Alternating Mathieu Series, Hilbert–Eisenstein Series and Their Generalized Omega Functions

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Dedicated to Professor Hari M. Srivastava

Abstract In this paper our aim is to generalize the complete Butzer–Flocke–Hauss (BFH) Ω-function in a natural way by using two approaches. Firstly, we introduce the generalized Omega function via alternating generalized Mathieu series by imposing Bessel function of the first kind of arbitrary order as the kernel function instead of the original cosine function in the integral definition of the Ω. We also study the following set of questions about generalized BFH Ω_ν-function: **(i)** two different sets of bounding inequalities by certain bounds upon the kernel Bessel function; **(ii)** linear ordinary differential equation of which particular solution is the newly introduced Ω_ν-function, and by virtue of the Čaplygin comparison theorem another set of bounding inequalities are given.

In the second main part of this paper we introduce another extension of BFH Omega function as the counterpart of generalized BFH function in terms of the positive integer order Hilbert–Eisenstein (HE) series. In this study we realize by exposing basic analytical properties, recurrence identities and integral representation formulae of Hilbert–Eisenstein series. Series expansion of these generalized BFH functions is obtained in terms of Gaussian hypergeometric function and some bridges are derived between Hilbert–Eisenstein series and alternating generalized Mathieu series. Finally, we expose a Turán-type inequality for the HE series $\mathfrak{h}_r(w)$.

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1 Invitation to Ω-Function and Alternating Mathieu Series

The *complex-index* Euler function $\mathfrak{E}_\alpha(z)$ is defined by [6, Definition 2.1]

$$\mathfrak{E}_\alpha(z) := \frac{\Gamma(\alpha+1)}{\pi \mathrm{i}} \int_{\mathfrak{C}_r} \frac{\mathrm{e}^{zu}}{\mathrm{e}^u + 1}\, u^{-\alpha-1}\, \mathrm{d}u \qquad \alpha \in \mathbb{C};\ z \in \mathbb{C} \setminus \mathbb{R}_0^-,$$

where $\mathfrak{C}_r$ denotes the positively oriented loop around the negative real axis $\mathbb{R}^-$, which is composed of a circle $C(0;r)$ centered at the origin and of radius $r \in (0,\pi)$ together with the lower and upper edges C_1 and C_2 of the complex plane cut along the negative real axis.

The *complex-index* Bernoulli function $\mathfrak{B}_\alpha(z)$ is given by [4, Definition 2.3(a)]

$$\mathfrak{B}_\alpha(z) := \frac{\Gamma(\alpha+1)}{2\pi \mathrm{i}} \int_{\mathfrak{C}_\rho} \frac{\mathrm{e}^{zv}}{\mathrm{e}^v - 1}\, v^{-\alpha}\, \mathrm{d}v, \qquad \alpha \in \mathbb{C};\ z \in \mathbb{C} \setminus \mathbb{R}_0^-.$$

Here, $\mathfrak{C}_\rho$ denotes the same shape integration contour as above with $\rho \in (0, 2\pi)$. For the connections between these two functions by way of their Hilbert transforms $\mathfrak{E}_\alpha^\sim(z)$ and $\mathfrak{B}_\alpha^\sim(z)$, the interested reader is referred to [4].

Almost twenty years ago, in their investigation of the complex-index Euler function $E_\alpha(z)$, Butzer, Flocke and Hauss [6] introduced the following special function:

$$\Omega(w) = 2\int_{0+}^{\frac{1}{2}} \sinh(wu)\cot(\pi u)\, \mathrm{d}u, \qquad w \in \mathbb{C},$$

which they called the *complete Omega function* [4, Definition 7.1], [6]. On the other hand, in view of the definition of the Hilbert transform, the complete Omega function $\Omega(w)$ is the Hilbert transform $\mathscr{H}[\mathrm{e}^{-wx}]_1(0)$ at 0 of the 1-periodic function $(\mathrm{e}^{-wx})_1$ defined by the periodic continuation of the following exponential function [4, p. 67]:

$$\mathrm{e}^{-x\,w}, \qquad |x| \leqslant \tfrac{1}{2};\ w \in \mathbb{C},$$

that is,

$$\mathscr{H}[\mathrm{e}^{-\cdot\, w}]_1(0) := \mathrm{PV}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{wu}\cot(\pi u)\, \mathrm{d}u = \Omega(w), \qquad w \in \mathbb{C},$$

where the integral is to be understood in the sense of Cauchy's principal value (PV) at zero. The highly important role of the Omega function in deep considerations and generating-function description of the Euler and Bernoulli functions was pointed out rather precisely by Butzer and his collaborators in their recent investigations [4–6]. There is also a basic association of the Omega function $\Omega(x)$ with the

Eisenstein series for circular functions and Hilbert–Eisenstein series introduced by Hauss [23]. For this matter as well as for some related open conjectures, see the work by Butzer [4, Sect. 9]. Further integral representations have been derived for the complete, real argument Omega function. So we mention the result by Butzer, Pogány and Srivastava [8, Theorem 2]:

$$\Omega(x) = \frac{2}{\pi} \sinh\left(\frac{x}{2}\right) \int_0^\infty \cos\left(\frac{xt}{2\pi}\right) \frac{\mathrm{d}t}{\mathrm{e}^t + 1}, \qquad x \in \mathbb{R}. \tag{1}$$

Tomovski and Pogány [46, Theorem 3.3] proved that

$$\Omega(x) = 2\sqrt{\frac{2}{\pi}} \sinh\left(\frac{x}{2}\right) \mathrm{PV} \int_0^\infty \sinh\left(\frac{xt}{\pi}\right) \tan t \,\mathrm{d}t, \qquad x \in \mathbb{R}. \tag{2}$$

Also, for the sake of completeness, we mention the Pogány–Srivastava integral representation [39, p. 589, Theorem 1]:

$$\Omega(x) = 16\pi^3 \sinh\left(\frac{x}{2}\right) \int_1^\infty \frac{\sin^2\left(\frac{\pi}{2}[\sqrt{u}]\right) - [\sqrt{u}]\cos\left(\pi[\sqrt{u}]\right)}{(4\pi^2 u + x^2)^2}\,\mathrm{d}u, \tag{3}$$

for all $x \in \mathbb{R}$. Here $[a]$ stands for the integer part of some real a. Secondly, bounding inequalities have been established for $\Omega(x)$ Butzer, Pogány and Srivastava [8, Theorem 3]:

$$\frac{1}{\pi}\sinh\left(\frac{x}{2}\right) \log\left(\frac{\zeta(3)x^2 + 8\pi^2}{3x^2 + 2\pi^2}\right) \leqslant \Omega(x) \leqslant \frac{1}{\pi}\sinh\left(\frac{x}{2}\right) \log\left(\frac{3x^2 + 8\pi^2}{\zeta(3)x^2 + 2\pi^2}\right).$$

The above inequalities are valid for $x > 0$, and for $x < 0$, the opposite inequalities hold true. Here $\zeta(3) = 1.20205690\ldots$ stands for the celebrated Apéry's constant. Thus, the Omega function behaves asymptotically like

$$\left(\frac{1}{\pi}\log\frac{\zeta(3)}{3}\right)\cdot \mathrm{e}^{\frac{x}{2}} \leqslant \Omega(x) \leqslant \left(\frac{1}{\pi}\log\frac{3}{\zeta(3)}\right)\cdot \mathrm{e}^{\frac{x}{2}}, \qquad x \to \infty.$$

Applying the Čaplygin comparison theorem [9, 10, 34], Pogány and Srivastava [39] obtained a bilateral bounding inequality for Ω function, by means of a linear ODE given earlier in [8]. Finally, different types of bounding inequalities were established by Alzer, Brenner and Ruehr; Draščić and Pogány; Mortici; Pogány, Srivastava and Tomovski; and others for the so-called generalized Mathieu series (see [41] and the references therein). Employing also the Čaplygin comparison theorem, Pogány, Tomovski and Leškovski [41] established very recently a set of bilateral inequalities for the real parameter complete Ω function considering the so-called alternating generalized Mathieu series' bounds; see [41] and the exhaustive companion references list.

The Omega function possesses an elegant and useful partial fraction representation [4, Theorem 1.3], [5, Theorem 1.24]:

$$\frac{\pi \Omega(2\pi w)}{\sinh(\pi w)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2n}{n^2 + w^2}, \qquad w \in \mathbb{C} \setminus \mathrm{i}\mathbb{Z}. \tag{4}$$

On the other hand, the generalized Mathieu series was introduced by Guo [21] in the form

$$S_\nu(r) = \sum_{n\geqslant 1} \frac{2n}{(n^2 + r^2)^\nu}, \qquad \nu > 1,\ r > 0,$$

posing the problem as to whether there is an integral representation for $S_\nu(r)$. The problem was solved by Cerone and Lenard [14, Theorem 1], who gave the integral representation

$$S_\nu(r) = \frac{\sqrt{\pi}}{(2r)^{\nu-\frac{3}{2}}\, \Gamma(\nu)} \int_0^\infty \frac{t^{\nu-\frac{1}{2}}}{\mathrm{e}^t - 1}\, J_{\nu-\frac{3}{2}}(rt)\, \mathrm{d}t, \qquad r > 0,\ \nu > 1.$$

In [40] Pogány, Srivastava and Tomovski proved that the alternating generalized Mathieu series

$$\tilde{S}_\nu(r) = \sum_{n\geqslant 1} \frac{(-1)^{n-1}\, 2n}{(n^2 + r^2)^\nu}, \qquad \nu > 0,\ r > 0, \tag{5}$$

posses the integral representation formula

$$\tilde{S}_\nu(r) = \frac{\sqrt{\pi}}{(2r)^{\nu-\frac{3}{2}}\, \Gamma(\nu)} \int_0^\infty \frac{t^{\nu-1/2}}{\mathrm{e}^t + 1}\, J_{\nu-\frac{3}{2}}(rt)\, \mathrm{d}t, \qquad r > 0,\ \nu > 0, \tag{6}$$

where J_a stands for the Bessel function of the first kind of order a. Obviously, letting here $\nu \to 1$, which, in view of the following relationship [1, p. 202, Eq. (4.6.4)]:

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, \cos z,$$

leads us easily to the integral representation (1). A number of other fashion integral representations for S_2, $\tilde{S}_2$ and for S_3 have been presented by Choi and Srivastava in [15, pp. 865–866, Theorem 3, Corollary 2]. The background of the relation (6) is very interesting. Namely, consider the classical Gegenbauer's formula [19] (see also [20, p. 712], [47, p. 386, Eq. 13.(6)]):

$$\int_0^\infty \mathrm{e}^{-\alpha x} x^{\mu+1} J_\nu\big(\beta x\big)\mathrm{d}x = \frac{2\alpha(2\beta)^\mu\, \Gamma\big(\mu + \frac{3}{2}\big)}{\sqrt{\pi}\, (\alpha^2 + \beta^2)^{\mu+3/2}}, \tag{7}$$

which is valid for all $\mathrm{Re}\{\mu\} > -1, \mathrm{Re}\{\alpha\} > \big|\mathrm{Im}\{\beta\}\big|$. Setting $\alpha = n,\ \mu = \nu - \frac{3}{2}$ and $\beta = r$ in (7), multiplying this relation with $(-1)^{n-1}$ and then summing up with respect to $n \in \mathbb{N}$, we clearly arrive at (6). But, noting (4) and Sect. 4 below,

$$\frac{\pi\,\Omega(2\pi x)}{\sinh(\pi x)} = \tilde{S}_1(x) = 2\int_0^\infty \frac{\cos(xt)}{\mathrm{e}^t + 1}\,\mathrm{d}t, \tag{8}$$

the integral representation only being valid for $w = x$ real therefore now follows (1) when we replace $2\pi x \mapsto x$.

Let us now introduce a new generalized Omega function, namely, $\Omega_\nu(\cdot)$, defined in terms of

$$\frac{\pi\,\Omega_\nu(2\pi w)}{\sinh(\pi w)} = \tilde{S}_\nu(w), \qquad w \in \mathbb{C}\setminus \mathrm{i}\mathbb{Z}\,, \tag{9}$$

an extensive counterpart of (8).

In order to apply the foregoing results of Pogány, Srivastava and Tomovski [40], we here need to restrict ourselves to $w = x \in \mathbb{R}$. Thus, (6) gives an analytic definition in matters of (9) as

$$\Omega_\nu(x) = \frac{\pi^{\nu-2}}{\Gamma(\nu)\,x^{\nu-\frac{3}{2}}}\,\sinh\left(\frac{x}{2}\right)\int_0^\infty \frac{t^{\nu-1/2}}{\mathrm{e}^t+1}\,J_{\nu-\frac{3}{2}}\left(\frac{xt}{2\pi}\right)\mathrm{d}t, \quad \Omega_1(x) \equiv \Omega(x)\,, \tag{10}$$

where $x \neq 0, \nu > 0$. In what follows, we call Ω_ν the *complete generalized* BFH *Omega function of the order* ν.

2 Bounds for Ω_ν by Using Results on J_μ

The main purpose here is to establish a bounding inequality of Ω_ν in terms of J_μ. Rearranging the integral representation (10) of the complete generalized BFH Omega function of the order ν, we deduce

$$\begin{aligned}\big|\Omega_\nu(x)\big| &\leqslant \frac{\pi^{\nu-2}}{\Gamma(\nu)\,|x|^{\nu-\frac{3}{2}}}\left|\sinh\left(\frac{x}{2}\right)\right|\int_0^\infty \frac{t^{\nu-1/2}}{\mathrm{e}^t+1}\left|J_{\nu-\frac{3}{2}}\left(\frac{xt}{2\pi}\right)\right|\mathrm{d}t \\ &= \frac{2^{\nu+\frac{1}{2}}\,\pi^{2\nu-\frac{3}{2}}}{\Gamma(\nu)\,|x|^{2\nu-1}}\left|\sinh\left(\frac{x}{2}\right)\right|\int_0^\infty \frac{t^{\nu-1/2}}{\mathrm{e}^{2\pi t/x}+1}\left|J_{\nu-\frac{3}{2}}(t)\right|\mathrm{d}t\,. \end{aligned} \tag{11}$$

Now, we are confronted with the problem of bounding $\big|J_\mu\big|$ by certain sharp bound on the positive real half-axis. Firstly, we will inspect the appropriate bounds' literature. Fortunately, there are numerous suitable bounds for the modulus of the Bessel function of the first kind, like Hansen's, valid for positive integral order Bessel function [22, pp. 107 et seq.], [47, p. 31]

$$|J_0(t)| \leqslant 1, \quad |J_r(t)| \leqslant \frac{1}{\sqrt{2}}, \qquad r \in \mathbb{N},\, t \in \mathbb{R}\,; \tag{12}$$

von Lommel extended [33, pp. 548–549], [47, p. 406] this results to

$$|J_\mu(t)| \leqslant 1, \quad |J_{\mu+1}(t)| \leqslant \frac{1}{\sqrt{2}}, \qquad \mu > 0,\, t \in \mathbb{R}\,. \tag{13}$$

Simple, efficient bounding inequality was proved by Minakshisundaram and Szász [35, p. 37, Corollary]:

$$|J_\mu(x)| \leqslant \frac{1}{\Gamma(\mu+1)} \left(\frac{|x|}{2}\right)^\mu, \qquad x \in \mathbb{R}\,; \tag{14}$$

obviously, this bound reduces Hansen's for $\mu = 0$.

More sophisticated bounds were given by Landau [31], who gave in a sense best possible bounds for the first kind Bessel function $J_\nu(x)$ with respect to ν and x, which read as follows:

$$|J_\mu(x)| \leqslant b_L\, \mu^{-1/3}, \qquad b_L = \sqrt[3]{2} \sup_{x \in \mathbb{R}_+} \mathrm{Ai}(x), \tag{15}$$

$$|J_\mu(x)| \leqslant c_L\, |x|^{-1/3}, \qquad c_L = \sup_{x \in \mathbb{R}_+} x^{1/3} J_0(x)\,, \tag{16}$$

where $\mathrm{Ai}(\cdot)$ stands for the familiar Airy function

$$\mathrm{Ai}(x) := \frac{\pi}{3}\sqrt{\frac{x}{3}}\Big(J_{-1/3}\{2(x/3)^{3/2}\} + J_{1/3}\{2(x/3)^{3/2}\}\Big)\,.$$

In fact Krasikov [30] pointed out that these bounds are sharp only in the transition region, i.e. for x around $j_{\mu,1}$, the first positive zero of $J_\mu(x)$.

In his recent article Olenko [37, Theorem 1] established the following sharp upper bound:

$$\sup_{x \geqslant 0} \sqrt{x}|J_\mu(x)| \leqslant b_L \sqrt{\mu^{1/3} + \frac{\alpha_1}{\mu^{1/3}} + \frac{3\alpha_1^2}{10\mu}} := d_O, \qquad \mu > 0\,, \tag{17}$$

where α_1 is the smallest positive zero of the Airy function $\mathrm{Ai}(x)$ and b_L is the Landau constant in (15). For further reading and detailed discussion, consult [37, Sect. 3].

Krasikov also established a uniform bound for $|J_\mu|$. Let $\mu > -1/2$, then

$$J_\mu^2(t) \leqslant \frac{4\big(4t^2-(2\mu+1)(2\mu+5)\big)}{\pi\big((4t^2-\lambda)^{3/2}-\lambda\big)} =: \mathfrak{K}_\mu(t), \tag{18}$$

for all

$$t > \tfrac{1}{2}\sqrt{\lambda+\lambda^{2/3}},\ \lambda := (2\mu+1)(2\mu+3)\,.$$

The estimate is sharp in certain sense, consult [30, Theorem 2]. Moreover, Krasikov mentioned that (18) provides sharp bound in the whole oscillatory region; however, in the transition region, this estimate becomes very poor and should be replaced with another estimate. Having in mind Krasikov's discussion, we propose to combine Krasikov's bound with Olenko's one. This approach was used by Srivastava and Pogány in [45]. Let us denote $\chi_S(x)$ the characteristic (or indicator) function of a set S, that is, $\chi_S(x) = 1$ for all $x \in S$ and $\chi_S(x) = 0$ otherwise. Since the integration domain coincides with the positive real half-axis, we need an efficient bound for $|J_\mu(t)|$ on $(0, A]$, $A > \sqrt{\lambda+\lambda^{2/3}}/2$. Therefore, we introduce the bounding function

$$|J_\mu(t)| \leqslant \mathscr{V}_\mu(t) := \frac{d_O}{\sqrt{t}}\,\chi_{(0,A_\lambda]}(t) + \sqrt{\mathfrak{K}_\mu(t)}\,\big(1-\chi_{(0,A_\lambda]}(t)\big)\,, \tag{19}$$

where, by simplicity reasons, our choice would be

$$A_\lambda = \tfrac{1}{2}\,\big(\lambda+(\lambda+1)^{2/3}\big),$$

because $\mathfrak{K}_\nu(t)$ is positive and monotonous decreasing for $t \in \frac{1}{2}\big((\lambda+\lambda^{2/3}),\infty\big)$, compare [45, Sect. 3]. Moreover, we point out that for A_λ, we can take any $\frac{1}{2}\big(\lambda + (\lambda+\eta)^{2/3}\big)$, $\eta > 0$.

Next, Pogány derived a different fashion bound for $|J_\mu|$, when the argument of the considered Bessel function is coming from a closed Cassinian oval from $\mathbb{C}$. To recall this result, we need the following definitions. Let us denote $\mathbb{D}_\eta = \{z\colon |z| \leqslant \eta\}$ the closed centered disc having diameter 2η, while the open unit disc $\mathbb{D} = \{z\colon |z| < 1\}$ and the closed Cassinian oval [38]

$$\mathfrak{C}_{\mu,\lambda} := \left\{z\colon \left|z^2 - j_{\mu,1}^2\right| \leqslant j_{\mu,1}^2\,\frac{1-\lambda}{1+\lambda}\right\}, \qquad \lambda \in [0,1]\,.$$

The famous von Lommel theorem *"$J_\nu(z)$ has an infinity of real zeros, for any given real value of ν"*, [47, p. 478], ensures the existence of such $j_{\nu,1}$. Thus [38, Theorem 1]

$$\left|J_\mu(z)\right| \leqslant \frac{|z|^\mu}{2^\mu\Gamma(\mu+1)}\,\exp\left\{-\frac{\lambda|z|^2}{4(\mu+1)}\right\}, \qquad \lambda \in (0,1),\ \mu > 0,\ z \in \mathfrak{C}_{\mu,\lambda}. \tag{20}$$

Here we mention the inequality [25, p. 215]

$$\left|J_\mu(t)\right| \leqslant \frac{t^\mu}{2^\mu \Gamma(\mu+1)}\, \exp\left\{-\frac{t^2}{4(\mu+1)}\right\}, \qquad t>0,\ \mu \geqslant 0\,. \tag{21}$$

Note that Watson [47, p. 16] originated back to Cauchy a weaker variant of this inequality (the exponential term contains $-t^2/4$), for integer order μ, see [11, p. 687], [12, p. 854].

Ifantis–Siafarikas improved (21) for the domain $t \in (0, j_{\mu,1}), \mu > -1$ in the following form [25, Eq. (3.15)]:

$$J_\mu(t) < \frac{t^\mu}{2^\mu \Gamma(\mu+1)}\, \exp\left\{-\frac{t^2}{4(\mu+1)} - \frac{t^4}{32(\mu+1)^2(\mu+2)}\right\}. \tag{22}$$

It is worth to mention Sitnik's paper [43], in which he reported stronger but more complicated bounds involving Rayleigh sums of negative powers of Bessel function zeros; his results concern Bessel function bounds inside the open unit disc $\mathbb{D}$. Interesting upper bound was established also by Lee and Shah for complex variable, integer order Bessel function $J_r(t)$; see [32, p. 148]. Finally, we refer to Cerone's book chapter [13, Sect. 2] for an inequality accomplished by bounds on a Čebyšev functional.

Theorem 2.1. *The following bounding inequalities hold true:*

a. *For all* $x \geqslant 0, \nu - \frac{3}{2} = r \in \mathbb{N}_0$, *we have*

$$|\Omega_{r+\frac{3}{2}}(x)| \leqslant \frac{\pi^{r-\frac{1}{2}}\,[1+\delta_{0r}(\sqrt{2}-1)]\,(r+1)!\,\eta(r+2)}{\sqrt{2}\,\Gamma(r+\frac{3}{2})\,x^r}\, \sinh\left(\frac{x}{2}\right), \tag{23}$$

where δ_{ab} *stands for the Kronecker delta, while*

$$\eta(p) = \sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n^p}, \qquad \mathrm{Re}\{p\} > 0,$$

denotes the Dirichlet Eta function.

b. *For all* $x \geqslant 0, \nu > \frac{1}{2}$, *it is*

$$|\Omega_\nu(x)| \leqslant \frac{2}{\pi}\,\eta(2\nu-1)\,\sinh\left(\frac{x}{2}\right). \tag{24}$$

c. *Let us denote* b_L, c_L *the Landau constants given in* (15), (16). *Then for all* $x > 0$, *we have*

$$|\Omega_\nu(x)| \leqslant \begin{cases} \dfrac{b_L \pi^{\nu-2}\,\Gamma\left(\nu+\frac{1}{2}\right)\,\eta\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\,\left(\nu-\frac{3}{2}\right)^{\frac{1}{3}}\,x^{\nu-\frac{3}{2}}}\,\sinh\left(\dfrac{x}{2}\right), & \nu > \frac{3}{2}, \\ \dfrac{c_L 2^{\frac{1}{3}}\,\pi^{\nu-\frac{5}{3}}\,\Gamma\left(\nu+\frac{1}{6}\right)\,\eta\left(\nu+\frac{1}{6}\right)}{\Gamma(\nu)\,x^{\nu-\frac{7}{6}}}\,\sinh\left(\dfrac{x}{2}\right), & \nu > -\frac{1}{6}. \end{cases} \tag{25}$$

d. *Let d_O be the Olenko coefficient in* (17). *Then*

$$|\Omega_\nu(x)| \leqslant \frac{d_O\,\sqrt{2}\,\pi^{\nu-\frac{3}{2}}\,\eta(\nu)}{x^{\nu-1}}\,\sinh\left(\frac{x}{2}\right), \qquad x>0,\ \nu>\tfrac{3}{2}\,. \tag{26}$$

Proof. **a.** Consider the integral [20, p. 349, Eq. 3.411.3]

$$\mathscr{I}(\alpha,\beta) := \int_0^\infty \frac{t^{\alpha-1}}{\mathrm{e}^{\beta t}+1}\,\mathrm{d}t = \beta^{-\alpha}\,\Gamma(\alpha)\,\eta(\alpha), \qquad \min\big(\mathrm{Re}\{\alpha\},\mathrm{Re}\{\beta\}\big)>0\,.$$

Applying Hansen's bound (12) to the Bessel function $J_{\nu-\frac{3}{2}}(t)$ appearing in (11), we conclude

$$|\Omega_\nu(x)| \leqslant \frac{2^\nu\,\pi^{2\nu-\frac{3}{2}}[1+\delta_{0,\nu-\frac{3}{2}}(\sqrt{2}-1)]}{\Gamma(\nu)\,x^{2\nu-1}}\,\sinh\left(\frac{x}{2}\right)\mathscr{I}\left(\nu+\frac{1}{2},\frac{2\pi}{x}\right) \qquad x>0.$$

Substituting $r=\nu-\frac{3}{2}\in\mathbb{N}_0$ and reducing the previous bound, we arrive at (23).
b. & **c.** & **d.** Similarly to the case **a**, estimating $|J_{\nu-\frac{3}{2}}|$ with the aid of the bounds (14), (15), (16) and (17), we derive appropriate respective specifications. For the four consequent bounding inequalities (24), (25) and (26). Observe that (25) consists from TWO upper bounds. □

Remark 2.1. We point out that von Lommel's extension (13) of Hansen's bounds (12) will give substantially more general but in form equivalent bound upon $\Omega_\nu(x)$; therefore, it is not necessary to consider this case separately.

Obviously, being the integration domain for $\Omega_\nu(x)$ the positive real half-axis, Krasikov's bound itself is automatically eliminated as a candidate to be employed in estimating the Bessel function in the kernel of the integrand of $\Omega_\nu(x)$. Therefore, instead of Krasikov's, the synthetized Olenko–Krasikov bound $\mathscr{V}_\mu(t)$ (19) shall we apply. The *lower incomplete Gamma function* $\gamma(s,\omega)$ [20, **8.350** 1.] one defines truncating the integration domain of Eulerian Gamma function to $[0,\alpha]$, i.e.

$$\gamma(s,\alpha) := \int_0^\alpha t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t\,.$$

Also, the *upper incomplete Gamma function* or *complementary incomplete Gamma function* [20, **8.350** 2.] is given by

$$\Gamma(s,\alpha) := \Gamma(s)-\gamma(s,\alpha) = \int_\alpha^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

For both incomplete Gamma functions, $\mathrm{Re}\{s\}>0$, $|\arg(\alpha)|\leqslant\pi-\epsilon$, $\epsilon\in(0,\pi)$. Let us remark that for certain fixed α, $\Gamma(s,\alpha)$ is an entire function of s, while $\gamma(s,\alpha)$ is a meromorphic function of α with simple poles at $s\in\mathbb{Z}_0^-$.

Theorem 2.2. *Let $\nu > 1, \lambda = 4\nu(\nu-1)$ and let x be positive real. Then we have the following bounding inequality:*

$$\big|\Omega_\nu(x)\big| \leqslant \frac{\sqrt{2}\,\pi^{\nu-\frac{3}{2}}}{\Gamma(\nu)\,x^{\nu-1}}\,\sinh\left(\frac{x}{2}\right)\left\{\frac{d_O\cdot\gamma\left(\nu,\frac{2\pi}{x}A_\lambda\right)}{1+\exp\{-\frac{2\pi}{x}A_\lambda\}} + \frac{2^{\nu-1}\sqrt{x\,\mathfrak{K}_{\nu-\frac{3}{2}}(A_\lambda)}\cdot\Gamma\left(\nu+\frac{1}{2},\frac{\pi}{x}A_\lambda\right)}{\sqrt{\pi}\,\cosh\left(\frac{\pi}{x}A_\lambda\right)}\right\}, \tag{27}$$

where $A_\lambda = \frac{1}{2}\left(\lambda+(\lambda+1)^{2/3}\right)$ *and*

$$\mathfrak{K}_{\nu-\frac{3}{2}}(A_\lambda) = \frac{4}{\pi}\left\{\left(\lambda+(\lambda+1)^{2/3}\right)^2 - 4(\nu^2-1)\right\}, \qquad x \geqslant A_\lambda\,.$$

Proof. By (11) and (19), it follows that

$$\begin{aligned}\big|\Omega_\nu(x)\big| &\leqslant \frac{2^{\nu+\frac{1}{2}}\,\pi^{2\nu-\frac{3}{2}}}{\Gamma(\nu)\,x^{2\nu-1}}\,\sinh\left(\frac{x}{2}\right)\int_0^\infty \frac{t^{\nu-1/2}}{\mathrm{e}^{2\pi t/x}+1}\,\mathscr{V}_{\nu-\frac{3}{2}}(t)\,\mathrm{d}t \\ &= \frac{2^{\nu+\frac{1}{2}}\,\pi^{2\nu-\frac{3}{2}}}{\Gamma(\nu)\,x^{2\nu-1}}\,\sinh\left(\frac{x}{2}\right)\left(d_O\int_0^{A_\lambda}\frac{t^{\nu-1}}{\mathrm{e}^{2\pi t/x}+1}\,\mathrm{d}t \qquad (=:\mathscr{I}_1)\right. \\ &\quad \left. + \int_{A_\lambda}^\infty \frac{t^{\nu-\frac{1}{2}}}{\mathrm{e}^{2\pi t/x}+1}\sqrt{\mathfrak{K}_{\nu-\frac{3}{2}}(t)}\,\mathrm{d}t\right) \qquad (=:\mathscr{I}_2)\,.\end{aligned} \tag{28}$$

Now, for the integral $\mathfrak{I}_1$, we calculate in the following manner:

$$\mathfrak{I}_1 = \int_0^{A_\lambda}\frac{t^{\nu-1}\,\mathrm{e}^{-\beta t}}{1+\mathrm{e}^{-\beta t}}\,\mathrm{d}t \leqslant \frac{1}{1+\mathrm{e}^{-\beta A_\lambda}}\int_0^{A_\lambda} t^{\nu-1}\,\mathrm{e}^{-\beta t}\,\mathrm{d}t = \frac{\gamma\left(\nu,\beta A_\lambda\right)}{\beta^\nu(1+\mathrm{e}^{-\beta A_\lambda})}\,. \tag{29}$$

The fact that $\mathfrak{K}_\mu$ decreases on $[A_\lambda,\infty)$ has been established already in [45, p. 199]; hence,

$$\mathfrak{K}_{\nu-\frac{3}{2}}(x) \leqslant \mathfrak{K}_{\nu-\frac{3}{2}}(A_\lambda) = \frac{4}{\pi}\left[\left(\lambda+(\lambda+1)^{2/3}\right)^2 - 4(\nu^2-1)\right], \qquad x \geqslant A_\lambda\,.$$

Accordingly,

$$\begin{aligned}\mathfrak{I}_2 = \frac{1}{2}\int_{A_\lambda}^\infty \frac{t^{\nu-\frac{1}{2}}\,\mathrm{e}^{-\frac{\beta}{2}t}}{\cosh\left(\frac{\beta}{2}t\right)}\,\mathrm{d}t &\leqslant \frac{1}{2\cosh\left(\frac{\beta}{2}A_\lambda\right)}\int_{A_\lambda}^\infty t^{\nu-\frac{1}{2}}\,\mathrm{e}^{-\frac{\beta}{2}t}\,\mathrm{d}t \\ &= \frac{2^{\nu-\frac{1}{2}}\,\Gamma\left(\nu+\frac{1}{2},\frac{1}{2}\beta\,A_\lambda\right)}{\beta^{\nu+\frac{1}{2}}\,\cosh\left(\frac{\beta}{2}A_\lambda\right)}\end{aligned} \tag{30}$$

in both integrals $\beta := 2\pi x^{-1}$. Now, obvious transformations of (28), (29) and (30) lead to the asserted bound (27). □

Considering further bounds (20), (21) and (22) upon the Bessel function of the first kind, we see that only the bound (21) possesses the property of direct applicability since the integration domain of $(0, \infty)$ in defining $\Omega_\nu(x)$. Here, and in what follows, ${}_p\Psi_q$ denotes the Fox–Wright generalization of the hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters defined by (cf. e.g. [44], [45, p. 197, Eq. (7)])

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_p),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle| x\right] = {}_p\Psi_q\left[\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\middle| x\right] := \sum_{m=0}^\infty \frac{\prod_{\ell=1}^p \Gamma\big(a_\ell+\alpha_\ell m\big)}{\prod_{\ell=1}^q \Gamma\big(b_\ell+\beta_\ell m\big)}\,\frac{x^m}{m!}, $$

where

$$ \alpha_\ell \in \mathbb{R}_+,\ \ell = \overline{1,p}; \quad \beta_j \in \mathbb{R}_+,\ j = \overline{1,q}; \quad \Delta := 1 + \sum_{\ell=1}^q \beta_\ell - \sum_{j=1}^p \alpha_j \geqslant 0\,, $$

and in the case of equality $\Delta = 0$, the absolute convergence holds for suitably bounded values of x given by

$$ |x| < \nabla = \prod_{j=1}^q \beta_j^{\beta_j} \prod_{j=1}^p \alpha_j^{-\alpha_j}\,, $$

while in the case $|x| = \nabla$, the condition

$$ \mathrm{Re}\left\{\sum_{\ell=1}^q b_\ell - \sum_{j=1}^p a_j\right\} + \frac{p-q-1}{2} > 0 $$

suffices for the absolute convergence of the series ${}_p\Psi_q[x]$.

Next, we introduce the Krätzel function, which is defined for $u > 0$, $\rho \in \mathbb{R}$ and $\nu \in \mathbb{C}$, being such that $\mathrm{Re}\{\nu\} < 0$ for $\rho \leqslant 0$, by the integral

$$ Z_\rho^\nu(u) = \int_0^\infty t^{\nu-1} \mathrm{e}^{-t^\rho - \frac{u}{t}}\, \mathrm{d}t\,. \tag{31} $$

For $\rho \geqslant 1$ the function (31) was introduced by E. Krätzel [29] as a kernel of the integral transform

$$ \left(K_\nu^\rho f\right)(u) = \int_0^\infty Z_\rho^\nu(ut)\, f(t)\, \mathrm{d}t, $$

which was applied to the solution of some ordinary differential equations. The study of the Krätzel function (31) and the above integral transform was continued, for example, in the paper by Kilbas, Saxena and Trujilló [27], in which the authors deduced explicit forms of Z_ρ^ν in terms of the generalized Wright function, or in the paper [2] by Baricz, Jankov and Pogány devoted among others to convexity property research and Laguerre- and Turán-type inequalities for the Krätzel function.

Theorem 2.3. *Let $\nu > \frac{1}{2}$. Then for all $x > 0$, the following inequality is valid:*

$$\begin{aligned}\left|\Omega_\nu(x)\right| &\leqslant \frac{\pi^{2\nu-\frac{3}{2}}\,(4\nu-2)^{\nu-\frac{1}{2}}\,\sinh\left(\frac{x}{2}\right)}{\Gamma(\nu)\Gamma(\nu-\frac{1}{2})\,x^{2\nu-1}}\;{}_1\Psi_0\left[\begin{matrix}\left(\nu-\frac{1}{2},\frac{1}{2}\right)\\ -\end{matrix}\,\middle|\,-\frac{\pi}{x}\sqrt{4\nu-2}\right]\\ &= \frac{2\pi^{2\nu-\frac{3}{2}}\,(4\nu-2)^{\nu-\frac{1}{2}}\,\sinh\left(\frac{x}{2}\right)}{\Gamma(\nu)\Gamma(\nu-\frac{1}{2})\,x^{2\nu-1}}\,Z_{-2}^{1-2\nu}\left(\frac{\pi}{x}\sqrt{4\nu-2}\right).\end{aligned}$$

Proof. By (11) and (21) we conclude the estimate

$$\left|\Omega_\nu(x)\right| \leqslant \frac{4\pi^{2\nu-\frac{3}{2}}\,\sinh\left(\frac{x}{2}\right)}{\Gamma(\nu)\Gamma(\nu-\frac{1}{2})\,x^{2\nu-1}}\int_0^\infty \frac{t^{2\nu-2}}{e^{2\pi t/x}+1}\,\exp\left\{-\frac{t^2}{2(2\nu-1)}\right\}\,dt.$$

Denoting

$$\mathcal{I}_3(\alpha,\beta,\gamma) = \int_0^\infty \frac{t^{\alpha-1}\,e^{-\gamma t^2}}{e^{\beta t}+1}\,dt,$$

we estimate the value of this integral:

$$\begin{aligned}\mathcal{I}_3(\alpha,\beta,\gamma) &\leqslant \frac{1}{2}\int_0^\infty \frac{t^{\alpha-1}}{\cosh\left(\frac{\beta}{2}t\right)}\,e^{-\frac{\beta}{2}t-\gamma t^2}\,dt\\ &\leqslant \frac{1}{2}\int_0^\infty t^{\alpha-1}\,e^{-\frac{\beta}{2}t-\gamma t^2}\,dt \qquad (=:\mathcal{I}_4)\\ &= \frac{1}{4}\sum_{n\geqslant 0}\frac{\left(-\frac{\beta}{2}\right)^n}{n!}\int_0^\infty t^{\frac{\alpha+n}{2}-1}\,e^{-\gamma t}\,dt\\ &= \frac{1}{4\gamma^{\frac{\alpha}{2}}}\sum_{n\geqslant 0}\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}n\right)\frac{\left(-\frac{\beta}{2\sqrt{\gamma}}\right)^n}{n!}\\ &= \frac{1}{4\gamma^{\frac{\alpha}{2}}}\,{}_1\Psi_0\left[\begin{matrix}\left(\frac{\alpha}{2},\frac{1}{2}\right)\\ -\end{matrix}\,\middle|\,-\frac{\beta}{2\sqrt{\gamma}}\right];\end{aligned} \tag{32}$$

the Fox–Wright function converges absolutely since $\Delta = \frac{1}{2} > 0$.

On the other hand, considering the integral $\mathcal{J}_4 = \mathcal{J}_4(\alpha, \beta, \gamma)$, we can express it *via* the Krätzel function:

$$\mathcal{J}_4 = \frac{1}{\gamma^{\frac{\alpha}{2}}}\, Z_{-2}^{-\alpha}\left(\frac{\beta}{2\sqrt{\gamma}}\right). \tag{33}$$

Indeed, the substitution $t^{-1} \mapsto t$ results in $Z_\rho^\nu(u) = \int_0^\infty t^{-\nu-1}\, \mathrm{e}^{-t^{-\rho}-ut}\, \mathrm{d}t$, and

$$\mathcal{J}_4 = \frac{1}{\gamma^{\frac{\alpha}{2}}} \int_0^\infty t^{\alpha-1}\, \mathrm{e}^{-t^2 - \frac{\beta}{2\sqrt{\gamma}}\, t}\, \mathrm{d}t = \frac{1}{\gamma^{\frac{\alpha}{2}}} \int_0^\infty t^{-(-\alpha)-1}\, \mathrm{e}^{-t^{-(-2)} - \frac{\beta}{2\sqrt{\gamma}}\, t}\, \mathrm{d}t\,,$$

so the relationship (33) is proved.

Now, it remains to specify in both formulae (32) and (33)

$$\alpha = 2\nu - 1, \quad \beta = \frac{2\pi}{x}, \quad \gamma = \frac{1}{4\nu - 2}\,,$$

because of

$$\begin{aligned}
\left|\Omega_\nu(x)\right| &\leqslant \frac{4\pi^{2\nu-\frac{3}{2}}\, \sinh\left(\frac{x}{2}\right)}{\Gamma(\nu)\Gamma(\nu-\frac{1}{2})\, x^{2\nu-1}}\, \mathcal{J}_3\left(2\nu-1, \frac{2\pi}{x}, \frac{1}{4\nu-2}\right) \\
&= \frac{4\pi^{2\nu-\frac{3}{2}}\, \sinh\left(\frac{x}{2}\right)}{\Gamma(\nu)\Gamma(\nu-\frac{1}{2})\, x^{2\nu-1}}\, \mathcal{J}_4\left(2\nu-1, \frac{2\pi}{x}, \frac{1}{4\nu-2}\right) \\
&= \frac{\pi^{2\nu-\frac{3}{2}}\, (4\nu-2)^{\nu-\frac{1}{2}}\, \sinh\left(\frac{x}{2}\right)}{\Gamma(\nu)\Gamma(\nu-\frac{1}{2})\, x^{2\nu-1}}\, {}_1\Psi_0\left[\left.\begin{matrix}(\nu-\frac{1}{2}, \frac{1}{2}) \\ -\end{matrix}\,\right|\, -\frac{\pi}{x}\sqrt{4\nu-2}\right] \\
&= \frac{2\pi^{2\nu-\frac{3}{2}}\, (4\nu-2)^{\nu-\frac{1}{2}}\, \sinh\left(\frac{x}{2}\right)}{\Gamma(\nu)\Gamma(\nu-\frac{1}{2})\, x^{2\nu-1}}\, Z_{-2}^{1-2\nu}\left(\frac{\pi}{x}\sqrt{4\nu-2}\right).
\end{aligned}$$

The proof is complete. □

Remark 2.2. As a by-product of the proof of Theorem 2.3 we get a relationship between Fox–Wright generalized hypergeometric function ${}_1\Psi_0[\cdot]$ and the Krätzel function $Z_\rho^\nu(\cdot)$:

$$\rho\, Z_\rho^\nu(u) = {}_1\Psi_0\left[\left.\begin{matrix}(\frac{\nu}{\rho}, -\frac{1}{\rho}) \\ -\end{matrix}\,\right|\, -u\right].$$

To prove this result we can apply the same methods as in the previous proof.

3 Bilateral Bounds Deduced via the Čaplygin Comparison Theorem

Two-sided bounding inequalities for the complete Butzer–Flocke–Hauss Omega function by the Čaplygin comparison theorem have been established for the first time by Pogány and Srivastava in [39, p. 591, Theorem 3]. Following this approach Pogány, Tomovski and Leškovski devoted the whole article [41] to this subject, deriving a few sets of bilateral inequalities for the BFH Omega function via alternating Mathieu series, which is closely connected in their proper fraction representation.

In this section we shall obtain some fashion bilateral bounding inequalities for the generalized BFH Omega function Ω_ν, adapting the Čaplygin differential inequality procedure developed in [39, 41]. Firstly, we consider a linear nonhomogeneous ordinary differential equation, of which a particular solution is Ω_ν, mentioning that the case $\nu = 1$ has been extensively studied by Butzer, Pogány and Srivastava [8, pp. 1074–1075, Theorem 1].

Theorem 3.1. *For all $\nu > \frac{1}{2}$, $x \in \mathbb{R}$, the generalized complete BFL $\Omega_\nu(x)$ function is a particular solution of the following ordinary differential equation:*

$$y' = \frac{1}{2}\coth\left(\frac{x}{2}\right) y - \frac{\nu x}{2\pi^3}\sinh\left(\frac{x}{2}\right) h(x)\,,$$

where

$$h(x) = \begin{cases} \tilde{S}_{\nu+1}\left(\dfrac{x}{2\pi}\right), & x \neq 0, \\ 2\,\eta(2\nu - 1), & x = 0. \end{cases}$$

Here $\tilde{S}_{\nu+1}(\cdot)$ stands for the generalized alternating Mathieu series of order $\nu + 1$, while $\eta(\cdot)$ denotes the Dirichlet Eta function.

Proof. Rewriting (9), we get

$$\Omega_\nu(x) = \frac{1}{\pi}\sinh\left(\frac{x}{2}\right)\tilde{S}_\nu\left(\frac{x}{2\pi}\right).$$

Differentiating this formula we obtain

$$\begin{aligned} \pi\,\Omega'_\nu(x) &= \frac{1}{2}\cosh\left(\frac{x}{2}\right)\tilde{S}_\nu\left(\frac{x}{2\pi}\right) - \frac{\nu x}{2\pi^2}\sum_{n\geqslant 1}\frac{(-1)^{n-1}2n}{\left(n^2 + \left(\frac{x}{2\pi}\right)^2\right)^{\nu+1}} \\ &= \frac{\pi}{2}\coth\left(\frac{x}{2}\right)\frac{1}{\pi}\sinh\left(\frac{x}{2}\right)\tilde{S}_\nu\left(\frac{x}{2\pi}\right) - \frac{\nu x}{2\pi^2}\tilde{S}_{\nu+1}\left(\frac{x}{2\pi}\right), \end{aligned}$$

which is in fact equivalent to the asserted ordinary differential equation, because

$$\lim_{\xi\to 0} \tilde{S}_\nu(\xi) = 2\,\eta(2\nu-1)\,.$$

The proof is complete. □

Consider the Cauchy problem given by

$$y' = f(x,y) \quad \text{and} \quad y(x_0) = y_0\,. \tag{34}$$

For a given interval $\mathcal{I} \subseteq \mathbb{R}$, let $x_0 \in \mathcal{I}$ and let the functions $\varphi, \psi \in \mathrm{C}^1(\mathcal{I})$. We say that φ and ψ are the *lower* and the *upper functions*, respectively, if

$$\varphi'(x) \leqslant f\big(x,\varphi(x)\big) \quad \text{and} \quad \psi'(x) \geqslant f\big(x,\psi(x)\big) \qquad x \in \mathcal{I};$$
$$\varphi(x_0) = \psi(x_0) = y_0\,.$$

Suppose also that the function $f(x,y)$ is continuous on some domain $\mathscr{D}$ in the (x,y)-plane containing the interval $\mathcal{I}$ with the lower and upper functions φ and ψ, respectively. Then the solution $y(x)$ of the Cauchy problem (34) satisfies the following two-sided inequality:

$$\varphi(x) \leqslant y(x) \leqslant \psi(x), \qquad x \in \mathcal{I}\,.$$

This is actually the so-called *Čaplygin-type differential inequality* or the *Čaplygin comparison theorem* [9, 10, 34] (also see [3, p. 202] and [36, pp. 3–4]).

We divide into four steps the derivation of two-sided bounds: **A.** obtaining guard functions couple $\tilde{L}_\nu(x), \tilde{R}_\nu(x)$ for $\tilde{S}_{\nu+1}(x)$, **B.** fixing the domain $\mathcal{I}$ of solution and the initial condition $\varphi_\nu(x_0) = \psi_\nu(x_0) = y_0$, **C.** solving the boundary ordinary differential equations for lower and upper guard functions and finally **D.** considering the particular solutions $\varphi_\nu(x), \psi_\nu(x)$.

A. Keeping in mind the definition (9) of $\Omega_\nu(x)$, the natural choice of domain is $\mathcal{I} = [0,\infty)$. On the other hand, since $\Omega_\nu(x)$ behaves near to the origin like

$$\Omega_\nu(x) = \frac{2}{\pi}\,\eta(2\nu-1)\,x\,(1+o(1)), \qquad x \to 0,$$

we pick up the initial condition of our Cauchy problem

$$\varphi_\nu(0) = \psi_\nu(0) = 0\,.$$

B. Let $L_\mu(x), R_\mu(x)$ denote the guard functions for the generalized Mathieu series $S_\mu(x)$, $x \in \mathcal{I}$. By the arithmetic mean–geometric mean inequality, we have

$$S_\mu(x) \leqslant \begin{cases} 2^{1-\mu}\,x^{-\mu}\,\zeta(\mu-1) & x>0 \\ 2\,\zeta(2\mu-1) & x=0 \end{cases} \qquad := R_\mu(x)\,,$$

and

$$S_\mu(x) \geqslant \int_1^\infty \frac{2t}{(t^2+x^2)^\mu}\,\mathrm{d}t = \frac{1}{(\mu-1)(1+x^2)^{\mu-1}} = L_\mu(x)\,.$$

Rewriting the fractional representation of the alternating generalized Mathieu series into

$$\begin{aligned}\tilde{S}_\mu(x) &= \sum_{n\geqslant 1} \frac{(-1)^{n-1}\,2n}{(n^2+x^2)^\mu} \\ &= \sum_{n\geqslant 1} \frac{2n}{(n^2+x^2)^\mu} - 4\cdot\sum_{n\geqslant 1}\frac{2n}{(4n^2+x^2)^\mu} = S_\mu(x) - 4^{1-\mu}\,S_\mu\left(\frac{x}{2}\right),\end{aligned}$$

we clearly deduce that

$$\tilde{L}_\mu(x) := L_\mu(x) - 4^{1-\mu}R_\mu\left(\frac{x}{2}\right) \leqslant \tilde{S}_\mu(x) \leqslant R_\mu(x) - 4^{1-\mu}L_\mu\left(\frac{x}{2}\right) =: \tilde{R}_\mu(x)\,. \tag{35}$$

Therefore, writing $\mu = \nu + 1$ throughout in (35), we get

$$\tilde{L}_{\nu+1}(x) = \frac{1}{\nu(1+x^2)^\nu} - \frac{\zeta(\nu)}{2^{2\nu-1}\,x^{\nu+1}}, \qquad \tilde{R}_{\nu+1}(x) = \frac{\zeta(\nu)}{2^\nu\,x^{\nu+1}} - \frac{1}{\nu(4+x^2)^\nu}\,.$$

C. Following the lines of Theorem 3.1, the lower function's ODE will be

$$\begin{aligned}\varphi_\nu' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\varphi_\nu &= -\frac{\nu x}{2\pi^3}\sinh\left(\frac{x}{2}\right)\tilde{R}_{\nu+1}\left(\frac{x}{2\pi}\right) \\ &= \pi^{\nu-2}\sinh\left(\frac{x}{2}\right)\left\{\frac{2^{2\nu-1}\pi^{\nu-1}x}{(16\pi^2+x^2)^\nu} - \frac{\nu\,\zeta(\nu)}{x^\nu}\right\}\;;\end{aligned}$$

hence

$$\varphi_\nu(x) = \sinh\left(\frac{x}{2}\right)\left\{c_\varphi + \pi^{\nu-2}\left(\frac{4^{\nu-1}\pi^{\nu-1}}{(1-\nu)(x^2+16\pi^2)^{\nu-1}} + \frac{\nu\,\zeta(\nu)}{(\nu-1)\,x^{\nu-1}}\right)\right\}\,.$$

The upper function's ODE reads as follows:

$$\begin{aligned}\psi_\nu' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\psi_\nu &= -\frac{\nu x}{2\pi^3}\sinh\left(\frac{x}{2}\right)\tilde{L}_{\nu+1}\left(\frac{x}{2\pi}\right) \\ &= \pi^{\nu-2}\sinh\left(\frac{x}{2}\right)\left\{\frac{\nu\,\zeta(\nu)}{2^{\nu-2}\,x^\nu} - \frac{2^{2\nu-1}\,\pi^{\nu-1}x}{(4\pi^2+x^2)^\nu}\right\}\,,\end{aligned}$$

accordingly,

$$\psi_\nu(x) = \sinh\left(\frac{x}{2}\right) \left\{ c_\psi + \frac{\pi^{\nu-2}\,\nu\,\zeta(\nu)}{2^{\nu-2}\,(1-\nu)\,x^{\nu-1}} + \frac{4^{\nu-1}\pi^{2\nu-3}}{(\nu-1)(x^2+4\pi^2)^{\nu-1}} \right\}.$$

Here $\zeta(\nu), \nu \in \left(\frac{1}{2}, 1\right)$ signifies the analytic continuation of the Riemann ζ function by the widely known formula

$$\zeta(\nu) = \frac{\eta(\nu)}{1-2^{1-\nu}}, \qquad \mathrm{Re}\{\nu\} > 0\,.$$

D. Bearing in mind the asymptotics of lower and upper functions and that of $\Omega_\nu(x)$ near to zero, we immediately get

$$\frac{\varphi_\nu(x)}{\sinh\left(\frac{x}{2}\right)} \sim c_\varphi + \frac{4}{\pi(1-\nu)} = \frac{2}{\pi}\,\eta(2\nu-1) = c_\psi - \frac{1}{\pi(1-\nu)} \sim \frac{\psi_\nu(x)}{\sinh\left(\frac{x}{2}\right)},$$

which determines the values of integration constants

$$\begin{aligned} c_\varphi &= \frac{2}{\pi}\,\eta(2\nu-1) - \frac{4}{\pi(1-\nu)}, \\ c_\psi &= \frac{2}{\pi}\,\eta(2\nu-1) + \frac{1}{\pi(1-\nu)}\,. \end{aligned}$$

Thus, the proof of the following result is given.

Theorem 3.2. *Let $\nu \in \left(\frac{1}{2}, 1\right)$. Then for all $x \in \mathfrak{I} = \mathbb{R}_+$, we have the following two-sided inequality:*

$$\varphi_\nu(x) \leqslant \Omega_\nu(x) \leqslant \psi_\nu(x),$$

where

$$\begin{aligned} \varphi_\nu(x) = \sinh\left(\frac{x}{2}\right) \Bigg\{ &\frac{2}{\pi}\,\eta(2\nu-1) - \frac{4}{\pi(1-\nu)} \\ &+ \frac{4^{\nu-1}\pi^{2\nu-3}}{(1-\nu)(x^2+16\pi^2)^{\nu-1}} - \frac{\pi^{\nu-2}\,\nu\,\zeta(\nu)}{(1-\nu)\,x^{\nu-1}} \Bigg\}, \end{aligned}$$

$$\begin{aligned} \psi_\nu(x) = \sinh\left(\frac{x}{2}\right) \Bigg\{ &\frac{2}{\pi}\,\eta(2\nu-1) + \frac{1}{\pi(1-\nu)} \\ &+ \frac{\pi^{\nu-2}\,\nu\,\zeta(\nu)}{2^{\nu-2}(1-\nu)x^{\nu-1}} - \frac{4^{\nu-1}\pi^{2\nu-3}}{(1-\nu)(x^2+4\pi^2)^{\nu-1}} \Bigg\}. \end{aligned}$$

Here $\zeta(\nu)$, stands for the analytic continuation of the Riemann ζ function to $\nu \in \left(\frac{1}{2}, 1\right)$.

4 Hilbert–Eisenstein Series and Their Basic Properties

4.1 In the sequel we give a brief introduction to certain important properties of the Hilbert–Eisenstein series. One of their bases is the Eisenstein theory of circular functions, founded by Gotthold Eisenstein [16] in 1847. They play an important role in number theory, especially their extension to elliptic functions; see, e.g. Weil [48, 49] and Iwaniec and Kowalski [26]. The series $\varepsilon_r(w)$ defined for all $w \in \mathbb{C} \setminus \mathbb{Z}$ and all integer $r \geqslant 2$ by

$$\varepsilon_r(w) = \sum_{k \in \mathbb{Z}} \frac{1}{(w+k)^r},$$

is called the *Eisenstein series of order* r. The $\varepsilon_r(w)$ are normally convergent and represent meromorphic functions in $\mathbb{C}$, are holomorphic in $\mathbb{C} \setminus \mathbb{Z}$ and posses poles in $k \in \mathbb{Z}$ (of order r and principal part $(w-k)^{-r}$). Recall that a series $\sum_k f_k$ of functions $f_k\colon X \mapsto \mathbb{C}$ is *normally convergent in* X if to each point $x \in X$ there exists a neighbourhood U such that $\sum_k |f_k| < \infty$. If the series is normally convergent in X, then $\sum |f_k|$ converges compactly in X. The converse is valid for domains $X = D \subseteq \mathbb{C}$, if all f_k are holomorphic in D. If such a series converges compactly in D, so does the series of its derivatives and, under weak assumptions, also the series of its primitives; see [42, pp. 92–95, 224].

For $r = 1$, the definition reads in Eisenstein's principal value notation

$$\begin{aligned}
\varepsilon_1(w) &= \sum_{k \in \mathbb{Z}}{}_e \frac{1}{w+k} := \lim_{N \to \infty} \sum_{|k| \leqslant N} \frac{1}{w+k} \\
&= \frac{1}{w} + \sum_{k \geqslant 1} \left(\frac{1}{w+k} + \frac{1}{w-k} \right) \\
&= \frac{1}{w} + \sum_{k \geqslant 1} \frac{2w}{w^2 - k^2} = \pi \cot(\pi w), \qquad w \in \mathbb{C} \setminus \mathrm{i}\mathbb{Z}. \qquad (36)
\end{aligned}$$

This partial fraction expansion of the cotangent function (due to Euler [17]), which is essentially the "alternating" generating function of the classical Bernoulli numbers $B_{2k} := B_{2k}(0)$ ($B_n(x)$ being the Bernoulli polynomials, $n \in \mathbb{N}_0$), namely,

$$\frac{w}{2} \cot \frac{w}{2} = \sum_{k \geqslant 0} \frac{(-1)^k}{(2k)!} B_{2k}\, w^{2k}, \qquad |w| < 2\pi,$$

is regarded by Konrad Knopp [28, p. 207] as the "*most remarkable expansion in partial fractions*".

Differentiating the normally convergent series (36), observing that $(\cot w)' = -\sin^{-2} w$, there follows

$$\varepsilon_2(w) = \frac{\pi^2}{\sin^2(\pi w)}, \qquad \varepsilon_3(w) = \frac{\pi^3\cot(\pi w)}{\sin^2(\pi w)},$$

giving the surprising result [42, p. 303]

$$\varepsilon_3(w) = \varepsilon_1(w) \cdot \varepsilon_2(w).$$

Eisenstein series' two essential properties are [48, pp. 6–13] and [42, p. 303]

$$\varepsilon_r'(w) = -r\varepsilon_{r+1}(w), \qquad \varepsilon_r(w) = \frac{(-1)^{r-1}}{(r-1)!}\,\varepsilon_1^{(r-1)}(w), \qquad r \in \mathbb{N}_2,$$

as well as their 1-periodicity in the sense that $\varepsilon_r(w+k) = \varepsilon_r(w)$ for all $w \in \mathbb{C}$, $k \in \mathbb{Z}$.

Differentiating the Fourier series expansion of $\cot \pi w$ [18, p. 386 et seq.], namely of

$$\cot \pi w = \mathrm{i}\begin{cases} -1 - 2\sum\limits_{k\geqslant 1} \mathrm{e}^{2\pi \mathrm{i} wk}, & \mathrm{Im}\{w\} > 0 \\ 1 + 2\sum\limits_{k\leqslant -1} \mathrm{e}^{2\pi \mathrm{i} wk}, & \mathrm{Im}\{w\} < 0 \end{cases},$$

iteratively, there follows that the $\varepsilon_r(w)$ posses Fourier expansions in the upper and lower half-planes with period π.

The basis to the Hilbert–Eisenstein series also includes the background to the so-called "Basler" problem, an open question since 1690, namely, whether there exists a counterpart of Euler's famous result on the closed expression for the Riemann Zeta function for even arguments, thus

$$\zeta(2m) = (-1)^{m+1} 2^{2m-1}\pi^{2m}\,\frac{B_{2m}}{(2m)!}, \qquad m \in \mathbb{N},$$

to the case of odd arguments, namely, $\zeta(2m+1)$.

Theorem A (Counterpart of Euler's formula for $\zeta(2m+1)$). *For $m \in \mathbb{N}$, there holds*

$$\zeta(2m+1) = (-1)^m 4^m \pi^{2m+1}\,\frac{B^{\sim}_{2m+1}(0)}{(2m+1)!}.$$

Thus, the solution consists in replacing the Bernoulli numbers B_{2m} in Euler's formula by the conjugate Bernoulli numbers $B^{\sim}_{2m+1}$ which are defined in terms of the Hilbert transform and were introduced by Butzer, Hauss and Leclerc in [7].

Starting with the 1-periodic Bernoulli polynomials $\mathscr{B}_n(x)$ defined as the periodic extension of $\mathscr{B}_n(x) = B_n(x), x \in (0, 1]$, one can—using Hilbert transforms—introduce 1-periodic conjugate Bernoulli "polynomials" $\mathscr{B}_n^{\sim}(x), x \in \mathbb{R}$ ($x \notin \mathbb{Z}$ if $n = 1$) by

$$\mathscr{B}_n^{\sim}(x) := \mathscr{H}\left[\mathscr{B}_n(\cdot)\right]_1(x), \qquad n \in \mathbb{N}.$$

These conjugate periodic functions $\mathfrak{B}_n^{\sim}(x)$ can be used to define the non-periodic functions $B_n^{\sim}(x)$ in a form such that their properties are similar to those of the classical Bernoulli polynomials $B_n(x)$. For details, see Butzer and Hauss [5, p. 22] and Butzer [4, pp. 37–56]. The conjugate Bernoulli numbers in question, the $B_{2m+1}^{\sim}$, are the $B_{2m+1}^{\sim}(0)(= B_{2m+1}^{\sim}(1))$ for which

$$B_{2m+1}^{\sim}\left(\frac{1}{2}\right) = \left(4^{-m} - 1\right) \cdot B_{2m+1}^{\sim}(1).$$

Some values of the conjugate Bernoulli numbers are

$$B_{2m+1}^{\sim}\left(\frac{1}{2}\right) = \begin{cases} -\dfrac{\log 2}{\pi} & m=0 \\ \dfrac{\log 2}{4\pi} - 2\displaystyle\int_{0+}^{\frac{1}{2}} u^2 \cot(\pi u)\mathrm{d}u & m=1 \\ \dfrac{11}{8}\displaystyle\int_{0+}^{\frac{1}{2}} u \cot \pi \mathrm{d}u + \dfrac{5}{3}\displaystyle\int_{0+}^{\frac{1}{2}} u^3 u \cot \pi \mathrm{d}u - 2\displaystyle\int_{0+}^{\frac{1}{2}} u^5 \cot \pi u \,\mathrm{d}u & m=2 \end{cases}.$$

Now, the indirect basis of the Hilbert–Eisenstein series is the Omega function $\Omega(\cdot)$. One arrives at it through the counterpart for the $B_n^{\sim}(x)$ of the exponential generating function of the classical polynomials $B_n(x)$, namely,

$$\sum_{n\geqslant 0} B_n(x)\,\frac{w^n}{n!} = \frac{w\,\mathrm{e}^{wx}}{\mathrm{e}^w - 1}, \qquad w \in \mathbb{C},\ |w| < 2\pi,\ x \in \mathbb{R}. \tag{37}$$

Theorem B (Exponential generating function of $B_k^{\sim}(\frac{1}{2})$). *For $|w| < 2\pi$, there holds*

$$\sum_{n\geqslant 0} B_n^{\sim}\left(\frac{1}{2}\right)\frac{w^n}{n!} = \frac{w\,\mathrm{e}^{wx}}{\mathrm{e}^w - 1} \cdot \Omega(w).$$

The proofs of Theorems A and B are connected with the Hilbert transform versions of the Euler–Maclaurin and Poisson summation formulae established by Hauss [24] (see also [5, p. 21–29] and [4, pp. 37–38, 78–80]). Observe that Theorem B tells us that the Hilbert transform of (37) essentially results in multiplying $w\,\mathrm{e}^{wx}(\mathrm{e}^w - 1)^{-1}$ by the Omega function $\Omega(w)$, taken at $x = \frac{1}{2}$.

4.2 The direct basis to what follows is the partial fraction expansion of $\Omega(w)$, thus a Hilbert-type version of the basic partial fraction expansion of $\cot \pi w$ in (36). It reads,

Theorem C (Partial fraction expansion of $\Omega(w)$). *For $w \in \mathbb{C} \setminus \mathrm{i}\mathbb{Z}$, one has*

$$\Omega(2\pi w) = \frac{1}{\pi}\,(\mathrm{e}^{-\pi w} - \mathrm{e}^{\pi w}) \sum_{k \geqslant 1} \frac{(-1)^k\, k}{w^2 + k^2} = \frac{\sinh(\pi w)}{\mathrm{i}w} \sum_{k \in \mathbb{Z}} \frac{(-1)^k\, \mathrm{sgn}(k)}{w + \mathrm{i}k}\,.$$

Definition 4.1. The Hilbert–Eisenstein (HE) series $\mathfrak{h}_r(w)$ are defined for $w \in \mathbb{C}\setminus\mathrm{i}\mathbb{Z}$ and $r \in \mathbb{N}_2$ by

$$\mathfrak{h}_r(w) := \sum_{k \in \mathbb{Z}} \frac{(-1)^k \mathrm{sgn}(k)}{(w + \mathrm{i}k)^r} = \sum_{k \geqslant 1} (-1)^k \left(\frac{1}{(w + \mathrm{i}k)^r} - \frac{1}{(w - \mathrm{i}} k)^r \right), \tag{38}$$

and

$$\mathfrak{h}_1(w) := \sum_{k \in \mathbb{Z}}{}_e \frac{(-1)^k \mathrm{sgn}(k)}{w + \mathrm{i}k} = \frac{\mathrm{i}\pi\; \Omega(2\pi w)}{\sinh \pi w} = \mathrm{i} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{w + \mathrm{i}k} \cdot \Omega(2\pi w)\,. \tag{39}$$

For $w = 0$, $\mathfrak{h}_1(0)$ can, since $\mathrm{sgn}(0) = 0$, be taken as

$$\mathfrak{h}_1(0) = \sum_{k \in \mathbb{Z}\setminus\{0\}}{}_e \frac{(-1)^k \mathrm{sgn}(k)}{\mathrm{i}k} = 2\mathrm{i}\ \log 2\,.$$

Observe that the partial fraction expansion of $\pi(\sinh \pi w)^{-1}$ follows by replacing w by $\mathrm{i}w$ and recalling $\sinh w = -\mathrm{i} \sin \mathrm{i}w$ in the well-known expansion

$$\frac{\pi}{\sin \pi w} = \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{w + k}, \qquad w \in \mathbb{C} \setminus \mathbb{Z}\,,$$

and $\pi(\sinh \pi w)^{-1}$ also possesses the Dirichlet series expansion for the right half-plane (see, e.g. [18, p. 405 et seq.])

$$\frac{\pi}{\sinh \pi w} = 2\pi \sum_{k \geqslant 0} \mathrm{e}^{-2\pi(2k+1)w}, \qquad \mathrm{Re}\{w\} > 0\,.$$

The case $r = 1$ of Definition 4.1 reveals that to achieve the Hilbert-type version of $\pi(\sinh \pi w)^{-1}$, that is, $\mathfrak{h}_1(w)$, one multiplies it (or its partial fraction expansion) by the complete Omega function $\Omega(2\pi w)$ and i. The counterparts of the corresponding properties of $\varepsilon_r(w)$ read

Proposition 4.1. *There holds for* $w \in \mathbb{C} \setminus \mathrm{i}\mathbb{Z}$ *and* $r \in \mathbb{N}_2$, $m \in \mathbb{N}$

$$\mathfrak{h}_r^{(m)}(w) = (-1)^m \, (r)_m \, \mathfrak{h}_{r+m}(w)\,, \tag{40}$$

where

$$(r)_m := \frac{\Gamma(r+m)}{\Gamma(r)} = r(r+1)\cdots(r+m-1)\,, \quad (r)_0 \equiv 1\,,$$

stands for the Pochhammer symbol (or shifted, rising factorial); moreover,

$$\mathfrak{h}_r(w) = \frac{(-1)^r}{\Gamma(r)}\, \mathfrak{h}_2^{(r-2)}(w), \qquad r \in \mathbb{N}_2\,, \tag{41}$$

as well as

$$\mathfrak{h}_r(w) + \mathfrak{h}_r(w+\mathrm{i}) = w^{-r} - (w+\mathrm{i})^{-r}\,. \tag{42}$$

Proof. Here the differentiability properties follow readily from Definition 4.1, and the i-periodicity-type formula we conclude from

$$\sum_{|k|\leqslant N} \frac{(-1)^k \operatorname{sgn}(k)}{(w+\mathrm{i}+\mathrm{i}k)^r} = \sum_{k=2}^{N+1} \frac{(-1)^{k-1} \operatorname{sgn}(k-1)}{(w+\mathrm{i}k)^r} + \frac{1}{w^r} + \sum_{k=-N+1}^{-1} \frac{(-1)^{k-1} \operatorname{sgn}(k-1)}{(w+\mathrm{i}k)^r}$$
$$= -\sum_{k=-N+1}^{N-1} \frac{(-1)^k \operatorname{sgn}(k)}{(w+\mathrm{i}k)^r} + \frac{1}{w^r} - \frac{1}{(w+\mathrm{i})^r} - \frac{(-1)^{N+1}}{(w+\mathrm{i}(N+1))^r} - \frac{(-1)^N}{(w+\mathrm{i}N)^r}\,.$$

Letting $N \to \infty$, we immediately arrive at (42). □

Theorem 4.1. *The* HE *series* $\mathfrak{h}_r(w)$ *possesses for* $x \in \mathbb{R}$ *and* $r \in \mathbb{N}$ *the integral representation*

$$\mathfrak{h}_r(x) = \frac{2\mathrm{i}(-1)^{r-2}}{\Gamma(r)} \int_0^\infty \frac{u^{r-1}}{\mathrm{e}^u+1} \sin\Big(\frac{r-2}{2}\,\pi + xu\Big)\mathrm{d}u\,. \tag{43}$$

Specifically, we have for $r = 1$ *and* 2,

$$\mathfrak{h}_1(x) = 2\mathrm{i} \int_0^\infty \frac{\cos(xu)}{\mathrm{e}^u+1}\, \mathrm{d}u \tag{44}$$

and

$$\mathfrak{h}_2(x) = 2\mathrm{i} \int_0^\infty u\, \frac{\sin(xu)}{\mathrm{e}^u+1}\, \mathrm{d}u\,. \tag{45}$$

Proof. Beginning with the representation

$$\mathfrak{h}_2(w) = \sum_{k\in\mathbb{Z}}^{e} \frac{(-1)^k \operatorname{sgn}(k)}{(w+\mathrm{i}k)^2} = 2\mathrm{i} \sum_{k\geqslant 1} \frac{(-1)^{k-1} 2kw}{(w^2+k^2)^2}\,, \tag{46}$$

we try to express the fraction $2kw(w^2+k^2)^{-2}$ as a Laplace transform, thus *via*

$$\frac{2sw}{(w^2+s^2)^2} = \mathscr{L}_u[u\,\sin(w\,u)](s) = \int_0^\infty \mathrm{e}^{-su}\,u\,\sin(wu)\,\mathrm{d}u\,.$$

Now, this transform is correct for $\operatorname{Re}\{s\} > |\operatorname{Im}\{w\}|$. But for the needed $s = k \in \mathbb{N}$, this inequality requires that $|\operatorname{Im}\{w\}| = 0$, so that w must be real, i.e. $w = x$.

Noting

$$\sum_{k\geqslant 1} (-1)^{k-1} \mathrm{e}^{-ku} = \frac{1}{\mathrm{e}^u+1}\,,$$

one has

$$\mathfrak{h}_2(x) = 2\mathrm{i} \sum_{k\geqslant 1} (-1)^{k-1} \int_0^\infty \mathrm{e}^{-ku}\,u\,\sin(xu)\,\mathrm{d}u = 2\mathrm{i} \int_0^\infty \frac{u\,\sin(xu)}{\mathrm{e}^u+1}\,\mathrm{d}u\,, \tag{47}$$

where, being $|\sin(xu)| \leqslant 1$, the interchange of sum and the integral is legitimate. This proves (45).

Repeated $r-2$-fold differentiation of $\mathfrak{h}_2(x)$ with respect to x according to (40), that is, (41), delivers

$$\begin{aligned}
\mathfrak{h}_r(x) &= \frac{2\mathrm{i}(-1)^r}{\Gamma(r)} \frac{\mathrm{d}^{r-2}}{\mathrm{d}x^{r-2}} \int_0^\infty u\,\frac{\sin(xu)}{\mathrm{e}^u+1}\,\mathrm{d}u \\
&= \frac{2\mathrm{i}(-1)^r}{\Gamma(r)} \int_0^\infty \frac{u^{r-1}}{\mathrm{e}^u+1}\,\sin\Big(\frac{r-2}{2}\,\pi + xu\Big)\mathrm{d}u,
\end{aligned}$$

which is (43), for all $r \in \mathbb{N}_2$.

It remains the case $r = 1$, which has to be considered separately. In turn, we have to connect the Hilbert–Eisenstein series $\mathfrak{h}_1(z)$, which converges in the sense of Eisenstein summation (but does not converges normally), and the normally convergent HE series $\mathfrak{h}_r(z), r \geqslant 2$, which is termwise integrable. Thus,

$$\begin{aligned}
\int_0^x \mathfrak{h}_2(t)\,\mathrm{d}t &= \sum_{k\in\mathbb{Z}}^{e} (-1)^k \operatorname{sgn}(k) \int_0^x \frac{\mathrm{d}t}{(t+\mathrm{i}k)^2} \\
&= \sum_{k\in\mathbb{Z}}^{e} \frac{(-1)^k \operatorname{sgn}(k)}{ik} - \sum_{k\in\mathbb{Z}}^{e} \frac{(-1)^k \operatorname{sgn}(k)}{x+ik} \\
&= 2\mathrm{i} \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k} - \mathfrak{h}_1(x) = 2\mathrm{i}\log 2 - \mathfrak{h}_1(x)\,.
\end{aligned}$$

On the other hand, the legitimate integration order exchange in (47) leads to

$$\begin{aligned}\int_0^x \mathfrak{h}_2(u)\mathrm{d}u &= 2\mathrm{i}\int_0^\infty \frac{u}{\mathrm{e}^u+1}\left\{\int_0^x \sin(xu)\,\mathrm{d}x\right\}\,\mathrm{d}u \\ &= 2\mathrm{i}\left\{\int_0^\infty \frac{1}{\mathrm{e}^u+1}\,\mathrm{d}u - \int_0^\infty \frac{\cos(xu)}{\mathrm{e}^u+1}\,\mathrm{d}u\right\} \\ &= 2\mathrm{i}\,\log 2 - 2\mathrm{i}\int_0^\infty \frac{\cos(xu)}{\mathrm{e}^u+1}\,\mathrm{d}u\,.\end{aligned}$$

The rest is clear. □

At this point let us recall integral representations which have been derived for the complete, real argument Omega function $\Omega(x)$. Among others we mentioned in Sect. 1 the results by Butzer, Pogány and Srivastava (1); consult [8, Theorem 2], by Pogány and Tomovski (2), and see [46, Theorem 3.3] and the Pogány–Srivastava integral representation (3) [39, p. 589, Theorem 1].

However, having in mind Theorem C and the differentiability property (41) in conjunction with Theorem 4.1, we now obtain a new integral representation formula for the complete Ω function and its derivatives *via* the rth order Hilbert–Eisenstein series $\mathfrak{h}_r$.

Theorem 4.2. *For $x \in \mathbb{R}$ and $r \in \mathbb{N}$, there holds true*

$$\Omega^{(r)}(x) = \frac{1}{2^r\,\pi}\int_0^\infty \frac{\mathrm{Re}\,\{\kappa_r(x;u)\}}{\mathrm{e}^u+1}\,\mathrm{d}u, \tag{48}$$

where

$$\kappa_r(x;u) = \left(\mathrm{e}^{\left(1+\frac{\mathrm{i}u}{\pi}\right)\frac{x}{2}} - (-1)^r\,\mathrm{e}^{-\left(1+\frac{\mathrm{i}u}{\pi}\right)\frac{x}{2}}\right)\left(1+\frac{\mathrm{i}u}{\pi}\right)^r\,.$$

Proof. Theorem C in conjunction with (44) implies

$$\Omega(2\pi x) = \frac{2}{\pi}\,\sinh(\pi x)\int_0^\infty \frac{\cos(x\,u)}{\mathrm{e}^u+1}\,\mathrm{d}u\,; \tag{49}$$

actually, by replacing $(2\pi x \mapsto x)$, we reobtained the integral expression (1). Differentiating this formula r times with respect to x, we get by virtue of property (41),

$$\Omega^{(r)}(x) = \frac{2}{\pi}\sum_{m=0}^r \binom{r}{m}\left\{\int_0^\infty \cos\left(\frac{xu}{2\pi}\right)\frac{\mathrm{d}u}{\mathrm{e}^u+1}\right\}^{(m)}\cdot\left\{\sinh\left(\frac{x}{2}\right)\right\}^{(r-m)}\,. \tag{50}$$

Differentiating the integral m times with respect to x, we get

$$\int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \cos\left(\frac{xu}{2\pi}\right) \frac{\mathrm{d}u}{\mathrm{e}^u+1} = \frac{\mathrm{i}^m}{2(2\pi)^m} \int_0^\infty \frac{u^m}{\mathrm{e}^u+1} \left(\mathrm{e}^{\mathrm{i}\frac{xu}{2\pi}} + (-1)^m \mathrm{e}^{-\mathrm{i}\frac{xu}{2\pi}}\right) \mathrm{d}u.$$

On the other hand,

$$\left\{\sinh\left(\frac{x}{2}\right)\right\}^{(r-m)} = \frac{1}{2^{r-m+1}} \left(\mathrm{e}^{\frac{x}{2}} - (-1)^{r-m}\mathrm{e}^{-\frac{x}{2}}\right).$$

Replacing both derivatives into (50), after suitable reduction of the material and summing up all terms inside the integrand, we arrive at the expression

$$\begin{aligned}\Omega^{(r)}(x) = \frac{1}{2^{r+1}\pi} \int_0^\infty \Big\{ &\left(\mathrm{e}^{\left(1+\frac{\mathrm{i}u}{\pi}\right)\frac{x}{2}} - (-1)^r\, \mathrm{e}^{-\left(1+\frac{\mathrm{i}u}{\pi}\right)\frac{x}{2}}\right) \left(1+\frac{\mathrm{i}u}{\pi}\right)^r \\ &+ \left(\mathrm{e}^{\left(1-\frac{\mathrm{i}u}{\pi}\right)\frac{x}{2}} - (-1)^r\, \mathrm{e}^{-\left(1-\frac{\mathrm{i}u}{\pi}\right)\frac{x}{2}}\right) \left(1-\frac{\mathrm{i}u}{\pi}\right)^r \Big\} \frac{\mathrm{d}u}{\mathrm{e}^u+1},\end{aligned}$$

which is equivalent to the statement. □

We mention that the HE series $\mathfrak{h}_r(w)$, or better still $w^r\, \mathfrak{h}_r(w)$, posses a Taylor series expansion, the coefficients even involving the Dirichlet Eta function values $\eta(2k+1)$, where the Dirichlet Eta function

$$\eta(s) = \sum_{k\geqslant 1} (-1)^{k-1} k^{-s}, \qquad \mathrm{Re}\{s\} > 0\,.$$

Indeed,

Theorem 4.3 ([4, p. 83, Theorem 9.1]). *For $w \in \mathbb{C}$, $|w| < 1$ and $r \in \mathbb{N}$, one has*

$$w\, \mathfrak{h}_1(w) = 2\mathrm{i} \sum_{k\geqslant 0} (-1)^k\, \eta(2k+1)\, w^{2k+1}$$

and

$$w^r\, \mathfrak{h}_r(w) = 2\mathrm{i}\, (-1)^{r-1} \sum_{k\geqslant [r/2]} (-1)^k \binom{2k}{r-1} \eta(2k+1)\, w^{2k+1},$$

with $\mathfrak{h}_1(0) = 2\mathrm{i}\, \log 2$.

Thus, $\mathfrak{h}_r(w)$ is holomorphic in $\mathbb{C} \setminus \mathrm{i}\mathbb{Z}$.

4.3 Let us briefly consider some connections between the $\mathfrak{h}_r(w)$ and alternating Mathieu series $\widetilde{S}_r(w)$.

Firstly, according to Proposition 4.1 (or evaluating $\mathfrak{h}_2(w)$ directly from its definition), having in mind (5) again, we have

$$\mathfrak{h}_2(w) = 2\mathrm{i}w \sum_{k\geqslant 1} \frac{(-1)^{k-1}2k}{(w^2+k^2)^2} = 2\mathrm{i}w\tilde{S}_2(w)\,.$$

As to the next step,

$$\mathfrak{h}_3(w) = -\mathrm{i} \sum_{k\geqslant 1} \frac{(-1)^{k-1}2k(k^2-3w^2)}{(w^2+k^2)^3}, \tag{51}$$

$$\tilde{S}_3(w) = \sum_{k\geqslant 1} \frac{(-1)^{k-1}2k}{(w^2+k^2)^3}\,.$$

Although the two look incomparable, see nevertheless Theorem 5.1 in Sect. 5.

For further systematic connections between $\tilde{S}_r(w)$ and $\mathfrak{h}_r(w)$, see Sect. 5.

4.4 We finally turn to a counterpart of the function $\Omega_\nu(w)$ introduced in (9). In regard to $\mathfrak{h}_r(w)$ we have seen that

$$\frac{\pi\,\Omega(2\pi w)}{\sinh(\pi w)} = -\mathrm{i}\mathfrak{h}_1(w)\,.$$

We define a new function, $\tilde{\Omega}_r(\cdot)$, say, in terms of the HE series $\mathfrak{h}_r(w)$ as follows. This answers a conjecture raised in [4, p. 82].

Definition 4.2. For all $w \in \mathbb{C} \setminus \mathrm{i}\mathbb{Z}$ and for all $r \in \mathbb{N}$, the extended Omega function $\tilde{\Omega}_r(\cdot)$ of positive integer order r is the function which satisfies equation

$$\frac{\pi\,\tilde{\Omega}_r(2\pi w)}{\sinh(\pi w)} = -\mathrm{i}\,\mathfrak{h}_r(w).$$

In fact, the new special function

$$\tilde{\Omega}_r(w) = -\frac{\mathrm{i}}{\pi}\,\sinh\left(\frac{w}{2}\right)\,\mathfrak{h}_r\left(\frac{w}{2\pi}\right), \qquad w \in \mathbb{C}\setminus \mathrm{i}\mathbb{Z},\;\; r\in\mathbb{N} \tag{52}$$

is the positive integer order counterpart of the function $\Omega_r(\cdot)$, defined already in terms of the alternating generalized Mathieu series $\tilde{S}_\nu(\cdot)$:

$$\frac{\pi\,\Omega_\nu(2\pi w)}{\sinh(\pi w)} = \sum_{k\geqslant 1} \frac{(-1)^{k-1}2k}{(k^2+w^2)^\nu} =: \tilde{S}_\nu(w)\,,$$

studied in [8] and here in Sect. 2 and Sect. 3, even for $\nu \in \mathbb{R}_+$.

Connecting Definition 4.2 and Theorem 4.1, we clearly arrive at the integral representation of the real variable alternating extended Omega function, recalling the integral expression for the Eta function

$$\eta(r) = \int_0^\infty \frac{u^{r-1}}{e^u + 1}\, du\,.$$

Theorem 4.4. *For any $r \in \mathbb{N}$ one has for $x \in \mathbb{R}$*

$$\tilde{\Omega}_r(x) = \frac{2(-1)^{r-1}}{\pi\, \Gamma(r)} \sinh\left(\frac{x}{2}\right) \int_0^\infty \frac{u^{r-1}}{e^u + 1} \sin\left(\frac{r-2}{2}\pi + \frac{xu}{2\pi}\right) du\,.$$

Moreover, in the same range of parameters we have the estimate

$$\left|\tilde{\Omega}_r(x)\right| \leqslant \frac{2}{\pi}\, \eta(r) \left|\sinh\left(\frac{x}{2}\right)\right|\,. \tag{53}$$

Remark 4.1. We recognize that the bound in the inequality (53) for $\tilde{\Omega}_r(x)$ is the same fashion result as the bound (24) achieved for the generalized BFH $\Omega_r(w)$.

5 Some Bridges Between $\boldsymbol{\Omega}_\nu(w)$, $\tilde{\boldsymbol{\Omega}}_r(w)$, $\tilde{S}_\nu(x)$ and $\mathfrak{h}_r(w)$

5.1 The alternating Mathieu series (5)

$$\tilde{S}_\nu(w) = \sum_{n\geqslant 1} \frac{(-1)^{n-1}\, 2n}{(n^2 + w^2)^\nu}, \qquad \nu > 0, w > 0\,,$$

and the Hilbert–Eisenstein series (38)

$$\mathfrak{h}_r(w) = \sum_{k\in\mathbb{Z}} \frac{(-1)^k \operatorname{sgn}(k)}{(w + ik)^r}, \qquad w \in \mathbb{C} \setminus i\mathbb{Z}, r \in \mathbb{N}$$

are intimately connected by (39), and relationship (9), e.g. we have seen in the previous section that

$$\mathfrak{h}_2(w) = 2i\, w\, \tilde{S}_2(w)\,.$$

However, higher-order alternating Mathieu and HE series do not coincide, but there is a close connection between them:

Theorem 5.1. *For all $w > 0, r \in \mathbb{N}_2$ there holds true*

$$\mathfrak{h}_r(w) = \frac{\mathrm{i}\,(-1)^{r-1}}{w^{r+1}} \sum_{\substack{0\leqslant m\leqslant r\\ r-m\equiv 1 (\mathrm{mod}\ 2)}} \sum_{j=\frac{r+m+1}{2}}^{r} \binom{r}{m}\binom{\frac{r+m+1}{2}}{r-j}(-w^2)^j\, \tilde{S}_j(w)\,. \tag{54}$$

Proof. Beginning with Definition 4.1, one transforms

$$\begin{aligned}\mathfrak{h}_r(w) &= \sum_{k\geqslant 1}\frac{(-1)^k}{(w+\mathrm{i}k)^r}\sum_{m=0}^{r}\binom{r}{m}\big((-1)^{r-m}-1\big)w^m(\mathrm{i}k)^{r-m}\\ &= \mathrm{i}\sum_{k\geqslant 1}\frac{2(-1)^{k-1}k}{(w^2+k^2)^r}\sum_{\substack{0\leqslant m\leqslant r\\ r-m\ \mathrm{odd}}}\binom{r}{m}w^m(-1)^{(r-m+1)/2}(k^2)^{(r-m-1)/2}\\ &= \mathrm{i}\sum_{\substack{0\leqslant m\leqslant r\\ r-m\ \mathrm{odd}}}\binom{r}{m}\sum_{j=0}^{(r-m-1)/2}\binom{\frac{r-m-1}{2}}{j}(-1)^{j-1}w^{r-1-2j}\,\tilde{S}_{r-j}(w)\,;\end{aligned}$$

changing the summation order, we get the asserted expression. □

The next few low-order particular cases, coming after $\mathfrak{h}_2(w)$, are

$$\mathfrak{h}_3(w) = -5w^2\mathrm{i}\,\tilde{S}_3(w) + 2\mathrm{i}\,\tilde{S}_2(w),$$

$$\mathfrak{h}_4(w) = -8w^3\mathrm{i}\,\tilde{S}_4(w) + 4w\mathrm{i}\,\tilde{S}_3(w).$$

The opposite question also arises, that is, how can we express alternating generalized Mathieu series *via* a linear combination of Hilbert–Eisenstein series of up to the same order? Collecting formulae for $\mathfrak{h}_j(w)$ in terms of $\tilde{S}_j(w)$, $j = 2,3,4$ and solving it with respect to Mathieu series $\tilde{S}_j(w)$, we get

$$\tilde{S}_2(w) = -\frac{\mathrm{i}}{2w}\,\mathfrak{h}_2(w),$$

$$\tilde{S}_3(w) = -\frac{\mathrm{i}}{5w^3}\,\mathfrak{h}_2(w) + \frac{\mathrm{i}}{5w^3}\,\mathfrak{h}_3(w),$$

$$\tilde{S}_4(w) = -\frac{\mathrm{i}}{10w^5}\,\mathfrak{h}_2(w) + \frac{\mathrm{i}}{10w^3}\,\mathfrak{h}_3(w) + \frac{\mathrm{i}}{8w^3}\,\mathfrak{h}_4(w), \quad \text{etc.}$$

As an immediate consequence of Theorem 5.1, we get by means of Definition 4.2 (of the HE series) the following connection between the generalized complete positive integer order BFH Omega function $\Omega_r(w)$ and its counterpart $\tilde{\Omega}_r(w)$.

Theorem 5.2. *Let the situation be the same as in previous* Theorem 5.1. *Then we have the following representation:*

$$\tilde{\Omega}_r(w) = \frac{(-1)^{r-1}}{w^{r+1}} \sum_{\substack{0\leqslant m\leqslant r\\ r-m\equiv 1 (\mathrm{mod}\ 2)}} \sum_{j=\frac{r+m+1}{2}}^{r} \binom{r}{m} \binom{\frac{r+m+1}{2}}{r-j} (-w^2)^j\, \Omega_j(w)\,.$$

Proof. Transforming representation (54) by Definition 4.2, (52) from one and Theorem 5.1 from another side, we derive

$$\begin{aligned}\tilde{\Omega}_r(w) &= -\frac{\mathrm{i}}{\pi}\, \sinh\left(\frac{w}{2}\right)\, \mathfrak{h}_r\left(\frac{w}{2\pi}\right)\\ &= \sum_{\substack{0\leqslant m\leqslant r\\ r-m\ \text{odd}}} \sum_{j=0}^{(r-m-1)/2} \binom{r}{m} \binom{\frac{r-m-1}{2}}{j} w^{r-1-2j} (-1)^{j-1}\\ &\quad\times \underbrace{\frac{1}{\pi}\, \sinh\left(\frac{w}{2}\right)\, \tilde{S}_{r-j}\left(\frac{w}{2\pi}\right)}_{\Omega_{r-j}(w)},\end{aligned}$$

which is equivalent to the assertion. □

Now, for $w > 0$ it is not hard to compile the formulae:

$$\begin{aligned}\tilde{\Omega}_2(w) &= \frac{2w}{\pi}\, \sinh\left(\frac{w}{2}\right)\, \tilde{S}_2\left(\frac{w}{2\pi}\right) = 2w\, \Omega_2(w),\\ \tilde{\Omega}_3(w) &= \frac{1}{\pi}\, \sinh\left(\frac{w}{2}\right) \left\{2\tilde{S}_2\left(\frac{w}{2\pi}\right) - 5w^2 \tilde{S}_3\left(\frac{w}{2\pi}\right)\right\}\\ &= 2\Omega_2(w) - 5w^2\, \Omega_3(w)\\ \tilde{\Omega}_4(w) &= \frac{4w}{\pi}\, \sinh\left(\frac{w}{2}\right) \left\{\tilde{S}_3\left(\frac{w}{2\pi}\right) - 2w^2 \tilde{S}_4\left(\frac{w}{2\pi}\right)\right\}\\ &= 4w\left\{\Omega_3(w) - 2w^2\, \Omega_4(w)\right\}.\end{aligned}$$

We point out that both Theorems 5.1 and 5.2 ensure a good tool for further bilateral bounding inequalities upon $\tilde{\Omega}_r(w)$. Namely, we establish in Sects. 2 and 3 numerous two-sided bounding inequalities for the generalized BFH Omega function $\Omega_j(w)$, $j \in \mathbb{N}$.

5.2 In the following, we study a series representation of the Hilbert–Eisenstein series in terms of the Gaussian hypergeometric function ${}_2F_1$. For the values w of the argument coming from the open unit disc $\mathbb{D} := \{w\colon |w| < 1\}$, we have the following expansion:

$$\begin{aligned}\mathfrak{h}_r(w) &= \mathrm{i}^{-r}\sum_{k\geqslant 1}\frac{(-1)^k}{k^r}\left\{\left(1+\frac{w}{\mathrm{i}k}\right)^{-r}-(-1)^r\left(1-\frac{w}{\mathrm{i}k}\right)^{-r}\right\}\\ &= \mathrm{i}^{-r}\sum_{k\geqslant 1}\frac{(-1)^k}{k^r}\sum_{j\geqslant 0}\binom{-r}{j}(1-(-1)^{r+j})\left(\frac{w}{\mathrm{i}k}\right)^j\\ &= \mathrm{i}^{-r}\sum_{k\geqslant 1}\frac{(-1)^k}{k^r}\sum_{j\geqslant 0}\frac{(-1)^j(1-(-1)^{r+j})\Gamma(r+j)}{\Gamma(r)\,j!}\left(\frac{w}{\mathrm{i}k}\right)^j.\end{aligned}$$

If r is either even or odd, we have more specific further results.

Theorem 5.3. *For all $|w| < 1$ and $r \in \mathbb{N}$, we have*

$$\mathfrak{h}_{2r-1}(w) = 2\mathrm{i}(-1)^r\sum_{k\geqslant 1}\frac{(-1)^{k-1}}{k^{2r-1}}\,{}_2F_1\left[\begin{matrix} r-\frac{1}{2},\, r\\ \frac{1}{2}\end{matrix}\,\middle|\, -\frac{w^2}{k^2}\right];$$

moreover,

$$\mathfrak{h}_{2r}(w) = 4wr\mathrm{i}(-1)^r\sum_{k\geqslant 1}\frac{(-1)^{k-1}}{k^{2r+1}}\,{}_2F_1\left[\begin{matrix} r+\frac{1}{2},\, r+1\\ \frac{3}{2}\end{matrix}\,\middle|\, -\frac{w^2}{k^2}\right].$$

Corollary 5.1. *For all $|w| < 1$ and $r \in \mathbb{N}$, there hold*

$$\mathfrak{h}_{2r-1}(w) = 2\mathrm{i}(-1)^r\sum_{k\geqslant 1}\frac{(-1)^{k-1}}{(w^2+k^2)^{r-\frac{1}{2}}}\cos\left((2r-1)\arctan\frac{w}{k}\right), \tag{55}$$

and

$$\mathfrak{h}_{2r}(w) = 2\mathrm{i}(-1)^r\sum_{k\geqslant 1}\frac{(-1)^{k-1}}{(w^2+k^2)^r}\sin\left(2r\arctan\frac{w}{k}\right). \tag{56}$$

Proof. By applying formulae

$${}_2F_1\left[\begin{matrix} r-\frac{1}{2},\, r\\ \frac{1}{2}\end{matrix}\,\middle|\, -\frac{w^2}{k^2}\right] = \frac{\cos\left((2r-1)\arctan\frac{w}{k}\right)}{\left(1+\frac{w^2}{k^2}\right)^{r-\frac{1}{2}}}$$

and

$$
{}_2F_1\left[\begin{matrix} r+\frac{1}{2},\, r+1 \\ \frac{3}{2} \end{matrix} \middle| -\frac{w^2}{k^2}\right] = \frac{k\,\sin\left(2r\,\arctan\dfrac{w}{k}\right)}{2rw\left(1+\dfrac{w^2}{k^2}\right)^r},
$$

we immediately conclude the asserted results (55) and (56), respectively. □

Setting $r = 2$ in the formula (56), we achieve the companion series representations to $\mathfrak{h}_3(w)$ (51), that is,

$$
\mathfrak{h}_4(w) = 8\mathrm{i}w\sum_{k\geqslant 1}\frac{(-1)^{k-1}k(k^2-w^2)}{(w^2+k^2)^4}.
$$

5.3 We introduce Dirichlet's Beta function as the series

$$
\beta(s) = \sum_{k\geqslant 0}\frac{(-1)^k}{(2k+1)^s}, \qquad \mathrm{Re}\{s\} > 0.
$$

We are now interested in a specific Hilbert–Eisenstein series, precisely in $\mathfrak{h}_r(\frac{\mathrm{i}}{2})$. Since

$$
\begin{aligned}
\mathfrak{h}_\nu\left(\frac{\mathrm{i}}{2}\right) &= \left(\frac{2}{\mathrm{i}}\right)^\nu\left\{\sum_{k\geqslant 0}\frac{(-1)^k}{(2k+1)^\nu}-1\right\} + \left(-\frac{2}{\mathrm{i}}\right)^\nu\sum_{k\geqslant 1}\frac{(-1)^{k-1}}{(2(k-1)+1)^\nu} \\
&= \left(\frac{2}{\mathrm{i}}\right)^\nu\left[(1+(-1)^\nu)\,\beta(\nu)-1\right],
\end{aligned}
$$

we have

$$
\frac{\mathfrak{h}_{\nu+\mu}\left(\frac{\mathrm{i}}{2}\right)}{\mathfrak{h}_\nu\left(\frac{\mathrm{i}}{2}\right)\mathfrak{h}_\mu\left(\frac{\mathrm{i}}{2}\right)} = \frac{(1+(-1)^{\nu+\mu})\,\beta(\nu+\mu)-1}{\left[(1+(-1)^\nu)\,\beta(\nu)-1\right]\left[(1+(-1)^\mu)\,\beta(\mu)-1\right]}.
$$

Now, choosing $\nu = 2r-1,\ \mu = 2s-1;\ r,s\in\mathbb{N}$, it follows

$$
\beta(2r+2s-2) = \frac{1}{2}\left\{\frac{\mathfrak{h}_{2r+2s-2}\left(\frac{\mathrm{i}}{2}\right)}{\mathfrak{h}_{2r-1}\left(\frac{\mathrm{i}}{2}\right)\mathfrak{h}_{2s-1}\left(\frac{\mathrm{i}}{2}\right)}+1\right\}, \qquad r,s\in\mathbb{N}.
$$

5.4 Finally, observe that the function

$$
\mu \mapsto \Gamma(2\mu)\mathfrak{h}_{2\mu}(\mathrm{i}w) = 2\int_0^\infty \frac{u^{2\mu-1}}{\mathrm{e}^u+1}\sinh(wu)\mathrm{d}u
$$

is logarithmically convex on $(0, \infty)$ for $w \in \mathbb{R} \setminus \mathbb{Z}$. This can be verified by using the classical Hölder-Rogers inequality for integrals or by using the fact that the integrand is logarithmically convex in μ and the integral preserves the logarithmical convexity. Consequently for all $\mu_1, \mu_2 > 0$, we have

$$\Gamma^2(\mu_1 + \mu_2)\mathfrak{h}^2_{\mu_1+\mu_2}(\mathrm{i}w) \leqslant \Gamma(2\mu_1)\mathfrak{h}_{2\mu_1}(\mathrm{i}w)\Gamma(2\mu_2)\mathfrak{h}_{2\mu_2}(\mathrm{i}w)$$

and choosing $\mu_1 = m - 1$ and $\mu_2 = m + 1$, we arrive at the Turán-type inequality

$$\mathfrak{h}^2_{2m}(\mathrm{i}w) \leq \frac{2m(2m+1)}{(2m-1)(2m-2)}\mathfrak{h}_{2m-2}(\mathrm{i}w)\mathfrak{h}_{2m+2}(\mathrm{i}w),$$

which holds for all $m \in \{2, 3, \dots\}$ and $w \in \mathbb{R} \setminus \mathbb{Z}$.

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