

Topics in Special Functions III

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Dedicated to Professor Hari M. Srivastava

Abstract The authors provide a survey of recent results in special functions of classical analysis and geometric function theory, in particular, the circular and hyperbolic functions, the gamma function, the elliptic integrals, the Gaussian hypergeometric function, power series, and mean values.

1 Introduction

The study of quasiconformal maps led the first two authors in their joint work with Vamanamurthy to formulate open problems or questions involving special functions [14, 16]. During the past two decades, many authors have contributed to the solution of these problems. However, most of the problems posed in [14] are still open.

The present paper is the third in a series of surveys by the first two authors, the previous papers [20, 23] being written jointly with the late Vamanamurthy. The aim of this series of surveys is to review the results motivated by the problems in [14, 16] and related developments during the past two decades. In the first of these we studied classical special functions, and in the next we focused on special functions occurring in the distortion theory of quasiconformal maps. Regretfully, Vamanamurthy passed

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away in 2009, and the remaining authors acknowledge his crucial role in our joint work. For an update to the bibliographies of [20, 23] the reader is referred to [12].

In 1993 the following monotone rule was derived [17, Lemma 2.2]. Though simple to state and easy to prove by means of the Cauchy Mean Value Theorem, this l'Hôpital Monotone Rule (LMR) has had wide application to special functions by many authors. Vamanamurthy was especially skillful in the application of this rule. We here quote the rule as it was restated in [21, Theorem 2].

Theorem 1.1 (l'Hôpital Monotone Rule). *Let $-\infty < a < b < \infty$, and let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous functions that are differentiable on (a, b) , with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x \in (a, b)$. If f'/g' is increasing (decreasing) on (a, b) , then so is f/g .*

Theorem 1.1 assumes that a and b are finite, but the rule can be extended easily by similar methods to the case where a or b is infinite. The LMR has been used effectively in the study of the monotonicity of a quotient of two functions. For instance, Pinelis' note [146] shows the potential of the LMR. As a complement to Pinelis' note, the paper [21] contains many applications of LMR in calculus. Also the history of LMR is reviewed there.

In this survey we give an account of the work in the special functions of classical analysis and geometric function theory since our second survey. In many of these results the LMR was an essential tool. Because of practical constraints, we have had to exclude many fine papers and have limited our bibliography to those papers most closely connected to our work.

The aim of our work on special functions has been to solve open problems in quasiconformal mapping theory. In particular, we tried to settle Mori's conjecture for quasiconformal mappings [127] (see also [118, p. 68]). For the formulation of this problem, let $K > 1$ be fixed and let $M(K)$ be the least constant such that

$$|f(z_1) - f(z_2)| \leq M(K)|z_1 - z_2|^{1/K}, \quad \text{for all } z_1, z_2 \in B,$$

for every K -quasiconformal mapping $f : B \rightarrow B$ of the unit disk B onto itself with $f(0) = 0$. A. Mori conjectured in 1956 that $M(K) \leq 16^{1-1/K}$. This conjecture is still open in 2013. Some of the open problems that we found will be discussed in the last section.

2 Generalizations of Jordan's Inequality

The LMR application list, begun in [21], led to the Master's thesis of Visuri, on which [109] is based. Furthermore, applications of LMR to trigonometric inequalities were given in [109]. Numerous further applications to trigonometric functions were found by many authors, and some of these papers are reviewed in this section and the next.

By elementary geometric methods one can prove that

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1, \quad 0 < x \leq \frac{\pi}{2},$$

a result known as *Jordan's inequality*. In a recent work, Klén et al. [109] have obtained the inequalities

$$\cos^2 \frac{x}{2} \leq \frac{\sin x}{x} \leq \cos^3 \frac{x}{3}, \quad x \in (-\sqrt{27/5}, \sqrt{27/5})$$

and

$$\cosh^{1/4} x < \frac{\sinh x}{x} < \cosh^{1/2} x, \quad x \in (0, 1).$$

Inspired by these results, Lv, Wang, and Chu [121] proved that, for $a = (\log(\pi/2))/\log \sqrt{2} \approx 1.30299$,

$$\cos^{4/3} \frac{x}{2} < \frac{\sin x}{x} < \cos^a \frac{x}{2}, \quad x \in (0, \pi/2),$$

where $4/3$ and a are best constants and that for $b = (\log \sinh 1)/(\log \cosh 1) \approx 0.372168$,

$$\cosh^{1/3} x < \frac{\sinh x}{x} < \cosh^b x, \quad x \in (0, 1),$$

where $1/3$ and b are best constants.

Many authors have generalized or sharpened Jordan's inequality, either by replacing the bounds by finite series or hyperbolic functions or by obtaining analogous results for other functions such as hyperbolic or Bessel functions. The comprehensive survey paper by Qi et al. [150] gives a clear picture of these developments as of 2009. For example, in 2008 Niu et al. [143] obtained the sharp inequality

$$\frac{2}{\pi} + \sum_{k=1}^n \alpha_k (\pi^2 - 4x^2)^k \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \sum_{k=1}^n \beta_k (\pi^2 - 4x^2)^k, \quad 0 < x \leq \pi/2,$$

for each natural number n , with best possible constants α_k and β_k . That same year Wu and Srivastava [198] obtained upper and lower estimates on $(0, \pi/2]$ for $(\sin x)/x$ that are finite series in powers of $(x - \theta)$, where $\theta \in [x, \pi/2]$, while Zhu [211] obtained bounds as finite series in powers of $(\pi^2 - 4x^2)$. Zhu [210] obtained bounds for $(\sin x)/x$ as finite series in powers of $(r^2 - x^2)$ for $0 < x \leq r \leq \pi/2$, yielding a new infinite series

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} a_n(r^2 - x^2)^n, \text{ for } 0 < |x| \leq r \leq \pi/2.$$

Yang [199] showed that a function f admits an infinite series expansion of the above type if and only if f is analytic and even.

In 2011 Huo et al. [97] obtained the following generalization of Jordan’s inequalities:

$$\sum_{k=1}^n \mu_k(\theta^t - x^t)^k \leq \frac{\sin x}{x} - \frac{\sin \theta}{\theta} \leq \sum_{k=1}^n \omega_k(\theta^t - x^t)^k$$

for $t \geq 2$, $n \in \mathbb{N}$, and $0 < x \leq \theta < \pi$, where the coefficients μ_k and ω_k are defined recursively and are best possible.

More recently, in 2012, Chen and Debnath [74] have proved that, for $0 < x \leq \pi/2$,

$$f_1(x) \leq \frac{\sin x}{x} \leq f_2(x),$$

where

$$f_1(x) = \frac{2}{\pi} + \frac{2\pi^{-\theta-1}}{\theta}(\pi^\theta - (2x)^\theta) + \frac{(-\pi^2 + 4 + 4\theta)\pi^{-2\theta-1}}{4\theta^2}(\pi^\theta - (2x)^\theta)^2$$

and

$$f_2(x) = \frac{2}{\pi} + \frac{2\pi^{-\theta-1}}{\theta}(\pi^\theta - (2x)^\theta) + \frac{((\pi - 2)\theta - 2)\pi^{-2\theta-1}}{\theta}(\pi^\theta - (2x)^\theta)^2,$$

for any $\theta \geq 2$, with equality when $x = \pi/2$.

In a recent work Sándor [164] (see also [165, p. 9]) proved that $h(x) \equiv [\log(x/\sin x)]/\log((\sinh x)/x)$ is strictly increasing on $(0, \pi/2)$. He used this result to prove that the best positive constants p and q for which

$$\left(\frac{\sinh x}{x}\right)^p < \frac{x}{\sin x} < \left(\frac{\sinh x}{x}\right)^q$$

is true are $p = 1$ and $q = [\log(\pi/2)]/\log((\sinh(\pi/2))/(\pi/2)) \approx 1.18$.

In an unpublished manuscript, Barbu and Pişcoran [28] have proved, in particular, that

$$(1 - x^2/3)^{-1/4} < \frac{\sinh x}{x} < 1 + \frac{x^2}{5}, \quad x \in (0, 1).$$

Kuo [116] has developed a method of obtaining an increasing sequence of lower bounds and a decreasing sequence of upper bounds for $(\sin x)/x$, and he has conjectured that the two sequences converge uniformly to $(\sin x)/x$.

Since there is a close connection between the function $(\sin x)/x$ and the Bessel function $J_{1/2}(x)$ (cf. [219]), it is natural for authors to seek analogs of the Jordan inequality for Bessel and closely related functions. Baricz and Wu [35, 40], Zhu [219, 220], and Niu et al. [144] have produced inequalities of this type. Zhu [221] has also obtained Jordan-type inequalities for $((\sin x)/x)^p$ for any $p > 0$. Wu and Debnath [195] have generalized Jordan's inequality to functions $f(x)/x$ on $[0, \theta]$ such that f is $(n + 1)$ -times differentiable, $f(0) = 0$, and either n is a positive even integer with $f^{(n+1)}$ increasing on $[0, \theta]$ or n is a positive odd integer with $f^{(n+1)}$ decreasing on $[0, \theta]$.

3 Other Inequalities Involving Circular and Hyperbolic Functions

3.1 Redheffer

In 1968 Redheffer [157] proposed the problem of showing that

$$\frac{\sin \pi x}{\pi x} \geq \frac{1 - x^2}{1 + x^2}, \text{ for all real } x \tag{1}$$

or, equivalently, that

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \text{ for all real } x. \tag{2}$$

A solution of this problem was provided by Williams [192], using infinite products, who also proved the stronger inequality

$$\frac{\sin \pi x}{\pi x} \geq \frac{1 - x^2}{1 + x^2} + \frac{(1 - x)^2}{x(1 + x^2)}, \text{ for } x \geq 1.$$

Later, using Erdős-Turán series and harmonic analysis, Li and Li [120] proved the double inequality

$$\frac{(1 - x^2)(4 - x^2)(9 - x^2)}{x^6 - 2x^4 + 13x^2 + 36} \leq \frac{\sin \pi x}{\pi x} \leq \frac{1 - x^2}{\sqrt{1 + 3x^4}}, \text{ for } 0 < x < 1.$$

They also found a method for obtaining new bounds from old for $(\sin x)/x$, but Kuo [116] gave an example to show that the new bounds are not necessarily stronger.

In 2003 Chen et al. [76], using mathematical induction and infinite products, found analogs of the Redheffer inequality for $\cos x$:

$$\cos x \geq \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}, \text{ for } |x| \leq \frac{\pi}{2},$$

and for hyperbolic functions

$$\frac{\sinh x}{x} \leq \frac{\pi^2 + x^2}{\pi^2 - x^2}, \text{ for } 0 < |x| \leq \pi; \quad \cosh x \leq \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}, \text{ for } |x| \leq \frac{\pi}{2}.$$

In 2008, inspired by the inequalities above, Zhu and Sun [224] proved that

$$\left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^\alpha \leq \cos x \leq \left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^\beta, \text{ for } 0 \leq x \leq \frac{\pi}{2},$$

with best possible constants $\alpha = 1$ and $\beta = \pi^2/16$, and

$$\left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^\gamma \leq \frac{\sin x}{x} \leq \left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^\delta, \text{ for } 0 < x < \pi,$$

with best possible constants $\gamma = 1$ and $\delta = \pi^2/12$. They obtained similar results for the hyperbolic sine and cosine functions. In 2009 Zhu [216] showed that

$$\left(\frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}}\right)^\alpha \leq \frac{\sin x}{x} \leq \left(\frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}}\right)^\beta, \quad 0 < x \leq \pi,$$

holds if and only if $\alpha \geq \pi^2/6$ and $\beta \leq 1$, with analogous results for $\cos x$ and $(\tan x)/x$. In 2009 Baricz and Wu [41] and in 2011 Zhu [222] proved Redheffer-type inequalities for Bessel functions.

3.2 Cusa-Huygens

The inequality

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}, \quad 0 < x < \pi/2$$

was discovered by N. de Cusa in the fifteenth century (cf. [71]) and proved rigorously by Huygens [98] in the seventeenth century. In 2009 Zhu [218] obtained the following inequalities of Cusa-Huygens type:

$$\left(\frac{\sin x}{x}\right)^\alpha < \frac{2}{3} + \frac{1}{3}(\cos x)^\alpha, \quad 0 < x < \frac{\pi}{2}, \quad \alpha \geq 1,$$

and

$$\left(\frac{\sinh x}{x}\right)^\alpha < \frac{2}{3} + \frac{1}{3}(\cosh x)^\alpha, \quad x > 0, \quad \alpha \geq 1.$$

That same year Zhu [214] discovered a more general set of inequalities of Cusa type, from which many other types of inequalities for circular functions can be derived. He proved the following: Let $0 < x < \pi/2$. If $p \geq 1$, then

$$(1 - \alpha) + \alpha(\cos x)^p < \left(\frac{\sin x}{x}\right)^p < (1 - \beta) + \beta(\cos x)^p \tag{3}$$

if and only if $\beta \leq 1/3$ and $\alpha \geq 1 - (2/\pi)^p$. If $0 \leq p \leq 4/5$, then (3) holds if and only if $\alpha \geq 1/3$ and $\beta \leq 1 - (2/\pi)^p$. If $p < 0$, then the second inequality in (3) holds if and only if $\beta \geq 1/3$. In a later paper [219] Zhu obtained estimates for $(\sin x)/x$ and $(\sinh x)/x$ that led to new infinite series for these functions. For some similar results see also [194].

In 2011 Chen and Cheung [71] obtained the sharp Cusa-Huygens-type inequality

$$\left(\frac{2 + \cos x}{3}\right)^\alpha < \frac{\sin x}{x} < \left(\frac{2 + \cos x}{3}\right)^\beta,$$

for $0 < x < \pi/2$, with best possible constants $\alpha = (\log(\pi/2))/\log(3/2) \approx 1.11$ and $\beta = 1$.

In 2011 Neuman and Sándor [142] discovered a pair of optimal inequalities for hyperbolic and trigonometric functions, proving that, for $0 < x < \pi/2$, the best positive constants p and q in the inequality

$$\frac{1}{(\cosh x)^p} < \frac{\sin x}{x} < \frac{1}{(\cosh x)^q}$$

are $p = (\log(\pi/2))/\log \cosh(\pi/2) \approx 0.49$ and $q = 1/3$ and that for $x \neq 0$ the best positive constants p and q in the inequality

$$\left(\frac{\sinh x}{x}\right)^p < \frac{2}{\cos x + 1} < \left(\frac{\sinh x}{x}\right)^q$$

are $p = 3/2$ and $q = (\log 2)/\log[(\sinh(\pi/2))/(\pi/2)] \approx 1.82$.

3.3 Becker-Stark

In 1978 Becker and Stark [49] obtained the double inequality

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}, \quad 0 < x < \frac{\pi}{2},$$

where the numerator constants 8 and π^2 are best possible.

In 2008 Zhu and Sun [224] showed that

$$\left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^\alpha \leq \frac{\tan x}{x} \leq \left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^\beta, \quad 0 < x < \frac{\pi}{2},$$

holds if and only if $\alpha \leq \pi^2/24$ and $\beta \geq 1$.

In 2010 Zhu and Hua [223] sharpened the Becker-Stark inequality by proving that

$$\frac{\pi^2 + \alpha x^2}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 + \beta x^2}{\pi^2 - 4x^2}, \quad 0 < x < \frac{\pi}{2},$$

where $\alpha = 4(8 - \pi^2)/\pi^2 \approx -0.76$ and $\beta = \pi^2/3 - 4 \approx -0.71$ are the best possible constants. They also developed a systematic method for obtaining a sequence of sharp inequalities of this sort.

In 2011 Ge [88] obtained

$$\frac{8}{\pi^2 - 4x^2} + \left(1 - \frac{8}{\pi^2}\right) < \frac{\tan x}{x} < \frac{\pi^4}{12} \frac{1}{\pi^2 - 4x^2} + \left(1 - \frac{\pi^2}{12}\right),$$

for $0 < x < \pi/2$. That same year Chen and Cheung [71] proved the sharp Becker-Stark-type inequality

$$\left(\frac{\pi^2}{\pi^2 - 4x^2}\right)^\alpha < \frac{\tan x}{x} < \left(\frac{\pi^2}{\pi^2 - 4x^2}\right)^\beta,$$

with best possible constants $\alpha = \pi^2/12 \approx 0.82$ and $\beta = 1$.

3.4 Wilker

In 1989 Wilker [190] posed the problem of proving that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad \text{for } 0 < x < \frac{\pi}{2} \tag{4}$$

and of finding

$$c \equiv \inf_{0 < x < \pi/2} \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x}. \tag{5}$$

Anglesio et al. [191] showed that the function in (5) is decreasing on $(0, \pi/2)$, that the value of c in (5) is $16/\pi^4$, and that, moreover, the supremum of the expression in (5) on $(0, \pi/2)$ is $8/45$. Hence

$$2 + \frac{16}{\pi^4}x^3 \tan x \leq \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} \leq 2 + \frac{8}{45}x^3 \tan x, \tag{6}$$

for $0 < x < \pi/2$, where $16/\pi^4 \approx 0.164$ and $8/45 \approx 0.178$ are best possible constants. (Note: [21] erroneously quoted [191] as saying that the function in (5) is increasing.) In 2007 Wu and Srivastava [197] proved the Wilker-type inequality

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \text{ for } 0 < x < \frac{\pi}{2}. \tag{7}$$

However, Baricz and Sándor [39] discovered that (7) is implied by (4).

In 2009 Zhu [218] generalized (4) and obtained analogs for hyperbolic functions, showing that, for $0 < x < \pi/2, \alpha \geq 1$,

$$\left(\frac{\sin x}{x}\right)^{2\alpha} + \left(\frac{\tan x}{x}\right)^\alpha > \left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\tan x}\right)^\alpha > 2$$

and that, for $x > 0, \alpha \geq 1$,

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} + \left(\frac{\tanh x}{x}\right)^\alpha > \left(\frac{x}{\sinh x}\right)^{2\alpha} + \left(\frac{x}{\tanh x}\right)^\alpha > 2.$$

These two results of Zhu are special cases of a recent lemma due to Neuman [138, Lemma 2].

In 2012 Sándor [162] has proved that, for $0 < x \leq \pi/2, \alpha > 0$,

$$\left(\frac{x}{\sin x}\right)^{2\alpha} + \left(\frac{x}{\sinh x}\right)^\alpha > \left(\frac{\sinh x}{x}\right)^{2\alpha} + \left(\frac{\sin x}{x}\right)^\alpha > 2.$$

Using power series, Chen and Cheung [72] obtained the following sharper versions of (6):

$$\frac{16}{315}x^5 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - \left[2 + \frac{8}{45}x^4\right] < \left(\frac{2}{\pi}\right)^6 x^5 \tan x, \tag{8}$$

and

$$\frac{104}{4725}x^7 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - \left[2 + \frac{8}{45}x^4 + \frac{16}{315}x^6\right] < \left(\frac{2}{\pi}\right)^8 x^7 \tan x. \tag{9}$$

The constants $16/315 \approx 0.051$ and $(2/\pi)^6 \approx 0.067$ in (8) and $104/4725 \approx 0.022$ and $(2/\pi)^8 \approx 0.027$ in (9) are best possible. For $0 < x < \pi/2$, Chen and Cheung also obtained upper estimates complementary to (7):

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \frac{2}{45}x^3 \tan x$$

and

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \frac{2}{45}x^4 + \frac{8}{945}x^5 \tan x,$$

where the constants $2/45$ and $8/945$ are best possible.

In 2012, Sándor [164] has shown that

$$\frac{\sin x}{x} + q \frac{\sinh x}{x} > q + 1, \quad x \neq 0$$

and

$$\left(\frac{\sinh x}{x}\right)^q + \frac{\sin x}{x} > 2, \quad 0 < x < \frac{\pi}{2},$$

where $q = [\log(\pi/2)] / \log[(\sinh(\pi/2))/(\pi/2)] \approx 1.18$.

Extensions of the generalized Wilker inequality for Bessel functions were obtained by Baricz and Sándor [39] in 2008.

3.5 Huygens

An older inequality due to Huygens [98] is similar in form to (4):

$$2 \left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3, \quad \text{for } 0 < |x| < \frac{\pi}{2} \quad (10)$$

and actually implies (4) (see [141]). In 2009, Zhu [217] obtained the following inequalities of Huygens type:

$$(1-p) \frac{\sin x}{x} + p \frac{\tan x}{x} > 1 > (1-q) \frac{\sin x}{x} + q \frac{\tan x}{x}$$

for all $x \in (0, \pi/2)$ if and only if $p \geq 1/3$ and $q \leq 0$;

$$(1-p) \frac{\sinh x}{x} + p \frac{\tanh x}{x} > 1 > (1-q) \frac{\sinh x}{x} + q \frac{\tanh x}{x}$$

for all $x \in (0, \infty)$ if and only if $p \leq 1/3$ and $q \geq 1$;

$$(1-p) \frac{x}{\sin x} + p \frac{x}{\tan x} > 1 > (1-q) \frac{x}{\sin x} + q \frac{x}{\tan x}$$

for all $x \in (0, \pi/2)$ if and only if $p \leq 1/3$ and $q \geq 1 - 2/\pi$; and

$$(1 - p) \frac{x}{\sinh x} + p \frac{x}{\tanh x} > 1 > (1 - q) \frac{x}{\sinh x} + q \frac{x}{\tanh x}$$

for all $x \in (0, \infty)$ if and only if $p \geq 1/3$ and $q \leq 0$.

In 2012 Sándor [162] has showed that, for $0 < x \leq \pi/2, \alpha > 0$,

$$2 \left(\frac{\sinh x}{x} \right)^\alpha + \left(\frac{\sin x}{x} \right)^\alpha > 2 \left(\frac{x}{\sin x} \right)^\alpha + \left(\frac{x}{\sinh x} \right)^\alpha > 3.$$

Chen and Cheung [72] also found sharper versions of (10) as follows: For $0 < x < \pi/2$,

$$3 + \frac{3}{20}x^3 \tan x < 2 \left(\frac{\sin x}{x} \right) + \frac{\tan x}{x} < 3 + \left(\frac{2}{\pi} \right)^4 x^3 \tan x \tag{11}$$

and

$$\frac{3}{56}x^5 \tan x < 2 \left(\frac{\sin x}{x} \right) + \frac{\tan x}{x} - \left[3 + \frac{3}{20}x^4 \right] < \left(\frac{2}{\pi} \right)^6 x^5 \tan x, \tag{12}$$

where the constants $3/20 = 0.15$ and $(2/\pi)^4 \approx 0.16$ in (11) and $3/56 \approx 0.054$ and $(2/\pi)^6 \approx 0.067$ in (12) are best possible.

Recently Hua [96] have proved the following sharp inequalities: For $0 < |x| < \pi/2$,

$$3 + \frac{1}{40}x^3 \sin x < \frac{\sin x}{x} + 2 \frac{\tan(x/2)}{x/2} < 3 + \frac{80 - 24\pi}{\pi^4} x^3 \sin x,$$

where the constants $1/40$ and $(80 - 24\pi)/\pi^4$ are best possible, and, for $x \neq 0$,

$$3 + \frac{3}{20}x^3 \tanh x < 2 \frac{\sinh x}{x} + \frac{\tanh x}{x} < 3 + \frac{3}{20}x^3 \sinh x,$$

where the constant $3/20$ is best possible.

3.6 Shafer

The problem of proving

$$\arctan x > \frac{3x}{1 + 2\sqrt{1 + x^2}}, \quad x > 0,$$

was proposed by Shafer [166] in 1966. Solutions were obtained by Grinstein, Marsh, and Konhauser [169] in 1967. In 2011 Chen, Cheung, and Wang [73] found, for each $a > 0$, the largest number b and the smallest number c such that the inequalities

$$\frac{bx}{1 + a\sqrt{1 + x^2}} \leq \arctan x \leq \frac{cx}{1 + a\sqrt{1 + x^2}}$$

are valid for all $x \geq 0$. Their answer to this question is indicated in the following table:

a	Largest b	Smallest c
$0 < a \leq \pi/2$	$b = \pi a/2$	$c = 1 + a$
$\pi/2 < a \leq 2/(\pi - 2)$	$b = 4(a^2 - 1)/a^2$	$c = 1 + a$
$2/(\pi - 2) < a < 2$	$b = 4(a^2 - 1)/a^2$	$c = \pi a/2$
$2 \leq a < \infty$	$b = 1 + a$	$c = \pi a/2$

In 1974, in a numerical analytical context [167], Shafer presented the inequality

$$\arctan x \geq \frac{8x}{3 + \sqrt{25 + (80/3)x^2}}, \quad x > 0,$$

which he later proved analytically [168]. In [213] Zhu proved that the constant $80/3$ in Shafer’s inequality is best possible and also obtained the complementary inequality

$$\arctan x < \frac{8x}{3 + \sqrt{25 + (256/\pi^2)x^2}}, \quad x > 0,$$

where $256/\pi^2$ is the best possible constant.

3.7 Fink

In [132, p. 247], there is a lower bound for $\arcsin x$ on $[0, 1]$ that is similar to Shafer’s for $\arctan x$. In 1995 Fink [87] supplied a complementary upper bound. The resulting double inequality is

$$\frac{3x}{2 + \sqrt{1 - x^2}} \leq \arcsin x \leq \frac{\pi x}{2 + \sqrt{1 - x^2}}, \quad 0 \leq x \leq 1, \tag{13}$$

and both numerator constants are best possible. Further refinements of these inequalities, along with analogous ones for $\operatorname{arcsinh} x$, were obtained by Zhu [212]

and by Pan with Zhu [145]. We note that, for $0 < x < 1$, this double inequality is equivalent to

$$\frac{2 + \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}, \quad 0 < x < \frac{\pi}{2},$$

in which the second relation is the Cusa inequality.

3.8 Carlson

In 1970 Carlson [67, (1.14)] proved the inequality

$$\frac{6\sqrt{1-x}}{2\sqrt{2} + \sqrt{1+x}} < \arccos x < \frac{\sqrt[3]{4} \cdot \sqrt{1-x}}{(1+x)^{1/6}}, \quad 0 \leq x < 1. \tag{14}$$

In 2012, seeking to sharpen and generalize (14), Chen and Mortici [75] determined, for each fixed $c > 0$, the largest number a and smallest number b such that the double inequality

$$\frac{a\sqrt{1-x}}{c + \sqrt{1+x}} \leq \arccos x \leq \frac{b\sqrt{1-x}}{c + \sqrt{1+x}}$$

is valid for all $x \in [0, 1]$. Their answer to this question is indicated in the following table:

c	Largest a	Smallest b
$0 < x < (2\pi - 4)/(4 - \pi)$	$(1 + a)\pi/2$	$2 + \sqrt{2}a$
$(2\pi - 4)/(4 - \pi) \leq x \leq (4 - \pi)/(\pi - 2\sqrt{2})$	$8(a^2 - 2)/a^2$	$2 + \sqrt{2}a$
$(4 - \pi)/(\pi - 2\sqrt{2}) < x < 2\sqrt{2}$	$4(a^2 - 1)/a^2$	$(1 + a)\pi/2$
$2\sqrt{2} \leq x < \infty$	$2 + \sqrt{2}a$	$(1 + a)\pi/2$

These authors also proved that, for all $x \in [0, 1]$, the inequalities

$$\frac{\sqrt[3]{4} \cdot \sqrt{1-x}}{a + (1+x)^{1/6}} \leq \arccos x \leq \frac{\sqrt[3]{4} \cdot \sqrt{1-x}}{b + (1+x)^{1/6}}$$

hold on $[0, 1]$, with best constants $a = (2\sqrt[3]{4} - \pi)/\pi \approx 0.01$ and $b = 0$.

Moreover, in view of the right side of (14), in 2011 Chen, Cheung, and Wang [73] considered functions of the form

$$f(x) \equiv \frac{r(1-x)^p}{(1+x)^q}$$

on $[0, 1]$ and determined the values of p, q, r such that $f(x)$ is the best third-order approximation of $\arccos x$ in a neighborhood of the origin. The answer is that, for $p = (\pi + 2)/\pi^2, q = (\pi - 2)/\pi^2, r = \pi/2$, one has

$$\lim_{x \rightarrow 0} \frac{\arccos x - f(x)}{x^3} = \frac{\pi^2 - 8}{6\pi^2}.$$

With the values of p, q, r stated above, the authors were led to a new lower bound for \arccos :

$$\arccos x \geq \frac{(\pi/2)(1-x)^{(\pi+2)/\pi^2}}{(1+x)^{(\pi-2)/\pi^2}}, \quad 0 < x \leq 1.$$

3.9 Lazarević

In [117] Lazarević proved that, for $x \neq 0$,

$$\left(\frac{\sinh x}{x}\right)^q > \cosh x$$

if and only if $q \geq 3$. Zhu improved upon this inequality in [215] by showing that if $p > 1$ or $p \leq 8/15$, then

$$\left(\frac{\sinh x}{x}\right)^q > p + (1-p) \cosh x$$

for all $x > 0$ if and only if $q \geq 3(1-p)$. For some similar results see also [194].

In 2008 Baricz [34] extended the Lazarević inequality to modified Bessel functions and also deduced some Turán- and Lazarević-type inequalities for the confluent hypergeometric functions.

3.10 Neuman

Neuman [137] has recently established several inequalities involving new combinations of circular and hyperbolic functions. In particular, he has proved that if $x \neq 0$, then

$$(\cosh x)^{2/3} < \frac{\sinh x}{\arcsin(\tanh x)} < \frac{1 + 2 \cosh x}{3},$$

$$[(\cosh 2x)^{1/2} \cosh^2 x]^{1/3} < \frac{\sinh x}{\operatorname{arcsinh}(\tanh x)} < \frac{(\cosh 2x)^{1/2} + 2 \cosh x}{3},$$

and

$$[(\cosh 2x) \cosh x]^{1/3} < \frac{\sinh x}{\arctan(\tanh x)} < \frac{2(\cosh 2x)^{1/2} + \cosh x}{3}.$$

4 Euler’s Gamma Function

For $\operatorname{Re} z > 0$ the *gamma function* is defined by

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} dt,$$

and the definition is extended by analytic continuation to the entire complex plane minus the set of nonpositive integers. This function, discovered by Leonhard Euler in 1729, is a natural generalization of the factorial, because of the functional identity

$$\Gamma(z + 1) = z\Gamma(z).$$

The gamma function is one of the best-known and most important special functions in mathematics and has been studied intensively.

We begin our treatment of this subject by considering an important special constant discovered by Euler and related to the gamma function.

4.1 The Euler-Mascheroni Constant and Harmonic Numbers

The *Euler-Mascheroni constant* $\gamma = 0.5772156649\dots$ is defined as

$$\gamma \equiv \lim_{n \rightarrow \infty} \gamma_n, \tag{15}$$

where $\gamma_n \equiv H_n - \log n$, $n \in \mathbb{N}$ and where H_n are the *harmonic numbers*

$$H_n \equiv \sum_{k=1}^n \frac{1}{k} = \int_0^1 \frac{1 - x^n}{1 - x} dx. \tag{16}$$

The number γ is one of the most important constants in mathematics and is useful in analysis, probability theory, number theory, and other branches of pure and applied mathematics. The numerical value of γ is known to 29,844,489,545 decimal places, thanks to computation by Yee and Chan in 2009 [201] (see [77, p. 273]).

The sequence γ_n converges very slowly to γ , namely with order $1/n$. By replacing $\log n$ in this sequence by $\log(n + 1/2)$, DeTemple [84] obtained quadratic convergence (see also [69]). In [130] Mortici made a careful study of how convergence is affected by changes in the logarithm term. He introduced new sequences

$$M_n \equiv H_n - \log \frac{P(n)}{Q(n)},$$

where P and Q are polynomials with leading coefficient 1 and $\deg P - \deg Q = 1$. By judicious choice of the degrees and coefficients of P and Q he was able to produce sequences M_n tending to γ with convergence of order $1/n^4$ and $1/n^6$. He also gave a recipe for obtaining sequences converging to γ with order $1/n^{2k+2}$, where k is any positive integer. This study is based on the author’s lemma, proved in [129], that connects the rate of convergence of a convergent sequence $\{x_n\}$ to that of the sequence $\{x_n - x_{n+1}\}$.

In 1997 Negoï [134] showed that if $T_n \equiv H_n - \log(n + 1/2 + 1/(24n))$, then $T_n + [4n^3]^{-1}$ is strictly decreasing to γ and $T_n + [48(n + 1)]^{-3}$ is strictly increasing to γ , so that $[48(n + 1)]^{-3} < \gamma - T_n < [48n^3]^{-1}$. In 2011 Chen [70] established sharper bounds for $\gamma - T_n$ by using a lemma of Mortici [129].

Using another approach, in 2011 Chlebus [77] developed a recursive scheme for modifying the sequence $H_n - \log n$ to accelerate the convergence to γ to any desired order. The first step in Chlebus’ scheme is equivalent to the DeTemple [84] approximation, while the next step yields a sequence that closely resembles the one due to Negoï [134].

In [8] Alzer studied the harmonic numbers (16), obtaining several new inequalities for them. In particular, for $n \geq 2$, he proved that

$$\alpha \frac{\log(\log n + \gamma)}{n^2} \leq H_n^{1/n} - H_{n+1}^{1/(n+1)} < \beta \frac{\log(\log n + \gamma)}{n^2},$$

where $\alpha = (6\sqrt{6} - 2\sqrt[3]{396})/(3 \log(\log 2 + \gamma)) \approx 0.014$ and $\beta = 1$ are the best possible constants and γ is the Euler-Mascheroni constant.

4.2 Estimates for the Gamma Function

In [14, Lemma 2.39] Anderson, Vamanamurthy, and Vuorinen proved that

$$\lim_{x \rightarrow \infty} \frac{\log \Gamma\left(\frac{x}{2} + 1\right)}{x \log x} = \frac{1}{2} \tag{17}$$

and that the function $(\log \Gamma(1 + x/2))/x$ is strictly increasing from $[2, \infty)$ onto $[0, \infty)$. In [13] Anderson and Qiu showed that $(\log \Gamma(x + 1))/(x \log x)$ is strictly increasing from $(1, \infty)$ onto $(1 - \gamma, 1)$, where γ is the Euler-Mascheroni constant defined by (15), thereby obtaining the strict inequalities

$$x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1}, \quad x > 1. \tag{18}$$

They also conjectured that the function $(\log \Gamma(x + 1))/(x \log x)$ is concave on $(1, \infty)$, and this conjecture was proved by Elbert and Laforgia in [85, Sect. 3]. One should note that in 1989 Sándor [159] proved that the function $(\Gamma(x + 1))^{1/x}$ is strictly concave for $x \geq 7$.

Later Alzer [4] was able to extend (18) by proving that, for $x \in (0, 1)$,

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{\beta(x-1)-\gamma}, \tag{19}$$

with best possible constants $\alpha = 1 - \gamma = 0.42278\dots$ and $\beta = (\pi^2/6 - \gamma)/2 = 0.53385\dots$. For $x \in (1, \infty)$ Alzer was able to sharpen (18) by showing that (19) holds with best possible constants $\alpha = (\pi^2/6 - \gamma)/2 \approx 0.534$ and $\beta = 1$. His principal new tool was the convolution theorem for Laplace transforms.

Another type of approximation for $\Gamma(x)$ was derived by Ivády [102] in 2009:

$$\frac{x^2 + 1}{x + 1} < \Gamma(x + 1) < \frac{x^2 + 2}{x + 2}, \quad 0 < x < 1. \tag{20}$$

In 2011 Zhao, Guo, and Qi [207] simplified and sharpened (20) by proving that the function

$$Q(x) \equiv \frac{\log \Gamma(x + 1)}{\log(x^2 + 1) - \log(x + 1)}$$

is strictly increasing from $(0, 1)$ onto $(\gamma, 2(1 - \gamma))$, where γ is the Euler-Mascheroni constant. As a consequence, they proved that

$$\left(\frac{x^2 + 1}{x + 1}\right)^\alpha < \Gamma(x + 1) < \left(\frac{x^2 + 1}{x + 1}\right)^\beta, \quad 0 < x < 1,$$

with best possible constants $\alpha = 2(1 - \gamma)$ and $\beta = \gamma$.

Very recently Mortici [133] has determined by numerical experiments that the upper estimate in (18) is a better approximation for $\Gamma(x)$ than the lower one when x is very large. Hence, he has sought estimates of the form $\Gamma(x) \approx x^{a(x)}$, where $a(x)$ is close to $x - 1$ as x approaches infinity. For example, he proves that

$$x^{(x-1)a(x)} < \Gamma(x) < x^{(x-1)b(x)}, \quad x \geq 20,$$

where $a(x) = 1 - 1/\log x + 1/(2x) - (1 - (\log 2\pi)/2)/(x \log x)$ and where $b(x) = 1 - 1/\log x + 1/(2x)$. The left inequality is valid for $x \geq 2$. Mortici has also obtained a pair of sharper inequalities of this type, valid for $x \geq 2$, and has showed how lower and upper estimates of any desired accuracy may be obtained. His proofs are based on an approximation for $\log \Gamma(x)$ in terms of series involving Bernoulli numbers [25, p. 29] and on truncations of an asymptotic series for the function $(\log \Gamma(x))/((x - 1) \log x)$. These results provide improvements of (18).

4.3 Factorials and Stirling’s Formula

The well-known *Stirling’s formula* for $n!$,

$$\alpha_n \equiv \left(\frac{n}{e}\right)^n \sqrt{2\pi n}, \tag{21}$$

discovered by the precocious homeschooled and largely self-taught eighteenth-century Scottish mathematician James Stirling, approximates $n!$ asymptotically in the sense that

$$\lim_{n \rightarrow \infty} \frac{n!}{\alpha_n} = 1.$$

Because of the importance of this formula in probability and statistics, number theory, and scientific computations, several authors have sought to replace (21) by a simple sequence that approximates $n!$ more closely (see the discussions in [47, 48]). For example, Burnside [63] proved in 1917 that

$$n! \sim \beta_n \equiv \sqrt{2\pi} \left(\frac{n + 1/2}{e}\right)^{n+1/2}, \tag{22}$$

that is, $\lim_{n \rightarrow \infty} (n!/\beta_n) = 1$. In 2008, Batir [47] determined that the best constants a and b such that

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-a}} \leq n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-b}} \tag{23}$$

are $a = 1 - 2\pi e^{-2} \approx 0.1497$ and $b = 1/6 \approx 0.1667$. Batir offers a numerical table illustrating that his upper bound formula $n^{n+1}e^{-n}\sqrt{2\pi}/\sqrt{n - 1/6}$ gives much better approximations to $n!$ than does either (21) or (22).

In a later paper [48] Batir observed that many of the improvements of Stirling’s formula take the form

$$n! \sim e^{-a} \left(\frac{n+a}{e}\right)^n \sqrt{2\pi(n+b)} \tag{24}$$

for some real numbers a and b . Batir sought the pair of constants a and b that would make (24) optimal. He proved that the best pairs (a, b) are (a_1, b_1) and (a_2, b_2) , where

$$a_1 = \frac{1}{3} + \frac{\lambda}{6} - \frac{1}{6}\sqrt{6 - \lambda^2 + 4/\lambda} \approx 0.54032, \quad b_1 = a_1^2 + 1/6 \approx 0.45861$$

and

$$a_2 = \frac{1}{3} + \frac{\lambda}{6} + \frac{1}{6}\sqrt{6 - \lambda^2 + 4/\lambda} \approx 0.95011, \quad b_2 = a_2^2 + 1/6 \approx 1.06937,$$

where $\lambda = \sqrt{2 + 2^{2/3} + 2^{4/3}} \approx 2.47128$ and a_1 and a_2 are the real roots of the quartic equation $3x^4 - 4x^3 + x^2 + 1/12 = 0$.

Ramanujan [156] sought to improve Stirling’s formula (21) by replacing $\sqrt{2n}$ in the formula by the sixth root of a cubic polynomial in n :

$$\Gamma(n + 1) \approx \sqrt{\pi} \left(\frac{n}{e}\right)^n \sqrt[6]{8n^3 + 4n^2 + n + \frac{1}{30}}. \tag{25}$$

In this connection there appears in the record also his double inequality, for $x \geq 1$,

$$\sqrt[6]{8x^3 + 4x^2 + x + \frac{1}{100}} < \frac{\Gamma(x + 1)}{\sqrt{\pi} \left(\frac{x}{e}\right)^x} < \sqrt[6]{8x^3 + 4x^2 + x + \frac{1}{30}}. \tag{26}$$

Motivated by this inequality of Ramanujan, the authors of [18] defined the function $h(x) \equiv u(x)^6 - (8x^3 + 4x^2 + x)$, where $u(x) = (e/x)^x \Gamma(x + 1)/\sqrt{\pi}$, and conjectured that $h(x)$ is increasing from $(1, \infty)$ into $(1/100, 1/30)$. In 2001 Karatsuba [106] settled this conjecture by showing that $h(x)$ is increasing from $[1, \infty)$ onto $[h(1), 1/30)$, where $h(1) = e^6/\pi^3 - 13 \approx 0.011$.

In an unpublished document, E. A. Karatsuba suggested modifying Ramanujan’s approximation formula (25) by replacing the radical with the $2k$ th root of a polynomial of degree k and determining the best such asymptotic approximation. Such a program was partially realized by Mortici [132] in 2011, who proposed formula (27) below for $k = 4$, but the more general problem suggested by Karatsuba remains an open problem. Mortici’s proposed Ramanujan-type asymptotic approximation is as follows:

$$\Gamma(n + 1) \approx \sqrt{\pi} \left(\frac{n}{e}\right)^n \sqrt[8]{16n^4 + \frac{32}{3}n^3 + \frac{32}{9}n^2 + \frac{176}{405}n - \frac{128}{1215}}. \tag{27}$$

In connection with (27), he defined the function

$$g(x) \equiv u(x)^8 - \left(16x^4 + \frac{32}{3}x^3 + \frac{32}{9}x^2 + \frac{176}{405}x\right),$$

where $u(x) = (e/x)^x \Gamma(x + 1)/\sqrt{\pi}$, and proved that $g(x)$ is strictly decreasing from $[3, \infty)$ onto $(g(\infty), g(3)]$, where $g(\infty) = -128/1215 \approx -0.105$ and $g(3) = 256e^{24}/(43046721\pi^4) - 218336/135 \approx -0.088$. Mortici's method for proving monotonicity was simpler than Karatsuba's, because he employed an excellent result of Alzer [3] concerning complete monotonicity (see Sect. 4.6 below for definitions). Mortici claimed that his method would also simplify Karatsuba's proof in [106]. Finally, he proved that, for $x \geq 3$,

$$R(x, \alpha) < \frac{\Gamma(x + 1)}{\sqrt{\pi} \left(\frac{x}{e}\right)^x} \leq R(x, \beta),$$

where $R(x, t) \equiv \sqrt[8]{16x^4 + \frac{32}{3}x^3 + \frac{32}{9}x^2 + \frac{176}{405}x - t}$, and $\alpha = 128/1215$, $\beta = g(3)$ are the best possible constants.

In 2012 Mahmoud, Alghamdi, and Agarwal [124] deduced a new family of upper bounds for $\Gamma(n + 1)$ of the form

$$\Gamma(n + 1) < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{M_n^{[m]}}, \quad n \in \mathbb{N},$$

$$M_n^{[m]} \equiv \frac{1}{2m + 3} \left[\frac{1}{4n} + \sum_{k=1}^m \frac{2m - 2k + 2}{2k + 1} 2^{-2k} \zeta(2k, n + 1/2) \right], \quad n \in \mathbb{N},$$

where ζ is the *Hurwitz zeta function*

$$\zeta(s, q) \equiv \sum_{k=0}^{\infty} \frac{1}{(k + q)^s}.$$

These upper bounds improve Mortici's inequality (27).

4.4 Volume of the Unit Ball

The volume Ω_n of the unit ball in \mathbb{R}^n is given in terms of the gamma function by the formula

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}, \quad n \in \mathbb{N}.$$

Whereas the volume of the unit cube is 1 in all dimensions, the numbers Ω_n strictly increase to the maximum $\Omega_5 = 8\pi^2/15$ and then strictly decrease to 0 as $n \rightarrow \infty$ (cf. [60, p.264]). Anderson, Vamanamurthy, and Vuorinen [14] proved that $\Omega_n^{1/n}$ is strictly decreasing and that the series $\sum_{n=2}^{\infty} \Omega_n^{1/\log n}$ is convergent. In [13] Anderson and Qiu proved that $\Omega_n^{1/(n \log n)}$ is strictly decreasing with limit $e^{-1/2}$ as $n \rightarrow \infty$.

In 2008 Alzer published a collection of new inequalities for combinations of different dimensions and powers of Ω_n [7, Sect. 3]. We quote several of them below:

$$a \frac{(2\pi e)^{n/2}}{n^{(n-1)/2}} \leq (n + 1)\Omega_n - n\Omega_{n+1} < b \frac{(2\pi e)^{n/2}}{n^{(n-1)/2}}, \quad n \geq 1, \tag{28}$$

where the best possible constants are $a = (4 - 9\pi/8)(2/(\pi e))^{1/2}/e = 0.0829\dots$ and $b = \pi^{-1/2} = 0.5641\dots$;

$$a \frac{(2\pi e)^n}{n^{n+2}} \leq \Omega_n^2 - \Omega_{n-1}\Omega_{n+1} < b \frac{(2\pi e)^n}{n^{n+2}}, \quad n \geq 2, \tag{29}$$

with best possible constant factors $a = (4/e^2)(1 - 8/(3\pi)) = 0.0818\dots$ and $b = 1/(2\pi) = 0.1591\dots$;

$$\frac{a}{\sqrt{n}} \leq \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \frac{b}{\sqrt{n}}, \quad n \geq 2, \tag{30}$$

with best possible constants $a = 3\sqrt{2\pi}/(6 + 4\pi) = 0.7178\dots$ and $b = \sqrt{2\pi} = 2.5066\dots$; and

$$\frac{a}{\sqrt{n}} \leq (n + 1) \frac{\Omega_{n+1}}{\Omega_n} - n \frac{\Omega_n}{\Omega_{n-1}} < \frac{b}{\sqrt{n}}, \quad n \geq 2, \tag{31}$$

with best possible constants $a = (4 - \pi)\sqrt{2} = 1.2139\dots$ and $b = \sqrt{2\pi}/2 = 1.2533\dots$

Alzer’s work in [7] includes a number of new results about the gamma function and its derivatives.

In 2010 Mortici [128], improving on some earlier work of Alzer [5, Theorem 1], obtained, for $n \geq 1$ on the left and for $n \geq 4$ on the right,

$$\frac{a}{\sqrt[2n]{2\pi}} \leq \frac{\Omega_n}{\Omega_{n+1}^{n/(n+1)}} < \frac{\sqrt{e}}{\sqrt[2n]{2\pi}},$$

where $a = 64 \cdot 720^{11/12} \cdot 2^{1/22} / (10395 \cdot \pi^{5/11}) = 1.5714\dots$. He sharpened the work of Alzer [5, Theorem 2] and Qiu and Vuorinen [154] in the following result, valid for $n \geq 1$:

$$\sqrt{\frac{2n + 1}{4\pi}} < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{2n + 1}{4\pi} + \frac{1}{16\pi n}}.$$

Mortici also proved, in [128, Theorem 4], that, for $n \geq 4$,

$$\left(1 + \frac{1}{n}\right)^{\frac{1}{2} - \frac{1}{4n}} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n}\right)^{\frac{1}{2}}.$$

This result improves a similar one by Alzer [5, Theorem 3, valid for $n \geq 1$], where the exponent on the left is the constant $2 - \log_2 \pi$. Very recently, Yin [202] improved Mortici’s result as follows: For $n \geq 1$,

$$\frac{(n + 1)(n + 1/6)}{(n + \beta)^2} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{(n + 1)(n + \beta/2)}{(n + 1/3)^2},$$

where $\beta = (391/30)^{1/3}$.

4.5 Digamma and Polygamma Functions

The logarithmic derivative of the gamma function, $\psi(x) \equiv \frac{d}{dx} \log \Gamma(x) = \Gamma'(x)/\Gamma(x)$, is known as the *digamma function*. Its derivatives $\psi^{(n)}, n \geq 1$, are known as the *polygamma functions* ψ_n . These functions have the following representations [1, pp. 258–260] for $x > 0$ and each natural number n :

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n=1}^\infty \frac{x}{n(x + n)}$$

and

$$\psi_n(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} dt = (-1)^{n+1} n! \sum_{k=0}^\infty (x + k)^{-n-1}.$$

Several researchers have studied the properties of these functions. In 2007, refining the left inequality in [6, Theorem 4.8], Batir [45] obtained estimates for ψ_n in terms of ψ or ψ_k , with $k < n$. In particular, he proved, for $x > 0$ and $n \in \mathbb{N}$:

$$(n - 1)! \exp(-n\psi(x + 1/2)) < |\psi_n(x)| < (n - 1)! \exp(-n\psi(x)),$$

and, for $1 \leq k \leq n - 1, x > 0$,

$$(n - 1)! \left(\frac{\psi_k(x + 1/2)}{(-1)^{k-1}(k - 1)!} \right)^{n/k} < |\psi_n(x)| < (n - 1)! \left(\frac{\psi_k(x)}{(-1)^{k-1}(k - 1)!} \right)^{n/k}.$$

He also proved, for example, the difference formula

$$\alpha < ((-1)^{n-1} \psi_n(x + 1))^{-1/n} - ((-1)^{n-1} \psi_n(x))^{-1/n} < \beta,$$

where $\alpha = (n! \zeta(n + 1))^{-1/n}$ and $\beta = ((n - 1)!)^{-1/n}$ are best possible, and the sharp estimates

$$-\gamma < \psi(x) + \log(e^{1/x} - 1) < 0,$$

where γ is the Euler-Mascheroni constant.

In 2010 Mortici [131] proved the following estimates, for $x > 0$ and $n \geq 1$, refining the work of Guo, Chen, and Qi [89]:

$$-\frac{1}{720} \frac{(n+3)!}{x^{n+4}} < |\psi_n(x)| - \left[\frac{(n-1)!}{x^n} + \frac{1}{2} \frac{n!}{x^{n+1}} + \frac{1}{12} \frac{(n+1)!}{x^{n+2}} \right] < 0.$$

4.6 Completely Monotonic Functions

A function f is said to be *completely monotonic* on an interval I if $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I$ and all nonnegative integers n . If this inequality is strict, then f is called *strictly completely monotonic*. Such functions occur in probability theory, numerical analysis, and other areas. Some of the most important completely monotonic functions are the gamma function and the digamma and polygamma functions. The Hausdorff-Bernstein-Widder theorem [189, Theorem 12b, p. 161] states that f is completely monotonic on $[0, \infty)$ if and only if there is a nonnegative measure μ on $[0, \infty)$ such that

$$f(x) = \int_0^\infty e^{-xt} d\mu(t)$$

for all $x > 0$. There is a well-written introduction to completely monotonic functions in [125].

In 2008 Batir [46] proved that the following function $F_a(x)$ related to the gamma function is completely monotonic on $(0, \infty)$ if and only if $a \geq 1/4$ and that $-F_a(x)$ is completely monotonic if and only if $a \leq 0$:

$$F_a(x) \equiv \log \Gamma(x) - x \log x + x - \frac{1}{2} \log(2\pi) + \frac{1}{2} \psi(x) + \frac{1}{6(x-a)}.$$

As a corollary he was able to prove, for $x > 0$, the inequality

$$\exp\left(-\frac{1}{2} \psi(x) - \frac{1}{6(x-\alpha)}\right) < \frac{\Gamma(x)}{x^x e^{-x} \sqrt{2\pi}} < \exp\left(-\frac{1}{2} \psi(x) - \frac{1}{6(x-\beta)}\right),$$

with best constants $\alpha = 1/4$ and $\beta = 0$, improving his earlier work with Alzer [9].

In 2010 Mortici [131] showed that for every $n \geq 1$, the functions $f, g : (0, \infty) \rightarrow \mathbb{R}$ given by

$$f(x) \equiv |\psi_n(x)| - \frac{(n-1)!}{x^n} - \frac{1}{2} \frac{n!}{x^{n+1}} - \frac{1}{12} \frac{(n+1)!}{x^{n+2}} + \frac{1}{720} \frac{(n+3)!}{x^{n+4}}$$

and

$$g(x) \equiv \frac{(n-1)!}{x^n} + \frac{1}{2} \frac{n!}{x^{n+1}} + \frac{1}{12} \frac{(n+1)!}{x^{n+2}} - |\psi_n(x)|$$

are completely monotonic on $(0, \infty)$. As a corollary, since $f(x)$ and $g(x)$ are positive, he obtained estimates for $|\psi_n(x)|$ as finite series in negative powers of x .

Anderson and Qiu [13], as well as some other authors (see [2]), have studied the monotonicity properties of the function $f(x) \equiv (\log \Gamma(x + 1))/x$. In 2011 Adell and Alzer [2] proved that f' is completely monotonic on $(-1, \infty)$.

In the course of pursuing research inspired by [13, 14] (see [53]), in 2012 Alzer [7] discussed properties of the function

$$f(x) \equiv \left(1 - \frac{\log x}{\log(1+x)}\right) x \log x,$$

which Qi and Guo [149] later conjectured to be completely monotonic on $(0, \infty)$. In [53] Berg and Pedersen proved this conjecture.

In 2001 Berg and Pedersen [50] proved that the derivative of the function

$$f(x) \equiv \frac{\log \Gamma(x+1)}{x \log x}, \quad x > 0$$

is completely monotonic (see also [51]). This result extends the work of [13, 85]. Very recently, Berg and Pedersen [52] showed that the function

$$F_a(x) \equiv \frac{\log \Gamma(x+1)}{x \log(ax)}$$

is a Pick function when $a \geq 1$, that is, it extends to a holomorphic function mapping the upper half plane into itself. The authors also considered the function

$$f(x) \equiv \left(\frac{\pi^{x/2}}{\Gamma(1+x/2)}\right)^{1/(x \log x)}$$

and proved that $\log f(x+1)$ is a Stieltjes function and hence that $f(x+1)$ is completely monotonic on $(0, \infty)$.

5 The Hypergeometric Function and Elliptic Integrals

The classical *hypergeometric function* is defined by

$$F(a, b; c; x) \equiv {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \quad |x| < 1,$$

where $(a, n) \equiv a(a + 1)(a + 2) \cdots (a + n - 1)$ for $n \in \mathbb{N}$ and $(a, 0) = 1$ for $a \neq 0$. This function is so general that for proper choice of the parameters a, b, c , one obtains logarithms, trigonometric functions, inverse trigonometric functions, elliptic integrals, or polynomials of Chebyshev, Legendre, Gegenbauer, Jacobi, and so on (see [1, Chap. 15]).

5.1 Hypergeometric Functions

The Bernoulli inequality [126, p. 34] may be written as

$$\log(1 + ct) \leq c \log(1 + t), \tag{32}$$

where $c > 1, t > 0$. In [111] some Bernoulli-type inequalities have been obtained.

It is well known that in the zero-balanced case $c = a + b$ the hypergeometric function $F(a, b; c; x)$ has a logarithmic singularity at $x = 1$ (cf. [18, Theorem 1.19(6)]). Moreover, as a special case [1, 15.1.3],

$$xF(1, 1, 2; x) = \log \frac{1}{1 - x}. \tag{33}$$

Because of this connection, Vuorinen and his collaborators [110] have generalized versions of (32) to a wide class of hypergeometric functions. In the course of their investigation they have studied monotonicity and convexity/concavity properties of such functions. For example, for positive a, b let $g(x) \equiv xF(a, b; a + b; x), x \in (0, 1)$. These authors have proved that $G(x) \equiv \log g(e^x/(1 + e^x))$ is concave on $(-\infty, \infty)$ if and only if $1/a + 1/b \geq 1$. And they have shown that, for fixed $a, b \in (0, 1]$ and for $x \in (0, 1), p > 0$, the function

$$\left(\frac{x^p}{1 + x^p} F \left(a, b; a + b; \frac{x^p}{1 + x^p} \right) \right)^{1/p}$$

is increasing in p . In particular,

$$\frac{\sqrt{r}}{1 + \sqrt{r}} F \left(a, b; a + b; \frac{\sqrt{r}}{1 + \sqrt{r}} \right) \leq \left(\frac{r}{1 + r} F \left(a, b; a + b; \frac{r}{1 + r} \right) \right)^{1/2}.$$

Motivated by the asymptotic behavior of $F(x) = F(a, b; c; x)$ as $x \rightarrow 1^-$, Simić and Vuorinen have carried the above work further in [170], finding best possible bounds, when $a, b, c > 0$ and $0 < x, y < 1$, for the quotient and difference

$$\frac{F(a, b; c; x) + F(a, b; c; y)}{F(a, b; c; x + y - xy)}, \quad F(x) + F(y) - F(x + y - xy).$$

In 2009 Karp and Sitnik [108] obtained some inequalities and monotonicity of ratios for the generalized hypergeometric function. The proofs hinge on a generalized Stieltjes representation of the generalized hypergeometric function.

5.2 Complete Elliptic Integrals

For $0 < r < 1$, the *complete elliptic integrals of the first and second kind* are defined as

$$\mathcal{K}(r) \equiv \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2 \sin^2 t}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2 t^2)}} \quad (34)$$

and

$$\mathcal{E}(r) \equiv \int_0^{\pi/2} \sqrt{1-r^2 \sin^2 t} dt = \int_0^1 \sqrt{\frac{1-r^2 t^2}{1-t^2}} dt, \quad (35)$$

respectively. Letting $r' \equiv \sqrt{1-r^2}$, we often denote

$$\mathcal{K}'(r) = \mathcal{K}(r'), \quad \mathcal{E}'(r) = \mathcal{E}(r').$$

These elliptic integrals have the hypergeometric series representations

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \quad \mathcal{E} = \frac{\pi}{2} F\left(\frac{1}{2}, -\frac{1}{2}; 1; r^2\right). \quad (36)$$

5.3 The Landen Identities

The functions \mathcal{K} and \mathcal{E} satisfy the following identities due to Landen [64, 163.01, 164.02]:

$$\begin{aligned} \mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) &= (1+r)\mathcal{K}(r), & \mathcal{K}\left(\frac{1-r}{1+r}\right) &= \frac{1}{2}(1+r)\mathcal{K}'(r), \\ \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) &= \frac{2\mathcal{E}(r) - r'^2\mathcal{K}(r)}{1+r}, & \mathcal{E}\left(\frac{1-r}{1+r}\right) &= \frac{\mathcal{E}'(r) + r\mathcal{K}'(r)}{1+r}. \end{aligned}$$

Using Landen's transformation formulas, we have the following identities [177, Lemma 2.8]: For $r \in (0, 1)$, let $t = (1-r)/(1+r)$. Then

$$\begin{aligned} \mathcal{K}(t^2) &= \frac{(1+r)^2}{4} \mathcal{K}'(r^2), \quad \mathcal{K}'(t^2) = (1+r)^2 \mathcal{K}(r^2), \\ \mathcal{E}(t^2) &= \frac{\mathcal{E}'(r^2) + (r+r^2+r^3)\mathcal{K}'(r^2)}{(1+r)^2}, \\ \mathcal{E}'(t^2) &= \frac{4\mathcal{E}(r^2) - (3-2r^2-r^4)\mathcal{K}(r^2)}{(1+r)^2}. \end{aligned}$$

Generalizing a Landen identity, Simić and Vuorinen [171] have determined the precise regions in the ab -plane for which a Landen inequality holds for zero-balanced hypergeometric functions. They proved that for all $a, b > 0$ with $ab \leq 1/4$ the inequality

$$F\left(a, b; a+b; \frac{4r}{(1+r)^2}\right) \leq (1+r)F(a, b; a+b; r^2)$$

holds for $r \in (0, 1)$, while for $a, b > 0$ with $1/a + 1/b \leq 4$, the following reversed inequality is true for each $r \in (0, 1)$:

$$F\left(a, b; a+b; \frac{4r}{(1+r)^2}\right) \geq (1+r)F(a, b; a+b; r^2) .$$

In the rest of the ab -plane neither of these inequalities holds for all $r \in (0, 1)$. These authors have also obtained sharp bounds for the quotient

$$\frac{(1+r)F(a, b; a+b; r^2)}{F(a, b; a+b; 4r/(1+r)^2)}$$

in certain regions of the ab -plane.

Some earlier results on Landen inequalities for hypergeometric functions can be found in [152]. Recently, Baricz obtained Landen-type inequalities for generalized Bessel functions [29, 37].

Inspired by an idea of Simić and Vuorinen [171], Wang, Chu, and Jiang [188] obtained some inequalities for zero-balanced hypergeometric functions which generalize Ramanujan’s cubic transformation formulas.

5.4 Legendre’s Relation and Generalizations

It is well known that the complete elliptic integrals satisfy the *Legendre relation* [64, 110.10]:

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}.$$

This relation has been generalized in various ways. Elliott [86] proved the identity

$$F_1 F_2 + F_3 F_4 - F_3 F_2 = \frac{\Gamma(1 + \lambda + \mu)\Gamma(1 + \mu + \nu)}{\Gamma(\lambda + \mu + \nu + \frac{3}{2})\Gamma(\frac{1}{2} + \mu)},$$

where

$$F_1 = F(\frac{1}{2} + \lambda, -\frac{1}{2} - \nu; 1 + \lambda + \mu; x), \quad F_2 = F(\frac{1}{2} - \lambda, \frac{1}{2} + \nu; 1 + \mu + \nu; 1 - x),$$

$$F_3 = F(\frac{1}{2} + \lambda, \frac{1}{2} - \nu; 1 + \lambda + \mu; x), \quad F_4 = F(-\frac{1}{2} - \lambda, \frac{1}{2} + \nu; 1 + \mu + \nu; 1 - x).$$

Elliott proved this formula by a clever change of variables in multiple integrals. Another proof, based on properties of the hypergeometric differential equation, was suggested without details in [25, p. 138], and the missing details were provided in [20]. It is easy to see that Elliott’s formula reduces to the Legendre relation when $\lambda = \mu = \nu = 0$ and $x = r^2$.

Another generalization of the Legendre relation was given in [19]. With the notation

$$u = u(r) = F(a - 1, b; c; r), \quad v = v(r) = F(a, b; c; r),$$

$$u_1 = u(1 - r), \quad v_1 = v(1 - r),$$

the authors considered the function

$$\mathcal{L}(a, b, c, r) = uv_1 + u_1v - vv_1,$$

proving, in particular, that

$$\mathcal{L}(a, 1 - a, c, r) = \frac{\Gamma^2(c)}{\Gamma(c + a - 1)\Gamma(c - a + 1)}.$$

This reduces to Elliott’s formula in case $\lambda = \nu = 1/2 - a$ and $\mu = c + a - 3/2$. In [19] it was conjectured that for $a, b \in (0, 1), a + b \leq 1 (\geq 1)$, $\mathcal{L}(a, b, c, r)$ is concave (convex) as a function of r on $(0, 1)$. In [107] Karatsuba and Vuorinen determined, in particular, the exact regions of abc -space in which the function $\mathcal{L}(a, b, c, r)$ is concave, convex, constant, positive, negative, zero, and where it attains its unique extremum.

In [27] Balasubramanian, Naik, Ponnusamy, and Vuorinen obtained a differentiation formula for an expression involving hypergeometric series that implies Elliott’s identity. This paper contains a number of other significant results, including a proof that Elliott’s identity is equivalent to a formula of Ramanujan [54, p. 87, Entry 30] on the differentiation of quotients of hypergeometric functions.

5.5 Some Approximations for $\mathcal{K}(r)$ by $\operatorname{arth} r$

Anderson, Vamanamurthy, and Vuorinen [15] approximated $\mathcal{K}(r)$ by the inverse hyperbolic tangent function arth , obtaining the inequalities

$$\frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{1/2} < \mathcal{K}(r) < \frac{\pi}{2} \frac{\operatorname{arth} r}{r}, \tag{37}$$

for $0 < r < 1$. Alzer and Qiu [11] refined (37) as

$$\frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{3/4} < \mathcal{K}(r) < \frac{\pi}{2} \frac{\operatorname{arth} r}{r}, \tag{38}$$

with the best exponents $3/4$ and 1 for $(\operatorname{arth} r)/r$ on the left and right, respectively. Seeking to improve the exponents in (38), they conjectured that the double inequality

$$\frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{3/4+\alpha r} < \mathcal{K}(r) < \frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{3/4+\beta r} \tag{39}$$

holds for all $0 < r < 1$, with best constants $\alpha = 0$ and $\beta = 1/4$. Very recently Chu et al. [81] gave a proof for this conjecture.

András and Baricz [24] presented some improved lower and upper bounds for $\mathcal{K}(r)$ involving the Gaussian hypergeometric series.

5.6 Approximations for $\mathcal{E}(r)$

In [90] Guo and Qi have obtained new approximations for $\mathcal{E}(r)$ as well as for $\mathcal{K}(r)$. For example, they showed that, for $0 < r < 1$,

$$\frac{\pi}{2} - \frac{1}{2} \log \frac{(1+r)^{1-r}}{(1-r)^{1+r}} < \mathcal{E}(r) < \frac{\pi-1}{2} + \frac{1-r^2}{4r} \log \frac{1+r}{1-r}.$$

In recent work [82, 178, 185] Chu et al. have obtained estimates for $\mathcal{E}(r)$ in terms of rational functions of the arithmetic, geometric, and root-square mean, implying new inequalities for the perimeter of an ellipse.

5.7 Generalized Complete Elliptic Integrals

For $0 < a < \min\{c, 1\}$ and $0 < b < c \leq a + b$, define the *generalized complete elliptic integrals of the first and second kind* on $[0, 1]$ by [95]

$$\mathcal{K}_{a,b,c} = \mathcal{K}_{a,b,c}(r) \equiv \frac{B(a,b)}{2} F(a,b;c;r^2), \quad (40)$$

$$\mathcal{E}_{a,b,c} = \mathcal{E}_{a,b,c}(r) \equiv \frac{B(a,b)}{2} F(a-1,b;c;r^2), \quad (41)$$

$$\mathcal{K}'_{a,b,c} = \mathcal{K}_{a,b,c}(r') \quad \text{and} \quad \mathcal{E}'_{a,b,c} = \mathcal{E}_{a,b,c}(r'), \quad (42)$$

for $r \in (0, 1)$, $r' = \sqrt{1-r^2}$. The end values are defined by limits as r tends to $0+$ and $1-$, respectively. Thus,

$$\mathcal{K}_{a,b,c}(0) = \mathcal{E}_{a,b,c}(0) = \frac{B(a,b)}{2}$$

and

$$\mathcal{E}_{a,b,c}(1) = \frac{1}{2} \frac{B(a,b)B(c,c+1-a-b)}{B(c+1-a,c-b)}, \quad \mathcal{K}_{a,b,c}(1) = \infty.$$

Note that the restrictions on the parameters a , b , and c ensure that the function $\mathcal{K}_{a,b,c}$ is increasing and unbounded, whereas $\mathcal{E}_{a,b,c}$ is decreasing and bounded, as in the classical case $a = b = 1/2$, $c = 1$.

Heikkala, Lindén, Vamanamurthy, and Vuorinen [94, 95] derived several differentiation formulas and obtained sharp monotonicity and convexity properties for certain combinations of the generalized elliptic integrals. They also constructed a conformal mapping $\text{sn}_{a,b,c}$ from a quadrilateral with internal angles $b\pi$, $(c-b)\pi$, $(1-a)\pi$, and $(1-c+a)\pi$ onto the upper half plane. These results generalize the work of [19]. For some particular parameter triples (a, b, c) , there are very recent results by many authors [37, 181, 206, 209].

With suitable restrictions on the parameters a, b, c , Neuman [135] has obtained bounds for $\mathcal{K}_{a,b,c}$ and $\mathcal{E}_{a,b,c}$ and for certain combinations and products of them. He has also proved that these generalized elliptic integrals are logarithmically convex as functions of the first parameter.

In 2007 Baricz [33, 36, 38] established some Turán-type inequalities for Gaussian hypergeometric functions and generalized complete elliptic integrals. He also studied the generalized convexity of the zero-balanced hypergeometric functions and generalized complete elliptic integrals [31] (see also [30, 32, 37]). Very recently, Kalmykov and Karp [103, 104] have studied log-convexity and log-concavity for series involving gamma functions and derived many known and new inequalities for the modified Bessel and Kummer and generalized hypergeometric functions and ratios of the Gauss hypergeometric functions. In particular, they improved and generalized Baricz's Turán-type inequalities.

5.8 The Generalized Modular Function and Generalized Linear Distortion Function

Let $a, b, c > 0$ with $a + b \geq c$. A *generalized modular equation of order* (or *degree*) $p > 0$ is

$$\frac{F(a, b; c; 1 - s^2)}{F(a, b; c; s^2)} = p \frac{F(a, b; c; 1 - r^2)}{F(a, b; c; r^2)}, \quad 0 < r < 1. \tag{43}$$

The *generalized modulus* is the decreasing homeomorphism $\mu_{a,b,c} : (0, 1) \rightarrow (0, \infty)$, defined by

$$\mu_{a,b,c}(r) \equiv \frac{B(a, b)}{2} \frac{F(a, b; c; 1 - r^2)}{F(a, b; c; r^2)}. \tag{44}$$

The generalized modular equation (43) can be written as

$$\mu_{a,b,c}(s) = p\mu_{a,b,c}(r).$$

With $p = 1/K$, $K > 0$, the solution of (43) is then given by

$$s = \varphi_K^{a,b,c}(r) \equiv \mu_{a,b,c}^{-1}(\mu_{a,b,c}(r)/K).$$

Here $\varphi_K^{a,b,c}$ is called the (a, b, c) -*modular function* with *degree* $p = 1/K$ [19, 94, 95]. Clearly the following identities hold:

$$\begin{aligned} \mu_{a,b,c}(r)\mu_{a,b,c}(r') &= \left(\frac{B(a, b)}{2}\right)^2, \\ \varphi_K^{a,b,c}(r)^2 + \varphi_{1/K}^{a,b,c}(r')^2 &= 1. \end{aligned}$$

In [94], the authors generalized the functional inequalities for the modular functions and Grötzsch function μ proved in [19] to hold also for the generalized modular functions and generalized modulus in the case $b = c - a$. For instance, for $0 < a < c \leq 1$ and $K > 1$, the inequalities

$$\mu_{a,c-a,c}(1 - \sqrt{(1-u)(1-t)}) \leq \frac{\mu_{a,c-a,c}(u) + \mu_{a,c-a,c}(t)}{2} \leq \mu_{a,c-a,c}(\sqrt{ut}) \tag{45}$$

hold for all $u, t \in (0, 1)$, with equality if and only if $u = t$, and

$$r^{1/K} < \varphi_K^{a,c-a,c}(r) < e^{(1-1/K)R(a,c-a)/2} r^{1/K}, \tag{46}$$

$$r^K > \varphi_{1/K}^{a,c-a,c}(r) > e^{(1-K)R(a,c-a)/2} r^K. \tag{47}$$

For the special case of $a = 1/2$ and $c = 1$ the readers are referred to [18]. Wang et al. [181] presented several sharp inequalities for the generalized modular functions with specific choice of parameters $c = 1$ and $b = 1 - a$.

A linearization for the generalized modular function is also presented in [94] as follows: Let $p : (0, 1) \rightarrow (-\infty, \infty)$ and $q : (-\infty, \infty) \rightarrow (0, 1)$ be given by $p(x) = 2 \log(x/x')$ and $q(x) = p^{-1}(x) = \sqrt{e^x/(e^x + 1)}$, respectively, and for $a \in (0, 1)$, $c \in (a, 1]$, $K \in (1, \infty)$, let $g, h : (-\infty, \infty) \rightarrow (-\infty, \infty)$ be defined by $g(x) = p(\varphi_K^{a,c-a,c}(q(x)))$ and $h(x) = p(\varphi_{1/K}^{a,c-a,c}(q(x)))$. Then

$$g(x) \geq \begin{cases} Kx, & \text{if } x \geq 0, \\ x/K, & \text{if } x < 0, \end{cases} \quad \text{and} \quad h(x) \leq \begin{cases} x/K, & \text{if } x \geq 0, \\ Kx, & \text{if } x < 0. \end{cases}$$

In the same paper the authors also studied how these generalized functions depend on the parameter c . Corresponding results for the case $c = 1$ can be found in the articles [19, 153, 204].

Recently Bhayo and Vuorinen [55] have studied the Hölder continuity and submultiplicative properties of $\varphi_K^{a,b,c}(r)$ in the case where $c = 1$ and $b = 1 - a$ and have obtained several sharp inequalities for $\varphi_K^{a,1-a,1}(r)$.

For $x, K \in (0, \infty)$, define

$$\eta_K^a(x) \equiv \left(\frac{s}{s'}\right)^2, \quad s = \varphi_K^{a,1-a,1}(r), \quad r = \sqrt{\frac{x}{1+x}},$$

and the *generalized linear distortion function*

$$\lambda(a, K) \equiv \left(\frac{\varphi_K^{a,1-a,1}(1/\sqrt{2})}{\varphi_{1/K}^{a,1-a,1}(1/\sqrt{2})}\right)^2, \quad \lambda(a, 1) = 1.$$

For $a = 1/2$, these two functions reduce to the well-known special case denoted by $\eta_K(x)$ and $\lambda(K)$, respectively, which play a crucial role in quasiconformal theory. Several inequalities for these functions have been obtained as an application of the monotonicity and convexity of certain combinations of these functions and some elementary functions; see [55, 80, 122, 123, 180, 203]. For instance, the following chain of inequalities is proved in [80]: for $a \in (0, 1/2]$, $K \in (1, \infty)$ and $x, y \in (0, \infty)$,

$$\begin{aligned} \max \left\{ \frac{2\eta_K^a(x)\eta_K^a(y)}{\eta_K^a(x) + \eta_K^a(y)}, \eta_K^a\left(\frac{2xy}{x+y}\right) \right\} &\leq \eta_K^a(\sqrt{xy}) \\ &\leq \sqrt{\eta_K^a(x)\eta_K^a(y)} \leq \min \left\{ \frac{\eta_K^a(x) + \eta_K^a(y)}{2}, \eta_K^a\left(\frac{x+y}{2}\right) \right\}, \end{aligned}$$

with equality if and only if $x = y$.

6 Inequalities for Power Series

The following theorem [95, Theorem 4.3] is an interesting tool in simplified proofs for monotonicity of the quotient of two power series:

Theorem 6.1. *Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two real power series converging on the interval $(-R, R)$. If the sequence $\{a_n/b_n\}$ is increasing (decreasing) and $b_n > 0$ for all n , then the function*

$$f(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}$$

is also increasing (decreasing) on $(0, R)$. In fact, the function

$$f'(x) \left(\sum_{n=0}^{\infty} b_n x^n \right)^2$$

has positive Maclaurin coefficients.

A more general version of this theorem appears in [58] and [147, Lemma 2.1]. This kind of rule also holds for the quotient of two polynomials instead of two power series (cf. [95, Theorem 4.4]):

Theorem 6.2. *Let $f_n(x) = \sum_{k=0}^n a_k x^k$ and $g_n(x) = \sum_{k=0}^n b_k x^k$ be two real polynomials, with $b_k > 0$ for all k . If the sequence $\{a_k/b_k\}$ is increasing (decreasing), then so is the function $f_n(x)/g_n(x)$ for all $x > 0$. In fact, $g_n f'_n - f_n g'_n$ has positive (negative) coefficients.*

In 1997 Ponnusamy and Vuorinen [147] refined Ramanujan’s work on asymptotic behavior of the hypergeometric function and also obtained many inequalities for the hypergeometric function by making use of Theorem 6.1. Many well-known results of monotonicity and inequalities for complete elliptic integrals have been extended to the generalized elliptic integrals in [94, 95].

Motivated by an open problem of Anderson et al. [16], in 2006 Baricz [30] considered ratios of general power series and obtained the following theorem. Note the similarity of the last inequality in Theorem 6.3 with the left-hand side of the inequality (45).

Theorem 6.3. *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_n > 0$ for all $n \geq 0$ is convergent for all $x \in (0, 1)$ and also that the sequence $\{(n + 1)a_{n+1}/a_n - n\}_{n \geq 0}$ is strictly decreasing. Let the function $m_f : (0, 1) \rightarrow (0, \infty)$ be defined as $m_f(r) = f(1 - r^2)/f(r^2)$. Then*

$$\sqrt[k]{\prod_{i=1}^k m_f(r_i)} \leq m_f \left(\sqrt[k]{\prod_{i=1}^k r_i} \right),$$

for all $r_1, r_2, \dots, r_k \in (0, 1)$, where equality holds if and only if $r_1 = r_2 = \dots = r_k$. In particular, for $k = 2$ the inequalities

$$\begin{aligned} \sqrt{m_f(r_1)m_f(r_2)} &\leq m_f(\sqrt{r_1r_2}), \\ \frac{1}{m_f(r_1)} + \frac{1}{m_f(r_2)} &\geq \frac{2}{m_f(\sqrt{r_1r_2})}, \\ m_f(r_1) + m_f(r_2) &\geq 2m_f \left(\sqrt{1 - \sqrt{(1-r_1^2)(1-r_2^2)}} \right) \end{aligned}$$

hold for all $r_1, r_2 \in (0, 1)$, and in all these inequalities equality holds if and only if $r_1 = r_2$.

The following Landen-type inequality for power series is also due to Baricz [29].

Theorem 6.4. *Suppose that the power series $f(x) = \sum_{n=0}^\infty a_n x^n$ with $a_n > 0$ for all $n \geq 0$ is convergent for all $x \in (0, 1)$ and that for a given $\delta > 1$ the sequence $\{n!a_n/(\log \delta)^n\}_{n \geq 0}$ is decreasing. If $\lambda_f(x) = f(x^2)$, then*

$$\lambda_f \left(\frac{2\sqrt{r}}{1+r} \right) < \rho \lambda_f(r)$$

holds for all $r \in (0, 1)$ and $\rho \geq \delta^{4\sqrt{2}-5}$.

Anderson, Vamanamurthy, and Vuorinen [22] studied generalized convexity and gave sufficient conditions for generalized convexity of functions defined by Maclaurin series. These results yield a class of new inequalities for power series which improve some earlier results obtained by Baricz. More inequalities for power series can be found in [37, 80].

In 1928 T. Kaluza gave a criterion for the signs of the power series of a function that is the reciprocal of another power series.

Theorem 6.5 ([105]). *Let $f(x) = \sum_{n \geq 0} a_n x^n$ be a convergent Maclaurin series with radius of convergence $r > 0$. If $a_n > 0$ for all $n \geq 0$ and the sequence $\{a_n\}_{n \geq 0}$ is log-convex, that is, for all $n \geq 0$*

$$a_n^2 \leq a_{n-1}a_{n+1}, \tag{48}$$

then the coefficients b_n of the reciprocal power series $1/f(x) = \sum_{n \geq 0} b_n x^n$ have the following properties: $b_0 = 1/a_0 > 0$ and $b_n \leq 0$ for all $n \geq 1$.

In 2011 Baricz, Vesti, and Vuorinen [42] showed that the condition (48) cannot be replaced by the condition

$$a_n \leq \left(\frac{a_{n-1}^t + a_{n+1}^t}{2} \right)^{1/t},$$

for any $t > 0$. However, it is not known whether the condition (48) is necessary.

In 2009 Koumandos and Pedersen [115, Lemma 2.2] proved the following interesting result, which deals with the monotonicity properties of the quotient of two series of functions.

Theorem 6.6. *Suppose that $a_k > 0$, $b_k > 0$ and that $\{u_k(x)\}$ is a sequence of positive C^1 -functions such that the series*

$$\sum_{k=0}^{\infty} a_k u_k^{(l)}(x) \quad \text{and} \quad \sum_{k=0}^{\infty} b_k u_k^{(l)}(x), \quad l = 0, 1,$$

converge absolutely and uniformly over compact subsets of $[0, \infty)$. Define

$$f(x) \equiv \frac{\sum_{k=0}^{\infty} a_k u_k(x)}{\sum_{k=0}^{\infty} b_k u_k(x)}.$$

1. *If the logarithmic derivatives $u'_k(x)/u_k(x)$ form an increasing sequence of functions and if a_k/b_k decreases (resp. increases), then $f(x)$ decreases (resp. increases) for $x \geq 0$.*
2. *If the logarithmic derivatives $u'_k(x)/u_k(x)$ form a decreasing sequence of functions and if a_k/b_k decreases (resp. increases), then $f(x)$ increases (resp. decreases) for $x \geq 0$.*

For inequalities of power series as complex functions, see [99–101] and the references therein.

7 Means

A *homogeneous bivariate mean* is defined as a continuous function $\mathcal{M} : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying $\min\{x, y\} \leq \mathcal{M}(x, y) \leq \max\{x, y\}$ and $\mathcal{M}(\lambda x, \lambda y) = \lambda \mathcal{M}(x, y)$ for all $x, y, \lambda > 0$. Important examples are the *arithmetic mean* $A(a, b)$, the *geometric mean* $G(a, b)$, the *logarithmic mean* $L(a, b)$, the *identric mean* $I(a, b)$, the *root-square mean* $Q(a, b)$, and the *power mean* $M_r(a, b)$ of order r defined, respectively, by

$$\begin{aligned}
 A(a, b) &= \frac{a + b}{2}, & G(a, b) &= \sqrt{ab}, \\
 L(a, b) &= \frac{a - b}{\log a - \log b}, & I(a, b) &= \frac{1}{e} \left(\frac{a^a}{b^b} \right)^{1/(a-b)}, \\
 Q(a, b) &= \sqrt{\frac{a^2 + b^2}{2}}, & M_r(a, b) &= \sqrt{\frac{a^r + b^r}{2}}.
 \end{aligned}$$

7.1 Power Means

The *weighted power means* are defined by

$$M_\lambda(\omega; a, b) \equiv (\omega a^\lambda + (1 - \omega)b^\lambda)^{1/\lambda} \quad (\lambda \neq 0),$$

$M_0(\omega; a, b) \equiv a^\omega b^{1-\omega}$, with *weights* $\omega, 1 - \omega > 0$. The *power means* are the equally weighted means $M_\lambda(a, b) = M_\lambda(1/2; a, b)$. As a special case, we have $M_0(1/2; a, b) = G(a, b)$.

In [114] Kouba studied the ratio of differences of power means

$$\rho(s, t, p; a, b) \equiv \frac{M_s^p(a, b) - G^p(a, b)}{M_t^p(a, b) - G^p(a, b)},$$

finding sharp bounds for $\rho(s, t, p; a, b)$ in various regions of *stp*-space with a, b positive and $a \neq b$. This work extends the results of Alzer and Qiu [10], Trif [175], Kouba [113], Wu [193], and Wu and Debnath [196]. Kouba also extended the range of validity of the following inequality, due to Wu and Debnath [196]:

$$\frac{2^{-p/r} - 2^{-p/s}}{2^{-p/t} - 2^{-p/s}} < \frac{M_r^p(a, b) - M_s^p(a, b)}{M_t^p(a, b) - M_s^p(a, b)} < \frac{r - s}{t - s}$$

to the set of real numbers r, t, s, p satisfying the conditions $0 < s < t < r$ and $0 < p \leq (4t + 2s)/3$.

7.2 Toader Means

If $p : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a strictly monotonic function, then define

$$f(a, b; p, n) \equiv \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} p((a^n \cos^2 \theta + b^n \sin^2 \theta)^{1/n}) d\theta & \text{if } n \neq 0, \\ \frac{1}{2\pi} \int_0^{2\pi} p(a^{\cos^2 \theta} b^{\sin^2 \theta}) d\theta & \text{if } n = 0, \end{cases}$$

where a, b are positive real numbers. The *Toader mean* [174] of a and b is defined as $T(a, b; p, n) \equiv p^{-1}(f(a, b; p, n))$. It is easy to see that the Toader mean is symmetric. For special choices of p , let $T_{q,n}(a, b) = T(a, b; p, n)$ if $p(x) = x^q$ with $q \neq 0$ and $T_{0,n}(a, b) = T(a, b; p, n)$ if $p(x) = \log x$. The means $T_{q,n}$ belong to a large family of means called the *hypergeometric means*, which have been studied by Carlson and others [62, 65, 68]. It is easy to see that $T_{q,n}$ is homogeneous. In particular, we have

$$T_{0,2}(a, b) = A(a, b), \quad T_{-2,2}(a, b) = G(a, b), \quad T_{2,2}(a, b) = Q(a, b).$$

Furthermore, the Toader means are related to the complete elliptic integrals: for $a \geq b > 0$,

$$T_{-1,2}(a, b) = \frac{\pi a}{2\mathcal{K}(\sqrt{1 - (b/a)^2})} \quad \text{and} \quad T_{1,2}(a, b) = \frac{2a}{\pi} \mathcal{E}(\sqrt{1 - (b/a)^2}).$$

In 1997 Qiu and Shen [151] proved that, for all $a, b > 0$ with $a \neq b$,

$$M_{3/2}(a, b) < T_{1,2}(a, b).$$

This inequality had been conjectured by Vuorinen [176]. Alzer and Qiu [10] proved the following best possible power mean upper bound:

$$T_{1,2}(a, b) < M_{\log 2 / \log(\pi/2)}(a, b).$$

Very recently, Chu and his collaborators [78, 79, 83] obtained several bounds for $T_{1,2}$ with respect to some combinations of various means.

7.3 Seiffert Means

The *Seiffert means* S_1 and S_2 are defined by

$$S_1(a, b) \equiv \frac{a - b}{2 \arcsin \frac{a-b}{a+b}}, \quad a \neq b, \quad S_1(a, a) = a,$$

and

$$S_2(a, b) \equiv \frac{a - b}{2 \arctan \frac{a-b}{a+b}}, \quad a \neq b, \quad S_2(a, a) = a.$$

It is well known that

$$\sqrt[3]{G^2 A} < L < \frac{2G + A}{3}.$$

Sándor proved similar results for Seiffert means [160, 161]:

$$\sqrt[3]{A^2G} < S_1 < \frac{G + 2A}{3} < I \quad (49)$$

and

$$\sqrt[3]{Q^2A} < S_2 < \frac{A + 2Q}{3}. \quad (50)$$

The inequalities (49) and (50) are special cases of more general results obtained by Neuman and Sándor [139, 140].

7.4 Extended Means

Let $a, b \in (0, \infty)$ be distinct and $s, t \in \mathbb{R} \setminus \{0\}$, $s \neq t$. We define the *extended mean* [172] with parameters s and t by

$$E_{s,t}(a, b) \equiv \left(\frac{t a^s - b^s}{s a^t - b^t} \right)^{1/(s-t)}$$

and also

$$E_{s,s}(a, b) \equiv \exp \left(\frac{1}{s} + \frac{a^s \log a - b^s \log b}{a^s - b^s} \right),$$

$$E_{s,0}(a, b) \equiv \left(\frac{a^s - b^s}{s \log(x/y)} \right)^{1/s} \quad \text{and} \quad E_{0,0}(a, b) \equiv \sqrt{ab}.$$

We see that all the classical means belong to the family of extended means. For example, $E_{2,1} = A$, $E_{0,0} = G$, $E_{-1,-2} = H$, and $E_{1,0} = L$ and, more generally, $M_\lambda = E_{2\lambda,\lambda}$ for $\lambda \in \mathbb{R}$. The reader is referred to the survey [148] for many interesting results on the extended mean.

In 2002 Hästö [91] studied a certain monotonicity property of ratios of extended means and Seiffert means, which he called a *strong inequality*. These strong inequalities were shown to be related to the so-called relative metric [92, 93].

7.5 Means and the Circular and Hyperbolic Functions

It is easy to check the following identities:

$$A(1 + \sin x, 1 - \sin x) = 1, \quad G(1 + \sin x, 1 - \sin x) = \cos x, \quad (51)$$

$$Q(1 + \sin x, 1 - \sin x) = \sqrt{1 + \sin^2 x}, \quad S_1(1 + \sin x, 1 - \sin x) = \frac{\sin x}{x}, \quad (52)$$

$$A(e^x, e^{-x}) = \cosh x, \quad G(e^x, e^{-x}) = 1, \quad Q(e^x, e^{-x}) = \sqrt{\cosh 2x}, \quad (53)$$

$$L(e^x, e^{-x}) = \frac{\sinh x}{x}, \quad I(e^x, e^{-x}) = e^{x \coth x - 1}, \quad (54)$$

$$S_1(e^x, e^{-x}) = \frac{\sinh x}{\arcsin(\tanh x)}, \quad S_2(e^x, e^{-x}) = \frac{\sinh x}{\arctan(\tanh x)}. \quad (55)$$

One can get many inequalities by combining the above identities and inequalities between means. For example, combining (49) and (52), we have

$$\sqrt[3]{\cos x} < \frac{\sin x}{x} < \frac{\cos x + 2}{3},$$

where the second inequality is the well-known Cusa-Huygens inequality, and combining (50), (53), and (55), we have

$$\sqrt[3]{(\cosh 2x)(\cosh x)} < \frac{\sinh x}{\arctan(\tanh x)} < \frac{\cosh x + 2\sqrt{\cosh 2x}}{3}.$$

More inequalities on mean values and trigonometric and hyperbolic functions can be found in [136, 163, 165, 200, 208] and references therein.

7.6 Means and Hypergeometric Functions

In 2005 Richards [158] obtained sharp power mean bounds for the hypergeometric function: Let $0 < a, b \leq 1$ and $c > \max\{-a, b\}$. If $c \geq \max\{1 - 2a, 2b\}$, then

$$M_\lambda(1 - b/c; 1, 1 - r) \leq F(-a, b; c; r)^{1/a}$$

if and only if $\lambda \leq \frac{a+c}{1+c}$. If $c \leq \min\{1 - 2a, 2b\}$, then

$$M_\mu(1 - b/c; 1, 1 - r) \leq F(-a, b; c; r)^{1/a}$$

if and only if $\mu \geq \frac{a+c}{1+c}$. These inequalities generalize earlier results proved by Carlson [66].

For hypergeometric functions of form $F(1/2 - s, 1/2 + s; 1; 1 - r^p)^q$, Borwein et al. [61] exhibited explicitly iterations similar to the arithmetic-geometric mean. Barnard et al. [43] presented sharp bounds for hypergeometric analogs of the arithmetic-geometric mean as follows: For $0 < \alpha \leq 1/2$ and $p > 0$,

$$M_\lambda(\alpha; 1, r) \leq F(\alpha, 1 - \alpha; 1; 1 - r^p)^{-1/(\alpha p)} \leq M_\mu(\alpha; 1, r)$$

if and only if $\lambda \leq 0$ and $\mu \geq p(1 - \alpha)/2$.

Some other inequalities involving hypergeometric functions and bivariate means can be found in the very recent survey [44].

For any two power means M_λ and M_μ , a function f is called $M_{\lambda,\mu}$ -convex if it satisfies

$$f(M_\lambda(x, y)) \leq M_\mu(f(x), f(y)).$$

Recently many authors have proved that the zero-balanced Gaussian hypergeometric function is $M_{\lambda,\lambda}$ -convex when $\lambda \in \{-1, 0, 1\}$. For details see [22, 26, 37, 80]. Baricz [31] generalized these results to the $M_{\lambda,\lambda}$ -convexity of zero-balanced Gaussian hypergeometric functions with respect to a power mean for $\lambda \in [0, 1]$. Zhang et al. [205] extended these results to the case of $M_{\lambda,\mu}$ -convexity with respect to two power means: For all $a, b > 0$, $\lambda \in (-\infty, 1]$, and $\mu \in [0, \infty)$ the hypergeometric function $F(a, b; a + b; r)$ is $M_{\lambda,\mu}$ -convex on $(0, 1)$.

The following interesting open problem is presented by Baricz [36]:

Open Problem. *If m_1 and m_2 are bivariate means, then find conditions on $a_1, a_2 > 0$ and $c > 0$ for which the inequality*

$$m_1(F_{a_1}(r), F_{a_2}(r)) \leq (\geq) F_{m_2(a_1, a_2)}(r)$$

holds true for all $r \in (0, 1)$, where $F_a(r) = F(a, c - a; c; r)$.

7.7 Means and Quasiconformal Analysis

Special functions have always played an important role in the distortion theory of quasiconformal mappings. Anderson, Vamanamurthy, and Vuorinen [18] have systematically investigated classical special functions and their extensive applications in the theory of conformal invariants and quasiconformal mappings. Some functional inequalities for special functions in quasiconformal mapping theory involve the arithmetic mean, geometric mean, or harmonic mean. For example, for the well-known Grötzsch ring function μ and the Hersch-Pfluger distortion function φ_K , the following inequalities hold for all $s, t \in (0, 1)$ with $s \neq t$:

$$\sqrt{\mu(s)\mu(t)} < \mu(\sqrt{st}),$$

and

$$\sqrt{\varphi_K(s)\varphi_K(t)} < \varphi_K(\sqrt{st}) \text{ for } K > 1.$$

Recently, Wang, Zhang, and Chu [182, 183] have extended these inequalities as follows:

$$M_\lambda(\mu(s), \mu(t)) < \mu(M_\lambda(s, t)) \text{ if and only if } \lambda \leq 0,$$

$$M_\lambda(\varphi_K(s), \varphi_K(t)) < \varphi_K(M_\lambda(s, t)) \text{ if and only if } \lambda \geq 0 \text{ and } K > 1,$$

and

$$M_\lambda(\varphi_K(s), \varphi_K(t)) > \varphi_K(M_\lambda(s, t)) \text{ if and only if } \lambda \geq 0 \text{ and } 0 < K < 1.$$

Some similar results for the generalized Grötzsch function, generalized modular function, and other special functions related to quasiconformal analysis can be found in [155, 179, 184, 186, 187].

8 Epilogue and a View Toward the Future

In earlier work we have listed many open problems. See especially [14, pp. 128–131] and [18, p. 478]. Many of these problems are still open. In Sects. 4, 6, and 7 above, we have also mentioned some open problems.

Finally, we wish to suggest some ideas for further research. In a frequently cited paper [119] Lindqvist introduced in 1995 the notion of *generalized trigonometric functions* such as \sin_p , and presently there is a large body of literature about this topic. For the case $p = 2$ the classical functions are obtained. In 2010, Biezuner et al. [59] developed a practical numerical method for computing values of \sin_p . Recently, Takeuchi [173] has gone a step further, introducing functions depending on two parameters p and q that reduce to the p -functions of Lindqvist when $p = q$. In [56, 57, 112] the authors have continued the study of this family of generalized functions and have suggested that many properties of classical functions have a counterpart in this more general setting. It would be natural to generalize the properties of trigonometric functions cited in this survey to the (p, q) -trigonometric functions of Takeuchi.

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