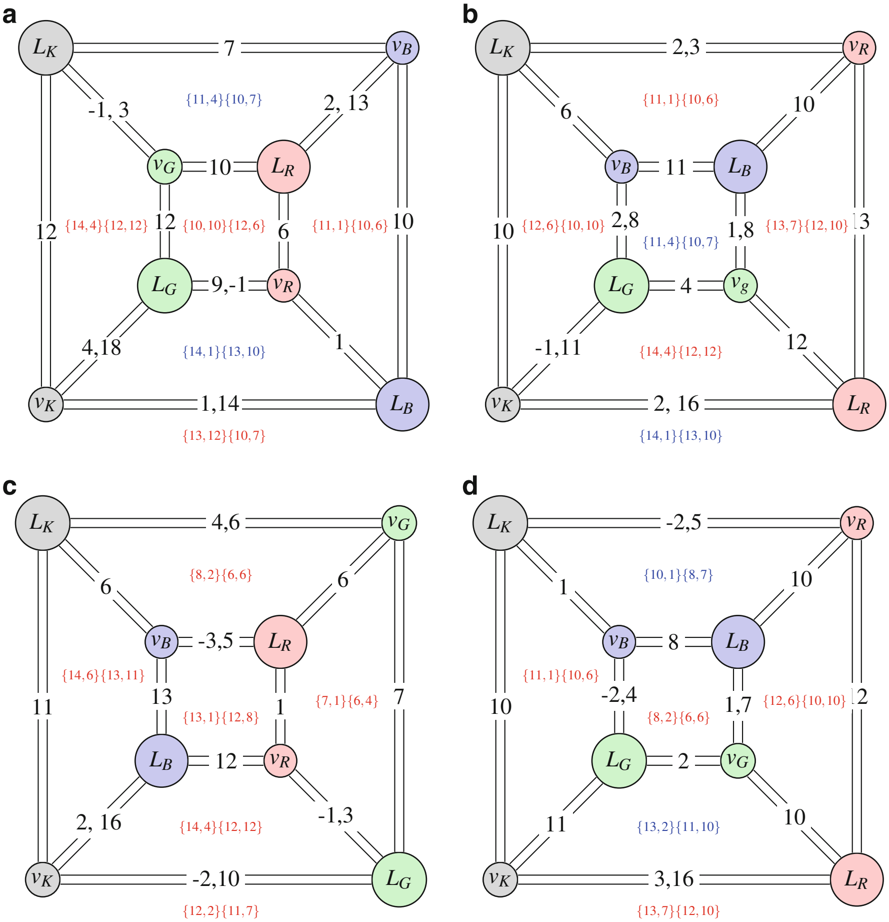

Fig. 9 Reduced Levi graphs for the other four known 4-astral 6-configurations. (**a**) Outside spans 12, 13, 14. (**b**) Outside spans 10, 12, 14. (**c**) Outside spans 11, 12, 14. (**d**) Outside spans 10, 11, 13

for the current work). Details are available in [11]. The method depends on the *Configuration Construction Lemma*, which may be phrased as follows:

Theorem 1. *Let $v_0, \ldots, v_{m-1}$ be the vertices of a regular m-gon centered at $\mathcal{O}$. Let $\mathcal{C}$ be the circle passing through v_d, $\mathcal{O}$ and v_{d-b}, and let w_0 be any point on $\mathcal{C}$. Finally, let w_i be the rotation of w_0 through $\frac{2\pi}{m}$ about $\mathcal{O}$. Then the points w_0, w_b and v_d are collinear.*

(In fact, the lemma is more broadly true: if $\mathcal{C}$ is a circle passing through Q, $\mathcal{O}$ and Q', where Q' is the rotation of Q through some angle β about $\mathcal{O}$, if R is any point on $\mathcal{C}$ and if R' is the rotation of R through angle β, then R, R' and Q are collinear. The proof is simple enough to be an exercise for a college geometry course, and uses only basic results about cyclic quadrilaterals.)

The basic construction technique described below was initially developed to analyze how to construct the 4-configuration shown in Fig. 2b. Adapted as follows, it will produce a usually chirally symmetric 6-configuration with $16 = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$ symmetry classes of points and lines. Given discrete parameters $m \geq 9$, $1 \leq a, b < \frac{m}{2}$, and d_1, d_2, d_3, d_4 where at least $d_i \neq 0, a, b, a+b$ (the exact constraints are tricky and technical), which may be decorated (d_i') or not, we construct a 6-configuration by the following sequence of steps. In subsequent figures, the 0th element of each symmetry class is shown larger.

Step 1: Begin with the vertices of a regular convex m-gon centered at $\mathscr{O}$, labelled cyclically as $(v_\emptyset)_0, \ldots, (v_\emptyset)_{m-1}$. Let $D = \{1, 2, 3, 4\}$ be the index set corresponding to the set of d_is. In subsequent figures these are black and slightly larger.

Step 2: Construct lines $(L_\emptyset)$ of span a with respect to the $(v_\emptyset)$: that is, $(L_\emptyset)_i = (v_\emptyset)_i \vee (v_\emptyset)_{i+a}$. In subsequent figures these are black.

Step 3: For each j in D, construct the circumcircle $\mathscr{C}_j$ passing through $(v_\emptyset)_{d_j}$, $\mathscr{O}$ and $(v_\emptyset)_{d_j-b}$, and let an intersection of $\mathscr{C}_j$ with $(L_\emptyset)_0$ be called $(v_j)_0$. The "decoration-status" (primed or not) of the d_j determines which intersection to choose: essentially, d_j corresponds to taking the left-most intersection, and d_j' to taking the right-most intersection. More precisely, if $(L_\emptyset)_0$ is parameterized as

$$(L_\emptyset)_0(t) := (1-t)(v_\emptyset)_0 + t(v_\emptyset)_a$$

and $\mathscr{C}_j$ intersects $(L_\emptyset)_0$ at parameter values λ_j, λ_j' with $\lambda_j < \lambda_j'$, then d_j indicates that $(v_j)_0 = (1-\lambda_j)(v_\emptyset)_0 + \lambda_j(v_\emptyset)_a$, while d_j' indicates that $(v_j)_0 = (1-\lambda_j')(v_\emptyset)_0 + \lambda_j'(v_\emptyset)_a$.

Let $(v_j)_i$ be the rotation of $(v_j)_0$ through $\frac{2\pi i}{m}$. In this step we have constructed four new vertex classes. In subsequent figures these are various shades of pink/magenta.

Step 4: Now, construct the lines $(L_j)_i := (v_j)_i \vee (v_j)_{i+b}$; these are each span b lines with respect to the just-created vertices, and by the Configuration Construction Lemma, each line $(L_j)_i$ also passes through vertex $(v_\emptyset)_{i+d_j}$. We have constructed 4 line classes at this step. In subsequent figures these are various shades of pink/magenta.

Step 5: For each pair of lines $(L_j)_i$ and $(L_k)_i$, define $(v_{jk})_i$ to be the intersection. This creates $\binom{4}{2}$ new vertex classes, one for each subset of size 2 from the index set D. In subsequent figures these are various shades of blue.

Step 6: Define $(L_{jk})_i := (v_{jk})_i \vee (v_{jk})_{i+a}$ to be lines of span a with respect to the vertices created in the previous step. In subsequent figures these are various shades of blue.

Step 7: Now, for each choice of 3-subset jkl from D, it turns out that the three lines $(L_{jk})_i$, $(L_{jl})_i$ and $(L_{kl})_i$ are all concurrent! (This is not obvious, and is part of the main content of [11].) We define $(v_{jkl})_i$ to be the common point of intersection. We have constructed $\binom{4}{3}$ symmetry classes of points at this step. In subsequent figures these are various shades of green.

Step 8: We define $(L_{jkl})_i := (v_{jkl})_i \vee (v_{jkl})_{i+b}$ to be lines of span b with respect to the just-defined vertices. It again is true but not obvious that line $(L_{jkl})_i$ passes through $(v_{jk})_{i+d_l}$, $(v_{jl})_{i+d_k}$ and $(v_{kl})_{i+d_j}$, as well as the vertices $(v_{jkl})_i$ and $(v_{jkl})_{i+b}$, for a total of 5 vertices so far. We have constructed $\binom{4}{3}$ symmetry classes of lines at this step. In subsequent figures these are various shades of green.

Step 9: Finally, the four lines $(L_{123})_i$, $(L_{124})_i$, $(L_{134})_i$ and $(L_{234})_i$ all intersect in a single point! We name this final class of points $(v_{1234})_i$. This class is shown in red.

Step 10: The last line class is $(L_{1234})_i := (v_{1234})_i \vee (v_{1234})_{i+a}$; in addition to $(v_{1234})_i$ and $(v_{1234})_{i+a}$, this passes through $(v_{123})_{i+d_4}$, $(v_{124})_{i+d_3}$, $(v_{134})_{i+d_2}$, and $(v_{234})_{i+d_1}$. This class is also shown in red.

Figure 10 shows a completed, chirally symmetric (272_6) configuration ($m = 17$), and Fig. 11 shows a (240_6) configuration ($m = 15$) constructed using this technique that has dihedral symmetry. While many 6-configurations constructed using the chiral construction technique are so cluttered that they are almost unintelligible, some are very attractive, and some are rather strange. The example shown in Fig. 11 is both very pretty and has the unexpected property that although there are 16 symmetry classes, there are only 9 distinct radii, and four distinct symmetry classes have the same radius.

6 Small 6-Configurations

The chiral construction technique for 6-configurations generalizes: given a collection of parameters m, a, b, $d_1, \dots, d_{k-2}$, with index set $D = \{1, 2, \dots, k-2\}$ constructing all the points v_σ and lines L_σ for each $\sigma \subseteq D$ analogously to the above construction produces a k-configuration called $\mathscr{A}(m; a, b; d_1, \dots, d_{k-2})$, where each d_j may be primed (as d'_j) or not, as described in Step 3 of the construction algorithm, to determine which intersection of the circle and line should be chosen. In particular, the configuration shown in Fig. 2b may be denoted as $\mathscr{A}(5; 2, 2; 1', 3)$.

If the same construction technique is used to construct $\mathscr{A}(12; 4, 4; 1', 3', 5)$ then the resulting configuration—which should be a 5-configuration—has extra incidences and actually produces a 6-configuration! This results because the configuration described as $\mathscr{A}(12; 4, 4; 1')$ actually produces the astral 4-configuration $12\#(4, 1; 4, 5)$, rather than a 3-configuration, which would be expected from the general construction technique. The configuration $\mathscr{A}(12; 4, 4; 1', 3', 5)$ is a (96_6) configuration, shown in Fig. 12, and this is the smallest known 6-configuration.

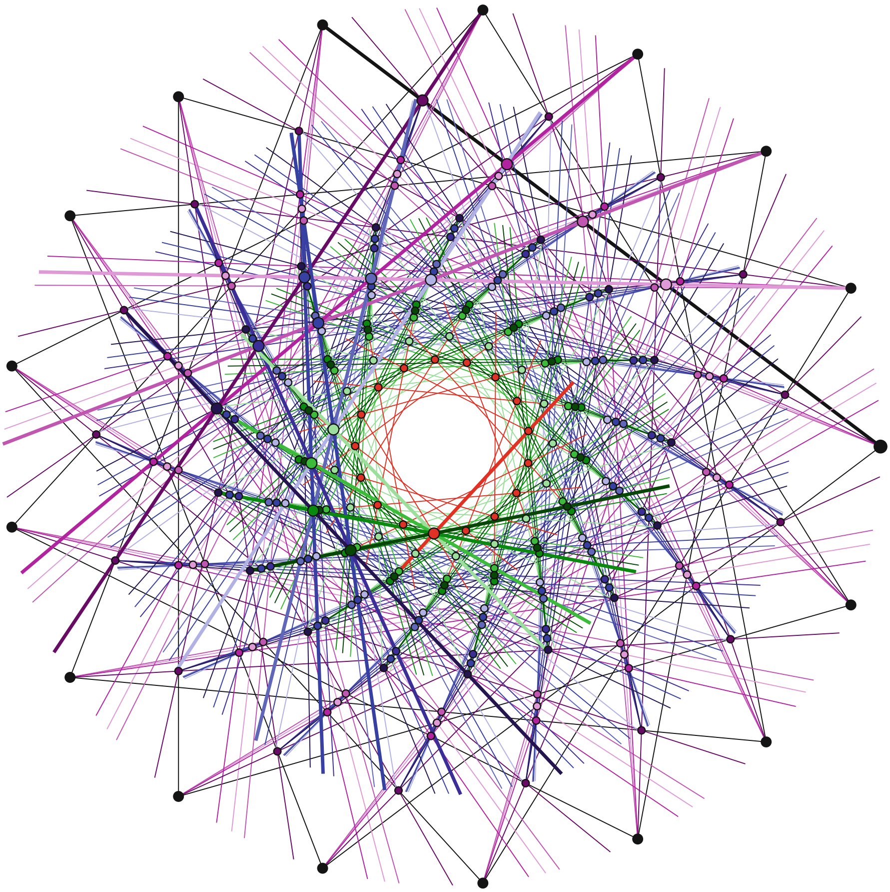

Fig. 10 A chirally symmetric 6-configuration, with $m = 17, a = 5, b = 5$, and $(d_1, d_2, d_3, d_4) = (1', 2', 3', 4')$. The vertices and lines $v_\emptyset$ and $L_\emptyset$ are *black* (and the *black vertices* are slightly larger than the other symmetry classes), vertices and lines v_j and L_j are various shades of *pink/magenta* ($j = 1, 2, 3, 4$), vertices v_{jk} and lines L_{jk} are various shades of *blue* ($jk = 12, 13, 14, 23, 24, 34$), vertices and lines v_{jkl} and L_{jkl} are various shades of *green* ($jkl = 123, 124, 134, 234$), and the vertices and lines v_{1234} and L_{1234} are *red*. The 0th element of each symmetry class is shown larger

The smallest topological 6-configuration known to the author is an (88_6) pseudoline configuration constructed analogously to the (96_6) configuration described above, only using an 11-gon as a starting point; it has several embedded topological astral (22_4) configurations. While there are other $\mathscr{A}(m; a, b; d_1, \ldots, d_{k-2})$ configurations that are "more incident" than expected because of the existence of embedded astral 4-configurations, none are smaller than (96_6).

Other small geometric 6-configurations are (110_6) from the nested celestial construction in Sect. 3, with $m = 11, (t_1, t_2, t_3) = (3, 2, 1), (s_1, s_2) = (4, 5)$ (shown in Fig. 5), (112_6) from the chiral construction, as $\mathscr{A}(7; 3, 3; 1, 2, 4, 5)$

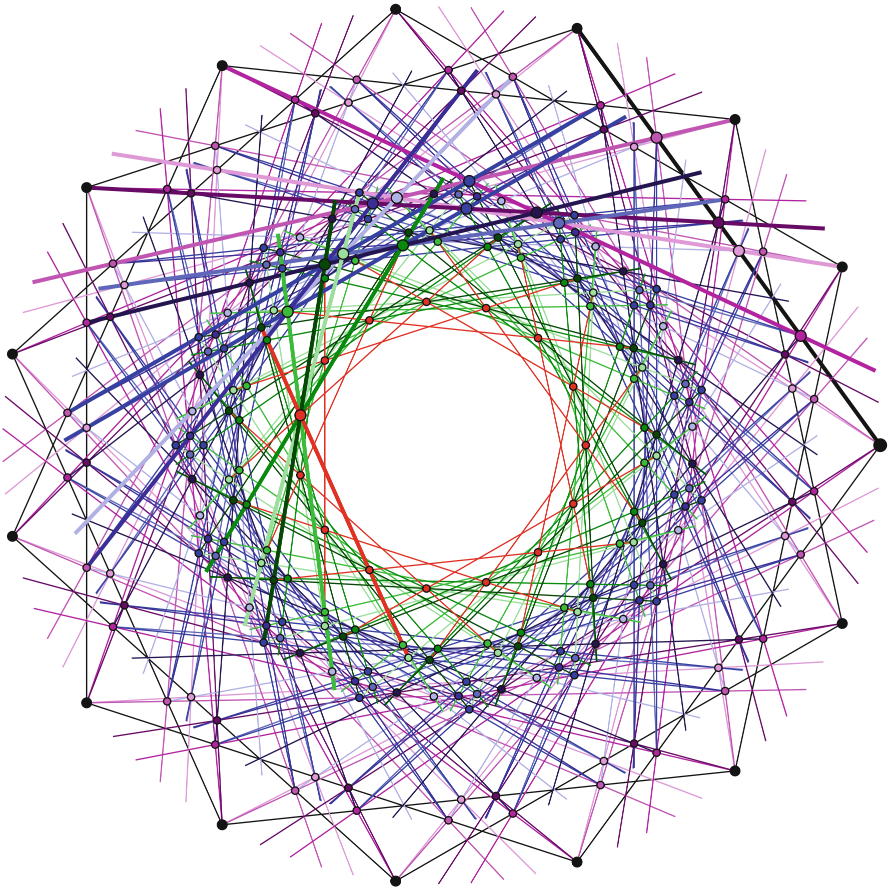

Fig. 11 The attractive, dihedrally symmetric configuration $\mathscr{A}(15; 3, 4; 1', 2', 5, 6)$ has four symmetry classes of vertices with the same radius, and a total of 9 distinct radii for the 16 symmetry classes of points

(unfortunately, the inner rings of points are clustered so close to the center of the configuration that the configuration is not really intelligible), and various (120_6) configurations (both the sporadic 4-astral 6-configurations described in Sect. 4 and systematic 10-astral configurations with $m = 12$).

7 Open Questions

Question 1. The construction method described in Sect. 3 essentially "nests" a collection of trivial 3-celestial 4-configurations to produce a 6-configuration. Is it possible to determine a new construction technique by considering other choices for

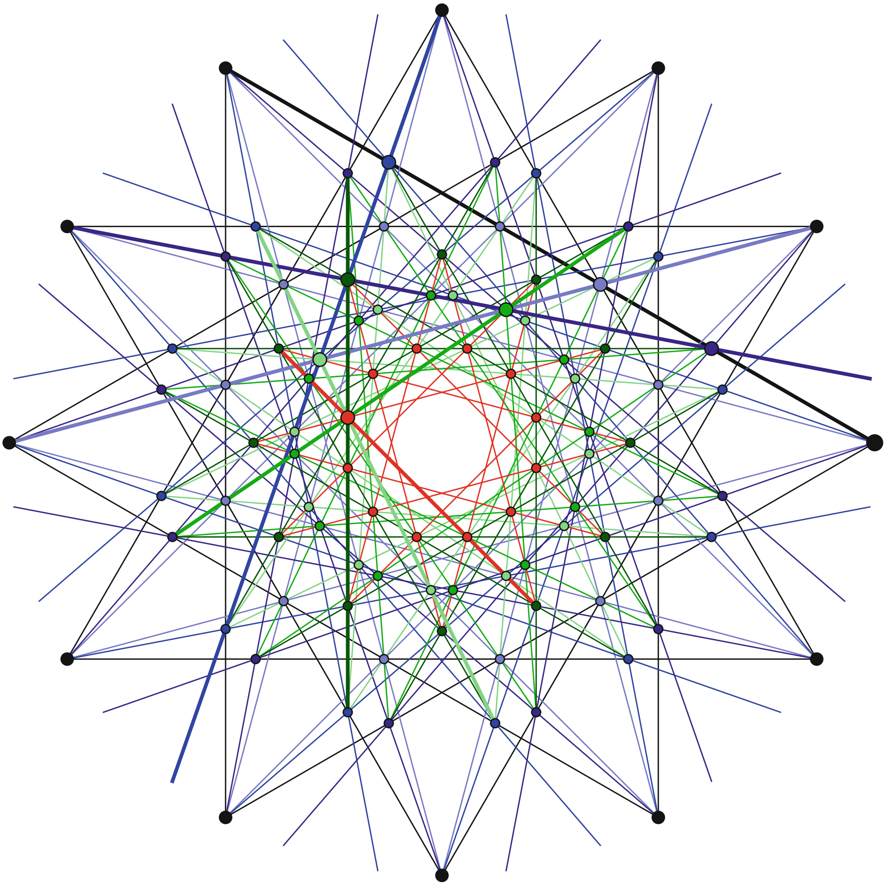

Fig. 12 The smallest known 6-configuration, a (96_6) configuration formed by applying the chiral configuration method to construct $\mathscr{A}(12; 4, 4; 1', 3', 5)$. In this figure, the vertices and lines v_j and L_j are *blue*, the vertices and lines v_{jk} and L_{jk} are *green*, and the vertices and lines v_{123} and L_{123} are *red*

labels for the reduced Levi graph? Is it possible to nest collections of other celestial 4-configurations to produce other 6-configurations? Is it possible to construct infinite families?

Question 2. Are the configurations shown in Fig. 7b and the configuration shown in Fig. 9d are polars? Are the configurations shown in Fig. 9a, b polars? Is the configuration in Fig. 9c self-polar?

Question 3. Is it possible to label the cubical reduced Levi graph using other combinations of sporadic and systematic astral 4-configurations than those described in Sect. 4?

Question 4. Does the construction method described in Sect. 4 generalize in some way? For example:

- The cubical graph is a 4-prism graph. Is it possible to construct a 6-configuration whose reduced Levi graph is a 6-prism graph whose edges are all doubled? This would correspond to a configuration which has two embedded 3-celestial 4-configurations and six 2-celestial 4-configurations. No such construction is known. What about q-prism graphs with doubled edges for some other q?
- Both the cubical graph and the Desargues graph are *bipartite Kneser graphs*. Is it possible to construct 6-configurations whose reduced Levi graphs are other bipartite Kneser graphs?
- Both the cubical graph and the Desargues graph are bipartite, symmetric, cubic graphs. There are other very nice bipartite, symmetric, cubic graphs, such as the Pappus graph. Is it possible to construct celestial 6-configurations in which the underlying reduced Levi graph is the Pappus graph with all edges doubled? Since the girth of the Pappus graph is 6 (like the Desargues graph), any such configurations would have 3-celestial 4-configurations as component graphs; unlike the 6-configurations described in Sect. 3, it is likely that such a configuration would need to use at least some systematic or sporadic configurations as components.

Question 5. What is the minimal n for the existence of a geometric (n_6) configuration. What is the minimal n for the existence of a topological (n_6) configuration?

Question 6. It is likely that there are more iterative construction methods for constructing highly incident geometric configurations, but so far no other methods have been discovered. What are some other iterative methods for producing 6-configurations?

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On External Symmetry Groups of Regular Maps

Marston D.E. Conder, Young Soo Kwon, and Jozef Širáň

Abstract Regular maps are embeddings of graphs or multigraphs on closed surfaces (which may be orientable or non-orientable), in which the automorphism group of the embedding acts regularly on flags. Such maps may admit *external symmetries* that are not automorphisms of the embedding, but correspond to combinations of well known operators that may transform the map into an isomorphic copy: duality, Petrie duality, and the 'hole operators', also known as 'taking exponents'. The group generated by the external symmetries admitted by a regular map is the *external symmetry group* of the map. We will be interested in external symmetry groups of regular maps in the case when the map admits both the above dualities (that is, if it has *trinity symmetry*) and all feasible hole operators (that is, if it is *kaleidoscopic*). Existence of finite kaleidoscopic regular maps was conjectured for every even valency by Wilson, and proved by Archdeacon, Conder and Širáň (2010).

It is well known that regular maps may be identified with quotients of extended triangle groups. In other words, these groups may be regarded as 'universal' for constructions of regular maps. It is therefore interesting to ask if similar 'universal' groups exist for kaleidoscopic regular maps with trinity symmetry. A satisfactory answer, however, is likely to be very complex, if indeed feasible at all. We demonstrate this (and other things) by a construction of an infinite family of finite kaleidoscopic regular maps with trinity symmetry, all of valency 8, such that

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the orders of their external symmetry groups are unbounded. Also we explicitly determine the external symmetry groups for the family of kaleidoscopic regular maps of even valency mentioned above.

Keywords Regular map • Group of external symmetries

Subject Classifications: 05E18, 20B25, 57M15

1 Introduction and Preliminaries

A regular map ia an embedding of a graph on a surface, such that the automorphism group of the embedding is transitive, and hence regular, on the flags of the embedding. Flags may be identified with mutually incident vertex-edge-face triples, except in the case of degenerate maps which are not considered here. Regular maps have been studied extensively, and we therefore assume that the basics are known, except for the association of regular maps with certain groups as described below. We refer the reader to the survey articles [3, 9, 13] for details concerning algebraic and topological representations of regular maps and for their further connections with group theory and the theory of Riemann surfaces.

Regular maps on surfaces can be identified with three-generator presentations $(G; a, b, c)$ of groups $G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^k = (bc)^m = (ca)^2 = \ldots = 1\rangle$. A corresponding surface embedding may be obtained by interpreting the left cosets of the subgroups $\langle a, b\rangle$, $\langle b, c\rangle$, $\langle c, a\rangle$ and $\langle c\rangle$ as vertices, face boundaries, edges and darts (ordered pairs of adjacent vertices), with their mutual incidence given by non-empty intersection of the cosets. In particular, k and m are the valency and the length of the face boundary walks of the map, and the pair (k, m) is called the *type* of the map. Although in most cases k and m will be finite, we do not exclude cases when one or both these parameters can be infinite. Observe that $\langle ab, bc\rangle$ is a subgroup of G of index at most 2, and it is well known that this index is equal to 2 if and only if the supporting surface of the map is orientable. It is easy to show [12] that two regular maps $(G; a, b, c)$ and $(G'; a', b', c',)$ are isomorphic if and only if there is an isomorphism $G \to G'$ that takes a, b and c to a', b' and c', respectively.

Such an algebraic setting immediately implies that regular maps of type (k, m) are smooth quotients of the *extended triangle group* $ET(k, m, 2)$ with presentation $\langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^k = (yz)^m = (zx)^2 = 1\rangle$. (Here 'smooth' simply means that the orders of x, y, z, xy, yz and zx are preserved in the quotient.) The group $ET(k, m, 2)$ itself is also the automorphism group of a regular map $U(k, m)$, which is a tessellation of a simply connected surface by congruent m-sided polygons, k of which meet at every vertex. The underlying surface for $U(k, m)$ is the sphere, the Euclidean plane, and the hyperbolic plane, depending on whether the quantity $1/k + 1/m$ is greater than $1/2$, equal to $1/2$, or smaller than $1/2$. In fact, a group epimorphism $ET(k, m, 2) \to G$ sending the ordered triple (x, y, z)

onto (a, b, c) corresponds to a smooth covering of the regular map $(G; a, b, c)$ by the regular map $U(k, m)$. The group $ET(k, m, 2)$ and the tessellation $U(k, m)$ may thus be regarded as 'universal' objects from which all regular maps of type (k, m) may be obtained by taking quotients. All these considerations are valid if one or both k and m are infinite; see [9].

The two most frequently studied operations on maps are duality and Petrie duality, which were thoroughly investigated in [16] and [10]. A regular map $(G; a, b, c)$ is *self-dual* if it is isomorphic to the map $(G; a', b', c')$ where $a' = c$, $b' = b$ and $c' = a$, and *self-Petrie-dual* if the map is isomorphic to $(G; a', b', c')$ where $a' = a$, $b' = b$ and $c' = ac$. In view of the above necessary and sufficient condition for map isomorphism, the map $(G; a, b, c)$ is self-dual if and only if G admits an automorphism D fixing b and interchanging a with c, and self-Petrie-dual if and only if G has an automorphism P fixing a and b and interchanging c with ac. Note that if a regular map of type (k, m) is self-dual, then necessarily $k = m$. Similarly, if we define the *'extended' type* of the map as $(k, m)_n$, where n is the order of abc (which is equal to the length of the *Petrie polygons* of the map), then necessarily $m = n$ if the map is self-Petrie-dual. If G admits both D and P as automorphisms, the regular maps is said to have *trinity symmetry*, by [1]. In such a case the group generated by D and P is isomorphic to S_3, inducing outer automorphisms of the group $\langle a, c\rangle = \{1, a, c, ac\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (and all six permutations of the parameters k, m and n), as already observed in [16].

Further operations on maps can be introduced in the context of studying different embeddings of the same graph. In a topological representation of a regular map $(G; a, b, c)$ in the form of a graph Γ embedded on a surface as described earlier, right multiplication by the element ab induces a cyclic permutation of darts around a vertex v, while multiplication by bc induces a cyclic permutation of the darts around a face f incident with v. These are known as 'rotations' of the map about v and f respectively. The order of ab is, of course, equal to the valency of the map, while the order of bc equals the face-size (or 'co-valency'). Note that if the supporting surface of the map is orientable, then these two rotations completely determine the way the underlying graph embeds. By the same token, one may take any power $(ab)^j$ of the rotation for j coprime to k (that is, $j \in \mathbb{Z}_k^*$), and embed the same graph Γ in an orientable surface with the new rotation $(ab)^j$, leaving a and c intact (and taking (ab) to $(ab)^j$ and therefore b to $a(ab)^j$). The new embedding may or may not be isomorphic to the original one; if it is, then j is said to be an *exponent* of the map.

Exponents were studied by Nedela and Škoviera in [11], but in fact the same concept was introduced earlier in [15, 16] by Wilson, who attributes it to Coxeter. Using Coxeter's terminology, for each $j \in \mathbb{Z}_k^*$ the mapping that takes the triple (a, b, c) to the triple $(a, a(ab)^j, c)$ is now known as the jth *hole operator* H_j. These 'hole operators' play an important role in determining the number of ways a given graph embeds on a surface.

In particular, for a regular map $(G; a, b, c)$ of valency k, the unit $j \in \mathbb{Z}_k^*$ is an exponent of the map precisely when the group G admits an automorphism H_j that fixes a and c and takes b to $a(ab)^j$. Any regular map of valency k that admits every unit $j \in \mathbb{Z}_k^*$ as an exponent is called *kaleidoscopic* [1], in which case the group generated by the automorphisms H_j is isomorphic to $\mathbb{Z}_k^*$.

Any one of the duality, Petrie duality and hole operators that preserves a regular map may be called an *external symmetry* of the map, as can any combination of these operators. Regular maps that are preserved by all of them are kaleidoscopic and have trinity symmetry. Such maps were investigated in [1], along with a construction that yields an example of such a finite map for every even valency. In this article, we ask how many external symmetries are possible. This question and its motivation are explained further in the next section.

2 Universal Groups and the Main Problem

In view of the 'universal' objects $ET(k,m)$ and $U(k,m)$ for construction of regular maps of type (k,m), one may ask what could be the analogous objects for construction of all regular maps with trinity symmetry. This question can be approached in two ways.

The first involves considering the tessellation $U(\infty,\infty)$ and the extended triangle group $ET(\infty,\infty,2) = \langle x, y, z \mid x^2 = y^2 = z^2 = (zx)^2 = 1\rangle$, which is isomorphic to the free product $(\mathbb{Z}_2 \times \mathbb{Z}_2) * \mathbb{Z}_2$. Since this group admits the duality and Petrie automorphisms, for which we will borrow the notation D and P as above, we know that all regular maps with trinity symmetry are quotients of $ET(\infty,\infty,2)$ by normal subgroups invariant under the action of D and P. Equivalently, one may consider the natural extension $ET(\infty,\infty,2) \rtimes \langle D, P\rangle$ containing $ET(\infty,\infty,2)$ as a normal subgroup of index 6, and then all regular maps with trinity symmetry correspond to subgroups of $ET(\infty,\infty,2)$ that are normal in $ET(\infty,\infty,2) \rtimes \langle D, P\rangle$. This principle was used in [12] to construct infinite families of finite regular maps with trinity symmetry with the help of residual finiteness of the group $ET(\infty,\infty,2)$.

The second approach is to add the necessary relations to the presentation of a group to admit the duality and the Petrie duality automorphisms. This means that the group G of a regular map $(G; a, b, c)$ of type (k,k) with trinity symmetry must, at the very least, have a presentation of the form $\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^k = (bc)^k = (ca)^2 = (abc)^k = \ldots = 1\rangle$. Equivalently, G must be a smooth quotient of the group $G^{k,k,k} = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^k = (yz)^k = (zx)^2 = (xyz)^k = 1\rangle$, admitting both the duality and Petrie duality automorphisms. The groups $G^{k,k,k}$, introduced and studied long time ago [6, 7], are finite if $k \le 5$ and are known to be infinite for all $k \ge 6$; see [8].

Comparing the two approaches, we note that the first one is easy to apply but does not offer control over the degree of the resulting maps, while in the second, we do not know the corresponding 'universal map' and we know very little about the structure of the 'universal group'; in particular, it appears to be not known if the groups $G^{k,k,k}$ are residually finite, even though they are known to have an abundance of quotients (see [4] for example).

Similarly, it is natural to ask what the 'universal' objects for constructions of all kaleidoscopic regular maps of a given valence k could be. The answer is, to some extent, similar to the one for dualities, but there are also important differences. One may begin with the universal tessellation $U(k, \infty)$ and its automorphism group $ET(k, \infty, 2) \cong (\mathbb{Z}_k * \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ with presentation $\langle x, y, z \,|\, x^2 = y^2 = z^2 = (xy)^k = (zx)^2 = 1 \rangle$. The group $ET(k, \infty, 2)$ clearly admits automorphisms $H_j : x \mapsto x,\ y \mapsto x(xy)^j,\ z \mapsto z$ for all $j \in \mathbb{Z}_k^*$, and one can form the split extension $ET(k, \infty, 2) \rtimes \mathbb{Z}_k^*$ in which $ET(k, \infty, 2)$ is normal of index $\varphi(k)$, the value of the Euler function at k. All kaleidoscopic regular maps of valency k are then quotients of $T(k, \infty, 2)$ by normal subgroups not containing any generator or any power $(xy)^i$ for $1 \le i < k$, but invariant under the automorphisms H_j for every $j \in \mathbb{Z}_k^*$; equivalently, such maps are quotients by subgroups of $ET(k, \infty, 2)$ normal in $ET(k, \infty, 2) \rtimes \mathbb{Z}_k^*$ and not containing any generator or any power $(xy)^i$ for $1 \le i < k$. In [14] this method was used in combination with residual finiteness to construct infinite families of finite kaleidoscopic regular maps of any valency $k \ge 3$. For the second approach, which also takes account of the face length m, one might consider adding to the presentation of a regular map $(G; a, b, c)$ of type (k, m) all the relations implied by the supposed existence of automorphisms H_j for all $j \in \mathbb{Z}_k^*$. This, however, is treacherous for at least two reasons. One reason is that for finite pairs (k, m) with $k \ge 7$, writing down the new relations yields presentations which do not appear to resemble any of the classes of groups studied in the past. Another reason is even worse: it is not true that kaleidoscopic regular maps exist for all types (k, m). For example, it follows from [14] that there are no such maps of type $(k, 3)$ for any $k \ge 5$ such that $k \equiv \pm 1 \bmod 6$.

The main and interesting problem now is to 'marry' the above approaches, and identify possible 'universal' kaleidoscopic regular maps having trinity symmetry, and the associated 'universal' groups. We do not see how this could be possible at the level of extended triangle groups, with (at least) one of the parameters being infinite. Indeed, a group $ET(k, m, 2)$ admits the duality automorphism if and only if $k = m$, which leaves us with considering $ET(\infty, \infty, 2)$; but for this group, when $j \ne \pm 1$ the assignment $(a, ab, c) \mapsto (a, (ab)^j, c)$ extends just to an endomorphism of $ET(\infty, \infty, 2)$, and not to an automorphism. From the previous analysis, it follows that the only viable option seems to be to consider implications of the assumption that $(G; a, b, c)$ is a kaleidoscopic regular map with trinity symmetry on the structure of G, for given finite valency k.

Before proceeding we introduce some terminology. In general, if $(G; a, b, c)$ is a regular map of valency k, then any of the operators D, P and H_j (for $j \in \mathbb{Z}_k^*$) that are admitted by the map will be called an *elementary external symmetry* of the map (and of the group G), and any combination of these admitted by the map called an *external symmetry* of the map. Also the group generated by the external symmetries will be called the *external symmetry group* of the map (and of G). Intuitively, one obtains the richest external symmetry groups from regular maps that admit both duality and Petrie duality, and all of the hole operators. This is the reason why we focus, in what follows, on external symmetry groups of kaleidoscopic regular maps with trinity symmetry.

Let $(G; a, b, c)$ be a kaleidoscopic regular map of valency k with trinity symmetry, and let E be the external symmetry group of the map—that is, the subgroup of the automorphism group of G generated by the operators D, P and H_j for all $j \in \mathbb{Z}_k^*$. Observe that every conjugate of H_j by an element of $\langle D, P\rangle \cong S_3$ fixes both a and c, while every non-trivial element of $\langle D, P\rangle$ fixes at most one of a and c. It follows that E is a semi-direct product $F \rtimes \langle D, P\rangle$ where F is the normal subgroup of E generated by all $\langle D, P\rangle$-conjugates of the hole operators H_j, for $j \in \mathbb{Z}_k^*$. In particular, E contains a subgroup isomorphic to $\mathbb{Z}_k^*$ generated by all the H_j, and even contains a subgroup isomorphic to $\mathbb{Z}_k^* \times \mathbb{Z}_2$, because each H_j commutes with P.

Also H_{k-1} acts on G like conjugation by a, while its conjugate $H_{k-1}^D = DH_{k-1}D$ acts like conjugation by c, and so commutes with H_{k-1}; moreover, P conjugates H_{k-1}^D to the product $H_{k-1}H_{k-1}^D$. In particular, if the valency k is 4 or 6, then the only non-trivial hole operator is H_{k-1}, and it follows that the largest number of external symmetries possible for valency 4 or 6 is 24, with external symmetry group isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3$. (Note: valency 3 is impossible, since the group $G^{3,3,3}$ is trivial.)

Apart from these observations, however, when writing down relations implied by the presence of the operators D, P and all the H_j, there appear to be no further general properties that can be exploited.

As we shall see, this is no accident, because in Sect. 3 we establish the existence of a kaleidoscopic map $(G; a, b, c)$ of valency 8 from an infinite quotient G of the group $G^{8,8,8}$, with an infinite external symmetry group. We also construct an infinite family of finite kaleidoscopic regular maps of valency 8 with trinity symmetry, the external symmetry groups of which are finite but of unbounded orders. These two results indicate that if there is a satisfactory answer to the question of existence of 'universal' objects for constructing kaleidoscopic regular maps with trinity symmetry, then it is likely to be very complex—even for the very restricted valency 8, let alone for general valency.

On a positive note, in Sect. 4 we determine the external symmetry group of each member of the family of kaleidoscopic maps of even degree that were constructed in [1]. This further illustrates the complexity of the general problem, from a different point of view.

3 Maps of Valency Eight with Unbounded Orders of External Symmetry Groups

The following observation destroys hope for existence of a 'nice' universal group for construction of all kaleidoscopic regular maps with trinity symmetry having a given valency. Part of the proof relies upon computations performed using the MAGMA package [2].

Theorem 1. *There exists an infinite kaleidoscopic regular map of valency* 8 *with trinity symmetry whose external symmetry group is infinite, and an infinite family of finite kaleidoscopic regular maps of valency* 8 *with trinity symmetry whose external symmetry groups have arbitrarily large orders.*

Proof. Consider the (infinite) group $G = G^{8,8,8}$ with presentation $G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^8 = (bc)^8 = (ca)^2 = (abc)^8 = 1\rangle$. With the help of MAGMA, we have identified a normal subgroup N of index 128 in G, such that N is the normal closure of the elements $(ab)^4(bc)^4$ and $(ab)^4(bac)^4$ (or equivalently, G/N has presentation $\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^8 = (bc)^8 = (ca)^2 = (abc)^8 = (ab)^4(bc)^4 = (ab)^4(bac)^4 = 1\rangle$). Based on this information, it can be checked that N is preserved by the duality operator D, the Petrie operator P, and by all of the hole operators for $k = 8$, that is, by H_j for $j \in \{1, 3, 5, 7\}$. Note that H_1 is the identity, and H_7 is simply conjugation by a, while H_3 takes the ordered triple (a, b, c) to $(a, babab, c)$, and H_5 takes (a, b, c) to $(a, abababa, c)$.

A further computation using MAGMA (but is also possible using the Reidemeister-Schreier process) reveals that N is actually generated by 16 elements subject to 8 relations, with each of the 16 generators appearing in each of the 8 relators with zero exponent sum. It follows that all the relations become trivial in the abelianisation N/N', and therefore $N/N' \cong \mathbb{Z}^{16}$. Since N' is characteristic in N, it is preserved by the elementary external symmetries D, P, and H_j for $j \in \{1, 3, 5, 7\}$. It follows that the (infinite) regular map $M = (G/N'; aN', bN', cN')$ of valency 8 is kaleidoscopic and has trinity symmetry.

Furthermore, we observe that for every positive integer m, the group $K_m = N'N^m$ (generated by all commutators and all mth powers of elements of N) is a normal subgroup of index $128m^{16}$ in G, with the quotient G/K_m being an extension of $(\mathbb{Z}_m)^{16}$ by G/N. Since K_m is characteristic in N, it is also preserved by each of the elementary external automorphisms. This gives, for every $m \geq 1$, a finite kaleidoscopic regular map $M_m = (G/K_m; aK_m, bK_m, cK_m)$ with trinity symmetry, of valency 8 and with automorphism group of order $128m^3$.

Now consider more carefully the effect of the external symmetries on the abelian subgroup $N/K_m \cong (\mathbb{Z}_m)^{16}$ of G/K_m. Each of D, P, H_3, H_5 and H_7 induces an automorphism of N/K_m order 2, while DP induces one of order 3. Similarly, each of PH_3, PH_5 and PH_7 induces an automorphism of N/K_m of order 2, while DH_7 induces one of order 4. On the other hand, each of DH_3 and DH_5 induces an automorphism of N/K_m of order $2m$ if m is even, and $4m$ if m is odd. Hence in particular, the order of the external symmetry group of the map M_m is at least $12m$.

We conclude that this family $(M_m)_{m\in\mathbb{N}}$ of finite kaleidoscopic regular maps of valency 8 with trinity symmetry contains infinitely many maps whose external symmetry groups have order larger than any given positive integer. This also implies that the infinite kaleidoscopic regular map M of valency 8 with trinity symmetry constructed above from the group G/N' has an infinite external symmetry group, despite its finite valency. □

4 External Symmetry Groups of a Family of Maps

As mentioned earlier, the definition of a kaleidoscopic regular map was introduced in [1], along with a construction of such a map $M_k^* = (G_k; a, b, c)$ of valency $2k$ for every $k \geq 1$, with $G_k = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^{2k} = (bc)^{2k} = (ca)^2 = (abc)^{2k} = (abacbcb)^2 = 1\rangle$. We sum up the relevant facts regarding the structure of this group, which were proved in (or easily follow from) [1]. The group G_k has order $8k^3$, and has a characteristic subgroup N_k isomorphic to $(\mathbb{Z}_k)^3$, generated by the three mutually commuting elements $u = (ab)^2$, $v = (bc)^2$ and $w = a(uv)^{-1}a$; indeed N_k is generated by the squares of all elements of G_k. In particular, G_k is isomorphic to an extension of $N_k \cong (\mathbb{Z}_k)^3$ by $G_k/N_k \cong (\mathbb{Z}_2)^3$.

For each k, the regular map M_k^* admits all the elementary external symmetries D, P and H_r (for $r \in \mathbb{Z}_{2k}^*$), and hence is kaleidoscopic with trinity symmetry. We now determine its external symmetry group E_k. In Sect. 2 we saw that $E_k = F_k \rtimes \langle D, P\rangle$, where F_k is the normal subgroup of E_k generated by conjugates of the hole operators H_r. It is not difficult to show (with the help of a coset diagram for N_k in G_k, or by direct verification) that conjugation by D, P and H_r of the generators u, v and w of N_k is given by

$$D: \begin{array}{l} u \mapsto v^{-1} \\ v \mapsto u^{-1} \\ w \mapsto w \end{array} \qquad P: \begin{array}{l} u \mapsto u \\ v \mapsto w^{-1} \\ w \mapsto v^{-1} \end{array} \qquad H_r: \begin{array}{l} u \mapsto u^r \\ v \mapsto u^{-i} v^{i+1} w^i \\ w \mapsto u^i v^i w^{i+1} \end{array},$$

where $r \equiv 2i+1 \bmod 2k$. Since P commutes with each H_r, it follows that the group F_k is generated by all automorphisms H_r, $H_s' = DH_sD$, and $H_t'' = PDH_tDP$, where r, s and t are relatively prime to $2k$. The above details showing the effects of the elementary external symmetries on the generators of N_k imply that F_k has a faithful representation as a subgroup L_k of $GL(3, \mathbb{Z}_k)$, given by $H_r \mapsto A_r$, $H_s' \mapsto A_s'$ and $H_t'' \mapsto A_t''$, where

$$A_r = \begin{bmatrix} r & -i & i \\ 0 & i+1 & i \\ 0 & i & i+1 \end{bmatrix}, \quad A_s' = \begin{bmatrix} j+1 & 0 & -j \\ -j & s & -j \\ -j & 0 & j+1 \end{bmatrix}, \quad A_t'' = \begin{bmatrix} \ell+1 & \ell & 0 \\ \ell & \ell+1 & 0 \\ \ell & -\ell & t \end{bmatrix},$$

for $r \equiv 2i+1 \bmod 2k$, $s \equiv 2j+1 \bmod 2k$, and $t \equiv 2\ell+1 \bmod 2k$.

It is a simple task to check that all of these matrices commute with each other, and hence that the subgroup $L_k \cong F_k$ is Abelian. To determine the order of L_k, it is sufficient to investigate conditions under which the product

$$A_r A_s' A_t'' = \begin{bmatrix} (j+\ell+1)r & (\ell-i)s & (i-j)t \\ (\ell-j)r & (i+\ell+1)s & (i-j)t \\ (\ell-j)r & (i-\ell)s & (i+j+1)t \end{bmatrix}$$

is equal to the identity matrix. This is easily seen to happen if and only if $r = s = t$ and $r^2 \equiv 1 \bmod k$. Then since there are $\varphi(2k)$ possibilities for each of r, s and t, it follows that the order of L_k is equal to $(\varphi(2k)^3)/m$, where m is the number of units r in $\mathbb{Z}_{2k}$ that are solutions of $r^2 \equiv 1 \bmod k$ (and φ is the Euler function).

By elementary number theory, if k has d distinct prime factors, then the number of distinct solutions mod k of the congruence $r^2 \equiv 1 \bmod k$ is equal to 2^{d+1}, 2^d or 2^{d-1}, depending on whether $k \equiv 0 \bmod 8$, or either $k \equiv 4 \bmod 8$ or k is odd, or $k \equiv 2 \bmod 4$, respectively. To find the number of r in $\mathbb{Z}_{2k}$ that satisfy $r^2 \equiv 1 \bmod k$, we observe the following: if k is odd, then the units mod $2k$ are in one-to-one correspondence with the units mod k, while on the other hand, if k is even, then for every unit r' mod k, both r' and $r' + k$ are units mod $2k$, and so the number of units mod $2k$ is twice the number of units mod k. Combining these facts, we find that $m = 2^{d+2}$ if $k \equiv 0 \bmod 8$, while $m = 2^{d+1}$ if $k \equiv 4 \bmod 8$, and $m = 2^d$ if either $k \equiv 2 \bmod 4$ or k is odd.

Taking into account these observations and the fact that $|E_k| = 6|F_k|$, we obtain the following:

Theorem 2. *Let d be the number of distinct prime factors of k. Then the external symmetry group E_k of the kaleidoscopic regular map M_k^* of valency $2k$ with trinity symmetry has order $6(\varphi(2k))^3/2^\beta$, where $\beta = d + 2$, $d + 1$ or d, depending on whether $k \equiv 0 \bmod 8$, $k \equiv 4 \bmod 8$, or $k \not\equiv 0 \bmod 4$, respectively. Moreover, E_k is isomorphic to a semi-direct product $F_k \rtimes \langle D, P \rangle$, where F_k is a quotient of $(\mathbb{Z}_{2k}^*)^3$.*

□

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Variance Groups and the Structure of Mixed Polytopes

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Abstract The natural mixing construction for abstract polytopes provides a way to build a minimal common cover of two regular or chiral polytopes. With the help of the chirality group of a polytope, it is often possible to determine when the mix of two chiral polytopes is still chiral. By generalizing the chirality group to a whole family of variance groups, we can explicitly describe the structure of the mix of two polytopes. We are also able to determine when the mix of two polytopes is invariant under other external symmetries, such as duality and Petrie duality.

Keywords Abstract regular polytope • Chiral polytope • Self-dual polytope • Chiral map • Petrie dual • External symmetry

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1 Introduction

The study of abstract polytopes, together with the study of maps on surfaces, is a vibrant area of current research. These fields bring together group theory, geometry, and combinatorics in a satisfying way, providing many fascinating structures to study. As in the classical theory of convex polytopes, the regular polytopes are particularly interesting. In our context, a polytope is regular if its automorphism group acts transitively on the flags. Also important are the chiral polytopes,

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whose defining features are that there are two flag orbits under the action of the automorphism group, and that flags that differ in only a single element lie in different orbits. Such polytopes occur in two mirror-image forms, but they have full rotational symmetry.

In addition to internal symmetries, which are represented by polytope automorphisms, there are a number of interesting external symmetries. Some, such as duality, have their roots in the study of convex polytopes. Others, such as Petrie duality, come more naturally from the study of maps on surfaces. Even the symmetry between a chiral polytope and its mirror image may be viewed as an external symmetry. Most of the work on external symmetries has focused on polyhedra and maps on surfaces (see [6, 12, 13, 17]), though some work has been done with polytopes in higher rank as well (see [10]).

Polytopes have a natural mixing construction, analogous to the join of two maps or hypermaps [9]. This construction lets us build the minimal common cover of two regular or chiral polytopes. Unlike joining maps, there is a significant hurdle when mixing polytopes; namely, there is no guarantee that the mix of two polytopes is itself a polytope. In some cases, we are able to determine whether the mix of two polytopes is polytopal based on simple combinatorial data.

By mixing a polytope with its images under an external symmetry or a group of symmetries, we can construct a polytope (or a slightly more general structure) that is invariant under that symmetry or symmetries. For example, we can build polytopes that are self-dual, self-Petrie, or both. Our goal then becomes determining the full structure of the mixed polytope, including its combinatorial data and its automorphism group.

Sometimes we are more interested in constructing polytopes that are *not* invariant under a given external symmetry. For example, we would like to know when the mix of two chiral polytopes is still chiral. By using the chirality group, which measures the degree to which a polytope is chiral, we can often make this determination (see [4, 7, 8]). The chirality group generalizes nicely, making it possible to measure how far a polytope is from being invariant under any external symmetry.

We start by giving background information on regular and chiral polytopes in Sect. 2. In Sect. 3, we introduce the mixing construction for polytopes and investigate the structure of mixed polytopes. Then we develop the theory of external symmetries in Sect. 4. The main result is Theorem 1, which uses a generalization of the chirality group to determine when the mix of two polytopes is invariant under an external symmetry. We then provide several consequences and examples.

2 Polytopes

General background information on abstract polytopes can be found in [15, Chs. 2, 3], and information on chiral polytopes specifically can be found in [18]. Here we review the concepts essential for this paper.

2.1 Definition of a Polytope

Let $\mathcal{P}$ be a ranked partially ordered set whose elements will be called *faces*. The faces of $\mathcal{P}$ will range in rank from -1 to n, and a face of rank j is called a *j-face*. The 0-faces, 1-faces, and $(n-1)$-faces are also called *vertices*, *edges*, and *facets*, respectively. A *flag* of $\mathcal{P}$ is a maximal chain. We say that two flags are *adjacent* if they differ in exactly one face, and that they are *j-adjacent* if they differ only in their j-face. If F and G are faces of $\mathcal{P}$ such that $F \le G$, then the *section* G/F consists of those faces H such that $F \le H \le G$.

We say that $\mathcal{P}$ is an *(abstract) polytope of rank n*, also called an *n-polytope*, if it satisfies the following four properties:

(a) There is a unique greatest face F_n of rank n and a unique least face F_{-1} of rank -1.
(b) Each flag of $\mathcal{P}$ has $n + 2$ faces.
(c) $\mathcal{P}$ is *strongly flag-connected*, meaning that if Φ and Ψ are two flags of $\mathcal{P}$, then there is a sequence of flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$ such that for $i = 0, \dots, k-1$, the flags Φ_i and Φ_{i+1} are adjacent, and each Φ_i contains $\Phi \cap \Psi$.
(d) (Diamond condition): Whenever $F < G$, where F is a $(j-1)$-face and G is a $(j+1)$-face for some j, then there are exactly two j-faces H with $F < H < G$.

Note that due to the diamond condition, any flag Φ has a unique j-adjacent flag (denoted Φ^j) for each $j = 0, 1, \dots, n-1$.

If F is a j-face and G is a k-face of a polytope with $F \le G$, then the section G/F is a $(k-j-1)$-polytope itself. We can identify a face F with the section F/F_{-1}, since if F is a j-face, then F/F_{-1} is a j-polytope. We call the section F_n/F the *co-face at F*; the co-face at a vertex is also called a *vertex-figure*.

We sometimes need to work with *pre-polytopes*, which are ranked partially ordered sets that satisfy the first, second, and fourth property above, but not necessarily the third. In this paper, all of the pre-polytopes we encounter will be *flag-connected*, meaning that if Φ and Ψ are two flags, there is a sequence of flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$ such that for $i = 0, \dots, k-1$, the flags Φ_i and Φ_{i+1} are adjacent (but we do not require each flag to contain $\Phi \cap \Psi$). When working with pre-polytopes, we apply all the same terminology as with polytopes.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes (or flag-connected pre-polytopes) of the same rank. A function $\gamma : \mathcal{P} \to \mathcal{Q}$ is called a *covering* if it preserves incidence of faces, ranks of faces, and adjacency of flags; then γ is necessarily surjective, by the flag-connectedness of $\mathcal{Q}$. We say that $\mathcal{P}$ *covers* $\mathcal{Q}$ if there exists a covering $\gamma : \mathcal{P} \to \mathcal{Q}$.

2.2 Regularity

For polytopes $\mathcal{P}$ and $\mathcal{Q}$, an *isomorphism* from $\mathcal{P}$ to $\mathcal{Q}$ is an incidence- and rank-preserving bijection on the set of faces. An isomorphism from $\mathcal{P}$ to itself is an *automorphism* of $\mathcal{P}$, and the group of all automorphisms of $\mathcal{P}$ is denoted $\Gamma(\mathcal{P})$.

We say that $\mathscr{P}$ is *regular* if the natural action of $\Gamma(\mathscr{P})$ on the flags of $\mathscr{P}$ is transitive. For convex polytopes, this definition is equivalent to any of the usual definitions of regularity (see [15, Sect. 1B]).

Given a regular polytope $\mathscr{P}$, fix a *base flag* Φ. Then the automorphism group $\Gamma(\mathscr{P})$ is generated by the *abstract reflections* $\rho_0, \ldots, \rho_{n-1}$, where ρ_i maps Φ to the unique flag Φ^i that is i-adjacent to Φ. These generators satisfy $\rho_i^2 = \epsilon$ for all i, and $(\rho_i \rho_j)^2 = \epsilon$ for all i and j such that $|i - j| \geq 2$. We say that $\mathscr{P}$ has (*Schläfli*) *type* $\{p_1, \ldots, p_{n-1}\}$ if for each $i = 1, \ldots, n-1$ the order of $\rho_{i-1}\rho_i$ is p_i (with $2 \leq p_i \leq \infty$).

For $I \subseteq \{0, 1, \ldots, n-1\}$ and a group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, we define $\Gamma_I := \langle \rho_i \mid i \in I \rangle$. The strong flag-connectivity of regular polytopes induces the following *intersection condition* in the group:

$$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J} \quad \text{for } I, J \subseteq \{0, \ldots, n-1\}. \tag{1}$$

In general, if $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a group such that each ρ_i has order 2 and such that $(\rho_i \rho_j)^2 = \epsilon$ whenever $|i - j| \geq 2$, then we say that Γ is a *string group generated by involutions* (or *sggi*). If Γ also satisfies the intersection condition (1) given above, then we call Γ a *string C-group*. There is a natural way of building a regular polytope $\mathscr{P}(\Gamma)$ from a string C-group Γ such that $\Gamma(\mathscr{P}(\Gamma)) = \Gamma$ and $\mathscr{P}(\Gamma(\mathscr{P})) = \mathscr{P}$ (see [15, Ch. 2E]). In particular, the i-faces of $\mathscr{P}(\Gamma)$ are taken to be the cosets of

$$\Gamma_i := \langle \rho_j \mid j \neq i \rangle,$$

where $\Gamma_i \varphi \leq \Gamma_j \psi$ if and only if $i \leq j$ and $\Gamma_i \varphi \cap \Gamma_j \psi \neq \emptyset$. This construction is also easily applied to any sggi (not just string C-groups), but in that case, the resulting poset is not necessarily a polytope.

If $\mathscr{P}$ and $\mathscr{Q}$ are regular n-polytopes, their automorphism groups are both quotients of the Coxeter group

$$\begin{aligned} W_n := [\infty, \ldots, \infty] = \langle \rho_0, \ldots, \rho_{n-1} \mid \rho_0^2 = \cdots = \rho_{n-1}^2 = \epsilon, \\ (\rho_i \rho_j)^2 = \epsilon \text{ when } |i - j| \geq 2 \rangle. \end{aligned} \tag{2}$$

Therefore there are normal subgroups M and K of W_n such that $\Gamma(\mathscr{P}) = W_n/M$ and $\Gamma(\mathscr{Q}) = W_n/K$. Then $\mathscr{P}$ covers $\mathscr{Q}$ if and only if $M \leq K$.

2.3 Direct Regularity and Chirality

If $\mathscr{P}$ is a regular polytope with automorphism group $\Gamma(\mathscr{P})$ generated by $\rho_0, \ldots, \rho_{n-1}$, then the *abstract rotations*

$$\sigma_i := \rho_{i-1}\rho_i \ (i = 1, \dots, n-1)$$

generate the *rotation subgroup* $\Gamma^+(\mathscr{P})$ of $\Gamma(\mathscr{P})$, which has index at most 2. We say that $\mathscr{P}$ is *directly regular* if this index is 2. This is essentially an orientability condition; for example, the directly regular polyhedra correspond to orientable regular maps. The convex regular polytopes are all directly regular.

We say that an n-polytope $\mathscr{P}$ is *chiral* if the action of $\Gamma(\mathscr{P})$ on the flags of $\mathscr{P}$ has two orbits such that adjacent flags are always in distinct orbits. For convenience, we define $\Gamma^+(\mathscr{P}) := \Gamma(\mathscr{P})$ whenever $\mathscr{P}$ is chiral. Given a chiral polytope $\mathscr{P}$ with base flag $\Phi = \{F_{-1}, F_0, \dots, F_n\}$, the automorphism group $\Gamma^+(\mathscr{P})$ is generated by elements $\sigma_1, \dots, \sigma_{n-1}$, where σ_i acts on Φ the same way that $\rho_{i-1}\rho_i$ acts on the base flag of a regular polytope. That is, σ_i sends Φ to $(\Phi^i)^{i-1}$ (which is usually denoted $\Phi^{i,i-1}$). For $i < j$, the product $\sigma_i \cdots \sigma_j$ is an involution. In analogy to regular polytopes, if the order of each σ_i is p_i, we say that the *type* of $\mathscr{P}$ is $\{p_1, \dots, p_{n-1}\}$.

The automorphism groups of chiral polytopes and the rotation groups of directly regular polytopes satisfy an intersection property analogous to that for string C-groups. Let $\Gamma^+ := \Gamma^+(\mathscr{P}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ be the automorphism group of a chiral polytope or the rotation subgroup of a directly regular polytope $\mathscr{P}$. For $1 \le i < j \le n-1$ define $\tau_{i,j} := \sigma_i \cdots \sigma_j$. By convention, we also define $\tau_{i,i} = \sigma_i$, and for $0 \le i \le n$, we define $\tau_{0,i} = \tau_{i,n} = \epsilon$. For $I \subseteq \{0, \dots, n-1\}$, set

$$\Gamma_I^+ := \langle \tau_{i,j} \mid i \le j \text{ and } i-1, j \in I \rangle.$$

Then the *intersection property* for Γ^+ is given by:

$$\Gamma_I^+ \cap \Gamma_J^+ = \Gamma_{I \cap J}^+ \quad \text{for } I, J \subseteq \{0, \dots, n-1\}. \tag{3}$$

If Γ^+ is a group generated by elements $\sigma_1, \dots, \sigma_{n-1}$ such that $(\sigma_i \cdots \sigma_j)^2 = \epsilon$ for $i < j$, and if Γ^+ satisfies the intersection property (3) above, then Γ^+ is either the automorphism group of a chiral n-polytope or the rotation subgroup of a directly regular n-polytope. In particular, it is the rotation subgroup of a directly regular polytope if and only if there is a group automorphism of Γ^+ that sends σ_1 to σ_1^{-1}, σ_2 to $\sigma_1^2\sigma_2$, and fixes every other generator.

Suppose $\mathscr{P}$ is a chiral polytope with base flag Φ and with

$$\Gamma^+(\mathscr{P}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle.$$

Let $\overline{\mathscr{P}}$ be the chiral polytope with the same underlying face-set as $\mathscr{P}$, but with base flag Φ^0. Then $\Gamma^+(\overline{\mathscr{P}}) = \langle \sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \dots, \sigma_{n-1} \rangle$. We call $\overline{\mathscr{P}}$ the *enantiomorphic form* or *mirror image* of $\mathscr{P}$. Though $\mathscr{P} \simeq \overline{\mathscr{P}}$, there is no automorphism of $\mathscr{P}$ that takes Φ to Φ^0.

Let $\Gamma^+ = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$, and let w be a word in the free group on these generators. We define the *enantiomorphic* (or *mirror image*) word $\overline{w}$ of w to be the word obtained from w by replacing every occurrence of σ_1 by σ_1^{-1} and σ_2 by $\sigma_1^2\sigma_2$, while keeping all σ_j with $j \ge 3$ unchanged. Then if Γ^+ is the rotation subgroup of a directly regular polytope, the elements of Γ^+ corresponding to w and $\overline{w}$ are conjugate in the full group Γ. On the other hand, if Γ^+ is the automorphism group of a chiral polytope, then w and $\overline{w}$ need not even have the same period. Note that $\overline{\overline{w}} = w$ for all words w.

The sections of a regular polytope are again regular, and the sections of a chiral polytope are either directly regular or chiral. Furthermore, for a chiral n-polytope, all the $(n-2)$-faces and all the co-faces at edges must be directly regular [18]. As a consequence, if $\mathscr{P}$ is a chiral polytope, it may be possible to extend it to a chiral polytope with facets isomorphic to $\mathscr{P}$, but it will then be impossible to extend that polytope once more to a chiral polytope.

Chiral polytopes only exist in ranks 3 and higher. The simplest examples are the torus maps $\{4,4\}_{(b,c)}$, $\{3,6\}_{(b,c)}$ and $\{6,3\}_{(b,c)}$, with $b, c \neq 0$ and $b \neq c$ (see [2]). These give rise to chiral 4-polytopes having toroidal maps as facets and/or vertex-figures. More examples of chiral 4- and 5-polytopes can be found in [1].

If a regular or chiral n-polytope $\mathscr{P}$ has facets $\mathscr{K}$ and vertex-figures $\mathscr{L}$, we say that $\mathscr{P}$ is of *type* $\{\mathscr{K}, \mathscr{L}\}$. If $\mathscr{P}$ is of type $\{\mathscr{K}, \mathscr{L}\}$ and it covers every other polytope of the same type, then we say that $\mathscr{P}$ is the *universal polytope of type* $\{\mathscr{K}, \mathscr{L}\}$, and we simply denote it by $\{\mathscr{K}, \mathscr{L}\}$.

If $\mathscr{P}$ and $\mathscr{Q}$ are chiral or directly regular n-polytopes, their rotation groups are both quotients of

$$W_n^+ := [\infty, \ldots, \infty]^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \mid (\sigma_i \cdots \sigma_j)^2 = \epsilon \text{ for } 1 \leq i < j \leq n-1 \rangle.$$

Therefore there are normal subgroups M and K of W_n^+ such that $\Gamma^+(\mathscr{P}) = W_n^+/M$ and $\Gamma^+(\mathscr{Q}) = W_n^+/K$. Then $\mathscr{P}$ covers $\mathscr{Q}$ if and only if $M \leq K$.

Let $\mathscr{P}$ be a chiral or directly regular polytope with $\Gamma^+(\mathscr{P}) = W_n^+/M$. We define

$$\overline{M} = \{\overline{w} \mid w \in M\}.$$

Note that $\overline{M} = \rho_0 M \rho_0$, where as before, ρ_0 is the first standard generator of W_n. If $\overline{M} = M$, then $\mathscr{P}$ is directly regular. Otherwise, $\mathscr{P}$ is chiral, and $\Gamma^+(\overline{\mathscr{P}}) = W_n^+/\overline{M}$.

2.4 Duality and Petrie Duality

For any polytope $\mathscr{P}$, we obtain the *dual of* $\mathscr{P}$ (denoted $\mathscr{P}^\delta$) by simply reversing the partial order. A *duality* from $\mathscr{P}$ to $\mathscr{Q}$ is an anti-isomorphism, that is, a bijection δ between the face sets such that $F < G$ in $\mathscr{P}$ if and only if $\delta(F) > \delta(G)$ in $\mathscr{Q}$. If a polytope is isomorphic to its dual, then it is called *self-dual*.

If $\mathscr{P}$ is of type $\{\mathscr{K}, \mathscr{L}\}$, then $\mathscr{P}^\delta$ is of type $\{\mathscr{L}^\delta, \mathscr{K}^\delta\}$. Therefore, in order for $\mathscr{P}$ to be self-dual, it is necessary (but not sufficient) that $\mathscr{K}$ be isomorphic to $\mathscr{L}^\delta$ (in which case it is also true that $\mathscr{K}^\delta$ is isomorphic to $\mathscr{L}$).

A self-dual regular polytope always possesses a duality that fixes the base flag. For chiral polytopes, this may not be the case. If a self-dual chiral polytope $\mathscr{P}$

possesses a duality that sends the base flag to another flag in the same orbit (but reversing its direction), then there is a duality that fixes the base flag, and we say that $\mathcal{P}$ is *properly self-dual* [11]. In this case, the groups $\Gamma^+(\mathcal{P})$ and $\Gamma^+(\mathcal{P}^\delta)$ have identical presentations. If a self-dual chiral polytope has no duality that fixes the base flag, then every duality sends the base flag to a flag in the other orbit, and $\mathcal{P}$ is said to be *improperly self-dual*. In this case, the groups $\Gamma^+(\overline{\mathcal{P}})$ and $\Gamma^+(\mathcal{P}^\delta)$ have identical presentations instead.

If $\mathcal{P}$ is a regular polytope with $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, then the group of $\mathcal{P}^\delta$ is $\Gamma(\mathcal{P}^\delta) = \langle \rho'_0, \ldots, \rho'_{n-1} \rangle$, where $\rho'_i = \rho_{n-1-i}$. If $\mathcal{P}$ is a directly regular or chiral polytope with $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, then the rotation group of $\mathcal{P}^\delta$ is $\Gamma^+(\mathcal{P}^\delta) = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle$, where $\sigma'_i = \sigma^{-1}_{n-i}$. Equivalently, if $\Gamma^+(\mathcal{P})$ has presentation

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid w_1, \ldots, w_k \rangle$$

then $\Gamma^+(\mathcal{P}^\delta)$ has presentation

$$\langle \sigma'_1, \ldots, \sigma'_{n-1} \mid \delta(w_1), \ldots, \delta(w_k) \rangle,$$

where if $w = \sigma_{i_1} \cdots \sigma_{i_j}$, then $\delta(w) = (\sigma'_{n-i_1})^{-1} \cdots (\sigma'_{n-i_j})^{-1}$.

Suppose $\mathcal{P}$ is a chiral or directly regular polytope with $\Gamma^+(\mathcal{P}) = W_n^+/M$. Then $\Gamma^+(\mathcal{P}^\delta) = W_n^+/\delta(M)$, where $\delta(M) = \{\delta(w) \mid w \in M\}$. If $\delta(M) = M$, then $\Gamma^+(\mathcal{P}) = \Gamma^+(\mathcal{P}^\delta)$, so $\mathcal{P}$ is properly self-dual.

If $\mathcal{P}$ is a chiral polytope, then $\overline{\mathcal{P}^\delta}$ is naturally isomorphic to $\overline{\mathcal{P}}^\delta$. Indeed, if w is a word in the generators $\sigma_1, \ldots, \sigma_{n-1}$ of $\Gamma^+(\mathcal{P})$, then

$$\delta(\overline{w}) = (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) \overline{\delta(w)} (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^{-1},$$

so we see that the presentation for $\overline{\mathcal{P}^\delta}$ is equivalent to that of $\overline{\mathcal{P}}^\delta$. In particular, if $\Gamma^+(\mathcal{P}) = W_n^+/M$, then

$$\delta(\overline{M}) = \overline{\delta(M)} := \{\overline{\delta(w)} \mid w \in M\},$$

since M is a normal subgroup of W_n^+, and thus $\overline{\delta(\overline{\delta(M)})} = M$.

There is also a second duality operation that is defined on abstract polyhedra. To start with, a *Petrie polygon* of a polyhedron is a maximal edge-path such that every two successive edges lie on a common face, but no three successive edges do. Given a polyhedron $\mathcal{P}$, its *Petrie dual* $\mathcal{P}^\pi$ consists of the same vertices and edges as $\mathcal{P}$, but its faces are the Petrie polygons of $\mathcal{P}$. Taking the Petrie dual of a polyhedron also forces the old faces to be the new Petrie polygons, so that $\mathcal{P}^{\pi\pi} \simeq \mathcal{P}$. If $\mathcal{P}$ is isomorphic to $\mathcal{P}^\pi$, then we say that $\mathcal{P}$ is *self-Petrie*.

The Petrie dual of an arbitrary polyhedron need not be a polyhedron itself. In particular, a Petrie polygon may visit a single vertex multiple times, causing there to be more than two edges incident on that Petrie polygon and vertex. When $\mathcal{P}$ is regular, however, the Petrie dual is a polyhedron except in rare cases; see [15, Sect. 7B].

3 Mixing Polytopes

The mixing operation on polytopes [15, 16] is analogous to the parallel product of groups [20], the tensor product of graphs, and the join of maps and hypermaps [9]. It gives us a natural way to find the minimal common cover of two regular or chiral polytopes. The basic method is to find the parallel product of the automorphism groups (or rotation groups) of two polytopes, and then to build a poset (usually a pre-polytope) from the resulting group. There are two main challenges. First, we want to determine how the structure of the mix depends on the two component polytopes. Second, we want to know when the mix of two polytopes is a polytope, and not just a pre-polytope. In a wide variety of cases, it is possible to easily determine the structure and polytopality of the mix.

3.1 Mixing Finitely Generated Groups

Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ and $\Gamma' = \langle x_1', \ldots, x_n' \rangle$ be finitely generated groups on n generators. Then the elements $z_i := (x_i, x_i') \in \Gamma \times \Gamma'$ (for $i = 1, \ldots, n$) generate a subgroup of $\Gamma \times \Gamma'$ that we call the *mix* of Γ and Γ', denoted $\Gamma \diamond \Gamma'$ (see [15, Ch.7A]).

If $\mathscr{P}$ and $\mathscr{Q}$ are regular polytopes, then we can mix their automorphism groups $\Gamma(\mathscr{P})$ and $\Gamma(\mathscr{Q})$. The result will be an sggi, but not necessarily a string C-group. In any case, we can always build a ranked poset from $\Gamma(\mathscr{P}) \diamond \Gamma(\mathscr{Q})$, and the result, which we emphasize might not be a polytope, is called the *mix of* $\mathscr{P}$ *and* $\mathscr{Q}$ and is denoted $\mathscr{P} \diamond \mathscr{Q}$. Similarly, if $\mathscr{P}$ and $\mathscr{Q}$ are chiral or directly regular, we can mix $\Gamma^+(\mathscr{P})$ and $\Gamma^+(\mathscr{Q})$ and build a poset from the resulting group. That poset is also called the mix of $\mathscr{P}$ and $\mathscr{Q}$ and denoted $\mathscr{P} \diamond \mathscr{Q}$. There is no chance for confusion, since if $\mathscr{P}$ and $\mathscr{Q}$ are both directly regular, then the poset built from $\Gamma(\mathscr{P}) \diamond \Gamma(\mathscr{Q})$ is the same as that built from $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$.

If the facets of $\mathscr{P}$ are isomorphic to $\mathscr{K}$ and the facets of $\mathscr{Q}$ are isomorphic to $\mathscr{L}$, then the facets of $\mathscr{P} \diamond \mathscr{Q}$ are isomorphic to $\mathscr{K} \diamond \mathscr{L}$. The vertex-figures of $\mathscr{P} \diamond \mathscr{Q}$ are analogously obtained. If $\mathscr{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$ and $\mathscr{Q}$ is of type $\{q_1, \ldots, q_{n-1}\}$, then $\mathscr{P} \diamond \mathscr{Q}$ is of type $\{\ell_1, \ldots, \ell_{n-1}\}$, where $\ell_i = \mathrm{lcm}(p_i, q_i)$ for $i \in \{1, \ldots, n-1\}$.

In order to avoid duplication, we shall usually assume that $\mathscr{P}$ and $\mathscr{Q}$ are chiral or directly regular, and we will work with $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ instead of $\Gamma(\mathscr{P}) \diamond \Gamma(\mathscr{Q})$. Most of our results are easily modified to work for $\Gamma(\mathscr{P}) \diamond \Gamma(\mathscr{Q})$ when necessary.

The automorphism group of a chiral or directly regular n-polytope can always be written as a natural quotient of W_n^+. The mix of two polytopes has a simple interpretation in terms of these quotients [8]:

Proposition 1. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes with $\Gamma^+(\mathscr{P}) = W_n^+/M$ and $\Gamma^+(\mathscr{Q}) = W_n^+/K$. Then $\Gamma^+(\mathscr{P} \diamond \mathscr{Q}) \simeq W_n^+/(M \cap K)$.*

Corollary 1. *Let $\mathscr{P}$, $\mathscr{Q}$, and $\mathscr{R}$ be chiral or directly regular n-polytopes. If $\mathscr{R}$ covers $\mathscr{P}$ and $\mathscr{Q}$, then it covers $\mathscr{P} \diamond \mathscr{Q}$.*

Dual to the mix is the *comix* of two groups. We define the comix $\Gamma \,\Box\, \Gamma'$ to be the amalgamated free product that identifies the generators of Γ with the corresponding generators of Γ'. That is, if Γ has presentation $\langle x_1, \ldots, x_n \mid R\rangle$ and Γ' has presentation $\langle x_1', \ldots, x_n' \mid S\rangle$, then $\Gamma \Box \Gamma'$ has presentation

$$\langle x_1, x_1', \ldots, x_n, x_n' \mid R, S, x_1^{-1}x_1', \ldots, x_n^{-1}x_n'\rangle.$$

Equivalently, we can just add the relations from Γ' to those of Γ, replacing each x_i' with x_i.

Just as the mix of two rotation groups has a simple description in terms of quotients of W_n^+, so does the comix of two rotation groups [3]:

Proposition 2. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes with $\Gamma^+(\mathscr{P}) = W_n^+/M$ and $\Gamma^+(\mathscr{Q}) = W_n^+/K$. Then $\Gamma^+(\mathscr{P})\Box\Gamma^+(\mathscr{Q}) \simeq W_n^+/MK$.*

The mixing and comixing operations on polytopes are commutative and associative in the sense that, for example, $\mathscr{P} \diamond \mathscr{Q}$ is naturally isomorphic to $\mathscr{Q} \diamond \mathscr{P}$. Furthermore, $\mathscr{P} \diamond \mathscr{P}$ and $\mathscr{P}\Box\mathscr{P}$ are both naturally isomorphic to $\mathscr{P}$. However, even if $\mathscr{P} \simeq \mathscr{Q}$, it may be the case that $\mathscr{P} \diamond \mathscr{Q} \not\simeq \mathscr{P}$. For example, if $\mathscr{P}$ is a chiral polytope, then $\mathscr{P} \simeq \overline{\mathscr{P}}$, but $\mathscr{P} \diamond \overline{\mathscr{P}}$ is not isomorphic to $\mathscr{P}$.

3.2 Variance Groups and the Structure of the Mix

There is a natural epimorphism from $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ to $\Gamma^+(\mathscr{P})$, sending each generator (σ_i, σ_i') to σ_i. By studying the kernel of this epimorphism and the analogous epimorphism to $\Gamma^+(\mathscr{Q})$, we can determine the structure of the $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$.

Definition 1. Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes. We denote the kernel of the natural epimorphism

$$f : \Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q}) \to \Gamma^+(\mathscr{P})$$

by $X(\mathscr{Q}|\mathscr{P})$, and we call it *the variance group of $\mathscr{Q}$ with respect to $\mathscr{P}$*. Similarly, we denote the kernel of the natural epimorphism

$$f' : \Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q}) \to \Gamma^+(\mathscr{Q})$$

by $X(\mathscr{P}|\mathscr{Q})$ and we call it the variance group of $\mathscr{P}$ with respect to $\mathscr{Q}$. In other words, $X(\mathscr{Q}|\mathscr{P})$ consists of the elements of $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ of the form (ϵ, w') (with $w' \in \Gamma^+(\mathscr{Q})$), and $X(\mathscr{P}|\mathscr{Q})$ consists of the elements of $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ of the form (w, ϵ) (with $w \in \Gamma^+(\mathscr{P})$).

By representing $\Gamma^+(\mathscr{P})$ as W_n^+/M and $\Gamma^+(\mathscr{Q})$ as W_n^+/K, we easily obtain the following:

Proposition 3. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes, with $\Gamma^+(\mathscr{P}) = W_n^+/M$ and $\Gamma^+(\mathscr{Q}) = W_n^+/K$. Then:*

(a) $X(\mathscr{P}|\mathscr{Q}) \simeq K/(M \cap K) \simeq MK/M$ *and* $X(\mathscr{Q}|\mathscr{P}) \simeq M/(M \cap K) \simeq MK/K$.
(b) *Let* $g : \Gamma^+(\mathscr{P}) \to \Gamma^+(\mathscr{P})\square\Gamma^+(\mathscr{Q})$ *and* $g' : \Gamma^+(\mathscr{Q}) \to \Gamma^+(\mathscr{P})\square\Gamma^+(\mathscr{Q})$ *be the natural epimorphisms. Then* $\ker g \simeq X(\mathscr{P}|\mathscr{Q})$ *and* $\ker g' \simeq X(\mathscr{Q}|\mathscr{P})$. *In particular,* $X(\mathscr{P}|\mathscr{Q})$ *and* $X(\mathscr{Q}|\mathscr{P})$ *can be viewed as normal subgroups of* $\Gamma^+(\mathscr{P})$ *and* $\Gamma^+(\mathscr{Q})$*, respectively.*

The fact that the natural epimorphisms $f' : \Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q}) \to \Gamma^+(\mathscr{Q})$ and $g : \Gamma^+(\mathscr{P}) \to \Gamma^+(\mathscr{P})\square\Gamma^+(\mathscr{Q})$ have isomorphic kernels allows us to use the comix of two polytopes to derive information about the mix. The following properties are immediate:

Proposition 4. *Let $\mathscr{P}$ and $\mathscr{Q}$ be finite chiral or directly regular n-polytopes. Then:*

(a) $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ *is finite, and*

$$|\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})| = |X(\mathscr{P}|\mathscr{Q})| \cdot |\Gamma^+(\mathscr{Q})| = |X(\mathscr{Q}|\mathscr{P})| \cdot |\Gamma^+(\mathscr{P})|.$$

(b) $\Gamma^+(\mathscr{P})\square\Gamma^+(\mathscr{Q})$ *is finite, and*

$$|\Gamma^+(\mathscr{P})\square\Gamma^+(\mathscr{Q})| = \frac{|\Gamma^+(\mathscr{P})|}{|X(\mathscr{P}|\mathscr{Q})|} = \frac{|\Gamma^+(\mathscr{Q})|}{|X(\mathscr{Q}|\mathscr{P})|}.$$

(c)

$$|\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})| \cdot |\Gamma^+(\mathscr{P})\square\Gamma^+(\mathscr{Q})| = |\Gamma^+(\mathscr{P})| \cdot |\Gamma^+(\mathscr{Q})|.$$

(d)

$$\frac{|X(\mathscr{P}|\mathscr{Q})|}{|X(\mathscr{Q}|\mathscr{P})|} = \frac{|\Gamma^+(\mathscr{P})|}{|\Gamma^+(\mathscr{Q})|}$$

Intuitively speaking, the group $X(\mathscr{Q}|\mathscr{P})$ tells us something about how many elements of $\Gamma^+(\mathscr{Q})$ do not correspond to elements of $\Gamma^+(\mathscr{P})$. If $\Gamma^+(\mathscr{P})$ covers $\Gamma^+(\mathscr{Q})$, then $X(\mathscr{Q}|\mathscr{P})$ is trivial. At the other extreme, if $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q}) = \Gamma^+(\mathscr{P}) \times \Gamma^+(\mathscr{Q})$, then $\Gamma^+(\mathscr{P})$ and $\Gamma^+(\mathscr{Q})$ have trivial overlap, and $X(\mathscr{Q}|\mathscr{P}) \simeq \Gamma^+(\mathscr{Q})$.

The group $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ is a subdirect product of $\Gamma^+(\mathscr{P})$ and $\Gamma^+(\mathscr{Q})$, and we can determine its structure explicitly:

Proposition 5. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes, with $\Gamma^+(\mathscr{P}) = \langle \sigma_1, \ldots, \sigma_n \rangle$ and $\Gamma^+(\mathscr{Q}) = \langle \sigma_1', \ldots, \sigma_n' \rangle$. Let $N = X(\mathscr{P}|\mathscr{Q})$ and $N' = X(\mathscr{Q}|\mathscr{P})$, and let $h : \Gamma^+(\mathscr{P})/N \to \Gamma^+(\mathscr{Q})/N'$ be the isomorphism sending $\sigma_i N$ to $\sigma_i' N'$. Then*

$$\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q}) = \{(u, v) \in \Gamma^+(\mathscr{P}) \times \Gamma^+(\mathscr{Q}) \mid h(uN) = vN'\}.$$

In particular, $X(\mathscr{P}|\mathscr{Q}) \times X(\mathscr{Q}|\mathscr{P})$ is a normal subgroup of $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$.

Proof. First of all, by Proposition 3,

$$\Gamma^+(\mathscr{P})/N \simeq \Gamma^+(\mathscr{P}) \Box \Gamma^+(\mathscr{Q}) \simeq \Gamma^+(\mathscr{Q})/N',$$

so that h really is an isomorphism. Let $f : \Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q}) \to \Gamma^+(\mathscr{P})$ and $f' : \Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q}) \to \Gamma^+(\mathscr{Q})$ be the natural epimorphisms, so that $N' = \ker f$ and $N = \ker f'$ (see Definition 1). Then the first part follows directly by Goursat's Lemma [14]. For the last part, note that if $u \in N$ and $v \in N'$, then $uN = N$ and $vN' = N'$. Therefore, $h(uN) = h(N) = N' = vN'$, so that $(u, v) \in \Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$. Then we see that $N \times N'$ is a subgroup of $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$, and since N is a normal subgroup of $\Gamma^+(\mathscr{P})$ and N' is a normal subgroup of $\Gamma^+(\mathscr{Q})$, it immediately follows that $N \times N'$ (i.e., $X(\mathscr{P}|\mathscr{Q}) \times X(\mathscr{Q}|\mathscr{P})$) is normal in $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$.

Now we are able to refine Proposition 4 to include infinite groups.

Proposition 6. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes. If $\Gamma^+(\mathscr{P}) \Box \Gamma^+(\mathscr{Q})$ is finite of order k, then the index of $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ in $\Gamma^+(\mathscr{P}) \times \Gamma^+(\mathscr{Q})$ is k.*

Proof. If $\Gamma^+(\mathscr{P})$ and $\Gamma^+(\mathscr{Q})$ are finite, this follows immediately from the third equation of Proposition 4. Now let $\Gamma^+(\mathscr{P})$ and $\Gamma^+(\mathscr{Q})$ be of arbitrary size, and set $N = X(\mathscr{P}|\mathscr{Q})$ and $N' = X(\mathscr{Q}|\mathscr{P})$, as in Proposition 5. If $\Gamma^+(\mathscr{P}) \Box \Gamma^+(\mathscr{Q})$ is finite of order k, then N' has index k in $\Gamma^+(\mathscr{Q})$. For each fixed $u \in \Gamma^+(\mathscr{P})$, Proposition 5 says that the element (u, v) is in $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ if and only if $h(uN) = vN'$. In other words, having fixed u we can pick any v that lies in the same (corresponding) coset. Then since N' has index k in $\Gamma^+(\mathscr{Q})$, the set of $(u, v) \in \Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ for a fixed u has "index" k in $\{u\} \times \Gamma^+(\mathscr{Q})$. Therefore, letting u range over all elements of $\Gamma^+(\mathscr{P})$, we see that $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ has index k in $\Gamma^+(\mathscr{P}) \times \Gamma^+(\mathscr{Q})$.

Corollary 2. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes. If $\Gamma^+(\mathscr{P}) \Box \Gamma^+(\mathscr{Q})$ is trivial, then $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q}) = \Gamma^+(\mathscr{P}) \times \Gamma^+(\mathscr{Q})$.*

3.3 *Polytopality of the Mix*

Our main goal is to use the mixing operation to construct new polytopes. In some cases, we can mix a polytope with a pre-polytope and still get a polytope:

Proposition 7. *Let $\mathscr{P}$ be a chiral or directly regular n-polytope with facets isomorphic to $\mathscr{K}$. Let $\mathscr{Q}$ be a chiral or directly regular n-pre-polytope with facets isomorphic to $\mathscr{K}'$. If $\mathscr{K}$ covers $\mathscr{K}'$, then $\mathscr{P} \diamond \mathscr{Q}$ is polytopal.*

Proof. Since $\mathcal{K}$ covers $\mathcal{K}'$, the facets of $\mathcal{P} \diamond \mathcal{Q}$ are isomorphic to $\mathcal{K}$. Therefore, the canonical projection from $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) \to \Gamma^+(\mathcal{P})$ is one-to-one on the subgroup of the facets, and by [8, Lemma 3.2], the group $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ has the intersection property. Therefore, $\mathcal{P} \diamond \mathcal{Q}$ is a polytope.

In general, there is no guarantee that the mix of a two polytopes is a polytope. For example, for $n \geq 4$, the mix of the n-cube with the n-orthotope is not a polytope [5]. In rank 3, however, polytopality is automatic [4]:

Proposition 8. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular polyhedra (3-polytopes). Then $\mathcal{P} \diamond \mathcal{Q}$ is a chiral or directly regular polyhedron.*

Theorem 3.7 in [4] is one example of a result that works in any rank:

Proposition 9. *Let $\mathcal{P}$ be a chiral or directly regular n-polytope of type $\{p_1, \ldots, p_{n-1}\}$, and let $\mathcal{Q}$ be a chiral or directly regular n-polytope of type $\{q_1, \ldots, q_{n-1}\}$. If p_i and q_i are relatively prime for each $i = 2, \ldots, n-2$ (but not necessarily for $i = 1$ or $i = n-1$), then $\mathcal{P} \diamond \mathcal{Q}$ is a chiral or directly regular n-polytope. Furthermore, if $n \geq 4$, then $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ is a subgroup of index 4 or less in $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$.*

We conclude this section with a negative result.

Proposition 10. *Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes. Suppose $\mathcal{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$, and that $\mathcal{Q}$ is of type $\{q_1, \ldots, q_{n-1}\}$. Let $r_i = \gcd(p_i, q_i)$ for $i \in \{1, \ldots, n-1\}$. If there is an integer $m \in \{2, \ldots, n-2\}$ such that $r_{m-1} = r_{m+1} = 1$ and $r_m \geq 3$, then $\mathcal{P} \diamond \mathcal{Q}$ is not a polytope.*

Proof. Let $\Gamma^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, $\Gamma^+(\mathcal{Q}) = \langle \sigma_1', \ldots, \sigma_{n-1}' \rangle$, and $\beta_i = (\sigma_i, \sigma_i')$ for each $i \in \{1, \ldots, n-1\}$. To show that $\mathcal{P} \diamond \mathcal{Q}$ is not polytopal, it suffices to show that

$$\langle \beta_{m-1}, \beta_m \rangle \cap \langle \beta_m, \beta_{m+1} \rangle \neq \langle \beta_m \rangle.$$

Now, since p_{m-1} and q_{m-1} are relatively prime, there is an integer k such that $kp_{m-1} \equiv 1 \pmod{q_{m-1}}$. Then since the order of σ_{m-1} is p_{m-1} and the order of σ_{m-1}' is q_{m-1}, we see that

$$\beta_{m-1}^{kp_{m-1}} = (\sigma_{m-1}^{kp_{m-1}}, (\sigma_{m-1}')^{kp_{m-1}}) = (\epsilon, \sigma_{m-1}'),$$

and therefore

$$(\beta_{m-1}^{kp_{m-1}} \beta_m)^2 = (\sigma_m^2, (\sigma_{m-1}' \sigma_m')^2) = (\sigma_m^2, \epsilon),$$

since we have $(\sigma_i' \sigma_{i+1}')^2 = \epsilon$ for any $i \in \{1, \ldots, n-2\}$. Thus, $(\sigma_m^2, \epsilon) \in \langle \beta_{m-1}, \beta_m \rangle$. Similarly, there is an integer k' such that $k' p_{m+1} \equiv 1 \pmod{q_{m+1}}$, and thus

$$(\beta_m \beta_{m+1}^{k' p_{m+1}})^2 = (\sigma_m^2, (\sigma_m' \sigma_{m+1}')^2) = (\sigma_m^2, \epsilon).$$

Therefore, $(\sigma_m^2, \epsilon) \in \langle \beta_m, \beta_{m+1} \rangle$ as well. So we see that

$$(\sigma_m^2, \epsilon) \in \langle \beta_{m-1}, \beta_m \rangle \cap \langle \beta_m, \beta_{m+1} \rangle.$$

Now, since $r_m \geq 3$, there is no integer k such that $\sigma_m^k = \sigma_m^2$ and $(\sigma_m')^k = \epsilon$. Therefore, $(\sigma_m^2, \epsilon) \notin \langle \beta_m \rangle$, and that proves the claim.

4 Measuring Invariance

In this section, we develop the theory of internal and external invariance of polytopes; the distinction is similar to that between inner and outer automorphisms of a group. Our framework provides a unified way to measure the extent to which a polytope is chiral, self-dual, or self-Petrie. Our goal is then to understand how the variance of $\mathcal{P} \diamond \mathcal{Q}$ depends on $\mathcal{P}$ and $\mathcal{Q}$, and to use this knowledge to build polytopes with or without specified symmetries.

4.1 External and Internal Invariance

Our study of invariance starts with the symmetries of

$$\begin{aligned} W_n := [\infty, \ldots, \infty] = \langle \rho_0, \ldots, \rho_{n-1} \mid \rho_0^2 = \cdots = \rho_{n-1}^2 = \epsilon, \\ (\rho_i \rho_j)^2 = \epsilon \text{ when } |i - j| \geq 2 \rangle, \end{aligned}$$

the automorphism group of the universal n-polytope. Let $\mathcal{P}$ be a regular n-polytope with base flag Φ. The group W_n acts on the flags of $\mathcal{P}$ by $\Phi^{j_1,\ldots,j_k} \rho_i = \Phi^{i,j_1,\ldots,j_k}$. If M is the stabilizer of the base flag Φ under this action, then M is normal in W_n and $\Gamma(\mathcal{P}) = W_n / M$.

Suppose φ is in $\mathrm{Aut}(W_n)$, the group of group automorphisms of W_n, and define $\mathcal{P}^\varphi$ to be the flagged poset built from $W_n / \varphi(M)$. If φ fixes M (globally), then $\mathcal{P}$ and $\mathcal{P}^\varphi$ are naturally isomorphic, and we shall consider them equal. On the other hand, if $\varphi(M) \neq M$, then the polytopes $\mathcal{P}$ and $\mathcal{P}^\varphi$ are distinct, and they need not be isomorphic, even though φ induces an isomorphism of their automorphism groups.

Similarly, if $\mathcal{P}$ is a chiral or directly regular n-polytope with base flag Φ, then

$$W_n^+ := [\infty, \ldots, \infty]^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \mid (\sigma_i \cdots \sigma_j)^2 = \epsilon \text{ for } 1 \leq i < j \leq n-1 \rangle$$

acts on the flags of $\mathcal{P}$ by $\Phi^{j_1,\ldots,j_k} \sigma_i = \Phi^{i,i-1,j_1,\ldots,j_k}$. If M is the stabilizer of the base flag under this action, then M is normal in W_n^+ and $\Gamma^+(\mathcal{P}) = W_n^+ / M$. Now, taking $\varphi \in \mathrm{Aut}(W_n^+)$, we similarly define $\mathscr{P}^\varphi$ to be the flagged poset built from $W_n^+/\varphi(M)$.

Definition 2. Let $\mathscr{P}$ be a regular or chiral n-polytope (or, more generally, a regular or chiral n-pre-polytope). Let φ be a group automorphism of W_n or W_n^+ (whichever is appropriate), and let $\mathscr{P}^\varphi$ be defined as above.

(a) If $\mathscr{P} = \mathscr{P}^\varphi$, we say that $\mathscr{P}$ is *internally φ-invariant*; otherwise we say that $\mathscr{P}$ is *internally φ-variant*.
(b) If $\mathscr{P} \simeq \mathscr{P}^\varphi$, we say that $\mathscr{P}$ is *externally φ-invariant*; otherwise we say that $\mathscr{P}$ is *externally φ-variant*.

Of course, if a polytope is internally φ-invariant, it must also be externally φ-invariant. Similarly, if a polytope is externally φ-variant, it must also be internally φ-variant.

Let us consider several applications. Let $\mathscr{P}$ be a regular polytope with $\Gamma(\mathscr{P}) = W_n/M$, and let $\chi_w \in \mathrm{Aut}(W_n)$ be conjugation by $w \in W_n$. Then since M is normal in W_n, χ_w fixes M. Therefore, every regular polytope is internally χ_w-invariant.

Similarly, for any element $w \in W_n$ there is an automorphism $\chi_w \in \mathrm{Aut}(W_n^+)$. If w is even (i.e., if $w \in W_n^+$), then every chiral polytope is internally χ_w-invariant. On the other hand, consider the automorphism $\chi := \chi_{\rho_0}$. This automorphism sends σ_1 to σ_1^{-1} and σ_2 to $\sigma_1^2\sigma_2$ while fixing all other generators σ_i. Then if $\mathscr{P}$ is a chiral polytope, $\mathscr{P}^\chi$ is the enantiomorphic form $\overline{\mathscr{P}}$ of $\mathscr{P}$. In particular, a chiral or directly regular polytope $\mathscr{P}$ is chiral if and only if it is not internally χ-invariant. (But note that in any case, if $\mathscr{P}$ is chiral or directly regular, it is *externally* χ-invariant.)

Moving on, let δ be the automorphism of W_n that sends each ρ_i to ρ_{n-i-1}, and let $\mathscr{P}$ be a regular n-polytope. Then $\mathscr{P}^\delta$ is the dual of $\mathscr{P}$ (and indeed, our notation for the dual was chosen in anticipation of this fact). The polytope $\mathscr{P}$ is externally δ-invariant if and only if it is self-dual. Every regular self-dual polytope has a duality that fixes the base flag while reversing the order [15], and therefore if $\mathscr{P}$ is regular and self-dual, the polytopes $\mathscr{P}$ and $\mathscr{P}^\delta$ have the same flag-stabilizer in W_n. Thus we see that a regular self-dual polytope is always internally δ-invariant.

Similarly, there is an automorphism δ^+ of W_n^+ that sends each σ_i to σ_{n-i}^{-1}. This is the automorphism induced by δ in the previous example (and by an abuse of notation, we frequently use δ to denote this automorphism of W_n^+ as well). Then a directly regular or chiral polytope $\mathscr{P}$ is externally δ^+-invariant if and only if it is self-dual. If $\mathscr{P}$ is properly self-dual (i.e., if there is a duality that fixes the base flag), then it is internally δ^+-invariant; otherwise $\mathscr{P}$ is improperly self-dual and internally δ^+-variant.

For our final example, let π be the automorphism of W_3 that sends ρ_0 to $\rho_0\rho_2$ and fixes every other ρ_i. If $\mathscr{P}$ is a regular polyhedron, then $\mathscr{P}^\pi$ is the Petrie dual of $\mathscr{P}$.

We will now explore the connection between invariance and polytope covers.

Proposition 11. *Let $\mathscr{P}$ be a chiral or regular n-polytope, and let φ be an automorphism of W_n or W_n^+, as appropriate. Suppose $\mathscr{Q}$ is a chiral or regular internally φ-invariant n-polytope that covers $\mathscr{P}$. Then $\mathscr{Q}$ covers $\mathscr{P}^\varphi$.*

Proof. Suppose $\mathscr{P}$ and $\mathscr{Q}$ are both regular; the proof is essentially the same in the other cases. We have $\Gamma(\mathscr{P}) = W_n/M$ and $\Gamma(\mathscr{Q}) = W_n/K$ for some normal subgroups M and K of W_n. Since $\mathscr{Q}$ covers $\mathscr{P}$, $K \le M$. Then $\varphi(K) \le \varphi(M)$ as well, and since $\mathscr{Q}$ is internally φ-invariant, $\varphi(K) = K$. Therefore $K \le \varphi(M)$, and so $\mathscr{Q}$ covers $\mathscr{P}^\varphi$.

Corollary 3. *Let $\mathscr{P}$ be a chiral or regular n-polytope, and let φ be an automorphism of W_n or W_n^+, as appropriate. Suppose that φ has finite order k, and that $\mathscr{Q}$ is a chiral or regular internally φ-invariant n-polytope that covers $\mathscr{P}$. Then $\mathscr{Q}$ covers $\mathscr{P} \diamond \mathscr{P}^\varphi \diamond \cdots \diamond \mathscr{P}^{\varphi^{k-1}}$.*

Proof. Repeated application of Proposition 11 shows that $\mathscr{Q}$ covers $\mathscr{P}^\varphi$, $\mathscr{P}^{\varphi^2}, \ldots,$ and $\mathscr{P}^{\varphi^{k-1}}$. Therefore, by Corollary 1, it covers their mix.

As we shall see shortly, the mix $\mathscr{P} \diamond \mathscr{P}^\varphi \diamond \cdots \diamond \mathscr{P}^{\varphi^{k-1}}$ is actually the minimal internally φ-invariant cover of $\mathscr{P}$. As such, we make the following definition.

Definition 3. Let $\mathscr{P}$ be a chiral or regular n-polytope, and let φ be an automorphism of W_n or W_n^+ (as appropriate) of finite order k. Then we define $\mathscr{P}^{\diamond\varphi}$ to be $\mathscr{P} \diamond \mathscr{P}^\varphi \diamond \cdots \diamond \mathscr{P}^{\varphi^{k-1}}$.

Proposition 12. *Let $\mathscr{P}$ be a chiral or regular n-polytope, and let φ be an automorphism of W_n or W_n^+ (as appropriate) of order finite k. Then $\mathscr{P}^{\diamond\varphi}$ is the minimal chiral or regular internally φ-invariant cover of $\mathscr{P}$.*

Proof. Since $\mathscr{P}^{\varphi^k} = \mathscr{P}$, it is clear that $(\mathscr{P}^{\diamond\varphi})^\varphi = \mathscr{P}^{\diamond\varphi}$. So $\mathscr{P}^{\diamond\varphi}$ is internally φ-invariant. By Corollary 3, every internally φ-invariant cover of $\mathscr{P}$ must cover $\mathscr{P}^{\diamond\varphi}$ as well. Thus it follows that $\mathscr{P}^{\diamond\varphi}$ is minimal.

In the rest of Sect. 4, we will usually assume that φ is an automorphism of W_n^+, and that any polytopes we deal with are chiral or directly regular. Note, however, that the definitions below all still make sense if we work with automorphisms of W_n instead and assume that our polytopes are regular.

Given an automorphism φ of W_n^+ and a chiral or directly regular polytope $\mathscr{P}$, we can consider the variance groups $X(\mathscr{P}|\mathscr{P}^\varphi)$ and $X(\mathscr{P}^\varphi|\mathscr{P})$. By Proposition 3, if $\Gamma^+(\mathscr{P}) = W_n^+/M$, then the former is isomorphic to $M\varphi(M)/M$, and the latter is isomorphic to $M\varphi(M)/\varphi(M)$. Since $M \simeq \varphi(M)$, the groups $X(\mathscr{P}|\mathscr{P}^\varphi)$ and $X(\mathscr{P}^\varphi|\mathscr{P})$ are isomorphic. We make the following definition:

Definition 4. Let $\mathscr{P}$ be a chiral or directly regular polytope of rank n. Let $\varphi \in \mathrm{Aut}(W_n^+)$. We define

$$X_\varphi(\mathscr{P}) := X(\mathscr{P}|\mathscr{P}^\varphi),$$

and we call this the *internal φ-variance group of $\mathscr{P}$* or simply the *φ-variance group of $\mathscr{P}$*.

In other words, $X_\varphi(\mathscr{P})$ is the kernel of the natural epimorphism from $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{P}^\varphi)$ to $\Gamma^+(\mathscr{P}^\varphi)$ (and also the kernel of the natural epimorphism from $\Gamma^+(\mathscr{P})$ to $\Gamma^+(\mathscr{P})\Box\Gamma^+(\mathscr{P}^\varphi)$).

The group $X_\varphi(\mathscr{P})$ gives us a measure of how different $\mathscr{P}$ is from $\mathscr{P}^\varphi$. At one extreme, $X_\varphi(\mathscr{P})$ might be trivial, in which case $\mathscr{P}$ is internally φ-invariant. At the other extreme, $X_\varphi(\mathscr{P})$ might coincide with the whole group $\Gamma^+(\mathscr{P})$; in that case, we say that $\mathscr{P}$ is *totally (internally) φ-variant*. (Again, we usually drop the word 'internally' for brevity.)

Let $\mathscr{P}$ be a chiral polytope and let χ be the automorphism of W_n^+ that sends σ_1 to σ_1^{-1} and σ_2 to σ_1^2 while fixing every other σ_i. Then the variance group $X_\chi(\mathscr{P})$ is identical to the *chirality group* $X(\mathscr{P})$, introduced in [8] for polytopes and earlier in [7] for maps and hypermaps. We can thus view φ-variance groups as a natural generalization of chirality groups.

4.2 Variance of the Mix

Using the tools we have developed, our goal now is to determine how $X_\varphi(\mathscr{P} \diamond \mathscr{Q})$ depends on $X_\varphi(\mathscr{P})$ and $X_\varphi(\mathscr{Q})$. We start with a simple result.

Proposition 13. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Then $(\mathscr{P} \diamond \mathscr{Q})^\varphi = \mathscr{P}^\varphi \diamond \mathscr{Q}^\varphi$ and $(\mathscr{P} \diamond \mathscr{Q})^{\diamond\varphi} = \mathscr{P}^{\diamond\varphi} \diamond \mathscr{Q}^{\diamond\varphi}$.*

The following lemma completely characterizes the invariance of $\mathscr{P} \diamond \mathscr{Q}$ in terms of polytope covers.

Lemma 1. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Then $\mathscr{P} \diamond \mathscr{Q}$ is internally φ-invariant if and only if it covers $\mathscr{P}^{\diamond\varphi}$ and $\mathscr{Q}^{\diamond\varphi}$.*

Proof. If $\mathscr{P} \diamond \mathscr{Q}$ is internally φ-invariant, then by Corollary 3, it covers $(\mathscr{P} \diamond \mathscr{Q})^{\diamond\varphi}$. Furthermore, by Proposition 13, the latter polytope is equal to $\mathscr{P}^{\diamond\varphi} \diamond \mathscr{Q}^{\diamond\varphi}$, which covers both $\mathscr{P}^{\diamond\varphi}$ and $\mathscr{Q}^{\diamond\varphi}$. Conversely, if $\mathscr{P} \diamond \mathscr{Q}$ covers $\mathscr{P}^{\diamond\varphi}$ and $\mathscr{Q}^{\diamond\varphi}$, then it covers $(\mathscr{P}\diamond\mathscr{Q})^{\diamond\varphi}$, which itself covers $\mathscr{P}\diamond\mathscr{Q}$. Then we must have that $(\mathscr{P}\diamond\mathscr{Q})^{\diamond\varphi} = \mathscr{P} \diamond \mathscr{Q}$; that is, $\mathscr{P} \diamond \mathscr{Q}$ must be internally φ-invariant.

Lemma 1 has several applications. For example, it tells us that $\mathscr{P} \diamond \mathscr{Q}$ is directly regular if and only if it covers $\mathscr{P} \diamond \overline{\mathscr{P}}$ and $\mathscr{Q} \diamond \overline{\mathscr{Q}}$. Similarly, $\mathscr{P} \diamond \mathscr{Q}$ is properly self-dual if and only if it covers $\mathscr{P} \diamond \mathscr{P}^\delta$ and $\mathscr{Q} \diamond \mathscr{Q}^\delta$, and it is self-Petrie if and only if it covers $\mathscr{P} \diamond \mathscr{P}^\pi$ and $\mathscr{Q} \diamond \mathscr{Q}^\pi$.

We now give the main theorem of this section.

Theorem 1. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Suppose $\mathscr{P} \diamond \mathscr{Q}$ is internally φ-invariant. Then there is a natural epimorphism from $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{Q})$ to $\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{P}^\varphi)$, and it* *restricts to an epimorphism from* $X(\mathcal{Q}|\mathcal{P})$ *to* $X_\varphi(\mathcal{P}^\varphi) = X(\mathcal{P}^\varphi|\mathcal{P})$. *Similarly, there is a natural epimorphism from* $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ *to* $\Gamma^+(\mathcal{Q}) \diamond \Gamma^+(\mathcal{Q}^\varphi)$ *that restricts to an epimorphism from* $X(\mathcal{P}|\mathcal{Q})$ *to* $X_\varphi(\mathcal{Q}^\varphi) = X(\mathcal{Q}^\varphi|\mathcal{Q})$.

Proof. Let $\Gamma^+(\mathcal{P}) = W_n^+/M$ and $\Gamma^+(\mathcal{Q}) = W_n^+/K$. By Lemma 1, since $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant, it covers $\mathcal{P}^{\diamond\varphi}$, which covers $\mathcal{P} \diamond \mathcal{P}^\varphi$. Therefore, $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ naturally covers $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\varphi)$. Since

$$\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) = W_n^+/(M \cap K)$$

and

$$\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\varphi) = W_n^+/(M \cap \varphi(M)),$$

this means that $M \cap K \leq M \cap \varphi(M)$. Thus, the group $M/(M \cap K)$ naturally covers $M/(M \cap \varphi(M))$. By Proposition 3, the former is the subgroup $X(\mathcal{Q}|\mathcal{P})$ of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$, and the latter is the subgroup $X(\mathcal{P}^\varphi|\mathcal{P})$ of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{P}^\varphi)$. The result then follows by symmetry.

Corollary 4. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be finite chiral or directly regular n-polytopes, and let* $\varphi \in Aut(W_n^+)$ *have finite order. If* $\mathcal{P} \diamond \mathcal{Q}$ *is internally* φ*-invariant, then* $|X_\varphi(\mathcal{P})|$ *divides* $|X(\mathcal{Q}|\mathcal{P})|$ *and* $|X_\varphi(\mathcal{Q})|$ *divides* $|X(\mathcal{P}|\mathcal{Q})|$.

Corollary 5. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be chiral or directly regular n-polytopes, and let* $\varphi \in Aut(W_n^+)$ *have finite order. Suppose that* $\mathcal{P}$ *has infinite* φ*-variance group* $X_\varphi(\mathcal{P})$ *and that* $\mathcal{Q}$ *is finite. Then* $\mathcal{P} \diamond \mathcal{Q}$ *is internally* φ*-variant (that is, not internally* φ*-invariant).*

Proof. Since $X(\mathcal{Q}|\mathcal{P})$ is isomorphic to a subgroup of $\Gamma^+(\mathcal{Q})$, it must be finite. Then there is no epimorphism from the finite group $X(\mathcal{Q}|\mathcal{P})$ to the infinite group $X_\varphi(\mathcal{P}^\varphi) \simeq X_\varphi(\mathcal{P})$, and thus by Theorem 1, $\mathcal{P} \diamond \mathcal{Q}$ must be internally φ-variant.

Corollary 6. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be chiral or directly regular n-polytopes, and let* $\varphi \in Aut(W_n^+)$ *have finite order. Suppose* $\mathcal{P}$ *is internally* φ*-variant and that* $\mathcal{Q}$ *has a rotation group* $\Gamma^+(\mathcal{Q})$ *that is simple. If* $X_\varphi(\mathcal{P})$ *is not isomorphic to* $\Gamma^+(\mathcal{Q})$*, then* $\mathcal{P} \diamond \mathcal{Q}$ *is internally* φ*-variant.*

Proof. Theorem 1 says that if $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant, then $X(\mathcal{Q}|\mathcal{P})$ covers $X_\varphi(\mathcal{P}^\varphi)$. Now, since $\mathcal{P}$ is internally φ-variant, $X_\varphi(\mathcal{P}^\varphi)$ is nontrivial, and since $\Gamma^+(\mathcal{Q})$ is simple, the normal subgroup $X(\mathcal{Q}|\mathcal{P})$ of $\Gamma^+(\mathcal{Q})$ is either trivial or the whole group $\Gamma^+(\mathcal{Q})$. The only way for $X(\mathcal{Q}|\mathcal{P})$ to cover $X_\varphi(\mathcal{P}^\varphi)$ is for $X(\mathcal{Q}|\mathcal{P})$ to be $\Gamma^+(\mathcal{Q})$, and then the only nontrivial group it covers is itself. Therefore, if $X_\varphi(\mathcal{P})$ (and thus $X_\varphi(\mathcal{P}^\varphi)$) is not isomorphic to $\Gamma^+(\mathcal{Q})$, then $X(\mathcal{Q}|\mathcal{P})$ cannot cover $X_\varphi(\mathcal{P}^\varphi)$, and the mix $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-variant.

We see that there are several simple tests that we can apply to determine whether $\mathcal{P} \diamond \mathcal{Q}$ is internally φ-invariant. We would like to extend the results to the mix of three or more polytopes. In order to do that, however, we need to know more about the size of $X_\varphi(\mathscr{P} \diamond \mathscr{Q})$. The following results are an easy generalization of Lemma 5.5 and Remark 5.1 in [8].

Proposition 14. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Then $X_\varphi(\mathscr{P} \diamond \mathscr{Q})$ is isomorphic to a subgroup of $X_\varphi(\mathscr{P}) \times X_\varphi(\mathscr{Q})$.*

Proposition 15. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Suppose that $\mathscr{Q}$ is internally φ-invariant. Then $X_\varphi(\mathscr{P} \diamond \mathscr{Q})$ is a normal subgroup of $X_\varphi(\mathscr{P})$.*

Next we generalize Corollary 4 to find a lower bound for $|X_\varphi(\mathscr{P} \diamond \mathscr{Q})|$.

Theorem 2. *Let $\mathscr{P}$ and $\mathscr{Q}$ be finite chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Then $|X_\varphi(\mathscr{P} \diamond \mathscr{Q})|$ is an integer multiple of $|X_\varphi(\mathscr{P})|/|X(\mathscr{Q}|\mathscr{P})|$.*

Proof. Since

$$\mathscr{P} \diamond \mathscr{Q} \diamond \mathscr{P}^\varphi \diamond \mathscr{Q}^\varphi = (\mathscr{P} \diamond \mathscr{Q}) \diamond (\mathscr{P} \diamond \mathscr{Q})^\varphi,$$

the latter covers $\mathscr{P} \diamond \mathscr{P}^\varphi$. Therefore, $|\Gamma^+(\mathscr{P}) \diamond \Gamma^+(\mathscr{P}^\varphi)|$ divides $|\Gamma^+(\mathscr{P} \diamond \mathscr{Q}) \diamond \Gamma^+((\mathscr{P} \diamond \mathscr{Q})^\varphi)|$. By Proposition 4, the former has size $|\Gamma^+(\mathscr{P})| \cdot |X_\varphi(\mathscr{P})|$, while the latter has size

$$|\Gamma^+(\mathscr{P} \diamond \mathscr{Q})| \cdot |X_\varphi(\mathscr{P} \diamond \mathscr{Q})| = |\Gamma^+(\mathscr{P})| \cdot |X(\mathscr{Q}|\mathscr{P})| \cdot |X_\varphi(\mathscr{P} \diamond \mathscr{Q})|.$$

Therefore, $|X_\varphi(\mathscr{P})|$ divides $|X(\mathscr{Q}|\mathscr{P})| \cdot |X_\varphi(\mathscr{P} \diamond \mathscr{Q})|$, and thus $|X_\varphi(\mathscr{P} \diamond \mathscr{Q})|$ is an integer multiple of $|X_\varphi(\mathscr{P})|/|X(\mathscr{Q}|\mathscr{P})|$.

Thus we see that, for instance, if $\mathscr{P}$ has a large φ-variance group $X_\varphi(\mathscr{P})$ and if $\mathscr{Q}$ is comparatively small (which forces $X(\mathscr{Q}|\mathscr{P})$ to be small), then $X_\varphi(\mathscr{P} \diamond \mathscr{Q})$ is still large.

A careful refinement lets us make a similar statement about infinite φ-variance groups:

Theorem 3. *Let $\mathscr{P}$ and $\mathscr{Q}$ be chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. If $X(\mathscr{Q}|\mathscr{P})$ is finite and $X_\varphi(\mathscr{P})$ is infinite, then $|X_\varphi(\mathscr{P} \diamond \mathscr{Q})|$ is infinite.*

Proof. Consider the commutative diagram below, where the maps are all the natural epimorphisms:

$$\begin{tikzcd}
\Gamma^+(\mathscr{P} \diamond \mathscr{P}^\varphi \diamond \mathscr{Q} \diamond \mathscr{Q}^\varphi) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \Gamma^+(\mathscr{P} \diamond \mathscr{P}^\varphi) \arrow[d, "g_1"] \\
\Gamma^+(\mathscr{P} \diamond \mathscr{Q}) \arrow[r, "g_2"'] & \Gamma^+(\mathscr{P})
\end{tikzcd}$$

Then $\ker(g_1 \circ f_1) = \ker(g_2 \circ f_2)$. Since $X_\varphi(\mathscr{P}) = \ker g_1$ is infinite by assumption, it follows that $\ker(g_1 \circ f_1)$ is infinite. Therefore, $\ker(g_2 \circ f_2)$ is infinite, and thus $\ker g_2$ and $\ker f_2$ cannot both be finite. Now, $\ker f_2 = X_\varphi(\mathscr{P} \diamond \mathscr{Q})$ and $\ker g_2 = X(\mathscr{Q}|\mathscr{P})$. Since $X(\mathscr{Q}|\mathscr{P})$ is finite by assumption, it follows that $X_\varphi(\mathscr{P} \diamond \mathscr{Q})$ must be infinite.

It is sometimes possible to fully determine $X_\varphi(\mathscr{P} \diamond \mathscr{Q})$:

Theorem 4. *Let $\mathscr{P}$ and $\mathscr{Q}$ be finite chiral or directly regular n-polytopes, and let $\varphi \in Aut(W_n^+)$ have finite order. Suppose that $\mathscr{P}$ is internally φ-variant, that $X_\varphi(\mathscr{P})$ is simple, and that $\mathscr{Q}$ is internally φ-invariant. If $|X_\varphi(\mathscr{P})|$ does not divide $|\Gamma^+(\mathscr{Q})|$, then $X_\varphi(\mathscr{P} \diamond \mathscr{Q}) = X_\varphi(\mathscr{P})$.*

Proof. Since $|X_\varphi(\mathscr{P})|$ does not divide $|\Gamma^+(\mathscr{Q})|$, the mix $\mathscr{P} \diamond \mathscr{Q}$ is internally φ-variant, by Corollary 4. Then by Proposition 15, $X_\varphi(\mathscr{P} \diamond \mathscr{Q})$ is a nontrivial normal subgroup of the simple group $X_\varphi(\mathscr{P})$. Therefore, $X_\varphi(\mathscr{P} \diamond \mathscr{Q}) = X_\varphi(\mathscr{P})$.

Now we will consider the interaction between two automorphisms φ and ψ of W_n^+.

Theorem 5. *Let $\mathscr{P}$ be a finite chiral or directly regular n-polytope, and let $\varphi, \psi \in Aut(W_n^+)$ have finite order. If $\mathscr{P} \diamond \mathscr{P}^\psi$ is internally φ-invariant, then $|X_\varphi(\mathscr{P})|$ divides $|X_\psi(\mathscr{P})|$.*

Proof. Apply Corollary 4 with $\mathscr{Q} = \mathscr{P}^\psi$.

Corollary 7. *Let $\mathscr{P}$ be a finite chiral n-polytope, and suppose that $|X(\mathscr{P})|$ (that is, $|X_\chi(\mathscr{P})|$) does not divide $|X_\delta(\mathscr{P})|$. Then $\mathscr{P} \diamond \mathscr{P}^\delta$ is a chiral pre-polytope.*

For example, let $\mathscr{P} = \{\{4,4\}_{(1,2)}, \{4,4\}_{(4,2)}\}$, a locally toroidal chiral polytope with $|\Gamma^+(\mathscr{P})| = 480$. Then a calculation with GAP [19] shows that $|X(\mathscr{P})| = 60$ and $|X_\delta(\mathscr{P})| = 4$. Therefore, by Corollary 7, $\mathscr{P} \diamond \mathscr{P}^\delta$ is a chiral pre-polytope.

Corollary 7 is essentially a restatement of [3, Thm. 5.2], and it highlights one of the principal uses of Theorem 5; namely, constructing chiral polytopes with certain external symmetries. Similar methods could be used to construct polyhedra $\mathscr{P}$ such that $\mathscr{P} = \mathscr{P}^{\pi\delta}$ but where $\mathscr{P}$ is neither self-dual or self-Petrie; see [13] for some work on constructing such polyhedra.

Finally, we note that the methods explored here could be somewhat more generalized by working with quotients of groups other than W_n and W_n^+. For example, given a polyhedron $\mathscr{P}$ of type $\{p,q\}$, the group $\Gamma(\mathscr{P})$ can be represented as a quotient of the Coxeter group $[p,q]$, or of $[p,\infty]$. These groups provide new automorphisms that W_n lacks, and would be a further source of external symmetries.

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Mobility in Symmetry-Regular Bar-and-Joint Frameworks

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Abstract In a *symmetry-regular* bar-and-joint framework of given point-group symmetry, all bars and joints occupy general positions with respect to the symmetry elements. The symmetry-extended form of Maxwell's Rule is applied to this simplest type of framework and is used to derive counts within irreducible representations for infinitesimal mechanisms and states of self stress. In particular, conditions are given for symmetry-regular frameworks to have at least one infinitesimal mechanism (respectively, state of self stress) within each irreducible representation of the point group of the framework. Similar conditions are found for symmetry-regular body-and-joint frameworks.

Keywords Infinitesimal rigidity • Mechanisms • Frameworks • Symmetry

Subject Classifications: 52C25, 70B15, 20C35

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1 Introduction

In 1864, James Clerk Maxwell published a counting rule that set out a necessary condition for a bar-and-joint framework to possess static and kinematic determinacy [22]. In the modern form due to Pellegrino and Calladine [23], the rule is

$$m - s = 3j - b - 6, \tag{1}$$

where m is the number of mechanisms, s the number of states of self stress, b is the number of bars and j the number of joints in the framework. The form (1) applies to a free-standing three-dimensional framework, but is easily altered for frameworks attached to fixed supports and/or frameworks with other dimensionality.

A symmetry-extended form of the rule has been stated by Fowler and Guest [6] in terms of point-group representations. The aim was to incorporate information not only on numbers of structural components and motifs, but also on the symmetries spanned by them in the point group of the framework, $\mathscr{G}$. In the language of point-group representations, the Maxwell Rule (1) becomes

$$\Gamma(m) - \Gamma(s) = \Gamma(j) \times \Gamma_T - \Gamma(b) - \Gamma_T - \Gamma_R, \tag{2}$$

where each Γ is the vector (or ordered set) of the traces of the corresponding representation matrices. Each such Γ is known in applied group theory as a representation of $\mathscr{G}$ [3], or in mathematical group theory as a character [17]. In this paper, the term character is reserved to denote an entry of a representation, i.e., the trace of a representation matrix. $\Gamma(m)$, $\Gamma(s)$, $\Gamma(j)$ and $\Gamma(b)$ are respectively the representations of mechanisms and states of self stress, and permutation representations of joints and bars. Γ_T and Γ_R are representations of the rigid-body translations and rotations. An equivalent statement in terms of the behaviour of the different objects under individual symmetry operations S is

$$\chi_m(S) - \chi_s(S) = \chi_j(S)\chi_T(S) - \chi_b(S) - \chi_T(S) - \chi_R(S), \tag{3}$$

where the various χ denote characters (i.e., traces of representation matrices) under operation S. For a permutation representation of a set of structureless objects, the character $\chi(S)$ is the number of objects left unshifted by S. For other types of object, the effects of the operation on signs and phases of the object must be taken into account. Techniques for calculation and manipulation of representations are described in many chemistry and physics texts, e.g., Bishop [3], and comprehensive sets of character tables are available [1, 2].

Equations (2) and (3) deal with the full set of mechanisms and states of self stress, and lead to a larger set of necessary conditions, row by row or column by column of the character table, which can often lead to the detection of mechanisms, states of self stress, or both, that may have escaped the pure counting approach embodied in (1) [5,7–11,13,14,18–21,25,26]. However, it has also proved fruitful in the study of, for example, protein mobility, to confine attention to mechanisms that are totally symmetric within the point group of the framework [27]. In such large systems, few if any structural elements occupy positions of non-trivial site symmetry, and useful global conclusions can be drawn from the study of frameworks under the restriction that all bars and joints are in general position. These restrictions, in particular the requirement that joints be in general position, also facilitate the construction of *orbit rigidity matrices* [27] with rows and columns indexed by the orbits of bars and joints, respectively, leading to practical and theoretical advantages for prediction of mechanisms and states of self stress belonging to given irreducible representations.

The present note shows how this restriction to components in general positions, when applied to the symmetry-extended Maxwell Rule (2), leads to general consequences for the distributions of mechanisms and states of self stress across the symmetries available within the point group.

2 Mobility of a Symmetry-Regular Framework

We call a bar-and-joint framework *symmetry-regular* if all joints and bars lie in general position with respect to the symmetry elements of the point group $\mathscr{G}$ that fixes the framework as a whole. In other words, no joint or bar in a symmetry-regular framework lies on any symmetry element of $\mathscr{G}$, and all joints and bars fall into (typically multiple) sets of equivalent objects permuted by all $|\mathscr{G}|$ operations of $\mathscr{G}$. These sets are *regular orbits* [24] of $\mathscr{G}$. If the framework has j joints and b bars, then

$$j = j_0|\mathscr{G}|, \tag{4}$$

$$b = b_0|\mathscr{G}|, \tag{5}$$

where j_0 and b_0 are the respective numbers of regular orbits of joints and bars, respectively. Realisation of some point groups requires the presence of multiple orbits and hence $b_0+j_0 > 1$ [16], but this condition is in fact trivially satisfied for the symmetry-regular frameworks of physical interest. The permutation representation of any single regular orbit of objects is Γ_{reg}, which has character $\chi_{\text{reg}}(E) = |\mathscr{G}|$ under the identity and $\chi_{\text{reg}}(S) = 0$ under all other operations. For all but the trivial point group $\mathscr{G} = C_1$, Γ_{reg} is reducible, and is the sum

$$\Gamma_{\text{reg}} = \sum_i g_i \Gamma_i, \tag{6}$$

where the summation runs over all the irreducible representations of the group, and g_i is the dimension of irreducible representation Γ_i, i.e., $\chi_i(E)$. Thus, Γ_{reg} contains one copy each of representations of types A and B, two of those of type E

(or only one if E is a separably degenerate representation in an Abelian point group with complex characters), and three, four and five for those of types T, G and H, respectively [3, 24].

From the expression for $\chi_{\mathrm{reg}}(S)$, it is clear that the product of Γ_{reg} with any reducible representation Γ is simply $g\Gamma_{\mathrm{reg}}$, where g is the dimension of Γ (equal to $\chi(E)$ for Γ).

For symmetry-regular frameworks, we have

$$\Gamma(j) = j_0 \Gamma_{\mathrm{reg}}, \tag{7}$$

$$\Gamma(b) = b_0 \Gamma_{\mathrm{reg}}, \tag{8}$$

and hence the symmetry-extended Maxwell Rule (2) reduces to

$$\Gamma(m) - \Gamma(s) = (3j_0 - b_0)\Gamma_{\mathrm{reg}} - \Gamma_T - \Gamma_R. \tag{9}$$

Three cases can be distinguished, according to the mobility $m - s$ computed with the scalar counting version of the Maxwell Rule. A symmetry-regular framework may have:

Case (i) $3j_0 - b_0 < 0$, and hence (by Eqs. (1), (4) and (5))

$$m - s = 3j - b - 6 = (3j_0 - b_0)|\mathscr{G}| - 6 < -6;$$

Case (ii) $3j_0 - b_0 = 0$, and hence

$$m - s = 3j - b - 6 = -6;$$

Case (iii) $3j_0 - b_0 > 0$, and hence

$$m - s = 3j - b - 6 = (3j_0 - b_0)|\mathscr{G}| - 6 > -6.$$

In Case (i), $|(3j_0 - b_0)||\mathscr{G}| + 6$ states of self stress are detectable by symmetry. Since $3j_0 - b_0 < 0$, it follows from Eqs. (6) and (9) that there are at least g_i states of self stress for each irreducible representation Γ_i, augmented by a further six states of self stress that match the symmetries of the translations and rotations in the point group. Thus, every irreducible representation of the point group occurs as the symmetry of a state of self stress in this case.

In Case (ii), Eq. (9) becomes $\Gamma(m) - \Gamma(s) = -\Gamma_T - \Gamma_R$, and hence the only states of self stress detectable by symmetry are six that match the translations and rotations in the point group, and again no mechanisms are detectable by symmetry counting alone.

Example 1. Consider the Case-(ii) symmetry-regular framework with point group $\mathscr{C}_2$ depicted in Fig. 1. This framework is a ring of four edge-sharing tetrahedra with four additional bars which correspond to the four diagonals of the cube formed by

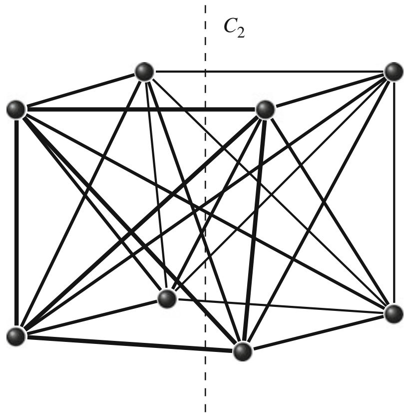

Fig. 1 A symmetry-regular framework with point group $\mathscr{C}_2$

the eight vertices. For this framework, the scalar counting version of the Maxwell Rule detects six states of self stress, because $3j - 6 - b = 3 \times 8 - 6 - 24 = -6$. The character table for the group $\mathscr{C}_2$ is

$\mathscr{C}_2$	E	C_2	Symmetry of rigid motions
A	1	1	z, R_z
B	1	-1	x, y, R_x, R_y

So, since $j_0 = 4$ and $b_0 = 12$, we have $3j_0 - b_0 = 0$, and Eq. (9) becomes

$$\Gamma(m) - \Gamma(s) = -\Gamma_T - \Gamma_R = -(A + 2B) - (A + 2B) = -2A - 4B.$$

Thus, there exist two states of self stress of symmetry A (fully symmetric self stresses) and four states of self stress of symmetry B (anti-symmetric self stresses). In particular, we detect a self stress for each irreducible representation of the group. For frameworks satisfying the condition of Case (ii), this is not true in general (since for certain point groups, $\Gamma_T + \Gamma_R$ does not contain each of the irreducible representations of the group [3]).

In Case (iii), the precise prediction for the mobility depends on the numerical value of $m - s$ and on the point group.

If an irreducible representation Γ_i has a positive weight on the LHS of Eq. (9), then every framework which is symmetric with the given group has an infinitesimal motion of symmetry Γ_i. Clearly, if the infinitesimal motion is totally symmetric (i.e., if Γ_i is the trivial irreducible representation which assigns the scalar 1 to each symmetry operation of the group), then it also extends to a finite symmetry-preserving mechanism, provided that the framework is 'generic' modulo the given symmetry constraints (or equivalently, if the orbit rigidity matrix of the framework has maximal rank). See Schulze [26] and Schulze and Whiteley [27] for details.

Similarly, an infinitesimal motion which is symmetric with respect to Γ_i, where Γ_i is not the totally symmetric irreducible representation, extends to a finite mechanism, provided that the framework is at a 'regular point' for the algebraic variety of all Γ_i-symmetric configurations (see again Schulze [26] for details). Such a mechanism preserves the sub-symmetry described by the kernel of the representation Γ_i [see 13, 26].

While it is in general difficult to check whether a framework is at a regular point for a given (not-totally-symmetric) irreducible representation Γ_i, there exist some special situations, where the presence of a finite mechanism can easily be deduced from the existence of an infinitesimal motion of symmetry Γ_i. As shown in Guest and Fowler [13], a Γ_i-symmetric motion will be finite if, in the point group of the undisplaced framework, there is neither a Γ_i-symmetric nor a totally symmetric self stress. This will be the case in many of the following examples, which do not possess a state of self stress.

As noted above, if all irreducible representations have positive weight on the LHS of (9), then there are symmetry-detectable mechanisms of all symmetries; if, however, some have zero or negative weight, there are gaps in the symmetries of the detectable mechanisms, and for those irreducible representations with negative weights, there are corresponding symmetry-detectable states of self stress. As the representation of the rigid-body motions, $\Gamma_T + \Gamma_R$, has fixed dimension 6, the detailed prediction for Case (iii) depends on the size and composition of Γ_{reg} in $\mathscr{G}$. Defining

$$\Gamma_{\text{rigid}} = \Gamma_T + \Gamma_R = \sum_i n_i \Gamma_i, \tag{10}$$

it is convenient to sub-divide Case (iii) according to the values in the sets $\{n_i\}$ *vs* $\{g_i\}$, as follows. In what follows, we will use 'detectable' as a synonym for 'detectable by symmetry'. The three sub-cases are:

Sub-case (iii)(a) $3j_0 - b_0 > 0$ and Γ_{rigid} is an exact multiple of Γ_{reg}, i.e., $\Gamma_{\text{rigid}} = k\Gamma_{\text{reg}}, k \in \mathbb{N}, k \geq 1$;

Sub-case (iii)(b) $3j_0 - b_0 > 0$ and Γ_{rigid} is contained in Γ_{reg}, i.e., $n_i \leq g_i$ for all irreducible representations Γ_i;

Sub-case (iii)(c) $3j_0 - b_0 > 0$ and Γ_{rigid} is contained in $k\Gamma_{\text{reg}}$, i.e., $n_i \leq kg_i$ for all i (where $k = 2$ or 4).

We next consider each case in turn.

Sub-case (iii)(a): Since Γ_{rigid} is an exact multiple of Γ_{reg}, it follows from the dimensions of the representations that $\Gamma_{\text{rigid}} = (6/|\mathscr{G}|)\Gamma_{\text{reg}}$, with $|\mathscr{G}| \leq 6$. The mobility representation obeys

$$\Gamma(m) - \Gamma(s) = (3j_0 - b_0 - \frac{6}{|\mathscr{G}|})\Gamma_{\text{reg}}, \tag{11}$$

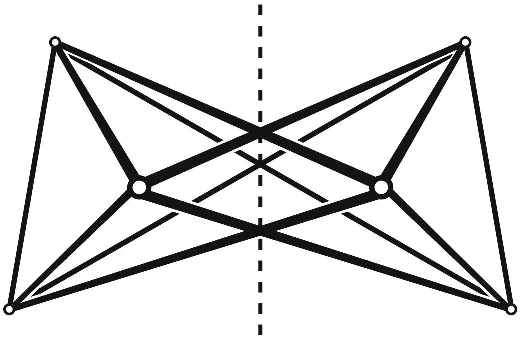

Fig. 2 An example of an isostatic framework with reflection symmetry (the mirror plane is indicated by a *dotted line*). This $\mathscr{C}_s$-symmetric framework is non-planar. *Larger* and *smaller* circles indicate joints that lie respectively in front of, and behind, the median plane of the framework parallel with the plane of the paper

with $|\mathscr{G}| = 1, 2, 3, 6$. The groups of this type are: $\mathscr{C}_1$ with $|\mathscr{G}| = 1$, $\mathscr{C}_\mathrm{s}$ and $\mathscr{C}_i$, with $|\mathscr{G}| = 2$, $\mathscr{C}_3$ with $|\mathscr{G}| = 3$, and $\mathscr{C}_{3\mathrm{v}}$, $\mathscr{C}_{3\mathrm{h}}$ and $\mathscr{S}_6$ with $|\mathscr{G}| = 6$.

For $3j_0 - b_0 < 6/|\mathscr{G}|$ (and $|\mathscr{G}| < 6$), there are detectable states of self stress spanning all symmetries. If $3j_0 - b_0 = 6/|\mathscr{G}|$, symmetry detects neither states of self stress nor mechanisms. If $3j_0 - b_0 > 6/|\mathscr{G}|$, symmetry detects mechanisms in all irreducible representations.

In particular, note that detectable states of self stress occur only for the groups $\mathscr{C}_1, \mathscr{C}_\mathrm{s}, \mathscr{C}_\mathrm{i}$ and $\mathscr{C}_3$.

Example 2. Consider the symmetry-regular $\mathscr{C}_\mathrm{s}$-symmetric framework with $j_0 = 3$ illustrated in Fig. 2. Here, b_0 has been chosen as 6, to achieve the isostatic count of $3j_0 - b_0 - 3 = 0$. Successive removal of orbits of bars, one orbit at a time, adds two mechanisms at each stage. By Eq. (11), the extra pair of mechanisms spans $\Gamma_\mathrm{reg} = A' + A''$, i.e., one of the extra mechanisms is symmetric and the other is anti-symmetric with respect to the mirror.

Sub-case (iii)(b): If the inequalities are strict for all i, (i.e., if $n_i < g_i$ for all i) then it follows from Eq. (9) that the framework has detectable mechanisms belonging to every irreducible representation of the group. The groups of this type are: $\mathscr{T}$, $\mathscr{T}_\mathrm{d}$, $\mathscr{T}_\mathrm{h}$, $\mathscr{O}$, $\mathscr{O}_\mathrm{h}$, $\mathscr{I}$, $\mathscr{I}_\mathrm{h}$.

Example 3. Consider a realisation of the cuboctahedron with point group symmetry $\mathscr{T}$ — the group of rotational symmetries of the regular tetrahedron (see also Fig. 3). This framework satisfies the condition of Case (iii) since $3j_0 - b_0 = 3 \times 1 - 2 = 1 > 0$. Moreover, from the character table of the group $\mathscr{T}$

$\mathscr{T}$	E	$4C_3$	$4C_3^2$	$3C_2$	Symmetry of rigid motions
A	1	1	1	1	
E	2	-1	-1	2	
T	3	0	0	-1	x, y, z, R_x, R_y, R_z

it follows that $\Gamma_\mathrm{reg} - \Gamma_\mathrm{rigid} = (A + E + 3T) - 2T = A + E + T$. Thus, we detect a mechanism for each of the irreducible representations of $\mathscr{T}$, one of symmetry

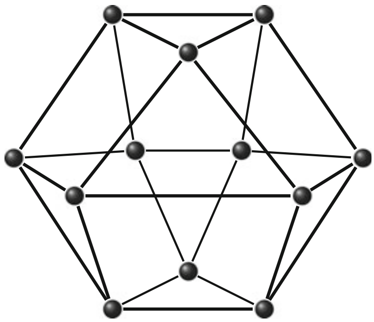

Fig. 3 A bar-and-joint framework connected as the skeleton of the cuboctahedron

Table 1 List of groups with $n_i = g_i$, and the list of missing irreducible representations in $\Gamma(m)$ for $(3j_0 - b_0) = 1$

Point group	Missing irreducible representations
$\mathscr{C}_{\text{nv}}$, with $n \geq 4$	A_1, A_2, $E_{(1)}$
$\mathscr{C}_{\text{nh}}$, with $n \geq 4$	A'/A_g, A''/A_u, E'/E_g, E''/E_u
$\mathscr{D}_{\text{2h}}$	all except A_g and A_u
$\mathscr{D}_{\text{nh}}$, with $n \geq 3$	A'_2/A_{2g}, A''_2/A_{2u}
$\mathscr{D}_{\text{2d}}$	A_2, B_2 and E
$\mathscr{D}_{\text{nd}}$, with $n \geq 3$	$A_{2(g)}$, $A_{2(u)}/B_2$
$\mathscr{S}_{\text{4n}}$, with $n > 1$	A, B, E_1, $E_{(n/2-1)}$
$\mathscr{S}_{\text{4n+2}}$, with $n \geq 1$	A_g, A_u, $E_{(1)g}$, $E_{(1)u}$

A, two of symmetry E, and three of symmetry T. Each of these mechanisms is finite since the framework clearly does not have any self stress (it is obtained from a triangulated convex polyhedron, which is isostatic by Cauchy's theorem, by removing six bars).

If, instead, $n_i = g_i$ for some i, there are systematic absences in the list of detectable mechanisms for $(3j_0 - b_0) = 1$: where $n_i = g_i$, Γ_i is missing from the list. For $(3j_0 - b_0) \geq 2$, however, all irreducible representations are present in $\Gamma(m)$. The groups of this type, and the irreducible representations missing from $\Gamma(m)$ are presented in Table 1.[1]

Example 4. The framework in Fig. 4 is symmetry-regular with respect to the symmetry group $\mathscr{C}_{4h}$ and satisfies $j_0 = 1$ and $b_0 = 2$, so that $(3j_0 - b_0) = 1$. (Note that the point group of the framework is actually the group $\mathscr{D}_{4h}$. However, the framework is not symmetry-regular with respect to $\mathscr{D}_{4h}$.) Since $\Gamma_{\text{reg}} = A_g + B_g + E_g + A_u + B_u + E_u$ and $\Gamma_{\text{rigid}} = A_g + E_g + A_u + E_u$, we detect two motions, one of symmetry B_g (which preserves the sub-group $\mathscr{C}_{2h}$) and one of symmetry B_u (which preserves the sub-group $\mathscr{S}_4$). Both of these mechanisms can be shown to be finite, although the framework also has a self stress (and hence an additional infinitesimal motion).

[1]Many published character tables for $\mathscr{S}_8$ correctly assign $\Gamma(x, y)$ to E_1 but also incorrectly assign $\Gamma(R_x, R_y)$ to E_1 instead of to E_3 [28]. The problem extends to some tables for $\mathscr{S}_{12}$, $\mathscr{S}_{16}$ and $\mathscr{S}_{20}$ [1], where the correct assignment is in fact $\Gamma(R_x, R_y) = E_{(n/2-1)}$.

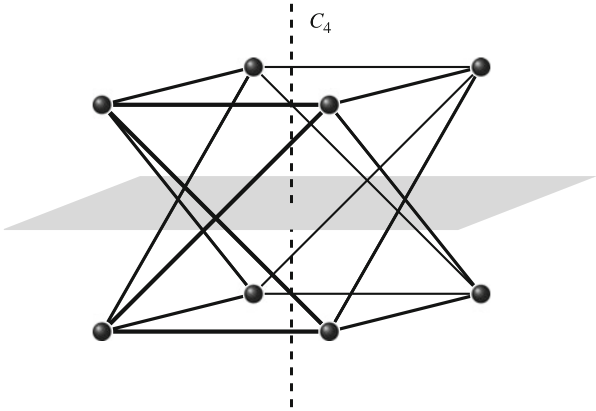

Fig. 4 A framework which is symmetry-regular with respect to $\mathscr{C}_{4h}$

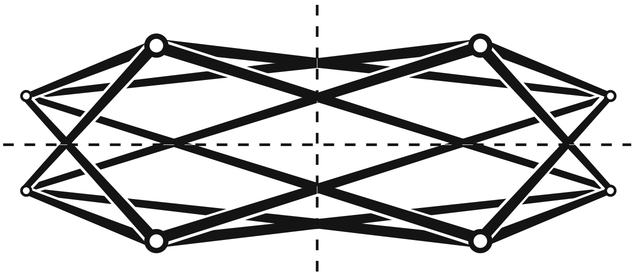

Fig. 5 An example of an over-constrained framework with $\mathscr{C}_{2v}$ symmetry based on a non-planar realisation of the complete bipartite graph $K_{4,4}$. As in Fig. 2, *dotted lines* indicate mirror planes and *larger and smaller circles* indicate joints that lie respectively in front of, and behind, the median plane of the framework parallel with the plane of the paper

Sub-case (iii)(c): The groups with $n_i \leq kg_i$ for all i (where $k = 2$ or 4) are: $\mathscr{C}_2$ ($k = 4$), $\mathscr{C}_{2v}$ ($k = 2$), $\mathscr{C}_{2h}$ ($k = 2$), $\mathscr{D}_n$ with $n \geq 2$ ($k = 2$), $\mathscr{S}_4$ ($k = 2$), and $\mathscr{C}_n$, with $n \geq 4$ ($k = 2$).

When $n_i = kg_i$, the representation Γ_i is the symmetry of a detectable state of self stress, a detectable mechanism, or absent from both lists, depending on whether $(3j_0 - b_0) < k$, $(3j_0 - b_0) > k$, or $(3j_0 - b_0) = k$, respectively.

When $n_i = (k/2)g_i$, Γ_i is either the symmetry of a detectable mechanism, or is absent from the lists of both mechanisms and self stresses, depending on whether $(3j_0 - b_0) > k/2$ or $(3j_0 - b_0) = k/2$.

In the groups $\mathscr{C}_n$, with $n \geq 4$, $n_i = 0$ for all but A and $E_{(1)}$. Hence for these groups, all irreducible representations, except A and $E_{(1)}$, are present in the list of detectable mechanisms for all positive $(3j_0 - b_0)$.

Example 5. The symmetry-regular framework with $\mathscr{C}_{2v}$ symmetry shown in Fig. 5 is a three-dimensional realisation of the complete bipartite graph, $K_{4,4}$. When considered in the 3D setting, this framework is underbraced by two bars, and hence has two finite mechanisms if realised generically without symmetry. For $\mathscr{C}_{2v}$, we have $k = 2$, $\Gamma_{\text{reg}} = A_1 + A_2 + B_1 + B_2$, and $\Gamma_{\text{rigid}} = A_1 + A_2 + 2B_1 + 2B_2$, so that

$$\begin{aligned}\Gamma(m) - \Gamma(s) &= (3j_0 - b_0)\Gamma_{\text{reg}} - \Gamma_{\text{rigid}} \\ &= (3j_0 - b_0 - 1)\Gamma_{\text{reg}} - B_1 - B_2.\end{aligned}$$

For the example shown, we have $j_0 = 2$, $b_0 = 4$ (and hence $(3j_0 - b_0) = 2 = k$). Thus, $\Gamma(m) - \Gamma(s) = A_1 + A_2$, i.e., the framework has two infinitesimal mechanisms, one totally symmetric and one preserving only the $\mathscr{C}_2$ rotational symmetry. Except at specific singular geometric configurations [27], the framework does not have a state of self stress, and hence these infinitesimal mechanisms are in fact finite. Projected into a horizontal plane, the A_1 motion corresponds to the Bottema mechanism [4]. The A_2 motion has quadrupolar character, and displacement along the A_2 path reduces the overall symmetry to $\mathscr{C}_2$, where $k = 4$, $\Gamma_{\text{reg}} = A + B$, and $\Gamma_{\text{rigid}} = 2A + 4B$, so that

$$\Gamma(m) - \Gamma(s) = (3j_0 - b_0 - 2)\Gamma_{\text{reg}} - 2B,$$

where now $j_0 = 4$, $b_0 = 8$, and $(3j_0 - b_0) = 4 = k$, as $|\mathscr{G}|$ has fallen to 2. Hence, in the lower symmetry group there are two totally symmetric mechanisms.

3 Restriction to the Plane

The discussion has concentrated on the Maxwell Rule for bar-and-joint frameworks in 3D, but similar conclusions are readily obtained for frameworks restricted to 2D.

The 2D restriction is made by deletion of terms from Γ_T and Γ_R. The counting rule, symmetry theorem and the theorem for symmetry-regular frameworks equivalent to (1), (2) and (9) are then:

$$m - s = 2j - b - 3, \tag{12}$$

$$\Gamma(m) - \Gamma(s) = \Gamma(j) \times \Gamma_T(x, y) - \Gamma(b) - \Gamma_T(x, y) - \Gamma_R(xy), \tag{13}$$

$$\Gamma(m) - \Gamma(s) = (2j_0 - b_0)\Gamma_{\text{reg}} - \Gamma_T(x, y) - \Gamma_R(xy), \tag{14}$$

where the framework is supposed to be confined to the xy plane, $\Gamma_T(x, y)$ is the representation of the two translations in the framework plane, and $\Gamma_R(xy)$ is the representation of the rotation in that plane.

In 2D, the possible point groups are $\mathscr{C}_{\mathrm{n}}$ and $\mathscr{C}_{\mathrm{nv}}$ (with $\mathscr{C}_{1\mathrm{v}} \equiv \mathscr{C}_{\mathrm{s}}$). It follows from the character tables of these groups that the representations of the rigid-body motions in the xy plane have the following form:

2D point group	$\Gamma_{\mathrm{rigid}}(x, y)$
$\mathscr{C}_1$	3A
$\mathscr{C}_2$	$A + 2B$
$\mathscr{C}_n, n \geq 3$	$A + E_{(1)}$
$\mathscr{C}_s$	$A' + 2A''$
$\mathscr{C}_{2v}$	$A_2 + B_1 + B_2$
$\mathscr{C}_{nv}, n \geq 3$	$A_2 + E_{(1)}$

In direct analogy with the $3D$ mobility analysis, the $2D$ analysis falls into the following three cases:

Case (i) $2j_0 - b_0 < 0$, and hence

$$m - s = 2j - b - 3 = (2j_0 - b_0)|\mathscr{G}| - 3 < -3;$$

Case (ii) $2j_0 - b_0 = 0$, and hence

$$m - s = 2j - b - 3 = -3;$$

Case (iii) $2j_0 - b_0 > 0$, and hence

$$m - s = 2j - b - 3 = (2j_0 - b_0)|\mathscr{G}| - 3 > -3.$$

As in $3D$, mechanisms can be detected only in Case (iii). Note that for the groups $\mathscr{C}_1$ and $\mathscr{C}_3$, $\Gamma_{\mathrm{rigid}}(x, y)$ is an exact multiple of Γ_{reg} (for $\mathscr{C}_3$, we even have $\Gamma_{\mathrm{rigid}}(x, y) = \Gamma_{\mathrm{reg}}$). The groups for which $\Gamma_{\mathrm{rigid}}(x, y)$ is contained in Γ_{reg} are $\mathscr{C}_n$, $n > 3$, and $\mathscr{C}_{nv}$, $n \geq 2$, and the groups for which $\Gamma_{\mathrm{rigid}}(x, y)$ is contained in $k\Gamma_{\mathrm{reg}}$, where $k = 2$, are $\mathscr{C}_2$ and $\mathscr{C}_s$.

4 Extension to Body-and-Joint Frameworks

Similar reasoning can be applied to the analysis of mobility of body-and-joint frameworks in 3D, where the joints may be of any type, e.g. revolute hinges, screw joints or spherical joints. The mobility criterion for a linkage consisting of v bodies connected by e joints, where joint i permits f_i relative freedoms, is [15]

$$m - s = 6(v - 1) - 6e + \sum_{i=1}^{e} f_i. \tag{15}$$

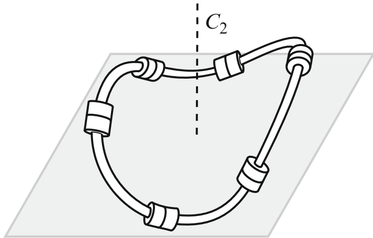

Fig. 6 A schematic view of a 6-loop, adapted from Guest and Fowler [14]. Six curved bodies are connected by six in-line revolute joints, each of which allows a single, twisting, degree of freedom

The symmetry-extended version of the mobility rule is then [12]

$$\Gamma(m) - \Gamma(s) = \Gamma(v, C) \times (\Gamma_T + \Gamma_R) - \Gamma_T - \Gamma_R - \Gamma_{\|}(e, C) \times (\Gamma_T + \Gamma_R) + \Gamma_{\mathrm{f}}, \quad (16)$$

where the notation is motivated by the association of the bodies with the vertices of a 'contact polyhedron' C and the hinges with the edges of that polyhedron. $\Gamma_{\|}(e, C)$ is the representation of a set of vectors along the edges of C, and Γ_{f} is the representation of the total set of freedoms allowed by the joints. Calculation of Γ_{f} requires specification of the types of hinges, but is straightforwardly calculated for each type.

In the case of a symmetry-regular body-and-joint framework with a contact polyhedron belonging to point group $\mathscr{G}$, the bodies and hinges are all in general position, and both $\Gamma(v, C)$ and $\Gamma_{\|}(e, C)$ consist of sets of copies of the regular representation. The hinges may admit different types and numbers of freedoms, but again Γ_{f} consists of a number of complete copies of the regular representation. The form of (16) applicable to symmetry-regular frameworks is therefore

$$\Gamma(m) - \Gamma(s) = (6v_0 - 6e_0 + F_0) \times \Gamma_{\mathrm{reg}} - \Gamma_T - \Gamma_R, \quad (17)$$

where the orbit counts are $v_0 = v/|\mathscr{G}|$ for bodies, $e_0 = e/|\mathscr{G}|$ for joints, and F_0 for total freedoms, where

$$F_0 = \sum_{i=1}^{5} i \; f_{0,i} \quad (18)$$

and $f_{0,i}$ is the number of orbits of hinges that admit i freedoms. Given this equation, the analysis follows the same course as for pin-jointed frameworks, with $6(v_0 - e_0) + F_0$ playing the role of $3j_0 - b_0$ in the arguments.

Example 6. Consider the body-and-joint framework shown in Fig. 6, which is a representative of the twist-boat conformation of cyclohexane, or the 6-loop [14]. The point-group symmetry is $\mathscr{C}_2$, where $\Gamma_T - \Gamma_R = 2A + 4B$ and all bodies and joints lie in general position with respect to the C_2 axis. We have 6 bodies, 6 joints and 6 freedoms spanning $v_0 = 3$, $e_0 = 3$ and $F_0 = 3$ copies of the regular representation $A + B$. Thus, the symmetry-extended mobility rule, (17), gives $\Gamma(m) - \Gamma(s) = (6v_0 - 6e_0 + F_0) \times \Gamma_{\text{reg}} - \Gamma_T - \Gamma_R = A - B$. Hence, $\Gamma(m)$ is Γ_{reg} minus the B representation, and $\Gamma(m)$ is Γ_{reg} minus the A representation. Simple scalar counting is consistent with an isostatic framework, but symmetry has revealed a totally symmetric (and hence finite) mechanism, and an antisymmetric state of self stress.

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Generic Global Rigidity in Complex and Pseudo-Euclidean Spaces

Steven J. Gortler and Dylan P. Thurston

Abstract In this paper we study the property of generic global rigidity for frameworks of graphs embedded in d-dimensional complex space and in a d-dimensional pseudo-Euclidean space ($\mathbb{R}^d$ with a metric of indefinite signature). We show that a graph is generically globally rigid in Euclidean space iff it is generically globally rigid in a complex or pseudo-Euclidean space. We also establish that global rigidity is always a generic property of a graph in complex space, and give a sufficient condition for it to be a generic property in a pseudo-Euclidean space. Extensions to hyperbolic space are also discussed.

Keywords Metric geometry

Subject Classifications: 52C25, 51M10, 05C62

1 Introduction

The property of generic global rigidity of a graph in d-dimensional Euclidean space has recently been fully characterized [4, 7]. It is quite natural to study this property in other spaces as well. For example, recent work of Owen and Jackson [8] has studied the number of equivalent realizations of frameworks in $\mathbb{C}^2$. In this paper

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we study the property of generic global rigidity of graphs embedded in $\mathbb{C}^d$ as well as graphs embedded in a pseudo Euclidean space ($\mathbb{R}^d$ equipped with an indefinite metric signature).

We show that a graph Γ is generically globally rigid (GGR) in d-dimensional Euclidean space iff Γ is GGR in d-dimensional complex space. Moreover, for any metric signature, s, We show that a graph Γ is GGR in d-dimensional Euclidean space iff Γ is GGR in d-dimensional real space under the signature s. Combining this with results from [5] also allows us to equate this property with generic global rigidity in hyperbolic space.

In the Euclidean and complex cases, global rigidity can be shown to be a generic property: a given graph is either generically globally rigid, or generically globally flexible. In the pseudo Euclidean (and equivalently the hyperbolic) case, though, we do not know this to be true. In this paper we do establish that global rigidity in pseudo Euclidean spaces is a generic property for graphs that contain a large enough GGR subgraph (such as a d-simplex).

2 Initial Definitions

Definition 1. We equip $\mathbb{R}^d$ with pseudo Euclidean metric in order to measure lengths. The metric is specified with a non negative integer s that determines how many of its coordinate directions are subtracted from the total. The squared length of a vector $\mathbf{w}$ is $|\mathbf{w}|^2 := -\sum_{i=1}^{s} \mathbf{w}_i^2 + \sum_{i=s+1}^{d} \mathbf{w}_i^2$. We will use the symbol $\mathbb{S}^d$ to denote the space $\mathbb{R}^d$ equipped with some fixed metric s. If $s = 0$, we have the Euclidean metric and the space may be denoted $\mathbb{E}^d$.

For complex space, The squared length of a vector $\mathbf{w}$ in $\mathbb{C}^d$ is $|\mathbf{w}|^2 := \sum_i \mathbf{w}_i^2$. Note here that we do not use conjugation, and thus vectors have complex squared lengths. (The use of conjugation would essentially reduce d-dimensional complex rigidity questions to 2d-dimensional Euclidean questions.)

Definition 2. A *graph* Γ is a set of v vertices $\mathscr{V}(\Gamma)$ and e edges $\mathscr{E}(\Gamma)$, where $\mathscr{E}(\Gamma)$ is a set of two element subsets of $\mathscr{V}(\Gamma)$. We will typically drop the graph Γ from this notation.

For $\mathbb{F} \in \{\mathbb{E}, \mathbb{S}, \mathbb{C}\}$, a *configuration* of the vertices $\mathscr{V}(\Gamma)$ of a graph in $\mathbb{F}^d$ is a mapping p from $\mathscr{V}(\Gamma)$ to $\mathbb{F}^d$. Let $C_{\mathbb{F}^d}(\mathscr{V})$ be the space of configurations in $\mathbb{F}^d$.

For $p \in C_{\mathbb{F}^d}(\mathscr{V})$ with $u \in \mathscr{V}(\Gamma)$, we write $p(u) \in \mathbb{F}^d$ for the image of u under p.

A *framework* $\rho = (p, \Gamma)$ of a graph is the pair of a graph and a configuration of its vertices. $C_{\mathbb{F}^d}(\Gamma)$ is the space of frameworks (p, Γ) with graph Γ and configurations in $\mathbb{F}^d$.

We may also write $\rho(u)$ for $p(u)$ where $\rho = (p, \Gamma)$ is a framework of the configuration p.

Definition 3. Two frameworks ρ and σ in $C_{\mathbb{F}^d}(\Gamma)$ are *equivalent* if for all $\{t, u\} \in \mathscr{E}$ we have $|\rho(t) - \rho(u)|^2 = |\sigma(t) - \sigma(u)|^2$.

Definition 4. Two configurations p and q in $C_{\mathbb{F}^d}(\mathscr{V})$ are *congruent* if for all vertex pairs, $\{t, u\}$, we have $|p(t) - p(u)|^2 = |q(t) - q(u)|^2$.

Two configurations p and q in $C_{\mathbb{F}^d}(\mathscr{V})$ are *strongly congruent* if they are related by a translation composed with an element of the orthogonal group of $\mathbb{F}^d$.

Remark 1. In $\mathbb{E}^d$, there is no difference between congruence and strong congruence. In other spaces, though, there can be some subtle differences. For the simplest example, in $\mathbb{C}^2$, the vectors $(0, 0)$ and $(i, 1)$ both have zero length, but are not related by a complex orthogonal transform. Such non-zero vectors with zero squared length are called *isotropic*. Thus the framework made up of a single edge connecting a vertex at the origin to a vertex at $(i, 1)$ is congruent to the framework with both vertices at the origin, but the two frameworks are not strongly congruent.

Fortunately, these differences are easy to avoid; for example, congruence and strong congruence coincide for points with a d-dimensional affine span. These notions will also coincide when there are fewer than $d + 1$ points, as long as the points are in affine general position. For more details, see Sect. 10.

We can now, finally, define global rigidity and flexibility.

Definition 5. A framework $\rho \in C_{\mathbb{F}^d}(\Gamma)$ is *globally rigid* in $\mathbb{F}^d$ if, for any other framework $\sigma \in C_{\mathbb{F}^d}(\Gamma)$ to which ρ is equivalent, we also have that ρ is congruent to σ. Otherwise we say that ρ is *globally flexible* in $\mathbb{F}^d$.

Definition 6. A configuration p in $C_{\mathbb{F}^d}(\mathscr{V})$ is generic if the coordinates do not satisfy any non-zero algebraic equation with rational coefficients. We call a framework generic if its configuration is generic. (See Sect. 9 for more background on (semi) algebraic sets and genericity.)

Definition 7. A graph Γ is generically globally rigid (resp. flexible) in $\mathbb{F}^d$ if all generic frameworks in $C_{\mathbb{F}^d}(\Gamma)$ are globally rigid (resp. flexible). These properties are abbreviated GGR and GGF.

Definition 8. A property is *generic* if, for every graph, either all generic frameworks in $C_{\mathbb{F}^d}(\Gamma)$ have the property or none do. For instance, global rigidity in $\mathbb{E}^d$ is a generic property of a graph [7]. So in this case, if a graph is not GGR, it must be GGF.

3 Complex Generic Global Rigidity

Our main theorem in this section is

Theorem 1. *A graph Γ is generically globally rigid in $\mathbb{C}^d$ iff it is generically globally rigid in $\mathbb{E}^d$.*

Remark 2. This fully describes the generic situation for complex frameworks as it is easy to see that generic global rigidity in $\mathbb{C}^d$ is a generic property of a graph.

Recall that a complex algebraically *constructible set* is a finite Boolean combination of complex algebraic sets. Also, an irreducible complex algebraic set V cannot have two disjoint constructible subsets with the same dimension as V.

Chevalley's theorem states that the image under a polynomial map of a complex algebraically constructible set, all defined over $\mathbb{Q}$, is also a complex algebraically constructible set defined over $\mathbb{Q}$ [1, Theorem 1.22]. Chevalley's theorem allows one to apply elimination, effectively replacing all quantifiers in a Boolean-algebraic expression with algebraic equations and Boolean set operations.

Now, let us assume Γ is locally rigid in $\mathbb{C}^d$. We can partition $C_{\mathbb{C}^d}(\Gamma)$ such that in each part, P_n, all of the frameworks have the same number, n, of equivalent and non-congruent frameworks. In light of Chevalley's theorem, each of these parts is constructible. And exactly one of them, P_{n_0}, must be of full dimension. This part contains all of the generic points and represents the generic behavior of the framework. If $n_0 = 1$ then the graph is GGR, while if $n_0 > 1$ then it must be GGF.

3.1 ⇒ of Theorem 1

The implication from Complex to Euclidean GGR follows almost directly from their definitions. For this argument we model each Euclidean framework ρ in $C_{\mathbb{E}^d}(\mathscr{V})$ as a Complex framework $\rho_{\mathbb{C}}$ in $C_{\mathbb{C}^d}(\mathscr{V})$ that happens to have all purely real coordinates. Clearly, for such configurations, the complex squared length measurement coincides with the Euclidean metric on real configurations.

Proof. Let ρ be a generic framework in $C_{\mathbb{E}^d}(\Gamma)$ and let $\rho_{\mathbb{C}}$ be its corresponding real valued framework in $C_{\mathbb{C}^d}(\Gamma)$. By our definitions, $\rho_{\mathbb{C}}$ is also generic when thought of as complex framework.

Since Γ is generically globally rigid in $\mathbb{C}^d$, $\rho_{\mathbb{C}}$ can have no equivalent and non-congruent framework in $C_{\mathbb{C}^d}(\Gamma)$, and thus it has no real valued, equivalent and non-congruent framework in $C_{\mathbb{C}^d}(\Gamma)$. Thus ρ has no equivalent and non-congruent framework in $C_{\mathbb{E}^d}(\Gamma)$.

3.2 ⇐ of Theorem 1

For the other direction of Theorem 1, we start with a complex version of a theorem by Connelly [4]:

Theorem 2. *Let ρ be a generic framework in $C_{\mathbb{C}^d}(\Gamma)$. If ρ has a complex equilibrium stress matrix of rank $v - d - 1$, then Γ is generically globally rigid in $\mathbb{C}^d$.*

Proof. The proof of the complex version of this theorem follows identically to Connelly's proof of the Euclidean version. In particular, the proof shows that any framework with the same complex squared edge lengths as ρ must be strongly congruent, and thus congruent to it.

(The interested reader can see [4] for the definition of an equilibrium stress matrix.)

Next, we recall a theorem from Gortler, Healy and Thurston [7]

Theorem 3. *Let ρ be a generic framework in $C_{\mathbb{E}^d}(\Gamma)$ with at least $d+2$ vertices. If ρ does not have a real equilibrium stress matrix of rank $v-d-1$, then Γ is generically globally flexible in $\mathbb{E}^d$. Moreover, there must be an even number of noncongruent frameworks with the same squared edge lengths as ρ in $\mathbb{E}^d$.*

And now we can prove this direction of our Theorem.

Proof. From Theorem 2, if Γ is not generically globally rigid in $\mathbb{C}^d$, there is no generic framework in $C_{\mathbb{C}^d}(\Gamma)$ that has a complex equilibrium stress matrix of rank $v-d-1$. Thus there can be no real valued and generic framework in $C_{\mathbb{C}^d}(\Gamma)$ with complex equilibrium stress matrix of rank $v-d-1$, and thus no generic framework in $C_{\mathbb{E}^d}(\Gamma)$ with a complex or real equilibrium stress matrix of rank $v-d-1$. Thus from Theorem 3, Γ is generically globally flexible in $\mathbb{E}^d$.

4 Pseudo Euclidean Generic Global Rigidity: Results

Our main theorem on pseudo Euclidean generic global rigidity is as follows:

Theorem 4. *For any pseudo Euclidean space $\mathbb{S}^d$, a graph Γ is generically globally rigid in $\mathbb{E}^d$ iff it is generically globally rigid in $\mathbb{S}^d$.*

Unfortunately we do not know if generic global rigidity is a generic property in $\mathbb{S}^d$. It is conceivable that there are some graphs that are not GGR in $\mathbb{S}^d$ but that do have *some* generic frameworks that are globally rigid in $\mathbb{S}^d$. We leave this as an open question. We do have the following partial result

Theorem 5. *If a graph Γ is not GGR in $\mathbb{S}^d$ and it has a GGR subgraph Γ_0 with $d+1$ or more vertices, then Γ must be GGF in $\mathbb{S}^d$.*

5 $\Rightarrow$ of Theorem 4

This argument is essentially identical to that of Sect. 3.1.

Definition 9. Given a pseudo Euclidean space $\mathbb{S}^d$ with signature s, we model each configuration $\rho \in C_{\mathbb{S}^d}(\mathcal{V})$ as a Complex configurations $\rho_{\mathbb{C}} \in C_{\mathbb{C}^d}(\mathcal{V})$ that happens to have the first s of its coordinates purely imaginary and the remaining $d-s$ of its coordinates purely real. We call this an *s-signature, real valued complex configuration*. We will shorten this to simply an *s-valued configuration*.

It is easy to verify that for such configurations, the complex squared length measurement coincides with the metric on $\mathbb{S}^d$.

And now we can prove this direction of our Theorem.

Proof. Let ρ be a generic framework in $C_{\mathbb{S}^d}(\Gamma)$. We model this with $\rho_{\mathbb{C}}$, an s-valued complex framework in $C_{\mathbb{C}^d}(\Gamma)$.

$\rho_{\mathbb{C}}$ must be a generic framework in $C_{\mathbb{C}^d}(\Gamma)$. For suppose there is a non-zero polynomial $\phi_{\mathbb{C}}$ with rational coefficients, that vanishes on $\rho_{\mathbb{C}}$. Then there is a polynomial ϕ with coefficients in $\mathbb{Q}(i)$ that vanishes on the real coordinates of ρ. Let $\bar{\phi}$ be the polynomial obtained by taking the conjugate of every coefficient in ϕ, and let $\psi := \phi * \bar{\phi}$. Then ψ is non zero and vanishes on ρ. Since ψ is fixed by conjugation, it has coefficients in $\mathbb{Q}$. This polynomial would make ρ non generic, leading to a contradiction.

Since Γ is generically globally rigid in $\mathbb{E}^d$, from Theorem 1 it is also generically globally rigid in $\mathbb{C}^d$. Thus $\rho_{\mathbb{C}}$ can have no equivalent and non-congruent framework in $C_{\mathbb{C}^d}(\Gamma)$, and thus it can have no s-valued, equivalent and non-congruent framework in $C_{\mathbb{C}^d}(\Gamma)$. Thus ρ can have no equivalent and non-congruent framework in $C_{\mathbb{S}^d}(\Gamma)$.

6 $\Leftarrow$ of Theorem 4

Remark 3. For this proof, we cannot apply the same reasoning as Sect. 3.2, as many of the stress matrix arguments and conclusions from [7] simply do not carry over to pseudo Euclidean spaces. Indeed, Jackson and Owen [8] have found a graph, they call G_3, that is GGF in $\mathbb{E}^2$, but for which there is always an *odd* number of equivalent realizations in 2-dimensional Minkowski space. Moreover, it is not even clear that for general pseudo Euclidean spaces of dimension 3 or greater, the "number of equivalent realizations mod 2" is even a generic property.

For this direction, we will show the contrapositive: namely, if there is a generic Euclidean framework that is not globally rigid, then there must be a generic framework in $\mathbb{S}^d$ that is not globally rigid. To do this, we will apply a basic construction by Saliola and Whiteley (personal communication, 2012) that takes a pair of equivalent Euclidean frameworks and produces a pair of equivalent frameworks in the desired space $C_{\mathbb{S}^d}(\Gamma)$. Whiteley refers to this recipe as a generalized Pogorelov map (Whiteley, W., personal communication, 2012).

Definition 10. Let P be the map from pairs of frameworks in $C_{\mathbb{E}^d}(\Gamma)$ to pairs of frameworks in $C_{\mathbb{S}^d}(\Gamma)$ defined as follows:

Step 1: Let ρ and σ be two frameworks in $\mathbb{E}^d$. Take their average to obtain $a := \frac{\rho+\sigma}{2}$. Take their difference to obtain $f := \frac{\rho-\sigma}{2}$.
Step 2: Let $\tilde{a}$ be the framework in $C_{\mathbb{S}^d}(\Gamma)$ with the same (real) coordinates of a. Let $\tilde{f}$ be defined by negating the first s of the coordinates in f.
Step 3: Finally, set $P(\rho, \sigma) := (\tilde{\rho}, \tilde{\sigma})$ where $\tilde{\rho} := \tilde{a} + \tilde{f}$ and $\tilde{\sigma} := \tilde{a} - \tilde{f}$.

The Pogorelov map is useful due to the following (Whiteley, W., personal communication, 2012):

Theorem 6. *Let ρ and σ be two equivalent frameworks in $C_{\mathbb{E}^d}(\Gamma)$. Then $P(\rho,\sigma)$ are a pair of equivalent frameworks in $C_{\mathbb{S}^d}(\Gamma)$.*

Proof. Using the notation of Definition 10 we see the following.

Step 1: From the *averaging principal* [3], a must be infinitesimally flexible with flex f.
Step 2: $\tilde{f}$ must be an infinitesimal flex for $\tilde{a}$ in $C_{\mathbb{S}^d}(\Gamma)$ [10].
Step 3: From the *flex-antiflex principal* [3] (also sometimes called the de-averaging principal), $\tilde{\rho}$ must be equivalent to $\tilde{\sigma}$ in $C_{\mathbb{S}^d}(\Gamma)$.

Remark 4. It is, perhaps, interesting to note that in our case, the map has the very simple form of "coordinate swapping". In particular, it is an easy calculation to see that $\tilde{\rho}$ will be made up of the first s coordinates of ρ and the remaining coordinates of σ, while $\tilde{\sigma}$ will be made up of the first s coordinates of σ and the remaining coordinates of ρ. It is also an simple calculation to directly verify, without using the averaging principle, that coordinate swapping will map pairs of equivalent Euclidean frameworks to pairs of equivalent frameworks in $C_{\mathbb{S}^d}(\Gamma)$.

Additionally, we can ensure that $\tilde{\rho}$ is not congruent to $\tilde{\sigma}$.

Lemma 1. *Let ρ and σ be two equivalent frameworks in $C_{\mathbb{E}^d}(\Gamma)$. And let $(\tilde{\rho},\tilde{\sigma}) := P(\rho,\sigma)$. Then ρ and σ are congruent in $C_{\mathbb{E}^d}(\Gamma)$ iff $\tilde{\rho}$ and $\tilde{\sigma}$ are congruent in $C_{\mathbb{S}^d}(\Gamma)$.*

Proof. Congruence between configurations is the same as equivalence between complete graphs over these configurations. Thus this property must map across the Pogorelov map (which does not depend on the edge set), and its inverse.

6.1 Genericity

The main (annoyingly) difficult technical issue left is to show that this construction can create a *generic* framework in $C_{\mathbb{S}^d}(\Gamma)$ that is globally flexible. A priori, it is conceivable that the image of the Pogorelov map, acting on all pairs of equivalent and non-congruent Euclidean frameworks, can only produce pseudo Euclidean configurations that lie on some subvariety of $C_{\mathbb{S}^d}(\Gamma)$. In this section, we rule this possibility out.

In this discussion, we will assume that Γ is generically locally rigid (otherwise we are done), but that it is not GGR in $\mathbb{E}^d$.

Definition 11. Let E^+ ('E' for 'equivalent') be the algebraic subset of $C_{\mathbb{E}^d}(\Gamma) \times C_{\mathbb{E}^d}(\Gamma)$ consisting of pairs of equivalent tuples. Let C^+ ('C' for 'congruent') be the algebraic subset of $C_{\mathbb{E}^d}(\Gamma) \times C_{\mathbb{E}^d}(\Gamma)$ consisting of pairs of congruent tuples. Let π_1 be the projection from a pair of frameworks onto its first factor.

Definition 12. Since Γ is not GGR in $\mathbb{E}^d$, $\dim(\pi_1(E^+ \backslash C^+)) = v * d$ and so E^+ must have at least one irreducible component E, with $\dim(\pi_1(E)) = v * d$ and such that it contains at least one tuple of non-congruent frameworks. We choose one such component and call it E. As per Remark 8, E must be defined over some algebraic extension of $\mathbb{Q}$. Thus if e is generic in E, then $\pi_1(e)$ is a generic framework in $C_{\mathbb{E}^d}(\Gamma)$.

Lemma 2. *Let $e := (\rho, \sigma) \in E$ be generic. Then ρ is not congruent to σ.*

Proof. Congruence is a relation that can be expressed with polynomials over $\mathbb{Q}$. By our assumptions on E, these polynomials do not vanish identically over E.

Lemma 3. *The (real) dimension of E is $v * d + \binom{d+1}{2}$. Moreover, if (ρ, σ) is generic in E, then for all σ^c in the congruence class $[\sigma]$, (ρ, σ^c) must be in E.*

Proof. We will pick a generic $e = (\rho, \sigma) \in E$, and look at the dimension of the fiber $\pi_1^{-1}(\rho)$ near this point e. (By considering only this neighborhood, we can avoid dealing with any non-smooth points of E, and thus can view this as a smooth map between manifolds.) The dimension of E must be the sum of the dimension of the span of $\pi_1(E)$, which is $v * d$, and the dimension of this fiber.

Since e is generic in E, ρ must be generic in $C_{\mathbb{E}^d}(\Gamma)$. Thus, from Lemma 11 (below), σ must be locally rigid and with non degenerate affine span. Thus its congruence class has dimension $\binom{d+1}{2}$.

Since e is generic in E, from Lemma 24, all nearby points in E^+ must, in fact, lie in E. In particular, for $\sigma^c \in [\sigma]$ and close to σ, the point (ρ, σ^c) must be in E. Thus the dimension of the fiber in E near e must be $\binom{d+1}{2}$. This gives us the desired dimension.

Moreover, since E is algebraic, for any $\sigma^c \in [\sigma]$, the point (ρ, σ^c) must be in E. This follows from the fact that the (Zariski) closure of a subset must be a subset of the closure.

Corollary 1. *Let π_2 be the projection of a pair onto its second factor. The (real) dimension of $\pi_2(E)$ is $v * d$. And if e is generic in E, $\pi_2(e)$ is generic in $C_{\mathbb{E}^d}(\Gamma)$.*

To study the behavior of P on E, we move our discussion over to complex space.

Definition 13. Let $E_{\mathbb{C}}^+$ be the algebraic subset of $C_{\mathbb{C}^d}(\Gamma) \times C_{\mathbb{C}^d}(\Gamma)$ consisting of pairs of equivalent tuples. Let $E_{\mathbb{C}}$ be any component of $E_{\mathbb{C}}^+$ that includes E. (This can be done as the complexification of E must be irreducible – see Definition 28.) From Corollary 2, below, we will also soon see that there is only one such component.

Lemma 4. *The (complex) dimension of $E_{\mathbb{C}}$ is $v * d + \binom{d+1}{2}$.*

Proof. $E_{\mathbb{C}}$ includes the complexification of E (see Definition 28), and so by assumption, the complex dimension of $\pi_1(E_{\mathbb{C}})$ must be at least $v * d$, and thus must be equal to $v * d$. We can then follow the proof of Lemma 3 to establish the complex dimension of the generic π_1 fibers of $E_{\mathbb{C}}$

Corollary 2. *$E_{\mathbb{C}}$ is* the *complexification of E. A generic point of E is generic in $E_{\mathbb{C}}$.*

Proof. By assumption, $E_{\mathbb{C}}$ is irreducible and contains E. Moreover the complex dimension of $E_{\mathbb{C}}$ equals the real dimension of E. Thus $E_{\mathbb{C}}$ cannot be larger than the complexification of E. Genericity carries across complexification (see Definition 28).

To study P, we will look at a complex Pogorelov map $P_{\mathbb{C}}$, that essentially reproduces the behavior of P when restricted to real input. In particular, this map will take real valued complex pairs, to s-valued complex pairs. We define $P_{\mathbb{C}}$ as the composition of some very simple maps.

Definition 14. Let $H_{\mathbb{C}}$, (a Haar like transform) be the invertible map from $(\rho_{\mathbb{C}}, \sigma_{\mathbb{C}})$, a pair of frameworks in $C_{\mathbb{C}^d}(\Gamma)$, to the pair $(\frac{\rho_{\mathbb{C}}+\sigma_{\mathbb{C}}}{2}, \frac{\rho_{\mathbb{C}}-\sigma_{\mathbb{C}}}{2})$.

Let $S_{\mathbb{C}}$ be the invertible map that takes $(a_{\mathbb{C}}, f_{\mathbb{C}})$, a pair of frameworks in $C_{\mathbb{C}^d}(\Gamma)$, to the pair $(\tilde{a}_{\mathbb{C}}, \tilde{f}_{\mathbb{C}})$, where the $\tilde{a}_{\mathbb{C}}$ is obtained from $a_{\mathbb{C}}$ by multiplying its first s coordinates by i, while $\tilde{f}_{\mathbb{C}}$ is obtained from $f_{\mathbb{C}}$ by multiplying its first s coordinates by $-i$.

$H_{\mathbb{C}}^{-1}(\tilde{a}_{\mathbb{C}}, \tilde{f}_{\mathbb{C}})$, the inverse Haar map, is simply $(\tilde{a}_{\mathbb{C}} + \tilde{f}_{\mathbb{C}}, \tilde{a}_{\mathbb{C}} - \tilde{f}_{\mathbb{C}})$.

Given this, $P_{\mathbb{C}} := H_{\mathbb{C}}^{-1} \circ S_{\mathbb{C}} \circ H_{\mathbb{C}}$.

This complex Pogorelov map coincides with the real map described above. In particular suppose ρ and σ are in $C_{\mathbb{E}^d}(\Gamma)$, and suppose $\rho_{\mathbb{C}}$ and $\sigma_{\mathbb{C}}$ are the corresponding real valued frameworks in $C_{\mathbb{C}^d}(\Gamma)$. Let $(\tilde{\rho}, \tilde{\sigma}) := P(\rho, \sigma)$ and $(\tilde{\rho}_{\mathbb{C}}, \tilde{\sigma}_{\mathbb{C}}) := P_{\mathbb{C}}(\rho_{\mathbb{C}}, \sigma_{\mathbb{C}})$. Then $\tilde{\rho}_{\mathbb{C}}$ and $\tilde{\sigma}_{\mathbb{C}}$ are the s-valued complex representations of $\tilde{\rho}$ and $\tilde{\sigma}$.

Clearly $P_{\mathbb{C}}$ maps $E_{\mathbb{C}}^{+}$ to itself. But a priori, it might map the component $E_{\mathbb{C}}$ to some other component of $E_{\mathbb{C}}^{+}$, and this other component might project under π_1 and π_2 onto a subvariety of (non generic) frameworks $C_{\mathbb{C}^d}(\Gamma)$. Our goal will be to show that this does not happen; instead $E_{\mathbb{C}}$ maps to itself under $P_{\mathbb{C}}$. As this map preservers genericity, and generic points of $E_{\mathbb{C}}$ project under π_1 to generic frameworks in $C_{\mathbb{C}^d}(\Gamma)$, we will then be done. (See Fig. 1.)

Definition 15. Let $B_{\mathbb{C}} := (H_{\mathbb{C}}(E_{\mathbb{C}}))$, ('B' for 'bundles' of flexes over frameworks). Since $B_{\mathbb{C}}$ is isomorphic to $E_{\mathbb{C}}$, it too must be an algebraic set. For any $(a_{\mathbb{C}}, f_{\mathbb{C}}) \in B_{\mathbb{C}}$, from the averaging principle, $f_{\mathbb{C}}$ is an infinitesimal flex for $a_{\mathbb{C}}$. $B_{\mathbb{C}}$ is irreducible (Lemma 22). And if $e_{\mathbb{C}}$ is generic in $E_{\mathbb{C}}$, $H_{\mathbb{C}}(e_{\mathbb{C}})$ (from Lemma 25) must be generic in $B_{\mathbb{C}}$.

Lemma 5. *Let $b_{\mathbb{C}} \in B_{\mathbb{C}}$ be generic. Let $b'_{\mathbb{C}} := (a'_{\mathbb{C}}, f'_{\mathbb{C}})$ be a nearby tuple in $C_{\mathbb{C}^d}(\Gamma) \times C_{\mathbb{C}^d}(\Gamma)$ such that $f'_{\mathbb{C}}$ is an infinitesimal flex for $a'_{\mathbb{C}}$. Then $b'_{\mathbb{C}} \in B_{\mathbb{C}}$.*

Proof. The tuple, $e_{\mathbb{C}} := H_{\mathbb{C}}^{-1}(b_{\mathbb{C}})$, is generic in E. From the flex/antiflex principal, $(\rho'_{\mathbb{C}}, \sigma'_{\mathbb{C}}) := e'_{\mathbb{C}} := H_{\mathbb{C}}^{-1}(a'_{\mathbb{C}}, f'_{\mathbb{C}})$ must be an equivalent pair of frameworks and thus in $E_{\mathbb{C}}^{+}$, and $e'_{\mathbb{C}}$ must be near $e_{\mathbb{C}}$. From Lemma 24, all nearby points in $E_{\mathbb{C}}^{+}$ must, in fact, lie in $E_{\mathbb{C}}$. Thus $e'_{\mathbb{C}}$ must be in $E_{\mathbb{C}}$, and from our definitions, $H_{\mathbb{C}}(e'_{\mathbb{C}}) = b'_{\mathbb{C}}$ must be in $B_{\mathbb{C}}$.

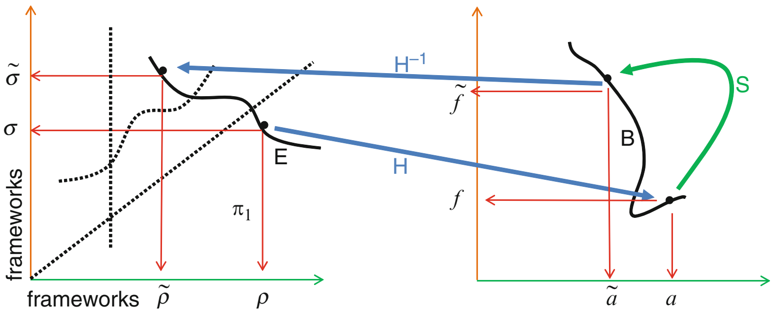

Fig. 1 *Left*: The space of pairs of complex frameworks. (All $\mathbb{C}$ subscripts are dropped for clarity.) The locus of equivalent pairs, $E^+_{\mathbb{C}}$, is shown in *solid* and *dotted black*. At least one component, $E_{\mathbb{C}}$, shown in *solid black*, has the property that $\dim(\pi_1(E_{\mathbb{C}})) = v * d$. *Right*: The space of pairs of complex frameworks. The variety $B_{\mathbb{C}} := H_{\mathbb{C}}(E_{\mathbb{C}})$ is made up of some frameworks and their flexes. (The image under $H_{\mathbb{C}}$ of the other components of $E^+_{\mathbb{C}}$ is not shown.) The map $S_{\mathbb{C}}$ maps $B_{\mathbb{C}}$ to itself, and thus the Pogorelov map is an automorphism of $E_{\mathbb{C}}$

Definition 16. Let $(a_{\mathbb{C}}, f_{\mathbb{C}}) = b_{\mathbb{C}}$ be a pair of framework in $C_{\mathbb{C}^d}(\Gamma)$. One can apply *coordinate scaling* to $b_{\mathbb{C}}$ by multiplying one chosen coordinate (out of the d coordinates in $\mathbb{C}^d$) of all the vertices in $a_{\mathbb{C}}$ by some complex scalar λ and the corresponding coordinate in all the vertices in $f_{\mathbb{C}}$ by $1/\lambda$.

Lemma 6. *The set $B_{\mathbb{C}}$ is invariant to coordinate scaling.*

Proof. Let $(a_{\mathbb{C}}, f_{\mathbb{C}}) = b_{\mathbb{C}} \in B_{\mathbb{C}}$ be generic. $f_{\mathbb{C}}$ is an infinitesimal flex for $a_{\mathbb{C}}$. Let us apply coordinate scaling to $b_{\mathbb{C}}$ with a scalar λ close to 1 and let us denote the result by $b'_{\mathbb{C}} = (a'_{\mathbb{C}}, f'_{\mathbb{C}})$. Looking at the effect of the rigidity matrix, we see that $f'_{\mathbb{C}}$ must be an infinitesimal flex for $a'_{\mathbb{C}}$, and from Lemma 5 must be in $B_{\mathbb{C}}$.

This means that $B_{\mathbb{C}}$ is invariant to nearly-unit coordinate scaling. Since $B_{\mathbb{C}}$ is algebraic, it must thus be invariant to all coordinate scaling. (This follows from the fact that the (Zariski) closure of a subset must be a subset of the closure.)

Corollary 3. *$S_{\mathbb{C}}$ is an automorphism of $B_{\mathbb{C}}$. Thus $P_{\mathbb{C}}$ is an automorphism of $E_{\mathbb{C}}$. Thus if $e_{\mathbb{C}} \in E_{\mathbb{C}}$ is generic, then $P_{\mathbb{C}}(e_{\mathbb{C}})$ is generic in $E_{\mathbb{C}}$ and both $\pi_1(P_{\mathbb{C}}(e_{\mathbb{C}}))$ and $\pi_2(P_{\mathbb{C}}(e_{\mathbb{C}}))$ are generic in $C_{\mathbb{C}^d}(\Gamma)$.*

With this we can finish the proof of this direction of Theorem 4.

Proof. Assume that Γ is not GGR in $\mathbb{E}^d$. Pick a generic $(\rho, \sigma) \in E$ (Definition 12).

From Theorem 6, $P(\rho, \sigma) =: (\tilde{\rho}, \tilde{\sigma})$ is a pair of equivalent frameworks $C_{\mathbb{S}^d}(\Gamma)$ which are not congruent from Lemma 2.

Let $\rho_{\mathbb{C}}$ and $\sigma_{\mathbb{C}}$ be the real valued complex frameworks corresponding to ρ and σ. From Corollary 2, $(\rho_{\mathbb{C}}, \sigma_{\mathbb{C}})$ is generic in $E_{\mathbb{C}}$. Meanwhile, $P_{\mathbb{C}}(\rho_{\mathbb{C}}, \sigma_{\mathbb{C}}) = (\tilde{\rho}_{\mathbb{C}}, \tilde{\sigma}_{\mathbb{C}})$, where $\tilde{\rho}_{\mathbb{C}}$ is the s-valued, complex representation of $\tilde{\rho}$, and $\tilde{\sigma}_{\mathbb{C}}$ is the s-valued, complex representation of $\tilde{\sigma}$. From Corollary 3, $\tilde{\rho}_{\mathbb{C}}$ is generic in $C_{\mathbb{C}^d}(\Gamma)$. Therefore $\tilde{\rho}$ must be generic in $C_{\mathbb{S}^d}(\Gamma)$, and we can conclude that Γ is not GGR in $\mathbb{S}^d$.

7 Proof of Theorem 5

We will prove the theorem by first showing that the existence of a large enough GGR subgraph Γ_0 is sufficient to rule out any "cross-talk" between different real signatures. In particular, if we have an s-valued framework of Γ_0, then Γ_0 cannot have a congruent framework that is s$'$-valued where $s \neq s'$. Thus, if we have an s-valued framework of Γ, then Γ cannot have an equivalent framework that is s$'$-valued where $s \neq s'$. With such cross talk ruled out, we will be able to apply an algebraic degree argument to show that Γ is GGF in $\mathbb{S}^d$.

In this section we will model congruence classes of frameworks in $C_{\mathbb{C}^d}(\mathscr{V})$ using complex symmetric matrices of rank d or less. First we spell out some basic facts about these matrices, and their relationship to configurations, as well as the notions of congruence and equivalence.

Definition 17. Let $\mathscr{G}$ be the set of symmetric $v-1$ by $v-1$ complex matrices of rank d or less. This is a determinantal variety which is irreducible. Assuming that $v \geq d+1$, $\mathscr{G}$ is of complex dimension $v*d-\binom{d+1}{2}$, and any generic $\mathbf{M} \in \mathscr{G}$ will have rank d.

For any configuration $p \in C_{\mathbb{C}^d}(\mathscr{V})$ (or framework $\rho \in C_{\mathbb{C}^d}(\Gamma)$) we associate its *g-matrix* $\mathbf{G}(p) \in \mathscr{G}$ as follows. We first translate p so its first vertex is at the origin. For any two remaining vertices t, u, we define the corresponding matrix entry as

$$\mathbf{G}(p)_{t,u} := \sum_{i=1}^{d} p(t)_i \; p(u)_i \tag{1}$$

(This is like a Gram matrix, but there is no conjugation involved.) Overloading this notation, if ρ is a framework with configuration p, we define $\mathbf{G}(\rho) := \mathbf{G}(p)$.

Definition 18. For any pair $\{t, u\}$, of distinct vertices in p, there is a linear map $\pi_{t,u}$ that computes the squared lengths between that pair using the entries in $\mathbf{G}(p)$. In the case where t is the first vertex (that was mapped to the origin), we have

$$\pi_{t,u}(\mathbf{G}(p)) = \mathbf{G}(p)_{u,u} \tag{2}$$

Otherwise, and in general,

$$\pi_{t,u}(\mathbf{G}(p)) = \mathbf{G}(p)_{t,t} + \mathbf{G}(p)_{u,u} - 2\mathbf{G}(p)_{t,u} \tag{3}$$

Applying this to all pairs of distinct vertices induces a linear map π_K from the set $\mathscr{G}$ to the set of symmetric v by v complex matrices with zeros on the diagonal.

Lemma 7. *The map π_K is injective.*

Proof. We just need to show that the kernel of π_K is 0. Let $\mathbf{M}$ be a matrix in the kernel of π_K. Starting with the first vertex at the origin, we find from Eq. (2) that all of the diagonal entries, $\mathbf{M}_{u,u}$ must vanish. Then, from Eq. (3), all the off diagonal entries of $\mathbf{M}$ must vanish as well.

Lemma 8. *p is congruent to q iff $\pi_K(\mathbf{G}(p)) = \pi_K(\mathbf{G}(q))$ and iff $\mathbf{G}(p) = \mathbf{G}(q)$.*

Proof. The first relation follows from the definition of congruence. The second follows from Lemma 7.

Corollary 4. *The map $\mathbf{G}$ acting on the quotient $C_{\mathbb{C}^d}(\mathscr{V})/\text{congruence}$ is injective.*

Lemma 9. *$\mathscr{G}$ is the Zariski closure of $\mathbf{G}(C_{\mathbb{C}^d}(\mathscr{V}))$. Moreover, if p is generic in $C_{\mathbb{C}^d}(\mathscr{V})$, then $\mathbf{G}(p)$ is generic in $\mathscr{G}$.*

Proof. Using Corollary 4, a dimension count verifies that the image $\mathbf{G}(C_{\mathbb{C}^d}(\mathscr{V}))$ must hit an open neighborhood of $\mathscr{G}$ (i.e. a subset of full dimension). The results follow as $\mathscr{G}$ is irreducible.

Equivalence of frameworks can be defined through their g-matrices as well:

Definition 19. Let $\pi_{\mathscr{E}}$ be the linear mapping from $\mathscr{G}$ to $\mathbb{C}^e$ defined by applying $\pi_{t,u}$ to each of the edges in $\mathscr{E}(\Gamma)$.

ρ is equivalent to σ, iff $\pi_{\mathscr{E}}(\mathbf{G}(\rho)) = \pi_{\mathscr{E}}(\mathbf{G}(\sigma))$.

If ρ is generic in $C_{\mathbb{C}^d}(\Gamma)$, then (assuming $v \geq d + 1$) $\pi_{\mathscr{E}}(\mathbf{G}(\rho))$ is generic in $\pi_{\mathscr{E}}(\mathscr{G})$.

The following Lemma will be useful when examining the cardinality of a fiber of $\pi_{\mathscr{E}}$.

Lemma 10. *Let $\mathbf{M}$ be any matrix in $\mathscr{G}$. If $\pi_{\mathscr{E}}(\mathbf{M})$ is real valued, there must be an even number of non real matrices in $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{M}))$.*

Proof. $\pi_{\mathscr{E}}$ is defined over $\mathbb{R}$ and thus if $\mathbf{M}_0$ is in $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{M}))$, so must its complex conjugate $\overline{\mathbf{M}}_0$. If such an $\mathbf{M}_0$ is not real, then it is not equal to its conjugate.

The following lemma is useful above in the proof of Lemma 3.

Lemma 11. *Let Γ be generically locally rigid (in $\mathbb{C}^d$). Let ρ be generic in $C_{\mathbb{C}^d}(\Gamma)$. Let σ be equivalent to ρ. Then σ is infinitesimally rigid.*

Proof. If Γ has less than $d + 2$ vertices and is generically locally rigid, it must be a simplex, and we are done.

From Corollary 4 and Lemma 9, the set of congruence classes of configurations has dimension $\dim(\mathscr{G})$, which is $v * d - \binom{d+1}{2}$. Due to local rigidity, its measurement set, $\pi_{\mathscr{E}}(\mathscr{G})$, has the same dimension.

Similarly, the set of frameworks with a degenerate affine span must map to g-matrices with rank no greater than $d - 1$, and thus their measurement set must have dimension at most $v * (d - 1) - \binom{d}{2}$. Thus such degenerate measurements are non generic in $\pi_{\mathscr{E}}(\mathscr{G})$.

Meanwhile, the set of infinitesimally flexible frameworks with non-degenerate span, is non generic in $C_{\mathbb{C}^d}(\mathscr{V})$, and so has dimension no larger than $v * d - 1$. Its measurement set has dimension no larger than $v * d - 1 - \binom{d+1}{2}$. Thus the infinitesimally flexible measurements are non generic in the measurement set.

Thus a generic ρ cannot map under the edge squared-length map to any measurement arising from an infinitesimally flexible framework.

A real valued matrix in $\mathscr{G}$ corresponds with an s-valued configuration. At the heart of this correspondence is Sylvester's law of inertia.

Law 1. Suppose $\mathbf{M}$ is a real valued symmetric matrix of size $v-1$ and rank d. Suppose that $\mathbf{M} = \mathbf{B}^t\mathbf{D}\mathbf{B}$, where $\mathbf{B}$ is a real non-singular matrix, and where $\mathbf{D}$ is a real diagonal matrix with s negative diagonal entries, $d-s$ positive diagonal entries, and $v-1-d$ zero diagonal entries. Let us call the triple $(s, d-s, v-1-d)$ the *signature* of $\mathbf{D}$.

Then $\mathbf{M}$ cannot be written as $\mathbf{M} = \mathbf{B}'^t\mathbf{D}'\mathbf{B}'$, where $\mathbf{B}'$ is real non-singular and $\mathbf{D}'$ is real diagonal with a different signature. Thus we can call $(s, d-s, v-1-d)$ the signature of $\mathbf{M}$.

Since every real symmetric matrix has an orthogonal eigen-decomposition, it must have a signature.

Lemma 12. *Suppose some $\mathbf{M} \in \mathscr{G}$ has all real entries and has signature $(s, d'-s, v-1-d')$ for some s and d' (with $d' \leq d$). There exists an s-valued configuration p with an affine span of dimension d' and with $\mathbf{G}(p) = \mathbf{M}$.*

Proof. By assumption $\mathbf{M} = \mathbf{B}^t\mathbf{D}\mathbf{B}$ where $\mathbf{D}$ has signature $(s, d'-s, v-1-d')$. Wlog, let us assume that the entries in $\mathbf{D}$ appear in an order that matches the signature. Let us drop the last $v-1-d'$ rows of $\mathbf{B}$. Let us divide the jth row of $\mathbf{B}$ by $\sqrt{|\mathbf{D}_{j,j}|}$ to obtain an d' by $v-1$ matrix $\mathbf{P}'$. Then we can write $\mathbf{M} = \mathbf{P}'^t\mathbf{S}\mathbf{P}'$, where $\mathbf{S}$ is an d' by d' diagonal "signature" matrix with its first s diagonal entries of -1 and remaining $d'-s$ diagonal entries of 1. Since $\mathbf{B}$ is non-singular, $\mathbf{P}'$ has rank d'.

Multiplying the first s rows of $\mathbf{P}'$ by $\sqrt{1}$, we can write $\mathbf{M} = \mathbf{P}^t\mathbf{P}$. The columns of $\mathbf{P}$ (along with the origin) then give us the complex coordinates of an s-valued configuration $p \in C_{\mathbb{C}^d}(\mathscr{V})$ with $\mathbf{G}(p) = \mathbf{M}$.

Remark 5. When $d' < d$, this does not rule out the possibility of other frameworks with a different dimensional affine span, and different real metric signature. When $d' = d$, Corollary 5 (below) will in fact rule out any other signatures and span dimensions.

Lemma 13. *Let $p \in C_{\mathbb{C}^d}(\mathscr{V})$ be an s-valued configuration, then $\mathbf{G}(p)$ is real. If p has an affine span of dimension $d' \leq d$, then $\mathbf{G}(p)$ has rank no more than d'. Moreover, if p has an affine span of dimension d, then $\mathbf{G}(p)$ has signature $(s, d-s, v-1-d)$.*

Proof. Since p is s-valued, $\mathbf{G}(p)$ can be written in coordinates as $\mathbf{P}'^t\mathbf{S}\mathbf{P}'$, where $\mathbf{P}'$ is a d by $v-1$ real matrix. And $\mathbf{S}$ is a diagonal matrix with s entries of -1 and $d-s$ entries of 1. The rank of $\mathbf{G}(p)$ cannot exceed the rank of $\mathbf{P}'$ which is d'.

If the affine span of p has dimension d, then $\mathbf{P}'$ has rank d. Since the rows of $\mathbf{P}'$ are linearly independent, we can use those rows as the first d rows of a non singular $v-1$ by $v-1$ matrix $\mathbf{B}$. We can use $\mathbf{S}$ as the upper left block of a diagonal matrix $\mathbf{D}$ with the rest of the entries zeroed out. Then we can write $\mathbf{M} = \mathbf{B}^t\mathbf{D}\mathbf{B}$ giving us the stated signature.

Corollary 5. *Let $p \in C_{\mathbb{C}^d}(\mathcal{V})$ be an s-valued configuration with an affine span of dimension d. Let $q \in C_{\mathbb{C}^d}(\mathcal{V})$ be an s'-valued configuration that is congruent to p. Then q has an affine span of dimension d and $s = s'$*

Proof. From Lemma 13, $\mathbf{G}(p)$ has signature $(s, d-s, v-1-d)$. By the congruence assumption and Corollary 4, , $\mathbf{G}(p) = \mathbf{G}(q)$. As $\mathbf{G}(q)$ has rank d, q must have an affine span no less than d, and thus equal to d. From Lemma 13, $\mathbf{G}(q)$ must have signature $(s', d-s', v-1-d)$. Thus $s = s'$.

Now we can establish that when there is a GGR subgraph, the signature of all real matrices in a fiber of $\pi_{\mathcal{E}}$ is fixed.

Lemma 14. *Let Γ be a graph and Γ_0 a GGR subgraph with v_0 vertices where $v_0 \geq d+1$. Let ρ be an s-valued framework in $C_{\mathbb{C}^d}(\Gamma)$ for some s, with configuration p. Suppose also that the affine span of the vertices of Γ_0 in p is all of $\mathbb{C}^d$. Then all of the real matrices in the fiber $\pi_{\mathcal{E}}^{-1}(\pi_{\mathcal{E}}(\mathbf{G}(\rho)))$ must have signature $(s, d-s, v-1-d)$.*

Proof. Wlog, let Γ_0 include the first vertex, and let its vertex set be $\mathcal{V}_0$. We denote by p_0 the configuration p restricted to $\mathcal{V}_0$. p_0, as a restriction of p, is s-valued.

Let $\mathbf{M}$ be any real matrix in the fiber, and let it have signature $(s', d'-s', v-1-d')$ for some s' and d'. From Lemma 12, there must be some q, an s$'$-valued configuration, with $\mathbf{G}(q) = \mathbf{M}$. When restricted to $\mathcal{V}_0$, the configuration q_0 must also be s$'$-valued. Since Γ_0 is complex GGR, p_0 must be congruent to q_0. Then from Corollary 5 q_0 must be s-valued and have affine span of dimension d. Thus $s = s'$. Since q, as a super-set of q_0, must have affine span of dimension d, then from Lemma 13, M must have signature $(s, d-s, v-1-d)$.

Definition 20. Let V and W be irreducible complex algebraic sets of the same dimension and $f : V \to W$ be a surjective (or just dominant) algebraic map, all defined over $\mathbf{k}$. Then the number of points in the fiber $f^{-1}(w)$ for any generic $w \in W$ is a constant. This constant is called the *algebraic degree* of f.

With this, we can complete the proof of Theorem 5 by applying a degree argument:

Proof. We will assume Γ is generically locally rigid, otherwise we are already done.

Let ρ be generic in $C_{\mathbb{E}^d}(\Gamma)$. From Lemma 13 $\mathbf{G}(\rho)$ is real with signature $(0, d, v-1-d)$ (i.e. it is PSD). Because of the existence of a GGR subgraph, from Lemma 14, all of the real matrices in the fiber $\pi_{\mathcal{E}}^{-1}(\pi_{\mathcal{E}}(\mathbf{G}(\rho)))$ must have the same signature. From Lemma 13 and Corollary 4, these matrices are in one to one correspondence with the congruence classes $[\rho_i]$ of equivalent frameworks in $C_{\mathbb{E}^d}(\Gamma)$. Since Γ is not GGR, from Theorem 3, there must be an even number of such classes and thus an even number of real matrices in the fiber.

From Lemma 10, there are an even number of non real matrices in the fiber and we see that the total cardinality of $\pi_{\mathcal{E}}^{-1}(\pi_{\mathcal{E}}(\mathbf{G}(\rho)))$ is even. Since $\pi_{\mathcal{E}}(\mathbf{G}(\rho))$ is generic in the image $\pi_{\mathcal{E}}(\mathcal{G})$, this means that the *algebraic degree* of $\pi_{\mathcal{E}}$ must be even.

Now suppose σ is generic in $C_{\mathbb{S}^d}(\Gamma)$, which we model as a generic s-valued framework in $C_{\mathbb{C}^d}(\Gamma)$. $\mathbf{G}(\sigma)$ is real valued and has signature $(s, d-s, v-1-d)$. From Lemma 14 all of the real matrices in the fiber $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{G}(\sigma)))$ must have the same signature $(s, d-s, v-1-d)$.

Since $\mathbf{G}(\sigma)$ is real, then so is $\pi_{\mathscr{E}}(\mathbf{G}(\sigma))$ so from Lemma 10 there must be an even number of non real matrices in the fiber $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{G}(\sigma)))$, and thus an even number of real matrices in the fiber, all with signature $(s, d-s, v-1-d)$.

From Lemma 13 and Corollary 4, these are in one to one correspondence with the congruence classes $[\sigma_i]$ of equivalent s-valued frameworks in $C_{\mathbb{C}^d}(\Gamma)$. Thus Γ is generically globally flexible in $\mathbb{S}^d$.

Remark 6. The reasoning in the above proof does not hold when Γ does not have the required GGR subgraph. In particular, the non-GGR graph G_3 of Jackson and Owen [8] generically has an *odd* number (namely 45) of equivalent complex realizations in $\mathbb{C}^2$.

8 Extension to Hyperbolic Space

Combining ideas from the previous section with results from Connelly and Whiteley [5], we can transfer the property of generic global rigidity to hyperbolic space $\mathbb{H}^d$ as well.

Corollary 6. *A graph Γ is generically globally rigid in $\mathbb{E}^d$ iff it is generically globally rigid in $\mathbb{H}^d$.*

This can be done using the coning operation explored in [5], and the proof is developed below.

Definition 21. Given a graph Γ and a new vertex u, the *coned graph* $\Gamma * \{c\}$ is the graph obtained starting with Γ, adding the vertex c and adding an edge connecting c to each vertex in Γ.

Theorem 7 (Connelly and Whiteley [5]). *A graph Γ is generically globally rigid in $\mathbb{E}^d$ iff $\Gamma * \{c\}$ is generically globally rigid in $\mathbb{E}^{d+1}$.*

(This theorem is proven using an argument about equilibrium stress matrices. See Fig. 2.)

By modeling spherical d-space within a Euclidean d + 1 space, Connelly and Whiteley then show the equivalence between Euclidean GGR of $\Gamma * \{c\}$ and spherical GGR of Γ.

In a similar manner, one can model hyperbolic space $\mathbb{H}^d$ within the d + 1 dimensional pseudo Euclidean space that has one negative coordinate in its signature. We denote this *Minkowski space* as $\mathbb{M}^{d+1}$. In particular, we model $\mathbb{H}^d$ as the subset of vectors $\mathbf{v} \in \mathbb{M}^{d+1}$ such that $|\mathbf{v}|^2 = -1$ under the Minkowski metric, and such

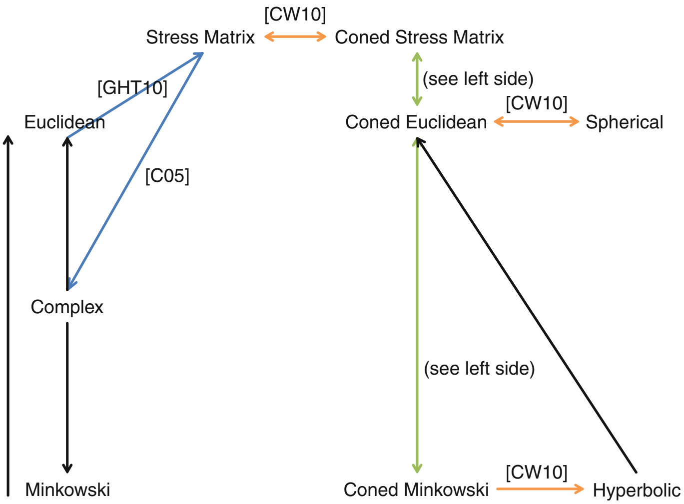

Fig. 2 Implications between generic global rigidity in various spaces. *Black lines* show implications proven in this paper

that $\mathbf{v}_1 > 0$, where $\mathbf{v}_1$ is the first coordinate of $\mathbf{v}$. For two vectors $\mathbf{v}$ and $\mathbf{w}$ on this "hyperbolic locus", their distance in $\mathbb{H}^d$ corresponds to the arcosh of their Minkowski inner product.

8.1 Proof of Corollary ⇒

We begin with a hyperbolic lemma that mirrors a spherical lemma in [5].

Lemma 15. *Let ρ and σ be two equivalent and non congruent frameworks of Γ in $\mathbb{H}^d$, then there is a corresponding pair $(\rho''_{\mathbb{M}}, \sigma''_{\mathbb{M}})$ of equivalent and non congruent frameworks of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$. Moreover, if ρ (or σ) is generic in $\mathbb{H}^d$, then we can find a corresponding $\rho''_{\mathbb{M}}$ (or $\sigma''_{\mathbb{M}}$) that is generic in $\mathbb{M}^{d+1}$.*

Proof. Given ρ and σ, we model these as $\rho_{\mathbb{M}}$ and $\sigma_{\mathbb{M}}$, two frameworks of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$, with the cone vertex c at the origin and the rest of the vertices on the hyperbolic locus. For each vertex $t \in \mathscr{V}(\Gamma)$, we pick a generic positive scale α_t and multiply all of the $d+1$ coordinates of $\rho_{\mathbb{M}}(t)$ and $\sigma_{\mathbb{M}}(t)$ by this α_t. Let us call the resulting pair, $\rho'_{\mathbb{M}}(t)$ and $\sigma'_{\mathbb{M}}(t)$. As in [5], $\rho'_{\mathbb{M}}(t)$ and $\sigma'_{\mathbb{M}}(t)$ are equivalent and non congruent in $\mathbb{M}^{d+1}$. By translating these frameworks by some generic offset, we obtain the desired pair $\rho''_{\mathbb{M}}$ and $\sigma''_{\mathbb{M}}$.

Proof (Proof of corollary ⇒). Suppose a graph Γ is not GGR in $\mathbb{H}^d$ then from Lemma 15, $\Gamma * \{c\}$ is not GGR in $\mathbb{M}^{d+1}$. Then from Theorem 4, $\Gamma * \{c\}$ is not GGR in $\mathbb{E}^{d+1}$. Then from Theorem 7, Γ is not GGR in $\mathbb{E}^d$. See Fig. 2.

8.2 Proof of Corollary $\Leftarrow$

In order to prove the other direction we restrict ourselves to Minkowski frameworks that can be moved to the hyperbolic locus using positive scaling.

Definition 22. We say that a framework ρ of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$ is *upper coned* if for all vertices $t \in \mathscr{V}(\Gamma)$, we have $|\rho(t) - \rho(c)|^2 < 0$ and $(\rho(t) - \rho(c))_1 > 0$. We say that ρ is *lower coned* if for all vertices $t \in \mathscr{V}(\Gamma)$, we have $|\rho(t) - \rho(c)|^2 < 0$ and $(\rho(t) - \rho(c))_1 < 0$.

The following lemma is the needed partial converse of Lemma 15.

Lemma 16. *Let ρ and σ be two equivalent and non congruent frameworks of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$. And let us also assume that ρ and σ are upper coned. Then there is a corresponding pair $(\rho_{\mathbb{H}}, \sigma_{\mathbb{H}})$ of equivalent and non congruent frameworks of Γ in $\mathbb{H}^d$. Moreover, if ρ (or σ) is generic in $\mathbb{M}^{d+1}$, then $\rho_{\mathbb{H}}$ (or $\sigma_{\mathbb{H}}$) is generic in $\mathbb{H}^d$.*

Proof. Given ρ and σ, we first translate the frameworks, moving the cone vertex, c, to the origin in $\mathbb{M}^{d+1}$. Let us call the resulting pair ρ' and σ'. For each vertex $t \in \mathscr{V}(\Gamma)$, we then divide all of the $d+1$ coordinates of $\rho'(t)$ and $\sigma'(t)$ by the positive quantity, $-|\rho(t) - \rho(c)|^2$ (which is the same as $-|\sigma(t) - \sigma(c)|^2$). Let us call the resulting pair, ρ'' and σ''. Due to our upper coned assumption, these vertices all lie on the hyperbolic locus and correspond to a pair of frameworks $\rho_{\mathbb{H}}$ and $\sigma_{\mathbb{H}}$ of Γ in $\mathbb{H}^d$. As in [5], the resulting frameworks, $\rho_{\mathbb{H}}$ and $\sigma_{\mathbb{H}}$, of Γ are equivalent, non congruent, and generic in $\mathbb{H}^d$.

In order to ultimately get upper coned Minkowski frameworks, we also define the following special framework classes.

Definition 23. We say that a framework ρ of $\Gamma * \{c\}$ in $\mathbb{E}^{d+1}$ is *spiky* if for one vertex $t_0 \in \mathscr{V}(\Gamma)$, we have $|\rho(t_0) - \rho(c)| > 2$ and for all edges $(t, u) \in \mathscr{E}(\Gamma)$, we have $|\rho(t) - \rho(u)| < \frac{1}{v}$.

Definition 24. We say that a framework ρ of $\Gamma * \{c\}$ in $\mathbb{F}^{d+1}$ is *upper cylindrical* if for all vertices $t \in \mathscr{V}(\Gamma)$, we have $(\rho(t) - \rho(c))_1 > 1$ and $\sum_{i=2}^{d+1} (\rho(t) - \rho(c))_i^2 < 1$.

Lemma 17. *Let Γ be connected. If a framework ρ of $\Gamma * \{c\}$ in $\mathbb{E}^{d+1}$ is spiky, then it is congruent to a framework which is upper cylindrical.*

Proof. We can find a rotation that moves $\rho(t_0) - \rho(c)$ onto the first axis, with a first coordinate greater than 2. Since Γ is connected, it has diameter no larger than v. From the triangle inequality, all of the coordinates of all of the vertices must satisfy the upper cylindrical conditions.

Lemma 18. *Let ρ and σ be two upper cylindrical frameworks of $\Gamma * \{c\}$ in $\mathbb{E}^{d+1}$. Then the resulting frameworks from the Pogorelov map to $\mathbb{M}^{d+1}$, $(\tilde{\rho}, \tilde{\sigma}) := P(\rho, \sigma)$, are both upper cylindrical.*

Proof. This follows from directly the "coordinate swapping" interpretation of the Pogorelov map from Remark 4.

Lemma 19. *If a framework ρ of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$ is upper cylindrical, then it is upper coned.*

Proof. By definition, the first coordinates of all vertices have the required sign. Moreover, for any $t \in \mathscr{V}(\Gamma)$,

$$|\rho(t) - \rho(c)|^2 = -(\rho(t) - \rho(c))_1^2 + \sum_{i=2}^{d+1} (\rho(t) - \rho(c))_i^2 < 0. \tag{4}$$

And thus it is upper coned.

With these simple facts established, we can now apply the machinery from Sect. 6 to the problem at hand.

Lemma 20. *Let $\Gamma * \{c\}$ be generically locally rigid in $\mathbb{E}^{d+1}$. Suppose $\Gamma * \{c\}$ is not GGR in $\mathbb{E}^{d+1}$, then $\Gamma * \{c\}$ has an pair of generic frameworks in $\mathbb{M}^{d+1}$, that are equivalent, non congruent, and upper coned.*

Proof. The proof follows that of Sect. 6. The only issue is ensuring the upper conedness of the result.

When picking the component E (see Definition 12) we choose a component of E^+ such that E contains some non-congruent pair, $\dim(\pi_1(E)) = v * d$, and such that $\pi_1(E)$ contains a framework ρ that is spiky.

Since the set of frameworks that are spiky is of dimension $v * d$, and by assumption, $\Gamma * \{c\}$ is not GGR in $\mathbb{E}^{d+1}$, and thus GGF in $\mathbb{E}^{d+1}$, the projection $\pi_1(E^+ \backslash C^+)$ must include a set of spiky frameworks with dimension $v * d$. Thus, at least one component with the stated properties must exist. We will chose one such component and will call it E.

Pick an $e := (\rho, \sigma) \in E$ in the fiber above ρ. Since ρ is spiky, and spikiness only depends on edge lengths, σ must be spiky as well. Next, we perturb e in E to get $e' =: (\rho', \sigma')$ that is generic in E. Since spikiness is an open property, for small enough perturbations, both ρ' and σ' will still be spiky.

Since $\Gamma * \{c\}$ is generically locally rigid in $\mathbb{E}^{d+1}$, Γ must be connected. Thus from Lemma 17, we can choose an upper cylindrical σ'^c that is congruent to σ' and an upper cylindrical ρ'^c that is congruent to ρ'. From Lemma 3, since e' is generic in E the point $e'^c := (\rho'^c, \sigma'^c)$ must be in E as well.

Next we perturb e'^c within E to get $e'^{c\prime} =: (\rho'^{c\prime}, \sigma'^{c\prime})$ which is generic in E. Since upper cylindricality is an open property, for small enough perturbations, both $\rho'^{c\prime}$ and $\sigma'^{c\prime}$ will still be upper cylindrical.

Now when we apply the Pogorelov map, $(\widetilde{\rho'^{c\prime}}, \widetilde{\sigma'^{c\prime}}) := P(e'^{c\prime})$. As in the proof of Theorem 4, $\widetilde{\rho'^{c\prime}}$ and $\widetilde{\sigma'^{c\prime}}$ are equivalent, non congruent and generic frameworks in $\mathbb{M}^{d+1}$. From Lemma 18 both $\widetilde{\rho'^{c\prime}}$ and $\widetilde{\sigma'^{c\prime}}$ must be upper cylindrical, and from Lemma 19, both $\widetilde{\rho'^{c\prime}}$ and $\widetilde{\sigma'^{c\prime}}$ must be upper coned,

Proof (Proof of corollary $\Leftarrow$). Suppose a graph Γ is not GGR in $\mathbb{E}^d$ then from Theorem 7, $\Gamma * \{c\}$ is not GGR in $\mathbb{E}^{d+1}$. Then from Lemma 20, $\Gamma * \{c\}$ has an pair of generic frameworks in $\mathbb{M}^{d+1}$ that are equivalent, non congruent, and upper coned. Then from Lemma 16, Γ is not GGR in $\mathbb{H}^d$.

Remark 7. In Section 7 of [5], there is a brief sketch describing how to directly use a Pogorelov type map to equate Euclidean GGR and hyperbolic GGR. That discussion does not go into the details showing that their construction hits an open neighborhood of frameworks (i.e. a generic framework), which is the main technical contribution of our Theorem 4.

8.3 Hyperbolic GGF

Using coning, we can also prove a hyperbolic version of Theorem 5, namely:

Corollary 7. *If a graph Γ is not GGR in $\mathbb{H}^d$, and it has a GGR subgraph Γ_0 with $d+1$ or more vertices, then Γ must be GGF in $\mathbb{H}^d$.*

Proof. Having established that generic global rigidity transfers between Pseudo Euclidean spaces and through coning, we know that $\Gamma * \{c\}$, is not GGR in $\mathbb{M}^{d+1}$. Likewise, it has a coned subgraph with at least $d+2$ vertices, $\Gamma_0 * \{c\}$, that is GGR in $\mathbb{M}^{d+1}$. Thus, from Theorem 5, $\Gamma * \{c\}$ must be GGF in $\mathbb{M}^{d+1}$.

Let ρ be a framework of Γ in $\mathbb{H}^d$. We model this as $\rho_{\mathbb{M}}$, a framework of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$, with the cone vertex c at the origin and the rest of the vertices on the hyperbolic locus. For each vertex $t \in \mathscr{V}(\Gamma)$, we pick a generic positive scale α_t and multiply all of the $d+1$ coordinates of $\rho_{\mathbb{M}}(t)$ by this α_t. Let us call the resulting framework $\rho'_{\mathbb{M}}(t)$. By translating this frameworks by some generic offset, we obtain $\rho''_{\mathbb{M}}$, a generic framework of the coned graph in $\mathbb{M}^{d+1}$. Since the α_t are all positive, $\rho''_{\mathbb{M}}$ must be upper coned.

Since $\Gamma * \{c\}$ is GGF in $\mathbb{M}^{d+1}$, $\rho''_{\mathbb{M}}$ must have an equivalent and non-congruent framework, $\sigma''_{\mathbb{M}}$. From Lemma 21 (below), we can choose $\sigma''_{\mathbb{M}}$ to be upper coned. Then from Lemma 16, there must be a framework, σ, of Γ in $\mathbb{H}^d$, that is equivalent and non congruent to ρ.

Lemma 21. *Let Γ be a connected graph. Let (ρ, σ) be a pair of equivalent frameworks of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$. Let us also assume that ρ is in general position. If ρ is upper coned, then either σ is upper coned or it is lower coned.*

Proof. Let t and u be two vertices of $\mathscr{V}(\Gamma)$ that are connected by an edge in Γ. Along with the edges $\{t, c\}$ and $\{u, c\}$, this defines a triangle T, which is a subgraph of $\Gamma * \{c\}$. Since σ is equivalent to ρ, these frameworks when restricted to T, must be, by definition, congruent.

Since ρ is in general position, from Corollary 8 these two frameworks of T must be strongly congruent. Thus, there is an orthogonal transform of $\mathbb{M}^{d+1}$ mapping $(\rho(t) - \rho(c))$ to $(\sigma(t) - \sigma(c))$ and mapping $(\rho(u) - \rho(c))$ to $(\sigma(u) - \sigma(c))$. An orthogonal transform either maps the entire upper cone to the upper cone, or it maps the entire upper cone to the lower cone. Since Γ is connected, this makes σ either upper coned or lower coned. (Moreover, by negating all of the coordinates in σ we can always obtain an upper coned equivalent framework.)

9 Algebraic Geometry Background

We start with some preliminaries from real and complex algebraic geometry, somewhat specialized to our particular case. For a general reference, see, for instance, the book by Bochnak, Coste, and Roy [2]. Much of this is adapted from [7].

Definition 25. An affine, real (resp. complex) *algebraic set* or *variety* V defined over a field $\mathbf{k}$ contained in $\mathbb{R}$ (resp. $\mathbb{C}$) is a subset of $\mathbb{R}^n$ (resp $\mathbb{C}^n$) that is defined by a set of algebraic equations with coefficients in $\mathbf{k}$.

An algebraic set is closed in the Euclidean topology.

An algebraic set is *irreducible* if it is not the union of two proper algebraic subsets defined over $\mathbb{R}$ (resp $\mathbb{C}$). Any reducible algebraic set V can be uniquely described as the union of a finite number of maximal irreducible subsets called the *components* of V.

A real (resp. complex) algebraic set has a real (resp. complex) *dimension* $\dim(V)$, which we will define as the largest t for which there is an open subset of V, in the Euclidean topology, that is isomorphic to $\mathbb{R}^t$ (resp. $\mathbb{C}^t$). Any algebraic subset of an irreducible algebraic set must be of strictly lower dimension.

A point x of an irreducible algebraic set V is *smooth* (in the differential geometric sense) if it has a neighborhood that is smoothly isomorphic to $\mathbb{R}^{\dim(V)}$ (resp. $\mathbb{C}^{\dim(V)}$). (Note that in a real variety, there may be points with neighborhoods isomorphic to $\mathbb{R}^n$ for some $n < \dim(V)$; we will not consider these points to be smooth.)

Definition 26. Let $\mathbf{k}$ be a subfield of $\mathbb{R}$. A *semi-algebraic set* S defined over $\mathbf{k}$ is a subset of $\mathbb{R}^n$ defined by algebraic equalities and inequalities with coefficients in $\mathbf{k}$; alternatively, it is the image of a real algebraic set (defined only by equalities) under an algebraic map with coefficients in $\mathbf{k}$. A semi-algebraic set has a well defined (maximal) dimension t.

The real *Zariski closure* of S is the smallest real algebraic set defined over $\mathbb{R}$ containing it. (Loosely speaking, we can get an algebraic set by keeping all algebraic equalities and dropping the inequalities. We may need to enlarge the field to cut out the smallest algebraic set containing S but a finite extension will always suffice.)

We call S *irreducible* if its real Zariski closure is irreducible. An irreducible semi-algebraic set S has the same real dimension as its real Zariski closure.

A point on S is smooth if it has a neighborhood in S smoothly isomorphic to $\mathbb{R}^{\dim(S)}$.

Lemma 22. *The image of an irreducible real algebraic or semi-algebraic set under a polynomial map is an irreducible semi-algebraic set. The image of an irreducible* *complex algebraic set under a polynomial map is an irreducible complex algebraic set, possibly with a finite number of subvarieties cut out from it.*

We next define genericity in larger generality and give some basic properties.

Definition 27. A point in a (semi-)algebraic set V defined over $\mathbf{k}$ is *generic* if its coordinates do not satisfy any algebraic equation with coefficients in $\mathbf{k}$ besides those that are satisfied by every point on V.

Almost every point in an irreducible (semi) algebraic set V is generic.

Remark 8. Note that the defining field might change when we take the real Zariski closure V of a semi-algebraic set S. For example, in $\mathbb{R}^1$, the single point $\sqrt{2}$ can be described using equalities and inequalities with coefficients in $\mathbb{Q}$, and thus it is semi-algebraic and defined over $\mathbb{Q}$. But as a real variety, the defining equation for this single-point variety requires coordinates in $\mathbb{Q}(\sqrt{2})$. Indeed, the smallest variety that contains the point $\sqrt{2}$ and that is defined over $\mathbb{Q}$ must also include the point $-\sqrt{2}$. However, this complication does not matter for the purposes of genericity.

Specifically, if $\mathbf{k}$ is a finite algebraic extension of $\mathbb{Q}$ and x is a generic point in an irreducible semi-algebraic set S defined over $\mathbf{k}$, then x is also generic in V, the real Zariski closure of S, defined over an appropriate field. This follows from a three step argument. First, a dimensionality argument shows that V must be a component of $V_{\mathbf{k}}^{+}$, the smallest real algebraic variety that is defined over $\mathbf{k}$ and contains S. Second, it is a standard algebraic fact that if a real (resp. complex) variety W^{+} is defined over $\mathbf{k}$, a subfield of $\mathbb{R}$ (resp. $\mathbb{C}$), then any of its components is defined over some field $\mathbf{k}'$, a subfield of $\mathbb{R}$ (resp. $\mathbb{C}$), which is a finite extension of $\mathbf{k}$. Finally, from Lemma 23 (below), any non generic point $x \in V$ (i.e. satisfying some algebraic equation with coefficients in $\mathbf{k}'$) must also satisfy some algebraic equation with coefficients in $\mathbf{k}$ (or even $\mathbb{Q}$) that is non-zero over V.

Lemma 23. *Let $\mathbf{k}'$ be some algebraic extension of $\mathbb{Q}$. Let V be an irreducible algebraic set defined over $\mathbf{k}'$. Suppose a point $x \in V$ satisfies an algebraic equation ϕ with coefficients in $\mathbf{k}'$ that is non-zero over V, then x must also satisfy some algebraic equation ψ with coefficients in $\mathbb{Q}$ that is non-zero over V.*

Proof. Let H be the Galois group of the (normal closure of) $\mathbf{k}'$ over $\mathbb{Q}$. For $h_i \in H$, denote $h_i(\phi)$ to be the polynomial where h_i is applied to each coefficient in ϕ. Let A be the (possibly empty) "annihilating set", such that $\forall h_i \in A$, $h_i(\phi)$ vanishes identically over V.

Let

$$\phi^{\Sigma} := \phi + \sum_{h_i \in A} \lambda_i h_i(\phi) \tag{5}$$

(Where the $\lambda_i \in \mathbb{Q}$ are simply an additional set of blending weights.)

ϕ^{Σ} has the following properties:

- $\phi^{\Sigma}(x) = 0$.

- (For almost every λ), for any $h \in H$, $h(\phi^{\Sigma})$ does not vanish identically over V. This follows since $h(\phi^{\Sigma})$ is made up of a sum of $|A| + 1$ polynomials, where no more than $|A|$ of them can vanish identically over V. Under almost any blending weights λ, their sum will not cancel.

  Let

$$\psi := \prod_{h_i \in H} h_i(\phi^{\Sigma}) \tag{6}$$

ψ has the following properties:

- $\psi(x) = 0$.
- ψ does not vanish over V.
- $h(\psi) = \psi$. Thus ψ has coefficients in the fixed field, $\mathbb{Q}$.

  The following propositions are standard [7]:

Proposition 1. *Every generic point of a (semi-)algebraic set is smooth.*

Lemma 24. *Let V^+ be a (semi) algebraic set, not necessarily irreducible, defined over* $\mathbf{k}$*. Let V be a component of V^+. Let x be generic in V. Then x does not lie on any other component of V^+. Moreover, any point $x' \in V^+$ that is sufficiently close to x cannot lie on any other component of V^+.*

Proof. As per Remark 8 any component must be defined over an algebraic extension of $\mathbf{k}$. The defining equations of any other component would produce an equation obstructing the genericity of x in V. Since a variety is a closed set in the Euclidean topology, no other component of V^+ can approach x.

Lemma 25. *Let V and W be (semi) algebraic sets with V irreducible, and let $f : V \to W$ be a surjective (or just dominant) algebraic map (i.e. where each of the coordinates of $f(x)$ is a some polynomial expression in the coordinates of x), all defined over* $\mathbf{k}$*. Then if $x \in V$ is generic, $f(x)$ is generic inside W.*

Definition 28. The *complexification* V^* of a real variety V is the smallest complex variety that contains V [11]. The complex dimension of V^* is equal to the real dimension of V. If V is irreducible, then so is V^*. If V is defined over $\mathbf{k}$, so is V^*. A generic point in V is also generic in V^*.

10 Congruence

The following material is standard and is included here for completeness. This presentation is adapted from [6, 9].

In *all* discussions in this section, we will assume that we have first translated any configuration, say $p \in C_{\mathbb{C}^d}(\mathcal{V})$ so that its first vertex lies at the origin. We then treat the rest of the vertices as vectors in $\mathbb{C}^d$, and call them the *vectors of* p.

Definition 29. We define the symmetric bilinear form $\beta(\mathbf{v}, \mathbf{w})$ over pairs of vectors, $\{\mathbf{v}, \mathbf{w}\}$ in $\mathbb{C}^d$ as $\beta(\mathbf{v}, \mathbf{w}) := \mathbf{V}^t\mathbf{W}$ where $\mathbf{V}$ is the d by 1 (canonical) coordinate vector of $\mathbf{v}$. (No conjugation is used here.) If O is an orthogonal transformation on $\mathbb{C}^d$, we have $\beta(\mathbf{v}, \mathbf{w}) = \beta(O(\mathbf{v}), O(\mathbf{w}))$.

β is non degenerate: there is no non-zero vector, $\mathbf{v}$, such that $\beta(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in \mathbb{C}^d$.

The squared length of a vector $\mathbf{v}$ is simply $\beta(\mathbf{v}, \mathbf{v})$

With this notation, the $v-1$ by $v-1$ g-matrix has entries $\mathbf{G}(p)_{t,u} = \beta(\overrightarrow{p(t)}, \overrightarrow{p(u)})$.

For the case of the pseudo Euclidean space $\mathbb{S}^d$ we define $\beta(\mathbf{v}, \mathbf{w}) := \mathbf{V}^t\mathbf{S}\mathbf{W}$, where $\mathbf{S}$ is the d by d diagonal "signature matrix" having its first s diagonal entries -1, and the remaining diagonal entries 1.

Lemma 26. *Let p_0 be a configuration of $d+1$ points in $\mathbb{C}^d$, with affine span of dimension d. Then $\mathbf{G}(p_0)$ has rank d. The same is true in a pseudo Euclidean space $\mathbb{S}^d$.*

Proof. The matrix $\mathbf{G}(p_0)$ represents the form β, over all of $\mathbb{C}^d$, expressed in the basis defined by the vectors of p_0. Since β is a non-degenerate form, $\mathbf{G}(p_0)$ must have rank d.

Lemma 27. *Let p_0 and q_0 be two congruent configurations of $a+1$ points in $\mathbb{C}^d$, both with affine span of dimension a. Then p_0 is strongly congruent to q_0. The same is true in a pseudo Euclidean space $\mathbb{S}^d$.*

Proof. Since the vectors of p_0 and q_0 are in general linear position, we can find an invertible linear transform O_0 such that, for all of the vectors of p_0 and q_0, indexed by a vertex t, we have $\overrightarrow{q(t)} = O_0(\overrightarrow{p(t)})$. (The action of O_0 is uniquely defined between $span(p_0)$ and $span(q_0)$, the a-dimensional linear spaces spanned by the vectors of p_0 and the vectors of q_0.)

The matrix $\mathbf{G}(p_0)$ represents the form β, restricted to $span(p_0)$, expressed in the basis defined by the vectors of p_0, while $\mathbf{G}(q_0)$ represents β, restricted to $span(q_0)$, expressed in the basis defined by the vectors of q_0, Since $\mathbf{G}(p_0) = \mathbf{G}(q_0)$, the map O_0 must act as an isometry between all of $span(p_0)$ and $span(q_0)$.

If $a = d$ we are done. Otherwise, from Witt's theorem (see [9]), the isometric action of O_0 between $span(p_0)$ and $span(q_0)$ can be can be extended to an isometry, O, acting on all of $\mathbb{C}^d$. Thus p_0 and q_0 must be strongly congruent.

Lemma 28. *Let p and q be two congruent configurations of v points in $\mathbb{C}^d$, both with affine span of dimension a. Suppose also that $\mathbf{G}(p) = \mathbf{G}(q)$ has rank a. Then p is strongly congruent to q. The same is true in a pseudo Euclidean space $\mathbb{S}^d$.*

Proof. Since $\mathbf{G}(p)$ has rank a, it must have some a by a non-singular principal submatrix, associated with a subset of a vertices. The vertices in this subset must have a linear span of dimension a in both p and q. We denote by p_0 the configuration p restricted to the $a+1$ vertices comprised of this subset together with the first vertex (at the origin). And likewise for q_0. From Lemma 27 there must be an isometry O of $\mathbb{C}^d$, such that for any vertex t in p_0, we have $\overrightarrow{q_0(t)} = O(\overrightarrow{p_0(t)})$.

For any vertex $u \in \mathscr{V}$, by our assumption on the dimension of the affine span of p and q, we have $\overrightarrow{p(u)} \in span(p_0)$ and $\overrightarrow{q(u)} \in span(q_0)$. Since $\mathbf{G}(p_0) = \mathbf{G}(q_0)$ is invertible, the coordinates of $\overrightarrow{p(u)}$ with respect to the basis p_0, can be determined from the appropriate entries in $\mathbf{G}(p)$. Likewise, the coordinates of $\overrightarrow{q(u)}$ with respect to the basis q_0, can be determined from $\mathbf{G}(q)$. Since $\mathbf{G}(p) = \mathbf{G}(q)$ these coordinates must be the same. Thus $\overrightarrow{q(u)} = O(\overrightarrow{q(u)})$, and p and q are strongly congruent.

Corollary 8. *Let p and q be two congruent configurations of $v \geq d+1$ points in $\mathbb{C}^d$, both with a d-dimensional affine span. Then p is strongly congruent to q. If $v < d+1$, and p and q are in general position, then p is strongly congruent to q.*

The same is true in a pseudo Euclidean space $\mathbb{S}^d$.

Proof. For the first statement, we can pick d vertices, together with the first vertex at the origin, to form a subset of size $d+1$, that has a linear span of dimension d in p. We denote by p_0 the configuration p restricted to this subset. From Lemma 26, the principal submatrix of $\mathbf{G}(p)$ associated with this basis must have rank d. The result then follows from Lemma 28.

If $v \leq d+1$ and the points are in general position, then the result follows directly from Lemma 27.

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Chiral Polytopes and Suzuki Simple Groups

Isabel Hubard and Dimitri Leemans

Abstract For each $q \neq 2$ an odd power of 2, we show that the Suzuki simple group $S = Sz(q)$ is the automorphism group of considerably more chiral polyhedra than regular polyhedra. Furthermore, we show that S cannot be the automorphism group of an abstract chiral polytope of rank greater than 4. For each almost simple group G such that $S < G \leq Aut(S)$, we prove that G is not the automorphism group of an abstract chiral polytope of rank greater than 3, and produce examples of chiral 3-polytopes for each such group G.

Keywords Abstract chiral polytopes • Suzuki simple groups

Subject Classifications: 52B11, 20D06

1 Introduction

Abstract polytopes are combinatorial structures that resemble the classical convex polytopes. Traditionally, the main interest in their study is that of their symmetries. In that vein, the regular ones are, by far, the most studied. For the last 6 years, abstract regular polytopes have been investigated by starting from some group G

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and determining all abstract polytopes having G as regular automorphism group. Thanks to experimental data collected over the years by Leemans, Vauthier, Hartley, Hulpke and Mixer on abstract regular polytopes, several general results have been proved for infinite families of (almost) simple groups. The aim of this paper is to give the state of the art in that very recent field of research and to provide the first similar results on chiral polytopes. In fact, our results deal with chiral polytopes and Suzuki groups. These results are summarized in the following theorem.

Theorem 1. *Let $S \leq G \leq Aut(S)$ where S is the Suzuki group $Sz(q)$, with $q = 2^{2e+1}$ and e is a positive integer. Then,*

1. *There exists at least one chiral polyhedron with automorphism group isomorphic to G;*
2. *There are no chiral polytopes of rank 5 or higher with automorphism group G;*
3. *Up to isomorphism and duality, there are*

$$\frac{\sum_{n|2e+1} \mu(\frac{2e+1}{n})(2^{2n})(2^n - 2)}{2e+1}$$

   chiral polyhedra whose automorphism group is the Suzuki group $Sz(q)$;
4. *If there exists a chiral polytope of rank 4 with automorphism group G, then $G = S$ and $2e + 1$ is not a prime number.*
5. *Let $\mathcal{P}$ be a chiral 4-polytope such that the automorphism group $\Gamma(\mathcal{P}) := \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is S. Then $\langle \sigma_1, \sigma_2 \rangle \cong \langle \sigma_2, \sigma_3 \rangle \cong Sz(q')$ for some q' such that $q = q'^m$ with m an odd integer.*

The techniques developed in this paper to prove this result are very likely to be applicable for other families of groups to obtain similar results.

The paper is organised as follows. Section 2 contains the basic theory about regular and chiral abstract polytopes. Section 3 surveys the state of the art with respect to regular abstract polytopes and almost simple groups. In Sect. 4 we give a brief introduction to Suzuki groups. In the last two sections we relate chiral polytopes with almost simple groups of Suzuki type. In particular, points 1 and 2 of Theorem 1 are shown in Sect. 5, points 3, 4 and 5 are proven in Sect. 6. Furthermore, the proof of the last point of Theorem 1, together with some experimental data leads us to conjecture that no chiral polytopes of rank 4 exist for Suzuki groups.

2 Regular and Chiral Abstract Polytopes

In this section, we review the basic properties about regular and chiral abstract polytopes. For details about this we refer the reader to [27] and [30].

An *(abstract) polytope* of rank n or an n-*polytope* is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function having range $\{-1, \dots, n\}$. The elements of $\mathcal{P}$ are called *faces*. Moreover, for $-1 \leq j \leq n$, a face of rank j is often called a j-*face* and the faces of rank $0, 1$ and $n-1$ are usually called the *vertices*, *edges* and *facets* of the polytope, respectively. We shall ask that $\mathscr{P}$ has a smallest face F_{-1}, and a greatest face F_n (called the *improper faces* of $\mathscr{P}$), and that each *flag* (that is, maximal chain of the order) of $\mathscr{P}$ contains exactly $n+2$ faces. Given two flags, we say that they are *adjacent* if they differ by exactly one face, or that they are j-adjacent, if the rank of the face they differ on is precisely j. We also require that $\mathscr{P}$ be strongly flag-connected, that is, any two flags $\Phi, \Psi \in \mathscr{F}(\mathscr{P})$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that each two successive flags Φ_{i-1} and Φ_i are adjacent with $\Phi \cap \Psi \subseteq \Phi_i$ for all i. Finally, we require the *diamond condition*, namely, whenever $F \leq G$, with $\mathrm{rank}(F) = j-1$ and $\mathrm{rank}(G) = j+1$, there are exactly two faces H of rank j such that $F \leq H \leq G$.

The diamond condition implies that, given a flag Φ of $\mathscr{P}$, for each $i \in \{0, \ldots, n-1\}$ there exists a unique i-adjacent flag to Φ, denoted by Φ^i. We extend this notation by induction by letting $(\Phi^{i_0,i_1,\ldots,i_{k-1}})^{i_k} =: \Phi^{i_0,i_1,\ldots,i_k}$. We further denote by $(\Phi)_i$ the i-face of the flag Φ and note that $(\Phi)_i = (\Phi^j)_i$ if and only if $i \neq j$.

Given two faces F and G of a polytope $\mathscr{P}$ such that $F \leq G$, the *section* G/F of $\mathscr{P}$ is the set of faces $\{H \in \mathscr{P} \mid F \leq H \leq G\}$. If F_0 is a vertex, then the section F_n/F_0 is called the *vertex-figure* of F_0. Note that every section G/F of a polytope $\mathscr{P}$ is also a polytope and has rank $\mathrm{rank}(G/F) = \mathrm{rank}(G) - \mathrm{rank}(F) - 1$.

Observe that a finite 3-polytope is a map (that is, a 2-cellular embedding of a connected graph on a compact surface); however the other way around is not true, as maps need not satisfy the diamond condition.

Let $\mathscr{P}$ and $\mathscr{Q}$ be two n-polytopes. An *isomorphism* from $\mathscr{P}$ to $\mathscr{Q}$ is a bijection $\gamma : \mathscr{P} \to \mathscr{Q}$ such that γ and γ^{-1} preserve order. An *anti-isomorphism* $\delta : \mathscr{P} \to \mathscr{Q}$ is an order-reversing bijection, in which case $\mathscr{P}$ and $\mathscr{Q}$ are said to be *duals* of each other, and the usual convention is to denote $\mathscr{Q}$ by $\mathscr{P}^*$. (Note that $(\mathscr{P}^*)^* \cong \mathscr{P}$.) An isomorphism from $\mathscr{P}$ onto itself is called an *automorphism* of $\mathscr{P}$. The set of all automorphisms of $\mathscr{P}$ forms a group, its automorphism group, denoted by $\Gamma(\mathscr{P})$. It is not difficult to see that $\Gamma(\mathscr{P})$ acts freely (that is, with trivial kernel) on $\mathscr{F}(\mathscr{P})$, the set of all flags of $\mathscr{P}$. An anti-isomorphism from $\mathscr{P}$ to itself is called a *duality*. When a duality of $\mathscr{P}$ exists, $\mathscr{P}$ is said to be *self-dual*. Note that the set of all dualities is not a group as the product of two dualities is in fact an automorphism. Hence, the set of all dualities and automorphisms of $\mathscr{P}$ form a group, the *extended group* of $\mathscr{P}$, denoted by $\bar{\Gamma}(\mathscr{P})$.

The main focus of the study of abstract polytopes has been that of symmetries, in particular that of highly symmetric polytopes. A polytope $\mathscr{P}$ is said to be *regular* if $\Gamma(\mathscr{P})$ is transitive on the flags of $\mathscr{P}$. The automorphism group of a regular polytope $\mathscr{P}$ is generated by n involutions $\rho_0, \rho_1, \ldots, \rho_{n-1}$, such that each ρ_i maps a given *(base) flag* Φ to the i-adjacent flag, Φ^i. These distinguished generators satisfy (among others) the relations

$$(\rho_i \rho_j)^{p_{ij}} = \epsilon \quad \text{for } 0 \leq i, j \leq n-1, \tag{1}$$

where ϵ is the identity element of $\Gamma(\mathscr{P})$, $p_{ii} = 1$ for all i, and $p_{ji} = p_{ij}$ whenever $|i-j| = 1$, and $p_{ij} = 2$ otherwise. Letting $p_i = p_{i-1,i} = p_{i,i-1}$ for $1 \leq i < n$, we say that $\mathscr{P}$ has *Schläfli type* $\{p_1, \ldots, p_{n-1}\}$.

Furthermore, the generators ρ_i for $\Gamma(\mathscr{P})$ satisfy an additional condition, often called the *intersection condition*, namely

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \quad \text{for every } I, J \subseteq \{0, 1, \ldots, n-1\}. \tag{2}$$

Conversely, if Γ is a permutation group generated by elements $\rho_0, \rho_1, \ldots, \rho_{n-1}$ which satisfy the relations (1) and condition (2), then there exists a regular polytope $\mathscr{P}$ with $\Gamma(\mathscr{P}) \cong \Gamma$.

A group Γ generated by involutions $\rho_0, \ldots, \rho_{n-1}$ that satisfy condition (2) is called a *C-group*. If moreover it satisfies relations (1), it is called a *string-C-group*.

When starting with a group Γ, determining all regular abstract polytopes of rank n with automorphism group isomorphic to Γ is equivalent to determining non-isomorphic (ordered) n-tuples $(\rho_0, \ldots, \rho_{n-1})$ of generating involutions of Γ satisfying relations (1) as well as the intersection condition (2). Two n-tuples $(\rho_0, \ldots, \rho_{n-1})$ and $(\rho'_0, \ldots, \rho'_{n-1})$ are said to be isomorphic if there exists an element $g \in Aut(\Gamma)$ such that $(\rho_0, \ldots, \rho_{n-1})^g = (\rho'_0, \ldots, \rho'_{n-1})$.

Every regular polytope $\mathscr{P}$ has a *rotation subgroup* $\Gamma^+(\mathscr{P})$ of $\Gamma(\mathscr{P})$ generated by

$$\sigma_i := \rho_{i-1}\rho_i, \quad i = 1, 2, \ldots, n-1.$$

These σ_i satisfy at least the relations

$$\sigma_i^{p_i} = \epsilon \text{ for } \quad 1 \le i \le n-1, \tag{3}$$

$$(\sigma_i \sigma_{i+1} \ldots \sigma_j)^2 = \epsilon \text{ for } \quad 1 \le i < j \le n-1. \tag{4}$$

Here again $\{p_1, p_2, \ldots, p_{n-1}\}$ is the Schläfli type of $\mathscr{P}$. The group $\Gamma^+(\mathscr{P})$ has index at most two in $\Gamma(\mathscr{P})$. A regular n-polytope $\mathscr{P}$ is called *directly regular* if $\Gamma^+(\mathscr{P})$ has index 2 in $\Gamma(\mathscr{P})$. Note that a regular polytope $\mathscr{P}$ is a directly regular polytope if and only if the flags of $\mathscr{P}$ can be coloured with two colours, in such a way that adjacent flags have different colours. In fact, this colouring divides the set of flags in two different sets, each of them being one orbit of flags under the action of $\Gamma^+(\mathscr{P})$.

An n-polytope $\mathscr{P}$ with base flag Φ is called *chiral* if it is not regular, but there exist automorphisms $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ such that each σ_i fixes all faces in Φ different from $(\Phi)_{i-1}$ and $(\Phi)_i$, and cyclically permutes consecutive i-faces of $\mathscr{P}$ in the rank 2 section $(\Phi)_{i+1}/(\Phi)_{i-2}$ of $\mathscr{P}$. Such automorphisms generate $\Gamma(\mathscr{P})$ and are called the *distinguished generators* of $\Gamma(\mathscr{P})$ with respect to Φ. Note that we may choose the orientation of the σ_i such that

$$\Phi\sigma_i = \Phi^{i,i-1}.$$

It follows that, for $i < j$,

$$\Phi\sigma_i\sigma_{i+1}\ldots\sigma_j = (\Phi\sigma_{i+1}\ldots\sigma_j)^{i,i-1} = \Phi^{j,j-1,j-1,\ldots,i+1,i,i,i-1} = \Phi^{j,i-1},$$

so that the σ_i satisfy the relations (4). Denoting by p_i the order of the generator σ_i, we have that $\{p_1, \ldots, p_{n-1}\}$ is the *Schläfli type* of $\mathscr{P}$. These relations imply that the automorphism group of a rank n chiral polytope is generated by involutions whenever $n \geq 4$. (However, this is not true for chiral maps, that is when $n = 3$. For example chiral maps on the torus are never generated by involutions, as it is proved in Section 6 of [16].)

Let Ψ be a flag of a polytope $\mathscr{P}$. We say that Ψ is *even with respect to* Φ if there exists a sequence of adjacent flags $\Phi = \Phi_0, \Phi_1, \ldots \Phi_{2k-1}, \Phi_{2k} = \Psi$. If $\Psi \in \mathscr{F}(\mathscr{P})$ is not even, then we say that it is *odd (with respect to* Φ*)*. It is not hard to see that the orbit of Φ under the automorphism group of a chiral polytope $\mathscr{P}$ is precisely the set of all even flags with respect to Φ (see [30] for details). This implies that odd flags exist in $\mathscr{P}$ and thus, the automorphism group of a chiral polytope has two orbits on the flags (the set of even flags and the set of odd flags). Furthermore, all the flags adjacent to an even flag are odd (and all flags adjacent to an odd flag are even). Hence, chiral polytopes are precisely those polytopes whose automorphism group has exactly two orbits on the flags, with adjacent flags in different orbits. This implies that the automorphism group of a chiral polytope is transitive on the faces of each rank. Furthermore, every section of a chiral polytope is itself either a chiral or a directly regular polytope. In particular the following proposition states a well-known property of chiral polytopes (see [30]) that we add as it will be of great use in our discussion in Sect. 6.

Proposition 1. *For every $k \in \{0, \ldots, n-2\}$, the k-faces of a chiral n-polytope are directly regular, as abstract polytopes.*

Chiral polytopes are said to occur in pairs of *enantiomorphic forms*, with one being the 'mirror image' of the other (see [30] for a precise discussion of this notion).

In a similar way as for the regular case, the distinguished generators of the automorphism group of a chiral polytope satisfy an *intersection condition*, arising from considering the stabilisers of the chains $\Phi_{\bar{J}} := \{(\Phi)_j \mid j \notin J\}$, for each $J \subseteq \{0, 1, \ldots, n-1\}$.

The intersection condition for chiral polytopes of rank 5 and higher is not straightforward to state and it depends on the so-called "half-turns" of the polytope (see [30]). For rank 3, the intersection condition states that

$$\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \{\epsilon\},$$

while for rank 4, it says that for $i, j, k \in \{1, 2, 3\}$ with $i \neq j \neq k \neq i$,

$$\langle \sigma_i \rangle \cap \langle \sigma_j \rangle = \{\epsilon\}, \tag{5}$$

and

$$\langle \sigma_i, \sigma_j \rangle \cap \langle \sigma_j, \sigma_k \rangle = \langle \sigma_j \rangle. \tag{6}$$

Conversely, if Γ is any permutation group generated by elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ which satisfy the relations (3) and (4), as well as the appropriate intersection condition, then there exists a polytope $\mathscr{P}$ of rank n which is either directly regular or chiral, such that $\Gamma(\mathscr{P}) \cong \Gamma$ if $\mathscr{P}$ is chiral, or $\Gamma^+(\mathscr{P}) \cong \Gamma$ if $\mathscr{P}$ is directly regular. Moreover, $\mathscr{P}$ is directly regular if and only if there exists an involutory group automorphism $\rho : \Gamma \to \Gamma$ such that $\rho(\sigma_1) = \sigma_1^{-1}$, $\rho(\sigma_2) = \sigma_1^2\sigma_2$, and $\rho(\sigma_i) = \sigma_i$ for $3 \leq i \leq n-1$ (that is, the group automorphism ρ acts like conjugation by the generator ρ_1 in the directly regular case).

The following lemma shall be of great help when dealing with chiral polytopes coming from Suzuki groups.

Lemma 1. *Let $\mathscr{P}$ be a chiral polytope of rank n, and let $\sigma_1, \ldots, \sigma_{n-1}$ be the distinguished generators of $\Gamma(\mathscr{P})$ with respect to some base flag Φ.*

1. *If $n \geq 5$, then σ_i and σ_i^{-1} are conjugate in $\Gamma(\mathscr{P})$ for every $i = 1, \ldots n-1$.*
2. *If $n = 4$, then σ_1 and σ_1^{-1} are conjugate and σ_3 and σ_3^{-1} are conjugate in $\Gamma(\mathscr{P})$.*

Proof. We prove both statements at the same time. We first recall that $\sigma_i\sigma_{i+1}\ldots\sigma_j$ is of order 2 for every $1 \leq i < j \leq n-1$. Hence, if $i \leq n-3$,

$$\sigma_i^{-1} = \sigma_{i+1}\ldots\sigma_{n-1}\sigma_i\sigma_{i+1}\ldots\sigma_{n-1} = (\sigma_{i+1}\ldots\sigma_{n-1})^{-1}(\sigma_i)(\sigma_{i+1}\ldots\sigma_{n-1});$$

and if $j \geq 3$,

$$\sigma_j^{-1} = \sigma_1\ldots\sigma_{j-1}\sigma_j\sigma_1\ldots\sigma_{j-1} = (\sigma_1\ldots\sigma_{j-1})^{-1}(\sigma_j)(\sigma_1\ldots\sigma_{j-1}).$$

Therefore if $n \geq 5$ every σ_i is conjugate to its inverse, while if $n = 4$ this is true for $i = 1, 3$ (but might not hold for $i = 2$).

3 Abstract Polytopes and Almost Simple Groups

The question of which finite simple groups occur as the automorphism group of regular maps appears already in the Kourovka Notebook [25] in 1980 and has naturally extended recently to that of regular and chiral polytopes. The aim of this section is to survey the state of the art with respect to abstract polytopes and almost simple groups, including what is known for regular maps.

We, however, start with a simple but important proposition about regular polytopes and simple groups.

Proposition 2. *If the automorphism group of an abstract regular polytope $\mathscr{P}$ is a simple group G, then $\mathscr{P}$ cannot be a directly regular polytope.*

Proof. Suppose otherwise. Then the rotational subgroup $\Gamma^+(\mathscr{P})$ has index 2 in $\Gamma(\mathscr{P}) = G$. This implies that $\Gamma^+(\mathscr{P})$ is a normal subgroup of G, contradicting the fact that G is a simple group.

3.1 Regular Maps and Almost Simple Groups

The first main results on regular maps with automorphism group A_n were obtained by Conder in his doctoral thesis, published in [3, 4]. These were the first results dealing with simple groups and regular maps. He proved that for every $k > 6$, all but finitely many A_n are the automorphism group of a regular map of type $\{3, k\}$. In fact, he also showed that all but few cases of symmetric groups can be generated by three involutions, two of which commute. Some years later Sjerve and Cherkassoff extended Conder's results by showing in [32] that S_n is a group generated by three involutions, two of which commute, provided that $n \geq 4$.

During the 1990s several people studied different families of almost simple groups in relation with regular maps. Jones and Silver (see [18]) took care of the Suzuki simple groups $Sz(q)$ and showed that each of them acts as the automorphism group of a regular map of type $\{4, 5\}$. In [32], Sjerve and Cherkassoff showed that, except for few cases, the $PSL(2, q)$ and $PGL(2, q)$ groups can be generated by three involutions, two of which commute.

Some years later, Nuzhin focused his attention on all simple groups except the sporadic groups (see [29] and its references). He completely determined which of them are automorphisms groups of regular maps. The sporadic groups were handled by Abasheec and Norton (see [26]).

Putting together the results of all of them, we now have the following theorem.

Theorem 2. *All finite simple groups are automorphism groups of regular maps except the following ones:* $A_6 = L_2(9)$, A_7, A_8, $L_2(7)$, $L_3(q)$, $L_4(2^n)$, $U_4(2^n)$, $U_3(q)$, $PSp_4(3)$, M_{11}, M_{22}, M_{23}, McL.

3.2 Regular Polytopes and Almost Simple Groups

Although already in 1896, E. H. Moore gave in [28] a set of involutions of S_n that corresponds to the $(n-1)$-simplex, showing that for every n, there is a regular polytope of rank $n-1$ whose automorphism group is S_n, the first results concerning regular abstract polytopes (and not only regular maps or convex polytopes) and almost simple groups are due to Leemans. In [20], he proved that if $G = Sz(q)$ with $q \neq 2$ an odd power of 2, then all the abstract regular polytopes having G as automorphism group are of rank 3 (and there exists at least one such polytope for each value of q). Moreover, if $Sz(q) < G \leq Aut(Sz(q))$, Leemans showed that G is not a C-group and therefore that there cannot exist an abstract regular polytope having G as automorphism group.

Leemans started the study of the Suzuki groups as automorphism groups of regular polytopes as a result of previous work on abstract polytopes. In 2003, Leemans and Hartley came to the conclusion that building atlases of polytopes would help understand how polytopes arise in some families of groups. To this end they made use of the computational software GAP and MAGMA [1].

Hartley then decided to build an atlas of regular polytopes whose automorphism group is a group of order less than 2,000, but not 1,024 nor 1,536 [13]. In parallel, Leemans, together with Laurence Vauthier, built an atlas of such polytopes for all groups G such that $S \leq G \leq Aut(S)$, with S a simple group in the Atlas of Finite Groups, of order less than 900,000 [24]. More recently in [14], Hartley and Alexander Hulpke classified all polytopes for the sporadic groups as large as the Held group (of order 4,030,387,200), and very recently, Leemans and Mark Mixer managed to compute all regular polytopes associated to Co_3 [21].

These collections of data have permitted to state a series of conjectures. Many of them are now proven. They have given rise to the study of several families of almost simple groups in association with regular polytopes. In the remainder of this section we survey this work.

In 2005, Leemans and Vauthier proved that if $G \cong PSL(2,q)$ is such that $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is a string C-group, then $n \leq 4$. Not long after that, together with Schulte, Leemans classified the rank 4 polytopes for groups $PSL(2,q)$ and $PGL(2,q)$ and showed that the highest rank is also 4 for the $PGL(2,q)$ groups. Finally, in 2010 Brooksbank and Vincinsky studied regular polytopes with automorphism group G with $PSL(3,q) \leq G \leq PGL(3,q)$. The known results with respect to almost simple groups of $PSL(n,q)$ type can be summarized the following theorem.

Theorem 3. *1. Let G* $\cong PSL(2,q)$. *If* $\mathscr{P}$ *is a polytope of rank 4 on which G acts regularly, then* $q = 11$ *or* 19 *[22].*

2. Let G $\cong PGL(2,q)$. *If* $\mathscr{P}$ *is a polytope of rank* ≥ 4 *on which G acts regularly, then* $n = 4$ *and* $q = 5$ *[23].*

3. If $PSL(3,q) \leq G \leq PGL(3,q)$*, then G is not the automorphism group of an abstract regular polytope [2].*

We finally turn our attention to almost simple groups of alternating type and their relation with abstract regular polytopes. As we pointed out before, Conder and Sjerve and Cherkassoff studied these groups in relation with regular maps. More recently Fernandes and Leemans studied the symmetric groups (see [10]), and then, together with Mixer took care of the alternating groups (see [11] and [12]).

Among other things, they showed that whenever $n \neq 4$, up to isomorphism, the $(n-1)$-simplex is the unique regular polytope of rank n with automorphism group S_n. In Hartley's atlas [13] we can see that for $n = 4$ there are three regular polyhedra, namely the tetrahedron, the hemicube and the hemioctahedron. Fernandes and Leemans furthermore showed that for $n \geq 7$, up to isomorphism, there exist exactly two regular polytopes of rank $n-2$ with automorphism group S_n, whose Schläfli types are $\{4, 6, 3^{n-5}\}$ and $\{3^{n-5}, 6, 4\}$, respectively and observe that the condition $n \geq 7$ is mandatory here. Indeed, there are four polytopes of rank 4, of respective Schläfli symbols $\{3,4,4\}$, $\{3,6,4\}$, $\{4,4,4\}$ and $\{4,6,4\}$ whose automorphism group is S_6; there are four polytopes of rank 3, of respective Schläfli symbols $\{4,5\}$, $\{4,6\}$, $\{5,6\}$ and $\{6,6\}$ whose automorphism group is S_5.

Finally, they proved that there is no gap in the ranks of polytopes associated to S_n, which can be stated as follows.

Theorem 4 ([10]). *For every positive integer $r \geq 3$, there are but finitely many groups S_n that are not automorphism groups of an abstract regular polytope of rank r.*

The results obtained for the symmetric groups by Fernandes and Leemans permitted them to show that the group A_n, with $n \geq 7$, is not the automorphism group of an abstract regular polytope of rank $n-2$. In fact, together with Mixer (see [11]), they showed that for each $n \notin \{3, 4, 5, 6, 7, 8, 11\}$, there is a rank $\lfloor \frac{n-1}{2} \rfloor$ string C-group representation of the alternating group A_n, and conjecture that, for $n \geq 12$, the highest rank of a string C-group having A_n as automorphism group is $\lfloor \frac{n-1}{2} \rfloor$. Finally, they also proved the following theorem.

Theorem 5 ([12]). *For each rank $k \geq 3$, there is a regular k-polytope $\mathscr{P}$ with automorphism group isomorphic to an alternating group A_n for some n. In particular, for each even rank $r \geq 4$, there is a regular polytope with Schläfli type $\{10, 3^{r-2}\}$ and group isomorphic to A_{2r+1}, and for each odd rank $q \geq 5$, there is a regular polytope with Schläfli type $\{10, 3^{q-4}, 6, 4\}$ and group isomorphic to A_{2q+3}.*

3.3 Chiral Polytopes and Almost Simple Groups

In contrast with regular polytopes, little is known with respect to chiral polytopes whose automorphism group is an almost simple group.

Conder studied chiral maps associated with symmetric and alternating groups [3,4], as well as those associated with then Matthieu groups [5]. Jones computed the symmetric genus of maps having a Suzuki [18] or a Ree [17] group as automorphism group, and gave families of chiral maps associated with such groups. Recently, Conder, Potočnik and Širáň [7] studied regular and chiral maps and hypermaps with automorphism group $PSL(2, q)$ and $PGL(2, q)$, for every prime power q.

For higher ranks, examples of chiral polytopes of rank 4, having $PSL(2, p^r)$, for $p \cong 1 \pmod 8$, and $p \cong 1 \pmod{12}$, were constructed by Schulte and Weiss in [31]. Recently, Conder, Hubard, O'Reilly and Pellicer (see [8, 9]) have constructed examples of rank 4 chiral polytopes for all but finitely many symmetric and alternating groups. They also found, for each $n \geq 5$, an example of a rank n chiral polytope with automorphism group isomorphic to either a symmetric or an alternating group.

On the other hand, some experimental data has been obtained. Hartley, Hubard and Leemans [15] used MAGMA to produce an Atlas of Chiral Polytopes from Almost Simple Groups, where they compute, for each almost simple group G such that $S \leq G \leq Aut(S)$, where S is a simple group of order less than 900,000 listed in the Atlas of Finite Groups, up to isomorphism, all the possible chiral polytopes that have G as their automorphism group.

4 Suzuki Groups

The aim of the remainder of this paper is to prove similar results as in [20], but in the chiral case. In this section we give a brief introduction to Suzuki groups and state here some properties of them that we shall make use of in the following sections. We refer the reader to [33] for the basic properties of Suzuki groups.

In the projective 3-space over the finite field $GF(q)$, for $q = 2^{2e+1}$, PG$(3, q)$, an *ovoid* D is a set of $q^2 + 1$ points satisfying the following axioms:

1. No three points are collinear;
2. For every $p \in D$, there exists a hyperplane E of PG$(3, q)$ such that $D \cap E = \{p\}$;
3. For each such $p \in D$ and E, for every line ℓ of PG$(3, q)$ through p that is not contained in E, there exists a point $p' \in D \cap \ell$ with $p' \neq p$.

For instance, quadrics are ovoids in PG$(3, q)$. Jacques Tits exhibited a class of ovoids that are not quadrics, but occur as the fixed points of an involutary automorphism of $PSp(4, q)$ [34]. Those ovoids are now called Tits ovoids or Suzuki ovoids. The *Suzuki group* $Sz(q)$ is defined as the subgroup of the collineations of PG$(3, q)$ that leave a Suzuki-Tits ovoid invariant. Tits showed that the choice of different Suzuki-Tits ovoids D (of the same projective space PG$(3, q)$) gives rise to conjugate groups in the group of all collineations of PG$(3, q)$. Thus, for each $q = 2^{2e+1}$, where e is a positive integer, $Sz(q)$ is a well defined group. Suzuki [33] showed that such a group is simple.

Throughout this paper we shall use several properties of Suzuki groups, their subgroups and their elements. Given $q = 2^{2e+1}$, the maximal subgroups of $Sz(q)$ have one of the following structures:

$$(E_q \hat{\cdot} E_q) : C_{q-1}, \quad D_{2(q-1)}, \quad C_{\alpha_q} : C_4, \quad C_{\beta_q} : C_4, \quad Sz(q'),$$

where the symbol $\hat{\cdot}$ stands for a non-split extension, $\alpha_q := q + \sqrt{2q} + 1$, $\beta_q := q - \sqrt{2q} + 1$ and $q' = 2^{2e'+1}$ such that $2e' + 1$ divides $2e + 1$. Note that $D_{2(q-1)}$ is the dihedral group of order $2(q - 1)$. Clearly, if $2e + 1$ is a prime number, then $Sz(q)$ has no proper subgroups of Suzuki type. In such case, the elements of $Sz(q)$ have order 2, 4 or d, where d is a divisor of either $q - 1, \alpha_q$ or β_q.

The following proposition encapsulates results about involutions and elements of order 4 in Suzuki groups and their action on ovoids that we shall need for this paper.

Proposition 3. *Let D be an ovoid and let $Sz(q)$, for $q = 2^{2e+1}$ be its Suzuki group. Then:*

1. *If τ is an involution of $Sz(q)$, then τ stabilizes exactly one point of D. The other q^2 points of D are divided into q sets of q points and τ stabilizes each of them, switching their elements pairwise.*
2. *All involutions are in the same conjugacy class of $Sz(q)$.*

3. *$Aut(Sz(q)) \cong Sz(q) : C_{2e+1}$; hence, for every involution ρ of $Aut(Sz(q))$ we have that $\rho \in Sz(q)$. In particular, if $Sz(q) < G \leq Aut(Sz(q))$, then G cannot be generated by involutions.*
4. *There is a unique conjugacy class of subgroups C_4.*
5. *There are two conjugacy classes of elements of order 4 in $Sz(q)$. An element g of order 4 and it inverse g^{-1} are never in the same conjugacy class.*

Each maximal subgroup of $Sz(q)$ can be described as the stabilizer of a set of point on the ovoid. Namely, a group $(E_q \cdot\!\!\!\cdot\ E_q) : C_{q-1}$ is the stabilizer of one point of the ovoid. $D_{2(q-1)}$ is the stabilizer of a pair of points, while the pointwise stabilizer of two points is either C_{q-1} or ϵ. Hence, if the pointwise stabilizer of two points is a C_{q-1}, this is the intersection of a $(E_q \cdot\!\!\!\cdot\ E_q) : C_{q-1}$ and a $D_{2(q-1)}$. The pointwise stabilizer of three different points is ϵ. Subgroups $C_{\alpha_q} : C_4$ and $C_{\beta_q} : C_4$ are the stabilizers of curves with α_q or β_q points, respectively. Finally, subgroups of Suzuki type are stabilizers of sub-ovoids.

Every C_{q-1} is the pointwise stabilizer of two points and hence the intersection of any two C_{q-1}'s is identity (as otherwise it would stabilize too many things).

The intersection of $D_{2(q-1)}$ and $C_{\alpha_q} : C_4$ (or $C_{\beta_q} : C_4$) is either C_2 or ϵ. The intersection of $C_{\alpha_q} : C_4$ and $C_{\beta_q} : C_4$ is either C_4, C_2 or ϵ.

The structure of the subgroups of a Suzuki group $Sz(q)$ implies the following lemma that is closely related with regular abstract polytopes.

Lemma 2. *Let ρ_0, ρ_2 be two commuting involutions of a Suzuki group $Sz(q)$, where $q = 2^{2e+1}$. If $\rho \in Sz(q)$ is an involution that does not commute with ρ_0, then $\langle \rho_0, \rho, \rho_2 \rangle$ is a group of Suzuki type. In particular if $2e + 1$ is a prime number, then $\langle \rho_0, \rho, \rho_2 \rangle = Sz(q)$.*

The $(E_q \cdot\!\!\!\cdot\ E_q) : C_{q-1}$ Subgroups

Let us now take a look at a group $G = (E_q \cdot\!\!\!\cdot\ E_q) : C_{q-1} \leq Sz(q)$. It has $q^2(q-1)$ elements and, as we said before, it is the stabilizer of a point, say $\mathcal{U}$.

The elements of this group are divided into three types: the involutions, the elements of order 4, and the elements of odd order. Note that the elements of odd order are all inside some C_{q-1} and hence they fix at least two points. Furthermore, the intersection of any two such C_{q-1} is the identity, as such an intersection fixes three points.

Now, any such C_{q-1} stabilizes two points, one of them being $\mathcal{U}$. Since the ovoid has $q^2 - 1$ points, the subgroups C_{q-1} are in one to one correspondence with the other q^2 points of the ovoid. And since each C_{q-1} has $q-2$ elements different than identity, then there are $q^2(q-2)$ elements of odd order.

The remaining $q^2 - 1$ elements are of order 2 or 4. More precisely, there are $q-1$ involutions and $q^2 - q$ elements of order 4.

5 Regular and Chiral Polyhedra and Almost Simple Groups of Suzuki Type

Recently Leemans [20] showed that there exist no regular polyhedra with automorphism group G, where $Sz(q) < G \leq Aut(Sz(q)) = Sz(q) : C_{2e+1}$. In [18] Jones and Silver studied the symmetric genus of maps having Suzuki simple groups as automorphism groups. They constructed, for each $Sz(q)$, with $q = 2^{2e+1}$ with e a positive integer, a rotary map of type $\{4, 5\}$ and showed that all such maps are indeed chiral. We now give, for each $q = 2^{2e+1}$ with e a positive integer, a construction for a chiral polyhedron with automorphism group G, for certain $Sz(q) < G \leq Aut(Sz(q))$.

Theorem 6. *Let $G := Sz(q) : \langle\sigma\rangle$ with σ an outer automorphism of order $d \mid 2e + 1$. There exists at least one chiral polyhedron having G as automorphism group.*

Proof. The group G is the semi-direct product $H \rtimes K$, where of $G \rhd H \cong Sz(q)$ and $K := \langle\sigma\rangle$ is a cyclic group. Its elements may be written as pairs (k, h) with $k \in K$ and $h \in H$. The product is defined by $(k_1, h_1) * (k_2, h_2) = (k_1 k_2, h_1^{k_2} h_2)$. We denote by 1_G, 1_K and 1_H respectively the identity elements of G, K and H.

Take the map of Jones and Silver. It is a map of type $\{4, 5\}$ generated by some elements σ_1 and σ_2 of H. The element of order 4 is conjugate to an element of order 4 of $Sz(2) \leq H$ that is obviously centralised by σ as $Sz(2)$ has all its matrices with entries in $GF(2)$. Therefore, we may assume that the map of type $\{4, 5\}$ is obtained by some generators σ_1 and σ_2 such that σ_1 is an element of order 4 of $Sz(2)$ and σ_2 is an element of order 5. Take $\tau_2 = (\sigma, \sigma_2)$. Now, let $\tau_1 = (\sigma^{-1}, \sigma_1)$. Clearly, $\tau_1\tau_2 = (1_K, \sigma_1^{\sigma}\sigma_2) = (1_K, \tau)$ is an involution of G. The group $\langle\tau_1, \tau_2\rangle$ contains $(1_K, \tau)$, $(\sigma, 1_H)$ and also $(1_K, \sigma_2)$, hence it is G itself. It remains to check that $\langle\tau_1\rangle \cap \langle\tau_2\rangle = 1_G$. We have that $\langle\tau_2\rangle = \langle(1_K, \sigma_2)\rangle \times \langle(\sigma, 1_H)\rangle$. Moreover, by construction, $\langle(1_K, \sigma_1)\rangle \cap \langle(1_K, \sigma_2)\rangle = 1_G$. So the only possibility for $\langle\tau_1\rangle \cap \langle\tau_2\rangle \neq 1_G$ is to have $(\sigma^{-1}, \sigma_1)^k = (\sigma, 1_H)^l$ for some integers k and l. But that requires both sides to be the identity which in not the case. Therefore, we can conclude that $\{\tau_1, \tau_2\}$ gives us a chiral polyhedron for G. Indeed, this cannot be regular as it is shown in [20] that G cannot be generated by three involutions.

5.1 Number of Non-isomorphic Regular and Chiral Polyhedra Associated to $Sz(q)$

The question "Are there more regular or chiral polytopes?" has been a discussion topic in the last few years. Although intuition hints us to think that there are more chiral polytopes, experimental data for maps tells us otherwise. In fact, for almost every genus g, with $g \leq 200$, Conder's database [6] shows that there are more regular than chiral maps on a surface of genus g.

In this section we count, for each $q = 2^{2e+1}$, the number of regular and chiral polyhedra that have $Sz(q)$ as its automorphism group. We then conclude that every Suzuki simple group $Sz(q)$ is the automorphism group of more chiral than regular polyhedra.

In [19], Kiefer and Leemans determined, up to isomorphism and duality, the number of abstract regular polyhedra having a Suzuki group $Sz(q)$ as automorphism group. None of these are self-dual and therefore, the number up to isomorphism is also known. It suffices to multiply by 2 the number given by Kiefer and Leemans.

Their computation may be greatly simplified by observing that outer automorphisms give orbits of maximum possible size in $Sz(q)$. Therefore, we can first count possibilities up to conjugacy and that is done in [19]. They obtain $(q-2)(q-1)$ possible triples of involutions, two of which commute, and that generate a Suzuki group. If $2e+1$ is not a prime, $Sz(q)$ has subgroups of Suzuki type. Therefore, to obtain the number of polyhedra up to isomorphism and duality, we have to remove those triples that generate only a sub-Suzuki group. We then get the following formula.

$$\frac{\sum_{n|2e+1} \mu(\frac{2e+1}{n})(2^n-1)(2^n-2)}{2e+1}.$$

Up to isomorphism we get the same number multiplied by 2, that is

$$2 \cdot \frac{\sum_{n|2e+1} \mu(\frac{2e+1}{n})(2^n-1)(2^n-2)}{2e+1}.$$

To count the number of regular or chiral polyhedra for a Suzuki group, we first pick an involution τ (there is a unique choice up to isomorphism and duality) and then we look for elements $\sigma \in Sz(q)$ such that $\langle \tau, \sigma \rangle = Sz(q)$. First we just ask τ and σ to generate a Suzuki group. The elements σ that do not generate a Suzuki group with τ generate a subgroup of one of $E_q{\cdot}E_q : (q-1)$, $D_{2(q-1)}$, $\alpha_q : 4$ or $\beta_q : 4$. We can count how many such elements there are. We get the following result.

Lemma 3. *Given an involution $\tau \in Sz(q)$, there are $q^2(4q-3)$ elements σ such that $\langle \tau, \sigma \rangle$ is not a Suzuki group.*

Proof. The possible orders of elements of a Suzuki group are 1, 2, 4 or divisors of $q-1$, α_q or β_q. There are $(q^2+1)(q-1)$ involutions that, taken with τ will only generate a dihedral group.

The elements of order $d \mid q-1$ are either in a $E_q{\cdot}E_q : (q-1)$ or a $D_{2(q-1)}$ containing τ. Simple computations show that in the first case, we have $q^3 - 2q^2$ such elements and in the second case, we have $q^3/2 - q^2$ such elements.

Similarly, by means of elementary computations, one can show that there are $q^2/4(\alpha_q + \beta_q - 2) = q^3/2$ elements of order $d \mid \alpha_q\beta_q$, and there are $q^2/2(\alpha_q + \beta_q - 2) + q^2 - q = q^3 + q^2 - q$ elements of order 4.

It remains to sum these numbers together and add 1 for the identity element. This gives

$$1 + (q^2 + 1)(q - 1) + q^3 - 2q^2 + q^3/2 - q^2 + q^3/2 + q^3 + q^2 - q = q^2(4q - 3).$$

The previous result leads to the following corollary.

Corollary 1. *Given an involution $\tau \in Sz(q)$, there are $q^4(q-1) - q^2(3(q-1)+1)$ elements σ such that $\langle \tau, \sigma \rangle$ is a Suzuki group.*

Proof. The group $Sz(q)$ has $(q^2+1)q^2(q-1)$ elements. Lemma 3 states that there are $q^2(4q-3)$ elements that, together with τ do not generate G. Hence, there are $(q^2+1)q^2(q-1) - q^2(4q-3)$ that do.

Lemma 4. *Up to conjugacy, there are $q^2(q-1) - (3(q-1)+1)$ pairs (τ, σ) of elements of $Sz(q)$ such that τ is an involution and $\langle \tau, \sigma \rangle$ is a Suzuki group.*

Proof. By Corollary 1, for a given τ, there are $q^4(q-1) - q^2(3(q-1)+1)$ elements σ such that $\langle \tau, \sigma \rangle$ is a Suzuki group. As the centralizer of τ in G is of order q^2 and does not stabilize any of the σ's, the number of pairs (τ, σ) that are pairwise nonconjugate in G is $\frac{q^4(q-1)-q^2(3(q-1)+1)}{q^2}$.

Some of these pairs of elements generate a sub-Suzuki group. Again, by Moebius inversion, we get the following Theorem.

Theorem 7. *Up to isomorphism, there are*

$$\frac{\sum_{n|2e+1} \mu(\frac{2e+1}{n})(2^{2n} + 2^n - 1)(2^n - 2)}{2e+1}$$

pairs (τ, σ) of elements of $Sz(q)$ such that τ is an involution and $\langle \tau, \sigma \rangle = Sz(q)$.

Proof. This is again a simple application of Moebius inversion.

Now, this result gives us either chiral polytopes or rotational subgroups of directly regular polytopes. However, Lemma 2 implies that all these polytopes are indeed chiral.

This construction may be applied to each abstract regular polytope of $Sz(q)$ to construct an abstract regular polytope of $Sz(q) \times 2$ whose rotational subgroup is $Sz(q)$. So, out of the

$$\frac{\sum_{n|2e+1} \mu(\frac{2e+1}{n})(2^{2n} + 2^n - 1)(2^n - 2)}{2e+1}$$

pairs that we get, there are

$$\frac{\sum_{n|2e+1} \mu(\frac{2e+1}{n})(2^n - 1)(2^n - 2)}{2e+1}$$

pairs that give the rotational subgroup of a regular polytope of $Sz(q) \times 2$. All the other ones give chiral polytopes. Subtracting the pairs that give rotational subgroups, we get the following result.

Theorem 8. *Up to isomorphism and duality, there are*

$$\frac{\sum_{n|2e+1} \mu(\frac{2e+1}{n})(2^{2n})(2^n - 2)}{2e+1}$$

chiral polyhedra whose automorphism group is the group $Sz(q)$.

We can now compare the numbers obtained in this section. It is clear that

$$\frac{\sum_{n|2e+1} \mu(\frac{2e+1}{n})(2^{2n})(2^n - 2)}{2e+1} > \frac{\sum_{n|2e+1} \mu(\frac{2e+1}{n})(2^n - 1)(2^n - 2)}{2e+1}$$

and therefore we may conclude that Suzuki groups are much more chiral than regular. Given a Suzuki group $Sz(q)$, let $f(q)$ and $g(q)$ be the number of regular and chiral polyhedra, respectively, that $Sz(q)$ has, up to isomorphism and duality. From the above discussion we observe that $g(q) = O(q \cdot f(q))$.

6 Almost Simple Groups of Suzuki Type and Chiral Polytopes

This section provides the first known results relating a family of almost simple groups and chiral polytopes by analysing almost simple groups of Suzuki type in relation with chiral polytopes.

Our aim is to construct chiral polytopes whose automorphism groups are (almost) simple groups of Suzuki type. To this end, we need to find generators $\sigma_1, \ldots, \sigma_{n-1}$ (for some appropriate n) of the group that satisfy the relations (3) and (4) as well as some intersection condition. In this way we would be constructing a chiral or directly regular polytope, whose rotational subgroup is the prescribed group. We start this section by showing that whenever we can find the required generators of a Suzuki group $Sz(q)$, the polytope constructed is always chiral.

Proposition 4. *A Suzuki group $Sz(q)$ (with $q = 2^{2e+1}$) is not the rotational subgroup of a directly regular polytope.*

Proof. We prove this proposition by contradiction. Suppose $Sz(q)$ is the rotational subgroup of a directly regular polytope $\mathscr{P}$ of rank n. Then, there exist $\sigma_1, \ldots \sigma_{n-1}$ that generate $Sz(q)$ and satisfy conditions (3) and (4), as well as the appropriate intersection condition. Furthermore, there exists an involutory group automorphism $\rho : Sz(q) \to Sz(q)$ such that $\rho(\sigma_1) = \sigma_1^{-1}$, $\rho(\sigma_2) = \sigma_1^2\sigma_2$, and $\rho(\sigma_i) = \sigma_i$ for $3 \le i \le n-1$. Hence, $\Gamma(\mathscr{P}) = \langle Sz(q), \rho\rangle \le Aut(Sz(q)) = Sz(q) : C_{2e+1}$, so that

$\Gamma(\mathscr{P}) = Sz(q) : C_k$, where $k > 2$ is an integer. We then get a contradiction as the automorphism group of a regular polytope can always be generated by involutions.

In the following two theorems we restrict the rank of a chiral polytope whose automorphism group is an almost simple group of Suzuki type.

Theorem 9. *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ where e is a strictly positive integer. Then, the rank of a chiral polytope having G as automorphism group is at most 4.*

Proof. Let $\mathscr{P}$ be a chiral polytope such that $\Gamma(\mathscr{P}) = Sz(q)$ and suppose the rank is $n \geq 5$. By Proposition 1, the k-faces or $\mathscr{P}$ must be directly regular, for every $k \leq n-2$. The fact that $n \geq 5$ implies in particular that all the 3-faces of $\mathscr{P}$ are directly regular.

Let $\mathscr{Q}$ be a 3-face of $\mathscr{P}$ and let $\rho_0, \rho_1, \rho_2 \in Sz(q)$ be the distinguished generators of $\Gamma(\mathscr{Q})$. Lemma 2 implies that $\langle \rho_0, \rho_1, \rho_2 \rangle$ is of Suzuki type. However this would imply that $\mathscr{Q}$ is a directly regular polytope whose group is a simple group, which is a contradiction to Proposition 2. Therefore $n \leq 4$.

Theorem 10. *Let $Sz(q) < G \leq Aut(Sz(q))$, with $q = 2^{2e+1}$ where e is a strictly positive integer. Then G is not the automorphism group of a regular polytope of rank ≥ 3 or a chiral polytope of rank ≥ 4.*

Proof. Since all the involutions of G are involutions of $Sz(q)$, the group G cannot be generated by involutions. In contrast, the automorphism group of regular polytopes as well as that of chiral polytopes of rank ≥ 4 can be always generated by involutions.

Observe that the regular part of this result was already proven in [20].

6.1 Rank 4 Chiral Polytopes from $Sz(q)$

Theorem 9 permits us to restrict ourselves to ranks 3 and 4. In this section we give restrictions to the possible Schläfli types that a chiral 4-polytope with an automorphism group of Suzuki type can have. We first show that no entry in the Schläfli type may be equal to 4.

Proposition 5. *Let $\mathscr{P}$ be a chiral 4-polytope of Schläfli type $\{p_1, p_2, p_3\}$ such that the automorphism group $\Gamma(\mathscr{P})$ is a Suzuki group $Sz(q)$. Then $p_i \neq 4$, for $i = 1, 2, 3$.*

Proof. By Lemma 1 and Proposition 3 it is immediate to see that $p_1, p_3 \neq 4$.

Suppose that $p_2 = 4$. That is, the order of σ_2 is 4. The group $C_4 := \langle \sigma_2 \rangle$ is then contained in $Sz(q') := \langle \sigma_1, \sigma_2 \rangle$ and $Sz(q'') := \langle \sigma_2, \sigma_3 \rangle$. Let us show that any subgroup C_4 lying in the intersection $Sz(q') \cap Sz(q'')$ is necessarily contained in a $C_4 \times C_2$ also lying in $Sz(q') \cap Sz(q'')$, contradicting the intersection condition.

There are subgroups $C_4 \times C_2$ in $Sz(q)$. Their normalizer $N_{Sz(q)}(C_4 \times C_2)$ is of order $4q$. The normalizer of a C_4 in $Sz(q)$ is, on the other hand, of order $2q$.

Suppose $q' \neq q''$ and count the triples $H \geq C_4 \leq J$ where H and J are isomorphic to $Sz(q')$ and $Sz(q'')$ respectively. There are $\frac{|Sz(q)|}{|Sz(q')|} \cdot \frac{|Sz(q')|}{|2q'|} \cdot \frac{q}{q'} = \frac{q \cdot |Sz(q)|}{2qq''}$ such triples. Now, count the triples $H \geq C_4 \times C_2 \leq J$ where H and J are isomorphic to $Sz(q')$ and $Sz(q'')$ respectively. There are $\frac{|Sz(q)|}{|Sz(q')|} \cdot \frac{|Sz(q')|}{|4q'|} \cdot \frac{q}{q'} = 1/2 \cdot \frac{q \cdot |Sz(q)|}{2qq''}$ such triples. But each $C_4 \times C_2$ contains exactly two C_4. Hence if $Sz(q') \cap Sz(q'') \geq C_4$, their intersection must contain a $C_4 \times C_2$ that contains the C_4. Suppose now that $q' = q''$ and count the triples $H \geq C_4 \leq J$ where H and J are isomorphic to $Sz(q')$. There are $\frac{|Sz(q)|}{|Sz(q')|} \cdot \frac{|Sz(q')|}{|2q'|} \cdot (\frac{q}{q'} - 1)$ such triples. Now, count the triples $H \geq C_4 \times C_2 \leq J$ where H and J are isomorphic to $Sz(q')$ and $H \neq J$. Again, there are $\frac{|Sz(q)|}{|Sz(q')|} \cdot \frac{|Sz(q')|}{|4q'|} \cdot (\frac{q}{q'} - 1)$ such triples. Hence if $Sz(q') \cap Sz(q'') \geq C_4$, their intersection must also contain a $C_4 \times C_2$ that contains the C_4.

Recall that $\alpha_q := q + \sqrt{2q} + 1$ and $\beta_q := q - \sqrt{2q} + 1$.

Proposition 6. *Let $G = Sz(q)$, with $q = 2^{2e+1}$ and $2e+1$ a prime number, be the automorphism group of a chiral 4-polytope $\mathscr{P}$. Let $\sigma_1, \sigma_2, \sigma_3$ be the distinguished generators of $\Gamma(\mathscr{P}) = G$ with respect to some base flag. Then, the order of σ_i cannot divide α_q nor β_q.*

Proof. We shall show this by analyzing the subgroup $H \leq G$ generated by σ_1 and σ_2. Note that:

$$(\sigma_1\sigma_2)^2 = \varepsilon; \tag{7}$$

$$H = \langle \sigma_1, \sigma_2 \rangle \neq G; \tag{8}$$

$$\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \{\varepsilon\},$$

and hence H is of even order.

Suppose that the order of σ_1, $o(\sigma_1)$, divides α_q. For (8) to be satisfied, $H \leq C_{\alpha_q} : C_4$ and hence the order of σ_2, $o(\sigma_2)$, must be either 4 or a divisor of α_q. Since α_q is odd, if $o(\sigma_2)$ divides α_q, then $H \leq C_{\alpha_q}$ and (7) is not satisfied. Therefore $o(\sigma_2) = 4$, a contradiction with Proposition 5. Hence $o(\sigma_1)$ cannot divide α_q. A similar argument can be used to show that $o(\sigma_1)$ cannot divide β_q.

Note that in the above argument we only used the structure of $C_{\alpha_q} : C_4 < Sz(q)$, as well as (7) and (8). Since the conditions (7) and (8) are symmetric with respect to σ_1 and σ_2, a similar argument as the one above shows that $o(\sigma_2)$ does not divide either α_q nor β_q.

Using the dual of $\mathscr{P}$ we can conclude that the order of σ_3 does not divide either α_q nor β_q.

Proposition 7. *Let $\mathscr{P}$ be a chiral 4-polytope, with Schläfli type $\{p_1, p_2, p_3\}$ and such that the automorphism group $\Gamma(\mathscr{P})$ is a Suzuki group $Sz(q)$. If p_1 and p_2 are divisors of $q - 1$ then $\langle \sigma_1, \sigma_2 \rangle$ is a Suzuki group $Sz(q')$ where $q'^a = q$ for some integer $a > 1$.*

Proof. By Proposition 5, all the p_i's have to be of odd order. Suppose that p_1 and p_2 divide $q-1$. Let $\sigma_1, \sigma_2, \sigma_3$ be the distinguished generators of $\Gamma(\mathscr{P})$ with respect to some base flag Φ. Then, for each $i = 1, 2$, the generator σ_i is an element of odd order that lies within a C_{q-1}; that is, $\langle\sigma_i\rangle \le C_{q-1} = Stab_{Sz(q)}(x_i, y_i)$, for some points x_i, y_i of the ovoid. Therefore $\langle\sigma_i\rangle \le Stab_{Sz(q)}(x_i) \cap Stab_{Sz(q)}(y_i) \cap Stab_{Sz(q)}\{x_i, y_i\} \le E_q\hat{\cdot}E_q : C_{q-1} \cap D_{2(q-1)}$.

Note that since $\mathscr{P}$ is of rank 4, the elements σ_1 and σ_2 do not generate the entire group $Sz(q)$; let $\langle\sigma_1, \sigma_2\rangle < Sz(q)$. Then $\langle\sigma_1, \sigma_2\rangle$ lies within a maximal subgroup H of $Sz(q)$. We now prove that H has to be isomorphic to some $Sz(q')$.

Suppose that $H \cong D_{2(q-1)}$. Then, since $\sigma_1\sigma_2$ is an involution, $\langle\sigma_1, \sigma_2\rangle = D_{2k}$, for some $k|q-1$. But since both σ_1 and σ_2 are of odd order, then $\sigma_1 \in C_k < D_{2k}$ and $\sigma_2 \in C_k < D_{2k}$, implying that $D_{2k} = \langle\sigma_1, \sigma_2\rangle \le C_k < D_{2k}$, which is a contradiction.

Suppose now that $\langle\sigma_1, \sigma_2\rangle \le E_q\hat{\cdot}E_q : C_{q-1}$.

Then σ_1 and σ_2 fix a point implying that without loss of generality we can assume that $x_1 = x_2$.

Now, since $\sigma_3^{\sigma_1\sigma_2} = \sigma_3^{-1}$,

$$\sigma_3^{-1}(x_1) = \sigma_1\sigma_2\sigma_3(x_1);$$
$$\sigma_3^{-2}(x_1) = \sigma_3^{-1}\sigma_1\sigma_2\sigma_3(x_1).$$

Since $\sigma_1\sigma_2$ is of order 2, then so is $\sigma_3^{-1}\sigma_1\sigma_2\sigma_3$, implying that $\sigma_3^{-4}(x_1) = x_1$. As σ_3 is of odd order, it necessarily fixes x_1 as well. Hence, p_3 must divide $q-1$ and $\langle\sigma_1, \sigma_2, \sigma_3\rangle \le E_q\hat{\cdot}E_q : C_{q-1}$, implying that $\Gamma(\mathscr{P}) \ne Sz(q)$, which is a contradiction. Therefore, $\langle\sigma_1, \sigma_2\rangle$ must be a group of Suzuki type.

Proposition 8. *Let $\mathscr{P}$ be a chiral 4-polytope, of Schläfli type $\{p_1, p_2, p_3\}$ and such that the automorphism group $\Gamma(\mathscr{P})$ is a Suzuki group $Sz(q)$ for some $q = 2^{2e+1}$. Then, $2e+1$ is not a prime.*

Proof. Suppose $2e+1$ is a prime. By Proposition 5 the p_i's are not 4, while by Proposition 6, they do not divide q^2+1. So they are all divisors of $q-1$. Suppose that p_1, p_2 and p_3 all divide $q-1$. Proposition 7 then shows that σ_1 and σ_2 generate a Suzuki group and as $2e+1$ is prime, we have $\langle\sigma_1, \sigma_2\rangle = Sz(q)$, a contradiction.

The following corollary is a direct consequence of Theorem 9 and Proposition 8.

Corollary 2. *Let $G \cong Sz(q)$ with $q = 2^{2e+1}$ and $2e+1$ prime. Then, the highest rank of a chiral polytope having G as automorphism group is 3.*

Lemma 5. *Let $\mathscr{P}$ be a chiral 4-polytope, of Schläfli type $\{p_1, p_2, p_3\}$ and such that the automorphism group $\Gamma(\mathscr{P})$ is a Suzuki group $Sz(q)$. If p_2 is a divisor of q^2+1, then $\langle\sigma_1, \sigma_2\rangle \cong \langle\sigma_2, \sigma_3\rangle \cong Sz(q')$ for some q' such that $q = q'^m$ with m an odd integer.*

Proof. Suppose p_2 is a divisor of q^2+1. Then it divides one of α_q or β_q, say γ_q. In order to satisfy the intersection condition, we must have $\langle\sigma_1, \sigma_2\rangle \cap \langle\sigma_2, \sigma_3\rangle = \langle\sigma_2\rangle$.

Also, we know that $H := \langle \sigma_1, \sigma_2 \rangle \cong Sz(q')$ and $J := \langle \sigma_2, \sigma_3 \rangle \cong Sz(q'')$ where q' and q'' may be equal. We now prove that if two non isomorphic sub-Suzuki groups both contain a given cyclic group C_d of order a divisor of $q^2 + 1$, they must have at least a subgroup of order $4d$ in common.

Suppose thus that $q' \neq q''$. We count the number of triples $H > C_d : 4 < J$ with $H \neq J$. There are $\frac{|Sz(q)|}{|Sz(q')|} \cdot \frac{|Sz(q')|}{4d} \cdot 1 = \frac{|Sz(q)|}{4d}$ such triples. Now, count the number of triples $H > C_d < J$ with $H \neq J$. There are $\frac{|Sz(q)|}{|Sz(q')|} \cdot \frac{|Sz(q')|}{4\gamma'_q} \cdot \frac{\gamma_q}{\gamma''_q} = \frac{|Sz(q)| \cdot \gamma_q}{4\gamma'_q \gamma''_q}$ such triples. But $d = gcd(\gamma'_q, \gamma''_q)$, for otherwise, there is a bigger cyclic group containing C_d and contained in $H \cap J$. That implies that $\gamma_q = d \cdot m_1 \cdot m_2$ and $\gamma'_q \gamma''_q = d^2 m_1 m_2$. Hence $\frac{\gamma_q}{\gamma'_q \gamma''_q} = \frac{1}{d}$. Therefore the number of triples $H > C_d : 4 < J$ is equal to the number of triples $H > C_d < J$ and hence $H \cap J \geq C_d : 4$. This is in contradiction with the intersection condition.

Lemma 6. *Let $\mathscr{P}$ be a chiral 4-polytope, of Schläfli type $\{p_1, p_2, p_3\}$ and such that the automorphism group $\Gamma(\mathscr{P})$ is a Suzuki group $Sz(q)$. If p_2 is a divisor of $q - 1$, then $\langle \sigma_1, \sigma_2 \rangle \cong \langle \sigma_2, \sigma_3 \rangle \cong Sz(q')$ for some q' such that $q = q'^m$ with m an odd integer.*

Proof. Let $p_2 \mid q - 1$. As in the previous Lemma, $H := \langle \sigma_1, \sigma_2 \rangle \cong Sz(q')$ and $J := \langle \sigma_2, \sigma_3 \rangle \cong Sz(q'')$ where q' and q'' may be equal. Suppose that $q' \neq q''$. We count the number of triples $H > C_d : 2 < J$ with $H \neq J$. There are $\frac{|Sz(q)|}{|Sz(q')|} \cdot \frac{|Sz(q')|}{2d} \cdot 1 = \frac{|Sz(q)|}{2d}$ such triples. Now, count the number of triples $H > C_d < J$ with $H \neq J$. There are $\frac{|Sz(q)|}{|Sz(q')|} \cdot \frac{|Sz(q')|}{2(q'-1)} \cdot \frac{q-1}{q''-1} = \frac{|Sz(q)| \cdot (q-1)}{2(q'-1)(q''-1)}$ such triples. But $d = gcd(q'-1, q''-1)$, for otherwise, there is a bigger cyclic group containing C_d and contained in $H \cap J$. That implies that $q - 1 = d \cdot m_1 \cdot m_2$ and $(q'-1)(q''-1) = d^2 m_1 m_2$. Hence $\frac{(q-1)}{(q'-1)(q''-1)} = \frac{1}{d}$. Therefore the number of triples $H > C_d : 2 < J$ is equal to the number of triples $H > C_d < J$ and hence $H \cap J \geq C_d : 2$. This is in contradiction with the intersection condition.

Corollary 3. *Let $\mathscr{P}$ be a chiral 4-polytope, of Schläfli type $\{p_1, p_2, p_3\}$ and such that the automorphism group $\Gamma(\mathscr{P})$ is a Suzuki group $Sz(q)$. Then $\langle \sigma_1, \sigma_2 \rangle \cong \langle \sigma_2, \sigma_3 \rangle \cong Sz(q')$ for some q' such that $q = q'^m$ with m an odd integer.*

Proof. This is just a combination of Proposition 5, Lemmas 5 and 6.

By the above corollary, if rank 4 polytopes exist for some Suzuki group $Sz(q)$, it is sufficient for it to have one conjugacy class of sub-Suzuki groups as $\langle \sigma_1 \sigma_2 \rangle$ and $\langle \sigma_2, \sigma_3 \rangle$ must be isomorphic sub-Suzuki groups. Hence, the smallest Suzuki group that has sub-Suzuki groups, that is $Sz(2^9)$, is a good case to consider when trying to construct a rank 4 chiral polytope. Computations with $Sz(2^9)$ where we tried to extend each chiral polyhedron of $Sz(2^3)$ to a chiral polytope of rank 4 of $Sz(2^9)$ showed us that this was impossible. Indeed, when σ_1 and σ_2 are chosen in $Sz(2^9)$ in such a way that $\sigma_1 \sigma_2$ is an involution and these two elements generate a sub-Suzuki group, it is not possible to find a σ_3 that generates a sub-Suzuki group with σ_2 in $Sz(2^9)$ and such that both $\sigma_1 \sigma_2 \sigma_3$ and $\sigma_2 \sigma_3$ are involutions. Indeed, adding a third generator σ_3 satisfying $(\sigma_1\sigma_2\sigma_3)^2 = (\sigma_2\sigma_3)^2 = \epsilon$ always ended up having $\langle\sigma_2, \sigma_3\rangle = Sz(2^9)$. We have a strong feeling that this will be the case in general. This leads us to conclude this paper with the following conjecture.

Conjecture 1. Let $Sz(q) \leq G \leq Aut(Sz(q))$ with $q = 2^{2e+1}$ where e is a strictly positive integer. Then, the rank of any chiral polytope having G as automorphism group is 3.

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Globally Linked Pairs of Vertices in Rigid Frameworks

Bill Jackson, Tibor Jordán, and Zoltán Szabadka

Abstract A 2-dimensional *framework* (G, p) is a graph $G = (V, E)$ together with a map $p : V \to \mathbb{R}^2$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^2$. Two realizations of G are *equivalent* if the corresponding edges in the two frameworks have the same length. A pair of vertices $\{u, v\}$ is *globally linked* in G if the distance between the points corresponding to u and v is the same in all pairs of equivalent generic realizations of G.

In this paper we extend our previous results on globally linked pairs and complete the characterization of globally linked pairs in minimally rigid graphs. We also show that the Henneberg 1-extension operation on a non-redundant edge preserves the property of being not globally linked, which can be used to identify globally linked pairs in broader families of graphs. We prove that if (G, p) is generic then the set of globally linked pairs does not change if we perturb the coordinates slightly. Finally, we investigate when we can choose a non-redundant edge e of G and then continuously deform a generic realization of $G - e$ to obtain equivalent generic realizations of G in which the distances between a given pair of vertices are different.

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1 Introduction

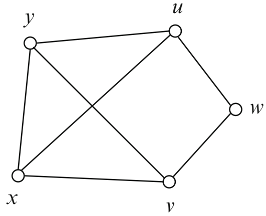
We shall consider finite graphs without loops, multiple edges or isolated vertices. A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line realization of G in $\mathbb{R}^d$. Two frameworks (G, p) and (G, q) are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$. This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of $\mathbb{R}^d$.

We say that (G, p) is *globally rigid* if every framework which is equivalent to (G, p) is congruent to (G, p). The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p). Intuitively, this means that if we think of a d-dimensional framework (G, p) as a collection of bars and joints where points correspond to joints and each edge to a rigid bar joining its end-points, then the framework is rigid if it has no non-trivial continuous deformations (see [9], [28, Section 3.2]). It seems to be a hard problem to decide if a given framework is rigid or globally rigid. Indeed Saxe [22] showed that it is NP-hard to decide if even a 1-dimensional framework is globally rigid and Abbot [1] showed that the rigidity problem is NP-hard for 2-dimensional frameworks. These problems become more tractable, however, if we consider *generic frameworks* i.e. frameworks in which there are no algebraic dependencies between the coordinates of the vertices.

It is known, see [28], that the rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic. We say that the graph G is *rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. The problem of characterizing when a graph is rigid in $\mathbb{R}^d$ has been solved for $d = 1, 2$ (and is a major open problem for $d \geq 3$). See Sect. 2 for more details.

A similar situation holds for global rigidity: the problem of characterizing when a generic framework is globally rigid in $\mathbb{R}^d$ has also been solved for $d = 1, 2$. A 1-dimensional generic framework (G, p) is globally rigid if and only if either G is the complete graph on two vertices or G is 2-connected. The characterization for $d = 2$ is as follows. We say that G is *redundantly rigid* in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for all edges e of G.

Fig. 1 A realization (G, p) of a rigid graph G in $\mathbb{R}^2$. The pair $\{u, v\}$ is globally linked in (G, p)



Theorem 1 ([15]). *Let (G, p) be a 2-dimensional generic framework. Then (G, p) is globally rigid if and only if either G is a complete graph on two or three vertices, or G is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

It follows that the global rigidity of d-dimensional frameworks is a generic property when $d = 1, 2$. Gortler, Healy and Thurston [10] proved that this holds for all $d \geq 1$. We say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. As for rigidity, it is an important open problem to characterize globally rigid graphs when $d \geq 3$. Hendrickson [12] showed that redundant rigidity and $(d + 1)$-connectivity are necessary conditions for all $d \geq 1$ (provided G has at least $d + 2$ vertices) but there are examples showing that these conditions are not sufficient when $d \geq 3$, see [5, 7].

We refer the reader to [11, 16, 28] for a detailed survey of rigid and globally rigid d-dimensional frameworks and their applications.

We will consider properties of 2-dimensional generic frameworks which are weaker than global rigidity. We assume henceforth that $d = 2$, unless specified otherwise. A pair of vertices $\{u, v\}$ in a framework (G, p) is *globally linked* in (G, p) if, in all equivalent frameworks (G, q), we have $||p(u)-p(v)|| = ||q(u)-q(v)||$. The pair $\{u, v\}$ is *globally linked* in G if it is globally linked in all generic frameworks (G, p). Thus G is globally rigid if and only if all pairs of vertices of G are globally linked. Unlike global rigidity, however, 'global linkedness' is not a generic property in $\mathbb{R}^2$. Figures 1 and 2 give an example of a pair of vertices in a rigid graph G which is globally linked in one generic realization, but not in another.

We initiated the study of globally linked pairs in [18]. We next summarize the main results and conjectures from this paper.

The Henneberg *1-extension* operation [13] (on edge xy and vertex w) deletes an edge xy from a graph H and adds a new vertex z and new edges zx, zy, zw for some vertex $w \in V(H) - \{x, y\}$. We showed that the 1-extension operation preserves the property that a pair of vertices is globally linked *as long as $H - xy$ is rigid*.

Theorem 2 ([18]). *Let G, H be graphs such that G is obtained from H by a 1-extension on edge xy and vertex w. Suppose that $H - xy$ is rigid and that $\{u, v\}$ is globally linked in H. Then $\{u, v\}$ is globally linked in G.*

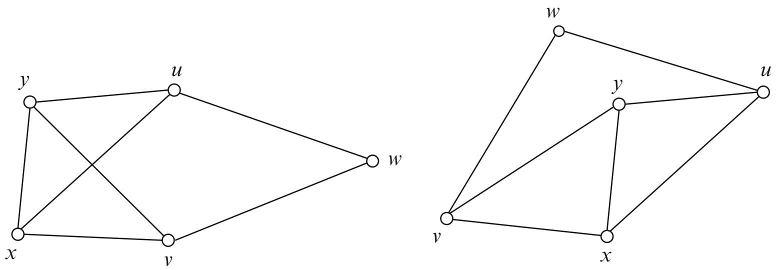


Fig. 2 Two equivalent realizations of the rigid graph G of Fig. 1, which show that the pair $\{u, v\}$ is not globally linked in G in $\mathbb{R}^2$

Let $H = (V, E)$ be a graph and $x, y \in V$. We use $\kappa_H(x, y)$ to denote the maximum number of pairwise openly disjoint xy-paths in H. Note that if $xy \notin E$ then, by Menger's theorem, $\kappa_H(x, y)$ is equal to the size of a smallest set $S \subseteq V(H) - \{x, y\}$ for which there is no xy-path in $H - S$.

Lemma 1 ([18]). *Let (G, p) be a generic framework, $x, y \in V(G)$, $xy \notin E(G)$, and suppose that $\kappa_G(x, y) \leq 2$. Then $\{x, y\}$ is not globally linked in (G, p).*

We used Theorem 2 and Lemma 1 to characterize globally linked pairs for the family of M-connected graphs, i.e graphs whose 2-dimensional rigidity matroid is connected (see Sect. 2 for formal definitions). This family lies strictly between the families of globally rigid graphs and redundantly rigid graphs.

Theorem 3 ([18]). *Let $G = (V, E)$ be an M-connected graph and $x, y \in V$. Then $\{x, y\}$ is globally linked in G if and only if $\kappa_G(x, y) \geq 3$.*

An *M-component* of a graph G is a maximal M-connected subgraph of G. Theorem 3 has the following immediate corollary.

Corollary 1 ([18]). *Let $G = (V, E)$ be a graph and $x, y \in V$. If either $xy \in E$, or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$, then $\{x, y\}$ is globally linked in G.*

We conjectured that the converse is also true.

Conjecture 2 ([18]). The pair $\{x, y\}$ is globally linked in a graph $G = (V, E)$ if and only if either $xy \in E$ or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$.

A *redundantly rigid component* of a graph G is a maximal redundantly rigid subgraph of G (see Sect. 2). We showed in [18] that Conjecture 2 is equivalent to the following pair of conjectures concerning the redundantly rigid components of G.

Conjecture 3 ([18]). Suppose that $\{x, y\}$ is a globally linked pair in a graph G. Then there is a redundantly rigid component R of G with $\{x, y\} \subseteq V(R)$.

Conjecture 4 ([18]). Let G be a graph. Suppose that there is a redundantly rigid component R of G with $\{x, y\} \subseteq V(R)$ and $\{x, y\}$ is globally linked in G. Then $\{x, y\}$ is globally linked in R.

The Henneberg 0-*extension* operation on vertices x, y in a graph H adds a new vertex z and new edges xz, yz to H. We showed that the 0-extension operation preserves the property that a pair of vertices is not globally linked.

Lemma 2 ([18]). *If $\{u, v\}$ is not globally linked in H and G is a 0-extension of H then $\{u, v\}$ is not globally linked in G.*

The purpose of this paper is to extend the results of [18] in several directions. In Sect. 3 we prove that the 1-extension operation preserves the property that a pair of vertices is not globally linked whenever it is applied to a non-redundant edge in an arbitrary rigid graph. We use this to deduce that Conjecture 2 holds for minimally rigid graphs. (Since the M-components of a minimally rigid graph are all isomorphic to K_2 this is equivalent to showing that a pair of vertices in a minimally rigid graph is globally linked if and only if they are adjacent.) We consider frameworks with the property that all equivalent frameworks are infinitesimally rigid in Sect. 4. We show that for such a framework (G, p), the number of equivalent pairwise non-congruent frameworks does not increase if we make small perturbations to the positions of its vertices. This extends a result of Connelly and Whiteley [6] on infinitesimally rigid globally rigid frameworks. We deduce that if p is generic, then the set of globally linked pairs in (G, p) does not change if we make small perturbations to the positions of its vertices. In Sect. 5 we investigate when we can choose a non-redundant edge e in a graph G and then continuously deform a generic realization of $G - e$ to obtain equivalent generic realizations of G in which the distances between a given pair of vertices are different.

2 Preliminaries

In this section we summarize the definitions and results from rigidity theory that we shall use later.

2.1 The Rigidity Matroid

The rigidity matroid of a graph G is a matroid defined on the set of edges of G which reflects the rigidity properties of all generic realizations of G. We will need basic definitions and results on this matroid to define M-connected graphs. The reader is referred to [21] for basic definitions and results of matroid theory.

Let (G, p) be a realization of a graph $G = (V, E)$. The *rigidity matrix* of the framework (G, p) is the matrix $R(G, p)$ of size $|E| \times 2|V|$, where, for each edge $v_i v_j \in E$, in the row corresponding to $v_i v_j$, the entries in the two columns corresponding to vertices v_i and v_j contain the two coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros. See [28] for more details. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by linear independence of rows of the rigidity matrix. Any two generic frameworks (G, p) and (G, q) have the same rigidity matroid. We call this the *rigidity matroid* $\mathscr{R}(G) = (E, r)$ of the graph G. We denote the rank of $\mathscr{R}(G)$ by $r(G)$. Gluck characterized rigid graphs in terms of their rank.

Theorem 4 ([9]). *Let $G = (V, E)$ be a graph. Then G is rigid if and only if $r(G) = 2|V| - 3$.*

We say that a graph $G = (V, E)$ is *M-independent* if E is independent in $\mathscr{R}(G)$. Knowing when subgraphs of G are M-independent allows us to determine the rank of G. This can be accomplished using the following characterization of M-independent graphs due to Laman. For $X \subseteq V$, let $E_G(X)$ denote the set, and $i_G(X)$ the number, of edges in G joining vertices in X.

Theorem 5 ([19]). *A graph $G = (V, E)$ is M-independent if and only if $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$.*

A graph $G = (V, E)$ is *minimally rigid* if G is rigid, but $G - e$ is not rigid for all $e \in E$. Theorems 4 and 5 imply that G is minimally rigid if and only if G is M-independent and $|E| = 2|V| - 3$. Note that, if G is rigid, then the edge sets of the minimally rigid spanning subgraphs of G form the bases in the rigidity matroid of G.

A pair of vertices $\{u, v\}$ in a framework (G, p) is *linked* in (G, p) if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(w) - q(w)|| < \epsilon$ for all $w \in V$, then we have $||p(u) - p(v)|| = ||q(u) - q(v)||$. Using Theorems 4 and 5, it can be seen that this is a generic property and that $\{u, v\}$ is linked in a generic framework (G, p) if and only if G has a rigid subgraph H with $\{u, v\} \subseteq V(H)$.

A compact characterization of all linked pairs can be deduced as follows. We define a *rigid component* of G to be a maximal rigid subgraph of G. It is well-known (see e.g. [15, Corollary 2.14]), that any two rigid components of G intersect in at most one vertex and hence that the edge sets of the rigid components of G partition the edge set of G. Thus $\{u, v\}$ is linked in a generic framework (G, p) if and only if $\{u, v\} \subseteq V(H)$ for some rigid component H of G.

Recall the definitions of the 0- and 1-extension operations from Sect. 1. The basic result about 0-extensions is the following.

Lemma 3 ([27]). *Let H be a graph and let G be obtained from H by a 0-extension. Then G is minimally rigid if and only if H is minimally rigid.*

It is known that the 1-extension operation preserves rigidity [27]. We shall need the following lemma about the inverse operation of 1-extension on minimally rigid graphs.

Lemma 4 ([27]). *Let $G = (V, E)$ be a minimally rigid graph and let $v \in V$ be a vertex with $d(v) = 3$. Then v has two non-adjacent neighbours u, w such that the graph $H = G - v + uw$ is minimally rigid.*

By observing that a minimally rigid graph on at least three vertices has a vertex of degree two or three, it follows that a graph is minimally rigid if and only if it can be constructed from an edge by a sequence of 0-extensions and 1-extensions.

2.2 *M*-Connected Graphs and *M*-Components

Given a graph $G = (V, E)$, a subgraph $H = (W, C)$ is said to be an *M-circuit* in G if C is a circuit (i.e. a minimal dependent set) in $\mathscr{R}(G)$. In particular, G is an *M-circuit* if E is a circuit in $\mathscr{R}(G)$. Using Theorem 5 we may deduce that G is an M-circuit if and only if $|E| = 2|V| - 2$ and $G - e$ is minimally rigid for all $e \in E$. Recall that a graph G is *redundantly rigid* if $G - e$ is rigid for all $e \in E$. Note also that a graph G is redundantly rigid if and only if G is rigid and each edge of G belongs to a circuit in $\mathscr{R}(G)$ i.e. an M-circuit of G.

Any two maximal redundantly rigid subgraphs of a graph $G = (V, E)$ can have at most one vertex in common, and hence are edge-disjoint (see [15]). Defining a *redundantly rigid component* of G to be either a maximal redundantly rigid subgraph of G, or a subgraph induced by an edge which belongs to no M-circuit of G, we deduce that the edge sets of the redundantly rigid components of G partition E. Since each redundantly rigid component is rigid, this partition is a refinement of the partition of E given by the rigid components of G. Note that the redundantly rigid components of G are induced subgraphs of G.

Given a matroid $\mathscr{M} = (E, \mathscr{I})$, we define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathscr{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathscr{M}$. If $\mathscr{M}$ has at least two elements and only one component then $\mathscr{M}$ is said to be *connected*.

We say that a graph $G = (V, E)$ is *M-connected* if $\mathscr{R}(G)$ is connected. Thus M-circuits are examples of M-connected graphs. Another example is the complete bipartite graph $K_{3,m}$, which is M-connected for all $m \geq 4$. (This follows because $K_{3,4}$ is an M-circuit and any pair of edges of $K_{3,m}$ are contained in a copy of $K_{3,4}$.) The *M-components* of G are the subgraphs of G induced by the components of $\mathscr{R}(G)$. Since each M-component with at least two edges is redundantly rigid, the partition of E given by the M-components is a refinement of the partition given by the redundantly rigid components of G. Note that the M-components of G are induced subgraphs. For more examples and basic properties of M-circuits and M-connected graphs see [3, 15].

Note that the rigid components, redundantly rigid components, and M-components of a graph can all be determined in polynomial time, see for example [4].

2.3 Rigidity, Infinitesimal Rigidity, and Flexes

In this subsection we consider d-dimensional frameworks for arbitrary $d \geq 1$. Let (G, p) be a d-dimensional framework. A *flexing* of the framework (G, p) is a continuous function $\pi : (-1, 1) \times V \to \mathbb{R}^d$ such that $\pi_0 = p$, and such that the frameworks (G, p) and (G, π_t) are equivalent for all $t \in (-1, 1)$, where $\pi_t : V \to \mathbb{R}^d$ is defined by $\pi_t(v) = \pi(t, v)$ for all $v \in V$. The flexing π is *trivial* if the frameworks (G, p) and (G, π_t) are congruent for all $t \in (-1, 1)$. A framework is said to be *flexible* if it has a non-trivial flexing. It is known [2, 9] that non-rigidity, flexibility and the existence of a non-trivial smooth flexing are all equivalent.

The first-order version of a flexing of the framework (G, p) is called an *infinitesimal motion*. This is an assignment of *infinitesimal velocities* to the vertices, $\tilde{p} : V \to \mathbb{R}^d$ satisfying

$$(p(u) - p(v))(\tilde{p}(u) - \tilde{p}(v)) = 0 \text{ for all pairs } u, v \text{ with } uv \in E. \tag{1}$$

If π is a smooth flexing of (G, p), then $\dot{\pi}_0 := \frac{d\pi}{dt}|_{t=0}$ is an infinitesimal motion of (G, p). A *trivial infinitesimal motion* of (G, p) has the form $\tilde{p}(v) = Ap(v) + b$, for all $v \in V$, for some $d \times d$ antisymmetric matrix A and some $b \in \mathbb{R}^d$. It is easy to see that these are indeed infinitesimal motions. A framework (G, p) is *infinitesimally flexible* if it has a non-trivial infinitesimal motion, otherwise it is *infinitesimally rigid*.

The set of infinitesimal motions of a framework (G, p) is a linear subspace of $\mathbb{R}^{d|V|}$, given by the system of $|E|$ linear equations (1). The matrix of this system of linear equations is the *rigidity matrix* $R(G, p)$ of (G, p) defined earlier. The *rigidity map* for a graph $G = (V, E)$ is the map $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$, given by

$$f_G(p) = (\ldots, ||p(u) - p(v)||^2, \ldots).$$

Note that the Jacobian of f_G at some point $p \in \mathbb{R}^{d|V|}$ is given by $2R(G, p)$.

Gluck [9] proved that if a framework (G, p) is infinitesimally rigid, then it is rigid. The converse of this is not true in general, but if we exclude certain 'degenerate' configurations, then rigidity and infinitesimal rigidity are equivalent. In order to establish this, let us recall some notions from differential topology. Given two smooth manifolds, M and N and a smooth map $f : M \to N$, we denote the derivative of f at some point $p \in M$ by $df|_p$, which is a linear map from T_pM, the tangent space of M at p, to $T_{f(p)}N$. Let k be the maximum rank of $df|_q$ over all $q \in M$. A point $p \in M$ is said to be a *regular point* of f, if rank $df|_p = k$, and a *critical point*, if rank $df|_p < k$. We say that a framework (G, p) is *regular*, if p is a regular point of f_G. Using the inverse function theorem, it can be shown (see for example [2, Proposition 2]) that if (G, p) is a regular framework, then there is a neighbourhood U_p of p, such that $f_G^{-1}(f_G(p)) \cap U_p$ is a manifold, whose tangent space at p is the kernel of df_p. This has the following corollary.

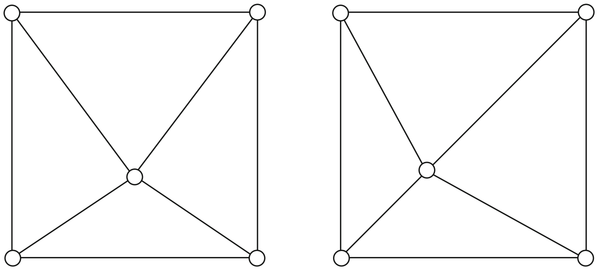


Fig. 3 Two regular realizations of a graph G. The first one is globally rigid, but the second is not, since it can fold around one of the diagonals

Theorem 6 ([2]). *Let (G, p) be a regular framework. If (G, p) is infinitesimally flexible, then it is flexible. Furthermore, if $\tilde{p}$ is a non-trivial infinitesimal motion of (G, p), then there is a non-trivial smooth flexing π of (G, p) such that $\dot{\pi}_0 = \tilde{p}$.*

Since the rank of the rigidity matrix for a given graph G is constant on the set of regular points of f_G and infinitesimal rigidity of a framework (G, p) depends only on the rank of $R(G, p)$ it follows that if a regular framework (G, p) is infinitesimally rigid, then all other regular frameworks (G, q) are infinitesimally rigid as well.

3 Extensions and Globally Linked Pairs

In this section we prove that the 1-extension operation preserves the property that a pair of vertices is not globally linked assuming that it is performed on a non-redundant edge. By using this result we shall complete the characterization of globally linked pairs in minimally rigid graphs.

Given a field $K \subseteq \mathbb{C}$ we use $\overline{K}$ to denote the algebraic closure of K. We say that a point $P = (x, y) \in \mathbb{C}^2$ is *generic over* K, if the set $\{x, y\}$ is algebraically independent over K. To prove the framework extension result of this section, we need a lemma concerning polynomials whose zeros are algebraically dependent over K. For a polynomial $f \in K[X_1, X_2]$, we denote the set of zeros of f in K^2 by $V(f, K)$. We will use the following facts concerning two polynomials $f, g \in K[X_1, X_2]$.

Fact 1: if g is irreducible over an algebraically closed subfield of $\mathbb{C}$ then g is irreducible over $\mathbb{C}$, see [14, page 76, Corollary to Theorem IV].

Fact 2: if $V(f, K) \cap V(g, K)$ is infinite then f and g have a non-trivial common factor in $K[X_1, X_2]$, see [8, Chapter 1, Proposition 2].

Lemma 5. *Let L be an algebraically closed subfield of $\mathbb{C}$ and $K = L \cap \mathbb{R}$. Suppose that $g \in K[X_1, X_2]$ is irreducible over K. Then g is irreducible over $\mathbb{R}$.*

Proof. Let $g = g_1 g_2 \dots g_m$ be the factorization of g into irreducible factors over L. Then $g = g_1 g_2 \dots g_m$ is also the factorization of g into irreducible factors over $\mathbb{C}$ by Fact 1. Now suppose that $g = h_1 h_2$ is a non-trivial factorization of g over $\mathbb{R}$. Relabeling if necessary and using the fact that $\mathbb{C}[X_1, X_2]$ is a unique factorization domain we have $h_1 = g_1 g_2 \dots g_s$ and $h_2 = g_{s+1} g_{s+2} \dots g_m$ for some $1 \le s \le m-1$. This implies that $h_1, h_2 \in L[X_1, X_2]$. Since we also have $h_1, h_2 \in \mathbb{R}[X_1, X_2]$ we get $h_1, h_2 \in K[X_1, X_2]$. This contradicts the irreducibility of g over K.

Lemma 6. *Let L be a countable algebraically closed subfield of $\mathbb{C}$, $K = L \cap \mathbb{R}$, and $f \in \mathbb{R}[X_1, X_2]$ be irreducible over $\mathbb{R}$. Suppose that $V(f, \mathbb{R})$ is uncountable and each $(x_1, x_2) \in V(f, \mathbb{R})$ is algebraically dependent over K. Then there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\lambda f \in K[X_1, X_2]$.*

Proof. Since each $(x_1, x_2) \in V(f, \mathbb{R})$ is algebraically dependent over K, each $(x_1, x_2) \in V(f, \mathbb{R})$ is a root of an irreducible polynomial in $K[X_1, X_2]$. Since $K[X_1, X_2]$ is countable and $V(f, \mathbb{R})$ is uncountable there exists an irreducible polynomial $g \in K[X_1, X_2]$ such that $V(f, \mathbb{R}) \cap V(g, \mathbb{R})$ is uncountable. Since L is algebraically closed, Lemma 5 implies that g is irreducible over $\mathbb{R}$. Since $V(f, \mathbb{R}) \cap V(g, \mathbb{R})$ is infinite, Fact 2 implies that f and g have a non-trivial common factor in $\mathbb{R}[X_1, X_2]$. Since they are both irreducible over $\mathbb{R}$, we have $f = \lambda g$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.

A framework (G, p) is *quasi-generic* if it is congruent to a generic framework. It is in *standard position with respect to two vertices* (v_1, v_2) if $p(v_1)$ lies at the origin and $p(v_2)$ lies on the second coordinate axis. We may use a translation and a rotation to transform every framework to a congruent framework which is in standard position with respect to any two given vertices. The next result determines what happens when we apply such a transformation to a quasi-generic framework.

Lemma 7 ([18, Lemma 3.5]). *Let (G, p) be a realisation of a graph $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_n\}$. Let (G, q) be a congruent realisation which is in standard position with respect to (v_1, v_2). Suppose $q(v_i) = (x_i, y_i)$ for $1 \le i \le n$ (so $x_1 = y_1 = x_2 = 0$). Then (G, p) is quasi-generic if and only if $\{y_2, x_3, y_3, \dots, x_n, y_n\}$ is algebraically independent over $\mathbb{Q}$.*

The following rather technical lemma is fundamental to our proof that constructing a 1-extension by deleting a non-redundant edge from a graph H preserves the property that two given vertices of H are not globally linked.

Lemma 8. *Let $G = (V, E)$ be a graph and v be a vertex of G of degree three with neighbour set $\{u, w, z\}$. Let $(G-v, p)$ and $(G-v, q)$ be equivalent frameworks which are in standard position with respect to (u, w). Suppose that p is quasi-generic, $q(u)$, $q(w)$ and $q(z)$ are not collinear, and $||q(u) - q(w)||^2 \notin \overline{\mathbb{Q}(p)}$. Then there are equivalent frameworks (G, p^*) and (G, q^*) where p^* is quasi-generic, $p^*|_{V-v} = p$ and $q^*|_{V-v} = q$.*

Proof. Let $L = \overline{\mathbb{Q}(p)}$ and $K = L \cap \mathbb{R}$. We have $p(u) = (0, 0)$, $p(w) = (0, p_3)$, $p(z) = (p_4, p_5)$, $q(u) = (0, 0)$, $q(w) = (0, q_3)$ and $q(z) = (q_4, q_5)$. Since $q(u)$,

$q(w)$ and $q(z)$ are not collinear, $q_3 \neq 0 \neq q_4$. Moreover $q_3^2 = ||q(u)-q(w)||^2 \notin K$. By reflecting the configuration q on the second coordinate axis, if necessary, we may assume that $q_4 \neq p_4$. By Lemma 7, $\{p_3, p_4, p_5\}$ is algebraically independent over $\mathbb{Q}$.

We call a point $(p_1, p_2) \in \mathbb{R}^2$ *feasible*, if there exists a point $(q_1, q_2) \in \mathbb{R}^2$, such that the extended frameworks (G, p^*) and (G, q^*) are equivalent, where $p^*|_{V-v} = p$, $q^*|_{V-v} = q$, $p^*(v) = (p_1, p_2)$ and $q^*(v) = (q_1, q_2)$. We will prove the lemma by finding a feasible point that is generic over K and then applying Lemma 7.

The set of feasible points can be described by the following system of equations:

$$q_1^2 + q_2^2 = p_1^2 + p_2^2 \tag{2}$$

$$q_1^2 + (q_2 - q_3)^2 = p_1^2 + (p_2 - p_3)^2 \tag{3}$$

$$(q_1 - q_4)^2 + (q_2 - q_5)^2 = (p_1 - p_4)^2 + (p_2 - p_5)^2 . \tag{4}$$

Equations (2) and (3) give

$$q_2 = \frac{q_3^2 - p_3^2 + 2p_2p_3}{2q_3}. \tag{5}$$

Equations (2), (4) and (5) now give

$$q_1 = \frac{q_4^2 + q_5^2 - p_4^2 - p_5^2 + 2p_1p_4 + 2p_2p_5 - q_5(\frac{q_3^2 - p_3^2 + 2p_2p_3}{q_3})}{2q_4} . \tag{6}$$

We can use (5) and (6) to eliminate q_2, q_1 from (2) and obtain

$$4q_3^2q_4^2(q_1^2+q_2^2-p_1^2-p_2^2) = a_{11}p_1^2+a_{22}p_2^2+a_{12}p_1p_2+a_1p_1+a_2p_2+a_0 = 0 \tag{7}$$

where

$$\begin{aligned}
a_{11} &= 4q_3^2(p_4^2 - q_4^2) \\
a_{22} &= 4q_4^2(p_3^2 - q_3^2) + 4(q_3p_5 - p_3q_5)^2 \\
a_{12} &= 8p_4q_3(q_3p_5 - p_3q_5) \\
a_1 &= 4p_4q_3(q_3(r + s) - q_5(q_3^2 - p_3^2)) \\
a_2 &= 4(q_3p_5 - p_3q_5)(q_3(r + s) - q_5(q_3^2 - p_3^2)) + 4p_3q_4^2(q_3^2 - p_3^2) \\
a_0 &= (q_3(r + s) - q_5(q_3^2 - p_3^2))^2 + q_4^2(q_3^2 - p_3^2)^2
\end{aligned}$$

taking $r = q_5^2 - p_5^2$ and $s = q_4^2 - p_4^2$. Note that any real solution (p_1, p_2) to (7) gives a real solution (q_1, q_2) to (5) and (6). Thus the set of feasible points is the set of points lying on the conic $f = 0$ where

$$f = a_{11}X_1^2 + a_{22}X_2^2 + a_{12}X_1X_2 + a_1X_1 + a_2X_2 + a_0 \in \mathbb{R}[X_1, X_2].$$

Note that since $q_3^2 \notin K$ we have $q_3^2 \neq p_3^2$, and since $q_4 \neq 0$, this gives $a_0 > 0$.

Claim 1. The conic $f = 0$ is not empty and is not a single point.

Proof. Since $q_4 \neq p_4$, the line segments $[p(w), q(w)]$ and $[p(z), q(z)]$ are not parallel. It is easy to see that the point of intersection of the perpendicular bisectors of these line segments is feasible. Thus the point

$$A = \left(\frac{r + s + (p_3 + q_3)(p_5 - q_5)}{2(q_4 - p_4)}, \frac{p_3 + q_3}{2} \right)$$

lies on $f = 0$. We may reflect (G, q) in the first coordinate axis and then apply the same construction to deduce that, if $q_4 \neq -p_4$, then $f = 0$ also contains the point

$$B = \left(\frac{r + s + (p_3 - q_3)(p_5 + q_5)}{-2(q_4 + p_4)}, \frac{p_3 - q_3}{2} \right),$$

and $A \neq B$ since $q_3 \neq 0$. Hence we may suppose that $q_4 = -p_4$. In this case $a_{11} = 0$. Thus $f \neq (b_1X_1 - b_2)^2 + (b_3X_2 - b_4)^2$ for all $b_1, b_2, b_3, b_4 \in \mathbb{R}$ with $b_1 \neq 0 \neq b_3$ so the conic cannot be a single point.

Let us suppose indirectly, that:

$$\text{no point on the conic } f = 0 \text{ is generic over } K. \tag{8}$$

Since this conic is not empty, and is not a single point, it follows from the classification of conics that it is either an ellipse, a parabola, a hyperbola or the union of two lines. Applying Lemma 6 to the irreducible components of f we deduce that there is a $\lambda \in \mathbb{R} \setminus \{0\}$, such that $\lambda f \in K[X_1, X_2]$. Hence

$$\{\lambda a_{11}, \lambda a_{12}, \lambda a_{22}, \lambda a_1, \lambda a_2, \lambda a_0\} \subset K. \tag{9}$$

Claim 2. $q_4^2 = p_4^2$ and $a_{11} = 0$.

Proof. Suppose that $q_4^2 \neq p_4^2$. Consider the following two polynomials $g, h \in \mathbb{R}[X]$:

$$g = a_{12}^2(p_3^2a_{12}^2 - 4a_1^2)X^3 + 8p_4^2a_{11}\left[p_3^2a_{22}a_{12}^2 + 2a_1^2(a_{11} - a_{22}) - 2a_{12}^2a_0\right]X^2$$
$$+16p_4^4a_{11}^2\left[4(a_{11} - a_{22})a_0 + p_3^2a_{22}^2 + a_1^2\right]X + 64p_4^6a_{11}^3a_0,$$

$$h = (4p_4^2q_4^2a_{11}^2 - 4sp_4^2a_{22}a_{11} - s^2a_{12}^2)X - 4p_3^2p_4^2q_4^2a_{11}^2.$$

Since $q_3^2 \neq 0$ and $q_4^2 \neq p_4^2$, $a_{11} \neq 0$. The fact that p_3, p_4, q_4, a_0 are non-zero now implies that the constant terms of both g and h are non-zero and hence neither h nor

g is identically zero. Substituting all coefficients with their appropriate expressions we see that $g(q_4^2 - p_4^2) = 0$ and $h(q_3^2) = 0$. We have $\lambda^4 g \in K[X]$ by (9). Since $g(q_4^2 - p_4^2) = 0$, $q_4^2 - p_4^2 \in \overline{\mathbb{Q}(p)}$ and hence $q_4 \in \overline{\mathbb{Q}(p)}$. Thus $q_4 \in K$. This and (9) imply that $\lambda^2 h \in K[X]$ and we may use a similar argument to deduce that $q_3^2 \in K$, which is a contradiction. Hence $q_4^2 = p_4^2$ and $a_{11} = 0$.

Claim 3. $a_{12} \neq 0$.

Proof. Suppose $a_{12} = 0$. Then $q_3 p_5 - p_3 q_5 = 0$, $a_{22} = 4p_4^2(p_3^2 - q_3^2)$, and $a_{22} \neq 0$ since $q_3^2 \neq p_3^2$. Consider the polynomial

$$g = p_5(p_3 - p_5)a_{22}X - p_3^2 p_4 a_1 \in \mathbb{R}[X].$$

Since $a_{22} \neq 0$, g is not identically zero. On the other hand $g(q_3^2) = 0$. We may now use (9) to deduce that $\lambda g \in K[X]$ and the argument of Claim 2 gives $q_3^2 \in K$, which is a contradiction.

Claim 4. Either $q_5 \in K$ or there is $\mu \in K$ such that $q_5 = \mu q_3$.

Proof. Suppose $2a_1 + p_3 a_{12} \neq 0$. Substituting all coefficients with their appropriate expressions we see that

$$\lambda\left([2p_4(a_2 + p_3 a_{22}) - p_5(2a_1 + p_3 a_{12})]\, q_3 + p_3(2a_1 + p_3 a_{12})q_5\right) = 0.$$

We may now use (9) to deduce that $q_5 = \mu q_3$ for some $\mu \in K$.

Hence we may suppose that

$$2a_1 + p_3 a_{12} = 8p_4 q_3^2(q_5^2 - q_3 q_5 - p_5^2 + p_3 p_5) = 0,$$

and thus $q_5^2 - q_3 q_5 = p_5^2 - p_3 p_5$. In this case q_5^2 is a zero of the following polynomial $g \in \mathbb{R}[X]$:

$$\begin{aligned} g = {} & \left[((p_3 - p_5)^2 - p_4^2)a_{12} + 2p_4(p_3 - p_5)a_{22}\right] X^2 \\ & + \left[(2p_5(p_3 - p_5)(p_5^2 - p_4^2 - 2p_3 p_5) + p_3^2 p_4^2)a_{12} \right. \\ & \quad \left. + \, 2p_5 p_4(p_3^2 - 3p_3 p_5 + 2p_5^2)a_{22}\right] X \\ & + p_5^2(p_3 - p_5)^2((p_5^2 - p_4^2)a_{12} - 2p_5 p_4 a_{22}). \end{aligned}$$

We may now use (9) and the argument of Claim 2 to deduce that $\lambda g \in K[X]$, and hence that $q_5 \in K$, as long as g is not identically zero. Let us suppose indirectly, that $g = 0$. Equating the coefficient of X^2 and the constant term of g to zero gives the following system of linear equations for a_{12}, a_{22}:

$$\begin{aligned} [(p_3 - p_5)^2 - p_4^2]\, a_{12} + 2p_4(p_3 - p_5)\, a_{22} &= 0 \\ (p_5^2 - p_4^2)\, a_{12} - 2p_5 p_4\, a_{22} &= 0\,. \end{aligned}$$

Since $a_{12} \neq 0$ by Claim 3, the determinant of this system, which is a non-zero polynomial in p_3, p_4, p_5 with integer coefficients, must be zero. This contradicts the fact that $\{p_3, p_4, p_5\}$ is algebraically independent over $\mathbb{Q}$.

We can now complete the proof of the lemma. Consider the following polynomial $g \in \mathbb{R}[X, Y]$:

$$g = \left[(p_5^2 - p_4^2)a_{12} - 2p_5p_4a_{22}\right] X^2 + 2p_3(p_4a_{22} - p_5a_{12})XY + p_3^2a_{12}Y^2 + p_3^2p_4^2a_{12}.$$

We have $g(q_3, q_5) = 0$.

Suppose $q_5 \in K$. Let $h = g(X, q_5)$. Then h is not identically zero since its constant term, $p_3^2a_{12}(q_5^2 + p_4^2) \neq 0$. On the other hand $h(q_3) = g(q_3, q_5) = 0$. We may use (9) to deduce that $\lambda h \in K[X]$ and then use the argument of Claim 2 to deduce that $q_3 \in K$, which is a contradiction. Thus $q_5 \notin K$.

By Claim 4, $q_3 = \mu q_5$ for some $\mu \in K$. Let $h' = g(X, \mu X)$. Then h' is not identically zero since its constant term, $p_3^2p_4^2a_{12} \neq 0$. On the other hand $h'(q_3) = g(q_3, q_5) = 0$. We may use (9) to deduce that $\lambda h' \in K[X]$. The argument of Claim 2 then gives $q_3 \in K$, which is a contradiction.

The only way out of this contradiction is that our assumption (8) must be false. Hence some point (p_1, p_2) on the conic $f = 0$ is generic over K. This gives us the required quasi-generic realisation (G, p^*) by Lemma 7.

We can use Lemma 8 to show that, if G is obtained by performing a 1-extension on a non-redundant edge, then the end-vertices of this edge are not globally linked in G.

Theorem 7. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Suppose the $H - uw$ is not rigid. Then $\{u, w\}$ is not globally linked in G.*

Proof. Let (H, p) be a quasi-generic framework which is in standard position with respect to (u, w). Since (H, p) is infinitesimally rigid, but $(H - uw, p)$ is not infinitesimally rigid, there is an infinitesimal motion $\tilde{p}$ of $(H - uw, p)$, such that

$$(p(u) - p(w))(\tilde{p}(u) - \tilde{p}(w)) \neq 0.$$

Theorem 6 gives a smooth flexing $\pi : (-1, 1) \times V \to \mathbb{R}^2$ of the framework $(H - uw, p)$ such that $\dot{\pi}_0 = \tilde{p}$.

Suppose that G is the 1-extension of H with a new vertex v with neighbour set $\{u, w, z\}$. Since p is quasi-generic, $p(u)$, $p(w)$ and $p(z)$ are not collinear. Since π is continuous, we can choose $t_1 > 0$ such that $\pi_{t_1}(u)$, $\pi_{t_1}(w)$ and $\pi_{t_1}(z)$ are not collinear for all $0 < t < t_1$. Let

$$f(t) = ||\pi_t(u) - \pi_t(w)||^2.$$

Then $\frac{df}{dt}|_{t=0} = 2(p(u) - p(w))(\tilde{p}(u) - \tilde{p}(w)) \neq 0$. Since $\overline{\mathbb{Q}(p)}$ is countable, it follows that $f(t_2) \notin \overline{\mathbb{Q}(p)}$ for some $0 < t_2 < t_1$. In particular,

$$||\pi_{t_2}(u) - \pi_{t_2}(w)|| \neq ||p(u) - p(w)||.$$

Let $(G - v, q)$ be a framework which is congruent to $(G - v, \pi_{t_2})$ and in standard position with respect to (u, w). Applying Lemma 8 to $(G - v, p)$ and $(G - v, q)$ we can find equivalent frameworks (G, p^*) and (G, q^*) such that p^* is quasi-generic, $p^*|_{V-v} = p$ and $q^*|_{V-v} = \pi_{t_2}$. This gives

$$||q^*(u) - q^*(w)|| = ||\pi_{t_2}(u) - \pi_{t_2}(w)|| \neq ||p(u) - p(w)|| = ||p^*(u) - p^*(w)||.$$

Hence $\{u, w\}$ is not globally linked in G.

We next use Lemma 8 to prove a counterpart of Theorem 2.

Theorem 8. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Suppose that $H - uw$ is not rigid and that $\{x, y\}$ is not globally linked in H for some $x, y \in V$. Then $\{x, y\}$ is not globally linked in G.*

Proof. Since $\{x, y\}$ is not globally linked in H, there are equivalent frameworks (H, p_1) and (H, p_2) in standard position with respect to (u, w) and such that p_1 is quasi-generic and

$$||p_1(x) - p_1(y)|| \neq ||p_2(x) - p_2(y)||.$$

Since H is rigid and (H, p_1) is quasi-generic, [18, Corollary 3.7] implies that (H, p_2) is quasi-generic. Hence (H, p_2) is infinitesimally rigid and $(H - uw, p_2)$ is not infinitesimally rigid. It follows that there is an infinitesimal motion $\tilde{p}$ of $(H - uw, p_2)$ such that

$$(p_2(u) - p_2(w))(\tilde{p}(u) - \tilde{p}(w)) \neq 0.$$

Theorem 6 gives a smooth flexing $\pi : [-1, 1] \times V \to \mathbb{R}^2$ of the framework $(H - uw, p_2)$ such that $\dot{\pi}_0 = \tilde{p}$.

Suppose that G is the 1-extension of H with a new vertex v with neighbour set $\{u, w, z\}$. Since p_2 is quasi-generic, $p_2(u)$, $p_2(w)$ and $p_2(z)$ are not collinear. Since π is continuous, we may choose $t_1 > 0$ such that $\pi_t(u)$, $\pi_t(w)$ and $\pi_t(z)$ are not collinear and $||\pi_t(x) - \pi_t(y)|| \neq ||p_1(x) - p_1(y)||$ for all $0 < t < t_1$. Let

$$f(t) = ||\pi_t(u) - \pi_t(w)||^2.$$

Then $\frac{df}{dt}|_{t=0} = 2(p(u) - p(w))(\tilde{p}(u) - \tilde{p}(w)) \neq 0$. Since $\overline{\mathbb{Q}(p)}$ is countable, it follows that $f(t_2) \notin \overline{\mathbb{Q}(p)}$ for some $0 < t_2 < t_1$.

Let $(G-v, q)$ be a framework which is congruent to $(G-v, \pi_{t_2})$ and in standard position with respect to (u, w). Applying Lemma 8 to $(G-v, p_1)$ and $(G-v, q)$ we can find equivalent frameworks (G, p^*) and (G, q^*) such that p^* is quasi-generic, $p^*|_{V-v} = p_1$ and $q^*|_{V-v} = q$. Therefore

$$||q^*(x) - q^*(y)|| = ||\pi_{t_2}(x) - \pi_{t_2}(y)|| \neq ||p_1(x) - p_1(y)|| = ||p^*(x) - p^*(y)||.$$

Hence $\{x, y\}$ is not globally linked in G.

We can use Theorems 7 and 8 to deduce that Conjectures 3 and 4 hold for graphs with at most one non-trivial redundantly rigid component.

Theorem 9. *Let $G = (V, E)$ be a rigid graph, $u, v \in V$, and $R = (U, F)$ be a redundantly rigid component of G. Suppose that $G-e$ is not rigid for all $e \in E-F$. Then $\{u, v\}$ is globally linked in G if and only if $uv \in E$ or $\{u, v\}$ is globally linked in R.*

Proof. Sufficiency is clear so we need only prove necessity. Suppose $\{u, v\}$ is globally linked in G. If $U = V$ then $G = R$ and the result is trivially true. Hence we may suppose that $U \neq V$.

We first show that there exists either a vertex of $V - U$ of degree two in G or at least three vertices of $V - U$ of degree three in G. Since G is rigid every vertex of G has degree at least two. We have $|E - F| = r(G) - r(R) = 2|V - U|$. If $|V - U| = 1$ this implies that the unique vertex of $V - U$ has degree two. Hence we may suppose that $|V - U| \geq 2$. The rigidity of G now implies that there are at least three edges between U and $V - U$. Hence

$$\sum_{x \in V-U} d_G(x) \leq 2|E - F| - 3 = 4|V - U| - 3.$$

The assertion about vertices of degree two or three in $V - U$ now follows.

Suppose there exists $x \in V - U$ with $d(x) = 2$. It is not difficult to see that x is only globally linked to its neighbours in G. Hence the theorem holds if $x \in \{u, v\}$ and we may suppose that this is not the case. Let $H = G - x$. Then (H, R) satisfies the hypotheses of the theorem. The result now follows by applying induction and Lemma 2.

Hence we may assume that there are at least three vertices of $V - U$ of degree three. Choose $x \in V - U$ with $d(x) = 3$ and $x \notin \{u, v\}$. By Lemma 4 there is a pair y, z of neighbors of x for which $H = G - x + yz$ is rigid. The rigidity of H implies that $\{y, z\} \nsubseteq U$ and that (H, R) satisfies the hypotheses of the theorem. The result now follows by applying induction and Theorem 8 when $\{y, z\} \neq \{u, v\}$, and by Theorem 7 when $\{y, z\} = \{u, v\}$.

Conjectures 3 and 4 follow for a (not necessarily rigid) graph G with at most one non-trivial redundantly rigid component by applying Theorem 9 to the rigid components of G (and using the fact that pairs of vertices belonging to different rigid components are not globally linked).

The special case of Theorem 9 when G has no non-trivial redundantly rigid components characterises globally linked pairs in minimally rigid graphs.

Corollary 2. *Let $G = (V, E)$ be a minimally rigid graph and $u, v \in V$. Then $\{u, v\}$ is globally linked in G if and only if $uv \in E$.*

Suppose we apply a 1-extension on a non-redundant edge xy of a rigid graph H. Then Theorem 7 implies that $\{x, y\}$ is not globally linked in the resulting graph G. On the other hand, Conjecture 2 would imply that this is the only pair of globally linked vertices of H which is not globally linked in G.

Conjecture 5. Suppose G is a 1-extension on a non-redundant edge xy of a rigid graph H and $\{u, v\} \neq \{x, y\}$ is globally linked in H. Then $\{u, v\}$ is globally linked in G.

4 Neighbourhood Stability

In this section we obtain analogues of the following result of Connelly and Whiteley for globally linked pairs.

Theorem 10 ([6, Theorem 13]). *Given a framework (G, p) which is globally rigid and infinitesimally rigid in $\mathbb{R}^d$, there is an open neighborhood U of p such that for all $q \in U$ the framework (G, q) is globally rigid and infinitesimally rigid.*

We will concentrate on the 2-dimensional case.[1] We will need the following well known 'averaging' result, see for example the proof of [6, Theorem 13].

Lemma 9. *Suppose that (G, p) and (G, q) are equivalent but non-congruent frameworks. Then $p - q$ is a non-trivial infinitesimal motion of $(G, p + q)$.*

We say that an infinitesimally rigid framework (G, p) is *regular valued* if all equivalent frameworks are infinitesimally rigid.[2] It is known that an infinitesimally rigid, regular valued framework (G, p) has only finitely many equivalent and pairwise non-congruent realisations.[3] We denote this number by $r(G, p)$. This parameter is related to global linkedness by the fact that two vertices u, v are globally

[1] We believe that our results extend to the d-dimensional case but the proofs become more complicated because of their reliance on 'special position' arguments. In particular we would need a d-dimensional version of Lemma 7.

[2] This is equivalent to saying that $f_G(p)$ is a *regular value* of the rigidity map of G i.e. q is a regular point of f_G for all $q \in f_G^{-1}(f_G(p))$.

[3] Since $f_G(p)$ is a regular value of f_G, $f_G^{-1}(f_G(p))$ is a 0-dimensional manifold. Compactness and the fact that (G, q) is infinitesimally rigid (and hence rigid) for all $q \in f_G^{-1}(f_G(p))$ now tells us that $f_G^{-1}(f_G(p))$ is finite.

linked in an infinitesimally rigid, regular valued framework (G, p) if and only if $r(G, p) = r(G + uv, p)$. Our first result shows that, for such a framework (G, p), $r(G, p)$ does not increase in some open neighbourhood of p.

Theorem 11. *Suppose that (G, p) is an infinitesimally rigid, regular valued framework. Then there exists an open neighbourhood U of p such that, for all $q \in U$, (G, q) is an infinitesimally rigid, regular valued framework with $r(G, q) \leq r(G, p)$.*

Proof. The theorem is trivially true if G has at most two vertices. Hence we may suppose that $|V(G)| \geq 3$. Since (G, p) is infinitesimally rigid we have $p(u) \neq p(v)$ for some edge uv of G.

We first show that there exists an open neighbourhood U of p such that (G, q) is an infinitesimally rigid, regular valued framework for all $q \in U$. Suppose not. Then there exists a sequence of realisations (G, p^k) with $p^k \to p$, and such that (G, p^k) is not infinitesimally rigid and regular valued for all k. Since (G, q) is infinitesimally rigid for q close enough to p, we may suppose that (G, p^k) is infinitesimally rigid and not regular valued for all k. Hence (G, p^k) has an equivalent realisation (G, q^k) which is not infinitesimally rigid. We may assume that each (G, q^k) is in standard position with respect to (u, v). By compactness q^k has a convergent subsequence $q^m \to q$. Since q^m is equivalent to p^m and $p^m \to p$, (G, q) is equivalent to (G, p). Since (G, p) is regular valued, (G, q) is infinitesimally rigid. This contradicts the fact that $q^m \to q$ and (G, q^m) is not infinitesimally rigid for all m.

We next show that there exists an open neighbourhood U of p such that $r(G, q) \leq r(G, p)$ for all $q \in U$. Suppose not. Then, by the previous paragraph, there exists a sequence of infinitesimally rigid, regular valued realisations (G, p^k) with $p^k \to p$ and $r(G, p^k) > r(G, p)$ for all $k \geq 1$. Let $S = \{(G, p_1), (G, p_2), \ldots, (G, p_s)\}$ be the set of all equivalent realisations which are in standard position with respect to (u, v). Since (G, p) is infinitesimally rigid and regular valued, each of the (G, p_i) is infinitesimally rigid and hence, in particular, does not have all its vertices on a line. This implies that each congruence class of (G, p) will be represented exactly four times in S and hence $s = 4r(G, p)$. Since $r(G, p^k) > r(G, p)$ for each $k \geq 1$, we may choose a set $\{q_1^k, q_2^k, \ldots, q_{s+1}^k\}$ of realisations which are equivalent to (G, p^k) and are in standard position with respect to (u, v). By compactness there exist convergent subsequences $q_i^m \to q_i$ for all $1 \leq i \leq s + 1$. Since q_i^m is equivalent to p^m and $p^m \to p$, each q_i is equivalent to p. Hence $q_i = p_j$ for some $1 \leq j \leq s$. By the pigeon hole principle, we may choose two sequences q_1^m, q_2^m say, converging to the same realisation, (G, p_1) say, of G. By Lemma 9, $q_1^m - q_2^m$ is a non-trivial infinitesimal motion of $(G, q_1^m + q_2^m)$, and hence $(G, q_1^m + q_2^m)$ it is not infinitesimally rigid. Since $q_1^m + q_2^m \to 2p_1$, $(G, 2p_1)$ is not infinitesimally rigid. This implies that (G, p_1) is not infinitesimally rigid and contradicts the hypothesis that all equivalent realisations of (G, p) are infinitesimally rigid.

Note that Theorem 11 generalises (the 2-dimensional version of) Theorem 10 since an infinitesimally rigid, globally rigid framework (G, p) is regular valued and has $r(G, p) = 1$.

We can have $r(G,q) < r(G,p)$ for any framework (G,p) satisfying the hypotheses of Theorem 11 and q arbitrarily close to p. Consider for example the realisation (G,p) of a wheel in which the central vertex and two nonconsecutive rim vertices are collinear, see Fig. 3. Then $r(G,p) = 2$ but $r(G,q) = 1$ for all generic q. We will show, however, that $r(G,p)$ is constant in some open neighbourhood of p if either G is minimally rigid or p is generic.

Theorem 12. *Suppose that (G,p) is an infinitesimally rigid, regular valued realisation of a minimally rigid graph $G = (V,E)$. Then there exists an open neighbourhood U of p such that, for all $q \in U$, (G,q) is infinitesimally rigid, regular valued, and has $r(G,q) = r(G,p)$.*

Proof. By Theorem 11, it will suffice to show that an arbitrary sequence of infinitesimally rigid realisations (G,p^k) with $p^k \to p$ has $r(G,p^k) \geq r(G,p)$ for k large enough. Since (G,p) is infinitesimally rigid we have $p(u) \neq p(v)$ for some $uv \in E$. Let $\{p_1, p_2, \ldots, p_s\}$ be the set of all realisations in standard position with respect to (u,v) which are equivalent to (G,p). Then $s = 4r(G,p)$ as in the proof of Theorem 11.

We consider $f_G : \mathbb{R}^{2|V|-3} \to \mathbb{R}^{|E|}$ by restricting the domain of f_G to realisations in standard position with respect to (u,v). By the inverse function theorem, we may choose disjoint neighbourhoods U_i of p_i in $\mathbb{R}^{2|V|-3}$ and W of $f_G(p)$ in $\mathbb{R}^{|E|}$ such that f_G maps U_i diffeomorphically onto W for all $1 \leq i \leq s$. We may also assume that $p = p_1$ and hence that $p^k \in U_1$ for k large enough, say $k \geq K$. Then $f_G(p^k) \in W$ and hence there exists $p_i^k \in U_i$ with $f_G(p_i^k) = f_G(p^k)$ for all $1 \leq i \leq s$ and all $k \geq K$. This implies that there are at least s distinct realisations (G,p_i^k) in standard position with respect to (u,v) which are equivalent to (G,p^k). Hence $r(G,p^k) \geq r(G,p)$ for $k \geq K$.

Corollary 3. *Suppose that (G,p) is an infinitesimally rigid, regular valued realisation of a minimally rigid graph $G = (V,E)$. Then there exists an open neighbourhood U of p such that, for all $u,v \in V$ and all $q \in U$, $\{u,v\}$ is not globally linked in (G,q) if $\{u,v\}$ is not globally linked in (G,p).*

Proof. This follows from Theorems 11 and 12 and the fact that, for any infinitesimally rigid, regular valued realisation (G,q), $\{u,v\}$ is globally linked in (G,q) if and only if $r(G,q) = r(G+uv,q)$.

The example in Fig. 4 shows that there can exist pairs of vertices which are globally linked in (G,p) but are not globally linked in (G,q) for q arbitrarily close to p.

We next show that $r(G,p)$ remains constant in an open neighbourhood of p for any rigid graph G when p is generic. Our proof uses the Tarski-Seidenberg theorem on semi-algebraic sets. A subset S of $\mathbb{R}^n$ is *semi-algebraic over* $\mathbb{Q}$ if it can be expressed as a finite union of sets of the form

$$\{x \in \mathbb{R}^n : P_i(x) = 0 \text{ for } 1 \leq i \leq s \text{ and } Q_j(x) > 0 \text{ for } 1 \leq j \leq t\},$$

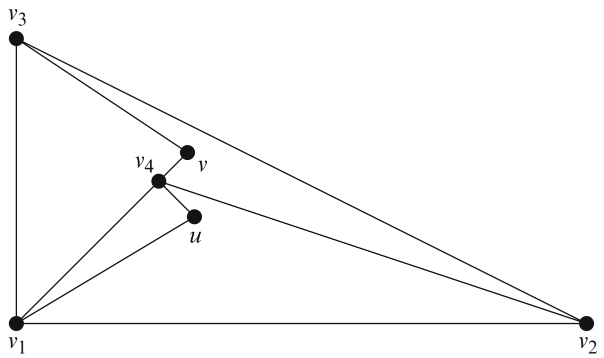


Fig. 4 A regular realisation (G, p) of a minimally rigid graph. The line through u, v_4 is perpendicular to the line through v_1, v and passes through v_3. There are exactly four equivalent realisations which keep the triangle $v_1v_2v_3$ fixed, and they can be obtained by reflecting v in the line through v_3v_4 and/or reflecting u in the line through v_1v_4. The distance between u and v is the same in all such realisations so (u, v) is globally linked in (G, p). On the other hand, (u, v) is not globally linked in any generic realisation (G, q)

where $P_i \in \mathbb{Q}[X_1, \ldots, X_n]$ for $1 \le i \le s$, and $Q_j \in \mathbb{Q}[X_1, \ldots, X_n]$ for $1 \le j \le t$.[4]

Theorem 13 ([26]). *Let $S \subseteq \mathbb{R}^{n+k}$ be semi-algebraic over $\mathbb{Q}$ and $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the projection onto the first n coordinates. Then $\pi(S)$ is semi-algebraic over $\mathbb{Q}$.*

Theorem 14. *Suppose that $G = (V, E)$ is rigid and (G, p) is generic. Then there exists an open neighbourhood U of p such that, for all $q \in U$, (G, q) is infinitesimally rigid, regular valued, and has $r(G, p) = r(G, q)$.*

Proof. The hypothesis that (G, p) is generic implies that $f_G(p)$ is a regular value of f_G by [18, Corollary 3.7]. Hence (G, q) is infinitesimally rigid and regular valued for q in some open neighbourhood of p by Theorem 11.

Let $V = \{v_1, v_2, \ldots, v_n\}$, $|E| = m$ and suppose that $v_1v_2 \in E$. Choose a realisation (G, p_1) which is congruent to p and is in standard position with respect to (v_1, v_2). We will consider the set of all realisations of G which are in standard position with respect to (v_1, v_2). For any such realisation (G, q) the first three coordinates of q are zero. We will abuse notation and consider $q \in \mathbb{R}^{2n-3}$. Similarly we will consider the rigidity map f_G to be a map from $\mathbb{R}^{2n-3}$ to $\mathbb{R}^m$.

Let $s = 4r(G, p)$, and let S be the set of all s-tuples of vectors $(q_1, q_2, \ldots, q_s)$ where $q_i \in \mathbb{R}^{2n-3}$ and $\{(G, q_1), (G, q_2), \ldots, (G, q_s)\}$ is a set of distinct pairwise equivalent realisations of G in standard position with respect to (v_1, v_2). Then $S \subseteq \mathbb{R}^{s(2n-3)}$ and we may represent S as

[4]The usual definition for a semi-algebraic set uses polynomials with coefficients in $\mathbb{R}$, or more generally in a real closed field. The fact that the Tarski-Seidenberg Theorem holds for semi-algebraic sets over $\mathbb{Q}$ follows from the original papers [23, 26].

$$S = \{(q_1, q_2, \ldots, q_s) : q_i \in \mathbb{R}^{2n-3}, f_G(q_i) = f_G(q_1), q_i \neq q_j \text{ for } 1 \leq i \neq j \leq s\}.$$

Hence S is semi-algebraic over $\mathbb{Q}$. Let

$$S_1 = \{q_1 \in \mathbb{R}^{2n-3} : (q_1, q_2, \ldots, q_s) \in S \text{ for some } q_2, q_3, \ldots, q_s \in \mathbb{R}^{2n-3}\}$$

be the projection of S onto the first $2n-3$ coordinates. Then S_1 is the set of all $q \in \mathbb{R}^{2n-3}$ such that (G, q) has s distinct pairwise equivalent realisations in standard position with respect to (v_1, v_2). Thus $p_1 \in S_1$.

By Theorem 13, S_1 is semi-algebraic over $\mathbb{Q}$. Since p is generic, the (non-zero) coordinates of p_1 are algebraically independent over $\mathbb{Q}$ by Lemma 7. Hence $P(p_1) \neq 0$ for all $P \in \mathbb{Q}[X_1, \ldots, X_{2n-3}]$. Since $p_1 \in S_1$ and S_1 is semi-algebraic over $\mathbb{Q}$, we must have $p_1 \in S_2$ for some $S_2 \subseteq S_1$ of the form

$$S_2 = \{q \in \mathbb{R}^{2n-3} : Q_j(q) > 0 \text{ for } 1 \leq j \leq t\},$$

where $Q_j \in \mathbb{Q}[X_1, \ldots, X_{2n-3}]$ for $1 \leq j \leq t$. We may choose an open neighbourhood U of p_1 in $\mathbb{R}^{2n-3}$ such that $Q_j(q) > 0$ for all $q \in U$ and all $1 \leq j \leq t$. Then $q \in S_2 \subseteq S_1$ for all $q \in U$. By the first paragraph of the proof, we may choose U small enough so that (G, q) is infinitesimally rigid and regular valued for all $q \in U$. Then $r(G, q) \geq s/4 = r(G, p)$ for all $q \in U$. Theorem 11 now implies that there exists a possibly even smaller open neighbourhood U' of p_1 in $\mathbb{R}^{2n-3}$ with $r(G, q) = r(G, p)$ for all $q \in U'$. We can now complete the proof by choosing an open neighbourhood U'' of p in $\mathbb{R}^{2n}$ such that, for each $q \in U''$, (G, q) is congruent to (G, q_1) for some $q_1 \in U'$.

Corollary 4. *Suppose that $G = (V, E)$ is rigid and (G, p) is generic. Then there exists an open neighbourhood U of p such that, for all $u, v \in V$ and all $q \in U$, $\{u, v\}$ is globally linked in (G, p) if and only if $\{u, v\}$ is globally linked in (G, q).*

Proof. This follows immediately from Theorem 14 and the fact that, for any infinitesimally rigid, regular valued realisation (G, q), $\{u, v\}$ is globally linked in (G, q) if and only if $r(G, q) = r(G + uv, q)$.

The realisation (G, p) of a wheel in which the central vertex and two nonconsecutive rim vertices are collinear (see Fig. 3) shows that Corollaries 3 and 4 become false if we remove the respective hypotheses that G is minimally rigid or p is generic. The problem is that there are pairs of vertices which are not globally linked in (G, p) but are globally linked in (G, q) for q arbitrarily close to p. The example after Corollary 3 shows that we can also have pairs of vertices which are globally linked in (G, p) but are not globally linked in (G, q) for q arbitrarily close to p, if we remove the hypothesis that (G, p) is generic.

5 Finding Equivalent Realizations by Flexing

In this section we describe a possible approach to verifying Conjecture 3 which is analogous to that used by Hendrickson [12] to show that redundant rigidity is a necessary condition for global rigidity. We need to show that if two vertices u, v are not contained in the same redundantly rigid component of a rigid graph G then they are not globally linked.[5] The idea is to find an edge $e = wx$ in G such that u, v do not belong to the same rigid component of $G - e$. We then choose a flexing of a generic realisation $(G-e, p)$ to find another realisation $(G-e, q)$ with the properties that $\|q(w) - q(x)\| = \|p(w) - p(x)\|$ and $\|q(u) - q(v)\| \neq \|p(u) - p(v)\|$. The equivalent realisations (G, p) and (G, q) will then certify that $\{u, v\}$ is not globally linked in G. The first step in this approach is to show that we can find a suitable edge e.

Lemma 10. *Let $G = (V, E)$ be a rigid graph and $u, v \in V$ with $uv \notin E$. Then $\{u, v\}$ is contained in a redundantly rigid component of G if and only if $\{u, v\}$ is contained in a rigid component of $G - e$ for all $e \in E$.*

Proof. We first prove necessity. Suppose u, v is contained in a redundantly rigid component H of G. Then $H \neq K_2$ and so $H - e$ is a rigid subgraph of G for all $e \in E$. Hence u, v is contained in a rigid component of $G - e$ for all $e \in E$.

We next prove sufficiency. Suppose u, v is not contained in a redundantly rigid component of G. Then G is not redundantly rigid so at least one edge of G is an M-bridge, that is, it does not belong to any M-circuit in G. Let $F = \{e_1, e_2, \ldots, e_m\}$ be the set of M-bridges of G. It is not hard to see that the rigid components of $G - F$ are exactly the non-trivial redundantly rigid components of G. Thus u, v is not contained in a rigid component of $G - F$. Let H' be a maximal M-independent subgraph of $G - F$. Note that the vertex sets of the rigid components of $G - F$ and H' are the same and $H' + F$ is an M-independent (and rigid) spanning subgraph of G.

Let F' be a maximal proper subset of F for which u, v is not contained in a rigid component of $H' + F'$. If $F - F' = \{f\}$ then we are done by choosing $e = f$. This follows from the fact that u, v is not contained in a rigid component of $H' - f$ and hence is not contained in a rigid component of $G - f$ as well. So we may suppose that we have two distinct edges $f_1, f_2 \in F - F'$. By the maximality of F' there is a rigid subgraph $G_i = (V_i, E_i)$ of $H' + F' + f_i$ which contains u and v, for $i = 1, 2$. Since $H' + F$ is M-independent, these subgraphs are induced subgraphs of $H' + F$ and we must have $f_1, f_2 \notin G_1 \cap G_2$. Then $G_1 \cap G_2$ is a rigid subgraph of $H' + F'$ which contains u and v. This contradicts the choice of F'.

Our next result implies that the 'flexing approach' to showing that $\{u, v\}$ is not globally linked in G works when $G + uv$ is an M-circuit.

[5] It is straightforward to reduce Conjecture 3 to rigid graphs since pairs of vertices which do not belong to the same rigid component of a graph cannot be globally linked.

Lemma 11. *Let (C, p_0) be a quasi-generic realisation of an M-circuit $C = (V, E)$, $e_1 = uv$ and $e_2 = wx$ be edges of C and $H = C - \{e_1, e_2\}$. Let $\mathscr{F}$ be the set of all frameworks which can be obtained by a flexing of (H, p_0). Then there exists $(H, p_1) \in \mathscr{F}$ with $\|p_0(u) - p_0(v)\| \neq \|p_1(u) - p_1(v)\|$ and $\|p_0(w) - p_0(x)\| = \|p_1(w) - p_1(x)\|$.*

Proof. We suppose that all realisations of H considered are in standard position with respect to (w, x). For each such realisation (H, q) we suppress the (zero) coordinates of q corresponding to $q(w)$ and the first coordinate of $q(x)$ and consider $q \in \mathbb{R}^{2|V|-3}$.

Let

$$S = \{p \in \mathbb{R}^{2|V|-3} : (H, p) \in \mathscr{F}\}.$$

Let $F : \mathbb{R}^{2|V|-3} \to \mathbb{R}$ be given by $F(q) = \|q(w) - q(x)\|^2$ (in the corresponding realisation (H, q)) and let f be the restriction of F to S. We can also view the rigidity maps f_{H+e_2}, f_H as maps on S. Note that the rigidity map f_{H+e_2} is obtained from f_H by adding an extra coordinate corresponding to e_2 i.e. the length of the edge e_2 in the realisation of $H + e_2$.

We can adapt the proof technique of [12] to show that S is a 1-dimensional manifold diffeomorphic to a circle. For each $p \in S$, [17, Lemma 3.4] gives

$$\text{rank } df|_p = \text{rank } df_{H+e_2}|_p - \text{rank } df_H|_p = \text{rank } R(H + e_2, p) - \text{rank } R(H, p).$$

Thus, for every generic point $p \in S$, we have rank $df|_p = 1$ so p is a regular point of f.

Choose a direction for traversing S and let p_1 be the first point after p_0 we reach when traversing S which satisfies $\|p_0(w) - p_0(x)\| = \|p_1(w) - p_1(x)\|$. We will show that $\|p_0(u) - p_0(v)\| \neq \|p_1(u) - p_1(v)\|$. Suppose to the contrary that $\|p_0(u) - p_0(v)\| = \|p_1(u) - p_1(v)\|$. Then (C, p_0) is equivalent to (C, p_1).

We first consider the case when C is 3-connected. Then C is globally rigid by [3] so (C, p_0) is congruent to (C, p_1). Since (C, p_0) and (C, p_1) are in standard position $(C, p_1) = \alpha(C, p_0)$, where α is a reflection in one of the two coordinate axes or a rotation of π about the origin. Let $a : [0, 1] \to S$ be the smooth path from p_0 to p_1 induced by the diffeomorphism from S to the circle, and let $b : [0, 1] \to S$ be obtained by putting $b(t) = \alpha(a(t))$ for all $0 \leq t \leq 1$. Then b is a smooth path in S from p_1 to p_0. Furthermore, we claim that a and b do not have the same image in S. For suppose to the contrary that a and b traverse some path P in S in opposite directions. Then by the intermediate value theorem there is some $t \in [0, 1]$ with $a(t) = b(t)$. This implies that $(H, a(t))$ has all vertices on one of the two coordinate axes, which is impossible since (H, p), and hence also $(H, a(t))$, has $2|V(H)| - 4$ algebraically independent edge-lengths. It follows that a and b trace out two paths that together form the entire manifold S. We can choose $t_1, t_2 \in [0, 1]$ with $f(a(t_1)) < f(p_0)$ and $f(a(t_2)) > f(p_0)$. Now the intermediate value theorem gives some t between t_1 and t_2 with $f(a(t)) = f(p_0)$. This contradicts the choice of p_1.

We next consider the case when C is not 3-connected. Let $C_1C_2 \dots C_m$ be the path in the cleavage unit tree of C with $e_1 \in E(C_1)$ and $e_2 \in E(C_m)$. (We refer the reader to [15, Section 3] for more details on cleavage unit trees of M-circuits.) Let $C' = C_1 \oplus_2 C_2 \oplus_2 \dots \oplus_2 C_m$. If $C' \neq C$ then we can apply induction to C'. Hence we may assume that $C' = C$. We will proceed by adapting the proof of the case when C is 3-connected.

Let $V(C_i) \cap V(C_{i+1}) = \{x_i, w_i\}$ for $1 \le i < m$. For each $p \in S$, let $\ell_i(p)$ be the line through $p(w_i), p(x_i)$ for $1 \le i < m$. Let $\theta_i(C, p)$ be the realisation of C obtained from (C, p) by reflecting $C_1, C_2, \dots, C_i$ in the line through $\ell_i(p)$. For each $T = \{i_1, i_2, \dots, i_s\} \subseteq \{1, 2, \dots, m\}$, let $\theta_T(C, p)$ be the realisation of C obtained recursively from (C, p) by applying θ_{i_s} to $\theta_{T-i_s}(C, p)$, and taking $\theta_T(C, p) = (C, p)$ when $T = \emptyset$.[6] Since (C, p_1) is equivalent to (C, p_0), it follows from the proof of [18, Theorem 8.2] that $(C, p_1) = \alpha\theta_T(C, p_0)$ for some $\emptyset \neq T' \subseteq \{1, 2, \dots, m\}$, where α is either the identity, a reflection in one of the two coordinate axes, or a rotation of π about the origin. Let $a : [0, 1] \to S$ be the smooth path from p_0 to p_1 induced by the diffeomorphism from S to the circle, and let $b : [0, 1] \to S$ be obtained by putting $b(t) = \overline{a(t)}$ for all $0 \le t \le 1$, where $(C, \overline{a(t)}) = \alpha\theta_{T'}(C, a(t))$. Then b is a smooth path in S from p_1 to p_0. Furthermore, we claim that a and b do not have the same image in S. For suppose to the contrary that a and b traverse some path P in S in opposite directions. Then by the intermediate value theorem there is some $t \in [0, 1]$ with $a(t) = b(t) = p_2$, say. But this implies that $(C, p_2) = \alpha\theta_{T'}(C, p_2)$, and in particular $(C_1, p_2) = \alpha\theta_{T'}(C_1, p_2)$. Since the action of $\theta_{T'}$ on (C, p_2) is a non-empty sequence of reflections through lines with algebraically independent slopes it is either a rotation or a reflection. Since (C_1, p_2) remains fixed under this action, all vertices of (C_1, p_2) must lie on the same line. This is impossible since $|V(C_1)| \geq 4$, and (H, p), and hence also (H, p_2), have $2|V(H)| - 4$ algebraically independent edge-lengths. It follows that a and b trace out two paths that together form the entire manifold S. We can choose $t_1, t_2 \in [0, 1]$ with $f(a(t_1)) < f(p_0)$ and $f(a(t_2)) > f(p_0)$. Now the intermediate value theorem gives some t between t_1 and t_2 with $f(a(t)) = f(p_0)$. This contradicts the choice of p_1.

Lemma 11 gives the following strengthening of Corollary 2 for a special family of minimally rigid graphs. We say that a pair of vertices $\{u, v\}$ is *globally loose* in a graph G if $\{u, v\}$ is not globally linked in all generic realisations of G.

Corollary 5. *Suppose G is minimally rigid and $G + uv$ is an M-circuit for two non-adjacent vertices u, v of G. Then $\{u, v\}$ is globally loose.*

The special case of Corollary 5, when $G + uv$ is a 3-connected M-circuit, follows from [18, Theorem 7.1]. The example in Fig. 1 shows that the stronger conclusion,

[6] It can be shown that $\theta_i(\theta_j(C, p)) = \theta_j(\theta_i(C, p))$ and hence $\theta_T(C, p)$ is independent of the ordering of the elements of T. We will not use this fact in our proof.

that $\{u, v\}$ is not globally linked in all generic realisations of G, may not hold when $G + uv$ is not an M-circuit. On the other hand, we can try to apply Lemma 11 to an arbitrary rigid graph as follows.

Given a framework (G, p) let $\mathscr{F}(G, p)$ be the set of all frameworks which can be obtained by a flexing of (G, p). We refer to $\mathscr{F}(G, p)$ as the *flex* of (G, p).

Suppose that G and H are minimally rigid graphs with at least three vertices and $H \subseteq G$. Let e be an edge of H and (G, p) be a realisation of G. We say that $(G - e, p)$ is *free* for $H - e$ if, for every $(H - e, q_0) \in \mathscr{F}(H - e, p|_H)$, there exists a $(G - e, q) \in \mathscr{F}(G - e, p)$ such that $q_0 = q|_H$. Intuitively $(G - e, p)$ is free for $H - e$ if the edges of $E(G) \setminus E(H)$ put no restriction on the flex of $(H - e, p|_H)$. We conjecture that such realisations always exist.

Conjecture 6. Let G be a minimally rigid graph, H be a minimally rigid subgraph of G with at least three vertices and e be an edge of H. Then there exists a generic realisation (G, p) of G such that $(G - e, p)$ is free for $H - e$.

We can use Lemmas 10 and 11 to show that Conjecture 3 would follow from Conjecture 6.

Lemma 12. *Suppose Conjecture 6 is true. Let $G = (V, E)$ be a rigid graph, and $u, v \in V$ be such that $\{u, v\}$ is not contained in any redundantly rigid component of G. Then $\{u, v\}$ is not globally linked in G.*

Proof. Let $e_1 = uv$ and let C be an M-circuit of $G+e_1$ containing e_1. By Lemma 10 we can find an edge $e_2 = wx$ such that $\{u, v\}$ is not contained in any rigid component of $G - e_2$. Then $e_2 \in E(C)$. Let G' be a minimally rigid spanning subgraph of G which contains $C - e_1$. By Conjecture 6, there exists a generic realisation (G, p) of G such that $(G' - e_2, p)$ is free for $C - e_1 - e_2$ By applying Lemma 11 to C, we may deduce that there exists $q \in \mathscr{F}(G' - e_2, p)$ such that $\|p(u) - p(v)\| \neq \|q(u) - q(v)\|$ and $\|p(w) - p(x)\| = \|q(w) - q(x)\|$. Since the distances between all pairs of vertices in the same rigid component of $G' - e_2$ remain constant for all $(G' - e_2, q) \in \mathscr{F}(G' - e_2, p)$, (G, q) is equivalent to (G, p). Since $\|p(u) - p(v)\| \neq \|q(u) - q(v)\|$, $\{u, v\}$ is not globally linked in (G, p).

5.1 Closing Remark

It is not difficult to show that if H is a minimally rigid subgraph of a minimally rigid graph G, then G can be obtained from H by a sequence of Henneberg extensions, see for example [15]. This fact encouraged us to try to prove Conjecture 6 recursively. Let $H = H_0, H_1, \ldots, H_s = G$ be a sequence of minimally rigid graphs with the property that H_i is a Henneberg extension of H_{i-1} for all $1 \leq i \leq s$ and let e be an edge of H. We could assume inductively that there exists a generic realisation $(H_{s-1} - e, p_{s-1})$ which is free for $H - e$ and try to extend it to a realisation $(H_s - e, p_s)$ which is free for $H - e$. A similar idea was outlined previously by Owen and Power [20, Problem 2]. It can be shown that $(H_{s-1} - e, p_{s-1})$ can be extended to a realisation $(H_s - e, p_s)$ which is free for $H - e$ when H_s is a 0-extension of H_{s-1}. We conjectured that the same should hold for 1-extensions at a workshop on rigidity held at BIRS (Banff, Canada) in 2012. Herman and Brigitte Servatius subsequently constructed an infinite family of counterexamples.

Lemma 13 ([24]). *There exist minimally rigid graphs H, K, L with $H \subset K$ and $H \subset L$ such that L is a 1-extension of K, e is an edge of H, $(K - e, p_0)$ is free for $H - e$ for some generic p_0, and $(L - e, p)$ is not free for $H - e$ for all generic p with $p|_K = p_0$.*

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Beauville Surfaces and Groups: A Survey

Gareth A. Jones

Abstract This is a survey of recent progress on Beauville surfaces, concentrating almost entirely on the group-theoretic and combinatorial problems associated with them. A Beauville surface $\mathscr{S}$ is a complex surface formed from two orientably regular hypermaps of genus at least 2 (viewed as compact Riemann surfaces and hence as algebraic curves), with the same automorphism group G acting freely on their product. The following questions are discussed: Which groups G (called Beauville groups) have this property? What can be said about the automorphism group and the fundamental group of $\mathscr{S}$? Beauville surfaces are defined (as algebraic varieties) over the field $\overline{\mathbb{Q}}$ of algebraic numbers, so how does the absolute Galois group $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ act on them?

Keywords Beauville surface • Beauville group • Hypermap • Compact Riemann surface • Absolute Galois group

Subject Classifications: Primary 20B25, Secondary 05C10, 11G32, 14J25, 14J50, 51M20.

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1 Introduction

The objects now known as Beauville surfaces[1] were introduced by the algebraic geometer Arnaud Beauville in [5, p. 159]. A Beauville surface $\mathscr{S}$ is a complex surface of general type [5, 28], constructed from a pair of orientably regular hypermaps (regular dessins, in Grothendieck's terminology [29]) of genus at least 2, with the same automorphism group G. The basic idea is that $\mathscr{S}$ can be designed to have certain properties by appropriate choices of G and its actions on the hypermaps. Since 2000, the geometric properties of Beauville surfaces, such as their rigidity (discussed in Sect. 9) have been intensively studied by Bauer, Catanese, Grunewald and others (see [3, 4, 9] for instance). More recently, group-theorists such as Guralnick, Lubotzky, Magaard, Malle and others have been interested in determining which groups G (known as Beauville groups) can be used in this construction. The rigidity properties of Beauville surfaces have been used by the author [34] to determine the structure of the automorphism group of a Beauville surface, and by González-Diez, Torres-Teigell and the author [25, 26] to extend an example of Serre [50], constructing arbitrarily large orbits of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ consisting of mutually non-homeomorphic algebraic varieties. This survey will describe some of these discoveries, and in addition, will suggest that there are interesting combinatorial questions to be investigated, including connections between Beauville surfaces and polytopes.

The paper is organised as follows. Section 2 explains how Belyĭ's Theorem gives a link between curves and hypermaps, used in Sect. 3 to give two equivalent definitions of a Beauville surface. These are translated into purely group-theoretic terms in Sect. 5, after a brief discussion of possible links with polytopes in Sect. 4. Various classes of Beauville groups are described in Sects. 6–8. The fundamental groups and automorphism groups of Beauville surfaces are described in Sects. 9 and 10, and the absolute Galois group and its action on Beauville surfaces are discussed in Sects. 10 and 11.

2 Curves and Hypermaps

Since Beauville surfaces are constructed from pairs of hypermaps on algebraic curves, this section will briefly summarise the connection between curves and hypermaps.

Compact Riemann surfaces are the same as algebraic curves (smooth, projective, defined over $\mathbb{C}$). This fact, first discovered by Riemann, is now expressed as an

[1]Here, as is customary in algebraic geometry, a 'surface' is an algebraic variety which is 2-dimensional over the field of coefficients; in this case, that field is $\mathbb{C}$ so these surfaces have dimension 4 as real manifolds. Rather confusingly, a complex algebraic curve, 1-dimensional over $\mathbb{C}$, can be regarded as a Riemann surface, where 'surface' now indicates 2-dimensionality over $\mathbb{R}$!

equivalence of categories: see [21, 48] for details. It is particularly interesting to know which compact Riemann surfaces are defined (as algebraic varieties) over various subfields of $\mathbb{C}$. Belyĭ's Theorem answers this question for the field $\overline{\mathbb{Q}}$ of algebraic numbers, by showing that the following conditions on a compact Riemann surface (or algebraic curve) $\mathscr{C}$ are equivalent:

(a) $\mathscr{C}$ is defined over $\overline{\mathbb{Q}}$;
(b) There is a meromorphic function $\beta : \mathscr{C} \to \mathbb{P}^1(\mathbb{C})$ branched over at most three points;
(c) $\mathscr{C}$ is uniformised by a subgroup K of finite index in a triangle group Δ;
(d) The complex structure on $\mathscr{C}$ is obtained, in a canonical way, from a hypermap $\mathscr{H}$ on $\mathscr{C}$.

A curve with these properties is called a *Belyĭ curve*. In fact, Belyĭ [6] gave an ingenious proof that (a) implies (b), and a two-line argument, referring to Weil's Rigidity Theorem [59], for the converse; full details (which are rather intricate) were later provided by Wolfart [61] and Köck [42]. Conditions (c) and (d) are straightforward reinterpretations of (b), due to Grothendieck, Wolfart and others (see [21, 37, 62], for instance). Belyĭ's Theorem has been extended to complex surfaces by González-Diez [22]. He and Girondo have written a very readable account of Belyĭ's Theorem and related matters in [21].

In (b), $\mathbb{P}^1(\mathbb{C})$ is the complex projective line (or Riemann sphere) $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; the three ramification points can be assumed, by applying a Möbius transformation, to be 0, 1 and ∞; a function β with these properties is called a *Belyĭ function*.

In (c), uniformisation means that $\mathscr{C} \cong \mathscr{U}/K$ where $\mathscr{U}$ is one of the three simply connected Riemann surfaces, namely $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or the hyperbolic plane $\mathbb{H}$, and K is a subgroup of a triangle group Δ acting as a group of automorphisms of $\mathscr{U}$. The inclusion $K \to \Delta$ induces a covering $\gamma : \mathscr{U}/K \to \mathscr{U}/\Delta$ corresponding to β:

$$\begin{array}{ccc} \mathscr{U}/K & \cong & \mathscr{C} \\ \gamma \downarrow & & \downarrow \beta \\ \mathscr{U}/\Delta & \cong & \mathbb{P}^1(\mathbb{C}) \end{array}$$

The degree (number of sheets) of this covering is equal to the index of K in Δ. We will be mainly interested in the case where $\mathscr{C}$ has genus at least 2, so that $\mathscr{U} = \mathbb{H}$.

In (d), a hypermap $\mathscr{H}$ on a curve $\mathscr{C}$ can be represented in several ways. Perhaps the most natural way is as a tripartite triangular map $\mathscr{T}$. This consists of a tripartite graph embedded in $\mathscr{C}$ with triangular faces; the three colour classes of vertices represent the hypervertices, hyperedges and hyperfaces of $\mathscr{H}$, and the edges correspond to incidences between them. This map can be constructed as the inverse image under β of the trivial triangulation of $\mathbb{P}^1(\mathbb{C})$; this has vertices at 0, 1 and ∞, joined by three edges along $\mathbb{R}$, and two triangular faces (the upper and lower half planes), so that its edges and faces lift to $\mathscr{C}$ without branching, which occurs only at the vertices. Thus $\mathscr{T}$ has triangular faces, and its vertices can be 3-coloured as they lie over 0, 1 or ∞.

A more economical and frequently-used representation of $\mathscr{H}$ is as a bipartite map $\mathscr{B}$ on $\mathscr{C}$, called the *Walsh map* of $\mathscr{H}$ [58]. This can be formed from $\mathscr{T}$ by deleting the vertices over ∞ and their incident edges; topologically, no information is lost since one can retrieve $\mathscr{T}$ (up to homeomorphisms fixing the graph) by stellating $\mathscr{B}$, placing a vertex in each face of $\mathscr{B}$, joined by mutually disjoint edges to the incident vertices. Equivalently, $\mathscr{B}$ is the inverse image under β of the trivial bipartite map on $\mathbb{P}^1(\mathbb{C})$; this consists of two vertices at 0 and 1, joined by an edge along the unit interval, and one face. Both $\mathscr{T}$ and $\mathscr{B}$ can also be formed as the quotients by K of Δ-invariant maps of the same type on the universal covering space $\mathscr{U}$ of $\mathscr{C}$ (see [38]).

The most symmetric Belyǐ curves $\mathscr{C}$ (and the only ones we will consider here) are the *quasiplatonic curves*, those which have a Belyǐ function β which is a regular covering, that is, there is a group G of automorphisms of $\mathscr{C}$ inducing the covering $\beta : \mathscr{C} \to \mathscr{C}/G \cong \mathbb{P}^1(\mathbb{C})$. This is equivalent to $\mathscr{C}$ being uniformised by a torsion-free normal subgroup K of finite index in a triangle group Δ, with $\Delta/K \cong G$; then K is a surface group, isomorphic to the fundamental group $\pi_1\mathscr{C}$ of $\mathscr{C}$. This is also equivalent to the hypermap $\mathscr{H}$ in (d) being (orientably) regular, with orientation-preserving automorphism group $\mathrm{Aut}^+\mathscr{H} \cong G$; this means that G is a group of orientation- and colour-preserving automorphisms of $\mathscr{T}$, with two orbits (necessarily regular) on the faces of $\mathscr{T}$, or equivalently one regular orbit on the edges of $\mathscr{B}$.

Example 1. Let $\mathscr{C}$ be the Fermat curve

$$\mathscr{F}_n = \{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) \mid x^n + y^n + z^n = 0\}$$

of degree n. This is a compact Riemann surface, visibly defined over $\mathbb{Q}$, and hence over $\overline{\mathbb{Q}}$. The meromorphic function

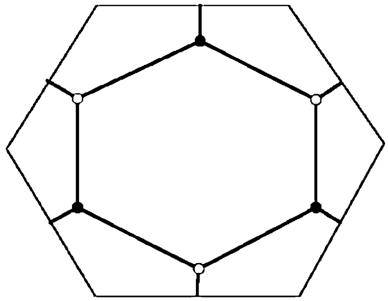
$$\beta : [x, y, z] \mapsto -\left(\frac{x}{z}\right)^n$$

is an n^2-sheeted covering $\mathscr{C} \to \mathbb{P}^1(\mathbb{C})$, branched where $xyz = 0$, that is, over 0, 1 and ∞, each of which lifts to n points on $\mathscr{C}$. The triangulation $\mathscr{T}$ therefore has $3n$ vertices, $3n^2$ edges and $2n^2$ faces, so $\mathscr{C}$ has Euler characteristic $n(3-n)$ and hence genus $(n-1)(n-2)/2$; the underlying graph of $\mathscr{T}$ is, in fact, the complete tripartite graph $K_{n,n,n}$, and this is a minimum genus embedding of that graph. Similarly, $\mathscr{B}$ is an embedding of the complete bipartite graph $K_{n,n}$: see Fig. 1 for the case $n = 3$, with opposite sides of the outer hexagon identified to form a torus.

As a Riemann surface, $\mathscr{C}$ is uniformised by the commutator subgroup $K = \Delta'$ of the triangle group $\Delta = \Delta(n, n, n)$, acting on $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or $\mathbb{H}$ as $n < 3$, $n = 3$ or $n > 3$. This is a normal subgroup of Δ, with $\Delta/K \cong G := \mathbb{Z}_n \oplus \mathbb{Z}_n$ acting as a group of automorphisms

$$(j, k) : [x, y, z] \mapsto [\zeta_n^j x, \zeta_n^k y, z] \quad (j, k \in \mathbb{Z}_n)$$

Fig. 1 $K_{3,3}$ embedded in the Fermat curve $\mathscr{F}_3$



of $\mathscr{C}$, where $\zeta_n = \exp(2\pi i/n)$, and inducing the regular covering β. Thus $\mathscr{C}$ is a quasiplatonic curve, and $\mathscr{H}$ is an orientably regular hypermap with $\mathrm{Aut}^+\mathscr{H} \cong G$. (In fact, if $n > 3$ then the full automorphism group of $\mathscr{C}$ is a semidirect product of G by S_3, permuting the coordinates x, y and z; this corresponds to K being normal in the maximal triangle group $\Delta(2, 3, 2n)$, which contains $\Delta(n, n, n)$ as a normal subgroup with quotient group S_3.)

3 Definition of a Beauville Surface

We say that $\mathscr{S}$ is a *Beauville surface* (of unmixed type) if

1. $\mathscr{S} = (\mathscr{C}_1 \times \mathscr{C}_2)/G$ where each $\mathscr{C}_i$ is a complex projective algebraic curve of genus $g_i > 1$, and G is a finite group acting faithfully as a group of automorphisms of each $\mathscr{C}_i$, so that it acts freely (i.e. without fixed points) on $\mathscr{C}_1 \times \mathscr{C}_2$;
2. For each i, $\mathscr{C}_i/G$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$, with the induced projection $\beta_i : \mathscr{C}_i \to \mathbb{P}^1(\mathbb{C})$ branched over three points.

Here we will ignore the technically more difficult case of Beauville surfaces of mixed type [4, §7], where half the elements of G transpose two isomorphic factors $\mathscr{C}_i$. The product $\mathscr{C}_1 \times \mathscr{C}_2$ is a complex manifold (in fact, an algebraic variety) of dimension 2, and hence so is the quotient $\mathscr{S}$ since G acts freely on $\mathscr{C}_1 \times \mathscr{C}_2$. In combinatorial terms, the above conditions can be restated as follows:

1. $\mathscr{S} = (\mathscr{C}_1 \times \mathscr{C}_2)/G$ where each $\mathscr{C}_i$ is a quasiplatonic curve of genus $g_i > 1$, carrying an orientably regular hypermap $\mathscr{H}_i$ with $\mathrm{Aut}^+\mathscr{H}_i \cong G$;
2. The induced action of G on $\mathscr{H}_1 \times \mathscr{H}_2$ is fixed-point-free.

Thus a Beauville surface is formed from a pair of orientably regular hypermaps of hyperbolic type, with the same automorphism group acting freely on their product.

4 Combinatorial Structures

If $\mathscr{S}$ is a Beauville surface $(\mathscr{C}_1 \times \mathscr{C}_2)/G$, then each curve $\mathscr{C}_i$ carries an orientably regular hypermap (or regular dessin) $\mathscr{H}_i$ with $\mathrm{Aut}^+\mathscr{H}_i \cong G$. As explained in Sect. 2, these hypermaps can be represented combinatorially in several ways, as triangulations $\mathscr{T}_i$ or bipartite maps $\mathscr{B}_i$, for instance. These combinatorial structures on the curves $\mathscr{C}_i$ induce further combinatorial structures on $\mathscr{C}_1 \times \mathscr{C}_2$, and hence, by means of the smooth covering $\mathscr{C}_1 \times \mathscr{C}_2 \to \mathscr{S} = (\mathscr{C}_1 \times \mathscr{C}_2)/G$, on the Beauville surface $\mathscr{S}$.

For example $\mathscr{T}_1 \times \mathscr{T}_2$ can be regarded as a 4-dimensional CW-complex structure on $\mathscr{C}_1 \times \mathscr{C}_2$: each 2-cell is either a triangle (the product of a vertex on one curve and a face on the other) or a square (the product of two edges), each 3-cell is a triangular prism (the product of a triangular face and an edge), and each 4-cell is the product of two triangles (a 3,3-duoprism). In addition, the 3-colourings of the vertices of the triangulations $\mathscr{T}_i$ induce a 9-colouring of the vertices of $\mathscr{T}_1 \times \mathscr{T}_2$. This structure, including its vertex-colouring, is invariant under the natural action of $G \times G$ on $\mathscr{C}_1 \times \mathscr{C}_2$. The free action of the diagonal subgroup means that the quotient surface $\mathscr{S}$ inherits the structure of a 4-dimensional CW-complex $(\mathscr{T}_1 \times \mathscr{T}_2)/G$, with the number of k-cells divided by $|G|$ for each dimension $k = 0, \ldots, 4$. This structure on $\mathscr{S}$ is preserved by the automorphisms of $\mathscr{S}$, which are described in Sect. 10. Similarly, the bipartite maps $\mathscr{B}_i$ induce a CW-complex $(\mathscr{B}_1 \times \mathscr{B}_2)/G$ on $\mathscr{S}$, with each k-cell a union of k-cells of $(\mathscr{T}_1 \times \mathscr{T}_2)/G$.

Although Beauville surfaces have been studied quite extensively from the points of view of algebraic geometry and group theory, this aspect of the theory seems not to have been investigated so far. It should be noted that although the curves $\mathscr{C}_i$ carry regular dessins, these maps need not be regular when viewed as 3-polytopes: they could be chiral, with automorphism groups having two orbits on flags: this happens for the Beauville surfaces in Example 3 when $f < e$ (see Sect. 7 and [36]), and also for those based on Ree groups and Suzuki groups in [17] (see Sect. 8). Moreover, although $\mathscr{C}_1 \times \mathscr{C}_2$ will have many automorphisms, as a surface or a polytope, taking a quotient by G may destroy most, and possibly all, of this symmetry: see Sect. 10, where automorphisms are discussed.

5 Beauville Groups

We call a finite group G a *Beauville group* if there is a Beauville surface $\mathscr{S} = (\mathscr{C}_1 \times \mathscr{C}_2)/G$. Here we translate that definition into purely group-theoretic terms.

A group G is a quotient of a triangle group

$$\Delta_i = \Delta(l_i, m_i, n_i) = \langle A_i, B_i, C_i \mid A_i^{l_i} = B_i^{m_i} = C_i^{n_i} = A_i B_i C_i = 1 \rangle$$

if and only if it has a presentation

$$G = \langle a_i, b_i, c_i \mid a_i^{l_i} = b_i^{m_i} = c_i^{n_i} = a_i b_i c_i = 1, \ldots \rangle, \quad (1)$$

with each a_i, b_i, c_i the image of A_i, B_i or C_i. The torsion elements of Δ_i are the conjugates of the powers of the generators A_i, B_i and C_i, so the kernel K_i of the natural epimorphism $\Delta_i \to G$ is torsion-free if and only if the generators a_i, b_i and c_i have orders

$$|a_i| = l_i, \ |b_i| = m_i, \ |c_i| = n_i. \tag{2}$$

The triangle group Δ_i acts on $\mathbb{H}$ if and only if

$$\frac{1}{l_i} + \frac{1}{m_i} + \frac{1}{n_i} < 1, \tag{3}$$

in which case there is an induced action of G on the Riemann surface $\mathbb{H}/K_i$, which is compact (and thus an algebraic curve $\mathscr{C}_i$) if and only if G is finite. The elements of G with fixed points in $\mathscr{C}_i$ are the conjugates of the powers of the generators a_i, b_i and c_i, forming a subset

$$\Sigma_i = \Sigma_i(G) = \bigcup_{g \in G} (\langle a_i \rangle \cup \langle b_i \rangle \cup \langle c_i \rangle)^g$$

of G. Then G acts freely on the product $\mathscr{C}_1 \times \mathscr{C}_2$ of two such curves $\mathscr{C}_i$ $(i = 1, 2)$ if and only if no non-identity element of G has fixed points on both curves, that is,

$$\Sigma_1 \cap \Sigma_2 = \{1\}. \tag{4}$$

Thus conditions (1)–(4) are necessary and sufficient for a finite group G to be a Beauville group. When these conditions are satisfied, we call the pair of generating triples (a_i, b_i, c_i) a *Beauville structure* of *type* $(l_1, m_1, n_1; l_2, m_2, n_2)$ on G. Such a structure on G uniquely determines the curves $\mathscr{C}_i$, and hence the Beauville surface $\mathscr{S}$. This equivalence between surfaces and structures means that one can study many aspects of Beauville surfaces entirely within the theories of finite groups or of regular hypermaps.

6 Beauville's Example

The original examples of Beauville surfaces are constructed as follows:

Example 2. Let $\mathscr{C}_1 = \mathscr{C}_2$ be the Fermat curve $\mathscr{F}_n$ of degree n, described in Example 1. There is a faithful action $\rho_1 : G \to \operatorname{Aut} \mathscr{F}_n$ of the group $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ on $\mathscr{F}_n$, given by

$$(j, k) : [x, y, z] \mapsto [\zeta_n^j x, \zeta_n^k y, z]$$

for all $j, k \in \mathbb{Z}_n$. In this action of G, the elements with fixed points are the multiples of the generating triple $a_1 = (1, 0)$ fixing points $[0, y, z] \in \mathscr{F}_n$, $b_1 = (0, 1)$ fixing points $[x, 0, z] \in \mathscr{F}_n$, and $c_1 = (-1, -1)$ fixing points $[x, y, 0] \in \mathscr{F}_n$. Thus

$$\Sigma_1 = \{(j, k) \in G \mid j = 0, k = 0 \text{ or } j = k\}.$$

We need a second action of G on this curve. If α is an automorphism of G then (composing from right to left) $\rho_2 := \rho_1 \circ \alpha^{-1} : G \to \operatorname{Aut} \mathscr{F}_n$ is a faithful action of G on $\mathscr{F}_n$ with $\Sigma_2 = \alpha(\Sigma_1)$. If we define $\alpha : (j, k) \mapsto (4j + 2k, j + k)$, then simple number theory shows that this is an automorphism of G with $\Sigma_1 \cap \Sigma_2 = \{(0, 0)\}$ if and only if n is coprime to 6.

In fact, Beauville set the case $n = 5$ as an exercise in [5], and then invited the reader to generalise this construction. In 2000 Catanese [9] showed that these are the only abelian examples:

Theorem 1 (Catanese). *The only abelian Beauville groups are the groups* $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ *where* $n > 1$ *and* n *is coprime to* 6.

The proof depends on simple applications of the structure theorems for finite abelian groups. This result raises the question of how many Beauville surfaces are associated with the group $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$. Bauer, Catanese and Grunewald gave asymptotic estimates in [3], and Garion and Penegini gave upper and lower bounds in [19]. The following argument, due to González-Diez, Torres-Teigell and the author [24], gives an exact formula.

Without loss of generality, one can assume that the first generating triple (a_1, b_1, c_1) is as above. The second triple differs from it by an automorphism of G, i.e. a matrix $A \in GL_2(\mathbb{Z}_n)$. It is shown in both [19] and [24] that the set $\mathfrak{F}_n$ of matrices A inducing automorphisms of G satisfying $\Sigma_1 \cap \Sigma_2 = \{(0, 0)\}$ has cardinality

$$|\mathfrak{F}_n| = n^4 \prod_{p|n} \left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right)\left(1 - \frac{3}{p}\right)\left(1 - \frac{4}{p}\right), \tag{5}$$

where p ranges over the distinct primes dividing n. (Notice that this expression is 0 unless n is coprime to 6.) One can prove this by using basic linear algebra in the case where n is prime, then lifting to powers of that prime by Hensel's Lemma, and finally using the Chinese Remainder Theorem for general integers n.

Now two matrices $A, A' \in GL_2(\mathbb{Z}_n)$ give isomorphic Beauville surfaces if and only if $A' = PA^{\pm 1}Q$ where P and Q are elements of a certain subgroup of $GL_2(\mathbf{Z}_n)$ isomorphic to S_3, permuting the standard triple $\{a_1, b_1, c_1\}$. We thus have an action on $\mathfrak{F}_n$ by the wreath product $W = S_3 \wr S_2$, a semidirect product of $S_3 \times S_3$ by S_2: here the two direct factors S_3 correspond to the matrices P and Q, permuting the three vertex colours on each curve $\mathscr{C}_i$, and the complement S_2 corresponds to inverting A and transposing the curves. The number of non-isomorphic Beauville surfaces obtained is equal to the number of orbits of W on $\mathfrak{F}_n$, and this can be found by applying the Cauchy-Frobenius Counting Lemma (otherwise known as Burnside's Lemma). This states that the number of orbits of a finite group on a finite set is equal to the average number of points fixed by the elements of the group. In our case, inspection shows that most of the elements of W act without fixed points on $\mathfrak{F}_n$, giving the following result (see [24] for details):

Theorem 2. *Let $n = p_1^{e_1} \cdot \ldots \cdot p_s^{e_s}$ be a natural number coprime to 6, where $p_1, \ldots, p_k$ are distinct primes. Then the number of isomorphism classes of Beauville surfaces with Beauville group $\mathbb{Z}_n \oplus \mathbb{Z}_n$ is*

$$\Theta(n) = \frac{1}{72}\left(\Theta_1(n) + 4\prod_{i=1}^{s}\Theta_2(p_i^{e_i}) + 6\prod_{i=1}^{s}\Theta_3(p_i^{e_i}) + 12\prod_{i=1}^{s}\Theta_4(p_i^{e_i})\right), \qquad (6)$$

where $\Theta_1(n) = |\mathfrak{F}_n|$,

$$\Theta_2(p^e) := \begin{cases} p^{2e}\left(1-\frac{1}{p}\right)\left(1-\frac{4}{p}\right) & \textit{if } p \equiv 1 \bmod (3), \\ p^{2e}\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right) & \textit{if } p \equiv 2 \bmod (3), \end{cases}$$

$$\Theta_3(p^e) := p^{2e}(1-3/p)(1-5/p),$$

and

$$\Theta_4(p^e) := \begin{cases} 2 & \textit{if } p \equiv 1 \bmod (3), \\ 0 & \textit{if } p \equiv 2 \bmod (3). \end{cases}$$

Here 72 is the order of W, while $\Theta_1(n)$ is the number $|\mathfrak{F}_n|$ of fixed points of its identity element, given by (5), and the terms in (6) involving Θ_2, Θ_3 and Θ_4 are the contributions to the average from conjugacy classes in W containing 4, 6, and 12 elements of orders 3, 2 and 6.

For large n the sum in (6) is dominated by $\Theta_1(n)$, so we have

$$\Theta(n) \sim \frac{1}{72}\Theta_1(n) = \frac{n^4}{72}\prod_{p|n}\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)\left(1-\frac{3}{p}\right)\left(1-\frac{4}{p}\right)$$

as $n \to \infty$ with n coprime to 6. (Note that, despite appearances, $\Theta(n)/n^4$ is not bounded away from 0: if we take n to be the product of the first k primes $p > 3$, then

$$\prod_{p|n}\left(1-\frac{1}{p}\right) \to 0$$

as $k \to \infty$ (see [35, Exercise 9.3]), and hence $\Theta(n)/n^4 \to 0$ for such integers n.)

7 Beauville p-Groups

It is natural to try to extend the classification of Beauville groups from abelian groups to wider classes, such as nilpotent groups. A finite group is nilpotent if and only if it is a direct product of its Sylow subgroups, and a direct product of Beauville groups of mutually coprime orders is clearly a Beauville group, so the main objective in such an extension is to study Beauville structures on p-groups for the various primes p. Barker, Boston, Peyerimhoff and Vdovina [2] have obtained Beauville 2-groups as quotients of the fundamental group of a certain simplicial complex, while Barker, Boston and Fairbairn [1] have constructed many examples for all p. For instance, they show that in addition to the abelian p-groups $C_{p^e} \times C_{p^e}$ with $p \geq 5$, given by Example 2, there is at least one nonabelian Beauville group of every prime-power order p^k provided $p \geq 7$ and $k \geq 3$. (For primes $p < 7$, the smallest nonabelian Beauville p-groups have orders 2^7, 3^5 and 5^4.)

Example 3. For each prime $p \geq 5$, let

$$G = G(e, f) = \langle x, y \mid x^{p^e} = y^{p^e} = 1,\ y^x = y^{1+p^f} \rangle$$

where $1 \leq f \leq e$. Thus G is a semidirect product of two cyclic groups $\langle x \rangle$ and $\langle y \rangle$ of order p^e, so G has order p^{2e}; it is abelian if and only if $f = e$. The Frattini subgroup of G is the normal subgroup $\Phi = \langle x^p, y^p \rangle$, with $G/\Phi \cong C_p \times C_p$. The Beauville structures of type $(p, p, p; p, p, p)$ on $C_p \times C_p$ constructed in Example 2 lift back to Beauville structures of type $(p^e, p^e, p^e; p^e, p^e, p^e)$ on G. These groups appeared in connection with the classification of orientably regular embeddings of complete bipartite graphs in [33,36], and their connections with dessins were studied in [40].

This example deals with even powers of primes $p \geq 5$. Barker, Boston and Fairbairn [1] give a similar construction for odd powers.

Example 4. Let G be a 2-generator finite group of prime exponent $p \geq 5$. As in Example 3, any Beauville structure on the quotient group $G/\Phi \cong C_p \times C_p$ lifts to a Beauville structure on G, this time of type $(p, p, p; p, p, p)$. By Kostrikin's solution [43] of the restricted Burnside problem for prime exponents, for each p there is a largest such 2-generator finite group G, denoted by $R(2, p)$, and all others are quotients of it. These groups $R(2, p)$ are in fact very large: for instance, Havas, Wall and Wamsley [31] have shown that $|R(2, 5)| = 5^{34}$, while O'Brien and Vaughan-Lee [47] have shown that $|R(2, 7)| = 7^{20416}$. For a detailed survey of the restricted Burnside problem, see [56].

Barker, Boston and Fairbairn show in [1] that the proportion of 2-generator groups of order p^5 which are Beauville groups tends to 1 as $p \to \infty$, but that this is not the case for groups of order p^6. The question raised by Fuertes, González-Diez and Jaikin-Zapirain in [16], namely whether, in any sense, most 2-generator p-groups are Beauville groups, remains open.

8 Simple Beauville Groups

It is easy to see that the alternating group A_5 is not a Beauville group. For instance, its non-identity elements have orders 2, 3 or 5. If $l, m, n \in \{2, 3\}$ then the triangle group $\Delta(l, m, n)$ is solvable, whereas A_5 is not, so any generating triple for A_5 must contain an element of order 5. Since the Sylow 5-subgroups of A_5 are cyclic, any two elements of order 5 are conjugate to powers of each other, so no two generating triples can satisfy the Beauville condition (4).

In 2005, Bauer, Catanese and Grunewald [3] made the following conjecture:

Every non-abelian finite simple group except A_5 is a Beauville group.

As evidence for this, they showed that A_n is a Beauville group for all sufficiently large n, as are the groups $PSL_2(p)$ for all primes $p > 5$ (note that $PSL_2(5) \cong A_5$) and the Suzuki groups $Sz(2^e)$ for all odd primes e. Fuertes and González-Diez [14] showed that A_n is a Beauville group for all $n \geq 6$. In [17], Fuertes and the author showed that various other simple groups are Beauville groups, namely $PSL_2(q)$ for all prime powers $q > 5$, and the Suzuki groups $Sz(2^e)$ and the Ree groups $R(3^e)$ for all odd $e \geq 3$. They also showed that certain quasisimple groups (perfect central extensions of simple groups) are Beauville groups, namely the groups $SL_2(q)$ for $q > 5$, again extending a result for prime q in [3].

Around the same time, Garion and Penegini [19] obtained the above result for $PSL_2(q)$, using results of Macbeath [45] on generating triples for this group. They also used probabilistic methods to show that $Sz(2^e)$ and $R(3^e)$ are Beauville groups for all sufficiently large odd e, with similar results for several other families of simple groups, including $PSL_3(q)$ and the unitary groups $U_3(q)$.

Soon, specialists in the study of finite simple groups became interested in this problem: the classification of such groups, announced around 30 years ago but not completely proved until 2004, allows conjectures such as this to be obtained by inspection. Several major advances were announced in 2010. Firstly, Garion, Larsen, and Lubotzky [18] used probabilistic methods to show that the conjecture is true with at most finitely many exceptions. Soon afterwards, Guralnick and Malle [30] gave a complete proof of the conjecture, while Fairbairn, Magaard and Parker [13] extended it further to all finite quasisimple groups except A_5 and its central cover $SL_2(5)$. In all three cases, the proofs require deep knowledge of the structure of finite simple groups, especially those of Lie type; see [8, 60] for detailed accounts of these groups, and [11] for a concise (but hardly pocket-sized) summary.

9 Fundamental Groups and Rigidity

Just as each Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ is constructed from a group G, it gives rise to two more groups: as a connected topological space it has a fundamental group $\pi_1\mathcal{S}$, and as an algebraic variety it has an automorphism group $\mathrm{Aut}\,\mathcal{S}$.

The fundamental group of $\mathcal{S}$ is easily described. We have a pair of triangle groups Δ_i, each with a normal subgroup $K_i \cong \pi_1\mathcal{C}_i$ such that $\mathcal{C}_i \cong \mathbb{H}/K_i$ and $\Delta_i/K_i \cong G$. Each Δ_i acts on $\mathbb{H}$, so there is an induced action of $\Delta_1 \times \Delta_2$ on the simply connected space $\mathbb{H} \times \mathbb{H}$. Let Π denote the inverse image of the diagonal subgroup in the natural epimorphism $\Delta_1 \times \Delta_2 \to G \times G$, that is, the subgroup of $\Delta_1 \times \Delta_2$ consisting of those pairs which map onto the same element of G. Beauville condition (4) implies that Π acts freely on $\mathbb{H} \times \mathbb{H}$, with $(\mathbb{H} \times \mathbb{H})/\Pi \cong \mathcal{S}$, so $\pi_1\mathcal{S}$ can be identified with Π. Thus $\pi_1\mathcal{S}$ has a normal subgroup $K_1 \times K_2 \cong \pi_1\mathcal{C}_1 \times \pi_1\mathcal{C}_2 \cong \pi_1(\mathcal{C}_1 \times \mathcal{C}_2)$, with quotient group G, corresponding to the regular covering $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{S}$ with covering group G. It also has a normal subgroup K_1, with quotient group Δ_2, corresponding to the regular covering $\mathcal{C}_1 \times \mathbb{H} \to \mathcal{S}$ with covering group Δ_2, and similarly for the normal subgroup K_2.

This leads to a property of Beauville surfaces called rigidity [9], meaning essentially that the topology determines the geometric structure. In a hyperbolic triangle group, the centraliser of each non-identity element is cyclic. Thus the centraliser in $\Pi = \pi_1\mathcal{S}$ of any element of K_i contains a surface group (namely K_{3-i}), and is therefore nonabelian, whereas any other element of Π has an abelian centraliser. It follows that if $\mathcal{S}' = (\mathcal{C}_1' \times \mathcal{C}_2')/G'$ is another Beauville surface, then any isomorphism $\pi_1\mathcal{S} \to \pi_1\mathcal{S}'$ induces isomorphisms $\Delta_i \to \Delta_i'$ between the corresponding triangle groups (possibly after transposing factors), and an isomorphism $G \to G'$ of their Beauville groups. Now any isomorphism of cocompact hyperbolic triangle groups is induced by an isometry of $\mathbb{H}$, since the corresponding triangles are isometric. It follows that homeomorphic Beauville surfaces are in fact isometric, and that $\mathcal{S}$ is uniquely determined, up to complex conjugation of either or both of the curves $\mathcal{C}_i$, by its fundamental group. Such rigidity properties help to explain why Beauville surfaces are so interesting to algebraic geometers. (The above argument, taken from a more detailed proof given by González-Diez and Torres-Teigell in [26], is a group-theoretic analogue of the arguments based on algebraic geometry given by Catanese in [9] and by Bauer, Catanese and Grunewald in [4].)

10 Automorphism Groups of Beauville Surfaces

This section summarises results of the author on automorphism groups of Beauville surfaces in [34]; some of these results have been obtained independently by Fuertes and González-Diez in [15], and have been extended to mixed Beauville surfaces by González-Diez and Torres-Teigell in [27].

The rigidity results outlined in the preceding section show that any automorphism of a Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ lifts to an automorphism of $\mathcal{C}_1 \times \mathcal{C}_2$, and this either preserves or transposes the curves $\mathcal{C}_i$; such automorphisms of $\mathcal{S}$ are called *direct* or *indirect* respectively. First we consider the group $\mathrm{Aut}^0\mathcal{S}$ of direct automorphisms of $\mathcal{S}$, a subgroup of index at most 2 in $\mathrm{Aut}\,\mathcal{S}$.

Let $A_i := \mathrm{Aut}\,\mathscr{C}_i$. There is a natural action of $A_1 \times A_2$ on $\mathscr{C}_1 \times \mathscr{C}_2$, and we can regard $\mathscr{S}$ as the quotient of $\mathscr{C}_1 \times \mathscr{C}_2$ by the diagonal subgroup D of the subgroup $G \times G$ of $A_1 \times A_2$. A simple calculation shows that an element $(\alpha_1, \alpha_2) \in A_1 \times A_2$, acting on $\mathscr{C}_1 \times \mathscr{C}_2$, permutes the orbits of D, and hence induces an automorphism of $\mathscr{S}$, if and only if

1. Each α_i is in the normaliser $N_i := N_{A_i}(G)$ of G in A_i, and
2. α_1 and α_2, acting by conjugation, induce the same automorphism of G.

Such elements (α_1, α_2) form a subgroup N of $N_1 \times N_2$, the inverse image of the diagonal subgroup of $\mathrm{Aut}\,G \times \mathrm{Aut}\,G$ under the natural homomorphism $N_1 \times N_2 \to \mathrm{Aut}\,G \times \mathrm{Aut}\,G$. The kernel of this action of N is D, so the group $A^0 = \mathrm{Aut}^0\mathscr{S}$ of direct automorphisms of $\mathscr{S}$ is isomorphic to N/D.

In particular, if each $\alpha_i \in G$ then condition (1) is satisfied, and (2) is satisfied if and only if $\alpha_1\alpha_2^{-1}$ is in the centre $Z := Z(G)$ of G. Thus N contains a normal subgroup

$$M = N \cap (G \times G) = \{(\alpha_1, \alpha_2) \in G \times G \mid \alpha_1\alpha_2^{-1} \in Z\} \cong D \times Z,$$

inducing on $\mathscr{S}$ a normal subgroup $I := \mathrm{Inn}\,\mathscr{S} \cong M/D \cong Z$ of A^0; the elements of I are called the *inner automorphisms* of $\mathscr{S}$, induced by compatible pairs of elements of G acting on the curves $\mathscr{C}_i$. Since I is isomorphic to the centre of G, it is finite and abelian. The quotient group $A^0/I \cong N/M$ is called the *direct outer automorphism group* $\mathrm{Out}^0\mathscr{S}$ of $\mathscr{S}$.

In many cases $G = N_i$ for each i (for instance if $G = A_i$), so that $M = N$ and hence $A^0 = I \cong Z$. If $G < N_i$ for some i, then Δ_i is a proper normal subgroup of a Fuchsian group $\tilde{\Delta}_i$, with $\tilde{\Delta}_i/K_i \cong N_i$. Singerman [52] has shown that any Fuchsian group containing a triangle group must also be a triangle group, and that any proper normal inclusion between them must be (up to permutations of the periods) of one of the forms

(a) $\Delta(s,s,t) \lhd \Delta(2,s,2t)$, (b) $\Delta(t,t,t) \lhd \Delta(3,3,t)$, (c) $\Delta(t,t,t) \lhd \Delta(2,3,2t)$,

with the quotient group isomorphic to C_2, C_3 or S_3 respectively. In all three cases, at least two of the three periods of Δ_i are equal, so we have:

Proposition 1. *If a Beauville structure on a group G has type $(l_1, m_1, n_1; l_2, m_2, n_2)$, and for each i the periods l_i, m_i and n_i are mutually distinct, then the direct automorphism group $\mathrm{Aut}^0\mathscr{S}$ of the corresponding Beauville surface $\mathscr{S}$ is isomorphic to the centre of G.* □

If there are repetitions among either or both of the triples l_i, m_i, n_i, then $\mathscr{S}$ may have direct outer automorphisms, arising from proper normal inclusions $\Delta_i \lhd \tilde{\Delta}_i$. In this case Singerman's results, stated above, allow us to deduce the following:

Proposition 2. *The direct automorphism group* $\mathrm{Aut}^0\mathcal{S}$ *of a Beauville surface* $\mathcal{S}$ *has a normal subgroup* $\mathrm{Inn}\,\mathcal{S} \cong Z(G)$ *with* $\mathrm{Aut}^0\mathcal{S}/\mathrm{Inn}\,\mathcal{S}$ *isomorphic to a subgroup of* $S_3 \times S_3$. *In particular,* $\mathrm{Aut}^0\mathcal{S}$ *is a finite solvable group, of derived length at most* 3. □

The direct factors S_3 can be regarded as permuting the fibres of β_i over 0, 1 and ∞.

Example 5. Let $\mathcal{S} = (\mathcal{F}_n \times \mathcal{F}_n)/G$ as in Example 3, with the Beauville group $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$. Since G is abelian we have $\mathrm{Inn}\,\mathcal{S} \cong G$. Then $\mathrm{Out}^0\mathcal{S} \cong C_3$ or C_1 as the automorphism of G induced by the 3-cycle (a_1, b_1, c_1) is or is not the same as that induced by (a_2, b_2, c_2). Thus $\mathrm{Aut}^0\mathcal{S}$ is isomorphic to an extension of G by C_3, or to G, depending on the choice of the matrix $A \in GL_2(\mathbb{Z}_n) = \mathrm{Aut}\,G$ linking the two representations ρ_i of G on $\mathcal{F}_n$.

Any indirect automorphism of $\mathcal{S}$ is induced by an automorphism of $\mathcal{C}_1 \times \mathcal{C}_2$ of the form

$$(p_1, p_2) \mapsto (p_2\phi_2, p_1\phi_1),$$

where $\phi_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $\phi_2 : \mathcal{C}_2 \to \mathcal{C}_1$ are isomorphisms of curves. It is not hard to prove:

Proposition 3. *A Beauville surface* $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ *has an indirect automorphism if and only if* $\mathcal{C}_1 \cong \mathcal{C}_2$ *and* G *has an automorphism* ζ *transposing the equivalence classes of its representations on* $\mathcal{C}_1$ *and* $\mathcal{C}_2$. □

Here two representations are defined to be equivalent if each is obtained from the other by composition with an isomorphism of curves.

Corollary 1. *If a Beauville surface* $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ *has an indirect automorphism, then the corresponding Beauville structure on* G *must consist of two triples of equivalent types.* □

Here two types are defined to be equivalent if each is a permutation of the other. The analogue of Proposition 10.3 is the following:

Proposition 4. *The automorphism group* $\mathrm{Aut}\,\mathcal{S}$ *of a Beauville surface* $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ *has a normal subgroup* $\mathrm{Inn}\,\mathcal{S} \cong Z(G)$ *with* $\mathrm{Aut}\,\mathcal{S}/\mathrm{Inn}\,\mathcal{S}$ *isomorphic to a subgroup of* $S_3 \wr S_2$. *In particular,* $\mathrm{Aut}\,\mathcal{S}$ *is a finite solvable group, of derived length at most* 4. □

By the above results, many Beauville surfaces (for instance, most of those with simple Beauville groups) have only the identity automorphism.

There are no restrictions on the centre of a Beauville group, and hence on $\mathrm{Inn}\,\mathcal{S}$, other than the obvious ones that it should be finite and abelian:

Theorem 3. *Given any finite abelian group* H, *there is a Beauville group* G *with centre* $Z(G) \cong H$. □

It immediately follows that there is a Beauville surface $\mathcal{S}$ with $\mathrm{Inn}\,\mathcal{S} \cong H$; this remains true, even if one requires $\mathrm{Out}\,\mathcal{S}$ to be as large (isomorphic to $S_3 \wr S_2$) or as small (the trivial group) as possible. A key ingredient of the proof adapts a method used by Conder [10] for constructing Hurwitz groups with large centres: we represent H as a direct product of cyclic groups C_{m_i}, each isomorphic to the centre of some group $SL_{n_i}(q_i)$, where $m_i = \gcd(n_i, q_i - 1)$, so that the direct product G of these groups $SL_{n_i}(q_i)$ has centre $Z(G) \cong H$. Results of Lucchini [44] on generators of special linear groups allow one to choose the groups $SL_{n_i}(q_i)$, and hence also their product G, to be quotients of $\Delta(2, 3, p)$ and hence of $\Delta(p, p, p)$, for two different primes $p = p_1, p_2$, thus giving a Beauville structure of type $(p_1, p_1, p_1; p_2, p_2, p_2)$ on G. Modifications of this construction provide some control over the outer automorphism group of the resulting Beauville surface $\mathcal{S}$. For details, see [34].

11 The Absolute Galois Group

Belyĭ's Theorem [6] implies that the curves $\mathcal{C}_i$ used in constructing a Beauville surface $\mathcal{S}$ are defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers, and it follows that $\mathcal{S}$ is also defined over this field. The *absolute Galois group* is the automorphism group

$$\Gamma = \mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$$

of this field. Since $\overline{\mathbb{Q}}$ is the direct limit (i.e. union) of the Galois (finite normal) extensions K of $\mathbb{Q}$, it follows that Γ is the inverse limit

$$\Gamma = \varprojlim \mathrm{Gal}\,K/\mathbb{Q}$$

of the Galois groups of these fields; the homomorphisms in this inverse system are the restriction mappings

$$\mathrm{Gal}\,L/\mathbb{Q} \to \mathrm{Gal}\,K/\mathbb{Q}$$

induced by inclusions $K \subseteq L$ between such fields. Since these are all epimorphisms between finite groups, Γ is in fact a profinite group, that is, a projective limit of finite groups: it can be identified with the subgroup of the cartesian product Π of all such groups $\mathrm{Gal}\,K/\mathbb{Q}$ consisting of the elements whose coordinates are compatible with the restriction mappings.

Giving the finite groups $\mathrm{Gal}\,K/\mathbb{Q}$ the discrete topology makes Π a topological group, compact by Tychonoff's Theorem, so Γ, as a closed subgroup of Π, is also a compact topological group (in fact, homeomorphic to a Cantor set). The Galois correspondence is then between the subfields of $\overline{\mathbb{Q}}$ and the closed subgroups of Γ. Understanding Γ is therefore critical to an understanding of algebraic number theory. There are many important open problems associated with this group. For instance the Inverse Galois Problem, Hilbert's question whether every finite group is isomorphic to a Galois group over $\mathbb{Q}$, is equivalent to asking whether every finite group is the quotient of Γ by some closed normal subgroup. Books by Malle and Matzat [46], Serre [51] and Völklein [57] describe progress on this.

In the mid-1980s, Grothendieck [29] proposed that one should study Γ through its actions on various geometric and combinatorial objects, the simplest of which are oriented hypermaps, or *dessins d'enfants* (children's drawings) as he called them, viewed as unbranched finite coverings of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. By Belyĭ's Theorem these are defined over $\overline{\mathbb{Q}}$, and there is a natural action of Γ on them, through its action on the coefficients of the polynomials and rational functions defining them. This action preserves the obvious numerical parameters of a dessin, such as the numbers and valencies of its vertices and faces, and hence its genus [39]. However, using elementary properties of the modular j-function it is easy to show that Γ acts faithfully on dessins of genus 1 (those on elliptic curves). Less obviously, Schneps [49] has shown that it acts faithfully on plane trees, while Girondo and González-Diez [20] have shown that it is faithful on dessins of each genus $g \geq 2$. González-Diez and Jaikin-Zapirain [23] have recently shown that Γ acts faithfully on regular dessins, i.e. orientably regular hypermaps.

Example 6. Hurwitz [32] showed that if $\mathscr{C}$ is a compact Riemann surface (or algebraic curve) of genus $g \geq 2$ then $\mathrm{Aut}\,\mathscr{C}$ has order at most $84(g-1)$. The finite groups G attaining this bound, namely the nontrivial finite quotients of the triangle group $\Delta = \Delta(2, 3, 7)$, are called *Hurwitz groups*. Macbeath [45] classified those groups $PSL_2(q)$ which are Hurwitz groups, and these include the groups $G = PSL_2(p)$ for all primes $p \equiv \pm 1 \bmod (7)$. For such groups G there are, in fact, three normal subgroups N of Δ with $\Delta/N \cong G$, corresponding to choosing elements from the three conjugacy classes of elements of order 7 as members of generating triples for G. We thus obtain three non-isomorphic Riemann surfaces $\mathscr{C} = \mathbb{H}/N$, of genus

$$g = 1 + \frac{p(p^2-1)}{168}$$

and with automorphism group $PSL_2(p)$, attaining Hurwitz's bound. Streit [53] showed that, as algebraic curves, these are defined over the cubic field $K = \mathbb{Q}(\zeta_7) \cap \mathbb{R}$, and are conjugate under the Galois group $\mathrm{Gal}\,K/\mathbb{Q} \cong C_3$ of that field. The normal inclusions of the subgroups N in Δ equip each $\mathscr{C}$ with a regular dessin, specifically an orientably regular 7-valent triangular map, inherited from the corresponding Δ-invariant tessellation of $\mathbb{H}$. These three algebraically conjugate maps are mutually non-isomorphic, and in fact so are their embedded graphs [41].

Example 7. In [53], Streit generalised the above example, replacing the integer 7 with an arbitrary integer $n \geq 7$. For any prime $p \equiv \pm 1 \bmod (2n)$ there are $\phi(n)/2$ conjugacy classes of elements of order n in the group $G = PSL_2(p)$, giving rise to $\phi(n)/2$ normal subgroups N of the triangle group $\Delta = \Delta(2, 3, n)$ with $\Delta/N \cong G$.

These in turn correspond to the same number of non-isomorphic curves $\mathscr{C} = \mathbb{H}/N$, all with automorphism group G and carrying orientably regular n-valent triangular maps. These curves are defined over the field $\mathbb{Q}(\zeta_n) \cap \mathbb{R}$, and are equivalent under the Galois group of that field, isomorphic to $\mathbb{Z}_n^*/\{\pm 1\}$. As before, these maps are mutually non-isomorphic.

12 Conjugate but Non-homeomorphic Varieties

The examples in the preceding section show how the action of Γ can change analytic and combinatorial structures defined over $\overline{\mathbb{Q}}$, but what about topology? The genus of an algebraic curve can be defined purely algebraically (using the Riemann-Roch Theorem, for example), so it is invariant under Γ; thus Galois conjugate curves are homeomorphic to each other. However, in 1964 Serre [50] showed that in each dimension greater than 1 there are pairs of algebraic varieties, defined over $\overline{\mathbb{Q}}$, which are conjugate under Γ but not homeomorphic to each other. Subsequently, further examples of such pairs have been constructed. In fact, there exist arbitrarily large Galois orbits consisting of mutually non-homeomorphic Beauville surfaces.

Example 8. The first such examples were given by Gonzaléz-Diez and Torres-Teigell [26], using Beauville structures of type $(2, 3, n; p, p, p)$ on the group $G = PSL_2(p)$, for integers $n \geq 7$ and primes $p \equiv \pm 1 \bmod (2n)$. As in Example 7, generating triples of type $(2, 3, n)$ in G give rise to a Galois orbit of $\phi(n)/2$ non-isomorphic curves $\mathscr{C}_1$. By using triples of type (p, p, p) for $\mathscr{C}_2$ they obtained an orbit of at least $\phi(n)/2$ mutually non-isomorphic Beauville surfaces. By rigidity, these have non-isomorphic fundamental groups, so they are mutually non-homeomorphic. For fixed n, Dirichlet's Theorem gives infinitely many suitable primes p, and elementary properties of Euler's function show that the size of these orbits of Γ tends to infinity as n increases.

Example 9. The authors of [26] were unable to determine the exact size of the orbits in Example 8 because of the technical difficulty of finding how the outer automorphism of $PSL_2(p)$, induced by conjugation in $PGL_2(p)$, acts on the associated Beauville surfaces. In [25], they and the present author avoided this problem by using a similar construction based on the Beauville group $G = PGL_2(p)$, which has only inner automorphisms.

If p is an odd prime then the non-identity elements of $PGL_2(p)$ are of three types: elliptic elements, of order dividing $p + 1$, with no fixed points on the projective line $\mathbb{P}^1(p)$; parabolic elements, of order p, with one fixed point; and hyperbolic elements, of order dividing $p - 1$, with two fixed points. An element of one type cannot be conjugate to a power of an element of another type. For any prime $p \equiv 19 \bmod (24)$ one can find generating triples for G of types $(2, 3, p - 1)$, consisting of hyperbolic elements, and $(2, 4, p + 1)$, consisting of elliptic elements; any such pair of triples forms a Beauville structure on G.

There are $\phi(p \pm 1)/2$ conjugacy classes of elements of order $p \pm 1$ in G. This is therefore the number of normal subgroups of the triangle group $\Delta_1 = \Delta(2, 3, p-1)$ or $\Delta_2 = \Delta(2, 4, p+1)$ with quotient group G, and hence also the number of non-isomorphic algebraic curves $\mathscr{C}_1$ or $\mathscr{C}_2$ uniformised by such subgroups. These curves all have automorphism group G since Singerman's results [52] show that Δ_1 and Δ_2 are maximal Fuchsian groups. These two families of curves $\mathscr{C}_i$ are defined over the field $K_i = \mathbb{Q}(\zeta_{p\pm1}) \cap \mathbb{R}$, and the members of each family are conjugate under the Galois group of that field. We thus obtain $\phi(p-1)\phi(p+1)/4$ Beauville surfaces $\mathscr{S} = (\mathscr{C}_1 \times \mathscr{C}_2)/G$, defined over the field $K = K_1K_2$; they are conjugate under $\mathrm{Gal}\, K/\mathbb{Q}$ and hence under Γ. By rigidity, these surfaces have mutually non-isomorphic fundamental groups, so they are mutually non-homeomorphic. As before, the size of this orbit of Γ tends to infinity as p increases.

In both of these examples, although the topological fundamental groups $\pi_1\mathscr{S}$ of the surfaces $\mathscr{S}$ in a given orbit are mutually non-isomorphic, the algebraic fundamental groups $\pi_1^{\mathrm{alg}}\mathscr{S}$, the profinite completions $\widehat{\pi_1\mathscr{S}}$ of the topological fundamental groups, are mutually isomorphic. This is because the finite quotients of the groups $\pi_1\mathscr{S}$ correspond to the finite regular unbranched coverings of $\mathscr{S}$, and these, being algebraically defined, are invariant under Γ (see [50]). By contrast with the groups $\pi_1\mathscr{S}$, Conder [7] has recently shown that triangle groups are determined, up to isomorphism, by their finite quotient groups.

In Examples 6–9, together with other similar examples in [12, 40, 41, 54, 55], for instance, the curves and surfaces in an orbit of Γ are all defined over some subfield of a cyclotomic field. The group of transformations induced by Γ on such an orbit is therefore abelian, so the commutator subgroup Γ' is contained in the kernel of the action. It would be interesting to have some nonabelian examples, which reveal more of the structure of Γ. In theory this should be possible, since it follows from a recent result of González-Diez and Jaikin-Zapirain [23] (see Sect. 11) that Γ acts faithfully on Beauville surfaces.

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Generic Rigidity with Forced Symmetry and Sparse Colored Graphs

Justin Malestein and Louis Theran

Abstract We review some recent results in the generic rigidity theory of planar frameworks with forced symmetry, giving a uniform treatment to the topic. We also give new combinatorial characterizations of minimally rigid periodic frameworks with fixed-area fundamental domain and fixed-angle fundamental domain.

Keywords Rigidity • Matroid • Colored graph • Periodic framework • Forced symmetry • Sparse graphs

Subject Classifications: 52C25, 52B40

1 Introduction

The Maxwell-Laman Theorem is the prototypical result of combinatorial rigidity theory.

Theorem 1 ([19, 29]). *A generic bar-joint framework in the plane is minimally rigid if and only if the graph defined by the frameworks edges has n vertices $m = 2n-3$ edges, and, for all subgraphs on n' vertices and m' edges, $m' \leq 2n'-3$.*

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The key feature of this, and all such "Maxwell-Laman-type results" is that, for almost all *geometric* data, rigidity is determined by the *combinatorial type* and can be decided by efficient *combinatorial* algorithms.

1.1 Some Generalizations

Finding generalizations of the Maxwell-Laman Theorem has been the motivation for a lot of progress in the field. The *body-bar* [49], *body-bar-hinge* [49, 51], and *panel-hinge* [17] frameworks have a rich generic theory in all dimensions. Here the "sparsity counts" are of the form $m' \leq Dn' - D$, where D is the dimension of the d-dimensional Euclidean group. On the other hand, various elaborations of the planar bar-joint model via *pinning* [6, 22, 35], *slider-pinning* [18, 46], *direction-length frameworks* [44], and other geometric restrictions like *incident vertices* [8] or, of more relevance here *symmetry* [41, 42], have all shed more light on the Maxwell-Laman Theorem itself.

In another direction, various families of graphs and hypergraphs defined by *heriditary sparsity counts* of the form $m' \leq kn' - \ell'$ have been studied in terms of *combinatorial structure* [20], *inductive constructions* [7, 20], *sparsity-certifying decompositions* [45, 51] and *linear representability* [47], [52, Appendix A] properties. Running through much of this work is a matroidal perspective first introduced by Lovász-Yemini [23].

While much is known about (k, ℓ)-sparse graphs and hypergraphs, the parameter settings that yield interesting *rigidity* theorems seem to be somewhat isolated, despite the uniform combinatorial theory and many operations connecting different sparsity families.

1.2 Forced Symmetry

For the past several years, the rigidity and flexibility of frameworks with additional *symmetry* has received much attention,[1] although it also goes back further. Broadly speaking, there are two approaches to this: *incidental symmetry*, in which one studies a framework that may move in unrestricted ways but starts in a symmetric position [10, 15, 16, 32, 41, 42]; and *forced* symmetry [4, 24–26, 37, 39] where a framework must maintain symmetry with respect to a specific group throughout its motion. Forced symmetry is particularly useful as a way to study *infinite* frameworks[2] arising in applications to crystallography [36, 50].

[1] See, e.g., the recent conferences [9, 21, 40].

[2] Infinite frameworks with no other assumptions can exhibit quite complicated behavior [33].

In a sequence of papers [24–26], we pioneered much of the generic and combinatorial rigidity theory for the forced-symmetric frameworks in the plane. The basic setup we consider is as follows: we are given a group Γ that acts discretely on the plane by Euclidean isometries, a graph $\tilde{G} = (\tilde{V}, \tilde{E})$, and a Γ-action φ on $\tilde{G}$ with finite quotient that is free on the vertices and edges. A (realized) *Γ-framework* $\tilde{G}(\mathbf{p}, \Phi)$ is given by a point set $\mathbf{p} = (\mathbf{p}_i)_{i \in \tilde{V}}$ and a representation Φ of Γ by Euclidean isometries, with the compatibility condition

$$\mathbf{p}_{\gamma \cdot i} = \Phi(\gamma) \cdot \mathbf{p}_i \tag{1}$$

holding for all $\gamma \in \Gamma$ and $i \in \tilde{V}$.

Intuitively, the allowed continuous motions through $\tilde{G}(\mathbf{p}, \Phi)$ are those that preserve the lengths and connectivity of the bars, and symmetry with respect to Γ, but the particular representation Φ is allowed to flex. When the only allowed motions are induced by Euclidean isometries, a framework is *rigid*, and otherwise it is *flexible*.

The combinatorial model for Γ-frameworks is *colored graphs*, which we describe in Sect. 2. These efficiently capture some canonical Γ-framework invariants relating to how much flexibility from the group representation Φ a sub-framework constrains. Our combined theorems can be described uniformly as follows:

Theorem 2 ([24–26]). *Let Γ be one of:*

- *$\mathbb{Z}^2$, acting on the plane by translation*
- *$\mathbb{Z}/k\mathbb{Z}$, for $k \in \mathbb{N}$, $k \geq 2$ acting on the plane by an order k rotation around the origin*
- *$\mathbb{Z}/2\mathbb{Z}$, acting on the plane by a reflection*
- *A crystallographic group generated by translations and a rotation.*

A generic Γ-framework $\tilde{G}(\mathbf{p}, \Phi)$ is minimally rigid if and only if the associated colored quotient graph $(G, \boldsymbol{\gamma})$ has n vertices, m edges and:

- $m = 2n + \text{teich}_{\Gamma}(\Gamma) - \text{cent}(\Gamma)$
- *For all subgraphs G' on n' vertices, m' edges, with connected components G_i that have ρ-image Γ_i,*

$$m' \leq 2n' + \text{teich}_{\Gamma}(\Lambda(G')) - \sum_i \text{cent}(\Gamma_i') \tag{2}$$

where $\Lambda(G')$ is the translation subgroup associated with Γ_i'.

(See Sect. 2 for definitions of teich_{Γ} and cent.) Theorem 2 gives a generic rigidity theory that is: (1) Combinatorial; (2) Computationally tractable; (3) Applicable to *almost all* frameworks; (4) Applicable to a small *geometric* perturbation of *all* frameworks. In other words, it carries *all* of the key properties of the Maxwell-Laman-Theorem to the forced symmetry setting.

1.3 Results and Roadmap

The classes of colored graphs appearing in Theorem 2 are a new, non-trivial, extension of the (k, ℓ)-sparse families that had not appeared before. The proof of Theorem 2 relies on a *direction network method* (cf. [46, 51]), and the papers [24–26] develop the required combinatorial theory for direction networks. In this paper, we focus more on frameworks, describing the colored graph invariants that correspond to "Maxwell-type heuristics" and showing how to explicitly compute them. Additionally, we study periodic frameworks in a bit more detail, and derive several new consequences of Theorem 2: conditions for a periodic framework to fix the representation of $\mathbb{Z}^2$ (Propositions 5 and 9), and, as a consequence, the Maxwell-Laman-type Theorem 4 for periodic frameworks with fixed area fundamental domain.

1.4 Some Related Works

We remark that Theorem 2 has subsequently been shown to hold in the case where Γ is any dihedral group of order $2k$ where k is odd [14]. From examples in the same preprint, it appears that the above heriditary sparsity condition, while necessary, fails to be sufficient when k is even. Another subsequent preprint of note by Tanigawa provides somewhat different characterizations of generic rigidity for the above frameworks in Theorem 2 when Γ is orientation-preserving [48]. Note that all these results are restricted to the plane, and in fact the problem of characterizing generic rigidity of symmetric bar-joint frameworks in higher dimensions is no easier than that in the nonsymmetric setting, a difficult and unsolved problem. However, some partial results in the periodic case in higher dimensions have been obtained [5].

1.5 Notation and Terminology

We use some standard terminology for (k, ℓ)-sparse graphs: a finite graph $G = (V, E)$ is (k, ℓ)-*sparse* if for all subgraphs on n' vertices and m' edges, $m' \le kn' - \ell$. If equality holds for all of G, then G is a (k, ℓ)-*graph*; a subgraph for which equality holds is a (k, ℓ)-*block* and maximal (k, ℓ)-blocks are (k, ℓ)-*components*. Edge-wise minimal violations of (k, ℓ)-sparsity are (k, ℓ)-circuits. If G contains a (k, ℓ)-graph as a spanning subgraph it is (k, ℓ)-*spanning*. A (k, ℓ)-*basis* of G is a maximal subgraph that is (k, ℓ)-sparse. We refer to $(2, 3)$-sparse graphs by their more conventional name: Laman-sparse graphs.

In the sequel, we will define a variety of hereditarily sparse *colored* graph families. We generalize the concepts of "*sparse*", "*block*", "*component*", "*basis*" and "*circuit*" in the natural way for any family of colored graphs defined by a sparsity condition.

2 The Model and Maxwell Heuristic

We now briefly review the degree of freedom heuristic that leads to the sparsity condition (2). As is standard, we begin with the desired form:

$$\#(\text{constraints}) \leq \#(\text{total d.o.f.}) - \#(\text{trivial motions}) \tag{3}$$

What distinguishes the forced symmetric setting is that the r.h.s. depends, in an essential way, on the representation Φ of the symmetry group. Thus, we modify (3) to

$$\#(\text{constraints}) \leq \#(\text{total non-trivial d.o.f.}) - \#(\text{rigid motions preserving } \Phi) \tag{4}$$

2.1 Flexibility of Symmetry Groups and Subgroups

Let Γ be a group as in Theorem 2. We define the *representation space* $\text{Rep}(\Gamma)$ to be the set of all faithful representations Φ of Γ by Euclidean isometries. The *Teichmüller space*[3] $\text{Teich}(\Gamma)$ is the quotient $\text{Rep}(\Gamma)/\,\text{Euc}(2)$ of the representation space by Euclidean isometries. We define $\text{teich}(\Gamma)$ to be the dimension of $\text{Teich}(\Gamma)$. For frameworks, the Teichmüller space plays a central role, since $\text{teich}(\Gamma)$ gives the total number of *non-trivial* degrees of freedom associated with representations of Γ.

Now let $\Gamma' < \Gamma$ be a subgroup of Γ. The *restricted Teichmüller space* $\text{Teich}_\Gamma(\Gamma')$ is the image of the restriction map from $\Gamma \to \Gamma'$ modulo Euclidean isometries. Equivalently it is the space of representations of Γ' that extend to representations of Γ. Its dimension is defined to be $\text{teich}_\Gamma(\Gamma')$.

The invariant $\text{teich}_\Gamma(\Gamma')$ measures how much of the flexibility of Γ can be "seen" by Γ'. In general, the restricted Teichmüller space of Γ' is *not* the same as its (unrestricted) Teichmüller space. For instance, the Teichmüller space $\text{Teich}(\mathbb{Z}^2)$ has dimension 3, but the restricted Teichmüller space $\text{Teich}_\Gamma(\mathbb{Z}^2)$ has dimension 1 if Γ contains a rotation of order 3.

2.2 Isometries of the Quotient

Now let Φ be a representation of Γ. The *centralizer* of Φ is the subgroup of Euclidean isometries commuting with $\Phi(\Gamma)$. We define $\text{cent}(\Gamma)$ to be the dimension of the centralizer, which is independent of Φ (see e.g. [24, Lemma 6.1]). An alternative interpretation of the centralizer is that it is the isometry group of the quotient orbifold $\mathbb{R}^2/\Gamma$.

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[3]We are extending the terminology "Teichmüller space" from its more typical usage for the group $\mathbb{Z}^2$ and lattices in $\text{PSL}(2,\mathbb{R})$. Our definition of $\text{Teich}(\mathbb{Z}^2)$ is non-standard since the usual one allows only unit-area fundamental domains.

2.3 Colored Graphs

The combinatorial model for a Γ-framework is a *colored graph* $(G, \boldsymbol{\gamma})$,[4] which is a finite, directed graph $G = (V, E)$ and an assignment $\boldsymbol{\gamma} = (\gamma_{ij})_{ij \in E}$ of a group element in Γ to each edge of G. The correspondence between colored graphs $(G, \boldsymbol{\gamma})$ and graphs with a Γ-action $(\tilde{G}, \varphi)$ is a straightforward specialization of covering space theory, and we have described the dictionary in detail in [24, Section 9]. The important facts are:

- The data $(\tilde{G}, \varphi)$ and a selection of a representative from each vertex and edge orbit determine a colored graph $(G, \boldsymbol{\gamma})$.
- Each colored graph $(G, \boldsymbol{\gamma})$ lifts to a graph $\tilde{G}$ with a free Γ-action by a natural construction.

Together these mean that the colored graph $(G, \boldsymbol{\gamma})$ captures *all* the information in $(\tilde{G}, \varphi)$.

2.4 The Homomorphism ρ

Let $(G, \boldsymbol{\gamma})$ be a connected colored graph, and select a base vertex b of G. The coloring on the edges then induces a natural homomorphism $\rho : \pi_1(G, b) \to \Gamma$. For a closed path P defined by the sequence of edges $bi_2, i_2i_3, \ldots, i_{\ell-1}b$, we have

$$\rho(P) = \gamma_{bi_2}\gamma_{i_2i_3}\cdots\gamma_{i_{\ell-1}b},$$

where γ_{ji} is taken to be γ_{ij}^{-1}. The key properties of ρ are [24, Lemmas 12.1 and 12.2]:

- The quantities $\operatorname{teich}_\Gamma(\rho(\pi_1(G, b)))$ and $\operatorname{cent}(\rho(\pi_1(G, b)))$ depend only on the lift $(\tilde{G}, \varphi)$, so, in particular, they are independent of the choice of b.
- If $G_1, G_2, \ldots, G_c$, are the connected components of a *disconnected* colored graph $(G, \boldsymbol{\gamma})$, there is a well-defined *translation subgroup* $\Lambda(G)$ of G.

2.5 Derivation of the Maxwell Heuristic

We are now ready to derive the degree of freedom heuristic for Γ-frameworks. Let $(G, \boldsymbol{\gamma})$ be a Γ-colored graph with n vertices, m edges, connected components $G_1, G_2, \ldots, G_i$, with ρ-images Γ_i'. We fill in the template (4) for the associated Γ-framework $\tilde{G}(\mathbf{p}, \Phi)$:

[4]Colored graphs are also known as "gain graphs" or "voltage graphs" [53]. The terminology of colored graphs originates from [36].

Non-trivial degrees of freedom. There are two sources of flexibility:

(A) The group representation Φ has, by definition, $\mathrm{teich}_\Gamma(\Lambda(G))$ degrees of freedom, up to Euclidean isometries. These are the non trivial degrees of Φ "seen" by G.

(B) The coordinates of the vertices are determined by the location of one representative of each Γ-orbit and Φ. There are n such orbits, for a total of $2n$ degrees of freedom

Here is the guiding intuition for **(A)**. We want to understand how the edge lengths can constrain the representation Φ. It is intuitively clear that if there is no pair of points $\tilde{\mathbf{p}}_i$ and $\tilde{\mathbf{p}}_{\gamma\cdot i}$ in the same Γ-orbit that are also connected in the lift $(\tilde{G}, \varphi)$, the framework cannot constrain Φ at all. Thus, we are interested in accounting for constraints arising from paths in $(\tilde{G}, \varphi)$ between pairs of points $\tilde{\mathbf{p}}_i$ and $\tilde{\mathbf{p}}_{\gamma\cdot i}$; in $(G, \boldsymbol{\gamma})$, this corresponds to a closed path P with $\rho(P) = \gamma$.

This reasoning leads us to consider $\mathrm{teich}_\Gamma(\cdot)$ of a subgroup generated by the ρ-images of some closed paths in $(G, \boldsymbol{\gamma})$. After some technical analysis, the correct subgroup is discovered to be $\Lambda(G)$.

Rigid motions independent of Φ. For each connected component of $\tilde{G}(\mathbf{p}, \Phi)$ induced by G_i: there is a $\mathrm{cent}(\Gamma'_i)$-dimensional space of these for each G_i, since any element of the centralizer of Γ'_i preserves all the edge lengths and compatibility with Φ. Because the components are disconnected, these motions are independent of each other.

3 Periodic Frameworks

A Γ-framework with symmetry group $\mathbb{Z}^2$ is called a *periodic framework* [4]. In this section, we specialize (2) to this case, and relate it to an alternate counting heuristic from [26, Section 3].

3.1 *Invariants for* $\mathbb{Z}^2$

Representations of $\mathbb{Z}^2$ by translations have very simple coordinates: they are given by mapping each of the generators $(1, 0)$ and $(0, 1)$ to a vector in $\mathbb{R}^2$. Thus, the space of (possibly degenerate) representations is isomorphic to the space of 2×2 matrices with real entries. Given such a matrix $\mathbf{L}$ and $\gamma \in \mathbb{Z}^2$, the translation representing γ is simply $\mathbf{L} \cdot \gamma$. Because of this identification, we denote realizations of periodic frameworks by $\tilde{G}(\mathbf{p}, \mathbf{L})$, and call $\mathbf{L}$ the *lattice representation*.

Subgroups of $\mathbb{Z}^2$ are always generated by $k = 0, 1, 2$ linearly independent vectors; given a subgroup, we define its *rank* to be the minimum size of a generating set. To specify a representation of a subgroup $\Gamma' < \mathbb{Z}^2$, we assign a vector in $\mathbb{R}^2$ to each of the k generators of Γ'. Such a representation always extends to a faithful representation of $\mathbb{Z}^2$. Thus, we see that the dimension of the space of representations of $\mathbb{Z}^2$ restricted to Γ' is $2k$.

The quotient of the representation space of $\mathbb{Z}^2$ by Euc(2) is also straightforward to describe. Each point has a representative $\mathbf{L}$ such that $\mathbf{L} \cdot (1,0) = (\lambda, 0)$ for some real scalar λ. From this, we get:

Proposition 1. *Let $\Gamma' < \mathbb{Z}^2$ be a subgroup of $\mathbb{Z}^2$ with rank k. Then* $\mathrm{teich}_{\mathbb{Z}^2}(\Gamma') = \max\{2k-1, 0\}$.

Finally, we compute the dimension of the centralizer of a subgroup Γ'. If Γ' is trivial, then the centralizer is the entire 3-dimensional Euclidean group. If Γ' is rank 1, then it is represented by a translation $t_1(\mathbf{p}) = \mathbf{p} + \mathbf{t}_1$, which commutes with other translations and reflections or glides fixing a line in the direction $\mathbf{t}_1$. For the rank 2 case, the centralizer is just the translation subgroup of Euc(2). We now have:

Proposition 2. *Let $\Gamma' < \mathbb{Z}^2$ be a subgroup of $\mathbb{Z}^2$ with rank k. Then*

$$\mathrm{cent}(\Gamma') = \begin{cases} 3 & \text{if } k = 0 \\ 2 & \text{if } k \geq 1 \end{cases}$$

3.2 The Homomorphism ρ for $\mathbb{Z}^2$

Now we turn to associating a colored graph $(G, \boldsymbol{\gamma})$ with a subgroup of $\mathbb{Z}^2$. This is simpler than the general case because $\mathbb{Z}^2$ is abelian, so we may define it as a map $\rho : H_1(G, \mathbb{Z}) \to \mathbb{Z}^2$, as was done in [26]. Here are the relevant facts:

Proposition 3. *Let $(G, \boldsymbol{\gamma})$ be a colored graph. Then the rank of the ρ-image is determined by the values of ρ on* any *homology (alternatively, cycle) basis of G, and thus ρ is well-defined when G has more than one connected component.*

3.3 Colored-Laman Graphs

With Propositions 1–3, the colored graph sparsity counts (2) from Theorem 2 specializes, for a $\mathbb{Z}^2$-colored graph to:

$$m' \leq 2n' + \max\{2k-1, 0\} - 3c_0' - 2c_{\geq 1}' \tag{5}$$

where k is the rank of the $\mathbb{Z}^2$-image of $(G, \boldsymbol{\gamma})$, c_0' is the number of connected components with trivial $\mathbb{Z}^2$-image and $c_{\geq 1}'$ is the number of connected components with non-trivial $\mathbb{Z}^2$-image (i.e., $k \geq 1$). This gives a matroidal family [26, Lemma 7.1], and we define the bases to be *colored-Laman graphs*.

3.4 An Alternative Sparsity Function

A slightly different counting heuristic for a periodic framework with colored quotient graph $(G, \boldsymbol{\gamma})$ having n vertices, m edges, c connected components and ρ-image with rank k is as follows:

- There are $2n$ variables specifying the points, and $2k$ variables giving a representation of the ρ-image.
- To remove Euclidean isometries that move the points and the lattice representation together, we pin down a connected component.
- Each of the remaining connected components may translate independently of each other.

Adding up the degrees of freedom and subtracting three degrees of freedom for pinning down one connected components and two each for translations of each other connected component yields the sparsity condition from [26, Section 3, p. 14]

$$m' \le 2(n' + k) - 3 - 2(c' - 1) \tag{6}$$

which is equivalent to the colored-Laman counts (5) by the following.

Proposition 4. *Let $(G, \boldsymbol{\gamma})$ be a $\mathbb{Z}^2$-colored graph. Then $(G, \boldsymbol{\gamma})$ satisfies (5) if and only if it satisfies (6).*

Proof. For convenience, we define the two functions:

$$f(G) = 2n + \max\{2k - 1, 0\} - 3c_0 - 2c_{\ge 1} \tag{7}$$

$$g(G) = 2(n + k - c) - 1 \tag{8}$$

where g is easily seen to be equal to the r.h.s. of (6). The definitions imply readily that $f(G) \le g(G)$, with equality when there is either one connected component in G or all connected components have ρ-images with rank at least one. Thus, it will be sufficient to show that, if $(G, \boldsymbol{\gamma})$ has n vertices, m edges, and ρ-image of rank k, and it is minimal with the property that $f(G) = m - 1$, then $g(G) = m - 1$.

Let $(G, \boldsymbol{\gamma})$ have these properties, and let G have connected components G_i with, n_i vertices, m_i edges, and ρ-images of rank k_i. The minimality hypothesis implies that for any G_i, the number of edges in $G \setminus G_i$ is

$$m - m_i \le f(G \setminus G_i) \tag{9}$$

but, if k_i is zero, the rank of the ρ-image of $G \setminus G_i$ is k, and $m_i \le 2n_i - 3$. Computing, we find that

$$\begin{aligned} m - m_i &\ge 2n + \max\{2k - 1, 0\} - 3c_0 - 2c_{\ge 1} + 1 - 2n_i + 3 \\ &= 2(n - n_i) + \max\{2k - 1, 0\} - 3(c_0 - 1) - 2c_{\ge 1} + 1 \\ &= f(G \setminus G_i) + 1 \end{aligned}$$

which is a contradiction to (9). We conclude that either there is one connected component in G or that none of the k_i were zero. In either of these cases $f(G) = g(G)$, which completes the proof. □

3.5 Example: Disconnected Circuits

The proof of Proposition 4 generalizes the folklore fact that, for Laman rigidity, we get the same class of graphs from "$m' \leq 2n' - 3$" and the more precise "$m' \leq 2n' - 3c'$". In the periodic setting the additional precision is *required*:

- There are periodic frameworks with *dependent* edges in different connected components of the colored quotient graph [26, Figure 20].
- There are *connected* $\mathbb{Z}^2$-colored graphs that are not colored-Laman sparse but satisfy (5) for *all* induced or connected subgraphs [26, Figure 8].

The intuition leading to the discovery of (6) is that connected components of a periodic framework's colored graph interact via the representation $\mathbf{L}$ when they have the same ρ-image.

3.6 Example: Disconnected Minimally Rigid Periodic Frameworks

Another phenomenon associated with periodic rigidity that is not seen in finite frameworks is that although the colored quotient graph $(G, \boldsymbol{\gamma})$ must be connected [26, Lemmas 4.2 and 7.3], the periodic framework $\tilde{G}(\mathbf{p}, \Phi)$ does not need to be as in [26, Figure 9]. To see this, we simply note that (5) depends only on the *rank* of the ρ-image, which is unchanged by multiplying the entries of the colors γ_{ij} on the edges $(G, \boldsymbol{\gamma})$ by an integer q. On the other hand, this increases the number of connected components by a factor of q^2. There is no paradox because periodic symmetry is being forced: once we know the realization of one connected component of $\tilde{G}(\mathbf{p}, \Phi)$, we can reconstruct the rest of them from the representation Φ of $\mathbb{Z}^2$.

3.7 Conditions for Fixing the Lattice

The definition of rigidity for periodic frameworks implies that a rigid framework fixes the representation $\mathbf{L}$ of $\mathbb{Z}^2$ up to a Euclidean isometry. It then follows that *any* periodic framework with a non-trivial *rigid component* must do the same. However, this is not the only possibility. Figure 1 shows a framework without a rigid

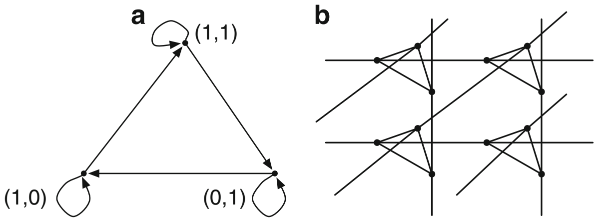

Fig. 1 A flexible periodic framework that determines the lattice representation: (**a**) the associated colored graph; (**b**) the periodic framework

component that fixes the lattice representation and its associated colored graph. The framework's non-trivial motion is a rotation of each of the triangles. This example is an instance of a more general phenomenon.

Proposition 5. *Suppose that $(G, \boldsymbol{\gamma})$ is a colored graph such that an associated generic periodic framework $\tilde{G}(\mathbf{p}, \Phi)$ fixes the lattice representation. Then $(G, \boldsymbol{\gamma})$ contains a subgraph G' with m edges and rank 2 ρ-image such that $m = f(G')$, where f is the sparsity function defined in (7).*

Proof. We may assume without loss of generality that $(G, \boldsymbol{\gamma})$ is colored-Laman sparse. Let $\eta \in \mathbb{Z}^2$ be a vector that is linearly independent of any ρ-image of any rank 1 subgraph of $(G, \boldsymbol{\gamma})$. Such an η exists since there are only finitely many subgraphs of $(G, \boldsymbol{\gamma})$. Because $\tilde{G}(\mathbf{p}, \Phi)$ is generic and fixes the lattice, Theorem 2 implies that adding a self-loop ℓ with color η leads to a colored graph that is not colored-Laman-sparse. This implies that there is a minimal subgraph $(G' + \ell, \boldsymbol{\gamma})$ of $(G + \ell, \boldsymbol{\gamma})$ that is not colored-Laman sparse. The ρ-image of G' must be rank 2 because, if it were not, the rank of the ρ-image of $G' + \ell$ would be strictly larger than that of G', thus $(G' + \ell, \boldsymbol{\gamma})$ would again be colored-Laman sparse. It follows that G' satisfies the conclusion of the Proposition. □

4 Specializations of Periodic Frameworks

Because Theorem 2 is quite general, we can deduce Laman-type theorems for many restricted versions of periodic frameworks from Theorem 2. In this section, we describe three of these in detail and discuss connections with some others.

4.1 The Periodic Rigidity Matrix

The proof of Theorem 2 relies on giving a combinatorial characterization of *infinitesimal rigidity* with forced symmetry constraints. The *rigidity matrix*, which is the formal differential of the length equations plays an important role. For periodic frameworks, this has the following form, which was first computed in [4]:

$$\begin{array}{cc} & \begin{array}{ccccccc} & i & & j & & L_1 & L_2 \end{array} \\ ij & \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -\eta_{ij} & \dots & \eta_{ij} & \dots & \gamma_{ij}^1 \eta_{ij} & \gamma_{ij}^2 \eta_{ij} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{array} \tag{10}$$

Here, $\eta_{ij} = \mathbf{p}_j + \mathbf{L} \cdot \gamma_{ij} - \mathbf{p}_i$ is the vector describing a representative of an edge orbit in $\tilde{G}(\mathbf{p}, \mathbf{L})$, which we identify with a colored edge of the quotient $(G, \boldsymbol{\gamma})$. There is one row for each edge in the quotient graph G. The column groups L_1 and L_2 correspond to the derivatives with respect to the variables in the rows of $\mathbf{L} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A framework is *infinitesimally rigid* if the rigidity matrix has corank 3, and *infinitesimally flexible* with d degrees of freedom if the rigidity matrix has corank $3 + d$. A framework is *generic* if the rank of the rigidity matrix is maximal over all frameworks with the same colored quotient graph. We will require some standard facts about infinitesimal rigidity that transfer from the finite to the periodic setting.

Proposition 6. *Let $\tilde{G}(\mathbf{p}, \mathbf{L})$ be a periodic framework with quotient graph $(G, \boldsymbol{\gamma})$. Then:*

- *For generic frameworks, infinitesimal rigidity and flexibility coincide with rigidity and flexibility [4, 26].*
- *Infinitesimal rigidity and flexibility are affinely invariant [4], with non-trivial infinitesimal motions mapped to non-trivial infinitesimal motions.*

4.2 One Flexible Period

A very simple restriction of the periodic model is to consider frameworks with one flexible period. The symmetry group is then $\mathbb{Z}$, acting on the plane by translations; we call such a framework a *cylinder framework*. We model the situation with $\mathbb{Z}$-colored graphs, and a single vector $\mathbf{l} \in \mathbb{R}^2$ representing the period lattice. In this case, the ρ-image of a colored graph always has rank 0 or 1.

We define a *cylinder-Laman graph* to be a $\mathbb{Z}$-colored graph $(G, \boldsymbol{\gamma})$ such that: G has n vertices, $2n - 1$ edges, and satisfies, for all subgraphs, on n' vertices, m' edges, ρ-image of rank k, c'_0 connected components with trivial ρ-image, and c'_1 connected components with non-trivial ρ-image:

$$m' \leq 2n' + k - 3c'_0 - 2c'_1 \tag{11}$$

Comparing (11) with (5), we see readily:

Proposition 7. *The family of cylinder-Laman graphs corresponds bijectively with the maximal colored-Laman sparse graphs that have colors of the form $\gamma_{ij} = (\cdot, 0)$.*

Theorem 3. *A generic cylinder framework is minimally rigid if and only if its associated colored graph is cylinder-Laman.*

Proof. The rigidity matrix for a cylinder framework has the same form as (10), except with the column group labeled L_2 discarded. Proposition 7 and then Theorem 2 yield the desired statement. □

4.3 Unit Area Fundamental Domain

Next, we consider the class of *unit-area frameworks*, for which the allowed motions preserve the area of the fundamental domain of the $\mathbb{Z}^2$-action on the plane induced by the $\mathbb{Z}^2$-representation $\mathbf{L}$.

We define a *unit-area-Laman graph* to be a $\mathbb{Z}^2$-colored graph $(G, \boldsymbol{\gamma})$ with n vertices, $m = 2n$ edges, and satisfying, for all subgraphs on n' vertices, m' edges, and c'_k connected components with ρ-image of rank k

$$m' \leq 2n' - 3c'_0 \qquad \text{if } c'_2 = c'_1 = 0 \tag{12}$$

$$m' \leq 2n' - 1 - 3c'_0 - 2(c'_1 - 1) \qquad \text{if } c'_2 = 0 \text{ and } c'_1 > 0 \tag{13}$$

$$m' \leq 2n' - 3c'_0 - 2(c'_1 + c'_2 - 1) \qquad \text{if } c'_2 > 0 \tag{14}$$

Theorem 4. *A generic unit-area framework is minimally rigid if and only if its associated colored graph $(G, \boldsymbol{\gamma})$ is unit-area-Laman.*

Proof of Theorem 4. The proof is comprised of three key propositions. The first is a combinatorial equivalence.

Proposition 8. *A $\mathbb{Z}^2$-colored graph $(G, \boldsymbol{\gamma})$ is unit-area-Laman if and only if it is colored-Laman-sparse and has n vertices, $2n$ edges, and no subgraph with rank 2 ρ-image and (5) holding with equality.*

Proof of Proposition 8. Comparing (5) with (12,13) and (14), we see that unit-area-Laman graphs are exactly those which, after following the construction used to prove Proposition 5, become colored-Laman. □

The Maxwell direction. For the geometric part of the proof, we first derive the rigidity matrix. If $\mathbf{L} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and we coordinatize infinitesimal motions as $(\mathbf{v}, \mathbf{M})$ with $\mathbf{M} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then this has the form of (10) plus one additional row corresponding to the equation

$$\langle (d, -c, -b, a), (p, q, r, s) \rangle = 0 \tag{15}$$

Violations of unit-area-Laman-sparsity come in two types, according to the rank k of the ρ-image. For $k = 0, 1$, these are all violations of colored-Laman sparsity, implying, by Theorem 2, a generic dependency in the unit-area rigidity matrix that does not involve the row (15). For $k = 2$, Proposition 8 implies a new type of violation: a subgraph $(G', \boldsymbol{\gamma})$ with n' vertices, ρ-image of rank 2, and $f(G')$ edges. If such a subgraph forces a generic periodic framework to fix the lattice representation $\mathbf{L}$, then Eq. (15) is dependent on the equations corresponding to edge lengths. The Maxwell direction then follows from the converse of Proposition 5.

Proposition 9. *Let $(G, \boldsymbol{\gamma})$ be a colored-Laman sparse graph with ρ-image of rank 2 and (5) met with equality. Then an associated generic framework has only motions that act trivially on the $\mathbb{Z}^2$-representation $\mathbf{L}$.*

Proof of Proposition 9. Let $(G, \boldsymbol{\gamma})$ have n vertices, and c connected components. It is sufficient to consider $(G, \boldsymbol{\gamma})$ that is minimal with respect to the hypotheses of the proposition, which forces every connected component to have ρ-image with rank at least one. In this case, there are $m = 2n + 3 - 2c$ edges. By Theorem 2, the kernel of the rigidity matrix has dimension $2n + 4 - m = 2c + 1$. Since the connected components can translate independently, and the whole framework can rotate, there are at least $2c + 1$ dimensions of infinitesimal motions acting trivially on the lattice. □

The Laman direction. Now let $(G, \boldsymbol{\gamma})$ be a unit-area-Laman graph. Theorem 2 implies that any generic periodic framework on $(G, \boldsymbol{\gamma})$ has a 4-dimensional space of infinitesimal motions, and that any non-trivial infinitesimal motion is a linear combination of 3 trivial ones and some other infinitesimal motion $(\mathbf{v}, \mathbf{M})$. Since the trivial infinitesimal motions act trivially on the lattice representation $\mathbf{L}$, if $(\mathbf{v}, \mathbf{M})$ does as well, then a generic framework on $(G, \boldsymbol{\gamma})$ fixes the lattice representation. By Propositions 8 and 5 this is impossible, implying that $(\mathbf{v}, \mathbf{M})$ does not act trivially on the lattice representation. However, it might preserve the area of the fundamental domain, which would make (15) part of a dependency in the unit-area rigidity matrix. The Laman direction will then follow once we can exhibit a generic periodic framework on $(G, \boldsymbol{\gamma})$ for which $(\mathbf{v}, \mathbf{M})$ does not preserve the area of the fundamental domain.

To do this, we recall, from Proposition 6, that a generic linear transformation

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{16}$$

preserves infinitesimal rigidity and sends the non-trivial infinitesimal motion $(\mathbf{v}, \mathbf{M})$ to another non-trivial infinitesimal motion $(\mathbf{v}', \mathbf{M}')$, which is given by

$$\mathbf{v}'_i = \mathbf{A}^* \cdot \mathbf{v}_i \qquad \text{for all } i \in V(G)$$
$$\mathbf{M}' = \mathbf{A}^* \cdot \mathbf{M}$$

where

$$\mathbf{A}^* = \det(\mathbf{A})^{-1} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \tag{17}$$

is the transpose of $\mathbf{A}^{-1}$. The main step is this next proposition which says that satisfying (15) is *not* affinely invariant.

Proposition 10. *Let* $(G, \boldsymbol{\gamma})$ *be a unit-area-Laman graph, and let* $\tilde{G}(\mathbf{p}, \mathbf{L})$ *be a generic realization with* $\mathbf{L}$ *being the identity matrix, let* $\mathbf{A}$ *be a generic linear transformation, and let the infinitesimal motions* $(\mathbf{v}, \mathbf{M})$ *and* $(\mathbf{v}', \mathbf{M}')$ *be defined as above. If* $(\mathbf{v}, \mathbf{M})$ *preserves the area of the fundamental domain, then* $(\mathbf{v}', \mathbf{M}')$ *does not.*

Proof of Proposition 10. Because $\mathbf{L}$ is the identity, $\mathbf{M}$ has the form

$$\mathbf{M} = \begin{pmatrix} \lambda & \mu \\ \nu & -\lambda \end{pmatrix} \tag{18}$$

either by direction computation, or by observing that it is an element of the Lie algebra sl(2), as discussed above, we know that $-\mu \neq \nu$, since $(\mathbf{v}, \mathbf{M})$ does not act trivially on $\mathbf{L}$. In particular, μ and ν are not both zero. Plugging in to (17) we get

$$\mathbf{M}' = \det(\mathbf{A})^{-1} \begin{pmatrix} d\lambda - c\nu & c\lambda + d\mu \\ a\nu - b\lambda & -a\lambda - b\mu \end{pmatrix} \tag{19}$$

Plugging entries of $\mathbf{M}'$ in to the l.h.s. of (15) to obtain:

$$\det(\mathbf{A})^{-1} \left(\lambda(d^2 + b^2 - a^2 - c^2) - (\mu + \nu)(ab + cd) \right) \tag{20}$$

which is generically non-zero in the entries of $\mathbf{A}$: the conditions for (20) to vanish are that its columns are the same length and orthogonal to each other. $\square$

We now observe that, by Proposition 6, there is a generic realization $\tilde{G}(\mathbf{p}, \mathbf{L})$ of a framework with unit-area-Laman colored quotient $(G, \boldsymbol{\gamma})$ in which $\mathbf{L}$ is the identity. If the non-trivial infinitesimal motion $(\mathbf{v}, \mathbf{M})$ does not satisfy (15), we are done. Otherwise, the hypothesis of Proposition 10 are met, and, thus, after applying a generic linear transformation, the proof is complete. $\square$

4.4 Fixed-Lattice Frameworks

Another restricted class of periodic frameworks are *fixed-lattice frameworks*. These are periodic frameworks, with the restriction that the allowed motions act trivially on the lattice representation. This model was introduced by Whiteley [51] in the first investigation of generic rigidity with forced symmetry. More recently, Ross discovered the following[5] complete characterization of minimal rigidity for fixed-lattice frameworks.

[5]The sparsity counts we describe here are slightly different from what is stated in [38, Theorem 4.2.1], but they are equivalent by an argument similar to that in the proof of Proposition 4. This presentation highlights the connection to colored-Laman graphs.

Theorem 5 ([37] and [38, Theorem 4.2.1]). *Let $\tilde{G}(\mathbf{p})$ be a generic fixed-lattice framework. Then $\tilde{G}(\mathbf{p})$ is minimally rigid if and only if the associated colored graph $(G, \boldsymbol{\gamma})$ has n vertices, $m = 2n - 2$ edges and, for all subgraphs G' of G with n' vertices, m' edges, c'_0 connected components with trivial ρ-image, and $c'_{\geq 1}$ connected components with non-trivial ρ-image:*

$$m' \leq 2n' - 3c'_0 - 2c'_{\geq 1} \tag{21}$$

We define the family of graphs appearing in Theorem 5 to be *Ross graphs*. In [26], we gave an alternate proof based on Theorem 2. The two steps are similar to the ones used to prove Theorem 4, except we can take a "shortcut" in the argument by simulating fixing the lattice by adding self-loops to the colored graph. The geometric step is:

Theorem 6 ([26, Section 19.1]). *Let $\tilde{G}(\mathbf{p})$ be a generic fixed-lattice framework. Then $\tilde{G}(\mathbf{p})$ is minimally rigid if and only if the associated colored graph $(G, \boldsymbol{\gamma})$ plus three self-loops colored $(1, 0)$, $(0, 1)$, $(1, 1)$ added to any vertex is colored-Laman.*

Theorem 5 then follows from the following combinatorial statement that generalizes an idea of Lovász-Yemini [23] and Recski [34] (cf. [12, 13]).

Proposition 11 ([26, Lemma 19.1]). *A colored graph $(G, \boldsymbol{\gamma})$ is a Ross graph if and only if adding three self-loops colored $(1, 0)$, $(0, 1)$, $(1, 1)$ to any vertex results in a colored-Laman graph.*

4.5 Further Connections

Theorems 3 and 4 suggest a more general methodology for obtaining Maxwell-Laman-type theorems for restrictions of periodic frameworks:

- Add an equation that restricts the allowed lattice representations $\mathbf{L}$.
- Identify which generic periodic frameworks are the maximal ones that do not imply the new restriction.

Our proof of Theorem 5 works this way as well: adding self-loops adds three equations constraining the lattice representation. Another perspective is that we are enlarging the class of trivial infinitesimal motions by forcing one or more vectors into the kernel of the periodic rigidity matrix. The most general form of this operation is known as the "Elementary Quotient" or "Dilworth Truncation", and it preserves *representability* of (k, ℓ)-sparsity matroids [47], but obtaining *rigidity* results (e.g., [23]) requires geometric analysis specific to each case. This section gives a family of examples where we find new *rigidity* matroids from each other using a specialized version of Dilworth Truncation.

Ross [38, Section 5] has studied some restrictions of periodic frameworks as generalizations of the fixed-lattice model. In this section we close the circle of ideas, showing how to study them as specializations of the flexible-lattice model.

4.6 One More Variant

We end this section with one more variation of the periodic model. A *fixed-angle framework* is defined to be a periodic framework where the allowed motions preserve the angle between the sides of the fundamental domain.

Theorem 7. *A generic fixed angle framework is minimally rigid if and only if its associated colored graph is unit-area-Laman.*

Proof Sketch. The steps are similar to the proof of Theorem 4. The new row in the rigidity matrix corresponds to (in the same notation) the partial derivatives of the equation:

$$\left\langle \frac{(a,c)}{||(a,c)||}, \frac{(b,d)}{||(b,d)||} \right\rangle = \text{const} \tag{22}$$

so the new row in the rigidity matrix corresponds to:

$$\left\langle \det(\mathbf{L}) \left(c||(b,d)||^2, -d||(a,c)||^2, a||(b,d)||^2, -b||(a,c)||^2 \right), (p,q,r,s) \right\rangle = 0 \tag{23}$$

The Maxwell direction's proof is exactly the same as for Theorem 4. For the Laman direction, we again start with a generic framework where $\mathbf{L}$ is the identity. If the non-trivial infinitesimal motion $(\mathbf{v}, \mathbf{M})$ does not preserve (23), then we are done. Otherwise, $\mathbf{M}$ has the form

$$\mathbf{M} = \begin{pmatrix} \mu & \lambda \\ -\lambda & \nu \end{pmatrix} \tag{24}$$

with μ and ν not both zero, because $\mathbf{M}$ does not act trivially on $\mathbf{L}$. We then construct a new generic framework by applying a linear map

$$\mathbf{A} = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \tag{25}$$

A computation, similar to that for (20) yields

$$-\frac{b\left(b^2\mu + d^2\mu + \nu\right)}{||(b,d)||^3} \tag{26}$$

which is, generically, not zero. □

5 Cone and Reflection Frameworks

The next cases of Theorem 2 are those of $\mathbb{Z}/2\mathbb{Z}$ acting on the plane by a single reflection and $\mathbb{Z}/k\mathbb{Z}$ acting on the plane by rotation through angle $2\pi/k$. The sparsity invariants are particularly easy to characterize in these two cases:

- The Teichmüller space is empty, since any rotation center or reflection axis can be moved on to another by an isometry.
- The centralizer is all of Euc(2) for the trivial subgroup and otherwise consists of rotation around a fixed center or translation parallel to the reflection axis.

Since the rank k of the ρ-image of a $\mathbb{Z}/k\mathbb{Z}$-colored graph is always zero or one, we specialize (2) to obtain the sparsity condition for subgraphs on n' vertices, m' edges, c'_0 connected components with trivial ρ-image and c'_1 connected components with non-trivial ρ-image.

$$m' \leq 2n' - 3c'_0 - c'_1 \tag{27}$$

We define the family of $\mathbb{Z}$-colored graphs $(G, \boldsymbol{\gamma})$ corresponding to minimally rigid frameworks to be *cone-Laman graphs*. The name *cone-Laman* comes from considering the quotient of the plane by a rotation through angle $2\pi/k$, which is a flat cone, as shown in Fig. 2. Cone-Laman graphs are closely related to $(2,1)$-sparse graphs [20], and in this section we use some sparse graph machinery to obtain combinatorial results on them.

5.1 Some Background in (k,ℓ)-Sparse Graphs

In this section, we relate cone-Laman graphs to Laman graphs, and we will repeatedly appeal to some standard results about (k,ℓ)-sparse graphs from [20]. In addition, we will require:

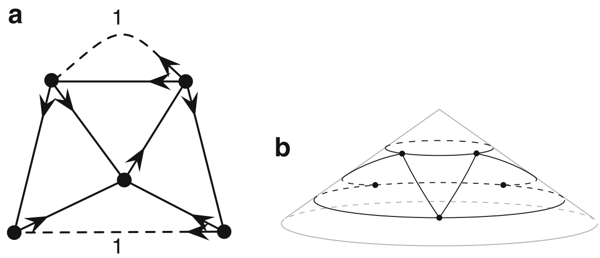

Fig. 2 Figures from [1]. (**a**) A cone-Laman graph. (**b**) A realization of (**a**) as a framework in a cone with opening angle $2\pi/3$

Proposition 12. *Let G be a $(2,1)$-graph. If there is exactly one $(2,2)$-circuit in G, then G is $(2,2)$-spanning. Otherwise, G is not $(2,2)$-spanning and the $(2,2)$-circuits in G are vertex-disjoint.*

Proof. Let G have n vertices. First assume that G has exactly one $(2,2)$-circuit. Then G is a $(2,2)$-sparse graph G' plus one edge; since G has $2n-1$ edges, G' is a $(2,2)$-graph. Otherwise there is more than one $(2,2)$-circuit. Pick a $(2,2)$-basis G' of G. In this case G' does not have enough edges to be a $(2,2)$-graph, so it decomposes, by [20, Theorem 5], into vertex-disjoint $(2,2)$-components that span all of the edges in $G \setminus G'$. Because G is $(2,1)$-sparse, it follows that each $(2,2)$-component spans at most one edge of $G \setminus G'$, and thus at most one $(2,2)$-circuit. We have now shown that the vertex sets of the $(2,2)$-circuits in G are each contained in a different $(2,2)$-component of G'. □

5.2 Cone-Laman vs. Cylinder-Laman

By comparing the cylinder-Laman counts (11) with the cone-Laman counts (27), we can see that every cylinder-Laman graph, interpreted as having $\mathbb{Z}/k\mathbb{Z}$ colors, is also cone-Laman for k large enough. However, the two classes are not equivalent. One can see this geometrically by considering a colored graph with two disconnected vertices and a self-loop with the color 1 on each vertex: this is evidently dependent in the cylinder, and independent in the cone. The conclusion is that the interplay between $\mathrm{teich}_\Gamma(\cdot)$ and $\mathrm{cent}(\cdot)$, can yield two different, geometrically interesting sparse colored families on $(2,1)$-graphs. The combinatorial relation is:

Theorem 8. *A $\mathbb{Z}$-colored graph $(G,\boldsymbol{\gamma})$ is cylinder-Laman if and only if it is cone-Laman when interpreted as having $\mathbb{Z}/k\mathbb{Z}$-colors for a sufficiently large k and G is $(2,2)$-spanning.*

Proof. The only difficult thing to check is that a cylinder-Laman graph $(G,\boldsymbol{\gamma})$ is $(2,2)$-spanning. Assuming that G is not $(2,2)$-spanning, Proposition 12 supplies two vertex-disjoint $(2,1)$-blocks. If the union spans n' vertices, there are $2n'-2$ edges, which violates (11). □

5.3 Connections to Symmetric Finite Frameworks

The following theorem of Schulze is superficially similar to Theorem 2 for $k=3$:

Theorem 9 ([42, Theorem 5.1]). *Let G be a Laman-graph with a free $\mathbb{Z}/3\mathbb{Z}$ action φ. Then a generic framework embedded such that φ is realized by a rotation through angle $2\pi/3$ is minimally rigid.*

We highlight this result to draw a distinction between forced and incidental symmetry: while Theorem 9 is related to Theorem 2, it is not implied by it. The issue is that while infinitesimal motions of the cone-framework lift to infinitesimal motions, only *symmetric* infinitesimal motions of the lift project to infinitesimal motions of the associated cone-framework. Thus, from Theorem 2, we learn that the lift of a generic minimally rigid cone framework for $k = 3$ has no *symmetric* infinitesimal motion as a finite framework, but there may be a *non-symmetric* motion induced by the added symmetry. An interesting question is whether the natural generalization of Schulze's Theorem holds:

Question 1. Let $k > 3$, and let (G, φ) be a graph with a free $\mathbb{Z}/k\mathbb{Z}$-action. Are generic frameworks with $Z/k\mathbb{Z}$-symmetry rigid if and only if G is Laman-spanning and its colored quotient is cone-Laman-spanning?

That G must be Laman-spanning is clear. On the other hand, the discussion above and Theorem 2 imply that to avoid a symmetric non-trivial infinitesimal motion, a generic $\mathbb{Z}/k\mathbb{Z}$-symmetric finite framework must have cone-Laman-spanning quotient. Ross, Schulze and Whiteley [39] and Schulze and Whiteley [43] use this same idea in a number of interesting 3-dimensional applications. The graphs described in the question are a family of simple $(2, 0)$-graphs; simple $(2, 1)$-graphs have recently played a role in the theory of frameworks restricted to lie in surfaces embedded in $\mathbb{R}^3$ [30, 31].

5.4 The Lift of a Cone-Laman Graph

The lift $(\tilde{G}, \varphi)$, defined in Sect. 2.3, of a $\mathbb{Z}/k\mathbb{Z}$-colored graph is itself a finite graph (G, φ) with a free action by $\mathbb{Z}/k\mathbb{Z}$. For $k \geq 3$ prime, cone-Laman graphs have a close connection to Laman graphs.

Proposition 13 ([1, Lemma 6]). *Let $k \geq 3$ be prime. A $\mathbb{Z}/k\mathbb{Z}$-colored graph is cone-Laman if and only if its lift (G, φ) has as its underlying graph a Laman-sparse graph G with $\tilde{n}$ vertices and $2\tilde{n} - k$ edges.*

As noted in [1], this statement is false for $k = 2$, so while we can relax the hypothesis somewhat at the expense of a more complicated statement, they cannot all be removed.

Although it is simple, Proposition 13 is surprisingly powerful, since it shows that one can study cone-Laman graphs using all the combinatorial tools related to Laman graphs. Proposition 13 depends in a fundamental way on the fact that cone-Laman graphs have $2n - 1$ edges, and it does not have a naive generalization to colored-Laman or unit-area-Laman graphs.

Question 2. What are the $\mathbb{Z}^2$-colored graphs $(G, \boldsymbol{\gamma})$ with the property that every *finite* subgraph of the periodic lift $(\tilde{G}, \varphi)$ is Laman-sparse?

We expect that this should be a more general family than unit-area-Laman graphs. On the other hand, it has been observed by Guest and Hutchinson [11] that (in our language) the lift of a colored-Laman graph is not Laman-sparse.

6 Groups with Rotations and Translations

The final case of Theorem 2 is that of crystallographic groups acting discretely and cocompactly by translations and rotations. It is a classical fact [2, 3] that all such groups other than $\mathbb{Z}^2$ are semi-direct products of the form

$$\Gamma_k := \mathbb{Z}^2 \rtimes \mathbb{Z}/k\mathbb{Z}$$

where $k = 2, 3, 4, 6$. The action on $\mathbb{Z}^2$ by the generator of $\mathbb{Z}/k\mathbb{Z}$ is given by the following table.

k	2	3	4	6
Matrix	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

6.1 *The Quantities* teich$(\cdot)$ *and* cent$(\cdot)$ *for Subgroups*

For any discrete faithful representation $\Phi : \Gamma_k \to \text{Euc}(2)$, it is a (non-obvious) fact that for any element $t \in \mathbb{Z}^2$ in Γ_k, the image $\Phi(t)$ is necessarily a translation, and for any $r \in \Gamma_k \setminus \mathbb{Z}^2$, the image $\Phi(r)$ is necessarily a rotation. Consequently, we respectively call such elements of Γ_k translations and rotations, and we call $\Lambda(\Gamma_k) = \mathbb{Z}^2$ the translation subgroup of Γ_k. For any subgroup $\Gamma' < \Gamma_k$, its translation subgroup is $\Lambda(\Gamma') = \Gamma' \cap \Lambda(\Gamma_k)$.

Let Φ be a representation of Γ_k. In the cases $k \neq 2$, we must have $\Phi(\Lambda(\Gamma_k))$ preserved by an order k rotation, and so the image of $\Lambda(\Gamma_k)$ is determined by the image of a single nontrivial $t \in \Lambda(\Gamma_k)$. Furthermore, by acting on Φ by a rotation in Euc(2), we can always obtain a new representation Φ' such that $\Phi(t)$ has translation vector $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$. Consequently, we have shown the following.

Proposition 14. *Let Γ' be a subgroup of Γ_k for $k = 3, 4, 6$. Then,* $\text{teich}_{\Gamma_k}(\Lambda(\Gamma')) = 1$ *if $\Lambda(\Gamma')$ is nontrivial and is* 0 *otherwise.*

In the case of $k = 2$, it turns out that since order 2 rotations preserve all lattices, this puts no constraint on how Φ embeds $\Lambda(\Gamma_2)$. Consequently, we have teich values similar to the periodic case.

Proposition 15. *Let Γ' be a subgroup of Γ_2. Then,* $\operatorname{teich}_{\Gamma_k}(\Lambda(\Gamma')) = max(2\ell - 1, 0)$ *where* $\ell = \operatorname{rk}(\Lambda(\Gamma'))$.

The dimension of the centralizer, similarly, is concrete and computable. If a subgroup contains a translation t, then $\Phi(t)$ commutes precisely with translations of Euc(2). If a subgroup Γ' of Γ_k is a cyclic subgroup of rotations, then $\Phi(\Gamma')$ is a group of rotations with the same rotation center, and it is easy to see that such a group commutes precisely with the (1-dimensional) subgroup in Euc(2) of rotations with that center. Consequently, we obtain the following characterization of cent.

Proposition 16. *Suppose that Γ' is a subgroup of Γ_k. Then,*

$$\operatorname{cent}(\Gamma') = \begin{cases} 3 & \text{if } \Gamma' \text{ is trivial} \\ 2 & \text{if } \Gamma' \text{ contains only translations} \\ 1 & \text{if } \Gamma' \text{ contains only rotations} \\ 0 & \text{if } \Gamma' \text{ contains both rotations and translations} \end{cases}$$

6.2 *The Quantities* teich(·) *and* cent(·) *for Colored Graphs*

For any Γ_k-colored graph $(G', \boldsymbol{\gamma})$, we associate subgroups of Γ_k. Suppose G has components $G'_1, \dots, G'_c$ and choose base vertices $b_1, \dots, b_c$. We set $\Gamma'_i = \rho(\pi_1(G'_i, b_i))$, and

$$\Lambda(G_1) = \langle \Lambda(\Gamma'_1), \Lambda(\Gamma'_2), \dots, \Lambda(\Gamma'_c) \rangle$$

The Γ-Laman sparsity counts are defined in terms of $\operatorname{teich}(\Lambda(G))$ and $\operatorname{cent}(\Gamma'_i)$. Since we chose base vertices b_i, one might worry that these quantities are not well-defined. However, changing the base vertex in G_i has the effect of conjugating Γ'_i. In Γ_k, conjugates of translations are translations and conjugates of rotations are rotations, so cent(·) is then well-defined by Proposition 16, and $\operatorname{teich}(\Lambda(G))$ for $k = 3, 4, 6$ by Proposition 14. Indeed, for $k = 3, 4, 6$, $\operatorname{teich}(\Lambda(G)) = 1$ if any $\Lambda(\Gamma'_i)$ is nontrivial and is 0 otherwise. In Γ_2, all translation subgroups are normal, so $\Lambda(\Gamma'_i)$ itself does not depend on the choice of base vertex.

6.3 *Computing* teich *and* cent *for Γ_2-Colored Graphs*

A quick and simple algorithm exists to compute $\operatorname{teich}(\Lambda(G))$ and $\operatorname{cent}(\Gamma'_i)$ which relies on finding a suitable generating set for Γ'_i. A generating set for $\pi_1(G'_i, b_i)$ can be constructed as follows. Find a spanning tree T_i of component G_i. Then for each edge $jk \in G_i - T_i$, let P_{jk} be the path traversing the (unique) path b_i to j in T_i, then jk, and then the (unique) path k to b_i in T_i. The P_{jk} ranging over $jk \in G_i - T_i$ generate $\pi_1(G'_i, b_i)$, and so $\eta_{jk} := \rho(P_{jk})$ ranging over the same set generates Γ'_i.

Next, relabel the generators of Γ_i' as $r_{j,i}, t_{j,i}$ where the $r_{j,i}$ are rotations and the $t_{j,i}$ are translations. If there are only translations, no modifications are required and $\Lambda(\Gamma_i') = \Gamma_i'$. Otherwise, set $t_{j,i}' = r_{1,i} r_{j,i}$ for $j \geq 2$. Since all rotations are order 2, the $t_{j,i}'$ are all translations and Γ_i' is generated by $r_{1,i}$, the $t_{j,i}'$ and the $t_{j,i}$. At this point, checking cent Γ_i' is straightforward. Furthermore, one can show that $\Lambda(\Gamma_i')$ is generated by the $t_{j,i}'$ and $t_{j,i}$, and so $\Lambda(G)$ is generated by the $t_{j,i}'$ and the $t_{j,i}$ over all i and j. Then, computing $\mathrm{rk}(\Lambda(G))$ is basic linear algebra, and $\mathrm{teich}(\Lambda(G))$ is given by Proposition 15.

6.4 *Computing* teich *and* cent *for Γ_k-Colored Graphs for $k \neq 2$*

In this case, all that needs to be determined is whether each Γ_i' contains rotations, translations, or both. Compute generators $r_{j,i}, t_{j,i}$ for Γ_i' as above. Then, Γ_i' contains a rotation if and only if there is at least one $r_{j,i}$. The only real difficulty is determining if Γ_i' contains translations when the generators are all rotations. Any group consisting entirely of rotations is cyclic (see, e.g., [24, Lemma 4.2]), and so it suffices to compute the commutators $r_{1,i} r_{j,i} r_{1,i}^{-1} r_{j,i}^{-1}$ for $j \geq 2$. The group contains no translations if and only if these commutators are all trivial.

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Rigidity of Regular Polytopes

Peter McMullen

Abstract A (geometric) regular polygon $\{p\}$ in a euclidean space can be specified by a *fine Schläfli symbol* $\{p\}$, where

$$p = \frac{r}{s_1, \ldots, s_k}$$

is a *generalized fraction*; here, $0 \leqslant s_1 < \cdots < s_k \leqslant \frac{1}{2}r$. This means that $\{p\}$ projects onto planar polygons $\{\frac{r}{s_j}\}$ (reduced to lowest terms) in orthogonal planes, with $\infty = \frac{1}{0}$ giving the linear apeirogon and 2 the digon (line segment). More generally, it may be possible to specify the *shape* or similarity class of a geometric regular polytope by means of a fine Schläfli symbol, whose data contain information about certain regular polygons occurring among its vertices in terms of generalized fractions. If so, then the fine Schläfli symbol is called *rigid*. This paper gives various criteria for rigidity; for instance, the classical regular polytopes are rigid. The theory is also illustrated by several examples. It is noteworthy, though, that a combinatorial description of a regular polytope – a presentation of its symmetry group – can differ considerably from its fine Schläfli symbol.

Keywords Polytope • Abstract • Regular • Realization • Fine Schläfli symbol • Rigidity

Subject Classifications: 51M20, 52C25

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1 Introduction

There are two strands to the theory of regular polytopes, the abstract and the geometric. Obviously, while the geometric regular polytopes came first, more recently they have inspired a vigorous and independent abstract theory; a summary of that theory up to around the beginning of this century is the monograph [16]. Nevertheless, geometric pictures remain an important aspect of the theory, if only because of their intrinsic appeal; realizations form the tool which leads from the abstract back to the geometric. In this arena, beginning with [15] with Schulte (which also revisited earlier work by Grünbaum [6] and Dress [4, 5] with a considerably more efficient treatment), the present author in [9–13] has classified the regular polytopes of full and nearly full rank in all dimensions (the terms used here will be explained later).

In some sense, though, the abstract and geometric theories can diverge quite drastically. To motivate what we do here, we begin with a well-known example. The great dodecahedron $\{5, \frac{5}{2}\}$ is never denoted in any other way in Coxeter's classical book [2]. As an abstract polyhedron, however, it is $\{5, 5 \mid 3\}$ (see, for example, [3]), with the last mark "3" indicating a certain triangular circuit of its edges – we shall explain this notation in Sect. 2. The dual small stellated dodecahedron $\{\frac{5}{2}, 5\}$ is also isomorphic to $\{5, 5 \mid 3\}$, as the symmetric form of the latter expression suggests. From the present viewpoint, the significant fact is that the notation $\{5, \frac{5}{2}\}$ clearly distinguishes the great dodecahedron from its dual $\{\frac{5}{2}, 5\}$ and, indeed, actually determines its geometry (up to similarity), if we make clear what we mean by the marks "5" and "$\frac{5}{2}$".

A prime motivation of the paper – at least originally – was to treat the 12 pure regular apeirohedra in $\mathbb{E}^3$ geometrically rather than abstractly, thus following the spirit of [2]. In [10] we introduced what we shall call a *fine Schläfli symbol* for regular polytopes, using generalized fractions (we refer here to Sect. 4), which was further developed in [11]. This specifies the geometry of a regular polytope (or apeirotope – we do not make the distinction at this stage) by means of certain of its induced regular polygons (with vertices among those of the polytope), albeit at first sight in a rather crude fashion. However, even though it may appear to carry too little information, we shall see that a somewhat abbreviated fine Schläfli symbol – dropping entries that are needed to determine its abstract group – will often suffice to determine the geometry of a regular polytope. On the other hand, we shall encounter cases where no fine Schläfli symbols (at least, in terms of the canonical generators of the symmetry groups) serve to specify their geometry.

Let us outline the rest of the paper. In Sect. 2 we give the necessary background to abstract regular polytopes, in Sect. 3 we outline the theory of their realizations, and in Sect. 4 we describe the notion of a fine Schläfli symbol. In Sect. 5 we introduce the new concept of rigidity of regular polytopes, and in Sect. 6 describe several criteria which ensure it. In the other two sections, we consider various examples to illustrate rigidity or its lack. In Sect. 7 we treat the twelve pure regular 3-dimensional apeirohedra in threes, classed according to their mirror vectors as 3-dimensional apeirohedra, and show that three of the four classes are rigid, explain why the fourth is not, and how to make it rigid by imposing an extra condition. In a similar way, in Sect. 8 we see that the free abelian apeirotopes (over the suitable classical regular polytopes of [2]) are not rigid as they stand, but that a natural extra specification will make them so.

We should emphasize that this paper is intended rather to introduce the concept of rigidity than to give a comprehensive account of it. Thus, while we do intend (in [14], for example) to revisit the regular polytopes and apeirotopes of full or nearly full rank to determine whether simple fine Schläfli symbols can specify their geometry, this is not a primary aim here.

It should be noted that we have slightly changed the notation of [16] and previous papers in the discussion of the abstract theory. It is hoped that these changes are not too disconcerting.

2 Regular Polytopes

An initial motivation for what we do here was a wish to characterize the regular apeirohedra of [15] in some geometric rather than combinatorial way; we shall assume that the reader is moderately familiar with that paper, or the appropriate sections of [16, Chapter 7]. The geometric notation which we employ lies at the centre of our approach. In order to introduce it, we need to say a few brief words about abstract regular polytopes and their realization theory; all that we require is taken from [7, 8] (see also [16, Chapter 5]).

An *abstract regular n-polytope* $\mathscr{P}$ (which may be infinite – we do not distinguish apeirotopes at this stage) is to be identified with its *automorphism group* $\boldsymbol{G}$. This is generated by n *canonical* involutions $\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1}$ (its *distinguished generators*), which are such that $\boldsymbol{r}_j$ and $\boldsymbol{r}_k$ commute if $|j-k| \geqslant 2$; they also satisfy the *intersection property*

$$\langle \boldsymbol{r}_i \mid i \in \mathsf{J} \rangle \cap \langle \boldsymbol{r}_i \mid i \in \mathsf{K} \rangle = \langle \boldsymbol{r}_i \mid i \in \mathsf{J} \cap \mathsf{K} \rangle \tag{1}$$

for all $\mathsf{J}, \mathsf{K} \subseteq \mathsf{N} := \{0, \ldots, n-1\}$. The *j-faces* of $\mathscr{P}$ are identified with the right cosets of the *distinguished* subgroup $\boldsymbol{G}_j := \langle \boldsymbol{r}_i \mid i \neq j \rangle$ for each $j = 0, \ldots, n-1$, with incidence given by non-empty intersection. The cases $j = 0, 1$ and $n-1$ give the *vertices, edges* and *facets* of $\mathscr{P}$, respectively. We take two copies $\boldsymbol{G}_{-1}$ and $\boldsymbol{G}_n$ of the whole group as the unique (-1)- and n-faces of $\mathscr{P}$; we then have a partial ordering $\boldsymbol{G}_j \boldsymbol{a} \leqslant \boldsymbol{G}_k \boldsymbol{b}$ if $\boldsymbol{G}_j \boldsymbol{a} \cap \boldsymbol{G}_k \boldsymbol{b} \neq \emptyset$ and $j \leqslant k$. Regarded as subgroups, $\boldsymbol{G}_{n-1}$ and $\boldsymbol{G}_0$ are abstract regular $(n-1)$-polytopes; as such, they are the (initial) *facet* and *vertex-figure* of $\mathscr{P}$, respectively.

If we drop the intersection property (1), then we obtain what we call a *pre-polytope*; we also refer to $\mathscr{P}$ as *non-polytopal*.

If we list the canonical generators $\boldsymbol{r}_j$ in the reverse order (that is, in effect, reverse the partial ordering), then we obtain the *dual* $\mathscr{P}^\delta$ of $\mathscr{P}$.

As usual, $[p_1, \ldots, p_{n-1}]$, with the $p_j \geqslant 3$ integers (at least in our context) denotes a *Coxeter group*, which is determined solely by the additional relations $(\boldsymbol{r}_{j-1}\boldsymbol{r}_j)^{p_j} = \varepsilon$ for $j = 1, \ldots, n-1$; there is a corresponding universal regular polytope, denoted by $\boldsymbol{\{}p_1, \ldots, p_{n-1}\boldsymbol{\}}$.

Remark 1. We use thick braces $\boldsymbol{\{}\cdots\boldsymbol{\}}$ to denote the *Schläfli type* of an *abstract* regular polytope; the entries in a Schläfli type are always integers (or ∞), because they stand for periods of group elements. The notation admits modifications as described immediately below. This distinguishes the abstract from the geometric, and enables us to use ordinary braces $\{\cdots\}$ for a fine Schläfli symbol, as we shall define it in Sect. 4.

With the regular polytope $\mathscr{P}$ are associated two important regular polygons. First, the *Petrie polygon* has group generators $(\boldsymbol{r}_0\boldsymbol{r}_2\cdots, \boldsymbol{r}_1\boldsymbol{r}_3\cdots)$; second, the *deep hole* has group generators $(\boldsymbol{r}_0, \boldsymbol{r}_1\boldsymbol{r}_2\cdots\boldsymbol{r}_{n-1}\boldsymbol{r}_{n-2}\cdots\boldsymbol{r}_1)$. If the Petrie polygon is an s-gon and the deep hole is a t-gon, then we can employ the notation $\boldsymbol{\{}p_1, \ldots, p_{n-1} : s \mid t\boldsymbol{\}}$ for $\mathscr{P}$, with s or t (and corresponding delimiter) omitted if unneeded. Of course, in general we shall need more relations than these to specify $\mathscr{P}$.

In the case of regular polyhedra (3-polytopes), we can often give a more exact description by specifying the types of its k*-zigzags* with group generators $(\boldsymbol{r}_0\boldsymbol{r}_2, (\boldsymbol{r}_1\boldsymbol{r}_2)^{k-1}\boldsymbol{r}_1)$, and k*-holes* with group generators $(\boldsymbol{r}_0, (\boldsymbol{r}_1\boldsymbol{r}_2)^{k-1}\boldsymbol{r}_1)$. Thus the 1-zigzag is the Petrie polygon, and the 2-hole (that is, deep hole in the general context) is usually just called the *hole*. If the k-zigzag is an s_k-gon and the k-hole is a t_k-gon, then a more general notation is

$$\boldsymbol{\{}p, q : s_1, \ldots \mid t_2, \ldots\boldsymbol{\}}; \tag{2}$$

we replace an unspecified s_j or t_j by $\cdot$, and terminate a string with the last specified s_k or t_k (with no indicator : or $\mid$ if the corresponding string is empty). The reader may also observe that, if $q = 2k$ is even, then the k-zigzag and k-hole appear to coincide; however, this is not true algebraically unless $s_k = t_k$ is even (one step along a zigzag reverses local orientation). Finally, we denote a corresponding entry for the dual $\mathscr{P}^\delta$ (with, we recall, group generators $(\boldsymbol{r}_2, \boldsymbol{r}_1, \boldsymbol{r}_0)$ in the reverse order) by preceding it with $*$. As a particular case of this last, we shall meet $\boldsymbol{\{}\infty, 4 : \cdot, *3\boldsymbol{\}}$ in Sect. 7.3; hence the extra relation corresponds to $((\boldsymbol{r}_0\boldsymbol{r}_1)^2\boldsymbol{r}_2)^3 = \boldsymbol{e}$.

In former times, a notation for zigzags has been employed using subscripts; thus, $\boldsymbol{\{}3, 5\boldsymbol{\}}_5$ rather than $\boldsymbol{\{}3, 5 : 5\boldsymbol{\}}$. However, when we come to realized regular polyhedra, integer marks corresponding to those in (2) are often replaced by generalized fractions (as defined in the next Sect. 3), in which case subscripts may be hard to read.

Related to Petrie polygons is the *Petrie operator* $\boldsymbol{\pi}$. For the moment, we only need it for polyhedra; we generalize it in Sect. 8 to polytopes of higher rank. This operator is

$$\boldsymbol{\pi}\colon (\boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{r}_2) \mapsto (\boldsymbol{r}_0\boldsymbol{r}_2, \boldsymbol{r}_1, \boldsymbol{r}_2) =: (\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2).$$

Observe that $\boldsymbol{\pi}$ is involutory. A polyhedron $\mathscr{Q}$ obtained from another $\mathscr{P}$ by $\boldsymbol{\pi}$ is denoted $\mathscr{Q} := \mathscr{P}^{\pi}$, and is called its *Petrial*. Only in very exceptional cases is $\mathscr{P}^{\pi}$ not actually polytopal; we shall not meet such cases here. If $\mathscr{P}^{\pi} \cong \mathscr{P}$ (with $\cong$ denoting isomorphism), then we call $\mathscr{P}$ *self-Petrie*.

It is useful to have notation which indicates extra relations of the above kind without needing to specify notation for the generators. Such notation was introduced in [16]. An element of the group $\boldsymbol{G} = \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1} \rangle$ is a word in the $\boldsymbol{r}_j$. It can be represented adequately by the string of the corresponding indices j; thus, for example,

$$((\boldsymbol{r}_0\boldsymbol{r}_1)^2\boldsymbol{r}_2)^3 \mapsto ((01)^2 2)^3 = 010120101201012$$

(to take that just mentioned), which now only retains the places of the generators. A number-string corresponding to a relation is called a *relator*; if we wish to impose certain additional relations on $\boldsymbol{G}$, given by the relators $\mathsf{J}_1, \ldots, \mathsf{J}_r$, then we write the quotient as $\boldsymbol{G}/\langle\langle \mathsf{J}_1, \ldots, \mathsf{J}_r \rangle\rangle$. We employ the same convention for the corresponding regular polytope, so that the example just encountered is

$$\{\infty, 4 : \cdot, *3\} = \{\infty, 4\}/\langle\langle ((01)^2 2)^3 \rangle\rangle = \{\cdot, 4\}/\langle\langle (01012)^3 \rangle\rangle.$$

The amalgamation problem asks whether, given two regular n-polytopes $\mathscr{P}$ and $\mathscr{Q}$ such that the vertex-figure of $\mathscr{P}$ and the facet of $\mathscr{Q}$ are isomorphic, there exists a regular $(n+1)$-polytope whose facet is $\mathscr{P}$ and vertex-figure is $\mathscr{Q}$. If one such does exist, then there is a universal one, denoted $\{\mathscr{P}, \mathscr{Q}\}$, which covers (as quotients) any other. In terms of the corresponding automorphism groups $\boldsymbol{G}$ and $\boldsymbol{H}$, we must think of $\boldsymbol{G} = \langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_{n-1} \rangle$ and $\boldsymbol{H} = \langle \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n \rangle$, so that the group of $\{\mathscr{P}, \mathscr{Q}\}$ is $\langle \boldsymbol{r}_0, \ldots, \boldsymbol{r}_n \rangle$, with no relators apart from $(0n)^2$ (which we need to make a string group) other than those arising from $\boldsymbol{G}$ and $\boldsymbol{H}$. A further important question is whether the universal amalgam is finite if $\mathscr{P}$ and $\mathscr{Q}$ are.

3 Realizations

A *realization* of $\mathscr{P}$ is induced by a representation $\Phi \colon \boldsymbol{G} \to \mathbf{M}$ as a subgroup $\mathbf{G} := \boldsymbol{G}\Phi$ of the group $\mathbf{M} = \mathbf{M}(\mathbb{E})$ of isometries of some euclidean space $\mathbb{E}$. In particular, each $\boldsymbol{r}_j$ corresponds to a *reflexion* or involutory isometry $R_j := \boldsymbol{r}_j\Phi$, or possibly the identity I. For most purposes, we can identify a reflexion R with its *mirror* of fixed points $\{x \in \mathbb{E} \mid xR = x\}$. The *mirror* or *dimension vector* of the realization is then $(\dim R_0, \ldots, \dim R_{n-1})$ (we have adopted the new term from [17]).

We identify faces of the realization with their vertex-sets, together with the partial ordering induced from that of $\mathscr{P}$. *Wythoff's construction* picks an *initial vertex* $v \in W := R_1 \cap \cdots \cap R_{n-1}$, the *Wythoff space*; the family of these realizations is denoted $\mathscr{P}(\boldsymbol{G}, \Phi)$. Thus the *vertex-set* $\operatorname{vert} P$ of the whole realization P is $v\mathbf{G}$.

In a similar way, the vertex-set $\operatorname{vert} F_j$ of the *initial j-face* F_j is $v\mathbf{G}_j$, with $\mathbf{G}_j := \boldsymbol{G}_j\Phi$. A general j-face F_jG (with $G \in \mathbf{G}$) then has vertex-set $(\operatorname{vert} F_j)G$. In practical terms, we think of 2-faces, in particular, as geometric polygons (or apeirogons, if infinite).

The realization Φ induces one of the vertex-figure of $\mathscr{P}$ as well. Let $w := vR_0$ be the other vertex of the initial edge of P. If we take the initial vertex to be $\frac{1}{2}(v+w)$ (which is fixed by R_0), then we obtain the *narrow* vertex-figure. However, it is usually more convenient to take the initial vertex to be w itself, so that the vertices of the vertex-figure sit among those of the whole realization P; this yields the *broad* vertex-figure. We shall make our usage clear in each context.

A realization Φ is *faithful* if the abstract and geometric incidence structures are isomorphic. Note that a non-faithful realization need not actually be polytopal.

Now suppose that L, M are orthogonal subspaces of $\mathbb{E}$, and that we have realizations P in L with mirrors $S_0, \ldots, S_{k-1}$ and initial vertex v and Q in M with mirrors $T_0, \ldots, T_{m-1}$ and initial vertex w; in fact, we need not require that the realizations be of the same polytope (see Remark 3 below). Then the *blend* $P \# Q$ has mirrors $R_j := S_j + T_j$ for $j = 0, \ldots, n-1$ with $n = \max\{k, m\}$ and $S_j = L$ if $j \geqslant k$ or $T_j = M$ if $j \geqslant m$, and initial vertex $(v, w) \in R_1 \cap \cdots \cap R_{n-1}$. We can also *scale* a realization by (for example) $R_j \mapsto \lambda R_j$ with (usually) $\lambda \neq 0$. Note that there are trivial expressions as blends, since

$$\lambda P \# \mu P = \nu P,$$

where $\nu^2 = \lambda^2 + \mu^2$. A realization which cannot be expressed as a non-trivial blend is called *pure*.

Remark 2. It was shown in [7] that the realization space of a fixed regular polytope $\mathscr{P}$ has the structure of a convex cone (denoted by the same symbol in the convention introduced in [7] and followed hitherto), which is closed if $\mathscr{P}$ is finite. Moreover, it need not be the case that $\mathscr{P}$ has a faithful realization. Henceforth, we shall depart from the previous convention, and use $\mathscr{P}$ to denote some specified subcone of the realization cone, rather than the whole cone.

Remark 3. The notion of blend extends in a natural way to different (abstract) regular polytopes $\mathscr{P}$ and $\mathscr{Q}$, although the blend of $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$ will not necessarily be polytopal. In this abstract context, we talk about the *mix* of $\mathscr{P}$ and $\mathscr{Q}$.

Remark 4. To avoid constant repetition, we take for granted from now on that all polytopes under discussion are regular. We often refer to apeirotopes, rather than infinite polytopes; in such cases, we shall further insist that the vertex-sets $v\mathbf{G} \subseteq \mathbb{E}$ be discrete. While non-discrete realizations play an important rôle in the general theory, we shall not treat them here.

4 Fine Schläfli Symbols

A fine Schläfli symbol specifies a realization subcone $\mathscr{P}$ by describing in general terms the geometry of certain regular polygons (that is, 2-polytopes) that occur among the vertices of $\mathscr{P}$. Thus $\mathscr{P}$ will now be a subcone of the whole realization cone of some abstract regular polytope, but the abstract type of $\mathscr{P}$ will generally be determined by geometric conditions which are not indicated in its notation.

In this sense, the realization space of a geometric regular polygon $\mathscr{P}$ is denoted by

$$\{p\} := \left\{\frac{s}{t_1, \ldots, t_k}\right\},$$

with s a positive integer and $t_1, \ldots, t_k$ non-negative integers such that $0 \leqslant t_1 < \cdots < t_k \leqslant \frac{1}{2}s$ (with strict inequalities to avoid trivial blends); moreover, their greatest common divisor is $(s, t_1, \ldots, t_k) = 1$. This means that

$$\{p\} = \{p_1\} \# \cdots \# \{p_k\}, \tag{3}$$

with $p_j := s/t_j$ in lowest terms; these $\{p_j\}$ are the *components* of $\{p\}$. There are two special cases: $\{\frac{1}{0}\}$ is the linear apeirogon (equally spaced points along the real line $\mathbb{R}$, and often denoted $\{\infty\}$ when there can be no ambiguity), while $\{2\}$ is the *digon* or (line) segment, with two vertices (previously – for example, in [16] – usually denoted $\{\ \}$). Then each $\{p_j\}$ for $p_j \neq \infty, 2$ is a planar regular polygon, which will be a star-polygon if p_j is not an integer. We call the mark p here a *generalized fraction*.

Remark 5. Observe that the family $\mathscr{P}$ of polygons with the fine Schläfli symbol $\{p\}$ of (3) forms a k-dimensional cone, because there are k degrees of freedom for the relative sizes of the k components.

We shall write $|p| := s$ when $t_1 > 0$, so that $\{p\}$ is then a finite $|p|$-gon, and $|p|$ is the period of the product of its generating reflexions.

We should say a few more words here about certain special regular polygons or apeirogons. First, we have *skew* or *zigzag* polygons, of the form $\{\frac{2k}{t,k}\}$; the most frequently occurring cases in this paper are $k = 2$ or 3 and $t = 1$. Note that the polygon $\{\frac{6}{1,3}\}$ is centrally symmetric. Second, we have the *r-helix* $\mathscr{H}_r = \{\frac{r}{0,1}\}$ with r an integer; its fundamental property is that, if the vertices of some $H \in \mathscr{H}_r$ are $\ldots, a_{-1}, a_0, a_1, a_2, \ldots$, then $t = t(H) := a_{r+j} - a_j$ (for any j) is a vector which yields a translational symmetry of H, though not necessarily of a regular apeirotope whose specification involves helices in $\mathscr{H}_r$. (More general helices will also occur, and we shall draw attention to them in the appropriate place.)

In general, a *fine Schläfli symbol* specifies a realization subcone by introducing generalized fractions into the abstract notation of Sect. 2 instead of integer entries or ∞. So, at the very least, a fine Schläfli symbol will look like $\{p_1, \ldots, p_{n-1}\}$, where the p_j are generalized fractions; in general, further geometric data are given. In particular, while in discrete realizations we may have infinite 2-faces, nevertheless in a fine Schläfli symbol the geometric type of these apeirogons will be specified.

Suppose that J and K are relators corresponding to involutory elements (reflexions) S and T in the realization $\mathscr{P}$. Then $\langle S, T\rangle$ is the symmetry group of a polygon $\{q\}$ with initial vertex a vertex of $\mathscr{P}$ lying in T. It is important to bear in mind that the order (S, T) is crucial here; while the periods of the products ST and TS are the same, reversing the order may yield a different regular polygon $\{r\}$. An example to illustrate this is the Petrie-Coxeter apeirohedron $\{4, \frac{6}{1,3} \mid 4\}$, whose (geometric) dual $\{6, \frac{4}{1,2} \mid 4\}$ is obtained by reversing the order of the canonical generating reflexions (R_0, R_1, R_2); thus (R_1, R_2) gives a skew hexagon $\{\frac{6}{1,3}\}$, while (R_2, R_1) gives a planar hexagon $\{6\}$. The resulting fine Schläfli symbol is then of the form

$$\{p_1, \dots, p_{n-1}\}/\langle\langle \dots, (\mathsf{J}\cdot\mathsf{K})^q, \dots\rangle\rangle,$$

where $p_1, \dots, p_{n-1}, q$ are now generalized fractions, and $\cdot$ is used as an indicatory separator. Abstractly, of course, we have a corresponding relator $(\mathsf{JK})^{|q|}$ when appropriate.

Remark 6. We continue to use notation such as $\{p_1, \dots, p_{n-1} : s\}$ or $\{p_1, \dots, p_{n-1} \mid t\}$ to specify Petrie polygons or deep holes. The main saving avoids having to employ such clumsy concatenations as we have had in the past. Thus, for $\{4, 3, 4\}^\pi$ (which turns up in Sect. 8), we could write

$$\{4, \tfrac{6}{1,3}, 4\}/\langle\langle (0\cdot 121)^4, (13\cdot 2)^3\rangle\rangle = \{\{4, \tfrac{6}{1,3} \mid 4\}, \{\tfrac{6}{1,3}, 4 : 3\}\},$$

although this is perhaps not a particularly good example.

We end the section with some useful notation. For rational $q \geqslant 2$ or $q = \infty$, define q'' by

$$\frac{1}{q} + \frac{1}{q''} = \frac{1}{2},$$

with the obvious conventions $2'' = \infty$ and $\infty'' = 2$. More generally, if $\{p\}$ is as in (3), set

$$\{p''\} = \{p''_k\} \# \cdots \# \{p''_1\}, \tag{4}$$

which we call the *supplement* of $\{p\}$; the reversal of order of the components is to accord with the generalized fraction notation. Supplementary polygons frequently arise from our constructions.

5 Rigidity

As we have said, a fine Schläfli symbol determines a subcone of the realization cone of a regular polytope by demanding that certain of its induced regular polygons be given by specific generalized fractions. We shall refer to the similarity class of a geometric regular polytope as its *shape*. We should emphasize from the outset that a given regular polytope may have different fine Schläfli symbols, each of which nevertheless determines its shape. We address this topic here (and give more examples in Sect. 7).

A fine Schläfli symbol therefore determines a subcone $\mathscr{P}$ of the realization cone of some abstract regular polytope, assuming (of course) that the corresponding group relations yield a string C-group $\boldsymbol{G}$. If this cone is a ray (that is, 1-dimensional), so that $\mathscr{P}$ consists of a single similarity class, then we say that $\mathscr{P}$ is *rigid*. Thus the polytopes in $\mathscr{P}$ must be pure (and faithful) realizations, but the converse need not hold; rigidity is a geometric rather than an algebraic property. For convenience, we shall usually talk about subcones $\mathscr{P}$ rather than their individual polytopes.

More generally, we can ask what subcone $\mathscr{P}$ is specified by a fine Schläfli symbol and, in particular (in the case of apeirotopes) whether $\mathscr{P}$ is finite-dimensional. In this context, it is worth reminding ourselves that the realization cones of (abstract) regular apeirotopes will generally have uncountably infinite algebraic dimension, as was shown in [8].

There is a complementary problem. Given a geometric regular polytope, we may ask whether there is a fine Schläfli symbol which prescribes its shape.

With the understanding that they are fine Schläfli symbols, we shall write $\mathscr{P} \preccurlyeq \mathscr{Q}$ to mean that the cone specified by $\mathscr{P}$ is a subcone of that specified by $\mathscr{Q}$, and $\mathscr{P} \approx \mathscr{Q}$ to mean that the two fine Schläfli symbols specify the same cone.

Before we go on to establish some general criteria, we give a couple of simple examples. The realization cone of the regular icosahedron is 3-dimensional; a general realization will have three pure components: the usual 3-dimensional icosahedron $\{3,5\}$ and great icosahedron $\{3,\frac{5}{2}\}$, and the 5-dimensional hemi-icosahedron $\{3,\frac{5}{1,2} : \frac{5}{1,2}\}$. The first two are isomorphic, while the third is (abstractly) $\boldsymbol{\{}3,5 : 5\boldsymbol{\}}$ ($=\boldsymbol{\{}3,5\boldsymbol{\}}_5$ in the natural adaptation of the old notation). The usual Schläfli symbols – their fine Schläfli symbols in our terms – distinguish the first two; the *abstract* type of the third is actually enough to determine the geometry of its realizations, because each pair of its six vertices forms an edge.

However, the case of the regular dodecahedron is a little different. Here, the realization cone is 5-dimensional; again, there are five pure components. Two of these are the familiar 3-dimensional dodecahedron $\{5,3\}$ and great stellated dodecahedron $\{\frac{5}{2},3\}$. There is also a 4-dimensional faithful realization, whose fine Schläfli symbol is $\{\frac{5}{1,2},3 : \frac{10}{1,3}\}$; this again determines its shape completely.

There are also two realizations – one 4-dimensional and the other 5-dimensional – of the hemi-dodecahedron $\boldsymbol{\{}5,3 : 5\boldsymbol{\}}$ ($=\boldsymbol{\{}5,3\boldsymbol{\}}_5$). However, these both have fine Schläfli symbol $\{\frac{5}{1,2},3 : \frac{5}{1,2}\}$, with no apparent way of separating them by regular edge-circuits and the like. In fact, they can only be distinguished by a more detailed description of their faces. The angle at a vertex of the general regular pentagon varies between $\pi/5 = \arccos(\tau/2)$ for the pentagram and $3\pi/5 = \arccos(-\tau^{-1}/2)$ for the (convex) pentagon (as usual, $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the golden section); that for the face of the general $\{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}$ varies from $\arccos(3/4)$ for the 4-dimensional pure realization to $\pi/2$ for the 5-dimensional one (in between, the realization is 9-dimensional and blended). The crucial point is that $\{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}$ does not determine a ray in the realization cone of the dodecahedron.

Remark 7. The polyhedron $\mathscr{P} = \{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}$ is self-Petrie, as its notation indicates. While $\mathscr{P}$ has certain regular hexagonal edge-circuits, these do not have their full symmetries in the group of $\mathscr{P}$. However, we obtain all their symmetries, if we adjoin an outer automorphism T of the symmetry group of $\mathscr{P}$ which interchanges its faces and Petrie polygons. In the 4-dimensional realization these hexagons are $\{\frac{6}{2,3}\}$, while in the 5-dimensional one they are $\{\frac{6}{1,3}\}$.

It should be noted as well that more than geometry is involved here. For instance, the edges and diagonals of the vertex figure of the hemi-icosahedron $\{3, \frac{5}{1,2} : \frac{5}{1,2}\}$ are equal. But this is also true of the realization of the icosahedron whose vertices are those of the regular 6-cross-polytope; this will have fine Schläfli symbol $\{3, \frac{5}{1,2}\}$, illustrating the fact that $\{3, \frac{5}{1,2} : \frac{5}{1,2}\} \preccurlyeq \{3, \frac{5}{1,2}\}$ (the latter, of course, not being rigid).

6 General Criteria

We now describe a number of general conditions which will ensure rigidity of a regular polytope. Bear in mind here that rigidity depends on a given fine Schläfli symbol, so that one that ensures rigidity may be rather different from one that describes its abstract type.

By definition, a rigid regular polytope $\mathscr{P}$ determines a single similarity class. The same is then true of all sections of $\mathscr{P}$, even though some of these may not be rigid themselves. Thus many of our arguments really depend less on rigidity itself than on working within fixed similarity classes. In this context, it is natural to raise

Question 1. Suppose that $\mathscr{P}$ is a geometric regular n-polytope such that, for some $2 \leqslant k \leqslant n-2$, the k-face and $(k-1)$-coface of $\mathscr{P}$ are constrained to lie in fixed similarity classes. Does $\mathscr{P}$ then consist of a single similarity class?

What we shall do in this section is establish rigidity in a variety of different general cases. The first covers the classical regular polytopes.

Theorem 1. *If* rank $\mathscr{P} \geqslant 2$ *and the fine Schläfli type* $\mathscr{P}$ *has planar* 2*-faces and rigid vertex-figure* $\mathscr{Q}$*, then* $\mathscr{P}$ *is rigid.*

Proof. This is clear, because the ratio $1 : 2\cos(\pi/p)$ between the edge-length and next diagonal of the planar regular polygon $\{p\}$ fixes the relative distance of the initial vertex from the adjacent vertices. Since $\mathscr{Q}$ is itself rigid, this fixes the whole geometry of $\mathscr{P}$.

Remark 8. It is worth pointing out that even more is determined in this case. If the distance forces the initial vertex v to be at the centre of the vertex-figure $\mathscr{Q}$, then $\mathscr{P}$ is necessarily infinite. If the distance is greater, then $\mathscr{P}$ must be finite, and its centre and circumradius can immediately be found.

Corollary 1. *The classical regular polytopes and honeycombs are rigid.*

We recall that the regular polytopes of [2] have symmetry groups generated by hyperplane reflexions; all the entries in their Schläfli symbols are simple fractions (and thus correspond to planar polygons).

Observe, in fact, that all 18 of the finite regular polyhedra in $\mathbb{E}^3$ are rigid, if we specify them appropriately. For instance, as we have seen, the great dodecahedron is $\{\frac{5}{2}, 3\}$. Its Petrial is $\{\frac{10}{3,5}, 3 : \frac{5}{2}\}$, which distinguishes it from the Petrial $\{\frac{10}{1,5}, 3 : 5\}$ of the Platonic dodecahedron $\{5, 3\}$. (Indeed, we could denote these Petrials by $\{\cdot, 3 : \frac{5}{2}\}$ and $\{\cdot, 3 : 5\}$, respectively, since the faces are determined by the Petrie polygons.) Thus we have another family of regular polytopes which are automatically rigid.

Theorem 2. *Suppose that* rank $\mathscr{P} \geqslant 3$ *and that the fine Schläfli type of* $\mathscr{P}$ *is such that the type of the vertex-figure is a blend* $\mathscr{Q}\,\#\,\{2\}$ *with* $\mathscr{Q}$ *rigid, and the* 2*-face and the hole of the* 3*-face of* $\mathscr{P}$ *are planar. Then* $\mathscr{P}$ *is rigid.*

Proof. Again, the geometry of the 2-face and hole fix the shape of the skew 2-faces of the vertex figure; in turn (as in the proof of the previous Theorem 1), this fixes the geometry of $\mathscr{P}$.

Theorem 2 enables us to deal with another wide range of examples in a categorical fashion, and so no individual treatment of such cases will be needed.

We end the section with a useful observation.

Remark 9. The faces and Petrie polygons of a geometric regular polyhedron have the same angle.

7 Apeirohedra in 3-Dimensions

In this section, we consider the 3-dimensional pure regular apeirohedra, with a view to determining which of them are rigid; for more details about these apeirohedra, see [15] or [16, Section 7E]. We treat them according to their mirror vectors. As we shall see, three of the four classes consist of rigid apeirohedra, while those of the fourth need an extra condition to ensure rigidity.

7.1 Mirror Vector $(2, 1, 2)$

The apeirohedra under discussion here are those discovered by Petrie and Coxeter; see [1]. The appropriate fine Schläfli symbols for these are

$$\{6, \tfrac{6}{1,3} \mid 3\} \cong \boldsymbol{\{}6, 6 \mid 3\boldsymbol{\}},$$

$$\{4, \tfrac{6}{1,3} \mid 4\} \cong \boldsymbol{\{}4, 6 \mid 4\boldsymbol{\}},$$

$$\{6, \tfrac{4}{1,2} \mid 4\} \cong \boldsymbol{\{}6, 4 \mid 4\boldsymbol{\}}.$$

The only difference between the fine Schläfli symbols and the abstract ones is that the geometry of the faces, vertex-figures and holes is specified; it is worth emphasizing that (for example) an entry '6' refers to a *planar* hexagon, whereas an entry '$\frac{6}{1,3}$' means a *skew* hexagon.

The classification here is a special case of Theorem 2.

Theorem 3. *The regular polyhedra* $\{6, \frac{6}{1,3} \mid 3\}$, $\{4, \frac{6}{1,3} \mid 4\}$ *and* $\{6, \frac{4}{1,2} \mid 4\}$ *are rigid, and hence are the corresponding* 3*-dimensional Petrie-Coxeter apeirohedra.*

Thus this class consists of rigid apeirohedra. We shall see further applications of Theorem 2 in the following Sect. 8.

7.2 Mirror Vector $(1, 1, 2)$

Theorem 3 was, perhaps, exactly what should have been expected. In this section, though, we show that the situation for the Petrials of the Petrie-Coxeter apeirohedra is a little different. In [15], we specified these Petrials by their 1- and 2-zigzags, the 1-zigzag being the Petrie polygon. However, the notation there gave abstract descriptions of the apeirohedra; in particular, the faces were merely indicated by $\{\infty\}$. If we restore the geometric information about the faces, then things change. In fine, we shall show that describing the faces by generalized fractions enables us to drop mention of the 2-zigzags.

So, assuming the result of Theorem 4, we shall have

$$\{\tfrac{4}{0,1}, \tfrac{6}{1,3} : 6\} \cong \boldsymbol{\{}\infty, 6 : 6, 3\boldsymbol{\}},$$

$$\{\tfrac{3}{0,1}, \tfrac{6}{1,3} : 4\} \cong \boldsymbol{\{}\infty, 6 : 4, 4\boldsymbol{\}},$$

$$\{\tfrac{3}{0,1}, \tfrac{4}{1,2} : 6\} \cong \boldsymbol{\{}\infty, 4 : 6, 4\boldsymbol{\}};$$

we remark here that the notation for these apeirohedra in [12] is incorrect.

The fine Schläfli symbols here provide a good illustration of the way that the notation differs from that of the abstract apeirohedra. So, in the first, the apeirogonal face is now specified geometrically as $\{\frac{4}{0,1}\} = \{\infty\} \# \{4\}$, a 4-helix. Similarly, the vertex-figure $\{\frac{6}{1,3}\} = \{6\} \# \{2\}$ is a skew hexagon. However, it is important to observe that the relative sizes of the components of the blends are *not* determined by the notation; compare Remark 5.

The key observation in this case is that the (planar) Petrie polygons determine the geometry of the helical faces completely. For example, the face $\{\frac{3}{0,1}\}$ of $\{\frac{3}{0,1}, \frac{6}{1,3} : 4\}$ has the same angle $\pi/2$ at a vertex as that of the Petrie polygon $\{4\}$ (see Remark 9), and this shows that it is congruent to the Petrie polygon of the tiling of $\mathbb{E}^3$ by cubes (with the same edge-length) or – more importantly – that of $\{4, \frac{6}{1,3} \mid 4\}$. In general, let $\ldots, a_{-1}, a_0, a_1, a_2, \ldots$ be the successive vertices of the initial face F, say, of the given apeirohedron P (we know from its faces that P must be infinite). For each j, let G_j be the Petrie polygon of P of which a_{j-1}, a_j, a_{j+1} are successive vertices; notice that each G_j lies in the 3-dimensional space $\mathbb{E}$ spanned by F. If $\{a_0, c_j\}$ is the other edge of G_j through a_0 for $j = -1, 1$, then c_{-1}, a_{-1}, a_1, c_1 are four successive vertices of the vertex-figure Q of P at a_0. We deduce that the whole of Q lies in the same 3-space $\mathbb{E}$, and that a_0 must be the centre of Q. Then every face of P through a_0 (and similarly though each a_j) also lies in $\mathbb{E}$; thus P lies in $\mathbb{E}$, and so is 3-dimensional. Since it is clear that P will coincide locally with the corresponding Petrial of the (geometric) Petrie-Coxeter apeirohedron, it will therefore coincide globally, which is what we want.

We conclude that we have established

Theorem 4. *The regular apeirohedra* $\{\frac{4}{0,1}, \frac{6}{1,3} : 6\}$, $\{\frac{3}{0,1}, \frac{6}{1,3} : 4\}$ *and* $\{\frac{3}{0,1}, \frac{4}{1,2} : 6\}$ *are rigid; they are the Petrials of the corresponding* 3*-dimensional Petrie-Coxeter apeirohedra.*

There is an interesting consequence of this characterization.

Corollary 2. *The three regular apeirohedra* $\{6, \frac{6}{1,3} : \frac{4}{0,1}\}$, $\{4, \frac{6}{1,3} : \frac{3}{0,1}\}$ *and* $\{6, \frac{4}{1,2} : \frac{3}{0,1}\}$ *are rigid; they are the corresponding* 3*-dimensional Petrie-Coxeter apeirohedra.*

Proof. This is clear from Theorem 4; we are just representing these apeirohedra as the Petrials of those characterized by that theorem.

In other words, we have alternative expressions for the (geometric) Petrie-Coxeter apeirohedra:

$$\{6, \tfrac{6}{1,3} : \tfrac{4}{0,1}\} \cong \boldsymbol{\{}6, 6 \mid 3\boldsymbol{\}},$$

$$\{4, \tfrac{6}{1,3} : \tfrac{3}{0,1}\} \cong \boldsymbol{\{}4, 6 \mid 4\boldsymbol{\}},$$

$$\{6, \tfrac{4}{1,2} : \tfrac{3}{0,1}\} \cong \boldsymbol{\{}6, 4 \mid 4\boldsymbol{\}}.$$

Observe that the geometric and abstract descriptions are now significantly different. Note also that we can write $\{6, \frac{6}{1,3} : \frac{4}{0,1}\} \approx \{6, \frac{6}{1,3} \mid 3\}$, in the notation introduced in Sect. 5, and so on.

7.3 Mirror Vector $(1, 1, 1)$

The three regular apeirohedra in this class are

$$\{\tfrac{3}{0,1}, 3 : \tfrac{4}{0,1}\} \cong \{\infty, 3\}^{(a)},$$

$$\{\tfrac{4}{0,1}, 3 : \tfrac{3}{0,1}\} \cong \{\infty, 3\}^{(b)},$$

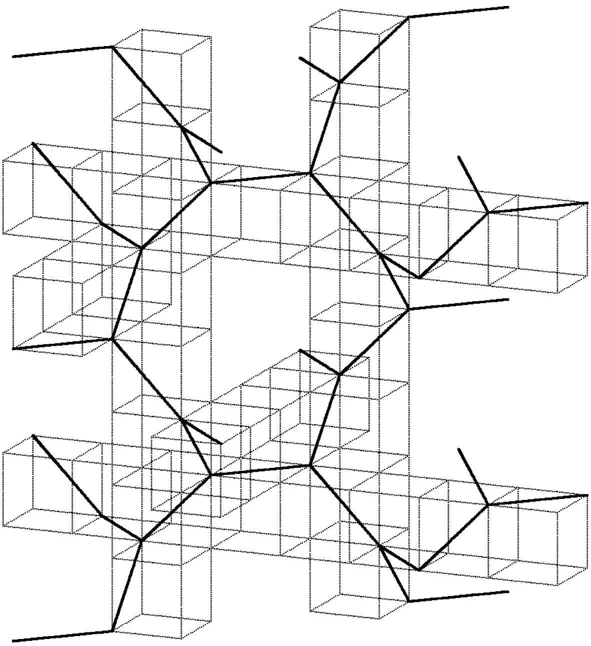
$$\{\tfrac{3}{0,1}, 4 : \tfrac{3}{0,1}\} \cong \{\infty, 4 : \cdot, *3\}.$$

The notation for the first two, which we shall explain shortly (at least, in general terms), is as in [15] (see also [16, Section 7B]); these are two of the regular apeirohedra found by Grünbaum [6], and form a Petrie pair. The third (the only one of the twelve under discussion which was missed by Grünbaum) was due to Dress [4, 5]; its notation was explained in Sect. 2.

We showed in [15, Theorem 7.2] that the two apeirohedra of type $\{\infty, 3\}$ are determined by the fact that the translations induced by the appropriate number of steps along a certain helical face and a certain Petrie apeirogon commute. In one sense, this is an obvious condition arising from the geometry. However, there are hidden assumptions here, which need to be brought out into the open and properly addressed. The basic one is that the translational symmetries of a helix (whether face or Petrie apeirogon) extend to ones of the whole apeirotope; this will be true if the apeirotope can be shown to be 3-dimensional, but will not necessarily hold otherwise. In fact, while the helical symmetries will be appealed to, the core of the proofs of the first two theorems of the section will depend on purely local properties.

We begin with

Theorem 5. *The regular apeirohedra* $\{\frac{3}{0,1}, 3 : \frac{4}{0,1}\}$ *and* $\{\frac{4}{0,1}, 3 : \frac{3}{0,1}\}$ *are rigid, and are isomorphic to* $\{\infty, 3\}^{(a)}$ *and* $\{\infty, 3\}^{(b)}$*, respectively.*



(5)

We picture part of these apeirohedra in (5) above; we have included some 'scaffolding' for 4-helices.

Proof. We begin the proof by observing that, while the notation demands that the faces and Petrie apeirogons be 3-dimensional, there is no initial requirement that this be true of the apeirohedron itself. As in [15], we shall work with the latter apeirohedron; to deal with the former, we merely swap the rôles of face and Petrie apeirogon. The key fact employed is that, as already mentioned in Sect. 4, k steps along a k-helix (here, $k = 3$ or 4) is a translational symmetry of the helix, though not necessarily of the whole apeirohedron.

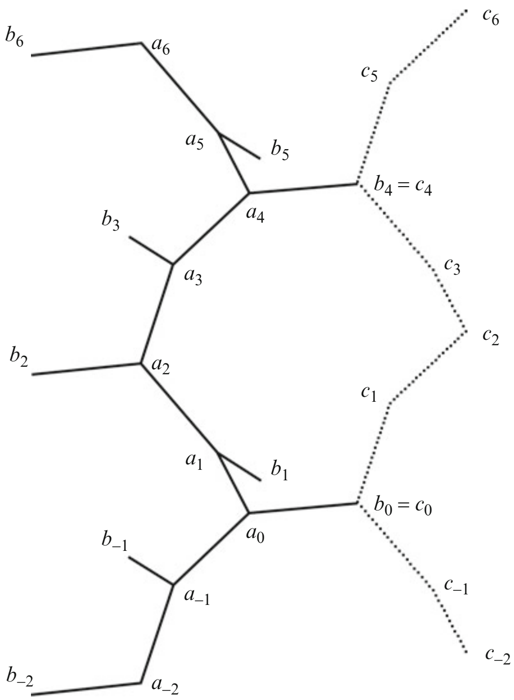
Let $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, \ldots$ be the successive vertices of an initial 4-helical face F_1 of $P \in \{\frac{4}{0,1}, 3 : \frac{3}{0,1}\}$ (as a realization cone, of course); the proof is illustrated in (6). For each such j, there is a third edge $\{a_j, b_j\}$ (that is, other than $\{a_{j-1}, a_j\}$ and $\{a_j, a_{j+1}\}$) containing a_j. For each j again, there is a Petrie apeirogon $G_j = \{\frac{3}{0,1}\}$ of P, with successive vertices $\ldots, b_{j-1}, a_{j-1}, a_j, a_{j+1}, b_{j+1}, \ldots$; thus $a_{j-1} - b_{j-1} = b_{j+1} - a_{j+1}$. Repeating this shows that

$$b_{j+4} - a_{j+4} = b_j - a_j$$

for each $j \in \mathbb{Z}$. Observe that we already have a strong suggestion here of a global translational symmetry of P.

Let F' be the face of P which contains b_0, but not the edge $E := \{a_0, b_0\}$; suppose that it has successive vertices $\ldots, c_{-1}, c_0, c_1, c_2, \ldots$, with $c_0 = b_0$. Consider the two Petrie apeirogons G_{-1} and G_1 of P which share the edge E. Up to changing signs of the indices in F', we will have successive vertices

$$G_{-1}\colon \ldots, b_{-2}, a_{-2}, a_{-1}, a_0, b_0 = c_0, c_1, c_2, \ldots$$
$$G_1\colon \ldots, b_2, a_2, a_1, a_0, b_0 = c_0, c_{-1}, c_{-2}, \ldots$$



(6)

Again using the local translational symmetries of G_{-1}, G_1 and F_1, we have

$$\begin{aligned} c_{-1} - c_{-2} &= a_1 - a_0, \\ c_0 - c_{-1} &= a_2 - a_1 \\ c_1 - c_0 &= a_{-1} - a_0 = a_3 - a_2 \\ c_2 - c_1 &= a_0 - a_{-1} = a_4 - a_3. \end{aligned}$$

We deduce three things at once. First, $F' = F_1 + s_{-1} = F_1 - s_1$, where

$$s_j := a_{j+1} - b_{j-1} = b_{j+1} - a_{j-1}$$

is the basic translational symmetry of G_j for $j \in \mathbb{Z}$. Second, the basic translational symmetry t_1 of F_1 (with $a_{j+4} = a_j + t_1$) is

$$t_1 = s_{j+1} - s_{j-1}$$

for each j. Third, it follows that $c_{4j} = b_{4j}$ for each j; that is, the sets of parallel edges $\{a_{4j}, b_{4j}\}$ all lead from F_1 to F'.

The same then holds for the other sets of parallel edges from F_1, and we easily conclude that the vertices of P fall into the vertex-sets of disjoint faces of P parallel to F_1. The same will then be true of the other two faces F_2 and F_3 of P which contain a_0. Finally, it follows that we have (commuting) translations S_j by s_j of P (with $S_{j+4} = S_j$) and T_k by t_k which satisfy

$$S_{j-1}^{-1} S_{j+1} = T_1$$

and similar relations, showing that the translation group has rank 3; moreover, all this shows that P is just 3-dimensional. We now really need no more description than this to recognize $\{\frac{4}{0,1}, 3 : \frac{3}{0,1}\}$ as the apeirohedron $\boldsymbol{\{}\infty, 3\boldsymbol{\}}^{(b)}$ of [15] or [16, Section 7B]).

Remark 10. Since a k-helix induces a k-fold rotation in the point or special group, we deduce that the special group of $P = \{\frac{4}{0,1}, 3 : \frac{3}{0,1}\}$ is the rotation group $[4, 3]^+ \cong \mathscr{S}_4$ of the cube. Moreover, the edges of P are translates of the six edges of the regular tetrahedron $\{3, 3\}$, perhaps better regarded in the present context as those of its Petrial $\{\frac{4}{1,2}, 3 : 3\}$.

The remaining case is the apeirohedron denoted $\boldsymbol{\{}\infty, 4 : \cdot, *3\boldsymbol{\}}$ in [15]. This apeirohedron is self-Petrie; its faces and Petrie polygons are 3-helices. We shall prove

Theorem 6. *The regular apeirohedron* $\{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$ *is* 3*-dimensional, and is isomorphic to the abstract apeirohedron* $\boldsymbol{\{}\infty, 4 : \cdot, *3\boldsymbol{\}}$.

Proof. We proceed in a somewhat different way from that in Theorem 5, but again – apart from appealing to the intrinsic translational symmetries of the faces or Petrie apeirogons, which we do not need to distinguish – we use purely local properties. Let $P \in \{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$, and let $a \in \operatorname{vert} P$ have adjacent vertices b_0, b_1, b_2, b_3 in this order around the square vertex-figure. For $j = 0, 1, 2, 3$, let the remaining vertices adjacent to b_j be c_j, h_j, d_j, so that a, c_j, h_j, d_j are again the vertices (in this order) of its vertex-figure (see (9), where we have suppressed h_1 and h_3 for clarity). For each $j = 0, \ldots, 3$ (with indices j modulo 4) there is a (unique) face F_j of P containing b_j, a, b_{j+1}; let this have vertices $\ldots, c_j, b_j, a, b_{j+1}, d_{j+1}, \ldots$. Then the Petrie apeirogon G_j of P which also contains b_j, a, b_{j+1} has vertices $\ldots, d_j, b_j, a, b_{j+1}, c_{j+1}, \ldots$.

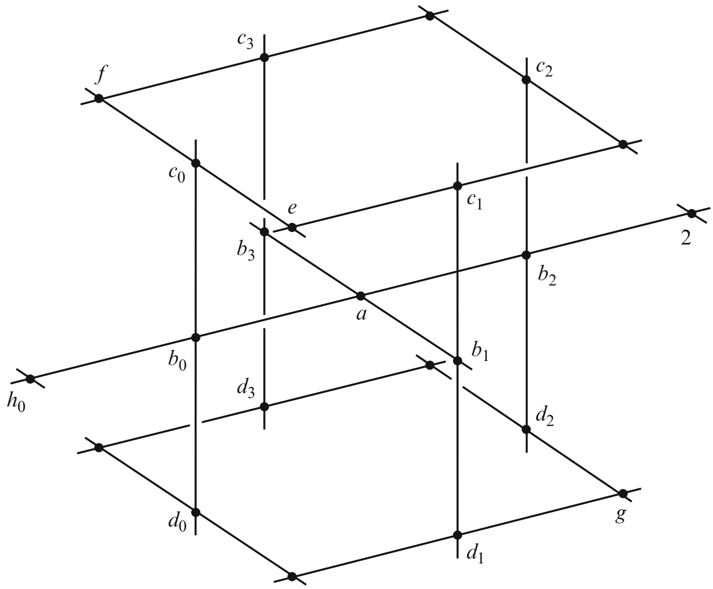
Bearing in mind that the faces are 3-helices, for each j we obtain

$$d_{j+1} - b_j = b_{j+1} - c_j, \quad c_{j+1} - b_j = b_{j+1} - d_j. \tag{7}$$

Subtracting one equation from the other yields

$$c_{j+1} - d_{j+1} = c_j - d_j =: t_3, \tag{8}$$

say.



(9)

The vertices $w_0, \ldots, w_3$ of P (in this order) adjacent to $v \in \operatorname{vert} P$ are coplanar by assumption, and so the *displacement vector* $d(v)$ of v from the centre of its broad vertex-figure is

$$d(v) = v - \tfrac{1}{2}(w_0 + w_s) = v - \tfrac{1}{2}(w_1 + w_3).$$

Adding the two equations of (7) shows that

$$d(b_{j+1}) = -d(b_j), \qquad d(b_{j+2}) = d(b_j)$$

for each j. The latter of these two equations shows that the hole $\ldots, h_0, b_0, a, b_2, h_2, \ldots$ must actually be a zigzag $\{\frac{2}{0,1}\}$, and thus

$$d(a) = -d(b_0).$$

But from the hole $\dots, h_1, b_1, a, b_3, h_3, \dots$ we similarly conclude that $d(a) = -d(b_1) = d(b_0)$, so that $d(a) = -d(a)$, and hence $d(a) = o$.

It follows that each vertex lies at the centre of its vertex-figure. Hence the holes are linear apeirogons $\{\infty\} = \{\frac{1}{0}\}$, the edges fall into three mutually orthogonal families, and thus P itself is 3-dimensional. As a consequence, the holes and hexagons link up, as in the picture, and we have $P \cong \mathbf{\{}\infty, 4 : \cdot, *3\mathbf{\}}$, as claimed. As in the previous case, the special group is $[3,4]^+$. Once more, of course, the translational symmetries of the 3-helices extend to ones of P itself.

Remark 11. It is worth noting that the abstract notation for $\{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$ results either from following through the operations in [15] which derive it from one of the Petrie-Coxeter apeirohedra, or from an application of the circuit criterion [16, Theorem 2F4] (which was actually introduced in [15]). The extra relation corresponds to the skew hexagon with vertices a, b_0, c_0, e, c_1, b_1, which we shall meet again in the proof of Theorem 8.

Remark 12. As we noted above, the sole regular apeirohedron in $\mathbb{E}^3$ which was missed by Grünbaum in [6] is $\{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$. In a way, this is surprising, since $\{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$ has the same edge-graph as the apeirohedron $\{\frac{6}{1,3}, 4 : \frac{6}{1,3}\}$ which we discuss below. On the other hand, $\{\frac{3}{0,1}, 4 : \frac{3}{0,1}\}$ cannot be derived from any of the other regular apeirohedra in $\mathbb{E}^3$ by immediately obvious operations such as duality, Petriality, and the like.

We used the planarity of the vertex-figures in an essential way in the proof of Theorem 6. It may be of interest to see what happens if we relax this condition.

Theorem 7. *The general regular apeirohedron in* $\{\frac{3}{0,1}, \frac{4}{1,2} : \frac{3}{0,1}\}$ *is* 6*-dimensional, and is a blend*

$$\{\tfrac{3}{0,1}, \tfrac{4}{1,2} : \tfrac{3}{0,1}\} = \{\tfrac{3}{0,1}, 4 : \tfrac{3}{0,1}\} \# \{\tfrac{1}{0}\} \# \{3\}.$$

Proof. We begin by observing that

$$\{\tfrac{3}{0,1}, 4 : \tfrac{3}{0,1}\},\ \{\tfrac{1}{0}\},\ \{3\} \preccurlyeq \{\tfrac{3}{0,1}, \tfrac{4}{1,2} : \tfrac{3}{0,1}\}$$

as realization cones, and that the blend of the geometric polytopes is 6-dimensional (note that we specify $\{\frac{1}{0}\}$ rather than just $\{\infty\}$ here). Thus we only have to show that each $P \in \{\frac{3}{0,1}, \frac{4}{1,2} : \frac{3}{0,1}\}$ is at most 6-dimensional. At this stage, we may also notice that

$$\{\tfrac{3}{0,1}, 4 : \tfrac{3}{0,1}\} \# \{3\} \cong \mathbf{\{}\infty, 4 : \cdot, *3\mathbf{\}},$$

but that this isomorphism fails if we further blend with $\{\infty\}$ (in any realization); indeed, the typical hexagon which provided the defining relation for $\mathbf{\{}\infty, 4 : \cdot, *3\mathbf{\}}$ opens up here into an apeirogon $\{\frac{6}{0,1,3}\}$.

Our proof carries on from the analysis in that of Theorem 6, when we obtained the vector t_3 in (8). We now proceed along the holes of P though a, and then along those though the h_j, and so on; that is, we move to any new hole which diverges after an even number of edges from a – we can think of these as 'horizontal' holes. We make no assumption about how these different holes might subsequently meet. However, what we do see is that the vertices of P adjacent to those at odd edge distance from a in the horizontal holes are paired up by the translation vector t_3.

It follows from this that t_3 actually induces a translational symmetry T_3 of P, and that the 'vertical' holes (like those through d_j, b_j, c_j) are actually zigzags. In a similar way, we see that the 'horizontal' holes give rise to translational symmetries T_1 and T_2 (say). Crucially, it follows that there are at most six translation classes of edges, so that P is, as claimed, at most 6-dimensional.

Remark 13. The translation subgroup here has rank 4, as one would expect. Notice that, even in this case, we have appealed to very little of the abstract structure of the apeirohedron.

7.4 Mirror Vector $(1, 2, 1)$

We have left this class until last, because there is a contrast between it and the others.

The apeirohedra in the class have (finite) skew polygonal faces, planar vertex-figures and (finite) skew polygonal Petrie polygons. They are

$$\{\tfrac{6}{1,3}, 6 : \tfrac{4}{1,2}\} \cong \{6, 6 : 4\},$$

$$\{\tfrac{4}{1,2}, 6 : \tfrac{6}{1,3}\} \cong \{6, 6 : 6\},$$

$$\{\tfrac{6}{1,3}, 4 : \tfrac{6}{1,3}\} \cong \{6, 4 : 6\}.$$

The fine Schläfli symbols imply the corresponding isomorphisms. However, it is immediately apparent that they cannot ensure that the apeirohedra be 3-dimensional. The reason is simple: the faces and Petrie polygons of each are blends $\{6\} \# \{2\}$ or $\{4\} \# \{2\}$, and so blending the apeirohedra with digons $\{2\}$ will not change their types. Observe that the edge-graphs of these apeirohedra are bipartite. Thus the best that we can achieve is

Theorem 8. *In general, the apeirohedra with fine Schläfli symbols* $\{\tfrac{6}{1,3}, 6 : \tfrac{4}{1,2}\}$, $\{\tfrac{4}{1,2}, 6 : \tfrac{6}{1,3}\}$ *and* $\{\tfrac{6}{1,3}, 4 : \tfrac{6}{1,3}\}$ *are* 4*-dimensional, being blends of a* 3*-dimensional pure component and a segment.*

Proof. We deal with $P \in \{\tfrac{6}{1,3}, 4 : \tfrac{6}{1,3}\}$ first, while the treatment of $\{\tfrac{3}{0,1}, 4 : \tfrac{3}{0,1}\}$ is still (perhaps) fresh in the mind. The two apeirohedra have isomorphic edge-graphs, and so we shall adopt the same initial notation; again refer to (9). Now, however, a typical skew hexagonal face (or Petrie polygon – we shall not bother to distinguish them) F of type $\{\frac{6}{1,3}\}$ has vertices a, b_0, c_0, e, c_1, b_1; since F is centrally symmetric, we see that $c_j - b_j = w_0$ and $d_j - b_j = w_1$ for each j. Then $t_1 := c_j - d_j$ (for each j) is a translation vector of P as in the proof of Theorem 7; we clearly have analogous translations t_2, t_3 as well, but now there is no translation s_1.

The same calculations of centres of vertex-figures yields

$$v_0 + v_2 = v_1 + v_3 = -(w_0 + w_1),$$

but no contradictions of signs. Indeed, we see that the displacements of vertices from the centres of their vertex-figures are equal but alternating in sign at adjacent vertices. It follows immediately that, if this displacement is non-zero, then P is a blend.

Similar considerations apply to the other two cases, which we can treat together; we take $P \in \{\frac{6}{1,3}, 6 : \frac{4}{1,2}\}$. Let a be the initial vertex, whose vertex-figure has vertices $b_0, \ldots, b_5$ (in cyclic order). Let a, b_0, c, b_1 be a skew tetragonal Petrie polygon G of type $\{\frac{4}{0,1}\}$ of P, and let b_5, a, b_0, c, d_5, e_5 and b_2, a, b_1, c, d_1, d_2 be the two skew hexagonal faces $F_5, F_2 = \{\frac{6}{1,3}\}$ of P which share two edges of G. Then $v_j = b_j - a = d_j - c$ for $j = 2, 5$ (again, parallel sides of centrally symmetric skew hexagons); consequently, the vectors from a and c to the centres of their vertex-figures are the same. Tracing edge-paths shows that the same holds for alternate vertices; once again, then, P will generally be a blend. Note that the displacement from the remaining vertices must be equal and opposite in sign.

Remark 14. In view of the following Theorem 9, this actually suffices for a proof. However, the interested reader might like to show directly that alternate vertices genuinely form 3-dimensional configurations. For example, from $\{\frac{4}{1,2}, 6 : \frac{6}{1,3}\}$ we obtain $\{6, \frac{6}{1,3} \mid 3\}$, as comparison with the abstract case demonstrates (see [15]). Actually, here it is even easier to show that the mid-points of the edges of $\{\frac{6}{1,3}, 6 : \frac{4}{1,2}\}$ yield $\{6, \frac{4}{1,2} \mid 4\}$, with the faces and vertex-figures providing the planar hexagons, and the Petrie polygons providing the planar holes. There is no associated *regular* figure in the third case, but there is enough symmetry to establish the 3-dimensionality of the alternate vertex configuration.

It is natural to ask if there is any way of imposing rigidity on these apeirohedra. In fact, there is.

Theorem 9. *Three of the Grünbaum apeirohedra are determined by the fine Schläfli symbols* $\{\frac{6}{1,3}, 6 : \frac{4}{1,2}, \cdot, \frac{1}{0}\}$, $\{\frac{4}{1,2}, 6 : \frac{6}{1,3}, \cdot, \frac{1}{0}\}$ *and* $\{\frac{6}{1,3}, 4 : \frac{6}{1,3}, \frac{1}{0}\}$ *as rigid regular apeirohedra.*

Proof. We deliberately write $\{\frac{1}{0}\}$ rather than $\{\infty\}$ here, to emphasize that we are talking about the linear apeirogon. The reason is straightforward. Specifying these apeirogons as linear ensures that a vertex is the centre of its vertex-figure (compare the previous proof), and hence that the corresponding apeirohedron is 3-dimensional.

Remark 15. We observed in Sect. 2 that, if the vertex-figure of a regular polyhedron $\mathscr{P}$ is a $2k$-gon, then k-holes and k-zigzags of $\mathscr{P}$ coincide if they have even length (or are infinite). Hence we can represent these rigid apeirohedra alternatively as

$$\{\tfrac{6}{1,3}, 6 : \tfrac{4}{1,2} \mid \cdot, \tfrac{1}{0}\}, \quad \{\tfrac{4}{1,2}, 6 : \tfrac{6}{1,3} \mid \cdot, \tfrac{1}{0}\}, \quad \{\tfrac{6}{1,3}, 4 : \tfrac{6}{1,3} \mid \tfrac{1}{0}\}.$$

8 Further Applications

In this final section, we present a miscellany of examples which illustrate rigidity or its lack.

8.1 *Free Abelian Apeirotopes*

The construction of the free abelian apeirotope $P := \operatorname{apeir} Q$ on a finite regular polytope Q is purely geometric; we introduce a new notation here, which fits in more uniformly with the notation for other operations on polytopes. For discreteness, we must have Q a *rational* regular polytope, by which we mean that the vertices of Q have rational coordinates with respect to some (affine) basis of its affine hull aff Q. We think of Q (which will be the narrow vertex-figure – thus, in the strict sense) as sitting in a linear hyperplane H of our ambient space $\mathbb{E}$ with its centre at o, and we take an initial vertex v to lie on the line L through o orthogonal to H (this is essentially the only freedom we have). Then the mid-points of the edges of P through v are the vertices of Q, and – in the general situation – the vertices of P fall into two hyperplanes in $\mathbb{E}$ parallel to H. The canonical generators $R_1, \dots, R_n$ of the group of P are those of Q, with indices increased by 1 (that is, as mirrors extended by the line L); R_0 is the point-reflexion in the initial vertex of Q.

As the construction shows, in general apeir Q will be a blend with one component $\{2\}$; we therefore use this notation to denote the class of such apeirotopes, rather than an individual member. To distinguish it from the general situation, we introduce the notation Q^α for the special case $v = o$, so that $\operatorname{apeir} Q = Q^\alpha \# \{2\}$ is the general case.

We start with an ancillary result.

Lemma 1. *If Q is a crystallographic classical regular polytope, then $\{\frac{2}{0,1}, Q\} =$ apeir Q.*

Proof. Let $P \in \{\frac{2}{0,1}, Q\}$. If we take Q to have centre o and let $H := \operatorname{lin} Q$, then (in general) the initial vertex v lies in the orthogonal complement of H in the ambient space $\mathbb{E}$. Since the 2-faces containing v are zigzag apeirogons $\{\frac{2}{0,1}\}$, we see that their vertices all lie alternately in two affine subspaces $H^\pm$, say, parallel to H.

Now consider any vertex w of P adjacent to v. Let us write Q_w for the narrow vertex-figure of P at w. Then $x := \frac{1}{2}(v+w) \in \operatorname{vert} Q$. If $E = \{x, y\}$ is any edge of Q through x, then $2x - y \in \operatorname{vert} Q_w$. So, if G is the broad vertex-figure of Q at x, then similarly $2x - G$ is the broad vertex-figure of Q_w at x. But x and G fix Q, so that x and $2x-G$ fix Q_w, and hence $Q_w = 2x-Q \subseteq H$ also. Indeed, it is clear from this fact that, if $\{v, w\}$ is the initial edge of P, then the symmetry $R_0 \in \mathbf{G}(P)$ which interchanges v and w is the point-reflexion in the mid-point $x = \frac{1}{2}(v + w)$ of that edge. We hardly need to deduce as well (from the connectedness of the edge-graph of P) that all vertex-figures at vertices of P lie in H, and therefore $\operatorname{vert} P \subseteq H^{\pm}$. In conclusion, $P \in \operatorname{apeir} Q$, as required.

Remark 16. Observe that we have actually only appealed to the fact that x and G determine Q, and so the same argument will apply in many more cases.

Since the examples in this section are only intended to be illustrative, we shall just consider the case of Theorem 1. As we have seen, in general P will be a blend with one component $\{2\}$. Thus, to make P rigid, we must force v to be the centre of Q. So far, P belongs to the fine Schläfli symbol $\{\frac{2}{0,1}, q_2, \ldots, q_n\}$, where $Q = \{q_2, \ldots, q_n\}$ (observe that we could use the notation $\{\frac{2}{0,1}, Q\}$ for any rational regular polytope Q). The entry $\frac{2}{0,1}$ for the 2-face will remain, and so some other (geometric) relation needs to be imposed.

Theorem 10. *If $Q = \{q_2, \ldots, q_n\}$ is a crystallographic classical regular polytope, whose Petrie polygon is $\{s\}$ (with s a generalized fraction), then the fine Schläfli symbol $\mathscr{P} = \{\frac{2}{0,1}, q_2, \ldots, q_n : s''\}$ is rigid, and determines the apeirotope* apeir Q.

We have explained the notation s'' in (4).

Proof. The reason is fairly simple. First, we observe that, if $v = o$ (the centre of Q), then $\{s''\}$ is indeed the Petrie polygon of apeir Q. If P is to be a blend, of which one component must be $\{2\}$, then the Petrie polygon will similarly have to have $\{2\}$ as a component. However, $\{s''\}$ cannot have a component $\{2\}$, because $\{s\}$ (being finite) does not have a component $\{\infty\}$.

In the present context, there are two cases where Q^{α} and the general apeir Q are not isomorphic. For these, Q has a diametral hexagon; apart from the hexagon $\{6\}$ itself, the other case is the 24-cell $\{3, 4, 3\}$. When $Q = \{6\}$, we have $s = 6$ (of course) and thus $s'' = 3$, giving P as the Petrial of the tessellation $\{3, 6\}$ of the plane by triangles.

The interesting case is therefore $\{3, 4, 3\}$, where $s = \frac{12}{1,5}$, so that $s'' = \frac{12}{1,5}$ also. The half-turn about the initial edge of the rigid $P = \{\frac{2}{0,1}, 3, 4, 3 : \frac{12}{1,5}\}$ is $(R_2 R_3 R_4)^3$ (the central symmetry of the edge-figure $\{4, 3\}$), so that the reflexion in the hyperplane perpendicular to the initial edge through its centre is $S_0 := R_0(R_2 R_3 R_4)^3$. With $S_j := R_j$ for $j = 1, \ldots, 4$, we have the symmetry group $\langle S_0, \ldots, S_4\rangle$ of the (rigid) regular honeycomb $\{3, 3, 4, 3\}$. Thus the rigidity can also be expressed by the relation $(R_0R_1(R_2R_3R_4)^3)^3 = I$ for the triangular 2-face $\{3\}$ of the honeycomb; however, this relation does not impose a natural geometric condition on P.

Remark 17. It sheds no more light on the situation to describe the group of the triangular face in terms of the generating reflexions $R_0(R_2R_3R_4)^3$ and R_1 of its symmetry group.

8.2 Polytopes of Full Rank

We briefly survey the remaining (that is, non-classical) regular polytopes and honeycombs of full rank, though not in an exhaustive way. We begin with the polytopes.

The general *Petrie operation* is

$$\boldsymbol{\pi}\colon (R_0,\ldots,R_{n-1}) \mapsto (R_0,\ldots,R_{n-4},R_{n-3}R_{n-1},R_{n-2},R_{n-1}) =: (S_0,\ldots,S_{n-1}).$$

This is always applicable to a classical regular polytope (or honeycomb) $\{p_1,\ldots,p_{n-1}\}$ when $n = 3$, but only if $p_{n-3} = 4$ when $n \geqslant 4$ (we have given a more detailed criterion in [12]). Thus the only applicable cases (apart from the polyhedra) are $P = \{4,3,4\}$, $\{4,3,3\}$, $\{3,4,3,3\}$ and apeir$\{4,3,3\}$. However, except in the last case, the way that the Petrial P^{π} is specified notationally leads at once to the original P, and so rigidity is ensured. For example,

$$\{4,3,4\}^{\pi} = \{\{4,\tfrac{6}{1,3} \mid 4\},\{\tfrac{6}{1,3},4:3\}\},$$

and – ignoring the entries $\frac{6}{1,3}$ and 4 for the hole – we see that the original entries in the Schläfli symbol $\{4,3,4\}$ are still present implying (and this is the crucial point) appropriate *planar* polygons.

The remaining operation which yields new regular polytopes from finite regular polytopes of full rank is $\boldsymbol{\zeta}$, which replaces the initial hyperplane reflexion R_0 in the symmetry group by $-R_0$ (assuming, as usual, that we take the centre to be o). Since the new 2-face is a skew polygon, we should not expect rigidity to hold. Indeed, in general it does not, without the imposition of additional specifications.

There are $n+1$ pure realizations of the n-cube $\mathbf{\{}4,3^{n-2}\mathbf{\}}$, including the trivial realization $\{1\}$ and the segment $\{2\}$. Of the others, when $n \geqslant 4$ only the classical convex cube $\{4,3^{n-2}\}$ is rigid: the general fine Schläfli symbol $\{\frac{4}{1,2},3^{n-2}\}$ does not distinguish among the rest. It might be thought that the situation could be different for the central quotient $\mathbf{\{}4,3^{n-2}\mathbf{\}}/2$. However, even for $n = 4$, when there is just one pure faithful realization, the fine Schläfli symbol $\{\frac{4}{1,2},3,3:\frac{4}{1,2}\}$ cannot exclude blends with the digon $\{2\}$.

8.3 Four-Dimensional Polyhedra

We recall from [10] that an important tool for the classification of the 4-dimensional regular polyhedra was the mirror vector, as defined in Sect. 3. As before, we shall merely consider representative examples. It may be helpful to bear in mind that such polyhedra with mirror vector (r_0, r_1, r_2) are finite analogues of (pure) 3-dimensional apeirohedra with mirror vector $(r_0 - 1, r_1 - 1, r_2 - 1)$.

The polyhedra with mirror vector $(3, 2, 3)$ have planar faces and holes and skew vertex-figures; they are therefore covered by Theorem 2. Nevertheless, a couple of cases deserve further mention. First, we have (as a fine Schlf̈li symbol) $\mathscr{P} := \{4, \frac{6}{1,3} \mid 3\} \cong \boldsymbol{\{4, 6 \mid 3\}}$, the universal polyhedron. However, there is also $\mathscr{Q} := \{4, \frac{6}{2,3} : \frac{5}{1,2} \mid 3\} \cong \boldsymbol{\{4, 6 : 5 \mid 3\}}$, which is doubly covered by $\mathscr{P}$. Theorem 2 tells us that $\mathscr{Q}$ is rigid, and is actually specifiable as $\mathscr{Q} = \{4, \frac{6}{2,3} \mid 3\}$, with no mention of the Petrie polygon. Notice that the sole difference between $\mathscr{P}$ and $\mathscr{Q}$ (as fine Schläfli symbols) now lies in the vertex-figures alone.

Associated with the mirror class $(3, 2, 3)$ are the classes $(2, 2, 3)$ and $(1, 2, 3)$, which are the results of applying $\boldsymbol{\pi}$, $\boldsymbol{\zeta}$ or both. Thus we should not expect straightforward ways of ensuring rigidity; we shall not discuss these classes here.

The mirror class $(2, 3, 2)$ consists of polyhedra with planar vertex-figures, but skew faces and Petrie polygons. This skewness enables us to blend them with $\{2\}$, while not affecting fine Schläfli symbols which only indicate the three features mentioned. With Sect. 7.4 in mind, we see that additional conditions will be needed to impose rigidity.

The final mirror class is $(2, 2, 2)$. In view of Sect. 7.3, we might hope that mere specification of planar vertex-figures and faces and Petrie polygons as helices (that is, blends of two polygons) would suffice to ensure rigidity. Unfortunately, we saw in Sect. 5 that the case of the hemi-dodecahedron $\{\frac{5}{1,2}, 3 : \frac{5}{1,2}\}$ refutes this (while the 4-dimensional realization $\{\frac{5}{1,2}, 3 : \frac{10}{1,3}\}$ of the dodecahedron is rigid, we showed this indirectly, rather than by extending the ideas of Sect. 7). Since things can go wrong with the simplest example, we should not expect anything better for the rest.

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Hereditary Polytopes

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With best wishes to our friend and colleague Peter McMullen.

Abstract Every regular polytope has the remarkable property that it inherits all symmetries of each of its facets. This property distinguishes a natural class of polytopes which are called hereditary. Regular polytopes are by definition hereditary, but the other polytopes in this class are interesting, have possible applications in modeling of structures, and have not been previously investigated. This paper establishes the basic theory of hereditary polytopes, focussing on the analysis and construction of hereditary polytopes with highly symmetric faces.

Keywords Regular polytope • Chiral polytope • Extension of automorphisms

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1 Introduction

In the classical theory of convex polyhedra, the Platonic and Archimedean solids form a natural class of highly symmetric objects. The symmetry group of each of these polyhedra acts transitively on its vertices. If we restrict to those solids whose symmetry groups also act transitively on their edges, only the regular polyhedra, the cuboctahedron, and the icosidodecahedron remain. These polytopes all have the distinguishing property that every symmetry of their polygonal faces extends to a symmetry of the solid. In fact, if we look for convex "hereditary" polyhedra (those having this property of inheriting all the symmetries of their faces) with regular faces, we find that vertex and edge transitivity is implied (as we shall see in a more general setting in Sect. 3).

It is natural to generalize this idea of hereditary polyhedra to the setting of abstract polytopes of any rank. In this paper we study those polytopes that have the property of inheriting all symmetries of their facets. The formal definition of a hereditary polytope can be found in Sect. 2, along with other basic notions required for the understanding of this paper.

An abstract polytope of rank 3 can be seen as a map, that is a 2-cell embedding of a connected graph into a closed surface. Regular and chiral maps have been studied extensively in the past (see for example [2, 6]), and form a natural class of highly symmetric maps. In some older literature, chiral maps are labeled as regular, as locally they are regular in the following sense. The symmetry group of a chiral map acts transitively on the vertices, edges, and faces, and the maps have the maximal possible rotational symmetry. However, none of the reflectional symmetry of any of the faces of a chiral map extends to a global symmetry. Therefore chiral maps, although highly symmetric, are not hereditary in our sense.

The non-regular hereditary maps are the 2-orbit maps which are vertex and edge transitive. This type of map has been extensively studied (see for example [9, 12, 24]). It will be shown that certain 2-orbit polytopes will always be hereditary (see Theorem 2). However, the characterization of hereditary polytopes of rank greater than three is complex.

In Sect. 3, we consider how various transitivity properties of the facets affect the transitivity properties of a hereditary polytope. Section 4 deals with polytopes with regular facets, with an emphasis on hereditary polyhedra. In Sect. 5, we consider polytopes with chiral facets, and prove the existence of certain hereditary polytopes of this type. In Sect. 6, some questions regarding the extensions of hereditary polytopes are considered. We conclude with a brief section which suggests some interesting problems for related work.

2 Basic Notions

Following [16], a *polytope* $\mathscr{P}$ of rank n, or an *n-polytope*, is a partially ordered set of faces, with a strictly monotone rank function having range $\{-1, \ldots, n\}$. The elements of $\mathscr{P}$ with rank i are called *i-faces*; typically F_i indicates an i-face.

A *chain* of type $\{i_1, \ldots, i_k\}$ is a totally ordered set faces of ranks $\{i_1, \ldots, i_k\}$. The maximal chains of $\mathscr{P}$ are called *flags*. We require that $\mathscr{P}$ have a smallest (-1)-face F_{-1}, a greatest n-face F_n and that each flag contains exactly $n+2$ faces. Also, $\mathscr{P}$ should be *strongly flag-connected*, that is, any two flags Φ and Ψ of $\mathscr{P}$ can be joined by a sequence of flags $\Phi := \Phi_0, \Phi_1, \ldots, \Phi_k =: \Psi$ such that each Φ_j and Φ_{j+1} are *adjacent* (in the sense that they differ by just one face), and $\Phi \cap \Psi \subseteq \Phi_j$ for each j. Furthermore, we require the following homogeneity property: whenever $F < G$, with $\mathrm{rank}(F) = i-1$ and $\mathrm{rank}(G) = i+1$, then there are exactly two i-faces H with $F < H < G$. Essentially, these conditions say that $\mathscr{P}$ shares many combinatorial properties with the face lattice of a convex polytope.

If Φ is a flag, then we denote the *i-adjacent flag* by Φ^i, that is the unique flag adjacent to Φ and differing from it in the face of rank i. More generally, we define inductively $\Phi^{i_1,\ldots,i_k} := (\Phi^{i_1,\ldots,i_{k-1}})^{i_k}$ for $k \geq 2$. Note that if $|i-j| \geq 2$, then $\Phi^{i,j} = \Phi^{j,i}$; otherwise in general, $\Phi^{i,j} \neq \Phi^{j,i}$.

The faces of rank 0, 1, and $n-1$ are called *vertices, edges, and facets*, respectively. We will sometimes identify a face F with the section F/F_{-1} when there is no chance for confusion. If F is a vertex, the section $F_n/F := \{G | F \leq G \leq F_n\}$ is called the *vertex-figure* of $\mathscr{P}$ at F. A polytope is said to be *equivelar* of (Schläfli) *type* $\{p_1, \ldots, p_{n-1}\}$ if each section G/F, with G an $(i+1)$-face and F an $(i-2)$-face with $F < G$, is combinatorially equivalent to a p_i-gon. Additionally, if the facets of $\mathscr{P}$ are all isomorphic to an $(n-1)$-polytope $\mathscr{K}$ and its vertex-figures are all isomorphic to an $(n-1)$-polytope $\mathscr{L}$, then we say $\mathscr{P}$ is of type $\{\mathscr{K}, \mathscr{L}\}$. (This is a small change of terminology from [16].)

The set of all automorphisms of $\mathscr{P}$ is a group denoted by $\Gamma(\mathscr{P})$ and called the *automorphism group* of $\mathscr{P}$. For $0 \leq i \leq n-1$, an n-polytope $\mathscr{P}$ is called *i-face transitive* if $\Gamma(\mathscr{P})$ acts transitively on the set of i-faces of $\mathscr{P}$. In addition, $\mathscr{P}$ is said to be $\{0, 1, \ldots, i\}$*-chain transitive* if $\Gamma(\mathscr{P})$ acts transitively on the set of chains of $\mathscr{P}$ of type $\{0, 1, \ldots, i\}$.

A polytope $\mathscr{P}$ is said to be *regular* if $\Gamma(\mathscr{P})$ acts transitively on the flags, that is if $\mathscr{P}$ is $\{0, 1, \ldots, n-1\}$-chain transitive. The automorphism group of a regular n-polytope is known to be a *string C-group* (a smooth quotient of a Coxeter group with a linear diagram, which satisfies a specified intersection condition), and is generated by involutions $\rho_0, \ldots, \rho_{n-1}$, which are called the *distinguished generators* associated with a flag Φ, and defined as follows. Each ρ_i maps Φ to Φ^i. These generators for a polytope of Schläfli type $\{p_1, \ldots, p_{n-1}\}$ satisfy relations of the form

$$(\rho_i \rho_j)^{p_{ij}} = \varepsilon \text{ for } i, j = 0, \ldots, n-1, \tag{1}$$

where $p_{ii} = 1$, $p_{ij} = p_{ji} := p_j$ if $j = i+1$, and $p_{ij} = 2$ otherwise. When the sections F/F_{-1} of a polytope $\mathscr{P}$ determined by facets F are themselves regular, we say that $\mathscr{P}$ is *regular-facetted*.

A regular polytope $\mathscr{P}$ is called *directly regular* if the even (or rotation) subgroup $\Gamma^+(\mathscr{P})$ of $\Gamma(\mathscr{P})$ has index 2 in $\Gamma(\mathscr{P})$. Recall that $\Gamma^+(\mathscr{P})$ consists of the elements of $\Gamma(\mathscr{P})$ that can be expressed as a product of an even number of distinguished generators ρ_i.

A polytope $\mathscr{P}$ is said to be *chiral* if there are two orbits of flags under the action of $\Gamma(\mathscr{P})$ and adjacent flags are in different orbits. In this case, given a flag $\Phi = \{F_{-1}, \ldots, F_n\}$ of $\mathscr{P}$ there exist automorphisms, which are also called *distinguished generators*, $\sigma_1, \ldots, \sigma_{n-1}$ of $\mathscr{P}$ such that each σ_i fixes all faces in $\Phi \setminus \{F_{i-1}, F_i\}$ and cyclically permutes consecutive i-faces of $\mathscr{P}$ in the rank 2 section of F_{i+1}/F_{i-2}. Each chiral polytope comes in two *enantiomorphic forms*; one associated with a base flag Φ and the other with any of its adjacent flags. When the sections F_{n-1}/F_{-1} of a polytope $\mathscr{P}$ determined by facets F are themselves chiral, we say that $\mathscr{P}$ is *chiral-facetted*.

A polytope $\mathscr{P}$ is said to be *k-orbit* if there are k orbits of flags under the action of $\Gamma(\mathscr{P})$. In the case of 2-orbit polytopes, if $I \subseteq \{0, \ldots, n-1\}$ is such that i-adjacent flags are in the same orbit for $i \in I$ and in different orbits for $i \notin I$, then we say that $\mathscr{P}$ is in the *class* 2_I.

Finally, a polytope $\mathscr{P}$ is called *hereditary* if for each facet F of $\mathscr{P}$ the group $\Gamma(F/F_{-1})$ of the corresponding section F/F_{-1} is a subgroup of $\Gamma(\mathscr{P})$; in fact, then $\Gamma(F/F_{-1})$ is a subgroup of $\Gamma_F(\mathscr{P})$, the stabilizer of F in $\Gamma(\mathscr{P})$. More informally, $\mathscr{P}$ is hereditary if every automorphism of every facet F extends to an automorphism of $\mathscr{P}$ which fixes F.

3 Transitivity on Faces

We begin with a number of basic properties of hereditary polytopes which have highly symmetric facets.

Proposition 1. *If an n-polytope $\mathscr{P}$ is hereditary and each facet is $\{0, 1, \ldots, i\}$-chain transitive for some i with $i \leq n-2$, then $\mathscr{P}$ is $\{0, 1, \ldots, i\}$-chain transitive, and hence the i-faces of $\mathscr{P}$ are mutually isomorphic regular i-polytopes.*

Proof. Let Φ and Ψ be two chains of $\mathscr{P}$ of type $\{0, 1, \ldots, i\}$. Since $\mathscr{P}$ is strongly flag-connected and $i \leq n-2$, there exists a sequence $\Phi := \Phi_0, \Phi_1, \ldots, \Phi_k := \Psi$ of chains of type $\{0, 1, \ldots, i\}$ such that, for $j = 1, \ldots, k$, all faces of Φ_{j-1} and Φ_j are incident to a common facet H_j. As each facet is transitive on chains of this type, there is an automorphism of H_j mapping Φ_{j-1} to Φ_j. These automorphisms of the facets H_j are also automorphisms of $\mathscr{P}$, and their composition maps Φ to Ψ. □

In much the same way we can also prove the following proposition, again appealing to the strong flag-connectedness.

Proposition 2. *If an n-polytope $\mathscr{P}$ is hereditary and each facet is i-face transitive for some i with $i \leq n-2$, then $\mathscr{P}$ is i-face transitive. In particular, if each facet is vertex transitive, then $\mathscr{P}$ is vertex transitive.*

Proof. Join any two i-faces of $\mathscr{P}$ by a sequence of i-faces in which any two consecutive i-faces lie in a common facet. Then proceed as in the previous proof. □

Proposition 1 also has the following immediate consequence.

Proposition 3. *If an n-polytope $\mathscr{P}$ is hereditary and each facet is regular, then the $(n-2)$-faces of $\mathscr{P}$ are all regular $(n-2)$-polytopes mutually isomorphic under isomorphisms induced by automorphisms of $\mathscr{P}$.*

Our main interest is in hereditary polytopes all of whose facets are either regular or chiral. The following theorem says that any such polytope must have its facets either all regular or all chiral. In other words, the "mixed-case" does not occur.

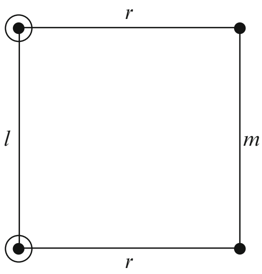
Theorem 1. *If $\mathscr{P}$ is a hereditary polytope with each facet either regular or chiral, then either each facet of $\mathscr{P}$ is regular or each facet of $\mathscr{P}$ is chiral.*

Proof. Suppose $\mathscr{P}$ has at least one regular facet. We prove that then each facet of $\mathscr{P}$ must be regular. By the connectedness properties of $\mathscr{P}$ it suffices to show that each facet adjacent to a regular facet must itself be regular.

Let H be a regular facet, and let H' be an adjacent facet meeting H in an $(n-2)$-face G. Let Ω be a flag of H/F_{-1} containing G. Since H is regular, its group $\Gamma(H/F_{-1})$ contains a "reflection" ρ_0^H which maps Ω to Ω^0, the 0-adjacent flag of Ω in H/F_{-1}. Since $\mathscr{P}$ is hereditary, ρ_0^H extends to an automorphism of $\mathscr{P}$, again denoted ρ_0^H, which takes the flag $\Psi := \Omega \cup \{F_n\}$ of $\mathscr{P}$ to Ψ^0. But ρ_0^H fixes H and G, so must necessarily fix H' as well and hence belong to $\Gamma(H'/F_{-1})$. Moreover, ρ_0^H maps the flag $\Omega' := (\Omega \setminus \{H\}) \cup \{H'\}$ of H'/F_{-1} to its 0-adjacent flag $(\Omega')^0$. Thus $\Gamma(H'/F_{-1})$ contains an element which takes a flag of H'/F_{-1} to an adjacent flag. On the other hand, each facet of $\mathscr{P}$ is regular or chiral, so H' must necessarily be regular. Bear in mind here that a chiral polytope does not admit an automorphism mapping a flag to an adjacent flag. □

A hereditary polytope with some of its facets regular, need not have all of its facets regular. This is illustrated by the example of the semiregular tessellation $\mathscr{T}$ of Euclidean 3-space by regular tetrahedra and (vertex) truncated regular tetrahedra. This tessellation is related to the Petrie-Coxeter polyhedron $\{6,6\,|\,3\}$. The facets (tiles) of $\mathscr{T}$ are of two kinds, namely (regular) Platonic solids and (semiregular) Archimedean solids, or more precisely, truncated Platonic solids. This tessellation is a hereditary 4-orbit polytope of rank 4.

More examples arise in a similar way from the semiregular tessellations of the 3-sphere $\mathbb{S}^3$ or hyperbolic 3-space $\mathbb{H}^3$ related to the regular skew polyhedra $\{2l,2m\,|\,r\}$ in these spaces. Their tiles are Platonic solids $\{r,m\}$ and (vertex) truncated Platonic solids $\{l,r\}$. These tessellations can be derived by Wythoff's construction applied to the spherical or hyperbolic 4-generator Coxeter group with square diagram.



(2)

More details, including a list of the various possible choices for l, m, r, can be found in [4, 14, 21]. The semiregular tessellation $\mathscr{T}$ in $\mathbb{E}^3$ mentioned earlier is obtained in a similar fashion from the Euclidean Coxeter group given by the diagram in (2) with $l = m = r = 3$.

4 Hereditary Polytopes with Regular Facets

In this section we investigate hereditary polytopes which are regular-facetted. We show that each such polytope is either itself regular or a 2-orbit polytope.

4.1 Flag-Orbits

We begin with the following observation.

Proposition 4. *Let $\mathscr{P}$ be a regular-facetted hereditary n-polytope. If there exists an $(n-3)$-face F such that its co-face F_n/F is a q-gon with q odd, then $\mathscr{P}$ is a regular n-polytope of Schläfli type $\{p_1, \ldots, p_{n-2}, q\}$, where $\{p_1, \ldots, p_{n-2}\}$ is the Schläfli type of any facet of $\mathscr{P}$.*

Proof. The proof exploits the fact that for odd q the dihedral group D_q has just one conjugacy class of reflections. Geometrically speaking this means that the reflection mirror bisecting an edge of a convex regular q-gon also bisects the angle at the opposite vertex. This conjugacy class then generates D_q.

First observe that, by Proposition 2, the group $\Gamma(\mathscr{P})$ is transitive on the $(n-3)$-faces of $\mathscr{P}$ since $\mathscr{P}$ has regular facets. (This already implies that *each* co-face of an $(n-3)$-face is a q-gon.) Now, if we can show that the stabilizer of an $(n-3)$-face in $\Gamma(\mathscr{P})$ acts transitively on the flags of $\mathscr{P}$ containing that $(n-3)$-face, then clearly $\Gamma(\mathscr{P})$ acts flag-transitively on $\mathscr{P}$ and hence $\mathscr{P}$ must be regular.

Now suppose F is an $(n-3)$-face such that F_n/F is a q-gon. Clearly, since the facets of $\mathscr{P}$ are regular, F is also regular and its group $\Gamma(F/F_{-1})$ can be viewed as a subgroup of the automorphism group of any facet H of $\mathscr{P}$ with $F < H$ that acts trivially on H/F. Moreover, since $\mathscr{P}$ is hereditary, $\Gamma(F/F_{-1})$ is also a subgroup of $\Gamma(\mathscr{P})$ acting flag-transitively on F/F_{-1} and trivially on F_n/F.

On the other hand, if H is any facet of $\mathscr{P}$ containing F, then there exists a unique involution ρ^H_{n-2} (say) in $\Gamma(H/F_{-1})$ which fixes a flag of F/F_{-1} and interchanges the two $(n-2)$-faces of H containing F. Now, since q is odd, the reflections ρ^H_{n-2}, with H a facet containing F, generate a subgroup isomorphic to the dihedral group D_q. Hence, since this subgroup acts flag-transitively on F_n/F and trivially on F/F_{-1}, it can be identified with $\Gamma(F_n/F)$.

Then $\Gamma_F(\mathscr{P}) = \Gamma(F/F_{-1}) \times \Gamma(F_n/F)$, and $\Gamma_F(\mathscr{P})$ acts transitively on the flags of $\mathscr{P}$ that contain F. □

The following theorem says that the hereditary polytopes with regular facets fall into two families.

Theorem 2. *A regular-facetted n-polytope is hereditary if and only if it is regular or a 2-orbit polytope in the class* $2_{\{0,1,\ldots,n-2\}}$.

Proof. Let $\mathscr{P}$ be a regular-facetted hereditary n-polytope. As before, $\mathscr{P}$ is $(n-3)$-face transitive. Let F be any $(n-3)$-face of $\mathscr{P}$. We must show that the stabilizer $\Gamma_F(\mathscr{P})$ has at most two orbits on the set of flags of $\mathscr{P}$ containing F. Since F is regular and $\mathscr{P}$ is hereditary, $\Gamma(F/F_{-1})$ can again be viewed a subgroup of $\Gamma_F(\mathscr{P})$ acting flag-transitively on F/F_{-1} and trivially on F_n/F.

As in the previous proof, for each facet H of $\mathscr{P}$ containing F, there exists a unique involution ρ^H_{n-2} (say) in $\Gamma(H/F_{-1})$ which fixes a flag of F/F_{-1} and interchanges the two $(n-2)$-faces of H containing F. Suppose the co-face F_n/F is a q-gon, allowing $q = \infty$. By Proposition 4, if q is odd, then $\mathscr{P}$ is regular and we are done.

Now suppose $\mathscr{P}$ is not regular. Then q is even or $q = \infty$. In this case the subgroup Λ of $\Gamma_F(\mathscr{P})$ generated by the involutions ρ^H_{n-2}, with H a facet containing F, is isomorphic to a dihedral group $D_{q/2}$; when restricted to the q-gonal co-face F_n/F, these involutions ρ^H_{n-2} generate a subgroup of index 2 in the full dihedral automorphism group D_q of F_n/F. Hence Λ, restricted to F_n/F, has two flag-orbits on F_n/F. It follows that $\Gamma_F(\mathscr{P}) = \Gamma(F/F_{-1}) \times \Lambda$, and that $\Gamma_F(\mathscr{P})$ has two orbits on the flags of $\mathscr{P}$ that contain F. Thus $\mathscr{P}$ is a 2-orbit polytope. Moreover, since $\mathscr{P}$ is hereditary and the facets of $\mathscr{P}$ are regular, $\Gamma(\mathscr{P})$ contains the distinguished generators for the group of any facet of $\mathscr{P}$, so $\mathscr{P}$ is necessarily of type 2_I with $\{0,\ldots,n-2\} \subseteq I$. On the other hand, since $\mathscr{P}$ itself is not regular, no flag can be mapped onto its $(n-1)$-adjacent flag by an automorphism of $\mathscr{P}$. Hence $\mathscr{P}$ must be a 2-orbit polytope in the class $2_{\{0,1,\ldots,n-2\}}$.

Conversely, if $\mathscr{P}$ is in the class $2_{\{0,1,\ldots,n-2\}}$, then it has regular facets, and since all flags that contain a mutual facet are in the same orbit, it is hereditary. □

Note that every 2-orbit polytope $\mathscr{P}$ in the class $2_{\{0,1,\ldots,n-2\}}$ necessarily has regular facets, generally of two different kinds. In particular, $\mathscr{P}$ has a generalized Schläfli symbol of the form

$$\left\{p_1, \ldots, p_{n-3}, \begin{matrix} p_{n-2} \\ q_{n-2} \end{matrix}\right\},$$

where $\{p_1, \ldots, p_{n-3}, p_{n-2}\}$ and $\{p_1, \ldots, p_{n-3}, q_{n-2}\}$ are the ordinary Schläfli symbols for the two kinds of facets (see [13]). This is a generalization of the classical Schläfli symbol used in Coxeter [5] for semiregular convex polytopes.

We now describe some examples of regular-facetted hereditary polytopes of low rank, concentrating mainly on rank 3. All regular polytopes are hereditary and (trivially) regular-facetted, so we consider only non-regular polytopes, which, as we just proved, must necessarily be 2-orbit polytopes in the class $2_{\{0,1,\ldots,n-2\}}$.

4.2 Hereditary Polyhedra

Since all abstract 2-polytopes (polygons) are regular, by Theorem 2, each hereditary polyhedron is (trivially) regular-facetted and hence is either regular or a 2-orbit polyhedron in the class $2_{\{0,1\}}$. Both the cuboctahedron and the icosidodecahedron can easily be seen to be hereditary polyhedra. In fact, these are the only hereditary polyhedra amongst the Archimedean solids. Similarly, the uniform Euclidean plane tessellation of type $(3.6)^2$ is an infinite hereditary polyhedron (see [10]).

Recall that the *medial* of a polyhedron (map) $\mathscr{P}$ on a closed surface is the polyhedron $\mathrm{Me}(\mathscr{P})$ on the same surface whose vertices are the "midpoints" of the edges of $\mathscr{P}$, and whose edges join two vertices if and only if the corresponding edges of $\mathscr{P}$ are adjacent edges of a face of $\mathscr{P}$. All three examples of hereditary polyhedra just mentioned can be constructed as medials of regular maps, namely of the cube $\{4, 3\}$, the dodecahedron $\{5, 3\}$, and the euclidean plane tessellation $\{6, 3\}$, respectively.

More generally, given a regular polyhedron $\mathscr{P}$ of type $\{p, q\}$, the medial $\mathrm{Me}(\mathscr{P})$ is a hereditary polyhedron, and $\mathrm{Me}(\mathscr{P})$ is regular if and only if $\mathscr{P}$ is self-dual (see [18, Theorem 4.1]). This can be quickly seen algebraically. If the automorphism group of the original polyhedron is $\Gamma(\mathscr{P}) = \langle \rho_0, \rho_1, \rho_2 \rangle$ (say), then $\Gamma(\mathrm{Me}(\mathscr{P}))$ is isomorphic to $\Gamma(\mathscr{P})$ if $\mathscr{P}$ is not self-dual, or $\Gamma(\mathscr{P}) \ltimes C_2$ if $\mathscr{P}$ is self-dual (this latter group is just the extended group of $\mathscr{P}$, consisting of all automorphisms and dualities of $\mathscr{P}$). When $\mathscr{P}$ is not self-dual, there are generally two kinds of facets of $\mathrm{Me}(\mathscr{P})$, namely p-gons corresponding to conjugate subgroups of $\langle \rho_0, \rho_1 \rangle$ in $\Gamma(\mathscr{P})$, and q-gons corresponding to conjugate subgroups of $\langle \rho_1, \rho_2 \rangle$ in $\Gamma(\mathscr{P})$; in particular, when $q = p$ all facets of $\mathrm{Me}(\mathscr{P})$ are p-gons, so $\mathrm{Me}(\mathscr{P})$ has facets of just one type (we describe an example below). This is also true when $\mathscr{P}$ is self-dual; however, in this case the two subgroups are conjugate in the extended group of $\mathscr{P}$ (under a polarity fixing the base flag). In either case, $\mathrm{Me}(\mathscr{P})$ is hereditary since the two kinds of conjugate subgroups in $\Gamma(\mathscr{P})$ are also subgroups of $\Gamma(\mathrm{Me}(\mathscr{P}))$.

Using the medial construction, we can find a finite hereditary polyhedron with only one isomorphism type of facet, which, although it has a Schläfli symbol, is not regular. Consider a non self-dual polyhedron of type $\{p, p\}$, for example the polyhedron $\mathscr{P}$ of type $\{5, 5\}_{12}$ denoted as "N98.6" by Conder [2] (or as $\{5, 5\} * 1920b$ by Hartley [11]). The medial of $\mathscr{P}$ is a polyhedron of type $\{5, 4\}$ with the same automorphism group, of order 1920, but with twice as many flags. Thus this polyhedron is not regular, but it is still hereditary and of type $2_{\{0,1\}}$.

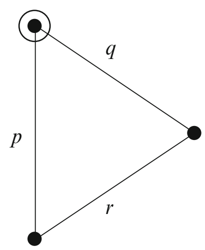
The previous example is also of independent interest with regards to the following problem about the lengths of certain distinguished paths in its edge graph.

Remark 1. In Problem 7 of [23], it is asked to what extent a regular or chiral polyhedron of type $\{p, q\}$ is determined by the lengths of its j-holes and the lengths of its j-zigzags. The polyhedron $\mathscr{P}$ with 1920 flags, mentioned above, has Petrie polygons (1-zigzags) of length 12, 2-zigzags of length 5, and 2-holes of length 12. Thus, we say it is of type $\{5, 5 \mid 12\}_{12,5}$ (see [16, p. 196]). However, calculation in MAGMA [1] shows that the universal polyhedron of this type has 30720 flags. This gives an example of a regular polyhedron which is not determined by the lengths of all of its j-holes and the lengths of all of its j-zigzags.

Every non-regular hereditary polyhedron $\mathscr{P}$, by Proposition 4, has vertex-figures which are q-gons with q even. In particular, by Theorem 2, $\mathscr{P}$ is a 2-orbit polyhedron in class $2_{\{0,1\}}$. Additionally, Theorem 4.2 of [18] shows that any 2-orbit polyhedron in class $2_{\{0,1\}}$ is the medial of a regular map if and only if $q = 4$.

However, there are non-regular hereditary polyhedra which are not medials of regular maps. We now define a class of such examples via a map operation which we call "generalized halving." The halving operation itself is described in Sect. 5.3.1. If $\mathscr{K}$ is a regular map of type $\{2p, q\}$ whose edge graph is bipartite, then we define a hereditary polyhedron $\mathscr{K}^a$ (on the same surface as $\mathscr{K}$) as follows; here "a" stands for "alternating vertices" (see also Sect. 5.3.1). Suppose that the vertices of $\mathscr{K}$ are colored *red* and *yellow* such that adjacent vertices have different colors. The vertex-figures at the red vertices of $\mathscr{K}$ (obtained by joining the yellow vertices adjacent to a given red vertex in cyclic order) form one class of facets of $\mathscr{K}^a$. The other class of facets of $\mathscr{K}^a$ is defined by joining the yellow vertices of a facet of $\mathscr{K}$ whenever they are adjacent to the same red vertex in that facet. The resulting polyhedron has facets of type $\{p\}$ and $\{q\}$, and vertex-figures of type $\{2q\}$. The polyhedron $\mathscr{K}^a$ is in the class $2_{\{0,1\}}$, and thus is hereditary.

Non-regular hereditary polyhedra with $2r$-valent vertices can be seen as quotients of the uniform tessellations $(p.q)^r := (p.q.p.q. \dots p.q.p.q)$, with r entries p and q, of the sphere, Euclidean plane, or hyperbolic plane, which can be derived by Wythoff's construction from the (p, q, r) extended triangle group as indicated below (see [5]).



(3)

In particular, the hereditary polyhedra arising as medials of regular maps of type $\{p, q\}$ are quotients of the above infinite tessellations constructed from the $(p, q, 2)$ extended triangle groups. Similarly, the polyhedra arising from our generalized halving construction of a regular map of type $\{2p, q\}$ are quotients of the infinite tessellations constructed from the (p, q, q) extended triangle groups.

Moving on to rank 4, we observe that the semi-regular tessellation of Euclidean 3-space by regular tetrahedra and octahedra gives a simple example of a regular-facetted hereditary polytope which is not regular. Its geometric symmetry group is a subgroup of index 2 in the symmetry group of the cubical tessellations of 3-space. Note that the combinatorial automorphism group of either tessellation is isomorphic to its symmetry group.

5 Hereditary Polytopes with Chiral Facets

When a hereditary polytope has chiral facets, its rank is at least 4. In this section we show that any such polytope has either two or four flag-orbits.

5.1 Flag-Orbits

Call an abstract polytope $\mathscr{P}$ *equifacetted* if its facets are mutually isomorphic. All regular or chiral polytopes are equifacetted. A 2-orbit n-polytope in a class 2_I with $n-1 \in I$ is also equifacetted.

Theorem 3. *A chiral-facetted hereditary n-polytope is a 2-orbit polytope which is either chiral or in class $2_{\{n-1\}}$ (and hence is equifacetted), or a 4-orbit polytope.*

Proof. Let $\mathscr{P}$ be a chiral-facetted hereditary n-polytope. First note that we must have $n \geq 4$, since the facets of polytopes of rank at most 3 are always regular, not chiral. By Proposition 2, the polytope $\mathscr{P}$ is $(n-3)$-face transitive since its facets are $(n-3)$-face transitive. In particular, any flag of $\mathscr{P}$ is equivalent under $\Gamma(\mathscr{P})$ to a flag containing a fixed $(n-3)$-face F of $\mathscr{P}$. Again we employ the action of the stabilizer $\Gamma_F(\mathscr{P})$ on the set of flags of $\mathscr{P}$ containing F.

Let F be an $(n-3)$-face of $\mathscr{P}$, and let Ω be a flag of the section F/F_{-1}. For each facet H of $\mathscr{P}$ containing F there exists a unique involution $\tau^H_{0,n-2}$ (say) in $\Gamma(H/F_{-1})$ which interchanges the two $(n-2)$-faces of H containing F while fixing all faces of Ω except the 0-face. Let Λ denote the subgroup of $\Gamma_F(\mathscr{P})$ generated by the involutions $\tau^H_{0,n-2}$, with H a facet containing F. Now suppose again that the 2-polytope F_n/F is a q-gon, allowing $q = \infty$. When restricted to the co-face F_n/F, the involutions $\tau^H_{0,n-2}$ act like reflections in perpendicular bisectors of edges of a convex regular q-gon, and so the restricted Λ is isomorphic to a dihedral group D_q or $D_{q/2}$ according as q is odd or even. Hence Λ, restricted to F_n/F, has one or two flag-orbits on the 2-polytope F_n/F, respectively; in the latter case the two flag-orbits can be represented by a pair of 1-adjacent flags of F_n/F. Note, however, that unlike in the case of hereditary polytopes with regular facets, Λ does not act trivially on the $(n-3)$-face F/F_{-1}. (In fact, each $\tau^{H}_{0,n-2}$ maps Ω to Ω^0, the 0-adjacent flag, so the restriction of Λ to F/F_{-1} is a group C_2.)

Now let G be an $(n-2)$-face of $\mathscr{P}$ incident with F, and let H and H' denote the two facets of $\mathscr{P}$ meeting at G. Then $\Phi := \Omega \cup \{G, H, F_n\}$ is a flag of $\mathscr{P}$ containing F. Note that $\{F, G, H, F_n\}$ and $\{F, G, H', F_n\}$ are 1-adjacent flags of the q-gon F_n/F which are contained in Φ or Φ^{n-1}, respectively. Now let Ψ be any flag of $\mathscr{P}$ containing F. Then two possible scenarios can occur.

First suppose q is odd. Then since Λ acts flag-transitively on F_n/F, the flag Ψ can be mapped by an element of Λ to a flag Ψ' containing $\{F, G, H, F_n\}$. Then $\Psi' \setminus \{F_n\}$ is a flag of the facet H/F_{-1}, and since H/F_{-1} is chiral, it can be taken by an automorphism of H/F_{-1} to either the flag $\Phi \setminus \{F_n\}$ of H/F_{-1} or the j-adjacent flag $(\Phi \setminus \{F_n\})^j$, for any $j = 0, \ldots, n-2$. But $\mathscr{P}$ is hereditary, so the extension of this automorphism to $\mathscr{P}$ then necessarily maps Ψ' to either Φ or Φ^j. On the other hand, the two flags Φ and Φ^j are not equivalent under $\Gamma(\mathscr{P})$, since otherwise the facets would be regular, not chiral. Thus $\Gamma(\mathscr{P})$ has two flag-orbits represented by any pair of j-adjacent flags, with $j = 0, \ldots, n-2$. Hence $\mathscr{P}$ is a 2-orbit polytope, either of type $2_{\emptyset}$ and then $\mathscr{P}$ is chiral, or of type $2_{\{n-1\}}$. (Note that our arguments do not require the above automorphisms to belong to $\Gamma_F(\mathscr{P})$; in fact, when $j = n-3$, and possibly when $j = n-2$ with $n \geq 5$, they will not lie $\Gamma_F(\mathscr{P})$.)

Next suppose q is even. Now Λ acts with two flag-orbits on F_n/F, so Ψ can be mapped under Λ to a flag Ψ' which either contains $\{F, G, H, F_n\}$ or $\{F, G, H', F_n\}$. In the former case, Ψ' is as above equivalent to Φ or Φ^j, for any $j = 0, \ldots, n-2$, again under the extension of a suitable automorphism of the chiral facet H/F_{-1} to $\mathscr{P}$. In the latter case, Ψ' is equivalent to Φ^{n-1} or $\Phi^{n-1,j}$, for any $j = 0, \ldots, n-2$, now under the extended automorphism of the $(n-1)$-adjacent facet H'/F_{-1} of H/F_{-1} in $\mathscr{P}$. As before, Φ and Φ^j cannot be equivalent under $\Gamma(\mathscr{P})$, and neither can Φ^{n-1} and $\Phi^{n-1,j}$. Moreover, Φ is equivalent to Φ^{n-1} or $\Phi^{n-1,j}$ respectively, if and only if Φ^j is equivalent $\Phi^{n-1,j}$ or Φ^{n-1}. Hence $\mathscr{P}$ has two or four flag-orbits. If there are four flag-orbits, then these can be represented by Φ, Φ^j, Φ^{n-1}, $\Phi^{n-1,j}$, and we are done. Otherwise $\mathscr{P}$ is a 2-orbit polytope with its two flag-orbits represented by Φ and Φ^j. In this case $\mathscr{P}$ is either of type $2_{\emptyset}$ and then $\mathscr{P}$ is chiral, or of type $2_{\{n-1\}}$; accordingly, Φ and Φ^{n-1} represent different, or the same, flag-orbits under $\Gamma(\mathscr{P})$. In either case we are done as well, and the proof is complete. □

Note that the proof of Theorem 3 shows that the four flag-orbits of a chiral-facetted hereditary 4-orbit n-polytope $\mathscr{P}$ can be represented by the four flags $\Psi, \Psi^0, \Psi^{n-1}, \Psi^{n-1,0}$, where Ψ is any flag of $\mathscr{P}$.

In rank 4, many examples of chiral polytopes with chiral facets are known (see [3, 8, 22]). These are chiral-facetted hereditary polytopes of the first kind mentioned in Theorem 3. By contrast, it is not at all clear that chiral-facetted hereditary polytopes of the two other kinds actually exist (for any rank $n \geq 4$). In the remainder of this section we establish the existence of such examples. We show that there is a wealth of chiral-facetted hereditary 2-orbit polytopes in the class $2_{\{n-1\}}$, at least for $n = 4, 5$ but most likely for any $n \geq 4$; and that chiral-facetted hereditary 4-orbit polytopes exist at least in rank 4.

5.2 Chiral-Facetted Hereditary n-Polytopes in Class $\mathbf{2}_{\{n-1\}}$

We begin by briefly reviewing the cube-like polytopes $2^{\mathcal{K}}$ originally due to Danzer (see [7, 20] and [16, Section 8D]).

Let $\mathcal{K}$ be a finite abstract $(n-1)$-polytope with vertex-set $V := \{1, \ldots, v\}$ (say). Suppose $\mathcal{K}$ is *vertex-describable*, meaning that its faces are uniquely determined by their vertex-sets. Thus we may identify the faces of $\mathcal{K}$ with their vertex-sets, which are subsets of V. Then $2^{\mathcal{K}}$ is a (vertex-describable) abstract n-polytope with 2^v vertices, each with a vertex-figure isomorphic to $\mathcal{K}$. The vertex-set of $2^{\mathcal{K}}$ is

$$2^V := \bigotimes_{i=1}^{v} \{0, 1\}, \tag{4}$$

the cartesian product of v copies of $\{0, 1\}$. When $j \geq 1$ we take as j-faces of $2^{\mathcal{K}}$, for any $(j-1)$-face F of $\mathcal{K}$ and any $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_v)$ in 2^V, the subsets $F(\varepsilon)$ of 2^V defined by

$$F(\varepsilon) := \{(\eta_1, \ldots, \eta_v) \in 2^V \mid \eta_i = \varepsilon_i \text{ if } i \notin F\}, \tag{5}$$

or, abusing notation, by the cartesian product

$$F(\varepsilon) := \left(\bigotimes_{i \in F} \{0, 1\}\right) \times \left(\bigotimes_{i \notin F} \{\varepsilon_i\}\right).$$

Then, if F, F' are faces of $\mathcal{K}$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_v)$, $\varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_v)$ are elements in 2^V, we have $F(\varepsilon) \subseteq F'(\varepsilon')$ in $2^{\mathcal{K}}$ if and only if $F \leq F'$ in $\mathcal{K}$ and $\varepsilon_i = \varepsilon'_i$ for each i not in F'. The least face of $2^{\mathcal{K}}$ (of rank -1) is the empty set. Note that the vertices ε of $2^{\mathcal{K}}$ arise here as singletons in the form $F(\varepsilon) = \{\varepsilon\}$ when $F = \emptyset$, the least face of $\mathcal{K}$. Notice that if $\mathcal{K}$ is the $(n-1)$-simplex, then $2^{\mathcal{K}}$ is the n-cube.

Each j-face of $2^{\mathcal{K}}$ is isomorphic to a j-polytope $2^{\mathcal{F}}$, where $\mathcal{F}$ is a $(j-1)$-face of $\mathcal{K}$. More precisely, if F is a $(j-1)$-face of $\mathcal{K}$ and $\mathcal{F} := F/F_{-1}$, then each j-face $F(\varepsilon)$ with ε in 2^V is isomorphic to $2^{\mathcal{F}}$.

The automorphism group of $2^{\mathcal{K}}$ is given by

$$\Gamma(2^{\mathcal{K}}) \cong C_2 \wr \Gamma(\mathcal{K}) \cong C_2^v \rtimes \Gamma(\mathcal{K}), \tag{6}$$

the wreath product of C_2 and $\Gamma(\mathcal{K})$ defined by the natural action of $\Gamma(\mathcal{K})$ on the vertex-set of $\mathcal{K}$. In particular, $\Gamma(2^{\mathcal{K}})$ acts vertex-transitively on $2^{\mathcal{K}}$ and the vertex stabilizers are isomorphic to $\Gamma(\mathcal{K})$. Moreover, each automorphism of every vertex-figure of $2^{\mathcal{K}}$ extends to an automorphism of the entire polytope $2^{\mathcal{K}}$.

The following theorem summarizes properties of $2^{\mathcal{K}}$ that are relevant for our discussion of hereditary polytopes.

Theorem 4. *Let $\mathscr{K}$ be a finite abstract $(n-1)$-polytope with v vertices, and let $\mathscr{K}$ be vertex-describable. Then $2^{\mathscr{K}}$ is a finite abstract n-polytope with the following properties.*

(a) If $\mathscr{K}$ is a k-orbit polytope for $k \geq 1$, then $2^{\mathscr{K}}$ is also a k-orbit polytope.
(b) If $\mathscr{K}$ is regular, then $2^{\mathscr{K}}$ is regular.
(c) If $\mathscr{K}$ is a 2-orbit polytope in class 2_I for $I \subseteq \{0, \dots, n-2\}$, then $2^{\mathscr{K}}$ is a 2-orbit polytope in class 2_J for $J := \{0\} \cup \{i+1 \mid i \in I\}$.
(d) If $\mathscr{K}$ is chiral, then $2^{\mathscr{K}}$ is a 2-orbit polytope in class $2_{\{0\}}$.

Proof. For part (a), since $\Gamma(2^{\mathscr{K}})$ acts vertex-transitively on $2^{\mathscr{K}}$, every flag-orbit under $\Gamma(2^{\mathscr{K}})$ can be represented by a flag containing the vertex $o := (0, \dots, 0)$ of $2^{\mathscr{K}}$. Moreover, since the vertex stabilizer of o is isomorphic to $\Gamma(\mathscr{K})$, two flags containing o are equivalent in $\Gamma(2^{\mathscr{K}})$ if and only if they are equivalent in $\Gamma(\mathscr{K})$. Thus the number of flag-orbits of $\mathscr{K}$ and $2^{\mathscr{K}}$ is the same. This proves part (a). For part (b), simply apply part (a) with $k = 1$.

For part (c), suppose $\mathscr{K}$ is a 2-orbit polytope in class 2_I. Then part (a) shows that $2^{\mathscr{K}}$ is also a 2-orbit polytope. Choose a flag $\Psi := \{F_0, F_1, \dots, F_{n-2}\}$ of $\mathscr{K}$ and consider the corresponding flag $\Phi := \{o, F_0(o), F_1(o), \dots, F_{n-2}(o)\}$ of $2^{\mathscr{K}}$ which contains o (we are suppressing the least face and the largest face). In $\mathscr{K}$, the i-adjacent flags Ψ, Ψ^i of $\mathscr{K}$ lie in the same orbit under $\Gamma(\mathscr{K})$ if and only if $i \in I$. Relative to $2^{\mathscr{K}}$, the adjacency levels of $\mathscr{K}$ are shifted by 1. Hence, if $j \geq 1$, then a pair of j-adjacent flags Φ, Φ^j of $2^{\mathscr{K}}$ lie in the same orbit under $\Gamma(2^{\mathscr{K}})$ if and only if $j \in \{i+1 \mid i \in I\}$. In addition, the 0-adjacent flags Φ, Φ^0 of $2^{\mathscr{K}}$ always are equivalent under $\Gamma(2^{\mathscr{K}})$; in fact, the mapping on 2^V defined by

$$(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_v) \longrightarrow (\varepsilon_1 + 1, \varepsilon_2, \dots, \varepsilon_v),$$

with addition mod 2 in the first component, induces an automorphism of $2^{\mathscr{K}}$ taking Φ to Φ^0. Thus, Φ and Φ^j are in the same flag-orbit of $2^{\mathscr{K}}$ if and only if $j \in J$. This proves part (c). For part (d), apply part (c) with $I = \emptyset$. $\square$

Appealing to duality, the previous theorem now allows us to settle the existence of chiral-facetted hereditary n-polytopes in class $2_{\{n-1\}}$. Call an abstract polytope $\mathscr{Q}$ *facet-describable* if each face of $\mathscr{Q}$ is uniquely determined by the facets of $\mathscr{Q}$ that are incident with it. Thus, $\mathscr{Q}$ is facet-describable if and only if its dual $\mathscr{Q}^*$ is vertex-describable. Just like vertex-describability, facet-describability is a relatively mild assumption on a polytope. Any polytope that is a lattice, is both vertex-describable and facet-describable.

Corollary 1. *Let $\mathscr{Q}$ be a finite chiral $(n-1)$-polytope, and let $\mathscr{Q}$ be facet-describable. Then $(2^{\mathscr{Q}^*})^*$ is a chiral-facetted hereditary 2-orbit n-polytope in class $2_{\{n-1\}}$ with facets isomorphic to $\mathscr{Q}$. Moreover,*

$$\Gamma((2^{\mathscr{Q}^*})^*) \cong C_2 \wr \Gamma(\mathscr{Q}) \cong C_2^f \rtimes \Gamma(\mathscr{Q}),$$

where f is the number of facets of $\mathscr{Q}$.

Proof. The dual $\mathcal{Q}^*$ of $\mathcal{Q}$ is chiral and vertex-describable. By Theorem 4, the polytope $2^{\mathcal{Q}^*}$ has 2 orbits and belongs to class $2_{\{0\}}$. Hence its dual, $(2^{\mathcal{Q}^*})^*$, is a 2-orbit polytope in class $2_{\{n-1\}}$. Its facets are the duals of the vertex-figures of $2^{\mathcal{Q}^*}$. Thus the facets of $(2^{\mathcal{Q}^*})^*$ are isomorphic to $\mathcal{Q}$ and hence are chiral. Moreover, $(2^{\mathcal{Q}^*})^*$ is hereditary, since every automorphism of every vertex-figure of $2^{\mathcal{Q}^*}$ extends to an automorphism of the entire polytope $2^{\mathcal{Q}^*}$. The second part of the corollary follows from (6), bearing in mind that f is just the number of vertices of $\mathcal{Q}^*$ and that dual polytopes have the same group. □

Chiral polytopes are known to exist for every rank greater than or equal to 3 (see Pellicer [19]). We strongly suspect that most polytopes constructed in [19] are also facet-describable. Corollary 1 provides an n-polytope of the desired kind for every $n \geq 4$ for which there exists a finite chiral $(n-1)$-polytope which is facet-describable. For $n = 4$ or 5 there are many such examples.

5.3 Chiral-Facetted Hereditary Polytopes with Four Orbits

In this section we describe a construction of "alternating" polytopes which is inspired by the methods in Monson and Schulte [17] and provides examples of chiral-facetted hereditary 4-polytopes with four flag-orbits.

5.3.1 Halving of Polyhedra

We begin by reviewing a construction of polyhedra which arises from the halving operation η of [16, Section 7B] described below; it can be considered as a special case of the construction given in 4.2.

Let $\mathcal{K}$ be an equivelar map of type $\{4, q\}$ whose edge graph is bipartite. Then every edge circuit in $\mathcal{K}$ has even length. Suppose that the vertices of $\mathcal{K}$ are colored *red* and *yellow* such that adjacent vertices have different colors. The vertex-figures at the red vertices of $\mathcal{K}$ (obtained by joining the vertices adjacent to a given red vertex in cyclic order) form the faces of a map of type $\{q, q\}$ (which is usually a polyhedron) on the same surface as the original map. Its vertices and "face centers" are the yellow and red vertices of $\mathcal{K}$, respectively; its edges are the "diagonals" in (square) faces of $\mathcal{K}$ that join yellow vertices. Notice that the original map $\mathcal{K}$ can be recovered from the new map.

When the two colors are interchanged, we similarly obtain another map of type $\{q, q\}$, the dual of the first map, which a priori need not be isomorphic to the first map. However, if $\mathcal{K}$ admits an automorphism swapping the two color-classes of vertices, then these maps are isomorphic; this holds, for example, if the original polyhedron $\mathcal{K}$ is vertex-transitive. In our applications this will always be the case, and in such instances we denote the map by $\mathcal{K}^a$ (with the "a" standing for "alternate vertices"). Note that $\mathscr{K}^a$ has half as many flags as $\mathscr{K}$; in fact, the number of vertices is halved and the vertex degrees are maintained.

We now impose symmetry conditions on $\mathscr{K}$. First let $\mathscr{K}$ be regular, and let $\Gamma(\mathscr{K}) = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$, where $\alpha_0, \alpha_1, \alpha_2$ are the distinguished generators. From the *halving operation*

$$\eta: \ (\alpha_0, \alpha_1, \alpha_2) \to (\alpha_0\alpha_1\alpha_0, \alpha_2, \alpha_1) =: (\beta_0, \beta_1, \beta_2), \tag{7}$$

we then obtain the generators $\beta_0, \beta_1, \beta_2$ for the automorphism group of a self-dual regular polyhedron $\mathscr{K}^\eta$ of type $\{q, q\}$, which is a subgroup of index 2 in $\Gamma(\mathscr{K})$ (see [16, Section 7B]); bear in mind here that the edge graph of $\mathscr{K}$ is bipartite and that $(\alpha_0\alpha_1)^4 = \varepsilon$. This polyhedron can be drawn as a map on the same surface as $\mathscr{K}$ by employing Wythoff's construction with generators $\beta_0, \beta_1, \beta_2$ and base vertex z (say) of $\mathscr{K}$. Then it is easily seen that $\mathscr{K}^\eta$ is just the polyhedron $\mathscr{K}^a$ described earlier, realized here with z as a yellow vertex of $\mathscr{K}$.

Notice that replacing η by

$$\eta^0: \ (\alpha_0, \alpha_1, \alpha_2) \to (\alpha_1, \alpha_2, \alpha_0\alpha_1\alpha_0) =: (\gamma_0, \gamma_1, \gamma_2) \tag{8}$$

results in another set of generators $\gamma_0, \gamma_1, \gamma_2$, which are conjugate under α_0 to $\beta_0, \beta_1, \beta_2$. When Wythoff's construction is applied with these new generators and base vertex $\alpha_0(z)$ adjacent to z, we similarly arrive at a regular polyhedron $\mathscr{K}^{\eta^0}$ on the same surface which is dually positioned to $\mathscr{K}^\eta$, has its vertices at the red vertices of $\mathscr{K}$, and is isomorphic to $\mathscr{K}^a$. Note that the new generators $\gamma_0, \gamma_1, \gamma_2$ in (8) can be found from $\alpha_0, \alpha_1, \alpha_2$ in one of two equivalent ways: either as in η^0 by first applying η and then conjugating the β_j's by α_0, or by first conjugating the α_j's by α_0 and then applying η to these new generators.

If $\mathscr{K}$ is chiral, we can work with corresponding operations at the rotation group level, again denoted by η and η^0. Suppose $\Gamma(\mathscr{K}) = \langle \sigma_1, \sigma_2 \rangle$, where σ_1, σ_2 are the distinguished generators. Then the two operations

$$\begin{aligned} \eta &: (\sigma_1, \sigma_2) \to (\sigma_1^2\sigma_2, \sigma_2^{-1}) =: (\varphi_1, \varphi_2) \\ \eta^0 &: (\sigma_1, \sigma_2) \to (\sigma_2, \sigma_2^{-1}\sigma_1^2) =: (\psi_1, \psi_2) \end{aligned} \tag{9}$$

give a pair of self-dual maps of type $\{q, q\}$ each isomorphic to $\mathscr{K}^a$. Now these maps are chiral, as can be seen as follows. First note that $\langle \varphi_1, \varphi_2 \rangle$ and $\langle \psi_1, \psi_2 \rangle$ are subgroups of $\Gamma(\mathscr{K})$ preserving the vertex partition of $\mathscr{K}$, and hence their index in $\Gamma(\mathscr{K})$ must be 2. Moreover, $\mathscr{K}^a$ has half as many flags as $\mathscr{K}$, so these groups must act on the new map $\mathscr{K}^a$ with two flag-orbits. Now suppose $\mathscr{K}^a$ was directly regular, rather than chiral. Then it is immediately clear from the geometry of $\mathscr{K}^a$ that any automorphism ρ of $\mathscr{K}^a$ which fixes a 2-face of $\mathscr{K}^a$ and interchanges the two edges at a vertex of this 2-face, must act on the original map $\mathscr{K}$ like a combinatorial reflection in the edge of $\mathscr{K}$ at that vertex invariant under ρ. More informally, an automorphism ρ of $\mathscr{K}^a$ which acts like a generator ρ_1 of $\Gamma(\mathscr{K}^a)$, becomes an automorphism of $\mathscr{K}$ which acts like a generator ρ_2 of $\Gamma(\mathscr{K})$. Alternatively this can be verified algebraically at the group level: any involutory group automorphism of $\langle\varphi_1, \varphi_2\rangle$ (or $\langle\psi_1, \psi_2\rangle$, respectively) that behaves just like conjugation by a generator ρ_1, gives rise to an involutory group automorphism of $\Gamma(\mathscr{K})$ that behaves just like conjugation by a generator ρ_2.

As in the case of regular maps, $\mathscr{K}^{\eta}$ and $\mathscr{K}^{\eta^0}$ can be drawn on the same underlying surface by employing a variant of Wythoff's construction, now applied with the new generators of (9) and with either z or $\sigma_1(z)$ as base vertex. The two maps are again dually positioned relative to each other. The vertex z of $\mathscr{K}$ is a vertex of $\mathscr{K}^{\eta}$, but not of $\mathscr{K}^{\eta^0}$. Hence, if z is a yellow vertex of $\mathscr{K}$, then $\mathscr{K}^{\eta}$ uses only yellow vertices of $\mathscr{K}$ while $\mathscr{K}^{\eta^0}$ uses only red vertices of $\mathscr{K}$. In analogy to what we said about the operations in (7) and (8), the new generators ψ_1, ψ_2 in η^0 of (9) can be found from σ_1, σ_2 in one of two equivalent ways: either as in η^0 by first applying η and then passing to the generators for the other enantiomorphic form of $\mathscr{K}^{\eta}$, or by first passing to the generators for the other enantiomorphic form of $\mathscr{K}$ and then applying η to these new generators.

5.3.2 Alternating Chiral-Facetted 4-Polytopes

Following [17], an n-polytope is said to be *alternating* if it has facets of possibly two distinct combinatorial isomorphism types appearing in alternating fashion around faces of rank $n-3$. We allow the possibility that the two isomorphism types coincide, although we are less interested in this case. The cuboctahedron is an example of an alternating polyhedron in which triangles and squares alternate around a vertex.

A more interesting example is the familiar semiregular tiling $\mathscr{T}$ of Euclidean 3-space $\mathbb{E}^3$ by regular octahedra and tetrahedra illustrated in Fig. 1, which is an alternating 4-polytope in which octahedra and tetrahedra alternate around an edge (see [5, 17]). Its vertex-figures are (alternating) cuboctahedra. More generally it is true that the vertex-figures of an alternating n-polytopes are alternating $(n-1)$-polytopes. From now on, we restrict ourselves to polytopes of rank 3 or 4.

The relationship of the semiregular tiling $\mathscr{T}$ with the (regular) cubical tiling $\mathscr{C} := \{4, 3, 4\}$ in $\mathbb{E}^3$ will serve as the blueprint for our construction. As the edge graph of $\mathscr{C}$ is bipartite, we can color the vertices *red* or *yellow* such that adjacent vertices receive different colors. Then the octahedral tiles of $\mathscr{T}$ can be viewed as the vertex-figures of $\mathscr{C}$ at the red vertices, each spanned by the yellow vertices adjacent to the corresponding red vertex. The complement in $\mathbb{E}^3$ of the union of all these octahedral tiles gives rise to the family of tetrahedral tiles of $\mathscr{T}$, each inscribed in a cube of $\mathscr{C}$; each cube contributes exactly one tetrahedral tile, such that the tetrahedral tiles in adjacent cubes share a common edge.

Now let $\mathscr{P}$ be any finite 4-polytope, let $\mathscr{K}$ be a vertex-transitive polyhedron of type $\{4, q\}$, and let $\mathscr{L}$ be a polyhedron of type $\{q, r\}$. Suppose that all facets of $\mathscr{P}$ are isomorphic to $\mathscr{K}$, and that all vertex-figures are isomorphic to $\mathscr{L}$. Thus $\mathscr{P}$ is equivelar of type $\{4, q, r\}$.

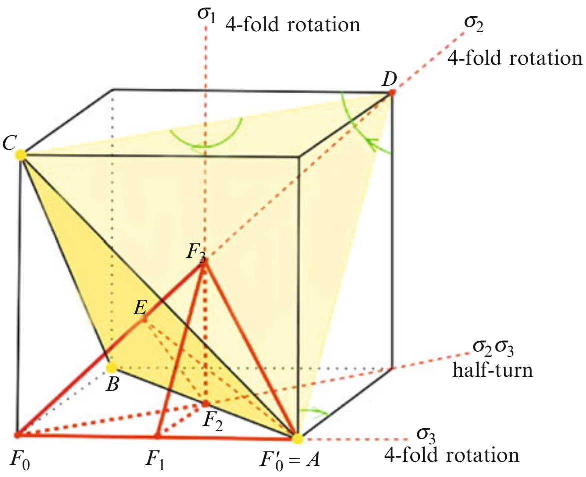


Fig. 1 A patch of the semiregular tiling $\mathscr{T}$ derived from the cubical tiling $\mathscr{C}$. Shown are a tetrahedal tile with vertices A, B, C, D, and one eighths of an octahedral tile (vertices A, B, C) centered at the base vertex $F_0 = z$. The axes of the three generating rotations $\sigma_1, \sigma_2, \sigma_3$ for the rotation subgroup of $\mathscr{C}$ are indicated, as is the fundamental tetrahedron for this subgroup with vertices F_0, F_0', F_2, F_3. The plane through A, B, C dissects this fundamental tetrahedron into two smaller tetrahedra, each becoming a fundamental tetrahedron for the full symmetry group of a tile, namely the tetrahedron with vertices F_0, F_0', F_2, E for the octahedral tile and the tetrahedron with vertices F_0', F_2, E, F_3 for the tetrahedral tile

Further, suppose the edge graph of $\mathscr{P}$ is bipartite, with vertices colored red or yellow such that adjacent vertices have different colors. Let R and Y, respectively, denote the sets of red or yellow vertices of $\mathscr{P}$. Then every edge circuit in $\mathscr{P}$ has even length, and the edge graph of $\mathscr{K}$ is also bipartite. It is convenient to require two additional "lattice-like" conditions to hold. First, both $\mathscr{P}$ and $\mathscr{L}$ should be vertex-describable, so that we may identify faces with their vertex-sets; then, as a facet of a vertex-describable polytope, $\mathscr{K}$ must also be vertex-describable. Second, any two opposite vertices of a 2-face of $\mathscr{P}$ should not be opposite vertices of another 2-face of $\mathscr{P}$. Later we impose strong symmetry conditions on $\mathscr{K}$, $\mathscr{L}$ and $\mathscr{P}$, but for now we work in the present generality.

We now derive from $\mathscr{P}$ a new 4-polytope $\mathscr{P}^a$, where "a" indicates "alternating". The vertex-set of $\mathscr{P}^a$ is the set Y of yellow vertices of $\mathscr{P}$. Our description of the faces of $\mathscr{P}^a$ is in terms of their vertex-sets, that is, subsets of Y. In particular, the edges of $\mathscr{P}^a$ are the diagonals of the (square) 2-faces of $\mathscr{P}$ that connect yellow vertices; more precisely, $\{v, w\}$ is a 1-face of $\mathscr{P}^a$ if and only if $v, w \in Y$ and v, w are opposite vertices in a 2-face of $\mathscr{P}$. Then, by our assumption on the 2-faces of $\mathscr{P}$, any two vertices of $\mathscr{P}^a$ are joined by at most one edge.

The 2-faces of $\mathscr{P}^a$ are the vertex-figures, within the facets of $\mathscr{P}$, at the red vertices of these facets; more precisely, $\{v_1, \ldots, v_{q'}\}$ is a 2-face of $\mathscr{P}^a$ if and only if there exists a facet F of $\mathscr{P}$ with a red vertex v such that $\{v_1, \ldots, v_{q'}\}$ is the set of (yellow) vertices, labeled in cyclic order, of the vertex-figure at v in F. Clearly, the 2-faces of $\mathscr{P}^a$ must be q-gons, that is, $q' = q$ in each case. Alternatively, we can describe the 2-faces of $\mathscr{P}^a$ as the 2-faces of the vertex-figures at red vertices in $\mathscr{P}$.

The facets of $\mathscr{P}^a$ are of two kinds and correspond to either a halved facet or the vertex figure at a red vertex of $\mathscr{P}$. Each facet F of $\mathscr{P}$ gives rise to a facet F^a of $\mathscr{P}^a$, of the *first kind*, obtained (as in Sect. 5.3.1) as the polyhedron whose 2-faces are the vertex-figures of F at the red vertices; when F is viewed as a map of type $\{4, q\}$ on a surface, F^a is a map of type $\{q, q\}$ that can be drawn on the same surface. Note here that, by the vertex-transitivity of $\mathscr{K}$, the combinatorial structure of F^a does not depend on which class of vertices in the bipartition of the vertex-set of F is used as the vertex-set for F^a (the two maps arising from the two possible choices of vertex-sets are related by duality, but they are isomorphic since $\mathscr{K}$ is vertex-transitive). Thus the facets F^a of the first kind are mutually isomorphic, each to the map $\mathscr{K}^a$ of Sect. 5.3.1. The facets of $\mathscr{P}^a$ of the *second kind* are the vertex-figures, $\mathscr{P}/v$, of $\mathscr{P}$ at the red vertices, v.

For example, if $\mathscr{P}$ is the cubical tessellation $\mathscr{C}$ described earlier, then the facets of the first kind are tetrahedra $F^a = \{3, 3\}$ inscribed in cubes F of $\mathscr{C}$, and the facets of the second kind are the octahedral vertex-figures $\mathscr{C}/v = \{3, 4\}$ of $\mathscr{C}$ at the red vertices. Thus, combinatorially, $\mathscr{P}^a = \mathscr{T}$, the semiregular tiling of $\mathbb{E}^3$ by tetrahedra and octahedra.

Incidence of faces in $\mathscr{P}^a$ is defined by inclusion of vertex-sets; that is, two faces of $\mathscr{P}^a$ are incident if and only if their vertex-sets (as subsets of the vertex-set of $\mathscr{P}$) are related by inclusion. Note that two facets of $\mathscr{P}^a$ can only be adjacent (share a 2-face) if they are of different kinds, and that a facet F^a of the first kind is adjacent to a facet $\mathscr{P}/v$ of the second kind if and only if v is a vertex of F. Each edge of $\mathscr{P}^a$ is surrounded by four facets of $\mathscr{P}^a$, occurring in alternating fashion; more explicitly, if $\{v, w\}$ is an edge of $\mathscr{P}^a$ given by the diagonal of a 2-face G of $\mathscr{P}$, then these four facets are F^a, $\mathscr{P}/u$, $(F')^a$ and $\mathscr{P}/u'$, in this order, where F and F' are the two facets of $\mathscr{P}$ meeting at G, and u, u' are the two vertices of G distinct from v and w. Thus $\mathscr{P}^a$ is alternating.

The vertex-set of the vertex-figure $\mathscr{P}^a/v$ of $\mathscr{P}^a$ at a vertex v (a yellow vertex of $\mathscr{P}$) consists of the vertices w of $\mathscr{P}^a$ such that $\{v, w\}$ is an edge of $\mathscr{P}^a$. Combinatorially, $\mathscr{P}^a/v$ is the medial $Me(\mathscr{L})$ of the vertex-figure $\mathscr{L}$ of $\mathscr{P}$. To see this, in the above, replace the vertex w of the edge $\{v, w\}$ by the "midpoint" of that edge (this is the "center" of the respective 2-face of $\mathscr{P}$ that determines that edge), and impose on this new vertex-set the same combinatorial structure as on the original vertex-set of $\mathscr{P}^a/v$. In the example of the semiregular tiling $\mathscr{T}$ of $\mathbb{E}^3$ the vertex-figures are cuboctahedra, occurring as medials of the octahedral vertex-figures of the cubical tiling $\mathscr{C}$ at yellow vertices.

Notice that the new polytope $\mathscr{P}^a$ has the same number of flags as the original polytope $\mathscr{P}$. In fact, the number of vertices of $\mathscr{P}^a$ is half that of $\mathscr{P}$, while the number of flags of the vertex-figures $Me(\mathscr{L})$ of $\mathscr{P}^a$ is twice that of the vertex-figures $\mathscr{L}$ or $\mathscr{P}$. Bear in mind our assumption that $\mathscr{P}$ is finite.

We now investigate the combinatorial symmetries of $\mathscr{P}^a$. First observe that $\mathscr{P}^a$ inherits all automorphisms of $\mathscr{P}$ that preserve colors of vertices. Observe here that, since the edge graph of $\mathscr{P}$ is bipartite and connected, an automorphism γ of $\mathscr{P}$ maps the full set of yellow vertices Y to itself if and only if γ maps any yellow vertex to a yellow vertex. Let $\Gamma^c(\mathscr{P})$ denote the subgroup of $\Gamma(\mathscr{P})$ mapping Y (and thus R) to itself. Clearly, $\Gamma^c(\mathscr{P})$ has index 1 or 2 in $\Gamma(\mathscr{P})$. Then it is immediately clear that $\Gamma^c(\mathscr{P})$ is a subgroup of $\Gamma(\mathscr{P}^a)$. In fact, the combinatorics of $\mathscr{P}^a$ is entirely derived from Y and has been described in a Y-invariant fashion.

With an eye on the hereditary property, we remark further that the vertex stabilizer $\Gamma_v(\mathscr{P})$ of a red vertex v in $\Gamma(\mathscr{P})$ becomes a subgroup of the automorphism group of the corresponding facet $\mathscr{P}/v$ of $\mathscr{P}^a$. Similarly, for any facet F of $\mathscr{P}$, the stabilizer of F in $\Gamma^c(\mathscr{P})$, which is given by $\Gamma^c(\mathscr{P}) \cap \Gamma_F(\mathscr{P})$, becomes a subgroup of the automorphism group of the corresponding facet F^a of $\mathscr{P}^a$.

Our remarks about $\Gamma^c(\mathscr{P})$ have immediate implications for the number of flag-orbits of $\mathscr{P}^a$.

In particular, if $\mathscr{P}$ is regular, then $\Gamma^c(\mathscr{P})$ must have index 2 as a subgroup of $\Gamma(\mathscr{P})$, and thus index 1 or 2 as a subgroup of $\Gamma(\mathscr{P}^a)$. To see this, note that the order of $\Gamma^c(\mathscr{P})$ is exactly half the number of flags of $\mathscr{P}$, and thus of $\mathscr{P}^a$. Hence $\mathscr{P}^a$ is regular or a 2-orbit polytope in class $2_{\{0,1,2\}}$. In either case, $\mathscr{P}^a$ is hereditary (and regular-facetted).

Similarly, if $\mathscr{P}$ is chiral, then $\Gamma^c(\mathscr{P})$ must have index 2 as a subgroup of $\Gamma(\mathscr{P})$, and thus index 1, 2 or 4 as a subgroup of $\Gamma(\mathscr{P}^a)$. Now the order of $\Gamma^c(\mathscr{P})$ is exactly a quarter of the number of flags of $\mathscr{P}$, and thus of $\mathscr{P}^a$. Now suppose $\mathscr{P}^a$ is hereditary. We show that then the facets and vertex-figures of $\mathscr{P}$ must be all regular or all chiral.

In fact, if the facets of the original polytope $\mathscr{P}$ are regular, each facet F^a of $\mathscr{P}^a$ of the first kind must also be regular and its full automorphism group must be a subgroup of $\Gamma(\mathscr{P}^a)$ (see Sect. 5.3.1); now since the combinatorial reflection symmetry in F^a that takes a flag of F^a to its 0-adjacent flag also gives a similar such reflection symmetry in the adjacent facet $\mathscr{P}/v$ (say) of $\mathscr{P}^a$ meeting F^a in the 2-face of the flag, it follows that the vertex-figures of $\mathscr{P}$ must actually also be regular since they already have (at least) maximal symmetry by rotation. Similarly, if the vertex-figures of the original polytope $\mathscr{P}$ are regular, then the hereditary property of $\mathscr{P}^a$ implies that the full automorphism group $\Gamma(\mathscr{P}/v)$ of a facet $\mathscr{P}/v$ of $\mathscr{P}^a$ is a subgroup of $\Gamma(\mathscr{P}^a)$ containing a combinatorial reflection symmetry of $\mathscr{P}/v$ that takes a flag of $\mathscr{P}/v$ to its 0-adjacent flag; as above, this reflection symmetry must induce a similar reflection symmetry in an adjacent facet F^a (say) of $\mathscr{P}^a$ and hence force this facet to be regular, since it already has (at least) maximal symmetry by rotation. Thus, if the original polytope $\mathscr{P}$ is chiral, then $\mathscr{P}^a$ can be hereditary only if the facets and vertex-figures of $\mathscr{P}^a$ are all regular or all chiral.

Conversely, if the facets and vertex-figures of a chiral polytope $\mathscr{P}$ are all regular or all chiral, then the new polytope $\mathscr{P}^a$ is hereditary, since each facet of $\mathscr{P}^a$ of either kind has all its automorphisms extended to the entire polytope $\mathscr{P}^a$. In particular, if the facets and vertex-figures of $\mathscr{P}$ are all regular, then $\mathscr{P}^a$ is regular-facetted and is either itself regular or a 2-orbit polytope of type $2_{\{0,1,2\}}$. Otherwise, $\mathscr{P}^a$ is chiral-facetted and has 1, 2 or 4 flag-orbits.

Now suppose $\mathscr{P}$ and all its facets and vertex-figures are chiral. Then recall from Sect. 5.1 that the flag-orbits of the corresponding hereditary polytope $\mathscr{P}^a$ can be represented by one, two, or four flags from among Ψ, Ψ^0, Ψ^3, $\Psi^{3,0}$, where Ψ is any flag of $\mathscr{P}^a$. First note that a pair of 0-adjacent flags of $\mathscr{P}^a$ cannot possibly be equivalent under $\Gamma(\mathscr{P}^a)$, since otherwise the facet of $\mathscr{P}^a$ common to both flags would have to be regular, not chiral. Thus Ψ, Ψ^0 (resp. $\Psi^3, \Psi^{3,0}$) are not equivalent under $\Gamma(\mathscr{P}^a)$, and $\mathscr{P}^a$ has 2 or 4 flag-orbits. Similarly, if the two kinds of facets of $\mathscr{P}^a$ are distinct (that is, non-isomorphic), then a pair of 3-adjacent flags of $\mathscr{P}^a$ cannot possibly be equivalent either, since any automorphism of $\mathscr{P}^a$ taking a flag to its 3-adjacent flag would provide an isomorphism between the two facets contained in these flags. Thus Ψ, Ψ^3 (resp. $\Psi^0, \Psi^{3,0}$) are non-equivalent and $\mathscr{P}^a$ must have 4 flag-orbits. Note that the non-isomorphism condition on the two kinds of facets of $\mathscr{P}^a$ holds, for example, if their numbers of flags are distinct, that is, if the number of flags of $\mathscr{K}$ is not exactly twice that of $\mathscr{L}$.

Our main findings are summarized in the following theorem.

Theorem 5. *Let $\mathscr{P}$ be a finite regular or chiral 4-polytope of type $\{\mathscr{K}, \mathscr{L}\}$, where $\mathscr{K}$ and $\mathscr{L}$ are polyhedra of type $\{4, q\}$ and $\{q, r\}$, respectively. Suppose that the edge graph of $\mathscr{P}$ is bipartite, that $\mathscr{P}$ and $\mathscr{L}$ are vertex-describable, and that any two opposite vertices of a 2-face of $\mathscr{P}$ are not opposite vertices of another 2-face of $\mathscr{P}$. Then $\mathscr{P}^a$ is a finite alternating hereditary 4-polytope with facets isomorphic to $\mathscr{L}$ or $\mathscr{K}^a$, and with vertex-figures isomorphic to the medial $Me(\mathscr{L})$ of $\mathscr{L}$. Every edge of $\mathscr{P}^a$ is surrounded by four facets, two of each kind occurring in an alternating fashion. Moreover, $\mathscr{P}^a$ has the following hereditary properties.*

(a) If $\mathscr{K}$ and $\mathscr{L}$ are regular, then $\mathscr{P}^a$ is a regular-facetted hereditary polytope and is either itself regular or a 2-orbit polytope of type $2_{\{0,1,2\}}$.
(b) If $\mathscr{K}$ and $\mathscr{L}$ are chiral, then $\mathscr{P}^a$ is a chiral-facetted hereditary polytope with 2 or 4 flag-orbits. If $\mathscr{L}$ and $\mathscr{K}^a$ are not isomorphic (for example, this holds when $|\Gamma(\mathscr{L})| \neq |\Gamma(\mathscr{K})|/2$), then $\mathscr{P}^a$ has 4 flag-orbits.

In either case (a) or (b), the group of all color preserving automorphisms $\Gamma^c(\mathscr{P})$ of $\mathscr{P}$ is a subgroup of $\Gamma(\mathscr{P}^a)$ of index 1 or 2, with the same or twice the number of flag-orbits as $\Gamma(\mathscr{P}^a)$.

The construction summarized in the previous theorem is a rich source for interesting examples of chiral-facetted hereditary 4-polytopes with 4 flag-orbits. To begin with, suppose $\mathscr{P}$ is a finite chiral 4-polytope of type $\{\mathscr{K}, \mathscr{L}\}$ such that $\mathscr{K}, \mathscr{L}$ are chiral and $\mathscr{K}^a, \mathscr{L}$ are non-isomorphic. There is a wealth of polytopes of this kind. Now, if the edge graph of $\mathscr{P}$ is bipartite, $\mathscr{P}$ and $\mathscr{L}$ are vertex-describable, and any two opposite vertices of a 2-face of $\mathscr{P}$ are not opposite vertices of another 2-face of $\mathscr{P}$, then Theorem 5 applies and yields a chiral-facetted alternating 4-polytope $\mathscr{P}^a$ which is hereditary and has 4 flag-orbits. Thus we need to assure that these three conditions hold; the requirement of a bipartite edge graph seems to be the most severe condition among the three. In our examples described below we verified these conditions with MAGMA.

For example, starting with the universal 4-polytope $\mathscr{P} = \{\{4,4\}_{1,3}, \{4,4\}_{1,3}\}$, which has 50 vertices, 50 facets, and an automorphism group of size 2000, our construction yields a hereditary 4-orbit polytope $\mathscr{P}^a$ which has two kinds of chiral facets, namely $\{4,4\}_{1,3}$ and $\{4,4\}_{1,2}$. It can be seen, for example using MAGMA, that the universal 4-polytope with the same facets but the enantiomorphic vertex-figures fails the conditions of Theorem 5, in that there exist two opposite vertices of a 2-face which are opposite vertices of another 2-face of that polytope.

6 Extensions of Hereditary Polytopes

In this section we briefly discuss extension problems for hereditary polytopes. We begin with a generalization of the notion of a hereditary polytope.

Let $1 \leq j \leq n-1$. An n-polytope $\mathscr{P}$ is said to be *j-face hereditary* if for each j-face F of $\mathscr{P}$, the automorphism group $\Gamma(F/F_{-1})$ of the section F/F_{-1} can be viewed as a subgroup of $\Gamma(\mathscr{P})$ (and hence of $\Gamma_F(\mathscr{P})$). Thus $\mathscr{P}$ is j-face hereditary if every automorphism of a j-face F extends to an automorphism of $\mathscr{P}$. Note that a hereditary polytope is $(n-1)$-face hereditary, or *facet hereditary*.

A j-face hereditary polytope is *strongly j-face hereditary* if for each j-face F of $\mathscr{P}$, the automorphism group $\Gamma(F/F_{-1})$ is a subgroup of $\Gamma(\mathscr{P})$ acting trivially on the *co-face* F_n/F; then $\Gamma(F/F_{-1})$ is the stabilizer of a flag of F_n/F in $\Gamma_F(\mathscr{P})$. Thus, for a strongly j-face hereditary polytope, every automorphism of a j-face F extends to a particularly well-behaved automorphism of $\mathscr{P}$, namely one which fixes every face of $\mathscr{P}$ in the co-face of F in $\mathscr{P}$.

The (vertex) truncated tetrahedron is a 1-face (or edge-) hereditary polyhedron which is not 2-face hereditary. The perpendicular bisectors of its edges are mirrors of reflection, but no geometric symmetry or combinatorial automorphism can rotate the vertices of a single face by one step. This example is a 3-orbit polyhedron.

Note that every 2-orbit n-polytope in a class 2_I with $\{0,1,\ldots,j-1\} \subseteq I$ is a strongly j-face hereditary polytope with regular j-faces. This follows directly from the definition of the class 2_I. For example, a 2-orbit polytope of rank 4 and type $2_{\{0,1\}}$ is 2-face hereditary; it may also be 3-face hereditary, but not a priori so.

Now the basic question arises whether or not each hereditary n-polytope occurs as a facet of an $(n-1)$-face hereditary $(n+1)$-polytope; or more generally, whether or not each j-face hereditary n-polytope occurs as a j-face of a k-face hereditary $(n+1)$-polytope, for any $j \leq k \leq n$.

In this context the following result is of interest.

Theorem 6. *Let $\mathscr{K}$ be a finite j-face hereditary n-polytope for some $j = 1,\ldots,n-1$, and let $\mathscr{K}$ be vertex-describable. Then $\mathscr{K}$ is the vertex-figure of a vertex-transitive finite $(j+1)$-face hereditary $(n+1)$-polytope.*

Proof. We employ the $2^{\mathscr{K}}$ construction described in Sect. 5.2. Since $\mathscr{K}$ is a vertex-describable finite n-polytope, $2^{\mathscr{K}}$ is a vertex-transitive finite $(n+1)$-polytope all of whose vertex-figures are isomorphic to $\mathscr{K}$. Every $(j+1)$-face of $2^{\mathscr{K}}$ is isomorphic to a $(j+1)$-polytope $2^{\mathscr{F}}$, where $\mathscr{F} := F/F_{-1}$ is the j-polytope given by a j-face

F as $\mathscr{K}$. Moreover, $\Gamma(2^{\mathscr{K}}) \cong C_2^v \rtimes \Gamma(\mathscr{K})$, where v is the number of vertices of $\mathscr{K}$; similarly, $\Gamma(2^{\mathscr{F}}) \cong C_2^{v(\mathscr{F})} \rtimes \Gamma(\mathscr{F})$, where $v(\mathscr{F})$ is the number of vertices of $\mathscr{F}$ (that is, the number of vertices of F in $\mathscr{K}$). In particular, the automorphism group of any $(j+1)$-face $2^{\mathscr{F}}$ of $2^{\mathscr{K}}$ is a subgroup of $\Gamma(2^{\mathscr{K}})$ if $\mathscr{K}$ is j-face hereditary, since then $\Gamma(\mathscr{F})$ is a subgroup of $\Gamma(\mathscr{K})$. Thus $2^{\mathscr{K}}$ is a $(j+1)$-face hereditary $(n+1)$-polytope if $\mathscr{K}$ is a j-face hereditary n-polytope. □

When $j = n-1$ we have the following immediate consequence.

Corollary 2. *Each finite vertex-describable hereditary n-polytope is the vertex-figure of a vertex-transitive finite hereditary $(n+1)$-polytope.*

Theorem 6 and its proof are good sources for interesting examples of hereditary polytopes. For instance, if $\mathscr{K}$ is the truncated tetrahedron, which is 1-face hereditary but not 2-face hereditary, then $2^{\mathscr{K}}$ is a 2-face hereditary 4-polytope which is not 3-face hereditary. In fact, the facets of $2^{\mathscr{K}}$ are of two kinds, 3-cubes $\{4,3\} = 2^{\{3\}}$ and orientable regular maps $\{4,6\,|\,4,4\} = 2^{\{6\}}$ of genus 9 (see [16, p. 261]); however, not all automorphisms of facets of the latter kind extend to automorphisms of $2^{\mathscr{K}}$ (otherwise $\mathscr{K}$ would have to be 2-hereditary). Similar examples of arbitrary higher ranks can be constructed by iterating the $2^{\mathscr{K}}$ construction. For example, when $\mathscr{K}$ is the truncated tetrahedron, $2^{2^{\mathscr{K}}}$ is a 3-face, but not 4-face, hereditary 5-polytope.

Note that a further generalization of hereditary polytopes employs sections rather than faces. For $0 \le i < j \le n-1$, an n-polytope $\mathscr{P}$ is said to be (i,j)-*section hereditary* (resp. *strongly* (i,j)-*section hereditary*) if for each section G/F, with F an i-face and G a j-face with $F < G$, the group $\Gamma(G/F)$ of G/F is a subgroup of $\Gamma(\mathscr{P})$ (resp. fixing, in addition, each face in both F/F_{-1} and F_n/G).

7 Conclusion

This paper established the basic theory of hereditary polytopes. One should pursue these ideas further by considering some of the following problems, which have been brought to light by our work.

As a first example, one could examine if there exist hereditary polytopes whose i-faces are all themselves non-regular hereditary polytopes ($i \ge 3$). A closely related question asks if, given any hereditary polytope $\mathscr{P}$, one can build another hereditary polytope which has $\mathscr{P}$ as its facets? This questions is open even when $\mathscr{P}$ is of rank 3.

In this paper we considered polytopes where the automorphism group of each facet is a subgroup of the full automorphism group of the polytope. It would also be of interest to study "chirally hereditary" polytopes, that is, those polytopes which are not hereditary, but have the property that each rotational symmetry of a facet extends to a global symmetry. For example, an interesting class of such objects is the chiral polytopes with regular facets – which includes all chiral maps.

Additionally, it would be of interest to investigate the idea of geometrically hereditary polytopes. For example in $\mathbb{E}^3$, can one classify the i-face transitive geometrically hereditary polyhedra, that is, those with symmetry group inheriting all isometries of their polygonal faces? The rhombic dodecahedron is an example of a 2-face transitive geometrically hereditary polyhedron. (For a survey on related questions for convex polyhedra see also [15].)

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One Brick at a Time: A Survey of Inductive Constructions in Rigidity Theory

A. Nixon and E. Ross

Abstract We present a survey of results concerning the use of inductive constructions to study the rigidity of frameworks. By inductive constructions we mean simple graph moves which can be shown to preserve the rigidity of the corresponding framework. We describe a number of cases in which characterisations of rigidity were proved by inductive constructions. That is, by identifying recursive operations that preserved rigidity and proving that these operations were sufficient to generate all such frameworks. We also outline the use of inductive constructions in some recent areas of particularly active interest, namely symmetric and periodic frameworks, frameworks on surfaces, and body-bar frameworks. As the survey progresses we describe the key open problems related to inductions.

Keywords Rigidity • Bar-joint framework • Inductive construction

Subject Classifications: 52C25

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1 Introduction

Rigidity theory probes the question, given a geometric embedding of a graph, when is there a continuous motion or deformation of the vertices into a non-congruent embedding without breaking the connectivity of the graph or altering the edge lengths? The geometric embeddings in question are typically bar-joint frameworks: collections of flexible joints and stiff bars that are permitted to pass through each other. The question of rigidity or flexibility is inherently dependent on the ambient space: in 1 and 2-dimensional Euclidean space there are complete combinatorial descriptions of the generic behaviour of a framework. In higher dimensions, however, there is no such characterisation; indeed there remains a number of challenging open problems.

In this survey we will concentrate on the most celebrated way of proving such a combinatorial description: an inductive construction. By an inductive construction we mean a constructive characterisation of a class of graphs or frameworks using simple operations. It is perhaps the simplicity of inductive constructions that make them so appealing, and helps to explain their widespread use. After all, the study of rigidity theory centres around highly intuitive concepts: building large rigid structures from smaller rigid components (e.g. building buildings from bricks). Inductive constructions provide an abstract analogue of this building-up process.

There are two key ways in which inductive constructions have been used in rigidity theory. First, to show that a certain list of operations is sufficient to generate all graphs in a particular class (e.g. generic rigidity in the plane). Second, to show that certain inductive moves preserve rigidity (e.g. vertex splitting). As a result, inductive constructions have been used as proof techniques without necessarily hoping for complete combinatorial characterisations (e.g. the proof of the Molecular Conjecture). When a complete combinatorial description is obtained, inductive characterisations typically do not make for fast algorithms. On the other hand, once we have an inductive sequence for a rigid framework, we have an instant certificate of its rigidity.

We begin the survey with a gentle introduction into rigidity and global rigidity theory in 2-dimensions from an inductive perspective. From there we outline the key open problems in extending inductive constructions to 3-dimensional frameworks before describing some purely graph theoretical inductive constructions in Sect. 5. The central topic of discussion in Sects. 6 and 7 is the rigidity of periodic and symmetric frameworks, two types of frameworks with special geometric features. Following that we discuss frameworks on surfaces and body-bar frameworks (Sects. 8 and 9) before finishing the survey by briefly outlining, in Sect. 10, a number of other avenues of rigidity theory which have benefitted from inductive techniques.

2 Basics of Rigidity

A *(bar-joint) framework* is an ordered pair (G, p) where G is a graph and $p : V \to \mathbb{R}^d$ is a realisation of the vertices into $\mathbb{R}^d$. We are interested in the typical behaviour of frameworks. Thus we say that a framework is *generic* if the coordinates of the framework points form an algebraically independent set (over $\mathbb{Q}$). Two frameworks on the same graph (G, p) and (G, q) are *equivalent* if the (Euclidean) edge lengths in (G, p) are the same as those in (G, q) and are *congruent* if the distance between pairs of points in (G, p) are the same as those in (G, q).

Definition 1. A framework (G, p) is *flexible* in $\mathbb{R}^d$ if there is a continuous motion $x(t)$ of the framework points such that $(G, x(t))$ is equivalent to (G, p) for all t but is not congruent to (G, p) for some t (where $x(t) \neq p$). (G, p) is *(continuously) rigid* if it is not flexible.

Understanding rigidity becomes more tractable after linearising the problem. The *rigidity matrix* $R_d(G, p)$ is a sparse matrix where each row corresponds to an edge, and with the appropriate ordering each d-tuple of columns corresponds to the coordinates of a framework vertex. The entries in row ij are zero except in the columns corresponding to i and j where the entries are $p_i - p_j$ and $p_j - p_i$ respectively. This matrix is (up to scaling) the Jacobean derivative matrix of the system of quadratic edge length equations. The (infinitesimal) rigidity matroid $\mathcal{R}_d$ (for a generic framework (G, p)) is the linear matroid induced by linear independence in the rows of the rigidity matrix $R_d(G, p)$.

Definition 2. Let $p = (p_1, \dots, p_{|V|})$. An infinitesimal flex $u = (u_1, \dots, u_{|V|}) \in \mathbb{R}^{d|V|}$ is a vector satisfying $(p_i - p_j).(u_i - u_j) = 0$ for all edges ij. A framework is *infinitesimally rigid* if there are no non-trivial infinitesimal flexes.

There are examples of frameworks that are infinitesimally flexible but continuously rigid, however all such examples occur for geometric reasons.

Theorem 1 (Asimow and Roth [1]). *Let (G, p) be a generic framework. Then (G, p) is (continuously) rigid if and only if it is infinitesimally rigid.*

2.1 Constructing Frameworks in the Plane

Let us, for now, restrict attention to frameworks in the plane and consider the following construction moves [16], see Fig. 1:

1. Add a 2-valent vertex with distinct neighbours,
2. Remove an edge xy and add a 3-valent vertex v adjacent to x, y and some $z \in V$.

In the literature, operation 1 may be referred to as a *Henneberg 1* move, a *0-extension* or a *vertex addition* and operation 2 may be referred to as a *Henneberg 2* move [14, 36, 50], a *1-extension* [13, 17, 23] or *edge splitting* [41, 45, 55]. All have

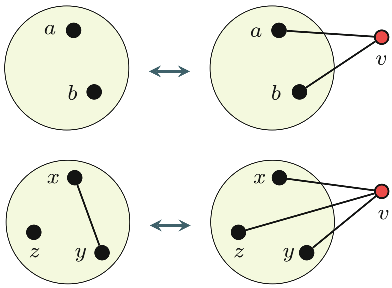

Fig. 1 Vertex addition and Edge splitting

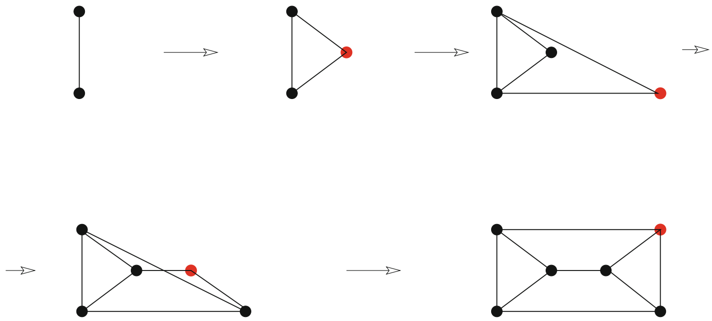

Fig. 2 A Henneberg-Laman sequence for the triangular prism

their advantages: Henneberg move gives credit to the original source; i-extension indicates the number of edges removed in the operation; vertex addition and edge splitting are the most accessible to newcomers to the subject. Since this is a survey we choose to use the last option from here on Fig. 2 shows 3 vertex additions followed by a single edge split on K_2 to produce the triangular prism.

Definition 3. A graph $G = (V, E)$ is $(2, 3)$-*sparse* if for every subgraph $G' = (V', E')$ with at least one edge, $|E'| \leq 2|V'| - 3$. G is $(2, 3)$-*tight* if G is $(2, 3)$-sparse and $|E| = 2|V| - 3$.

Theorem 2 (Henneberg [16] and Laman [25]). *A graph G is $(2, 3)$-tight if and only if it can be derived recursively from K_2 (the single edge) by vertex additions and edge splitting.*

An infinitesimally rigid graph G is called *isostatic* or *minimally rigid* if deleting any edge will destroy its rigidity.

Theorem 3 (Laman's theorem [25]). *A graph G is generically minimally rigid in the plane if and only if G is $(2, 3)$-tight.*

Maxwell [30] proved that any generically minimally rigid graph must be $(2, 3)$-tight. The harder sufficiency direction relies on Theorem 2. Given the inductive construction, and since K_2 clearly has a generically rigid realisation, it remains only to show that the result of applying vertex addition and edge splitting to a generically minimally rigid graph is a generically minimally rigid graph. Vertex addition is trivial; we have a rigidity matrix with rank equal to the number of rows and genericness ensures the two rows and two columns, that we add, increases the rank by two. Edge splitting is slightly more involved. Let G' be formed from G by edge splitting. Then a typical proof uses the fact that for a graph $H = (V, E)$ and two maps $q_1, q_2 : V \to \mathbb{R}^2$ with (H, q_1) generic, $\text{rank} R_2(H, q_1) \geq \text{rank} R_2(H, q_2)$ see, for example, [54]. Using this, choose the new vertex in G' to be on the line through the (not yet) removed edge in (G', p'). The collinear triangle created corresponds to a minimal set of linearly dependent rows in the rigidity matrix (i.e. a circuit in the rigidity matroid). We can remove any edge from this triangle without reducing the rank of $R_2(G + v, p')$. Since it is clear that $\text{rank} R_2(G, p) + 2 = \text{rank} R_2(G + v, p')$, it follows that $\text{rank} R_2(G', p') = \text{rank} R_2(G, p) + 2$.

3 Global Rigidity

There are a number of applications in which rigidity is not strong enough due to the possibility of multiple distinct realisations with the same edge lengths. Global rigidity corresponds exactly to there being a unique realisation, up to congruence, with the given edge lengths. For a full survey on global rigidity see [18], we give only a brief description of the use of inductive constructions for global rigidity.

Definition 4. A framework (G, p) is *globally rigid* if for all equivalent choices of q the frameworks (G, p) and (G, q) are congruent.

In characterising global rigidity we will also use the following strong form of rigidity.

Definition 5. Let $G = (V, E)$. A framework (G, p) is *redundantly rigid* if (G, p) is rigid and for all $e \in E$ the framework $(G - e, p)$ is rigid.

Theorem 4 (Hendrickson [15]). *Let (G, p) be a generic globally rigid framework in $\mathbb{R}^d$. Then G is a complete graph on at most $d + 1$ vertices or G is $(d + 1)$-connected and (G, p) is redundantly rigid in $\mathbb{R}^d$.*

3.1 Circuits

By Laman's theorem the minimal number of edges needed for a graph to be generically globally rigid in the plane is $2|V| - 2$. By Theorem 4 the graph must also be redundantly rigid. This implies that if G is generically globally rigid with $2|V| - 2$ edges then G is a $(2, 3)$-circuit; that is a graph with $2|V| - 2$ edges in which every proper subgraph (with at least one edge) is $(2, 3)$-sparse.

The beauty of Theorem 2 is that for every $(2, 3)$-tight graph containing a vertex of degree 3, there is always an inverse edge splitting operation resulting in a smaller

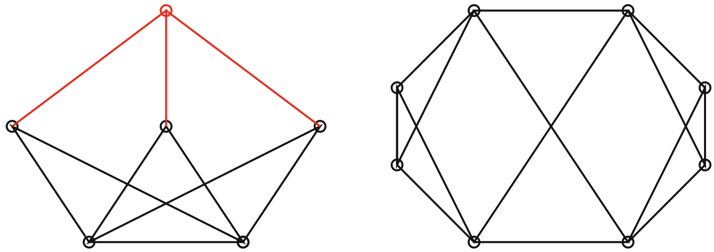

Fig. 3 Two examples of $(2,3)$-circuits. On the left the *red vertex* cannot be reduced as the result will be a copy of K_4 with a degree 2 vertex adjoined. On the right there is no inverse edge splitting move that results in a $(2,3)$-circuit

$(2,3)$-tight graph. This is not the case for $(2,3)$-circuits; hence it is significantly more challenging to prove an inductive construction. For example it is possible for a degree 3 vertex v in a $(2,3)$-circuit to have all neighbours x, y, z of degree 3. Here any inverse edge splitting operation results in a graph with $2|V|-2$ edges which is not a circuit since at least one of x, y, z has degree 2, see Fig. 3.

Theorem 5 (Berg and Jordán [2]). *Let G be a 3-connected $(2,3)$-circuit. Then there is an inverse edge splitting move on some vertex of G that results in a smaller $(2,3)$-circuit.*

Combining this with the well known 2-sum operation from matroid theory allowed them to inductively characterise $(2,3)$-circuits. The 2-sum operation glues two $(2,3)$-circuits together along an edge and deletes the common edge. The inverse operation separates along a 2-vertex cutset. This operation has further been examined from the rigidity perspective in [46].

Theorem 6 (Berg and Jordán [2]). *A graph G is a $(2,3)$-circuit if and only if G can be generated from copies of K_4 by applying edge splitting moves within connected components and taking 2-sums of connected components.*

While it was easy to see that the edge splitting operation preserves rigidity, showing that it preserves global rigidity is more intricate. This was originally proved by Connelly [8] as a corollary to his sufficient condition for global rigidity in terms of the rank of the stress matrix. An alternative proof was later given by Jackson, Jordán and Szabadka [21] during their analysis of globally linked vertices.

3.2 Characterising Global Rigidity

The characterisation of $(2,3)$-circuits was extended by Jackson and Jordán to M-connected graphs; these are graphs in which there is a $(2,3)$-circuit containing any pair of edges, i.e. the rigidity matroid is connected. They showed using ear decompositions that all 3-connected, M-connected graphs could be generated from

K_4 by edge splitting operations and edge additions. Part of the subtlety here is that they had to be able to alternate between the operations, see [17, Figure 6].

Theorem 7 (Hendrickson [15], Connelly [8], and Jackson and Jordán [17]). *A framework (G, p) is generically globally rigid in the plane if and only if G is a complete graph on at most 3 vertices or G is 3-connected and (G, p) is redundantly rigid.*

4 Rigidity in 3-Space

As in the plane the necessity of combinatorial counts for minimal rigidity was shown by Maxwell [30]. The appropriate graphs are the $(3, 6)$-tight graphs, see Definition 6. However it is no longer true that these graphs are sufficient for minimal rigidity; there exist $(3, 6)$-tight graphs which are generically flexible in 3-dimensions, see Fig. 4 for an example. Thus the outstanding open problem in rigidity theory is to find a good combinatorial description of generic minimal rigidity in 3-dimensions.

From an inductive construction perspective it is known that the analogues of vertex addition and edge splitting preserve rigidity. In fact Tay and Whiteley [52] proved that, in dimension d, the addition of a vertex of degree d (vertex addition) or the subdivision of an edge combined with adding $d-1$ additional edges incident to the new vertex (edge splitting) preserves rigidity. However the average degree in a $(3, 6)$-tight graph approaches 6. Thus we require new operations to deal with degree 5 vertices, see Fig. 5.

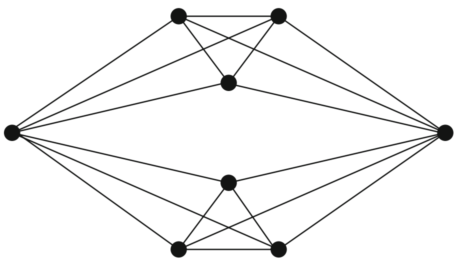

Fig. 4 The double banana; a flexible circuit in the 3-dimensional rigidity matroid

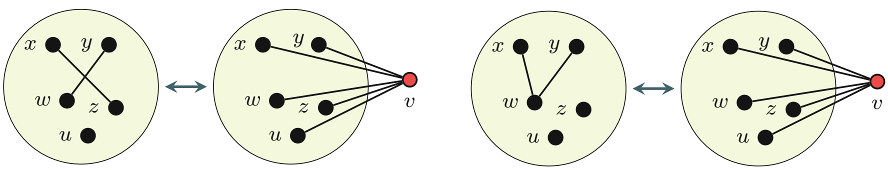

Fig. 5 The X- and V-replacement operations in three dimensions

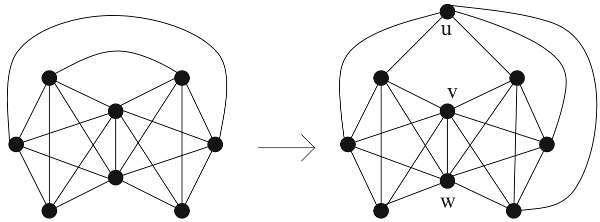

Fig. 6 An example due to Tibor Jordán showing X-replacement does not necessarily preserve global rigidity in 3-dimensions ([38])

4.1 Degree 5 Operations

Conjecture 1 (Whiteley [55]). Let G be generically rigid in $\mathbb{R}^3$ and let G' be the result of an X-replacement applied to G. Then G' is generically rigid in $\mathbb{R}^3$.

The conjecture is intuitively appealing since for the variant in the plane, similarly to the edge splitting argument, it is easy to establish the preservation of rigidity. Let G' be formed from G by an X-replacement. Then for some pair of edges in G, say uv and xy, G' is formed from $G - \{uv, xy\}$ by adding a single vertex z and edges uz, vz, xz, yz. We choose a realisation p^* of $G+z$ such that z lies on the unique point defining the intersection of the lines through uv and xy (since we may assume G was generic these lines are not parallel). Now in the rigidity matroid for $(G+z, p^*)$ there are two collinear circuits, defined by the edge sets $\{uv, uz, vz\}$ and $\{xy, xz, yz\}$ respectively. Thus the deletion of uv and xy does not reduce the rank of the rigidity matrix. The statement follows since it is not hard to argue that $\operatorname{rank} R_2(G+z, p^*)$ is 2 more than $\operatorname{rank} R_2(G, p)$.

However this argument easily breaks down in higher dimensions; generically two lines do not intersect. Going against the conjecture are the following two facts; the analogue of X-replacement in 4-dimensions fails and X-replacement in 3-dimensions does not preserve global rigidity.

The first fact is based on a general argument [13], which in particular shows that $K_{6,6}$ is dependent in the 4-dimensional rigidity matroid.

The second fact is illustrated in Fig. 6. The first graph is generically globally rigid in 3-dimensions. This is easily seen since it can be formed from K_5 by a sequence of (3-dimensional) edge splitting moves and edge additions, both of which preserve global rigidity. The second graph, obtained by an X-replacement on the first graph, contains a 3-vertex-cutset $\{u, v, w\}$. Thus Theorem 4 implies it is not globally rigid.

We also mention that [13] gives a nice discussion of the problem including several special cases where X-replacement is known to preserve rigidity.

It is quickly apparent that the V-replacement operation presents a new difficulty; the earlier inductive operations were easily seen to preserve the relevant vertex/edge counts on the graph and all subgraphs. It is not true, however, that V-replacement always preserves the subgraph counts, we may make a bad choice of vertex w. Tay and Whiteley [52] have made a double-V conjecture but this has

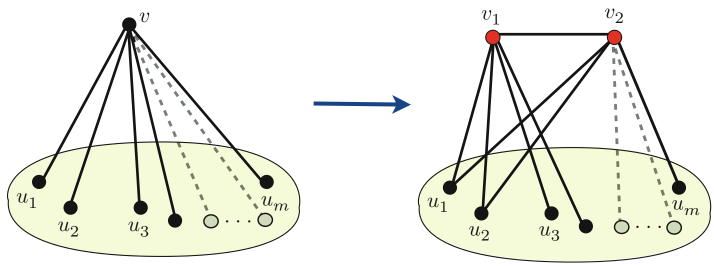

Fig. 7 The 3-dimensional vertex splitting operation

an immediate problem from an algorithmic perspective. The conjecture implies an inductive construction of minimally rigid graphs in 3-space. Using this inductive construction to check the rigidity of a given framework would require recording both graphs each time the V-replacement is applied. Thus for worst case graphs the generating sequence of inductive operations requires remembering exponentially many different graphs.

4.2 Vertex Splitting

Let $v \in V$ have $N(v) = \{u_1, \ldots, u_m\}$. A *vertex splitting* operation (in 3-dimensions) on v removes v and its incident edges, adds vertices v_0, v_1 and edges $u_1v_0, u_2v_0, u_1v_1, u_2v_1, v_0v_1$ and re-arranges the edges $u_3v, \ldots, u_mv$ in some way into edges u_iv_j for $i \in \{3, \ldots, m\}$ and $j \in \{0, 1\}$. See Fig. 7 and also [53] where the operation was introduced for d-dimensional frameworks.

Theorem 8 (Whiteley [53]). *Let G have a generically minimally rigid realisation in $\mathbb{R}^d$ and let G' be formed from G by a vertex splitting operation. Then G' has a generically minimally rigid realisation in $\mathbb{R}^d$.*

For a globally rigid graph in the plane it can be derived from Theorem 7, see [22], that applying a vertex splitting operation, in which each new vertex is at least 3-valent, results in a globally rigid graph.

Conjecture 2 (Cheung and Whiteley [7]). Let G be globally rigid in $\mathbb{R}^d$ and let G' be formed from G by a vertex splitting operation such that each new vertex is at least $d + 1$-valent. Then G' is globally rigid in $\mathbb{R}^d$.

Vertex splitting has also been used to prove a variety of results for restricted classes of three-dimensional frameworks. In particular, Finbow and Whiteley recently used vertex splitting to prove that *block and hole frameworks* are isostatic [11]. A block and hole framework is a triangulated sphere (known to be isostatic by early results of Cauchy and Dehn) where some edges have been removed to create *holes*, while others added to create isostatic subframeworks called *blocks*, all the while maintaining the general $|E| = 3|V| - 6$ count. An example of such a framework is a geodesic dome. The base of the dome can be considered as a block. It becomes possible to remove some edges from the rest of the dome, perhaps to create windows and doors. The result of Finbow and Whiteley will identify which edges may be removed. The proof of this result relies on vertex splitting in a central way.

5 Inductive Constructions for (k, l)-Tight Graphs

Up until now it has been obvious that we concentrated on simple graphs i.e. graphs with no loops or multiple edges. From here on graphs will allow loops or multiple edges and we will specify that graphs which do not are simple.

Definition 6. Let $k, l \in \mathbb{N}$ and $l < 2k$. A graph $G = (V, E)$ is (k, l)-sparse if for every subgraph $G' = (V', E')$, with $|V'| \geq k$, $|E'| \leq k|V'| - l$. G is (k, l)-tight if G is (k, l)-sparse and $|E| = k|V| - l$.

We choose to restrict to the range $l < 2k$ since in this range (k, l)-tight graphs are the bases of matroids [54] and [26]. Observe that $(3, 6)$-tight graphs are outside this range; indeed they do not form the bases of a matroid. Since we now allow multiple edges there are more possibilities for vertex additions and edge splitting operations. Throughout the rest of the paper, when we consider graphs these operations will be understood to allow the graph variants, see Figs. 9 and 10.

In [12], Frank and Szegő prove inductive characterisations of graphs which are *nearly k-tree connected*, which naturally extend the combinatorial elements of Henneberg's original result.

Definition 7. A graph G is called *k-tree connected* if it contains k edge-disjoint spanning trees. A graph is *nearly k-tree connected* if it is not k-tree connected, but the addition of any edge to G results in a k-tree connected graph.

Note that Theorem 2 can be rephrased as follows: A graph G is nearly 2-tree connected if and only if it can be constructed from a single edge by a sequence of vertex additions and edge splitting operations.

Theorem 9 (Frank and Szegő [12]). *A graph G is nearly k-tree-connected if and only if G can be constructed from the graph consisting of two vertices and $k-1$ parallel edges by applying the following operations:*

1. *Add a new vertex z and k new edges ending at z so that there are no k parallel edges,*
2. *Choose a subset F of i existing edges ($1 \leq i \leq k-1$), pinch the elements of F with a new vertex z, and add $k-i$ new edges connecting z with other vertices so that there are no k parallel edges in the resulting graph.*

We recall a result of Nash-Williams [31], which states that a graph $G = (V, E)$ is the union of k edge-disjoint forests if and only if $|E'| \leq k|V'| - k$ for all nonempty subgraphs $G' = (V', E') \subseteq G$. Continuing the theme of extending Henneberg's theorem, by this result of Nash-Williams, Frank and Szegő show that a graph G is $(k, k+1)$-tight if and only if it is nearly k-tree connected. We point the interested reader to a book of Recski [39] where a number of connections between minimally rigid graphs and tree decompositions are proved.

Fekete and Szegő have established a Henneberg-type characterisation theorem of (k, l)-sparse graphs for the range $0 \leq l \leq k$. The following definition extends vertex addition and edge splitting to arbitrary dimension.

Definition 8. Let G be a graph, and let $0 \leq j \leq m \leq k$. Choose j edges of G and pinch into a new vertex z. Put $m - j$ loops on z, and link it with other existing vertices of G by $k - m$ new edges. This move is called an edge pinch, and will be denoted $K(k, m, j)$.

The graph on a single vertex with l loops will be denoted P_l. The main result of [10] is the following.

Theorem 10 (Fekete and Szegő [10]). *Let $G = (V, E)$ be a graph and let $1 \leq l \leq k$. Then G is a (k, l)-tight graph if and only if G can be constructed from P_{k-l} with operations $K(k, m, j)$ where $j \leq m \leq k - 1$ and $m - j \leq k - l$.*

G is a $(k, 0)$-tight graph if and only if G can be constructed from P_k with operations $K(k, m, j)$, where $j \leq m \leq k$ and $m - j \leq k$.

This result has subsequently been applied to periodic body-bar frameworks [42], see Sect. 9.2. Inductive moves for (k, l)-tight graphs have also been considered using an algorithmic perspective in [26].

6 Periodic Frameworks

Over the past decade, the topic of periodic frameworks has witnessed a surge of interest in the rigidity theory community [4, 5, 29, 40, 41], in part due to questions raised about the structural properties of zeolites, a type of crystalline material with numerous practical applications. Inductive constructions have been used to provide combinatorial characterisations of certain restricted classes of periodic frameworks, which we describe below.

A *periodic framework* can be described by a locally finite infinite graph $\tilde{G}$, together with a periodic position of its vertices $\tilde{p}$ in $\mathbb{R}^d$ such that the resulting (infinite) framework is invariant under a symmetry group Γ, which contains as a subgroup the d-dimensional integer lattice $\mathbb{Z}^d$ [5]. A *periodic orbit framework* $(\langle G, m\rangle, p)$ consists of a *periodic orbit graph* $\langle G, m\rangle$ together with a position of its vertices onto the "flat torus" $\mathcal{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The periodic orbit graph is a finite graph G which is the quotient of $\tilde{G}$ under the action of Γ, together with a labeling of the directed edges of G, $m : E(G)^+ \to \mathbb{Z}^d$. This periodic orbit framework provides a "recipe" for the larger periodic framework, but does so with a finite graph G, which we can then consider using inductive constructions (Fig. 8). In addition, it is possible to define a *generic* position of the framework vertices on the torus $\mathcal{T}^d$.

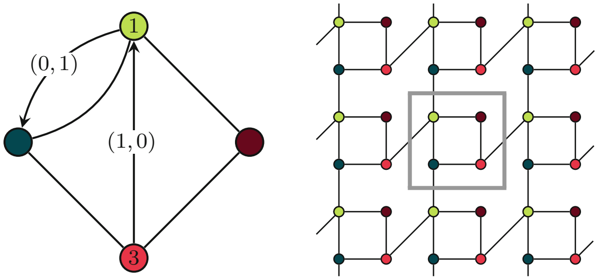

Fig. 8 A periodic orbit graph $\langle G, m\rangle$ on the left, where $m : E \to \mathbb{Z}^2$, and the corresponding periodic framework. Any labeled edge in $\langle G, m\rangle$, on the right, corresponds to an edge in the periodic framework which crosses the boundary of the "unit cell" (*grey box*)

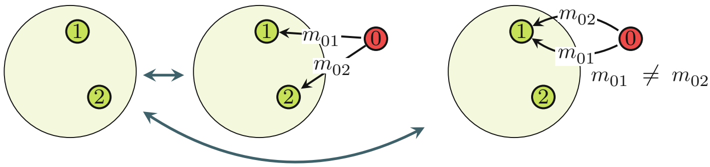

Fig. 9 Periodic vertex addition. The large circular region represents a generically rigid periodic orbit graph

6.1 Fixed Torus

The torus $\mathcal{T}^2$ in 2 dimensions can be seen as being generated by two lengths and an angle between then. When we do not allow the lengths or angle to change, we call the resulting structure the *fixed torus*, and denote it $\mathcal{T}_0^2$. In [41], a Laman-type characterisation of graphs which are minimally rigid on the fixed torus is obtained. The proof depended on the development of inductive constructions on periodic orbit graphs $\langle G, m\rangle$. These moves require an additional layer of complexity over the usual vertex addition and edge splitting operations. The directed, labeled edges of $\langle G, m\rangle$ are recorded by $e = \{v_1, v_2; m_e\}$. We have the following moves:

Definition 9. Let $\langle G, m\rangle = (V\langle G, m\rangle, E\langle G, m\rangle)$ be a periodic orbit graph $\langle G, m\rangle$, a periodic vertex addition is the addition of a single new vertex v_0 to $V\langle G, m\rangle$, and the edges $\{v_0, v_{i_1}; m_{01}\}$ and $\{v_0, v_{i_2}; m_{02}\}$ to $E\langle G, m\rangle$, such that $m_{01} \neq m_{02}$ whenever $v_{i_1} = v_{i_2}$ (see Fig. 9).

Let $e = \{v_{i_1}, v_{i_2}; m_e\}$ be an edge of $\langle G, m\rangle$. A periodic edge split $\langle G', m'\rangle$ of $\langle G, m\rangle$ results in a graph with vertex set $V \cup \{v_0\}$ and edge set consisting of all of the edges of $E\langle G, m\rangle$ except e, together with the edges

$$\{v_0, v_{i_1}; (0, 0)\}, \{v_0, v_{i_2}; m_e\}, \{v_0, v_{i_3}; m_{03}\}$$

where $v_{i_1} \neq v_{i_3}$, and $m_{03} \neq m_e$ if $v_{i_2} = v_{i_3}$ (see Fig. 10).

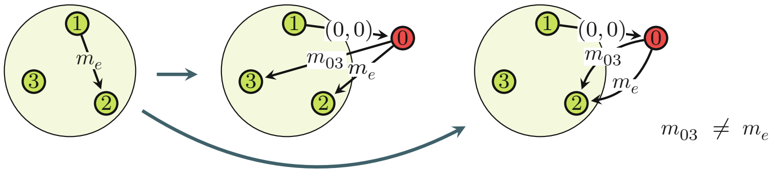

Fig. 10 Periodic edge split. The net gain on the edge connecting vertices 1 and 2 is preserved

Together the periodic vertex addition and edge split characterise generic rigidity on the fixed two-dimensional torus $\mathscr{T}_0^2$. Note that the single vertex graph $\langle G, m\rangle$ is generically rigid on $\mathscr{T}_0^2$.

Theorem 11 (Ross [41]). *A periodic orbit framework* $(\langle G, m\rangle, p)$ *on* $\mathscr{T}_0^2$ *is generically minimally rigid if and only if it can be constructed from a single vertex on* $\mathscr{T}_0^2$ *by a sequence of periodic vertex additions and edge splits.*

6.2 Partially Variable Torus

In [41], a characterisation was established of the generic rigidity of periodic frameworks on a partially variable torus (allowing one degree of flexibility). Recently, the authors of the present paper have outlined an inductive proof of this result [34].

Theorem 12 (Nixon and Ross [34]). *A framework* $(\langle G, m\rangle, p)$ *is generically minimally rigid on the partially variable torus (with one degree of freedom) if and only if it can be constructed from a single loop by a sequence of gain-preserving Henneberg operations.*

The operations referred to in Theorem 12 contain the periodic vertex addition and edge split operations described above. However, we also require one additional move, which is only used in a particular special case. It is an infinite but controllable class of graphs for which vertex addition and edge splitting is insufficient. In addition, while all the generically rigid graphs on the partially variable torus are $(2,1)$-tight, in fact the class of generically rigid graphs is strictly smaller. It is the set of graphs which can be decomposed into an edge-disjoint spanning tree and a *connected* spanning map-graph (a connected graph contained exactly one cycle). This hints at the subtlety involved when moving from graphs on the fixed torus to graphs on a partially variable torus, and suggests some challenges which may exist in trying to inductively characterise graphs on the fully flexible torus.

6.3 Fully Flexible Torus

Generic minimal rigidity on the fully variable torus has been completely characterised by Malestein and Theran [29]. Their proof is non-inductive, however, and there remain significant challenges to providing such a constructive characterisation, since the underlying orbit graph may have minimum degree 4. As we have seen the X- and V-replacement moves, are known to be problematic in other settings [55].

It may be possible to define somewhat weaker versions of inductive constructions in these settings, by relaxing our focus on "gain-preservation". That is, we can perform vertex addition and edge splitting on the orbit graph, but allow relabeling of the edges. This is, in some ways, a less satisfying if easier approach, as the moves no longer correspond to the "classical" inductive moves on the (infinite) periodic framework.

7 Symmetric Frameworks

A second class of frameworks which have experienced increased attention over the past decade is symmetric frameworks [27,40,44,45], and there are connections with the study of protein structure. Like periodic frameworks, symmetric frameworks are frameworks which are invariant under the action of certain symmetry groups, in this case, finite point groups.

Inductive constructions played a key role in Schulze's work on symmetric frameworks [45]. A *symmetric framework* is a finite framework (G, p) which is invariant under some symmetric point group. In 2-dimensions, this could be for example $\mathscr{C}_2$, half-turn symmetry or $\mathscr{C}_{\mathscr{S}}$, mirror symmetry. Schulze used symmetrized versions of vertex addition and edge splitting to prove Henneberg and Laman-type results for several classes of symmetric frameworks in $\mathbb{R}^2$, namely $\mathscr{C}_2, \mathscr{C}_3$, and $\mathscr{C}_s$. Furthermore, these results are stronger than the analogous results in the periodic setting, in that they are concerned with frameworks which are either forced to be symmetric, or frameworks which are simply incidentally symmetric. That is, the symmetry-adapted moves preserve the rank of both the (symmetry) orbit matrix, and of the original rigidity matrix of any given symmetric framework.

As an example, we consider incidental symmetry for frameworks with three-fold rotational symmetry (the group $\mathscr{C}_3$).

Theorem 13 (Schulze [45]). *A C_3-symmetric framework (G, p) is generically (symmetric)-isostatic if and only if it can be generated through three inductive moves, a three-fold vertex addition (one vertex is added symmetrically to each of the three orbits), a three-fold edge split (one edge is "split" symmetrically in each of the three orbits) and the Δ-move pictured in Fig. 11.*

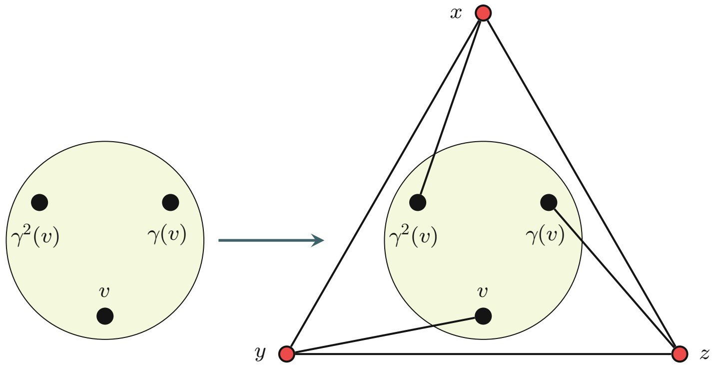

Fig. 11 One of the three $\mathscr{C}_3$-symmetric edge splitting operations, where γ represents rotation through $2\pi/3$. Henneberg proved that the natural generalization of this move preserves rigidity for arbitrary n-gons [16]. It should be noted, however, that Schulze proved the $\mathscr{C}_3$ move for the non-generic "special geometric" position shown above, where $y = \gamma(x), z = \gamma^2(x)$, and his arguments could easily be extended to cover non-generic n-gons (under $\mathscr{C}_n$ symmetry) as well

Schulze proves analogous results for $\mathscr{C}_2$ and $\mathscr{C}_s$ [44]. In the case of $\mathscr{C}_s$ (mirror symmetry), X-replacement is also required to handle certain special cases. Schulze also proves tree-covering results for these groups.

We remark that it would be possible to rework these results of Schulze using the language of *gain graphs* (graphs whose edges are labeled by group elements), as for periodic frameworks. In that scenario, we would capture the symmetric graph using an orbit graph whose edges were labeled with elements of the symmetry group (e.g. $\mathscr{C}_3$ etc.). The symmetric inductive moves could then be defined on this symmetric orbit graph. This is exactly the approach taken in a very recent work of Jordán, Kaszanitsky and Tanigawa [23] on forced-symmetric rigidity for the groups $\mathscr{C}_s$ (the reflection group), and the dihedral groups D_h, where h is odd. We mention here their results for D_h.

The authors define a D_h sparsity type of the gain graphs (G, ϕ), where ϕ is a labeling of the edges by elements of the group D_h. They then prove that all D_h-tight graphs can be constructed from the disjoint union of a few 'basic' graphs by a sequence of Henneberg-type moves on the underlying gain graph. In particular, they use vertex addition, edge splitting and X-replacements; including *loop vertex addition* (adding a 'lolipop'), and *edge splitting plus adding a loop on the new vertex*. This leads to the following combinatorial characterisation of rigid frameworks with D_h symmetry (A similar result is established for $\mathscr{C}_s$):

Theorem 14 (Jordán, Kaszanitsky and Tanigawa [23]). *(G, ϕ), $\phi : E(G) \to D_h$, where h is odd, is the gain graph of a rigid framework with D_h symmetry if and only if (G, ϕ) has a D_h-tight subgraph.*

Note that the work of Schulze provides combinatorial characterisations for $\mathscr{C}_2$, $\mathscr{C}_3$, and $\mathscr{C}_s$ only, but his results are for both incidental and forced symmetry. On the other hand, Jordán, Kaszanitsky and Tanigawa's results are for forced symmetry only but cover any cyclic group and odd order dihedral groups. Thus, there are a number of outstanding questions about symmetric frameworks, including the characterisation of the rigidity of frameworks with forced dihedral D_h (h even) symmetry, and the characterisation of incidental rigidity for the dihedral groups.

We remark that for the cyclic groups C_n and C_s (rotations and reflections) the characterisations of forced rigidity can also be obtained using direction networks and linear representability [28].

8 Frameworks on Surfaces

Inductive constructions have also played a big role in recent work on frameworks supported on surfaces. Here characterisations of minimal rigidity require us to remain within the class of simple graphs at each step of the induction. Hence the results of Fekete and Szegő, allowing multiple edges and loops, are not sufficient. For example a $(2, 2)$-tight graph may contain an arbitrarily large number of copies of K_4 and there is no inverse edge splitting operation on a degree 3 vertex in a copy of K_4 that preserves simplicity.

This motivates the vertex-to-K_4 move, in which we remove a vertex v (of any degree) and all incident edges $vx_1, \dots, vx_n$ and insert a copy of K_4 along with edges $x_1 y_1, \dots, x_n y_n$ where each $y_i \in V(K_4)$, see Fig. 12.

Theorem 15 (Nixon and Owen [33]). *A simple graph G is $(2, 2)$-tight if and only if G can be generated from K_4 by vertex addition, edge splitting, vertex-to-K_4 and (2-dimensional) vertex splitting operations.*

Similarly when dealing with $(2, 1)$-tight graphs, all low degree vertices may be contained in copies of $K_5 - e$ (the graph formed from K_5 by deleting any single edge). For these graphs, vertex-to-K_4 and vertex splitting moves are not sufficient so we introduce the edge joining move. This is the joining of two $(2, 1)$-tight graphs by a single edge. In the following theorem $K_4 \sqcup K_4$ is the unique graph formed from two copies of K_4 intersecting in a single edge.

Theorem 16 (Nixon and Owen [33]). *A simple graph G is $(2, 1)$-tight if and only if G can be generated from $K_5 - e$ or $K_4 \sqcup K_4$ by vertex addition, edge splitting, vertex-to-K_4, vertex splitting and edge joining operations.*

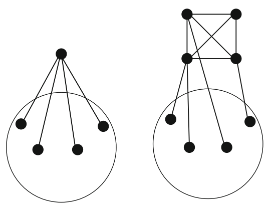

Fig. 12 An example of the vertex-to-K_4 operation

The first of these results has led to an analogue of Laman's theorem for an infinite circular cylinder and the second to analogues for surfaces admitting a single rotational isometry (such as the cone and torus).

Theorem 17 (Nixon, Owen and Power [36]). *Let $G = (V, E)$ with $|V| \geq 4$. Then the framework (G, p) is generically minimally rigid on a cylinder if and only if G is simple and $(2, 2)$-tight.*

Theorem 18 (Nixon, Owen and Power [35]). *Let $G = (V, E)$ with $|V| \geq 5$. Then the framework (G, p) is generically minimally rigid on a surface of revolution if and only if G is simple and $(2, 1)$-tight.*

The next extension would be to frameworks on a surface admitting no ambient isometries (such as an ellipsoid). However, this is known to be false, [36].

The insistence on simplicity also makes the characterisation of (k, l)-circuits more challenging. Similarly to Berg and Jordán's theorem the following inductive result is a step towards characterising global rigidity on the cylinder. The 1-, 2- and 3-join operations, defined in [32], are similar in spirit to the 2-sum operation used in Theorem 6.

Theorem 19 (Nixon [32]). *A simple graph G is a $(2, 2)$-circuit if and only if G can be generated from copies of $K_5 - e$ and $K_4 \sqcup K_4$ by applying edge splitting within connected components and taking 1-, 2- and 3-joins of connected components.*

It is an open problem to extend this characterisation to give an inductive construction for generically globally rigid frameworks on a cylinder.

9 Body-Bar Frameworks

Body-bar frameworks are a special class of frameworks where there is a more complete understanding in arbitrary dimension. Roughly speaking, a body-bar framework is a set of bodies (each spanning an affine space of dimension at least $d - 1$), which are linked together by stiff bars.

Theorem 20 (Tay [49]). *Let G be a graph. Then (G, p) is generically minimally rigid as a body-bar framework in $\mathbb{R}^d$ if and only if G is (D, D)-tight, where $D = \binom{d+1}{2}$ is the dimension of the Euclidean group.*

Tay subsequently proved an inductive characterisation of body-bar frameworks.

Theorem 21 (Tay [50]). *A graph G is (D, D)-tight if and only if G can be formed from K_1 by Henneberg operations.*

The Henneberg operations referred to in Theorem 21 are essentially the loopless versions of the edge-pinches of Fekete and Szegő (see Sect. 5).

Recently, Katoh and Tanigawa proved the *Molecular Conjecture*, a longstanding open question due to Tay and Whiteley, which is concerned with body-bar frameworks which are geometrically special:

Theorem 22 (Katoh and Tanigawa [24]). *Let $G = (V, E)$ be a graph. Then, G can be realised as an infinitesimally rigid body-and-hinge framework in $\mathbb{R}^d$ if and only if G can be realised as an infinitesimally rigid panel-and-hinge framework in $\mathbb{R}^d$.*

The settling of this conjecture is of particular significance to the materials science community, who use rigidity analysis for the modeling of molecular compounds. The proof of this result is quite involved, so we will not include many details here. However, one of the ingredients in the proof is inductive constructions. In particular, the authors use a type of *splitting off* operation, which removes a two-valent vertex v, and then inserts a new edge between the pair of vertices formerly adjacent to v. A second type of induction used is a *contraction* operation, which contracts a proper rigid subgraph to a vertex.

Along the way Katoh and Tanigawa also obtain a Henneberg-type characterisation of minimally rigid body-and-hinge graphs. In particular, they show that for any minimally rigid body-and-hinge framework, there is a sequence of graphs ending with the two vertex, two edge graph, where each graph in the sequence is obtained from the previous graph by a splitting off operation or a contraction operation (see Theorem 5.9, [24]).

9.1 Global Rigidity

Inductive constructions have also played a role in the proof of the following result concerning generic global rigidity of body-bar frameworks:

Theorem 23 (Connelly, Jordán and Whiteley [9]). *A body-bar framework is generically globally rigid in $\mathbb{R}^d$ if and only if it is generically redundantly rigid in $\mathbb{R}^d$.*

In particular, the authors' proof used Theorem 9 to produce an inductive construction of redundantly rigid body-bar graphs. One of the interesting elements of this proof is that the construction sequence specified by Theorem 9 may involve loops. However, no (globally) rigid finite framework will involve loops. The proof of Theorem 23 involved allowing for the possibility of loops, which would later be eliminated. In this way, the induction used here stepped outside of the class of frameworks under study, but eventually achieved the desired result.

9.2 Periodic Body-Bar Frameworks

It is possible to define periodic body-bar frameworks in much the same way as periodic bar-joint frameworks (see Sect. 6). A recent result of Ross characterises the generic rigidity of periodic body-bar frameworks on a three dimensional fixed torus [42]. It is based on the following sparsity condition which depends on the dimension of the *gain space* $\mathcal{G}_{\mathcal{C}}$: the vector space generated by the net gains on all of the cycles of a particular edge set Y.

Theorem 24 (Ross [42]). *$\langle H, m\rangle$ is a periodic orbit graph corresponding to a generically minimally rigid body-bar periodic framework in $\mathbb{R}^3$ if and only if $|E(H)| = 6|V(H)| - 3$ and for all non-empty subsets $Y \subset E(H)$ of edges*

$$|Y| \leq 6|V(Y)| - 6 + \sum_{i=1}^{|\mathcal{G}_{\mathcal{C}}(Y)|} (3 - i).$$

The proof relies on a careful modification of the edge-pinching results of Fekete and Szegő [10] to include labels on the edges of the graphs. It is interesting to note that the results of Fekete and Szegő cover the class of minimally rigid frameworks on the fixed torus, but will not assist us with the flexible torus. That is, for minimal rigidity on the fixed torus, we are considering $\left(\binom{d+1}{2}, d\right)$-tight graphs, whereas for minimal rigidity on the flexible torus, we are considering $\left(\binom{d+1}{2}, -\binom{d}{2}\right)$-tight graphs, which are not in the range covered by existing inductive results. Periodic body-bar frameworks with a flexible lattice have recently been considered in [6] using non-inductive methods.

10 Further Inductive Problems

Aside from the conjectures already discussed, a number of other problems, especially in 3-dimensions, remain open, see [13, 51, 54].

There are a number of connections between two-dimensional minimally rigid frameworks and the topic of pseudo-triangulations. A *pseudo-triangulation* is a tiling of a planar region into *psuedo-triangles*: simple polygons in the plane with exactly three convex vertices [43]. It is called a *pointed* pseudo-triangulation if every vertex is incident to an angle larger than π. Streinu proved that the underlying graph of a pointed pseudo-triangulation of a point set is minimally rigid [48]. As a converse, there is the following result:

Theorem 25 (Haas et al. [14]). *Every planar infinitesimally rigid graph can be embedded as a pseudo-triangulation.*

The proof uses vertex addition and edge splitting. Pseudo-triangulations are the topic of an extensive survey article [43], and further details on the inductive elements of the proof can be found there.

In [37] Pilaud and Santos consider an interesting application of rigidity in even dimensions to multitriangulations. In particular they use Theorem 8 to show that every 2-triangulation is generically minimally rigid in 4-dimensions and conjecture the analogue for k-triangulations in $2k$-dimensions.

If a framework is not globally rigid then the number of equivalent realisations of the graph is not unique. For $d \geq 2$ this is not a generic property, nevertheless bounds on the number of realisations were established by Borcea and Streinu [3] and recent work of Jackson and Owen [20], motivated by applications to Computer Aided Design (CAD), considering the number of complex realisations made use of vertex addition and edge splitting.

Servatius and Whiteley [47], again motivated by CAD, used the Henneberg operations to understand the rigidity of direction-length frameworks. Jackson and Jordán [19] established the analogue of Theorem 6 for direction-length frameworks however a characterisation of globally rigid direction-length frameworks remains an open problem.

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Polygonal Complexes and Graphs for Crystallographic Groups

Daniel Pellicer and Egon Schulte

Abstract The paper surveys highlights of the ongoing program to classify discrete polyhedral structures in Euclidean 3-space by distinguished transitivity properties of their symmetry groups, focussing in particular on various aspects of the classification of regular polygonal complexes, chiral polyhedra, and more generally, two-orbit polyhedra.

Keywords Regular polyhedron • Regular polytope • Abstract polytope • Complex • Crystallographic group

Subject Classifications: 51M20, 52B15

1 Introduction

The study of highly symmetric discrete structures in ordinary Euclidean 3-space $\mathbb{E}^3$ has a long and fascinating history tracing back to the early days of geometry. With the passage of time, various notions of discrete structures with properties similar to those of convex polyhedra have attracted attention and have brought to light new exciting figures intimately related to finite or infinite groups of isometries.

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A radically new "skeletal" approach to polyhedra in $\mathbb{E}^3$ was pioneered by Grünbaum [16] in the 1970s, building on Coxeter's work [5, 6]. A polyhedron is viewed as a finite or infinite periodic geometric (edge) graph in space equipped with additional structure imposed by the faces, and its symmetry is measured by transitivity properties of its geometric symmetry group. For example, the geometric graph of the cube carries four *Petrie polygons*, that is, polygons for which any two, but no three, consecutive edges belong to the same square of the cube. The geometric graph of the cube with its four hexagonal Petrie polygons constitutes one of the new regular polyhedra introduced by Grünbaum. Throughout this paper we shall adopt this notion of polyhedron.

Since the mid 1970s, there has been a lot of activity in this area, beginning with the full enumeration of the "new" regular polyhedra by Grünbaum [16] and Dress [13, 14] by around 1980 (see also McMullen-Schulte [24, Ch. 7E] or [23] for a faster method for arriving at the complete list); moving on to the full enumeration of the chiral polyhedra in [35, 36] by around 2005; and continuing with the enumeration of certain classes of regular polyhedra and polytopes in higher-dimensional spaces by McMullen [21, 22] (see also [1, 2]).

While all these structures have the essential characteristics of polyhedra and polytopes, the more general class of discrete "polygonal complexes" in 3-space is a hybrid of polyhedra and incidence geometries (see [3]). Every edge of a polyhedron belongs to precisely two edges, whereas an edge of a polygonal complex is surrounded by any number at least two. For example, the geometric edge graph of the cube endowed with the six squares and four Petrie polygons as faces constitutes a polygonal complex where every edge belongs to precisely four faces. In very recent joint work, we obtained a complete enumeration of the regular polygonal complexes in $\mathbb{E}^3$ (see [29, 30]). These are periodic structures with crystallographic symmetry groups exhibiting interesting geometric, combinatorial, and algebraic properties.

The purpose of this paper is to exhibit some of the highlights of the ongoing program to classify discrete structures built from vertices, edges and faces in Euclidean 3-space according to transitivity properties of their symmetry groups. We center our attention on the recent classification of regular polygonal complexes, chiral polyhedra, and more generally, two-orbit polyhedra.

In Sects. 2 and 3, we review basic terminology about polygonal complexes and describe structure results for the symmetry group of regular polygonal complexes. This is followed, in Sect. 4, by a brief description of the complete enumeration of regular polyhedra, seen from the perspective of regular polygonal complexes. Then Sects. 5 and 6 give an account of the regular polygonal complexes which are not polyhedra. In the last two sections we study certain kinds of two-orbit polyhedra in $\mathbb{E}^3$, beginning with a review of the enumeration of chiral polyhedra. Finally, Sect. 8 briefly summarizes the recent classification of regular polyhedra of index 2, obtained in Cutler [8] and [9]; these form a distinguished class of two-orbit polyhedra in $\mathbb{E}^3$.

2 Some Terminology

Informally, a polygonal complex is a discrete structure in $\mathbb{E}^3$ consisting of vertices (points), joined by edges (line segments) assembled in careful fashion into faces (polygons, allowed to be finite or infinite), with at least two faces on each edge. Our concepts of face (polygon) and polyhedron generalize those of convex polygon and convex polyhedron.

For our purposes, a *finite polygon* $(v_1, v_2, \dots, v_n)$ in $\mathbb{E}^3$ is a figure consisting of n distinct points $v_1, \dots, v_n$, together with the line segments (v_i, v_{i+1}), for $i = 1, \dots, n-1$, and (v_n, v_1). Similarly, an *infinite polygon* is a figure made up from an infinite sequence of distinct points $(\dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots)$, and line segments (v_i, v_{i+1}) for each i, such that each compact subset in $\mathbb{E}^3$ meets only finitely many line segments. In either case the points and line segments are referred to as the *vertices* and *edges* of the polygon, respectively.

By a *regular* polygon we mean a finite or infinite polygon such that its geometric symmetry group, restricted to the affine hull of the vertices, is a finite or infinite dihedral group acting transitively on the set of incident vertex-edge pairs, called *flags*. This definition covers not only the traditional (planar) convex regular polygons but also permits star-polygons, skew polygons, zigzags, or helices as regular polygons. A *star polygon* has the same vertices as a convex regular polygon; its edges connect vertices of the convex regular polygon that are a fixed number of steps apart on the boundary. A *skew polygon* lives properly in $\mathbb{E}^3$ and can be obtained from a planar finite (convex or star-) polygon by raising every other vertex perpendicularly by the same amount (thus doubly covering the original polygon if the number of vertices was odd); the vertex set then is contained in two parallel planes and every edge goes from a vertex in one plane to a vertex in the other plane. A *linear apeirogon* is an infinite polygon obtained by tessellating a line with line segments (usually of the same size). Linear apeirogons will not occur as faces of the geometric objects described in this paper, since no non-trivial connected structure can be assembled only from linear building blocks. A *zigzag* is a planar infinite polygon obtained from a linear apeirogon in a similar way as a skew polygon is obtained from a planar finite polygon; its vertices lie on two parallel lines, and its edges connect vertices on different lines. Finally, a *helix* is an infinite non-planar polygon and it can be thought as a spring rising above a finite planar (convex or star) polygon; more precisely, the orthogonal projection onto its axis gives a linear apeirogon, and the orthogonal projection along its axis gives a finite planar (convex or star-) polygon.

A *polygonal complex*, or simply *complex*, $\mathscr{K}$ in $\mathbb{E}^3$ consists of a set $\mathscr{V}$ of points, called *vertices*, a set $\mathscr{E}$ of line segments, called *edges*, and a set $\mathscr{F}$ of polygons, called *faces*, such that the following properties are satisfied. The graph defined by $\mathscr{V}$ and $\mathscr{E}$, called the *edge graph* of $\mathscr{K}$, is connected. Moreover, the vertex-figure of $\mathscr{K}$ at each vertex of $\mathscr{K}$ is connected. Here the *vertex-figure* of $\mathscr{K}$ at a vertex v is the graph, possibly with multiple edges, whose vertices are the neighbors of v in the edge graph of $\mathscr{K}$ and whose edges are the line segments (u, w), where (u, v) and

(v,w) are edges of a common face of $\mathscr{K}$. (Note that this is a small change over [29, 30], where a complex was required to have exactly r faces on each edge, for some fixed number $r \geqslant 2$. However, for regular complexes with at least two faces meeting at an edge the two definitions are equivalent.) All polygonal complexes studied in this paper have at least two faces on each edge. Finally, $\mathscr{K}$ is *discrete*, in the sense that each compact subset of $\mathbb{E}$ meets only finitely many faces of $\mathscr{K}$. A complex with exactly two faces on each edge (that is, $r = 2$) is also called a *polyhedron*. Note that this definition extends the notion of convex polyhedron, where the faces are convex (finite and planar) and the polyhedron itself is also finite. Polyhedra (finite or infinite) with high symmetry properties have been extensively studied in [24, Ch. 7E] and [1, 2, 5, 6, 16–18, 23, 26, 31, 35, 36]. The edge graphs of highly symmetric polyhedra frequently occur as nets in crystal chemistry [12, 27, 28, 37].

A polygonal complex $\mathscr{K}$ is said to be *regular* if its geometric symmetry group $G := G(\mathscr{K})$ is transitive on the incident vertex-edge-face triples, called *flags*. The faces of a regular complex are necessarily regular polygons. The vertex-figures are finite (flag-transitive) graphs with single or double edges. (A flag of a graph is just an incident vertex-edge pair.) Double edges occur precisely when any two adjacent edges of a face of $\mathscr{K}$ are adjacent edges of another (then uniquely determined) face of $\mathscr{K}$. If $\mathscr{K}$ is not a polyhedron, then G is infinite and affinely irreducible, that is, G is a standard crystallographic group (see [29]). In particular, there are no finite regular complexes other than polyhedra. The Platonic solids are the most natural examples of regular polyhedra, and the 2-skeleton of the tessellation of $\mathbb{E}^3$ by cubes is the most natural example of a regular polygonal complex which is not a polyhedron.

Regular polygonal complexes in $\mathbb{E}^3$ can be viewed as 3-dimensional (discrete faithful) Euclidean realizations of regular incidence complexes of rank 3 with polygonal faces (see [11, 33]). Our description of the symmetry groups will exploit this fact. In particular, the regular polyhedra in $\mathbb{E}^3$ are precisely the 3-dimensional discrete faithful Euclidean realizations of abstract regular polyhedra (abstract regular 3-polytopes); for more details, see [24, Ch. 7E] and [25].

Every regular polyhedron has the property that all its faces have the same number p of edges, and all its vertices have the same degree q. Polyhedra with this property are called *equivelar*, and their *Schläfli type* (or *Schläfli symbol*) is defined to be $\{p,q\}$. When the faces of an equivelar polyhedron are zigzags or helices, the first entry p is ∞; however, since we only consider discrete structures, q is always finite. Similarly, in the *Schläfli symbol* $\{p,q,r\}$ of a regular rank 4 polytope (a combinatorial structure constructed from vertices, edges, polygons and polyhedra) the first two entries give the Schläfli type $\{p,q\}$ of any of its rank 3 faces, while the last entry r is the number of rank 3 faces meeting around each edge (so that the vertex-figures have Schläfli type $\{q,r\}$).

In later sections we also meet various kinds of less symmetric polygonal complexes (in fact, polyhedra) in $\mathbb{E}^3$. These have more than one flag orbit under the symmetry group. A particularly interesting case arises when there are just two flag orbits. We say that a polygonal complex $\mathscr{K}$ is a *2-orbit polygonal complex* if

K has precisely two flag-orbits under G; in this case, if $\mathcal{K}$ is also a polyhedron, we call $\mathcal{K}$ a 2-*orbit polyhedron*. The cuboctahedron and icosidodecahedron are simple examples of two-orbit polyhedra.

There are different kinds of 2-orbit polyhedra in $\mathbb{E}^3$. Recall that two flags of a polyhedron $\mathcal{K}$ are called i-*adjacent*, with $i = 0$, 1, or 2 respectively, if they differ precisely in their vertices, edges, or faces (see [24, Ch. 2]). Thus, two flags are 1-adjacent if they have the same vertices and same faces, but different edges. Note that flags of polyhedra have unique i-adjacent flags for each i; for polygonal complexes which are not polyhedra, this still is true for $i = 0, 1$ but not for $i = 2$. Now 2-orbit polyhedra naturally fall into different classes indexed by proper subsets I of $\{0, 1, 2\}$ (see Hubard [19] and [20]). In particular, a 2-orbit polyhedron $\mathcal{K}$ is said to belong to the *class* 2_I if I consists precisely of those indices i such that any two i-adjacent flags lie in the same flag-orbit under G. The cuboctahedron and the icosidodecahedron are examples of two orbit polyhedra in class $2_{\{0,1\}}$. When $I = \emptyset$ this gives the class $2_\emptyset$ of *chiral* polyhedra. Thus a polyhedron $\mathcal{K}$ is chiral if and only if $\mathcal{K}$ has two flag orbits under G such that any two adjacent flags lie in distinct orbits. (The case $I = \{0, 1, 2\}$ is excluded here, as it describes the regular polyhedra.)

3 The Symmetry Group

The symmetry group $G = G(\mathcal{K})$ of a regular complex $\mathcal{K}$ in $\mathbb{E}^3$ either acts regularly on the set of flags or has flag-stabilizers of order 2. We call $\mathcal{K}$ *simply flag-transitive* if its (full) symmetry group G acts regularly on the flags of $\mathcal{K}$; in other words, G is simply transitive on the flags of $\mathcal{K}$. Note that a regular complex that is not simply flag-transitive can (in fact, always does) have a subgroup (of index 2) that acts simply flag-transitively. Each regular polyhedron, finite or infinite, is a simply flag-transitive regular polygonal complex.

The group G always has a well-behaved system of generators or generating subgroups, regardless of whether $\mathcal{K}$ is simply flag-transitive or not. Suppose $\Phi := \{F_0, F_1, F_2\}$ is a fixed, or *base*, flag of $\mathcal{K}$, consisting of a vertex F_0, an edge F_1, and a face F_2. For each $\Psi \subseteq \Phi$ we let G_Ψ denote the stabilizer of Ψ in G. Moreover, for $i = 0, 1, 2$ we set $G_i := G_{\{F_j, F_k\}}$, where i, j, k are distinct, and write $G_{F_i} := G_{\{F_i\}}$ for the stabilizer of F_i in G. Then G_Φ is the stabilizer of Φ and has order 1 or 2; in particular,

$$G_\Phi = G_0 \cap G_1 = G_0 \cap G_2 = G_1 \cap G_2.$$

The stabilizers G_0, G_1, G_2 form a generating set of subgroups for G, with the property that $G_0 \cdot G_2 = G_2 \cdot G_0 = G_{F_1}$ and $G_\Psi = \langle G_j \mid F_j \notin \Psi \rangle$ for each $\Psi \subseteq \Phi$. Moreover,

$$\langle G_j \mid j \in I \rangle \cap \langle G_j \mid j \in J \rangle = \langle G_j \mid j \in I \cap J \rangle \quad (I, J \subseteq \{0, 1, 2\}).$$

These statements about generating subgroups of G are particular instances of similar such statements about flag-transitive subgroups of automorphism groups of regular incidence complexes of rank 3 (or higher) obtained in [33, §2] (and also described in [24, pp. 33,34] for polyhedra).

From the base vertex F_0 and the symmetry group G of a regular complex $\mathscr{K}$, with generating subgroups G_0, G_1, G_2, we can reconstruct $\mathscr{K}$ by the following procedure, often called *Wythoff's construction*. First observe that the base edge F_1 of $\mathscr{K}$ is determined by the pair of vertices $\{F_0, F_0G_0\}$. Similarly, the vertex- and edge-sets, respectively, of the base face F_2 of $\mathscr{K}$ are just $\{F_0S \mid S \in \langle G_0, G_1\rangle\}$ and $\{F_1S \mid S \in \langle G_0, G_1\rangle\}$. This recovers the base flag of $\mathscr{K}$. Finally, the set of i-faces of $\mathscr{K}$ is just $\{F_iS \mid S \in G\}$ for each $i = 0, 1, 2$.

Most regular complexes $\mathscr{K}$ in $\mathbb{E}^3$ are infinite and have an affinely irreducible infinite discrete group of isometries as a symmetry group. In this case G is a crystallographic group (that is, G admits a compact fundamental domain). Then the Bieberbach theorems tell us that G contains a translation subgroup (of rank 3) such that the quotient of G by this subgroup is finite (see [32, §7.4]). If $R : x \mapsto xR' + t$ is a general element of G, with R' in O(3), the orthogonal group of $\mathbb{E}^3$, and t a translation vector in $\mathbb{E}^3$ (that we may also view as a translation), then the mappings R' clearly form a subgroup G_* of O(3), called the *special group* of G. Now if $T(G)$ denotes the full translation subgroup of G (consisting of all translations in G), then

$$G_* = G/T(G),$$

so in particular, G_* is a finite group. Thus G_* is among the finite subgroups of O(3), which are known (see [15]). The special group of any irreducible infinite discrete group of isometries in $\mathbb{E}^2$ or $\mathbb{E}^3$ never contains rotations of periods other than 2, 3, 4, or 6, and period 6 only occurs for $\mathbb{E}^2$ (see [24, p. 220] and [36, Lemma 3.1]).

The full translation subgroup of the symmetry group G of a regular complex $\mathscr{K}$ (and often the vertex set of $\mathscr{K}$ itself) is given by a 3-dimensional lattice in $\mathbb{E}^3$. We frequently meet the lattices $\Lambda_{\mathbf{a}}$ that are generated by a single vector $\mathbf{a} := (a^k, 0^{3-k})$ and its images under permutations and changes of sign of coordinates; here $a > 0$ and $k = 1, 2, 3$ (and $\mathbf{a}$ has k entries a and $3 - k$ entries 0). When $a = 1$ and $k = 1$, 2 or 3, respectively, these are the standard *cubic lattice* $\mathbb{Z}^3$, the *face-centered cubic lattice*, and the *body-centered cubic* lattice.

4 Regular Polyhedra

The regular polyhedra in space are also known as the *Grünbaum-Dress polyhedra* (see [34]). It is convenient to separate them from the simply flag-transitive regular complexes that are not polyhedra, and discuss them first. We follow [24, Ch. 7E].

For a regular polyhedron $\mathscr{K}$ in $\mathbb{E}^3$ with symmetry group $G(\mathscr{K})$, each subgroup G_j of $G(\mathscr{K})$ has order 2 and is generated by a reflection R_j in a point, line, or plane (a reflection in a line is a half-turn about the line). Thus $G(\mathscr{K})$ is generated by R_0, R_1, R_2. We let $dim(R_j)$ denote the dimension of the mirror (fixed point set) of the reflection R_j for each j, and call the vector $(dim(R_0), dim(R_1), dim(R_2))$ the *complete mirror vector* of $\mathscr{K}$; this is just the dimension vector of [24, Ch. 7E]. The use of the qualification "complete" will become clear in the next section. The *distinguished generators* R_0, R_1, R_2 of $G(\mathscr{K})$ satisfy (at least) the Coxeter-type relations

$$R_0^2 = R_1^2 = R_2^2 = (R_0 R_1)^p = (R_1 R_2)^q = (R_0 R_2)^2 = I, \tag{1}$$

the identity mapping, where p and q determine the *type* $\{p, q\}$ of $\mathscr{K}$.

The complete enumeration of the regular polyhedra naturally splits into four steps of varying degrees of difficulty: the finite polyhedra, the planar apeirohedra, the blended apeirohedra, and the pure (non-blended) apeirohedra. An *apeirohedron* is simply an infinite polyhedron.

There are just 18 finite regular polyhedra: the five (convex) Platonic solids

$$\{3,3\},\ \{3,4\},\ \{4,3\},\ \{3,5\},\ \{5,3\};$$

the four Kepler-Poinsot star-polyhedra

$$\{3,\tfrac{5}{2}\},\ \{\tfrac{5}{2},3\},\ \{5,\tfrac{5}{2}\},\ \{\tfrac{5}{2},5\},$$

where faces and vertex-figures are planar, but are allowed to be star polygons; and the Petrie-duals of these nine polyhedra. (Recall that the *Petrie dual* of a regular polyhedron $\mathscr{P}$ has the same vertices and edges as $\mathscr{P}$; however, its faces are the *Petrie polygons* of $\mathscr{P}$, whose defining property is that two successive edges, but not three, are edges of a face of $\mathscr{P}$. Thus the new faces are "zig-zags", leaving a face of $\mathscr{P}$ after traversing two of its edges.)

The 6 planar regular apeirohedra comprise the three familiar regular plane tessellations by squares, triangles, or hexagons,

$$\{4,4\},\ \{3,6\},\ \{6,3\},$$

and their Petrie-duals.

The remaining regular apeirohedra are genuinely 3-dimensional and fall into two families.

There are exactly 12 regular apeirohedra that in some sense are reducible and have components that are regular figures of dimensions 1 and 2. These apeirohedra are *blends* of a planar regular apeirohedron, and a line segment $\{\,\}$ or linear apeirogon $\{\infty\}$. This explains why there are $12 = 6 \cdot 2$ blended (or non-pure) aperiohedra. For example, the blend of the standard square tessellation $\{4,4\}$ and the infinite apeirogon $\{\infty\}$, denoted $\{4,4\}\#\{\infty\}$, is an apeirohedron whose faces are helical apeirogons (over squares), rising above the squares of $\{4,4\}$, such that 4 meet at each vertex; the orthogonal projections of $\{4,4\}\#\{\infty\}$ onto their component subspaces recover the original components, the square tessellation and linear apeirogon.

Table 1 The 12 pure apeirohedra in $\mathbb{E}^3$

Mirror vector	$\{3,3\}$	$\{3,4\}$	$\{4,3\}$	Faces	Vertex-fig.
(2,1,2)	$\{6,6\|3\}$	$\{6,4\|4\}$	$\{4,6\|4\}$	Planar	Skew
(1,1,2)	$\{\infty,6\}_{4,4}$	$\{\infty,4\}_{6,4}$	$\{\infty,6\}_{6,3}$	Helical	Skew
(1,2,1)	$\{6,6\}_4$	$\{6,4\}_6$	$\{4,6\}_6$	Skew	Planar
(1,1,1)	$\{\infty,3\}^{(a)}$	$\{\infty,4\}_{\cdot,*3}$	$\{\infty,3\}^{(b)}$	Helical	Planar

Note that each blended polyhedron really represents an entire family of polyhedra of the same kind, where the polyhedra in a family are determined by a parameter describing the relative scale of the two component figures. Thus there are infinitely many polyhedra of each kind, up to similarity, and our original count really refers to the 12 kinds rather than individual polyhedra.

Finally there are 12 regular apeirohedra that are irreducible, or *pure* (non-blended). In a sense, they fall into a single family, derived from the standard regular cubical tessellation. The 12 polyhedra in this family naturally are interrelated by a net of geometric operations (on polyhedra) and algebraic operations (on symmetry groups), which include the following: the duality operation; the previously mentioned Petrie-operation (of passing to the Petrie-dual); the *facetting* operation (of replacing the faces of a regular polyhedron by its *holes*, which are edge paths that successively take the second exit on the right at each vertex, while keeping all the vertices and edges unchanged); two lesser known operations called *halving* and *skewing*; and certain combinations of these operations.

We list these 12 pure apeirohedra in Table 1 taken from [24, p. 225], which also highlights the fact that there are just 12 polyhedra of this kind.

In this table, the first column gives the complete mirror vector, and the last two describe if the faces and vertex-figures are planar, skew or helical regular polygons; the geometric nature of the faces and vertex-figures only depends on the mirror vector. The second, third, and fourth column are indexed by the finite Platonic polyhedra whose rotation or full symmetry group is intimately related to the special group.

The three polyhedra along the top row are the famous Petrie-Coxeter polyhedra, which along with those in the third row comprise the pure regular polyhedra with finite faces. The pure polyhedra with infinite, helical faces are listed in the second and last row; those in the last row occur in two enantiomorphic (mirror image) forms, since their symmetry group is generated by half-turns and consists only of proper isometries. The fine Schläfli symbols for the polyhedra in the table signify defining relations for the symmetry groups; for example, extra relations often specify the orders of the elements $R_0R_1R_2$, $R_0R_1R_2R_1$ or $R_0(R_1R_2)^2$. These orders correspond to the lengths of the Petrie paths, of the holes (paths traversing edges where the new edge is chosen to be the second on the right according to some local orientation), and of the 2-zigzags (paths traversing edges where the new edge is chosen to be the second on the right, but reversing orientation on each step).

The regular polyhedron $\{\infty,3\}^{(b)}$ is illustrated in Fig. 1; three helical faces meet at each vertex. Some faces have a vertical axis; they are helices over squares, like

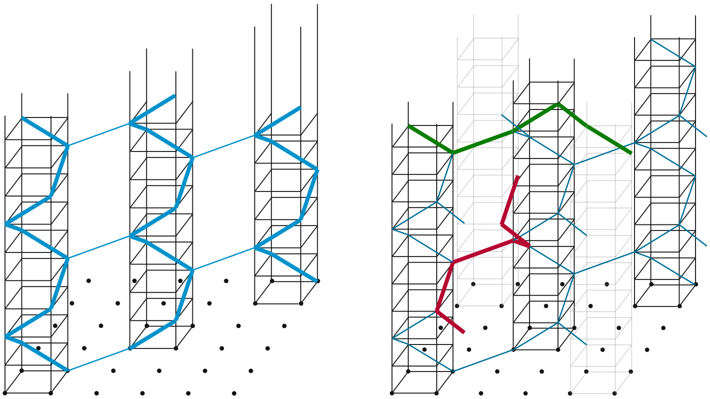

Fig. 1 The helix-faced regular polyhedron $\{\infty, 3\}^{(b)}$, with symmetry group requiring the single extra relation $(R_0R_1)^4(R_0R_1R_2)^3 = (R_0R_1R_2)^3(R_0R_1)^4$

the ones shown on the left, and are joined by horizontal edges. The remaining faces have axes parallel to the remaining two coordinate axes; one copy of each is shown on the right.

In summary we have

Theorem 1. *There are precisely* 48 *regular polyhedra in* $\mathbb{E}^3$*, up to similarity and scaling of components (when applicable). The list comprises* 18 *finite polyhedra and* 30 *apeirohedra.*

5 Non-simply Flag-Transitive Complexes

In order to complete the classification of regular polygonal complexes in $\mathbb{E}^3$ it remains to consider complexes with three or more faces around each edge. For convenience we split the discussion into two cases according to the size of the flag stabilizers. Throughout this and the next section we follow [29, 30].

Quite surprisingly, up to similarity, there are just four regular polygonal complexes that are not simply flag-transitive. They can be characterized as the regular complexes $\mathscr{K}$ that occur as 2-skeletons of regular 4-apeirotopes $\mathscr{P}$ in $\mathbb{E}^3$ (see [24, Ch. 7F]). The 2-*skeleton* of a 4-apeirotope is the incidence structure determined by its vertices, edges and polygons. These 4-apeirotopes in $\mathbb{E}^3$ are, by definition, the discrete faithful realizations of abstract regular polytopes of rank 4 in $\mathbb{E}^3$, so their combinatorial rank is 1 higher than the dimension of the ambient space.

There are precisely eight regular 4-apeirotopes $\mathscr{P}$ in $\mathbb{E}^3$, occurring in pairs of Petrie-duals as shown in (2). The Petrie-dual of a regular 4-apeirotope $\mathscr{P}$ is obtained by replacing the distinguished involutory generators T_0, T_1, T_2, T_3 of its symmetry group $G(\mathscr{P})$ by the new involutory generators

$$T_0, T_1T_3, T_2, T_3$$

of $G(\mathscr{P})$, and then applying Wythoff's construction with these new generators and with the same initial vertex as for $\mathscr{P}$ itself. Every pair of Petrie-duals contributes just one regular polygonal complex $\mathscr{K}$, since Petrie-duals have isomorphic 2-skeletons. Thus there are just four such complexes $\mathscr{K}$. Two of the eight apeirotopes $\mathscr{P}$ have (finite convex) square 2-faces, 4 occurring at each edge; and six have (infinite planar) zigzag 2-faces, with either 3 or 4 at each edge. Our notation follows [24, Ch. 7F].

$$\begin{array}{cc} \{4,3,4\} & \{\{4,6\,|\,4\},\{6,4\}_3\} \\ \{\{\infty,3\}_6\#\{\,\},\{3,3\}\} & \{\{\infty,4\}_4\#\{\infty\},\{4,3\}_3\} \\ \{\{\infty,3\}_6\#\{\,\},\{3,4\}\} & \{\{\infty,6\}_3\#\{\infty\},\{6,4\}_3\} \\ \{\{\infty,4\}_4\#\{\,\},\{4,3\}\} & \{\{\infty,6\}_3\#\{\infty\},\{6,3\}_4\} \end{array} \tag{2}$$

The two apeirotopes in the top row are the standard cubical tessellation $\{4,3,4\}$ in $\mathbb{E}^3$; and its Petrie-dual $\{\{4,6\,|\,4\},\{6,4\}_3\}$, whose rank 3 faces are Petrie-Coxeter polyhedra $\{4,6\,|\,4\}$ and whose vertex-figures are Petrie-duals $\{6,4\}_3$ of octahedra $\{3,4\}_6$. The 2-skeleton of the cubical tessellation is the simplest regular polygonal complex that is not simply flag-transitive.

The other six apeirotopes have finite crystallographic regular polyhedra as vertex-figures, namely either tetrahedra $\{3,3\}$, octahedra $\{3,4\}$, or cubes $\{4,3\}$, or Petrie-duals of one of those; their rank 3 faces are blends, namely of the Petrie-duals $\{\infty,3\}_6$ or $\{\infty,4\}_4$ of the plane tessellations $\{6,3\}$ or $\{4,4\}$, respectively, with the line segment $\{\,\}$ or linear apeirogon $\{\infty\}$ (see [24, Ch.7E]).

The number of faces r around an edge of the 2-skeleton $\mathscr{K}$ is just the last entry in the Schläfli symbol (the basic symbol $\{p,q,r\}$) of the underlying 4-apeirotope $\mathscr{P}$ (or, equivalently, of the Petrie dual of $\mathscr{P}$). Hence, $r = 4$, 3, 4 or 3, respectively.

Among the regular polygonal complexes $\mathscr{K}$, the non-simply transitive complexes can also be characterized as those that have face mirrors. A *face mirror* of $\mathscr{K}$ is an affine plane in $\mathbb{E}^3$ that contains a face of $\mathscr{K}$ and is the mirror of a plane reflection in $G(\mathscr{K})$. Clearly, regular complexes $\mathscr{K}$ with face mirrors must have planar faces, and every face must span a face mirror; moreover, the plane reflection in a face mirror of $\mathscr{K}$ fixes every flag of $\mathscr{K}$ that lies in this face mirror, and hence generates the corresponding flag stabilizer.

In summary we have

Theorem 2. *Up to similarity, there are just four non-simply flag-transitive regular polygonal complexes in $\mathbb{E}^3$, each given by the common 2-skeleton of the two regular 4-apeirotopes from a pair of Petrie-duals. These infinite complexes are precisely the regular polygonal complexes in $\mathbb{E}^3$ that have face mirrors.*

6 Simply Flag-Transitive Complexes

The class of simply flag-transitive regular polygonal complexes in $\mathbb{E}^3$ is much richer and comprises all finite or infinite regular polyhedra. As we have already described the regular polyhedra in Sect. 4, we can confine ourselves here to those complexes that are not polyhedra.

Thus let $\mathscr{K}$ be an (infinite) simply flag-transitive complex such that $G = G(\mathscr{K})$ is affinely irreducible, let $r \geqslant 3$, and let $\{F_0, F_1, F_2\}$ denote the base flag. Then the two subgroups G_0 and G_1 of G are again of order 2, and are generated by some point, line, or plane reflection R_0, and some line or plane reflection R_1, respectively; however, G_2 is a cyclic or dihedral group of order r. The *mirror vector* $(dim(R_0), dim(R_1))$ of $\mathscr{K}$ now has only two components recording the dimensions $dim(R_0)$ and $dim(R_1)$ of the mirrors of R_0 and R_1, respectively. (For polyhedra, G_2 is also generated by a reflection, and the complete mirror vector records the dimensions of all three mirrors.)

The vertex-stabilizer subgroup G_{F_0} in G of the base vertex F_0 is called the *vertex-figure group* of $\mathscr{K}$ at F_0, and is a finite group since $\mathscr{K}$ is discrete. In particular, $G_{F_0} = \langle R_1, G_2\rangle$, and G_{F_0} acts simply flag-transitively on the graph that forms the vertex-figure of $\mathscr{K}$ at F_0. Similarly, the face-stabilizer G_{F_2} in G of the base face F_2 is given by $G_{F_2} = \langle R_0, R_1\rangle$ and is isomorphic to a (finite or infinite) dihedral group acting simply transitively on the flags of $\mathscr{K}$ containing F_2.

The enumeration of the simply flag-transitive regular complexes for a given mirror vector is typically rather involved. A good number of complexes must be discovered by direct geometric or algebraic methods. Others then can be derived by operations applied to these complexes; that is, the new complexes are obtained by suitably modifying R_0 and R_1 while keeping the base vertex and preserving the group $\langle G_0, G_1, G_2\rangle$ as a (possibly proper) subgroup of symmetries. In this vein, the explicit enumeration of the simply flag-transitive complexes begins in [29] with the determination of the complexes with mirror vector $(1, 2)$, and then proceeds in [30] with the description of those for the remaining mirror vectors, accomplished by a mix of direct methods, applications of operations, and elimination of certain cases. At the end, we arrive at the following theorem.

Theorem 3. *Up to similarity, there are exactly* 21 *simply flag-transitive regular polygonal complexes in* $\mathbb{E}^3$ *that are not regular polyhedra.*

Thus, counting also the regular polyhedra from Theorem 1, there is total of 69 simply flag-transitive regular complexes, up to similarity and scaling of components for blended polyhedra.

Table 2 lists the 21 simply flag-transitive complexes by mirror vector, and records their data concerning the pointwise edge stabilizer G_2, the number r of faces surrounding an edge, the structure of the faces and vertex-figures, the vertex-set, and the structure of the special group G_*. In the face column we have used the symbols p_c, p_s, ∞_2, or ∞_k with $k = 3$ or 4, respectively, to indicate that the faces are $\underline{c}$*onvex* p-gons, $\underline{s}$*kew* p-gons, planar zigzags, or helical polygons over *k-gons*.

Table 2 The 21 simply flag-transitive regular complexes in $\mathbb{E}^3$ which are not regular polyhedra

Mirror vector	Complex	G_2	r	Face	Vertex-figure	Vertex-set	Special group
$(1,2)$	$\mathscr{K}_1(1,2)$	D_2	4	4_s	Cuboctahedron	$\Lambda_{(a,a,0)}$	[3, 4]
	$\mathscr{K}_2(1,2)$	C_3	3	4_s	Cube	$\Lambda_{(a,a,a)}$	[3, 4]
	$\mathscr{K}_3(1,2)$	D_3	6	4_s	Double cube	$\Lambda_{(a,a,a)}$	[3, 4]
	$\mathscr{K}_4(1,2)$	D_2	4	6_s	Octahedron	$a\mathbb{Z}^3$	[3, 4]
	$\mathscr{K}_5(1,2)$	D_2	4	6_s	Double square	V_a	[3, 4]
	$\mathscr{K}_6(1,2)$	D_4	8	6_s	Double octahedron	$a\mathbb{Z}^3$	[3, 4]
	$\mathscr{K}_7(1,2)$	D_3	6	6_s	Double tetrahedron	W_a	[3, 4]
	$\mathscr{K}_8(1,2)$	D_2	4	6_s	Cuboctahedron	$\Lambda_{(a,a,0)}$	[3, 4]
$(1,1)$	$\mathscr{K}_1(1,1)$	D_3	6	∞_3	Double cube	$\Lambda_{(a,a,a)}$	[3, 4]
	$\mathscr{K}_2(1,1)$	D_2	4	∞_3	Double square	V_a	[3, 4]
	$\mathscr{K}_3(1,1)$	D_4	8	∞_3	Double octahedron	$a\mathbb{Z}^3$	[3, 4]
	$\mathscr{K}_4(1,1)$	D_3	6	∞_4	Double tetrahedron	W_a	[3, 4]
	$\mathscr{K}_5(1,1)$	D_2	4	∞_4	ns-cuboctahedron	$\Lambda_{(a,a,0)}$	[3, 4]
	$\mathscr{K}_6(1,1)$	C_3	3	∞_4	Tetrahedron	W_a	$[3,4]^+$
	$\mathscr{K}_7(1,1)$	C_4	4	∞_3	Octahedron	$a\mathbb{Z}^3$	$[3,4]^+$
	$\mathscr{K}_8(1,1)$	D_2	4	∞_3	ns-cuboctahedron	$\Lambda_{(a,a,0)}$	[3, 4]
	$\mathscr{K}_9(1,1)$	C_3	3	∞_3	Cube	$\Lambda_{(a,a,a)}$	$[3,4]^+$
$(0,1)$	$\mathscr{K}(0,1)$	D_2	4	∞_2	ns-cuboctahedron	$\Lambda_{(a,a,0)}$	[3, 4]
$(0,2)$	$\mathscr{K}(0,2)$	D_2	4	∞_2	Cuboctahedron	$\Lambda_{(a,a,0)}$	[3, 4]
$(2,1)$	$\mathscr{K}(2,1)$	D_2	4	6_c	ns-cuboctahedron	$\Lambda_{(a,a,0)}$	[3, 4]
$(2,2)$	$\mathscr{K}(2,2)$	D_2	4	3_c	Cuboctahedron	$\Lambda_{(a,a,0)}$	[3, 4]

(A planar zigzag is viewed as a helix over a 2-gon, hence our notation. Clearly, the subscript in 3_c is redundant.) We also set

$$V_a := a\mathbb{Z}^3\backslash((0,0,a)+\Lambda_{(a,a,a)}),\quad W_a := 2\Lambda_{(a,a,0)} \cup ((a,-a,a)+2\Lambda_{(a,a,0)}),$$

to have a short symbol available for the vertex-sets of some complexes. The vertex-figures of polygonal complexes are finite geometric graphs, so an entry in the vertex-figure column describing a solid figure is meant to represent the edge-graph of this figure, with "double" indicating the double edge-graph. The abbreviation "ns-cuboctahedron" stands for the edge graph of a certain "non-standard cuboctahedron", a realization in $\mathbb{E}^3$ of the (abstract) cuboctahedron with non-planar square faces.

As an example, the faces of the complex $\mathscr{K}_6(1,2)$ are the Petrie polygons of all cubes of the cubical tessellation of $\mathbb{E}^3$; so in particular, the vertices and edges of $\mathscr{K}_6(1,2)$, respectively, comprise all vertices and edges of the cubical tessellation. Recall that every edge of a cube belongs to precisely two Petrie polygons of the same cube. Since every edge belongs to four cubes in the cubical tessellation, every edge must belong to eight Petrie polygons of cubes in $\mathscr{K}_6(1,2)$. The complex $\mathscr{K}_4(1,2)$ is a proper subcomplex of $\mathscr{K}_6(1,2)$ obtained by taking only the Petrie polygons of alternate cubes. The complex $\mathscr{K}_5(1,2)$ is another subcomplex of $\mathscr{K}_6(1,2)$ consisting only of the Petrie polygons with vertices in the set V_a defined above.

7 Chiral Polyhedra

Chiral polyhedra in $\mathbb{E}^3$ are the most interesting kind of nearly regular polyhedra; their geometric symmetry groups have two orbits on the flags, such that adjacent flags are in distinct orbits.

The structure results for the symmetry groups of regular polygonal complexes carry over to chiral polyhedra as follows (see [35,36]). Let $\mathscr{K}$ be a chiral polyhedron in $\mathbb{E}^3$ with symmetry group $G = G(\mathscr{K})$, let $\Phi := \{F_0, F_1, F_2\}$ be a base flag of $\mathscr{K}$, and let F_0', F_1', F_2' denote the faces of $\mathscr{K}$ with $F_0' < F_1$, $F_0 < F_1' < F_2$, $F_1 < F_2'$ and $F_j' \neq F_j$ for $j = 0, 1, 2$. Then G is generated by symmetries S_1, S_2 of $\mathscr{K}$, called the *distinguished generators* of G (relative to Φ), where S_1 leaves the base face F_2 invariant and cyclically permutes the vertices of F_2 such that $F_1 S_1 = F_1'$ (and thus $F_0' S_1 = F_0$), and S_2 leaves the base vertex F_0 invariant and cyclically permutes the vertices in the vertex-figure at F_0 such that $F_2 S_2 = F_2'$ (and thus $F_1' S_2 = F_1$). Then, in analogy to (1),

$$S_1^p = S_2^q = (S_1 S_2)^2 = I, \tag{3}$$

where $\{p, q\}$ is the Schläfli type of $\mathscr{K}$. The involutory symmetry $T := S_1 S_2$ interchanges the two end vertices of F_1 as well as the two faces meeting at F_1; that is, combinatorially, T acts like a half-turn about the midpoint of an edge. This symmetry T plays a critical role, in that it allows to employ a variant of Wythoff's construction (see [6]) to reconstruct a chiral polyhedron from its symmetry group.

Note that the symmetry groups of regular polyhedra in $\mathbb{E}^3$ have a subgroup of index at most 2 with properties very similar to those of the group of a chiral polyhedron. In fact, if $\mathscr{P}$ is a regular polyhedron and R_0, R_1, R_2 are the distinguished generators of its symmetry group $G(P)$ (relative to Φ), then $\hat{S}_1 := R_0 R_1$ and $\hat{S}_2 := R_1 R_2$ generate the *combinatorial rotation subgroup*, or *even subgroup*, $G^+(\mathscr{P}) := \langle \hat{S}_1, \hat{S}_2 \rangle$ of $G(P)$, of index 1 or 2. Now $\hat{T} := \hat{S}_1 \hat{S}_2 = R_0 R_2$ has properties similar to T. Whenever $G^+(\mathscr{P})$ has index 2 in $G(\mathscr{P})$ we say that $\mathscr{P}$ is *directly regular* or *orientable*.

Combinatorially speaking, chiral polyhedra have maximal "rotational" symmetry but no "reflexive" symmetry. (This does not mean that S_1 and S_2 are actually geometric rotations!) Thus our term "chiral" really means "maximal chiral". By contrast, again combinatorially speaking, regular polytopes have maximal "reflexive" symmetry. (Here R_0, R_1, R_2 are actually reflections, in points, lines, or planes.)

Chirality, in this sense of "maximal chirality", does not make any appearance in the classical theory of highly-symmetric figures in Euclidean spaces. This may explain why chiral polyhedra were only described and enumerated quite recently, in [35, 36].

The complete classification starts off with the observation that chiral polyhedra are necessarily pure apeirohedra; that is, infinite polyhedra that are not naturally "blends" of two lower-dimensional structures, and hence have an affinely irreducible

Table 3 The finite-faced chiral polyhedra, along with their related regular polyhedra

Type	$\{6,6\}$	$\{4,6\}$	$\{6,4\}$
Notation	$P(a,b)$	$Q(c,d)$	$Q(c,d)^*$
Parameters	$a,b \in \mathbb{Z},\ (a,b)=1$	$c,d \in \mathbb{Z},\ (c,d)=1$	$c,d \in \mathbb{Z},\ (c,d)=1$
Chiral	$b \neq \pm a$	$c,d \neq 0$	$c,d \neq 0$
Regular polyhedra	$P(a,-a)=\{6,6\}_4$	$Q(a,0)=\{4,6\}_6$	$Q(a,0)^*=\{6,4\}_6$
	$P(a,a)=\{6,6\|3\}$	$Q(0,a)=\{4,6\|4\}$	$Q(0,a)^*=\{6,4\|4\}$
	Geom. self-dual,		
	$P(a,b)^* \cong P(a,b)$		
Special group	$[3,3]^+ \times \langle -I \rangle$	$[3,4]$	$[3,4]$

Table 4 The helix-faced chiral polyhedra, along with their related regular polyhedra

Type	$\{\infty,3\}$	$\{\infty,3\}$	$\{\infty,4\}$
Notation	$P_1(a,b)$	$P_2(c,d)$	$P_3(c,d)$
Parameters	$a,b \in \mathbb{R}$	$c,d \in \mathbb{R}$	$c,d \in \mathbb{R}$
Chiral	$b \neq \pm a$	$c,d \neq 0$	$c,d \neq 0$
Regular polyhedra	$P_1(1,-1)=\{\infty,3\}^{(a)}$	$P_2(1,0)=\{\infty,3\}^{(b)}$	$P_3(0,1)=\{\infty,4\}_{\cdot,*3}$
	$P_1(1,1)=\{3,3\}$	$P_2(0,1)=\{4,3\}$	$P_3(1,0)=\{3,4\}$
Helices over	Triangles	Squares	Triangles
Special group	$[3,3]^+$	$[3,4]^+$	$[3,4]^+$

symmetry group. In short, unlike regular polyhedra, chiral polyhedra can neither be finite nor planar or blended.

The classification of chiral apeirohedra is quite elaborate and naturally breaks down into analyzing the finite-faced and the helix-faced polyhedra (see [35, 36]). The possible apeirohedra fall into six infinite 2-parameter families (up to congruence). In each family, all but two polyhedra are chiral; the two exceptional polyhedra are regular and are among those described in Sect. 4. Tables 3 and 4 list the families of polyhedra by Schläfli type, along with the two regular polyhedra occurring in each family; in the three families in Table 4, one exceptional polyhedron is finite. Also included is data about the *special group* of a polyhedron, that is, the quotient of the geometric symmetry group by its translation subgroup; here $[3,3]^+$ and $[3,4]^+$ denote the tetrahedral or octahedral rotation group, respectively, and $[3,4]$ the full octahedral group.

It is quite remarkable that a regular polyhedron cannot have both skew faces and skew vertex-figures. However, finite-faced chiral polyhedra must necessarily have both skew faces and skew vertex-figures. In fact, the generators S_1, S_2 of the symmetry group must be rotatory reflections in this case, resulting in skew faces and skew vertex-figures. Note, however, that the rotation subgroups for the regular polyhedra occurring in the three families of finite-faced polyhedra of Table 3 also have generators S_1, S_2 which are rotatory reflections, but here the position of the base vertex forces planarity of faces or vertex-figures.

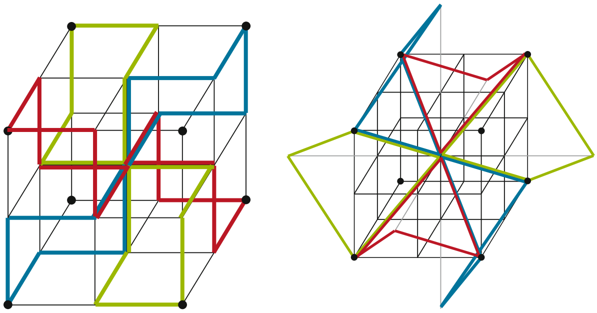

Fig. 2 The vertex-neigborhoods of the finite-faced chiral apeirohedra $P(1,0)$ and $Q(1,1)$, of types $\{6,6\}$ and $\{4,6\}$, respectively. Depicted is the neighborhood of a single vertex, where 6 skew hexagonal faces or 6 skew square faces meet. Each apeirohedron expands in a consistent manner throughout space such that all vertex neighborhoods are congruent to the one shown

Chiral apeirohedra with infinite faces must necessarily have helical faces spiraling over triangles or squares, as well as planar vertex-figures. The symmetry group is generated by a screw motion S_1 and a rotation S_2 in this case. Chiral helix-faced polyhedra unravel, in a sense, a "crystallographic" Platonic polyhedron, namely the finite regular polyhedron in their respective family.

The regular polyhedra listed in Tables 3 and 4 comprise 9 of the 12 pure regular apeirohedra in $\mathbb{E}^3$, namely those listed in Table 1 with complete mirror vectors $(1,2,1)$, $(1,1,1)$ or $(2,1,2)$, as well as the three crystallographic Platonic polyhedra. The three remaining pure regular apeirohedra $\{\infty,6\}_{4,4}$, $\{\infty,4\}_{6,4}$ and $\{\infty,6\}_{6,3}$ all have complete mirror vector $(1,1,2)$ and do not occur in families alongside chiral polyhedra.

These six families of chiral (or regular) polyhedra have some amazing properties. For example, any two distinct finite-faced polyhedra of the same type are combinatorially non-isomorphic. In fact, $P(a,b)$ and $P(a',b')$ are isomorphic if and only if $(a',b') = \pm(a,b), \pm(b,a)$; and similarly, $Q(c,d)$ and $Q(c',d')$ are isomorphic if and only if $(c',d') = \pm(c,d), \pm(-c,d)$. Thus there are very many combinatorially distinct finite-faced chiral polyhedra (Fig. 2). By contrast, as shown in Pellicer-Weiss [31], every helix-faced chiral polyhedron $P_1(a,b)$ or $P_2(c,d)$ is combinatorially isomorphic to the infinite regular polyhedron in its family. On the other hand, since the polyhedron $\{\infty,4\}_{\cdot,*3}$ is not orientable, it cannot have chiral realizations. Every chiral polyhedron $P_3(c,d)$ is then isomorphic to the (combinatorial) orientable double cover of $\{\infty,4\}_{\cdot,*3}$. Thus, up to isomorphism, there are just three helix-faced chiral polyhedra, each represented by a helix-faced regular polyhedron. But even more is true: in a sense that can be made precise, the helix-faced chiral polyhedra can be thought of as continuous "chiral deformations" of helix-faced regular polyhedra (see [31]). This surprising phenomenon is illustrated for the helix-faced polyhedra $P_2(c,d)$ in Fig. 3; shown is the effect on the location of the "vertical" helical faces, as a result of continuously changing the parameters c, d.

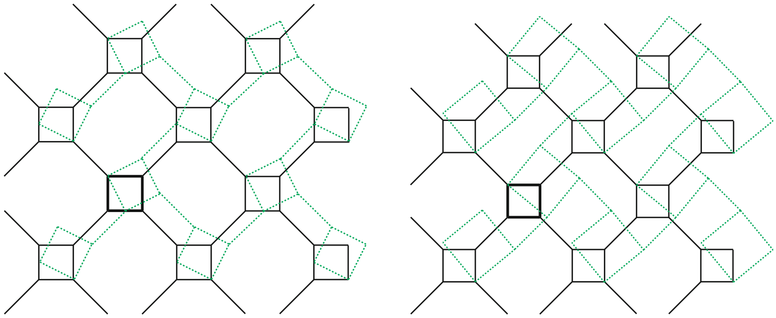

Fig. 3 The helix-faced polyhedron $P_2(1,0)$ and its deformations $P_2(1,d)$. The *solid black* or *green dotted lines* show the projection of the entire polyhedron $P_2(1,0)$ or $P_2(1,d)$, respectively, onto a "horizontal" plane perpendicular to the axis of a "vertical" helical face. The vertical helical faces of $P_2(1,0)$ or $P_2(1,d)$, respectively, then project onto the *small black squares* or *small green squares*; *one black square*, resulting from one vertical helical face of $P_2(1,0)$, is emphasized. As the parameter d is changed continuously, the vertical and "horizontal" helical faces move in such a way that the axes of corresponding faces remain parallel throughout the process. Accordingly, the projections of $P_2(1,d)$ move continuously as well. The figures on the *left* and *right*, respectively, show projections of $P_2(1,d)$ when d is small or when d gets larger

Finally, helix-faced chiral polyhedra are combinatorially regular, as they are isomorphic to regular polyhedra. However, by contrast, finite-faced chiral polyhedra are *combinatorially chiral*, meaning that the combinatorial automorphism group has two flag-orbits such that adjacent flags are in distinct orbits (see [31]).

In summary, we have the following

Theorem 4. *Up to congruence, the chiral polyhedra in $\mathbb{E}^3$ fall into six infinite,* 2-*parameter families of apeirohedra, each containing alongside chiral apeirohedra also two regular polyhedra. Three families consist of finite-faced apeirohedra, and three of helix-faced polyhedra. The finite-faced polyhedra are also combinatorially chiral, but the helix-faced polyhedra are combinatorially regular.*

8 Two-Orbit Polyhedra

The chiral polyhedra in $\mathbb{E}^3$ are by definition the 2-orbit polyhedra in $\mathbb{E}^3$ in the class 2_I with $I = \emptyset$. It is desirable to extend the classification of chiral polyhedra to 2-orbit polyhedra in arbitrary classes 2_I, with $I \subsetneq \{0,1,2\}$. We saw that chirality cannot occur among finite polyhedra; however, as the example of the cuboctahedron (in class $2_{\{0,1\}}$) shows, finite 2-orbit polyhedra already occur among the familiar convex polyhedra. Thus a good first step would be the complete enumeration of the finite 2-orbit polyhedra in $\mathbb{E}^3$.

Significant progress towards this goal has already been made for regular polyhedra of index 2. A polyhedron $\mathscr{K}$ is said to be a *regular polyhedron of index* 2 if its combinatorial automorphism group $\Gamma(\mathscr{K})$ acts flag-transitively on $\mathscr{K}$ and contains the geometric symmetry group $G(\mathscr{K})$ as a subgroup of index 2. In other words, $\mathscr{K}$ is combinatorially regular but "fails geometric regularity by a factor of 2". For any such polyhedron, the symmetry group has two orbits on the flags, and at most two orbits on the vertices, edges, and faces. Note that the helix-faced chiral polyhedra discussed in the previous section are examples of infinite regular polyhedra of index 2.

The finite regular polyhedra of index 2 were recently enumerated in Cutler-Schulte [9] and Cutler [8] (see also Wills [38]). The following theorem summarizes the results.

Theorem 5. *Up to similarity, there are exactly* 22 *infinite families of regular polyhedra of index* 2 *with vertices on two orbits under the symmetry group, where two polyhedra belong to the same family if they differ only in the relative size of the spheres containing their vertex orbits. In addition, up to similarity, there are exactly* 10 *(individual) polyhedra with vertices on one orbit under the symmetry group.*

In describing the polyhedra, we slightly abuse terminology and say that a polyhedron $\mathscr{K}$ is of *type* $\{p,q\}_r$ if the underlying regular map has (Schläfli) type $\{p,q\}$ and Petrie polygons of length r. Note here that we are not requiring the map to be the universal regular map of type $\{p,q\}$ with Petrie polygons of length r (denoted $\{p,q\}_r$ in [7]). However, in some case the map actually is universal (see [10]).

Table 5 records the 22 infinite families of polyhedra by combinatorial isomorphism type. For example, the last entry in row 5 indicates that there are 2 infinite families with polyhedra isomorphic to Gordan's (universal) map $\{4,5\}_6$. The third column gives the name of the map in the notation of Conder [4] (when applicable), with R or N, respectively, indicating an orientable or non-orientable regular map; the number before the period is the genus, and an asterisk indicates the dual. The polyhedra in these 22 families have their vertices located at those of a pair of similar, aligned or opposed, Platonic solids with the same symmetry group. There are respectively 4, 2 and 16 families with full tetrahedral, octahedral, and icosahedral symmetry. The symmetry group is face-transitive in each case, and each polyhedron is orientable. Among all polyhedra (in all families), there are just two polyhedra with planar faces. Figure 4 shows one face of a regular polyhedron of index 2 and type $\{10,5\}_6$ belonging to one of the four families in the last row of Table 5.

The 10 (individual) regular polyhedra of index 2 with vertices on one orbit are listed in Table 6. Each has full icosahedral symmetry. There are orientable and non-orientable examples. Figure 4 depicts one face of the planar-faced regular polyhedron of index 2 and type $\{6,6\}_6$ listed in the first row of Table 6.

Table 5 The 22 infinite families of regular polyhedra of index 2 with two vertex orbits, listed by combinatorial isomorphism type. The polyhedra in the first two rows have full tetrahedral symmetry, and those in the third row full octahedral symmetry; all others have full icosahedral symmetry

Type $\{p,q\}_r$	Face vector (f_0, f_1, f_2)	Map	# Families
$\{4,3\}_6$	$(8,12,6)$	Sphere	2
$\{6,3\}_4$	$(8,12,4)$	Torus	2
$\{6,4\}_6$	$(12,24,8)$	$R3.4^*$	2
$\{10,3\}_{10}$	$(40,60,12)$	$R5.2^*$	4
$\{4,5\}_6$	$(24,60,30)$	$R4.2$	2
$\{6,5\}_4$	$(24,60,20)$	$R9.16^*$	2
$\{6,5\}_{10}$	$(24,60,20)$	$R9.15^*$	4
$\{10,5\}_6$	$(24,60,12)$	$R13.8^*$	4

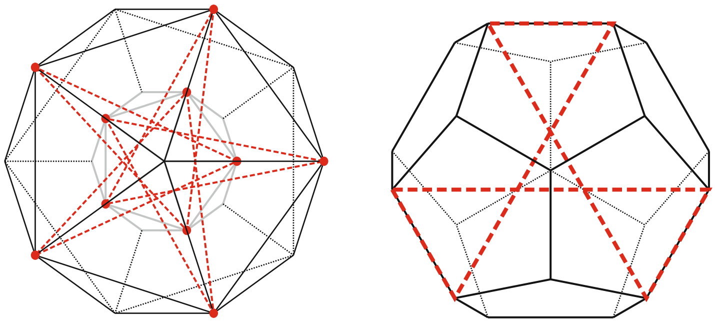

Fig. 4 Two regular polyhedra of index 2. The polyhedron on the left is a representative of one of the four infinite families of type $\{10,5\}_6$ with two vertex-orbits; its vertices lie on a pair of concentric icosahedra. The planar-faced polyhedron on the right has type $\{6,6\}_6$ and one vertex-orbit; its vertices are those of a dodecahedron. Only one face is shown in each case; the other faces are obtained by applying all icosahedral or dodecahedral symmetries

Table 6 The 10 (individual) regular polyhedra of index 2 with one vertex orbits. Each has full icosahedral symmetry

Type $\{p,q\}_r$	Face vector (f_0, f_1, f_2)	Map	Notes
$\{6,6\}_6$	$(20,60,20)$	$R11.5$	Planar faces, self-dual map
$\{6,6\}_6$	$(20,60,20)$	$N22.3$	Face transitive
$\{4,6\}_5$	$(20,60,30)$	$N12.1$	
$\{5,6\}_4$	$(20,60,24)$	$R9.16$	Planar faces
$\{6,4\}_5$	$(30,60,20)$	$N12.1^*$	
$\{5,4\}_6$	$(30,60,24)$	$R4.2^*$	Planar faces
$\{4,6\}_{10}$	$(20,60,30)$	$R6.2$	
$\{10,6\}_4$	$(20,60,12)$	$N30.11^*$	
$\{6,4\}_{10}$	$(30,60,20)$	$R6.2^*$	
$\{10,4\}_6$	$(30,60,12)$	$N20.1^*$	

9 Conclusions

The recent history of symmetric structures in Euclidean 3-space $\mathbb{E}^3$ suggests a rich variety of objects yet to be discovered. All geometrically regular polygonal complexes (including polyhedra), and all regular 4-polytopes, in Euclidean 3-space have now been classified; by contrast, little is known about polygonal complexes and 4-polytopes with slightly less symmetry. Two natural open questions concern the enumeration of all 2-orbit polyhedra and all edge-transitive polyhedra in $\mathbb{E}^3$. It appears more challenging to widen the scope of these problems to general polygonal complexes. A good starting point in this direction is a detailed classification of the finite 2-orbit, or edge-transitive, polygonal complexes that are not polyhedra; or a proof that such complexes cannot exist.

Significant progress has been made in the theory of realizations (in any dimension) for regular polytopes of any rank, mostly by McMullen; the state of the art will be summarized in his forthcoming monograph on "Geometric Regular Polytopes" [22], but many results can also be found in [24, Chapter 5]. However, little is known about realizations of other kinds of polytopes or polygonal complexes. The complete enumeration of particularly interesting families of such objects will greatly contribute to our basic understanding of geometric realizations of incidence structures.

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Two Notes on Maps and Surface Symmetry

Thomas W. Tucker

Abstract The first note of this paper determines for which g the orientable surface of genus g can be embedded in euclidean 3-space so as to have prismatic, cubical/octahedral, tetrahedral, or icosahedral/dodecahedral symmetry. The second note proves, through entirely elementary methods, that the clique number of the graph underlying a regular map is $m = 2, 3, 4, 6$; for $m = 6$ the map must be non-orientable and for $m = 4, 6$ the graph has a K_m factorization. Here a regular map is one having maximal symmetry: reflections in all edges and full rotational symmetry about every vertex, edge and face.

Keywords Riemann-Hurwitz equation • Regular map • Clique

Subject Classifications: 05C10, 57M15, 57M60

In this note, we prove two unrelated results about maps and surface symmetry.

The first concerns the possible finite symmetry, under euclidean isometry, of a surface of genus g embedded in 3-space. The theorem was inspired by a question from Bojan Mohar asking why the sculpture "The group of genus two" by DeWitt Godfrey [5], which appears on the cover of the journal *Ars Combinatorica Mathematica*, shows almost none of the rotational symmetry of the map.

Any finite group A of euclidean isometries of 3-space fixes the barycenter O of an orbit of A and hence leaves invariant the unit sphere centered at O. Thus the possibilities for A are just the symmetry groups of the n-prism, the Platonic solids (cube/octahedron, tetrahedron, and icosahedron/dodecahedron), and their subgroups. For each of these four types of symmetry, we show that for all but finitely many g, the surface of genus g can be embedded so as to have the given

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symmetry type, and we give the finite list of excluded g. Given a finite set X of natural numbers, let $L(X)$ be the set of all linear combinations of elements of X with nonnegative integral coefficients. Note that if $gcd(X) = 1$, then $L(X)$ contains all sufficiently large integers, by the "postage stamp" problem.

Theorem A (Surface Symmetry in 3-Space). *Let S be a surface of genus g. Then S can be embedded in 3-space so as to have the symmetry of:*

- *The n-prism if and only if $g \equiv 1 \pmod n$ or $g \in L(n, n-1)$;*
- *The cube or octahedron if and only if $g \in L(16, 18, 12) + k$, where $k = 0, 5, 7, 11, 13$;*
- *The tetrahedron if and only if $g \in L(8, 6) + k$, where $k = 0, 3, 5, 7$;*
- *The isosahedron or dodecahedron if and only if $g \in L(40, 48, 30) + k$ where $k = 0, 11, 19, 29, 31$.*

Moreover, S can be embedded with n-prism symmetry if and only if it can be embedded with n-fold rotational symmetry. Similarly, S can be embedded with full cubical (respectively, tetrahedral, icosahedral) symmetry if and only if it can be embedded with orientation-preserving cubical (respectively, tetrahedral, icosahedral) symmetry.

The second result concerns the clique number of a regular map. A *map* M is an embedding of a finite graph G in a closed surface S such that the interior of each face (component of $S - G$) is homeomorphic to an open disk; we call G the underlying graph of the map and S the underlying surface. An automorphism of the map is an automorphism of the graph that can be extended to a homeomorphism of the surface (combinatorially, the automorphism must take any cycle in G bounding a face to another such cycle). The collection of all such automorphisms forms a group, denoted $Aut(M)$. A map M is *regular* if $Aut(M)$ acts transitively on vertex-edge-face incidence triples (usually call flags). Intuitively, a regular map generalizes the Platonic solids in having full rotational symmetry about each vertex and face, as well as reflective symmetry. In particular, the stabilizer of any vertex acts on its d neighbors as the dihedral group D_d acting on the vertices of a regular d-sided polygon; we call such an action of D_d *naturally dihedral*. The study of regular maps goes back to the 1920s and Coxeter and Moser [4] has a whole chapter on them. The survey article [11] covers most of the history of regular maps, including recent advances like [3].

Note that our use of "regular" in the case of orientable maps, is sometimes called *reflexibly* regular. By contrast, a map M on an orientable surface that has full rotational symmetry about each vertex and face center, but not necessarily any orientation-reversing symmetry, is called *orientably regular*, and if there is no orientation-reversing symmetry, it is called *chiral*.

The *clique number* of a graph G is the largest m such that the complete graph K_m is a subgraph of G. We say that G has an H-factorization if there is a collection of edge-induced subgraphs G_i, all isomorphic to H, such that every edge is an exactly one G_i.

Theorem B (Cliques in Regular Maps). *The clique number of the graph G underlying a regular (reflexible) map M is $m = 2, 3, 4$ or 6. Moreover, if $m = 6$, then M must be non-orientable and for $m = 4, 6$ the graph G has a K_m-factorization.*

We also give the following purely graph-theoretic version:

Theorem C (Cliques in Graphs with Dihedral Vertex Stabilizers). *Let G be a graph with $A \subset Aut(G)$ such that for each vertex v, the action of the vertex stabilizer A_v on edges incident to v is naturally dihedral. Then the clique number of G is $m = 2, 3, 4, 6$. If $m \geq 3$, the action of A is vertex-transitive and if $m = 4, 6$, then G has a K_m-factorization.*

The proofs for the clique results are astonishingly simple and depend on the measure of an angle, which appears to be a new concept for maps. Corollaries of Theorem B are classical theorems on the possible complete graphs underlying regular orientable and non-orientable maps, obtained using entirely algebraic methods, especially Frobenius groups.

1 Surface Symmetry in 3-Space

Our proof of Theorem A is by cases. We first recall some facts from [6] about a finite group acting A on a closed orientable surface S by orientation-preserving homeomorphisms. If $x \in S$, let A_x be the stabilizer of x in A. Then x has a neighborhood N_x that is equivariant under A, that is if $a \in A_x$, then $a(N_x) = N_x$ and otherwise N_x and $a(N_x)$ are disjoint. Moreover, A_x is cyclic with a generator a_x that on N_x looks like the map $z \to z^r$ in the complex plane, where $r = |A_x|$.

Associated with the action of A on S is the quotient map $p : S \to S/A$, where S/A is the surface obtained by identifying each orbit under A to a single point. Note that S/A is a surface since $p(N_x)$ is a disk about $p(x)$. Let $X = \{x \in S : |A_x| > 1\}$ and let $Y = p(X)$. Then p is a local homeomorphism except at $x \in X$, making p a (regular) branched covering with branch set $Y = p(X)$. For each $y \in Y$, the common number $r_y = |A_x|$ for any $x \in p^{-1}(y)$ is called the order of the branch point y. If S has genus g and S/A has genus h, then Euler's formula $2g - 2 = E - V - F$ gives us the Riemann-Hurwitz equation:

$$2g - 2 = |A| \left((2h - 2) + \Sigma_{y \in Y} \left(1 - \frac{1}{r_y} \right) \right),$$

For later use, we observe that if $h = 0$, then a generating set for A is obtained by choosing, for each $y \in Y$, one $x \in p^{-1}(y)$ and a generator for A_x.

Prismatic symmetry. We first consider a surface S with n-fold rotational symmetry about an axis in 3-space. Since the axis intersects S in an even number of points, the number of branch points is even and each has order $n-1$. Thus:

$$2g-2 = n\left((2h-2)+2b\frac{n-1}{n}\right) \text{ so } g = 1+(h-1)n+b(n-1).$$

If $b=0$, then $g \equiv 1 \pmod{n}$. Otherwise,

$$g = 1+(h-1)n-1+(b-1)(n-1)+(n-1) = hn+(b-1)(n-1)$$

so in this case $g \in L(n, n-1)$.

To show these conditions on g are sufficient, we construct for each case a model of surface S in 3-space having the required symmetry. For $g \equiv 1 \pmod{n}$, we take the standard torus in 3-space and attach n surfaces of genus h along n disks invariant under the rotation. For $g = hn+(b-1)n$, where $b>0$, we begin with a surface P_n in 3-space obtained from the boundary of a thickening of a dipole consisting of two vertices with n edges connecting the vertices, so that the dipole is invariant under an n-fold rotation about the axis through the two vertices. The genus of P_n is $n-1$. We can then string together $b-1$, copies of P_n to obtain a surface of genus $(b-1)(n-1)$ having n-fold rotational symmetry about a central axis with $2b$ branch points of order n (for $b-1=0$, we have simply a sphere with branch points of order n at the north and south poles). Then we can add n surfaces of genus h at n disks symmetrically placed either on the midpoints of edges of the central dipole (if $b-1$ is odd) or around a neck dividing the surface in half (if $b-1$ is even). The result is a surface of genus $g = hn+(b-1)n$ with the required symmetry.

We observe that the models we have constructed also have antipodal and reflective symmetry on n planes passing through the axis of rotation. Thus these models have full n-prism symmetry. Conversely, if any surface has n-prism symmetry, it also must also have n-fold rotational symmetry, and hence must satisfy $g \equiv 1 \pmod{n}$ or $g = hn+(b-1)n$ for $b>0$.

Cubical symmetry. We first assume that the surface S embedded in 3-space is invariant under the orientation-preserving automorphism group A of a cube centered at the origin O; it is well known that A is isomorphic to the full symmetric group S_4. The cube has four axes of 3-fold rotational symmetry, three of 4-fold rotational symmetry, and six of 2-fold symmetry. Each axis passes through O and pierces the surface S in the same number of points in each half. If O is inside the solid bounded by S, this number must be odd; if O is outside the solid, then this number is even. Thus, if O is inside S, we have:

$$2g-2 = 24\left(2h-2+(2b+1)\frac{2}{3}+(2c+1)\frac{3}{4}+(2d+1)\frac{1}{2}\right)$$

Simplifying, we get $g = 16b + 18c + 12(2h + d)$ so $g \in L(16, 18, 12)$.

If O is outside S, then

$$2g - 2 = 24\left((2h - 2) + 2b\frac{2}{3} + 2c\frac{3}{4} + 2d\frac{1}{2}\right)$$

Simplifying, we get $g = -23 + 16b + 18c + 12(2h + d)$. In this second case, if at least two of the coefficients of $b, c, (2h + d)$ is nonzero, then it is easily checked that $g \in L(16, 18, 12) + k$, where $k = 5, 7, 11$. If $h = 0$, it is impossible for only one of b, c, d to be nonzero, since otherwise A is generated by A_x with all x on the same axis, making A cyclic. Thus we can assume that $h > 0$. The only cases for $g \notin L(16, 18, 12) + k$, for $k = 5, 7, 11$ are $g = 1$ for $(h, d) = (1, 0)$ and $g = 13$ for $(h, d) = (1, 1)$. But the only groups acting without fixed points on the torus are abelian [6] so $g = 1$ is impossible.

We conclude that for orientation-preserving cubical symmetry, we need $g \in L(16, 18, 12) + k$, where $k = 0, 5, 7, 11, 13$. Now we build a model surface S for all these cases, based on the branch point information in the coefficients b, c, d, h. We begin with a cube centered at O and consider first the cases where O is inside the surface S. We attach to each vertex a string of $2b$ thickened dipoles P_3, to the center of each face a string of $2c$ thickened dipoles P_4, to the midpoint of each edge a string of $2d$ thickened dipoles P_2, and to each point in the orbit of a nonbranch point a string of h thickened dipoles P_2. The boundary of the resulting solid has genus

$$2b(8) + 3c(6) + (2h + d)12 = 16b + 18c + 12(2h + d).$$

If we take the resulting solid and drill a hole between antipodal vertices through the center O, we add $8-1 = 7$ to the genus. Holes between antipodal face-centers adds $6 - 1 = 5$ and holes between antipodal edge-midpoints, adds $12 - 1 = 11$. If we drill holes between both vertices and face-centers, we add $8 + 6 - 1 = 13$ to the genus. Thus we get all:

$$g \in L(16, 18, 12) + k \text{ where } k = 0, 5, 7, 11, 13.$$

Tetrahedral symmetry. Again, we start with a tetrahedron centered at O and consider only orientation-preserving symmetries; the group in this case is the alternating group A_4. There are four axes of 3-fold symmetry between each vertex and the center of the opposing face and three axes between midpoints of opposite edges. If O is inside the surface, there are an odd number $2b' + 1$ and $2b'' + 1$ of intersection points on each half of a vertex-face axis and $2c + 1$ on each edge-edge axis. Thus if $b = b' + b'' + 1$, we have:

$$2g - 2 = 12\left(2h - 2 + 2b\frac{2}{3} + (2c + 1)\frac{1}{2}\right) \text{ so } g = 8b + 6(2h + c) - 8.$$

Since $b \geq 1$, we have $g \in L(8,6)$. If instead C is outside the surface, we get:

$$2g - 2 = 12\left(2h - 2 + 2b\frac{2}{3} + 2c\frac{1}{2}\right) \text{ so } g = 8b + 6(2h + c) - 11.$$

Then $g = 1$ for $h = 1, b = 0, c = 0$ or $b = 0, h = 0, c = 2$. The first is again impossible since any group acting without fixed points on the torus is abelian. The second is impossible since then the group action would be generated by rotations around only one axis. In all other cases, $g \in L(8,6) + k$ for $k = 3, 5, 7$.

For the models, we start with the tetrahedron and attach a string of b dipoles P_3 at each of the four vertices and a string of c dipoles P_2 at the midpoint of each edge to make a surface of genus $g = 8b + 6c$ with orientation-preserving tetrahedral symmetry. Drilling holes from each vertex to the center or each edge midpoint to the center or both, gives, as desired, all;

$$g \in L(8,6) + k \text{ where } k = 0, 3, 5, 7.$$

Icosahedral symmetry. We start again with an icosahedron centered at O and consider only orientation-preserving symmetries; the group in this case is A_5. From the Riemann-Hurwitz equation, the situation is exactly the same as for the cube, only with branch points of order 3, 5 and 2. If the center is inside the surface, we get $g = 40b + 48c + 30(2h + d)$. If the center is outside the surface, we get

$$g = 40b + 48c + 30(2h + d) - 59.$$

In this case, as long as at least two of $b, c, 2h + d$ is nonzero, then $g \in L(40, 48, 30) + k$, where $k = 11, 19, 29$. As with the cube, if $h = 0$, it is impossible for only one of b, c, d to be nonzero. Then the only remaining case is $g = 1$ for $(h, d) = (1, 0)$ and $g = 31$ for $(h, d) = (1, 1)$. Again, $g = 1$ is impossible since A_5 is not abelian, so we have $g \in L(40, 48, 30) + k$, where $k = 11, 19, 29, 31$.

For models, we attach b dipoles P_3 at vertices, c dipoles P_5 at face centers, and c dipoles P_2 at edge midpoints. We can also drill 6 tunnels between antipodal vertices, 10 between antipodal face centers, and 15 between antipodal edge midpoints, or any combination, giving all

$$g \in L(40, 48, 30) + k, \text{ where } k = 0, 11, 19, 29, 31.$$

For the cube, tetrahedron, and icosahedron, our models all can be constructed to have reflective symmetry, so our conditions on g guarantee not only orientation-preserving symmetry of the desired type, but also the full symmetry. Conversely, any surface of genus g having full symmetry automatically has orientation-preserving symmetry so g must satisfy our conditions. $\square$

For the cube and tetrahedron, the given formulas for g lead to a list of excluded g. For the cube, it is $g = 1, 2, 3, 4, 6, 8, 9, 10, 14, 15, 20, 22, 26, 38$. For the tetrahedron, the excluded list is $g = 1, 4, 10$. For the icosahedron, the list is long, but finite.

For prismatic symmetry, we have $g \equiv 1 \pmod{n}$ or $b > 0$ and

$$g = hn + (b-1)(n-1)n = qn - r \text{ where } q = h + b - 1 \geq r = b - 1$$

Notice if $r > n$, we can write instead $g = (q-1)n - (r-n)$ with $q - 1 \geq (n-r)$ so we can assume $r \leq n$. If we fix n, the pattern for the genera g allowing n-fold rotational symmetry is clear. For example, when $n = 6$, we first have all $g \equiv 0, 1 \pmod{6}$. The we slowly fill in the remaining residues classes as g increases. In the following sequence we have put the missing g in parenthesis:

$$g = 0, 1, (2-4)5, 6, 7(8-9), 10, 11, 12, 13, (14), 15, 16, \cdots$$

Since we can always handle $r = n - 1$ using $g \equiv 1 \pmod{n}$, the largest r we have to worry about is $r = n - 2$. Thus we have:

Corollary 1. *Given $n > 1$, all surfaces of genus $g > (n-3)n-(n-2) = (n-2)^2-2$ can be embedded with n-fold rotational symmetry in 3-space.*

In general, given a group A, we can ask for the genera g such that A acts, preserving orientation, on the surface of genus g, where now we do not require the action come from an embedding in 3-space. Kulkarni's Theorem [8] shows that there is a number $n(A)$ such that if A acts on the surface of genus g preserving orientation, then $g \equiv 1 \pmod{n(A)}$ and that there is an action for all but finitely many such g. The number $n(A)$ follows from the Riemann-Hurwitz equation and is easily computed from the exponent of the Sylow p-subgroups of A, with an extra technical condition for $p = 2$. In particular, a group A acts on almost all surfaces if and only if it is almost Sylow cyclic and does not contain $Z_2 \times Z_4$ [12]; the group A is almost Sylow cyclic if its Sylow p-subgroup A_p is cyclic when p is odd and has a cyclic group of index two, when $p = 2$. On the other hand, for any A, determining the finite exceptions is almost impossible, even for the case of the cyclic group of order n, when n is highly composite (see for example, [10]).

In addition to changing the group A, we can also consider immersed surfaces, which would allow non-orientable surfaces in 3-space. That problem is considered in [9]. The situation for bordered surfaces is considered by Cavendish and Conway [2].

2 The Clique Number of a Regular Map

Let M be a map. If u and w are vertices adjacent to v, we call uvw an *angle* at v. A local orientation of the map at a vertex v of valence d defines a cyclic order $u_1, u_2, \cdots u_d$ to the vertices adjacent to v We define the *measure* of angle $u_i v u_j$, denoted $m(u_i v u_j)$, as the smaller of $|i - j|$ and $d - |i - j|$; in particular, $m(u_i v u_j) \leq d/2$. Map automorphisms preserve angle measure, since they preserve or reverse local orientations. If M is regular, because of the dihedral action of the stabilizer of v, given any angle uvw, there is an automorphism fixing v and interchanging u and w. It follows that if uvw is a triangle (3-cycle) in G, then all its angles have the same measure.

Theorem B. *Let M be a regular map whose underlying graph G has no multiple edges. Then the clique number of G is $m = 2, 3, 4, 6$. In the case $m = 4$, any 4-clique H is invariant under a 3-fold rotation about any vertex in H and under a reflection in any edge of H; in particular, the valence d of G is divisible by 3. For $m = 6$, a 6-clique H is invariant under a 5-fold rotation about any vertex of H and under a reflection in any edge of H; in particular, d is divisible by 5. Moreover, for $m = 6$, the map must be non-orientable. For both $m = 4$ and $m = 6$, the graph G has a K_m-factorization.*

Proof. Suppose that G has a K_4 subgraph H with vertices u, v, w, x with u, v, w consecutive around x. Let $m(uxv) = a, m(vxw) = b$, and $m(uxw) = c$. Without loss of generality, we can assume that $a \leq b \leq c$. There are two possibilities: either $a + b > c$, in which case $a + b + c = d$, or $a + b = c$.

Suppose first that $a + b + c = d$. There are four triangles in H. One has all angles a, one b and one c. The fourth triangle has angles $d - (a + b), d - (b + c)$ and $d - (c + a)$. Thus

$$d - (a + b) = d - (b + c) = d - (c + a).$$

Since $a + b + c = d$, we have $a = b = c = d/3$. Note that in this case, H is invariant under 3-fold rotation about x and reflection in the edge xv. Since x and v are arbitrary vertices of H, the same is true for all vertices and edges.

Suppose instead that $a + b = c$. Then again, of the four triangles in H, one has all angles a, one b, and one c. The fourth triangle has one angle $a + b = c$, so all angles in the triangle have measure c. At the second angle, where angles a and c meet, we have $c = d - (a + c)$, since $c = a + c$ is impossible. Similarly, at the third angle we have $c = d - (b + c)$. Thus

$$a = d - 2c \text{ and } b = d - 2c,$$

so $a = b = d - 2c$. Since $a + b = c$, we have $(d - 2c) + (d - 2c) = c$, so $c = d/5$. In particular, d is divisible by 5. Let $u_1 = u$ and let $u_2, \cdots u_5$ be vertices in cyclic order about x making consecutive angles of $d/5$, so that $u_1 \cdots u_5$ are invariant under a 5-fold rotation about x. We can assume that $u_2 = v$ and $u_3 = w$. By the 5-fold symmetry about x, there are edges between all the vertices in $u_1, \cdots u_5$, so the subgraph induced by those vertices together with x is a 6-clique and is invariant under 5-fold rotations about any vertex in H and under reflection in any edge.

We claim for the case $m = 6$, the map M is non-orientable. Suppose not. Let H is a 6-clique and B the subgroup of $Aut(M)$ leaving H invariant. As we have observed, B includes a reflection in each edge and 5-fold rotations about every vertex, so B acts transitively on H with vertex stabilizers D_5. Thus $|B| = 6 \cdot 10 = 60$. Let $C \subset B$ be the subgroup generated by orientation-preserving automorphisms. Since B contains reflections, C has index two in B, so $|C| = 30$. Since C contains the rotations about each of the 6 vertices, it has 24 elements of order 5 and hence is generated by these vertex rotations, which as elements of the symmetric group S_6 are even permutations. Thus $C \subset A_6$. Any involution in A_6 fixes two vertices u, v and hence the edge uv in H. Since $|C|$ is even, it has an involution, but no orientation-preserving automorphism can fix an edge. We conclude that M is not orientable.

We have shown that any K_4 subgraph H has either all angles measure $d/3$ or all measure $d/5, 2d/5$. Since any clique of maximal size m has many different K_4 subgraphs for $m > 4$, they cannot all have angles $d/3$ or $d/5, 2d/5$, so the only possibility for m is 4 or 6.

For $m = 4, 6$, we have described completely the m-cliques containing any vertex v and shown that each edge incident to m is in one and only one clique. Thus the m-cliques give a K_m-factorization of G. □

Orientably regular maps with underlying graph K_n have been studied for many years. By the work of Biggs [1] and James and Jones [7] (see also [11]), such maps only occur for $n = p^e$ for prime p and are in one-to-one correspondence with generators of the cyclic multiplicative group of the finite field $GF(p^e)$. With this information, it is not hard to show all such maps are chiral except for $n = 2^2$. The methods used are entirely algebraic. Theorem B is entirely geometric and provides:

Corollary 2. *Any orientably regular map with underlying graph $K_n, n > 4$, is chiral.*

Wilson [13] investigated non-orientable regular maps with underlying graph K_n. HIs main result again follows immediately from Theorem B:

Corollary 3. *The only non-orientable regular maps with underlying graph K_n are for $n = 3, 4, 6$.*

We have assumed that our underlying graph G has no multiple edges. On the other hand, multiple edges arise naturally in an algebraic treatment of maps, as in [3]. Note that loops in G are not an issue when M is regular: by the rotational symmetry at any vertex, if one edge is a loop, then all are, so M has only one vertex. Our result for clique numbers also applies to maps with multiple edges:

Theorem 1. *Let M be any regular map, possibly with multiple edges. Then the clique number of M is $2, 3, 4, 6$.*

Proof. Suppose that M is a regular map with multiple edges and automorphism group A. Let the cyclic order of edges incident to vertex v be $e_1, \cdots e_d$ and let the other endpoint of edge e_i be u_i, for $i = 1, \cdots d$; if there are multiple edges, the vertices $u_1, \cdots u_d$ are not all distinct. Let k be the smallest value such that $u_1 = u_k$. Then by the rotational symmetry about v, we have $u_{i+k} = u_i$ for all i, where subscripts are treated as residues mod d; moreover, $u_i \neq u_j$ if $|i - j| < k$. Let f be the automorphism that rotates about v by the angle of measure k (so f is a rotation about v of order d/k). Since $u_i = u_{i+k}$, then f fixes not only v but all vertices adjacent to v. In addition, since f take e_i at vertex u_i to e_{i+k} also at vertex u_i, we must have that f also performs a rotation by angle k (of order d/k) at all vertices adjacent to v. Thus f fixes all vertices and performs a rotation of order d/k about all vertices.

In particular, the subgroup B generated by f in A is normal, since it fixes all vertices and is normal in A_v for each vertex v. Thus the quotient map M/B is regular with $Aut(M/B) = A/B$. The underlying graph G/B for M/B has the same vertices as G, since f fixes all vertices, and each set of n/k multiple edges between v and u_i is identified to a single edge. In particular, the clique number of G/B is the same as the clique number of G. Since M/B is regular, that clique number is $2, 3, 4, 6$. □

Note that in the case of multiple edges, the edges incident to v in a particular clique H may not be symmetrically located around v, since we may choose any edge we want from each set of multiple edges.

There are infinitely many regular orientable maps with clique number 4, and their Petrie duals [11] give non-orientable regular maps. For example, the family:

$$\langle X, Y : X^{3n} = Y^{3n} = (XY)^2 = 1, X^{12}Y^{12} = 1\rangle,$$

from [3] gives regular maps where the underlying graph G is K_4 with each edge replaced by n multiple edges. A natural question to ask is whether there are infinitely many with simple underlying graphs. Computer evidence suggests the answer is yes (Conder, personal communication).

There are also infinitely many regular (necessarily non-orientable) maps with clique number 6. Again, from [3], the family:

$$\langle X, Y : X^{3n} = Y^{3n} = (XY)^2 = 1, X^{60}Y^{60} = 1\rangle,$$

gives orientably regular, reflexible maps where the underlying graph is the icosahedron with each edge replaced by n multiple edges. There is a natural antipodal automorphism (orientation-reversing involution fixing no vertices) such that the orbit map is regular, non-orientable, with underlying graph K_6 with each edge replaced by n multiple edges. Again, a natural question to ask is whether there are infinitely many with simple underlying graphs and the computer evidence suggests the answer is again yes (Conder, personal communication).

Theorem B also applies to graphs, rather than maps:

Theorem C. *Let G be a graph and $A \subset Aut(G)$ such that the action of each vertex stabilizer A_v on edges incident to v is naturally dihedral. Then the clique number of G is $m = 2, 3, 4, 6$. If $m \geq 3$, then A is vertex-transitive. If $m = 4, 6$, then G has a K_m factorization.*

Proof. Suppose that the clique number is at least 3. We claim that A is vertex-transitive. Indeed, by the dihedral actions of vertex stabilizers, the action of A is edge-transitive. Moreover, G has a triangle uvw, and the dihedral action of A_v reverses the edge uw. Thus for every edge there is an $a \in A$ reversing the edge, making A vertex-transitive.

We can then use A to define an angle measure at every vertex that is preserved by A. First, fix a vertex v and choose a generator b of the index-two cyclic subgroup B_v of A_v. Since all other vertex stabilizers are conjugate to A_v and B is characteristic in A_v, we can use conjugates of b to define a cyclic ordering around every vertex that is preserved by A, which can then be used to define angle measure.

The proof then proceeds in exactly the same way as for regular maps. □

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Buildings and s-Transitive Graphs

Richard M. Weiss

Abstract A graph is s-transitive if its automorphism group acts transitively on the set of paths of length s. This is a notion due to William Tutte who showed in 1947 that a finite trivalent graph can never be 6-transitive. We examine connections between the theory of s-transitive graphs and the classification of Moufang polygons, a class of graphs exhibiting "local" s-transitivity for large values of s. Moufang polygons are examples of buildings. Both of these notions were introduced by Jacques Tits in the study of algebraic groups. We give an overview of Tits' classification results in the theory of spherical buildings (which include the classification of Moufang polygons as a special case) and describe, in particular, the classification of finite buildings.

Keywords Building • s-transitive graph • Moufang polygon

Subject Classifications: 20E42, 51E24

1 Introduction

Jacques Tits' classification of spherical buildings [8], published in 1974, is one of the great accomplishments in group theory. Starting with only a Coxeter group and a few combinatorial/geometrical axioms, he succeeded with this result in characterizing a large class of simple groups which includes, as a special case, all the finite simple groups of Lie type of rank at least 3.

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In subsequent work (mainly [10] and [12]), it emerged that Tits' theory of spherical buildings can be described purely in terms of graph theory and that there are great advantages in taking this point of view. In these notes, we describe the main steps in the classification of spherical buildings in the language of graph theory and highlight a connection to the theory of s-transitive graphs which had been introduced earlier by W. T. Tutte in [13].

2 s-Transitive Graphs

A graph Γ is called s-transitive if its automorphism group acts transitively on the set of paths of length s in Γ but intransitively on the set of paths of length $s+1$ (in which case the valency of Γ must be greater than 2). The notion of an s-transitive graph was introduced and developed by Tutte in [13] and [14]. If a graph Γ is s-transitive and has girth γ (i.e. γ is the length of a shortest circuit in Γ), then

$$\gamma \geq 2s - 2.$$

Tutte defined a γ-cage to be a connected s-transitive graph of girth γ such that either $\gamma = 2s-2$ or $\gamma = 2s-1$. If Γ is a γ-cage, then the diameter n of Γ is $s-1$ and if γ is even, then

$$\Gamma \text{ is bipartite and } \gamma = 2n. \tag{1}$$

A complete graph on a set X is a 3-cage if $|X| > 3$ and the complete bipartite graph on a pair of sets X and Y is a 4-cage as long as $|X| = |Y| > 2$. The Petersen graph is a 5-cage.

Let V be a 3-dimensional right vector space over a field or skew-field K, let X be the set of 1-dimensional subspaces of V, let Y be the set of 2-dimensional subspaces of V and let E be the set of pairs $\{x, y\}$ such that $x \in X$, $y \in Y$ and $x \subset y$. The graph with vertex set $X \cup Y$ and edge set E is a 6-cage. It is also the incidence graph of the projective plane associated with V. If $|K| = 2$, this graph is called the Heawood graph.

Let $W = \{1, 2, 3, 4, 5, 6\}$, let X be the set of all 2-element subsets of W, let Y be the set of partitions $u/v/w$ of W into three blocks u, v and w of size 2 and let E be the set of pairs $\{x, u/v/w\}$ such that $x \in X$, $u/v/w \in Y$ and $x \in \{u, v, w\}$. The graph with vertex set $X \cup Y$ and edge set E is an 8-cage. This graph is often called Tutte's 8-cage.

Tutte showed in [13] that the only trivalent cages are the complete graph, the complete bipartite graph, the Petersen graph, the Heawood graph and his 8-cage. In fact, his proof showed much more (see [6]):

Theorem 1. *Let Γ be an arbitrary connected trivalent graph—even a tree—and let u be a vertex of Γ. Suppose that G is a group acting transitively on paths of* *length s in Γ for some $s \geq 1$ but not on paths of length $s+1$ and that the stabilizer G_u is finite. Then $s \leq 5$ and for each value of s, the structure of G_u is uniquely determined.*

This was a result of great originality and, in some sense, far ahead of its time. It has been generalized in a number of directions. One generalization (proved in [16] and [18]) is the following:

Theorem 2. *Let Γ be an arbitrary connected graph and let u be a vertex of Γ. Suppose that G is a group acting transitively on paths of length s in Γ for some $s \geq 4$ but not on paths of length $s+1$ and that the stabilizer G_u is finite. Then $s = 4$, 5 or 7 and for each value of s, the structure of G_u is uniquely determined in an appropriate sense.*

An important ingredient in the proof of Theorem 2 is the Theorem of Thompson-Wielandt. Here is the relevant version of this result; see [4] or [15] for a proof.

Theorem 3. *Let Γ be an arbitrary connected graph and let $\{u, v\}$ be an edge of Γ. Suppose that G is a group acting transitively on the vertex set of Γ and that the stabilizer G_u is finite and acts primitively on Γ_u, the set of neighbors of u in Γ. Then $|G_{u,v}^{[1]}|$ is divisible by at most one prime, where $G_{u,v}^{[1]}$ denotes the pointwise stabilizer in G of $\Gamma_u \cup \Gamma_v$.*

Here now is how the proof of Theorem 2 begins. Let $\{u, v\}$ be an edge of Γ. Replacing G by the free amalgamated product $G_u *_{G_{u,v}} G_v$, we can assume without loss of generality that Γ is a tree. Let H denote the permutation group induced by G_u on Γ_u. The hypotheses of Theorem 2 imply that the group H acts 2-transitively and hence primitively on Γ_u. It can also be deduced from the hypotheses that the subgroup $G_{u,v}^{[1]}$ is non-trivial. By Theorem 3, therefore, there is a unique prime p dividing the order of this subgroup. It follows from this that the stabilizer H_v of a vertex $v \in \Gamma_u$ has a non-trivial normal subgroup of order a power of p. At this point the classification of finite 2-transitive groups (which rests on the classification of finite simple groups) is invoked. From this it can be deduced that H contains a normal subgroup isomorphic to the group $PSL_2(q)$ in its natural action on $|\Gamma_u| = q + 1$ points. This is the conclusion reached in [18].

With this conclusion as an hypothesis, it is shown in [16] (see also [2, 3.6]) that there is a G-compatible local isomorphism φ from the tree Γ to a $2(s-1)$-cage $\hat{\Gamma}$. By 'local isomorphism' we mean that φ is a map from the vertex set of Γ to the vertex set of $\hat{\Gamma}$ such that if $\varphi(x) = \hat{x}$, then φ induces a bijection from Γ_x to $\hat{\Gamma}_{\hat{x}}$, and by 'G-compatible' we mean that if $\varphi(x) = \varphi(y)$, then $\varphi(x^g) = \varphi(y^g)$ for all $g \in G$. It follows that G induces a group of automorphisms $\hat{G}$ of $\hat{\Gamma}$ and that if $\hat{u} = \varphi(u)$, then φ induces an isomorphism from the stabilizer G_u to the stabilizer $\hat{G}_{\hat{u}}$. At this point, the proof is concluded by citing the classification of $2(s-1)$-cages. This is a special case of the classification of Moufang polygons which we describe in the next section.

3 Moufang Polygons

The following notion was introduced by Tits in [7].

Definition 1. A *generalized* n*-gon* is a bipartite graph Γ with diameter $n \geq 2$ and girth γ such that

$$\gamma = 2n.$$

A *generalized polygon* is a generalized n-gon for some $n \geq 2$. A *generalized triangle*, *quadrangle*, etc., is a generalized n-gon for $n = 3$, $n = 4$, etc.

By (1), a γ-cage for γ even is automatically a generalized n-gon for $n = \gamma/2$. Note, however, that there is no group in the definition of a generalized polygon.

We say that a graph Γ is *thick* if $|\Gamma_u| \geq 3$ for all vertices u, respectively, Γ is *thin* if $|\Gamma_u| = 2$, for all vertices u (so many graphs are neither thick nor thin).

Let Γ be a generalized n-gon for some $n \geq 2$. Since Γ is bipartite, it can be regarded as the incidence graph of a geometry $\mathcal{G}$ consisting of points X and lines Y. If Γ is thin, the geometry $\mathcal{G}$ is just the set of points and lines of an ordinary n-gon, hence the name generalized n-gon.

Suppose that $n = 2$ and choose $x \in X$ and $y \in Y$. Then the distance from x to y in Γ is odd because Γ is bipartite. Since the diameter of Γ is 2, it follows that x and y are adjacent in Γ. We conclude that Γ is a complete bipartite graph.

Suppose that $n = 3$ and choose distinct vertices x, y both in X. Then the distance from x to y is even and bounded by the diameter 3 of Γ. Hence there is a vertex $z \in Y$ adjacent to both x and y. Since the girth of Γ is $2n = 6$, the vertex z is unique. Thus any two points of the geometry $\mathcal{G}$ are incident with a unique line. By a similar argument, any two lines of $\mathcal{G}$ are incident with a unique point. We conclude that Γ is the incidence graph of a projective plane. Conversely, the incidence graph of an arbitrary projective plane is a thick generalized triangle. Thus a generalized triangle is essentially the same thing as a projective plane.

This means, in particular, that there is no hope of classifying generalized polygons. In the appendix of [8], however, Tits introduced the notion of a *Moufang polygon* and asked whether it might be possible to classify them.

Definition 2. A *root* of a generalized n-gon Γ is a path of length n. To each root $\alpha = (x_0, x_1, \ldots, x_n)$, there is a corresponding *root group* U_α, the pointwise stabilizer of $\Gamma_{x_1} \cup \cdots \cup \Gamma_{x_{n-1}}$ in $G := \mathrm{Aut}(\Gamma)$. Thus

$$U_\alpha = G^{[1]}_{x_1,\ldots,x_{n-1}}.$$

A generalized n-gon Γ is said to satisfy the *Moufang condition* if it is thick, $n \geq 3$ and for each root $\alpha = (x_0, x_1, \ldots, x_n)$ in Γ, the corresponding root group U_α acts transitively on $\Gamma_{x_n} \backslash \{x_{n-1}\}$. A *Moufang polygon* is a generalized polygon that satisfies the Moufang condition.

Moufang triangles, in the guise of Moufang projective planes, were first studied by Ruth Moufang in the 1930s, hence the name Moufang.

The automorphism group of a Moufang n-gon Γ does not necessarily act transitively on the vertex set of Γ. If it does, however, then Γ is automatically $(n+1)$-transitive. It follows that a Moufang n-gon is a $2n$-cage if and only if its automorphism group acts transitively on the vertex set of Γ. The $2(s-1)$-cage $\hat{\Gamma}$ at the end of the previous section is, in fact, a Moufang n-gon with $n = s-1$, and a more accurate statement of Theorem 2 (as proved in [16] and [18]) is as follows:

Theorem 4. *Let Γ be a thick tree and let u be a vertex of Γ. Suppose that G is a group acting transitively on paths of length s in Γ for some $s \geq 4$ but not on paths of length $s+1$ and that the stabilizer G_u is finite. Then there exists a Moufang $(s-1)$-gon $\hat{\Gamma}$, a G-compatible local isomorphism φ from Γ to $\hat{\Gamma}$ and a subgroup $\hat{G}$ of* $\mathrm{Aut}(\hat{\Gamma})$ *containing all the root groups of $\hat{\Gamma}$ such that $G_u \cong \hat{G}_{\varphi(u)}$ for all vertices u of Γ.*

Corollary 1. *Let Γ be a $2n$-cage for some $n \geq 2$. Then either $n = 2$ and Γ is a complete bipartite graph or $n \geq 3$ and Γ is a Moufang n-gon.*

The classification of Moufang polygons was carried out in [12]. We use the rest of this section to sketch how this classification works. The first step is the following result [9]:

Theorem 5. *Moufang n-gons exist only for $n = 3, 4, 6$ and 8.*

It was shown in [17] that, in fact, the following holds:

Theorem 6. *Let $n \geq 3$, let Γ be a thick graph and let G be a subgroup of* $\mathrm{Aut}(\Gamma)$ *such that*

1. $G^{[1]}_{x_1,\ldots,x_{n-1}}$ acts transitively on $\Gamma_{x_n} \backslash \{x_{n-1}\}$ and
2. $G^{[1]}_{x_0,x_1} \cap G_{x_0,\ldots,x_n} = 1$

for all paths $(x_0, x_1, \ldots, x_n)$ of length n in Γ. Then $n = 3, 4, 6$ and 8.

It is easy to see that condition (2) in Theorem 6 holds automatically if Γ is a generalized n-gon. Thus Theorem 5 is a corollary of Theorem 6. If Γ, G and s are as in Theorem 1 and $s \geq 4$, then the hypotheses of Theorem 6 hold with $n = s-1$. In contrast to Theorem 2, the proof of Theorem 6 does not depend on the classification of finite simple groups (or on anything else, for that matter). Note, too, that it is not assumed in Theorems 5 and 6 that the stabilizers are finite.

The next step in the classification of Moufang polygons is to choose a circuit Σ of length $2n$ in a Moufang n-gon Γ and label its vertices by integers modulo $2n$. We then let α_i denote the root $(i, i+1, \ldots, i+n)$ and let U_i denote the root group U_{α_i} as defined in Definition 2 for all i. We set

$$U_{[i,j]} = \langle U_i, U_{i+1}, \ldots, U_j \rangle$$

for all i, j such that $i \leq j$ and $U_{[i,j]} = 1$ if $j < i$. Let $W = U_{[1,n]}$. By [12, 4.11 and 5.3], the group W acts regularly on the set of circuits of length $2n$ containing the edge $\{n, n+1\}$. Since every two edges of Γ are contained in a circuit of length $2n$, it follows that every vertex of Γ is in the W-orbit of a unique vertex of Σ and every edge of Γ is in the W-orbit of a unique edge of Σ. By [12, 7.3], the stabilizers in W of the vertices $1, \ldots, n$ of Σ are

$$U_1, U_{[1,2]}, \ldots, U_{[1,n]}$$

and the stabilizers in W of the vertices $n+1, \ldots, 2n$ are

$$U_{[1,n]}, U_{[2,n]}, \ldots, U_n.$$

It follows from these observations that the graph Γ can be reconstructed from the cosets of these subgroups in W and hence that Γ can be reconstructed from the $(n+1)$-tuple

$$\Omega = (W, U_1, \ldots, U_n).$$

See [12, 7.5] for the details. We call the $(n+1)$-tuple Ω a *root group sequence* of Γ. It depends on the choice of Σ and its labeling, but it is independent of these choices up to conjugation and opposites. The *opposite* of the root group sequence Ω is the root group sequence

$$(W, U_n, \ldots, U_1).$$

Every element of W can be written uniquely as a product $a_1 a_2 \cdots a_n$ with $a_i \in U_i$ for all $i \in [1, n]$. Another crucial observation is that

$$[U_i, U_j] \subset U_{[i+1, j-1]} \tag{2}$$

for all i, j such that $1 \leq i < j \leq n$, where $[U_i, U_j]$ denotes the subgroup generated by the commutators $[a, b] = a^{-1}b^{-1}ab$ for all $a \in U_i$ and all $b \in U_j$. (Thus, in particular,

$$[U_i, U_{i+1}] = 1 \tag{3}$$

for all i such that $1 \leq i < n$.) The commutator relations (2) determine the structure of W uniquely. We conclude that to describe the root group sequence Ω and thus the graph Γ, it suffices to give the structure of the individual root groups $U_1, \ldots, U_n$ and the commutator relations (2). In each case, these things are given in terms of certain algebraic data.

Suppose first that $n = 3$. In this case the classification of Moufang polygons tells us that there is an invariant K of Γ that is either a field, a skew-field or an octonion division algebra and there exist isomorphisms x_i from the additive group of K to U_i for $i = 1, 2$ and 3 such that

$$[x_1(s), x_3(t)] = x_2(st)$$

for all $s, t \in K$. By (3), the root group sequence Ω and hence the generalized triangle Γ are uniquely determined by K. (See [12, 9.3–9.4 and 9.11] for the definitions of quaternion and octonion division algebras.)

There are six different families of Moufang quadrangles. In the first, there is a triple (K, L, q), where K is a field, L is a vector space over K and q is an anisotropic quadratic form on L, as well as isomorphisms x_i from the additive group of K to U_i for $i = 1$ and 3 and isomorphisms x_i from the additive group of L to U_i for $i = 2$ and 4 such that $[U_1, U_3] = 1$,

$$[x_1(s), x_4(v)^{-1}] = x_2(sv)x_3(sq(v)) \tag{4}$$

for all $s \in K$ and all $v \in L$ and

$$[x_2(u), x_4(v)^{-1}] = x_3(f(u, v)) \tag{5}$$

for all $u, v \in L$, where f is the bilinear form associated with q. By (3), Ω and hence Γ are uniquely determined by (K, L, q).

In the second family of Moufang quadrangles, there is a triple (K, K_0, σ), where K is a skew-field, σ is an involution of K (i.e. an anti-automorphism of order 2) and K_0 is an additive subgroup of K containing 1 such that $K_\sigma \subset K_0 \subset K^\sigma$, where

$$K_\sigma = \{a + a^\sigma \mid a \in K\}$$

and

$$K^\sigma = \{a \mid a^\sigma = a\},$$

and $s^\sigma K_0 s \subset K_0$ for all $s \in K$ as well as isomorphisms x_i from K_0 to U_i for $i = 1$ and 3 and isomorphisms x_i from the additive group of K to U_i for $i = 2$ and 4 such that $[U_1, U_3] = 1$,

$$[x_1(s), x_4(t)^{-1}] = x_2(st)x_3(t^\sigma st)$$

for all $s \in K_0$ and all $t \in K$ and

$$[x_2(r), x_4(t)^{-1}] = x_3(r^\sigma t + t^\sigma r)$$

for all $r, t \in K$. By (3), Ω and hence Γ are uniquely determined by (K, K_0, σ). If $\mathrm{char}(K) \neq 2$, then $a = (a + a^\sigma)/2$ for all $a \in K^\sigma$, so $K_\sigma = K^\sigma$, but if $\mathrm{char}(K) = 2$, this is not true.

The algebraic structures for the remaining Moufang quadrangles and the Moufang hexagons and octagons (which, by Theorem 5, is all there is) are more complicated. We refer the reader to Chapter 16 in [12] and all the references there to earlier chapters of [12] for details. The general pattern is, however, the same: In each case, there is a recipe that turns an algebraic structure of a suitable type via commutator relations into a root group sequence Ω from which, in turn, a unique Moufang polygon can be constructed using cosets.

Let $\Omega = (W, U_1, \ldots, U_n)$ be a root group system of a Moufang polygon Γ. Then $G = \mathrm{Aut}(\Gamma)$ acts transitively on the vertex set of Γ and hence Γ is a $2n$-cage if and only if there is an automorphism φ of W such that $U_i^{\varphi} = U_{n+1-i}$ for each $i \in [1, n]$, i.e. if and only if Ω is isomorphic to its opposite. If $n = 3$ and K is the invariant of Γ described above, then Γ is a 6-cage if and only if K is isomorphic to its opposite. In particular, Γ is always a 6-cage if K is a field or an octonion division algebra (which always has involutions). If Γ is the Moufang quadrangle determined by an anisotropic quadratic space (K, L, q), then Γ is an 8-cage non-zero, if the characteristic of K is 2, $L = K$ and $q(t) = t^2$ for each $t \in K$, and if K is the field with two elements, then Γ is, in fact, isomorphic to Tutte's 8-cage as described in Sect. 2 above. A complete list of Moufang polygons whose root group system is isomorphic to its opposite is given in [12, 37.5]. In particular, we can observe that there exist 6-, 8 and 12-cages but no 16-cages.

Tits discovered the notion of a generalized polygon and the Moufang condition by studying the structure of "absolutely simple" algebraic groups. In particular, every absolutely simple algebraic groups "of relative rank 2" has a Moufang polygon associated to it. Not all Moufang polygons come from absolutely simple algebraic groups, however. For example, a Moufang triangle defined by a skew-field K comes from an absolutely simple algebraic group if and only if K is finite-dimensional over its center.

An important notion in the theory of buildings is that of a (B, N)-pair. By the results in Chapter 5 of [5], a group G has a *spherical* (B, N)*-pair of rank* 2 if and only if

1. There is a generalized n-gon Γ for some $n \geq 2$ on which G acts; and
2. G acts transitively on the set of pairs (Σ, e), where Σ is circuit of length $2n$ in Γ and e is an edge of Σ.

Given (1), condition (2) holds if and only if for each vertex u of Γ, the group G acts transitively on the set of paths of length $n + 1$ in Γ beginning at u. In other words, G has a (B, N)-pair of rank 2 if and only if it acts *locally* $(n + 1)$*-transitively* on a generalized n-gon. If Γ is a Moufang polygon and G is a subgroup of $\mathrm{Aut}(\Gamma)$ containing all the root groups of Γ, then condition (2) holds. It does not follow from conditions (1) and (2), however, that Γ is Moufang.

In Sect. 5 we will describe the connection between Moufang polygons and buildings.

4 Coxeter Complexes

Before we can introduce buildings in the next section, we need to say a few things about Coxeter groups.

A *Coxeter diagram* is a graph Π whose edges are labeled with elements of the set $\{3, 4, \ldots, \infty\}$. Let Π be a Coxeter diagram with vertex set S and edge set E and for each edge $\{i, j\}$ of Π let m_{ij} be its label. Let $m_{ij} = 2$ for all 2-element subsets $\{i, j\}$ of V that are not in E, let $m_{ii} = 1$ for all $i \in S$, let R be a collection of symbols r_i, one for each $i \in S$, and let W denote the corresponding Coxeter group

$$\langle R \mid (r_i r_j)^{m_{ij}} = 1 \text{ for all } i, j \in S\rangle.$$

By [19, Thm. 2.3], the elements of R have order 2 and $r_i \neq r_j$ if $i \neq j$. In particular, we can identify R with its image in W.

Let Σ_Π denote the Cayley graph associated with this data. Thus Σ_Π is the graph whose vertices are the elements of W and whose edges are the 2-element subsets $\{x, y\}$ of W such that $x^{-1}y \in R$. We endow Σ_Π with the edge coloring obtained by assigning to each edge $\{x, y\}$ of W the color $i \in S$ if and only if $x^{-1}y = r_i$. Thus S is simultaneously the set of vertices of Π and the set of colors on the edges of Σ_Π. We call Σ_Π the *Coxeter complex* associated with Π. (Σ_Π is not, in fact, a complex, but it is easy to see that our definition of the Coxeter complex is equivalent to the more traditional notion of the Coxeter complex.)

The edge-colored graphs Σ_Π have many remarkable properties. We mention only one, a proof of which can be found in [19, 3.11]:

Proposition 1. *Let $\Sigma = \Sigma_\Pi$ be a Coxeter complex and let $\{x, y\}$ be an edge of Σ. Let α be the set of vertices nearer to x than to y in Σ and let β be the set of vertices nearer to y than to x. Then the vertex set W of Σ is partitioned by α and β. A* root *of Σ is a subset of W of the form α or β for some edge $\{x, y\}$.*

Suppose, for example, that $|S| = 2$ and $|W| < \infty$. In this case, W is generated by two elements of order 2 and $\Sigma = \Sigma_\Pi$ is a circuit of length $2n$ for some $n \geq 2$ with two alternating colors on its edges. Let e and f be an opposite pair of edges of Σ. The roots associated with e are the vertex sets of the two connected components of the graph obtained from Σ by deleting e and f (but without deleting any vertices). Note that if Σ is the edge-graph of a $2n$-circuit in a generalized $2n$-gon, then these two roots are roots in the sense of Definition 2.

A Coxeter diagram Π is called *spherical* if the corresponding Coxeter complex Σ_Π is finite. Coxeter himself classified spherical Coxeter diagrams. Their connected components are the Coxeter diagrams underlying Dynkin diagrams, all Coxeter diagrams with at most 2 vertices but without the label ∞, plus two more Coxeter diagrams, H_3 and H_4. These last two are the Coxeter diagrams obtained from the Coxeter diagrams B_3 and B_4 by replacing the unique label 4 by a 5.

5 Spherical Buildings

We fix an arbitrary Coxeter diagram Π and let S and Σ_Π be as in the previous section.

Suppose Δ is an arbitrary graph whose edges are colored by elements of the set S, one color per edge. For each subset J of S, we let Δ_J denote the graph Δ with all edges deleted whose color is *not* in J (but with no vertices deleted). A J*-residue* of Δ for some $J \subset S$ is a connected component of Δ_J. Thus for each subset J of S and each vertex x of Δ, there is a unique J-residue of Δ containing x. In particular, for a given J, any two J-residues of Δ are disjoint. A *residue* of Δ is a J-residue for some $J \subset S$.

A path $(u_0, u_1, \ldots, u_m)$ of length m in a graph Γ is called *minimal* if there is no path of length less than m in Γ from the vertex u_0 to the vertex u_m. We say that a subset X of the vertex set of a graph Γ is *convex in* Γ if for every two vertices x, y in X and every minimal path $(u_0, u_1, \ldots, u_m)$ from $u_0 = x$ to $u_m = y$, all the intermediate vertices $u_1, \ldots, u_{m-1}$ also lie in X.

There are several ways to define a building. Given [19, 4.2 and 8.9], it is not difficult to show that the following is equivalent to the definition given in [19, 7.1]:

Definition 3. A *building of type* Π (for a given Coxeter diagram Π with vertex set S) is a graph Δ whose edges are colored by elements of the set S satisfying the following properties:

1. The $\{i\}$-residues of Δ are complete graphs with at least 2 vertices for each $i \in S$.
2. Every two vertices of Δ are contained in some subgraph that is isomorphic to Σ_Π (as an edge-colored graph).
3. The vertex set of every subgraph isomorphic to Σ_Π is convex in Δ.

The subgraphs of Δ that are isomorphic to Σ_Π are called *apartments* of Δ. The $\{i\}$-residues for some $i \in S$ are called *panels* or i*-panels*.

Let Δ be a building of type Π. The vertices of Δ are, for historical reasons, called *chambers*. By condition (1) of Definition 3, J is precisely the set of colors that appear on the edges of an arbitrary J-residue. Thus J is an invariant of a J-residue. The set J is called the *type* of a J-residue (or of a J-panel) and the cardinality $|J|$ is called the *rank* of a J-residue. Thus panels are residues of rank 1. It follows from condition (2) in Definition 3 that Δ is connected and hence Δ itself is the only S-residue and its rank is $|S|$. If $|S| = 1$, then Δ consists of a single panel and thus Δ is just a complete graph with all its edges painted the same color.

Suppose that $|S| = 2$ and let n be the label on the one edge of the Coxeter diagram Π if there is an edge; otherwise let $n = 2$. Suppose first that $n < \infty$. In this case, an apartment of Δ is a circuit of length $2n$ whose edges display, alternatingly, the two colors in S. Let Γ be the graph with vertex set the set of panels of Δ, where two panels are adjacent in Γ if and only if they contain a chamber in common. Distinct panels of the same type are disjoint. Therefore Γ is a bipartite graph. Apartments of Δ correspond to circuits of length $2n$ in Γ. From condition (2) in

Definition 3, it follows that the diameter of Γ is at most n and from condition (3), that the girth of Γ is $2n$. This is enough to deduce that Γ is a generalized n-gon. This construction is also reversible. In other words, a building of rank 2 with finite apartments is essentially the same thing as a generalized polygon. If the label n equals ∞, on the other hand, then Δ is a tree and the apartments are the connected subgraphs of valency 2.

For an elementary example of a building of arbitrary rank r with finite apartments arising from a vector space of dimension $r + 1$, see [19, 7.4].

A building Δ is called *irreducible* if its Coxeter diagram Π is connected. All buildings are direct products of irreducible buildings in an appropriate sense, so it is, for most purposes, sufficient to study irreducible buildings. A building Δ is called *thick* if all its panels contain at least 3 chambers, *thin* if all its panels contain exactly 2 chambers. Thus Σ_Π is an example of a thin building of type Σ_Π and, in fact, it is the only thin building of type Σ_Π. A building Δ is called *spherical* if its diagram Π is spherical. In other words, a spherical building is a building whose apartments are finite.

Every J-residue of a building of type Π (for a given subset J of S) is itself a building. The type of this building is Π_J, where Π_J denotes the subdiagram of Π spanned by J. Thus it makes sense to say whether a residue is irreducible or spherical: A J-residue is irreducible if and only if the diagram Π_J is connected, and a J-residue is spherical if and only if the diagram Π_J is spherical. Thus every residue of a spherical building is spherical, but it is, of course, not true that every residue of an irreducible building is irreducible.

If $|J| = 2$, then a J-residue is irreducible if and only if the two elements of J are joined by an edge in Π and a J-residue is spherical if and only if the two elements of J are not joined by an edge labeled ∞.

For each chamber x of Δ, there is a unique irreducible residue of rank 2 containing x corresponding to each edge of Π. We denote by

$$\Delta_2(x) \tag{6}$$

the subgraph of Δ spanned by the union of the chamber sets of all these irreducible rank 2 residues.

Suppose that $|J| = 2$ and that Π_J is spherical. Let n be the label on the unique edge of Π_J if Π_J is connected; if Π_J is not connected, we set $n = 2$. Then every J-residue is a building of rank 2 and of type Π_J. Hence every J-residue is the building associated with a generalized n-gon, as explained above. In fact, we can simply think of these residues as generalize n-gons.

A *root* of a building Δ of type Π is the image under an isomorphism from Σ_Π into Δ of a root of Σ_Π as defined in Proposition 1. Thus a root is always contained in an apartment; in fact, roots are contained in many apartments as long as Δ is thick. For each root α of Δ, the *root group* U_α is the pointwise stabilizer in $\mathrm{Aut}(\Delta)$ of the set of chambers contained in some panel containing two chambers of α. A building is called *Moufang* if it is irreducible, thick, spherical and of rank at least 2 and if for each root α, the root group U_α acts transitively on the set of apartments of Δ that contain α. If $|S| = 2$, the notion of a root group and the Moufang property coincide with the notions introduced in Sect. 3.

The main result of [8] is a classification of thick, irreducible spherical buildings of rank at least 3. (We have already observed that buildings of rank 2, namely generalized polygons, include all projective planes and are not, therefore, in any reasonable sense classifiable without additional hypotheses.) This classification rests on the following deep result [8, 4.1.2] of Tits:

Theorem 7. *Let Δ and $\hat{\Delta}$ be two thick irreducible spherical buildings of the same type Π and let x and $\hat{x}$ be chambers of Δ and $\hat{\Delta}$. Suppose that φ is an isomorphism from $\Delta_2(x)$ to $\hat{\Delta}_2(\hat{x})$—as defined in (6)—mapping x to $\hat{x}$. Then φ extends to an isomorphism from Δ to $\hat{\Delta}$.*

In other words, a building Δ with all the adjectives in Theorem 7 is uniquely determined by the subgraph $\Delta_2(x)$. The following is an important consequence of Theorem 7:

Theorem 8. *Every thick irreducible spherical building of rank $n \geq 3$ as well as all the irreducible residues of rank at least 2 of such a building are Moufang.*

For a self-contained proof of these remarkable results carried out entirely in the language of graph theory, see [19, 10.2, 11.6 and 11.8].

Theorems 7 and 8 tell us that, given the classification of Moufang polygons, to classify thick irreducible spherical buildings of rank $n \geq 3$, it suffices for each spherical Coxeter diagram Π to examine how Moufang polygons, one for each edge of Π, can be assembled to form the subgraph $\Delta_2(x)$ of a building Δ. (Note that two rank 2 residues contained in $\Delta_2(x)$ overlap in a panel if their types have a nonempty intersection, so the various rank 2 residues really do have to be "assembled" to form a viable $\Delta_2(x)$.)

Suppose first that Π is the Coxeter diagram A_n with $n \geq 3$. Then all the irreducible rank 2 residues are Moufang triangles. It turns out that they must all be isomorphic to each other, that there is just one way to assemble them into a $\Delta_2(x)$ and that the division ring K defining these triangles must be a field or a skew-field (i.e. not an octonion division algebra). In other words, for each field or skew-field K, there is just one building of type A_n "defined over K." The situation is even simpler when Π is the one of the diagrams D_n, E_6, E_7 or E_8. In these cases, there is exactly one building of a given type Π for each *commutative* field K.

Suppose next that Π is the Coxeter diagram B_n (which is the same as the Coxeter diagram C_n) with $n \geq 3$. In this case, there is exactly one irreducible residue R_1 containing a given chamber x which is a generalized quadrangle and one irreducible rank 2 residue R_2 containing x and intersecting R_1 in a panel P. Suppose that R_1 is isomorphic to the quadrangle defined by an anisotropic quadratic space (K, L, q). It turns out that there is exactly one way to assemble a $\Delta_2(x)$ starting with the root group sequence

$$\Omega = (W, U_1, U_2, U_3, U_4) \tag{7}$$

determined by (K, L, q) and Eqs. (4) and (5) in its "standard orientation," in which $[U_1, U_3] = 1$. In the corresponding building, which we denote by $B_n(K, L, q)$, the root groups acting on the panel P are those parametrized by the additive group of K and all the other irreducible rank 2 residues containing the chamber x, including R_2, are isomorphic to the Moufang triangle defined by K. If, on the other hand, we use Ω^{op} rather than Ω to assemble a $\Delta_2(x)$, we obtain as second building of type B_n, which we call $C_n(K, L, q)$, in which the root groups acting on the panel P are those parametrized by the additive group of L, but this second building $C_n(K, L, q)$ exists only when (K, L, q) is in one of the following cases:

1. L is a field of characteristic 2, K is a subfield containing L^2 and $q(x) = x^2$ for all $x \in L$.
2. $L = K$ and $q(x) = x^2$ for all $x \in L$.
3. L/K is a separable quadratic extension and q is the norm of this extension.
4. L is a quaternion division algebra with center K and q is the reduced norm of L.
5. L is an octonion division algebra with center K and q is the reduced norm of L.

In each of these five cases, all the irreducible rank 2 residues of $C_n(K, L, q)$ containing x other than R_1, including R_2, are isomorphic to the Moufang triangle defined by L. Furthermore, case (5) can only occur when $n = 3$, there is a unique building $F_4(\Lambda)$ of type F_4 for each anisotropic quadratic space $\Lambda = (K, L, q)$ in one of the cases (1)–(5), the building $F_4(\Lambda)$ has residues isomorphic to $B_3(K, L, q)$ and others isomorphic to $C_3(K, L, q)$, and these are the only buildings of type F_4. For some anisotropic quadratic spaces $\Lambda = (K, L, q)$, the root group sequences Ω and Ω^{op} are isomorphic. In these cases, Λ is in case (1) or (2), $\text{char}(K) = 2$ and $C_n(\Lambda)$ is isomorphic to $B_n(\Lambda)$. In general, however, $B_n(\Lambda)$ and $C_n(\Lambda)$ are different.

There is just one further family of buildings of type B_n for $n \geq 3$. The buildings in this family are defined by pseudo-quadratic spaces (possible of dimension $m = 0$) over a skew-field with involution. This is the family in [20, 30.14(iv)] for $m = 0$ and [20, 30.14(vii)] for $m > 0$. See also Chapter 11 of [12].

This completes the list of thick irreducible spherical buildings of rank $n \geq 3$. In particular, there are no thick buildings of type H_3 and H_4. If there were, then by Theorem 8, there would be residues which are Moufang 5-gons, but by Theorem 5, no such Moufang polygons exist.

When he published [8], Tits still had no proof that thick buildings of type H_3 or H_4 do not exist. A short time later, he introduced the Moufang property and showed that there are no Moufang pentagons precisely in order to eliminate these two diagrams. Carrying this out, Tits noticed that he could extend his methods to yield Theorem 5 and it was this success that led him to conjecture that Moufang polygons could, in fact, be classified.

In every case, the relevant algebraic structure is defined in terms of a field or a skew-field or an octonion division algebra K. The ring K is an invariant of the corresponding building Δ. We call it the *defining field* of Δ, even though it is not always a field. (This does not coincide entirely with the notion of defining field as it is used in the theory of algebraic groups.) See [20, 30.29–30.31] for details. We will refer to the defining field K in Sect. 7.

6 Finite Buildings

Let Δ be a thick irreducible spherical building of rank $n \geq 2$ and suppose that Δ is Moufang if $n = 2$. (By Theorem 8, the Moufang property is automatic if $n > 2$.) Let $G^\dagger$ be the subgroup of $\mathrm{Aut}(\Delta)$ generated by all the root groups of Δ. With only three exceptions, $G^\dagger$ is a simple group. In the three exceptions, $n = 2$ and $G^\dagger$ is isomorphic to $\mathrm{PSp}_4(2)$, ${}^2F_4(2)$ or $G_2(2)$.

Now suppose that G is an arbitrary non-abelian simple group. Then the classification of finite simple groups tells us that one of the following holds:

1. $G \cong \mathrm{PSL}_2(q)$, $\mathrm{U}_3(q)$, $\mathrm{Suz}(q)$ or $\mathrm{Ree}(q)$ for some prime power q.
2. G is isomorphic to the group $G^\dagger$ generated by all the root groups for some finite thick irreducible building of rank $n \geq 2$ satisfying the Moufang condition if $n = 2$.
3. G is an alternating group or one of the 26 sporadic groups.

Each group in case (1) is a 2-transitive permutation group on a set X having a conjugacy class of subgroups $\{U_x \mid x \in X\}$ such that for each $x \in X$, U_x fixes x and acts sharply transitively on $X \backslash \{x\}$. A permutation group satisfying these properties is called a *Moufang set*. Moufang sets have not been classified without the assumption of finiteness. Although considerable progress toward a classification of arbitrary Moufang sets has been made in recent years (see, in particular, [3]), this seems to be a very difficult problem.

The finite simple groups in cases (1) and (2) are the groups of *Lie type*. More precisely, the groups in case (1) are the groups of Lie type *of rank* $n = 1$ and those in case (2) are the groups of Lie type *of rank* n. The groups in case (1) also have associated buildings, but these buildings are of rank 1. As we observed in the previous section, buildings of rank 1 are simply complete graphs. The underlying set of chambers of this building is precisely the set X on which the group forms a Moufang set.

A finite building has, of course, finite apartments and is thus automatically spherical. As we have seen in the previous section, thick irreducible buildings of rank $n \geq 2$ satisfying the Moufang condition if $n = 2$ can all be described in terms of suitable algebraic data. The assumption that the building (and hence the algebraic data) is finite imposes severe restrictions on the types of algebraic structures that can occur, as we now explain.

Suppose, to start, that the Coxeter diagram Π of our building Δ is A_n, D_n or E_n for $n = 6$, 7 or 8. There are no finite non-commutative skew-fields, no finite octonion division algebras and just one field $\mathbb{F}_q$ for each prime power q. Thus Δ is uniquely determined by Π and a prime power q. The corresponding simple groups $G^\dagger$ are

$$A_n(q) = \mathrm{PSL}_{n+1}(q),\ D_n(q) = \mathrm{O}_{2n}^{+}(q) \text{ and } E_n(q).$$

Suppose that $\Pi = F_4$. We saw in the last section that Δ is determined by an anisotropic quadratic space (K, L, Q) in one of five cases (1)–(5). (We use an uppercase Q here rather than q for the quadratic form, since in this section q is always a prime power.) In fact, Δ depends only on the similarity class of this quadratic space. By finiteness $K = \mathbb{F}_q$ for some prime power q. Furthermore, there are only two anisotropic quadratic spaces over $\mathbb{F}_q$ up to similarity: either $L = K$ and $Q(x) = x^2$ for all $x \in L$ or $L = \mathbb{F}_{q^2}$ and Q is the norm of the extension L/K. Each of them does, in fact, give rise to a unique building of type F_4. The corresponding simple groups are

$$F_4(q) \text{ and } {}^2E_6(q^2).$$

(The superscript in the name ${}^2E_6(q^2)$ indicates that this group can be constructed as the centralizer in the group $E_6(q^2)$ of an outer automorphism of order 2 that involves the element of order 2 in the Galois group of the extension L/K. In fact, we can interpret the rank 1 groups $U_3(q)$, $\mathrm{Suz}(q)$ and $\mathrm{Ree}(q)$ analogously as ${}^2A_2(q^2)$, ${}^2B_2(q)$ and ${}^2G_2(q)$, where $B_2(q)$ and $G_2(q)$ are as described below.)

Suppose that $\Pi = G_2$. The relevant algebraic structure in this case is a "quadratic Jordan division algebra of degree 3;" see Chapter 15 of [12] for details. Let J be the underlying vector space over a field K of one of these algebras. In the finite case, $K = \mathbb{F}_q$ for some prime power q, either $J = K$ or $J = \mathbb{F}_{q^3}$ and in both cases the Jordan algebra is uniquely determined by q. The corresponding simple groups are

$$G_2(q) \text{ and } {}^3D_4(q^3).$$

Suppose that $n = 2$ and that the Moufang polygon associated with Δ is an octagon. In this case, Δ is defined by a pair (K, σ), where K is a field of characteristic 2 and σ is an endomorphism of K whose square is the Frobenius map $x \mapsto x^2$. In the finite case, K is the field with $q = 2^m$ elements for some odd m and for each odd m, the endomorphism σ is unique. The corresponding simple group is

$${}^2F_4(q).$$

Suppose next that Δ is one of the two buildings $B_n(K, L, Q)$ or $C_n(K, L, Q)$ for some anisotropic quadratic space (K, L, Q) described in Sect. 5. Then $K = \mathbb{F}_q$ for some prime power q and, as in the case F_4, either $L = K$ and $Q(x) = x^2$ or $L = \mathbb{F}_{q^2}$ and Q is the norm of the extension L/K. If $|L| = q$, the simple groups coming from $B_n(K, L, Q)$ and $C_n(K, L, Q)$ are

$$\mathrm{O}_{2n+1}(q) \text{ and } \mathrm{PSp}_{2n}(q).$$

and when $|L| = q^2$, they are

$$\mathrm{O}^-_{2n+2}(q) = {}^2D_{n+1}(q^2) \text{ and } \mathrm{U}_{2n}(q) = {}^2A_{2n-1}(q^2).$$

If $L = K$ and $\mathrm{char}(K) = 2$, then the root group sequence Ω in (7) is isomorphic to its opposite and hence $B_n(K, L, Q)$ is isomorphic to $C_n(K, L, Q)$. Therefore $\mathrm{O}_{2n+1}(q) \cong \mathrm{PSp}_{2n}(q)$ for q even. When $n = 2$, this is the group referred to as $B_2(q)$ above.

There is just one more family of finite buildings with Coxeter diagram $\Pi = B_n$, one for each prime power q. These are the buildings described in [20, 30.14(vii)] that are finite. They yield the simple groups

$$\mathrm{U}_{2n+1}(q) = {}^2A_{2n}(q^2).$$

These are the only finite buildings of type B_n not all of whose root groups are abelian. Let q be a prime power and let Δ be the corresponding building. The non-abelian root groups of Δ can be described as follows. Let L/K be the unique quadratic extension with $|K| = q$, let N be the norm of this extension, let σ be the unique non-trivial element in $\mathrm{Gal}(L/K)$, let $\beta = \alpha - \alpha^\sigma$ for some $\alpha \in L$ not contained in K and let S be the set $\{(a,b) \in L \times L \mid \alpha N(a) - b \in K\}$ endowed with the multiplication

$$(a,b) \cdot (c,d) = (a + c, b + d + \beta a^\sigma c).$$

Then S is a group of order q^3 whose center is $(0, K)$, where

$$(a,b)^{-1} = (-a, -b^\sigma)$$

for all $(a,b) \in S$. Every root group of Δ is isomorphic either to S or to the additive group of K.

7 Affine Buildings

Another celebrated result of Tits is his classification of affine buildings in [11]. A connected Coxeter diagram is called affine if it is the Coxeter diagram $\tilde{X}_n$ underlying the extended Dynkin diagram attached to the Dynkin diagram X_n for $X = A, B, C, D, E, F$ or G and for some $n \geq 1$, and a building with Coxeter diagram Π is affine if each connected component of Π is affine. (See the figure on page 1 of [20].)

Suppose that Ξ is a thick irreducible affine building. Thus its type Π is $\tilde{X}_n$ for some X and n. The apartments of Ξ have a natural embedding into a Euclidean (or affine) space of dimension n. For this reason, affine buildings are sometimes called Euclidean buildings. The principle structural feature of the building Ξ (apart from its apartments and residues) is its *building at infinity*, $\Delta := \Xi^\infty$. The building Δ is thick and of type X_n and thus spherical and irreducible. (If $X_n = A_1$, then Ξ is a tree, Δ, a building of rank 1, is the set of ends of this tree and the apartments of Ξ, which are the connected subgraphs of valency 2, can be thought of as Euclidean spaces of dimension 1.) We call Ξ a *Bruhat-Tits* building if its building at infinity Δ satisfies the Moufang condition (which requires that $n \geq 2$). These are the buildings studied systematically in [1]. By Theorem 8, Ξ is automatically a Bruhat-Tits building if $n \geq 3$, but not if $n = 2$. What Tits showed is that Ξ is uniquely determined by Δ if $n \geq 2$ and that a given thick, irreducible spherical building satisfying the Moufang condition is the building at infinity of a Bruhat-Tits buildings if and only if the defining field K of Δ as defined at the end of Sect. 5 above is complete with respect to a discrete valuation. (This statement is not quite accurate if the algebraic data determining Δ is infinite dimensional or in certain other ways "exotic", but it is not so far from begin accurate. See Chapter 27 of [20] for exact statements.) Thus the theory of Bruhat-Tits buildings brings out deep connections not just between group theory and geometry but number theory as well.

Of particular interest both to number theorists and to differential geometers is the case that Ξ is locally finite (in the usual sense of graph theory). This corresponds to the special case that the field K is a local field, by which we mean not only that K is complete with respect to a discrete valuation, but also that the residue field of K is finite. Every local field is a finite extension of a p-adic field for some prime p or a field of Laurent series over a finite field. In the locally finite case, it is possible to carry out a precise classification of all the possible algebraic structures that can occur in the spirit of the previous section. For example, if (K, L, q) is an anisotropic quadratic space and K is a local field, then $\dim_K L \leq 4$ and if $\dim_K L = 4$, then q is the reduced norm of a quaternion division algebra with center K and, furthermore, there is only one such quaternion division algebra. See Chapter 28 of [20] for all the details.

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