Chapter 7
Decomposing Efficiency and Returns to Scale in Two-Stage Network Systems

Biresh K. Sahoo, Joe Zhu, and Kaoru Tone

Abstract Most of real-life production technologies are multi-stage in nature. Characterization of such technologies via concept like *network returns to scale* is considered important to firm managers for the stage-specific analysis of their business decisions concerning expansion or contraction so as to improve their firms' overall performance. Similarly, depicting such multi-stage technologies via *network efficiency* is important in identifying the sources of network inefficiency. It is, therefore, imperative to estimate both efficiency and returns to scale of a firm not only for the network technology but also for the sub-technologies so as to locate the sources of efficiency and scale economies. The primary purpose of constructing a network technology is to address allocative efficiency that is associated with the choice of how much of intermediate products to produce and consume, in addition to the economic use of primary inputs and the maximal production of final outputs. Therefore, it is necessary that not only the intermediate products are explicitly modeled, but also their optimal values are considered in the construction of sub-technologies' frontiers so that the issue of allocative efficiency, if exists, can be addressed. Based on the premise concerning whether a network technology considers allocative inefficiency, two approaches are suggested for the estimation of network technology.

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The first approach makes use of a single network technology for two *interdependent* sub-technologies. The second approach, however, assumes complete allocative efficiency by considering two *independent* sub-technology frontiers, one for each sub-technology. These two approaches are, however, necessary, in modeling the output loss of a network firm suffering from allocative inefficiency, which arises due to any possible sub-optimal decision as to how much of intermediate products to produce and consume in the world of changing prices.

Keywords Data envelopment analysis • Network DEA • Returns to scale decomposition • Efficiency decomposition • Modeling output loss due to allocative inefficiency

7.1 Introduction

Most of real-life production technologies are multi-stage in nature. Characterization of such technologies via concept like *returns to scale* (RTS) or *scale elasticity* (SE) is considered important to firm managers for the stage-specific analysis of their business decisions concerning expansion or contraction. Therefore, it is imperative to estimate the SE of a firm not only for the network technology but also for its sub-technologies so as to locate the sources of scale economies. This chapter presents the idea of Sahoo et al. (2014), and develop new approaches in non-parametric data envelopment analysis (DEA) for the decomposition of efficiency and RTS of a network firm into its stage-specific efficiencies and RTS, which are of practical use to firm managers in improving the overall performance of their firms.

Data envelopment analysis (DEA), a linear programming (LP) based technique, has been widely accepted as a competent methodology to estimate the structure of production technology in both *primal* (production) and *dual* (cost) environments. See Scarf (1990) for a discussion on the analogy between economic institutions and algorithms for solving the LP problems where the *simplex method* is interpreted as a search for market prices that equilibrate demand for factors of production with their supply. Much of DEA literature that considers the evaluation of SE treats production technology as a black-box (see, e.g., Sahoo et al. 1999; Fukuyama 2003; Tone and Sahoo 2003; Banker et al. 2004; Sahoo and Tone 2013; Zelenyuk 2013; among others), thus completely ignoring the literature on production control problems dealing with multi-stage production technologies (see, e.g., Aburzzi 1965; Bakshi and Arora 1969; among others).

The DEA literature that considers modeling of multi-stage technology by linking its sub-technologies is fairly recent. To the best of our knowledge, the network structure that links sub-technologies with intermediate products in the DEA framework was first introduced by Färe (1991); was, subsequently, extended in Färe and Grosskopf (2000), and Tone and Tsutsui (2009, 2010); and was, finally, applied in Tone and Sahoo (2003), Prieto and Zofio (2007), Yu and Lin (2008), and Lewis et al. (2013), among others.

A special variant of Färe and Grosskopf's multi-stage technology, i.e., a two-stage technology was developed in a different way by several scholars under multiplier DEA models (see, e.g., Chen and Zhu 2004; Chen et al. 2006, 2009a, b, 2010, 2013; Liang et al. 2006, 2008; Kao and Hwang 2008, 2011; Kao 2009, 2013; Cook et al. 2010; among others). In this set up, sub-technology I consumes *input* resources to produce *intermediate* products, which are all, in turn, used as *inputs* to sub-technology II to produce *final* outputs. A further restricted variant of this two-stage structure is developed by Seiford and Zhu (1999) and Zhu (2000) where sub-technologies are treated *independent*, and network as well as its sub-technologies' efficiencies are estimated independently.

The two-stage DEA literature (Kao and Hwang 2008, 2011, 2014; Liang et al. 2008; Kao 2009, 2013; Chen et al. 2009a, b, 2010, 2013) that addresses the evaluation of the decomposition of network efficiency into the sub-technology specific efficiencies is fairly recent. This decomposition is done under the assumption of *constant returns to scale* (CRS). What seems to be more intriguing but has completely been overlooked is whether this decomposition can be made under the assumption of *variable returns to scale* (VRS). And, if the answer to this question is yes, but at a cost, then it is worth investigating what this cost amounts to, i.e., allocative inefficiency due to any sub-optimal decision by the sub-technology managers as to how much of intermediate products to produce in the world of changing prices. The first objective of this chapter is to address the aforementioned issue.

Another important issue related to the first one, which has also not been addressed in the two-stage DEA literature, is the decomposition of network SE into the sub-technology specific SEs. This issue is related because the SE estimation can be done only under VRS. This decomposition will help a firm manager to not only determine the scale economies of network technology but also locate their sources, which lie in the sub-technologies. To our best knowledge, Kao and Hwang (2011) are the first to propose a scheme to determine only the scale efficiency of independent sub-technologies under the two-stage setting. Therefore, the second objective of this chapter is to propose a scheme to analytically show the SE of network technology as the product of those of its two sub-technologies.

For network SE estimation, two approaches may be considered based on the premise concerning whether the VRS-based network technology construct considers allocative inefficiency. In economics, the primary purpose of constructing a technology is to address allocative efficiency associated with the economic choice of how much of intermediate products to produce and consume, in addition to the economic use of primary inputs and the maximal production of final outputs. Therefore, it is necessary that not only the intermediate products are explicitly modeled, but also their optimal values are considered in the construction of sub-technologies' frontiers so that the issue of allocative efficiency, if exists, can be addressed.

Under the first approach (Approach I), which is ours, one network frontier is constructed for the two *interdependent* sub-technology frontiers, which are linked through optimal values of intermediate products. The dual pricing interpretation of the constraint that the intermediate products are freely determined in our envelopment-based network technology is that the weights for intermediate products as inputs and outputs in our multiplier-based network technology are the same. We maintain that our multiplier-based network technology is *additive*.

The construct of our proposed additive network technology holds under two conditions: (1) weights for intermediate products as inputs and outputs are the same, and (2) intercept multiplier of network technology is the sum of those of the two sub-technologies. The first condition holds due to our constraint that the intermediate products are freely determined in our envelopment-based network technology. The second condition holds under the assumption that the additive network technology can inherit the properties of its sub-technologies, i.e., if the sub-technologies satisfy the properties such as no free lunch, free disposability in inputs and outputs, compactness, convexity, and returns to scale, then so does the additive network technology. The proof of this is made in the spirit of the proof of Proposition 2.3.2 in Färe and Grosskopf (1996, p. 23, pp. 44–45).

The network technical efficiency (TE) decomposition based on Approach I reveals that allocative inefficiency arises only under the VRS specification, but disappears under the CRS specification. It can, therefore, be argued that interpreting the 'same weights' assumption for the intermediate products as outputs and inputs as a perfect coordination between the two sub-technologies, as in Liang et al. (2008), is not sufficient to rule out allocative inefficiency in the VRS environment. Allocative inefficiency is a broader concept that includes inefficiencies arising from possible sub-optimal decisions as to how much of intermediate products to produce and consume in the world of changing prices. Our additive network technology can be used in identifying such inefficiency when optimal values of intermediate products are less than their observed values. Our network TE decomposition reveals that a network firm is fully efficient only when it is efficient in both of its sub-technologies.

The second approach (Approach II), which is due to Kao and Hwang (2011), requires the two sub-technologies to be *independent* for the construction of network frontier. To keep the sub-technologies independent, the input-orientation in the sub-technology I and the output-orientation in the sub-technology II are maintained to keep the level of intermediate products unaltered. This way of modeling network technology assumes the current uses of intermediate products as optimal, thereby effectively rules out allocative inefficiency arising from their possible sub-optimal uses. However, allocative inefficiency of this kind, if exists, may question the very TE estimates estimated against the two assumed independent sub-technology frontiers.

Note that the choice of a particular approach adopted implies whether assuming allocative inefficiency in the underlying technology construct, and hence, yields a distinct set of TE estimates. The distinction between the two approaches is important from a policy point of view as the factors attributing to the network's inefficiency in each approach are distinct. For example, a lower network TE may be due to allocative inefficiency in Approach I as against the same due to lower sub-technologies' efficiencies in Approach II. In this case, policies to remove allocative inefficiency may be more effective in improving the network efficiency in Approach I than the policies directed at improving the sub-technology specific TEs. However, a comparison between the two approaches can be worth revealing in modeling the output loss of a network firm suffering from allocative inefficiency that may arise due to sub-optimal decision as to how much of intermediate products to produce and consume by the sub-technology managers in the world of changing prices. This is the third objective of this chapter.

The remainder of the chapter proceeds as follows. Section 7.2 deals with a discussion on the development of variants of two-stage network DEA models to estimate the TE and SE of firms in the network technology as well as sub-technologies. Section 7.3 provides an illustrative empirical application, showing how the TE and SE estimates of a firm yielded from the two approaches are different due to allocative inefficiency. Section 7.4 provides some concluding remarks.

7.2 Model Development

7.2.1 Two-Stage Network Technology

Consider a two-stage technology in which sub-technologies are connected in a network to form a network technology (T^N) (see Fig. 7.1). Further, assume that there are n firms, and each firm ($h = 1, 2, \ldots, n$) in the first sub-technology (T^I) uses *inputs* x_i ($i = 1, 2, \ldots, m$) to produce *intermediate* outputs z_d ($d = 1, 2, \ldots, p$) and the same firm in the second sub-technology (T^{II}) uses these *intermediate* outputs as *inputs* to produce *final* outputs y_r ($r = 1, 2, \ldots, s$). These z_d are called intermediate measures by Chen and Zhu (2004) and Liang et al. (2008).

7.2.2 TE Estimation

We now discuss the *TE* evaluation using Approach I.

7.2.2.1 TE Estimation Using Approach I

One can evaluate the TE of a network firm either in input-oriented manner or in output-oriented manner or in non-oriented manner. In this study we, however, concentrate on TE evaluation in input-oriented manner. We set up the following

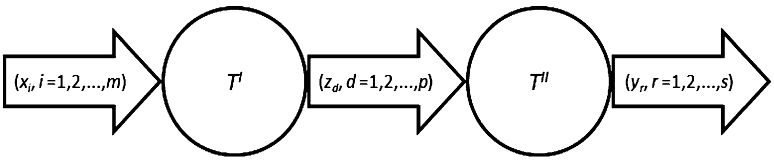

Fig. 7.1 Two-stage network technology

input-oriented VRS-based network DEA model for estimating the input *TE* of firm h ($TE_{ih}^{N(I)}$) in *envelopment* form as

$$
\begin{aligned}
TE_{ih}^{N(I)} &= \min_{\beta,\lambda,\widetilde{z}} \beta_h \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij}\lambda_j \le \beta_h x_{ih} (\forall i),\ \sum_{j=1}^{n} z_{dj}\lambda_j - \widetilde{z}_{dh} \ge 0\ (\forall d),\ \sum_{j=1}^{n} \lambda_j = 1, \quad \text{(sub-technology I)} \\
& \sum_{j=1}^{n} z_{dj}\mu_j - \widetilde{z}_{dh} \le 0\ (\forall d),\ \sum_{j=1}^{n} y_{rj}\mu_j \ge y_{rh} (\forall r),\ \sum_{j=1}^{n} \mu_j = 1, \text{(sub-technology II)} \\
& \beta_h \le 1, \lambda_j, \mu_j \ge 0\ (\forall j), \widetilde{z}_{dh} : \text{free } (\forall d)
\end{aligned} \tag{7.1}
$$

Let $(\beta^*, \lambda^*, \widetilde{z}^*)$ be optimal solution vector of model (7.1), which is based on the following VRS-based network technology set ($T_{VRS}^{N(I)}$) defined as

$$
T_{VRS}^{N(I)} = \left\{ (x,y,z) \left| \begin{array}{l} \sum_{j=1}^{n} x_{ij}\lambda_j \le x_i (\forall i),\ \sum_{j=1}^{n} z_{dj}\lambda_j - z_d \ge 0\ (\forall d),\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \ge 0\ (\forall j) \\ \sum_{j=1}^{n} z_{dj}\mu_j - z_d \le 0\ (\forall d),\ \sum_{j=1}^{n} y_{rj}\mu_j \ge y_r (\forall r), \sum_{j=1}^{n} \mu_j = 1, \mu_j \ge 0\ (\forall j) \end{array} \right. \right\} \tag{7.2}
$$

$T_{VRS}^{N(I)}$ uses λ and μ as intensity weights to form a linear combinations of n observed firms. Since both $T_{VRS}^{I(I)}$ and $T_{VRS}^{II(I)}$ satisfy VRS (i.e., $\sum_{j=1}^{n} \lambda_j = 1$ and $\sum_{j=1}^{n} \mu_j = 1$), $T_{VRS}^{N(I)}$ satisfies VRS. Similarly, $T_{VRS}^{I(I)}$ and $T_{VRS}^{II(I)}$ satisfy the assumption of *strong* (*free*) *disposability* of inputs and outputs by the use of inequality constraints, and so is the case with $T_{VRS}^{N(I)}$. The most distinguishing feature of $T_{VRS}^{N(I)}$ is that the intermediate products are explicitly modeled to be *freely* determined so as to make the sub-technologies *interdependent*. Chen and Zhu (2004), Liang et al. (2008), and Chen et al. (2010) also used this feature to reveal the frontier points of the two-stage technology.

β_h^* can be regarded as representing the *minimum* input proportion possible in $T_{VRS}^{N(I)}$ to produce y_h. Firm h is technically efficient, i.e., $TE_{ih}^{N(I)} = 1$ if and only if $(\beta_h^* x_h, \widetilde{z}_h^*, y_h) \in \partial T_{VRS}^{N(I)}(\cdot)$ where $\partial T_{VRS}^{N(I)}(\cdot)$ represents the boundary of $T_{VRS}^{N(I)}(\cdot)$, and $(\beta_h^* x_h, z_h, y_h) \notin \partial T_{VRS}^{N(I)}(\cdot)$ when $z_h \ne \widetilde{z}_h^*$.

One can also set up the *input-oriented* VRS-based network DEA model for estimating the input *TE* of firm h ($TE_{ih}^{N(I)}$) in *multiplier* form as

$$
TE_{ih}^{N(I)} = \max \sum_{r=1}^{s} u_r y_{rh} - \omega_I - \omega_{II} \tag{7.3}
$$

$$\text{s.t.} \quad \sum_{i=1}^{m} v_i x_{ih} = 1, \tag{7.3.1}$$

$$\sum_{d=1}^{p} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \omega_I \leq 0 \ (\forall j), \tag{7.3.2}$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{p} w_d z_{dj} - \omega_{II} \leq 0 \ (\forall j), \tag{7.3.3}$$

$$v_i, u_r, w_d \geq 0 \ (\forall i, r, d); \omega_I, \omega_{II} : \text{free} \tag{7.3.4}$$

where v_i, w_{dI} and ω_I are the dual decision variables to the respective constraints of sub-technology I, and w_{dII}, u_r and ω_{II} are the dual decision variables to the respective constraints of sub-technology II, in (7.1). Here $w_{dI} = w_{dII}(=w_d)$, which is due to the constraint that $\widetilde{z}_d^*$ are free in (7.1). Otherwise, w_{dI} would have been no less than $w_{dII}(w_{dI} \geq w_{dII})$, had $\widetilde{z}_d^*$ been non-negative. Note that Liang et al. (2008) model the 'same weights' assumption on z_d as a perfect coordination between the two sub-technologies under the CRS specification.

Constraints (7.3.2) and (7.3.3) correspond to the sub-technologies $T_{VRS}^{I(I)}$ and $T_{VRS}^{II(I)}$, respectively whose respective intercept multipliers are ω_I and ω_{II}. The construct of our network technology is such that the network technology constraint is the sum of the two sub-technology constraints, i.e., $T_{VRS}^{N(I)}$ is *additive*. This proposed additive structure holds under two conditions: (1) weights for the intermediate measures (products as inputs and outputs) are the same, and (2) intercept multiplier of $T_{VRS}^{N(I)}$ is the sum of those of its two sub-technologies. The first condition is satisfied due to the fact that $\widetilde{z}_d^*$ are free in (7.1). The second condition holds under the assumption that the additive $T_{VRS}^{N(I)}$ can inherit the properties of its sub-technologies, i.e., if the sub-technologies satisfy the properties such as no free lunch, free disposability in inputs and outputs, compactness, convexity, and returns to scale, then so does the additive network technology. The proof of this is made in the spirit of the proof of Proposition 2.3.2 in Färe and Grosskopf (1996, p. 23, pp. 44–45).

Using optimal multipliers from (7.3), one can obtain the input-oriented TE of firm h in $T_{VRS}^{N(I)}$ ($TE_{ih}^{N(I)}$) and the sub-technologies (($TE_{ih}^{I(I)}$) and $TE_{ih}^{II(I)}$) as:

$$TE_{ih}^{N(I)} = \frac{\sum_{r=1}^{s} u_r^* y_{rh} - \omega_I^* - \omega_{II}^*}{\sum_{i=1}^{m} v_i^* x_{ih}}, TE_{ih}^{I(I)} = \frac{\sum_{d=1}^{p} w_d^* z_{dh} - \omega_I^*}{\sum_{i=1}^{m} v_i^* x_{ih}} \text{ and } TE_{ih}^{II(I)} = \frac{\sum_{r=1}^{s} u_r^* y_{rh} - \omega_{II}^*}{\sum_{d=1}^{p} w_d^* z_{dh}} \tag{7.4}$$

One can express $TE_{ih}^{N(I)}$ as the product of three terms: the first two terms representing the TEs in the sub-technologies – $TE_{ih}^{I(I)}$ and $TE_{ih}^{II(I)}$, respectively, and the third term representing an index (I_h^{VRS}) indicating whether the decision concerning the use of *observed* intermediate products (z) as intermediate measures (outputs and inputs) is optimal, i.e., whether z_h equals $\widetilde{z}_h$. The proposed TE decomposition is given below.

$$TE_{ih}^{N(I)} = \left(\sum_{r=1}^{s} u_r^* y_{rh} - \omega_I^* - \omega_{II}^*\right) \Big/ \sum_{i=1}^{m} v_i^* x_{ih}$$
$$= \frac{\sum_{d=1}^{p} w_d^* z_{dh} - \omega_I^*}{\sum_{i=1}^{m} v_i^* x_{ih}} \cdot \frac{\sum_{r=1}^{s} u_r^* y_{rh} - \omega_{II}^*}{\sum_{d=1}^{p} w_d^* z_{ih}} \cdot \frac{\left(\sum_{r=1}^{s} u_r^* y_{rh} - \omega_I^* - \omega_{II}^* \Big/ \sum_{r=1}^{s} u_r^* y_{rh} - \omega_{II}^*\right)}{\left(\sum_{r=1}^{s} w_d^* z_{dh} - \omega_I^* \Big/ \sum_{r=1}^{s} w_d^* z_{dh}\right)}$$
$$= TE_{ih}^{I(I)} \cdot TE_{ih}^{II(I)} \cdot \frac{1 - \frac{1}{TE_{ih}^{II}}\left(\omega_I^* \Big/ \sum_{r=1}^{s} w_d^* z_{dh}\right)}{1 - \left(\omega_I^* \Big/ \sum_{r=1}^{s} w_d^* z_{dh}\right)} = TE_{ih}^{I(I)} \cdot TE_{ih}^{II(I)} \cdot I_h^{VRS} \tag{7.5}$$

Assuming unique optimal solutions in (7.3), we have three remarks based on the TE decomposition in (7.5).

Remark 1 I_h^{VRS} represents a proxy for the indication of allocative inefficiency, in which case $I_h^{VRS} > (<) \, 1$. Allocative inefficiency arises under the VRS specification but disappears under the CRS specification. One can therefore infer that maintaining the 'same weight' assumption on z as outputs and inputs under the VRS specification is not sufficient to rule out allocative inefficiency. Allocative inefficiency is a broader concept that includes inefficiencies arising from any possible sub-optimal decision as to how much z to produce and consume in the light of changing prices, i.e.,$\widetilde{z}_h < z_h$, in which case $I_h^{VRS} \neq 1$. Our proposed additive $T_{VRS}^{N(I)}$ is helpful in identifying such inefficiency when the optimal intermediate products ($\widetilde{z}^*$) is less than its observed counterparts (z), i.e., $\widetilde{z}_h < z_h$ when $T_{VRS}^{II(I)}$ turns inefficient.

Remark 2 $I_h^{VRS} = 1$ when $TE_{ih}^{II(I)} = 1$, implying the decision concerning the use of *observed* intermediate products (z_h) as outputs and inputs as optimal, i.e., $z_h = \widetilde{z}_h$. This means that there is no allocative inefficiency in the use of observed z_h. In this case, $TE_{ih}^{N(I)} = TE_{ih}^{I(I)}$. Therefore, the TE decomposition under the additive network structure reveals that a network firm is fully efficient only when it is efficient in both of its sub-technologies.

Remark 3 When $TE_{ih}^{II(I)} < 1$, $I_h^{VRS} > (<)\ 1$. (A) $I_h^{VRS} > 1$ when (1) $\omega_I^* < 0$ and (2) $TE_{ih}^{I(I)} > |\omega_I^*|$ in which case firm h exhibits *increasing returns to scale* (IRS) in $T_{VRS}^{I(I)}$. (B) $I_h^{VRS} < 1$ when $\omega_I^* > 0$ in which case firm h exhibits *decreasing returns to scale* (DRS) in $T_{VRS}^{I(I)}$.

To prove the statement (A) in Remark 3, let us redefine $I_h^{VRS} = 1 - \frac{1}{TE_{ih}^{II}}\left(\omega_I^* \Big/ \sum_{r=1}^{s} w_d^* z_{dh}\right) \Big/ 1 - \left(\omega_I^* \Big/ \sum_{r=1}^{s} w_d^* z_{dh}\right)$ as $\left(1 - \frac{1}{TE_{ih}^{II}} \cdot \frac{\omega_I^*}{\omega_I^* + TE_{ih}^{I(I)}}\right) \Big/ \left(1 - \frac{\omega_I^*}{\omega_I^* + TE_{ih}^{I(I)}}\right) \left(= I^1/I^2, \text{ say}\right)$. $I_h^{VRS} > 1$ implies that $I^1 - I^2 > 0$. This means that $\frac{\omega_I^*}{\omega_I^* + TE_{ih}^{I(I)}} \cdot \frac{TE_{ih}^{II(I)} - 1}{TE_{ih}^{II(I)}} > 0$. One can see that for this strict inequality to hold, two conditions need to hold: (1) $\omega_I^* < 0$ and (2) $TE_{ih}^{I(I)} > |\omega_I^*|$ since $TE_{ih}^{II(I)} < 1$; and firm h exhibits IRS since $\omega_I^* < 0$. Similarly, one can prove the statement (B) by examining the value of I_h^{VRS} when it is less than 1. $I_h^{VRS} < 1$ when $I^1 - I^2 < 0$, i.e., $\frac{\omega_I^*}{\omega_I^* + TE_{ih}^{I(I)}} \cdot \frac{TE_{ih}^{II(I)} - 1}{TE_{ih}^{II(I)}} < 0$. This inequality holds only when $\omega_I^* > 0$ irrespective of the values of $TE_{ih}^{I(I)}$ since $TE_{ih}^{II(I)} < 1$; and firm h exhibits DRS since $\omega_I^* > 0$. Note that the issue of determination of returns to scale will be dealt with in Sect. 7.2.3.

Note that optimal multipliers obtained from (7.3) may not be unique, implying that $TE_{ih}^{I(I)}$ and $TE_{ih}^{II(I)}$ are not unique. Therefore, in the spirit of Kao and Hwang (2008), assuming $T_{VRS}^{I(I)}$ to be more important, we first determine the maximum value of $TE_{ih}^{I(I)}$ via

$$TE_{ih}^{I} = \max \sum_{d=1}^{p} w_d z_{dh} - \omega_I \tag{7.6}$$

$$\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rh} - \omega_I - \omega_{II} = TE_{ih}^{N(I)}, \sum_{i=1}^{m} v_i x_{ih} = 1, \sum_{d=1}^{p} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \omega_I \le 0\ (\forall j),$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{p} w_d z_{dj} - \omega_{II} \le 0\ (\forall j), v_i, u_r, w_d \ge 0\ (\forall i, r, d); \omega_I, \omega_{II} : \text{free}$$

One can then compute the minimum of $TE_{ih}^{II(I)}$ by using optimal multipliers obtained from model (7.6). However, if $T_{VRS}^{II(I)}$ is considered more important, we first determine the maximum value of $TE_{ih}^{II(I)}$, and then the minimum value of $TE_{ih}^{I(I)}$ in an analogous manner.

We now illustrate how to measure TE using Approach II.

7.2.2.2 TE Estimation Using Approach II

As shown in Chen et al. (2013), since the sub-technology specific TEs can be computed independently of the overall efficiencies, we set up the network technology set ($T_{VRS}^{N(II)}$):

$$T_{VRS}^{N(II)} = \left\{(x, z, y)|T_{VRS}^{I(II)} \cup T_{VRS}^{II(II)}\right\} \tag{7.7}$$

where

$$T_{VRS}^{I(II)} = \left\{(x, z)\middle|\sum_{j=1}^{n} x_{ij}\alpha_j \leq x_i(\forall i),\ \sum_{j=1}^{n} z_{dj}\alpha_j \geq z_d(\forall d),\ \sum_{j=1}^{n} \alpha_j = 1,\ \alpha_j \geq 0\ (\forall j)\right\} \tag{7.7.1}$$

$$T_{VRS}^{II(II)} = \left\{(z, y)\middle|\sum_{j=1}^{n} z_{dj}\beta_j \leq z_d(\forall d),\ \sum_{j=1}^{n} y_{rj}\beta_j \geq y_r(\forall r),\ \sum_{j=1}^{n} \beta_j = 1,\ \beta_j \geq 0\ (\forall j)\right\} \tag{7.7.2}$$

For the construction of $T_{VRS}^{N(II)}$, Kao and Hwang (2011) maintains input-orientation in $T_{VRS}^{I(II)}$ and output-orientation in $T_{VRS}^{II(II)}$. The input-oriented TE of firm h in $T_{VRS}^{I(II)}(TE_{ih}^{I(II)})$ can be computed by setting up the following linear problem:

$$TE_{ih}^{I(II)} = \min_{\delta, \alpha}\left\{\delta_h : (\delta_h x_h, z_h) \in T_{VRS}^{I(II)}\right\} \tag{7.8}$$

Similarly, the output-oriented TE of firm h in $T_{VRS}^{II(II)}(TE_{oh}^{II(II)})$ can be obtained from the following linear problem:

$$\left(TE_{oh}^{II(II)}\right)^{-1} = \max_{\mu, \beta}\left\{\mu_h : (z_h, \mu_h y_h) \in T_{VRS}^{II(II)}\right\} \tag{7.9}$$

Kao and Hwang (2011) have shown that the network TE of firm h ($TE_h^{N(II)}$) is the product of $TE_{ih}^{I(II)}$ and $TE_{oh}^{II(II)}$, i.e.,

$$TE_h^{N(II)} = TE_{ih}^{I(II)} \times TE_{oh}^{II(II)} \tag{7.10}$$

We now discuss the evaluation of SE.

7.2.3 SE Evaluation

7.2.3.1 Estimating SE Using Approach I

To compute the input-oriented SE of network firm h, we first need to compute its TE using the model (7.1). Let its optimal solution vector be $\left(\beta_h^*, \lambda^*, \mu^*, \ \widetilde{z}^*\right)$. Firm h is (input-oriented) technically efficient if $\beta_h^* = 1$, $z_h = \widetilde{z}_h^*$ and input and output slacks are all zero. If it is not, then it needs to be projected onto the network frontier by applying the following formulae:

$$x_h^* \leftarrow \beta_h^* x_h - s^-, \ \ \widetilde{z}_h^* \leftarrow \widetilde{z}_h^* \text{ and } y_h^* \leftarrow y_h + s^+ \tag{7.11}$$

where s^- and s^+ are respectively vectors of input and output slacks under (7.1).

Due to duality theory, the following transformation function $F^{N(I)}\left(x_h^*, y_h^*; \widetilde{z}_h^*\right) = 0$ holds:

$$F^{N(I)}\left(x_h^*, y_h^*; \widetilde{z}_h^*\right) \equiv \sum_{r=1}^{s} u_r^*\left(y_{rh} + s_r^+\right) - \sum_{i=1}^{m} v_i^*\left(\beta_h^* x_{ih} - s_i^-\right) - \omega_I^* - \omega_{II}^* = 0 \tag{7.12}$$

where u_r^*, v_i^*, w_d^*, ω_I^* and ω_{II}^* are assumed to be the *unique* optimal multipliers obtained from (7.3); otherwise $F^{N(I)}(\cdot)$ is not differentiable at extreme points.

To define the SE in $T_{VRS}^{N(I)}$, $T_{VRS}^{I(I)}$ and $T_{VRS}^{II(I)}$, we consider, respectively, the following input–output vectors from (7.11): (x_h^*, y_h^*), $\left(x_h^*, \widetilde{z}_h^*\right)$ and $\left(\widetilde{z}_h^*, y_h^*\right)$. Following Baumol et al. (1982), we define the input-oriented (local) SE of firm h in $T_{VRS}^{N(I)}$, $\varepsilon_{ih}^{N(I)}\left(x_h^*, y_h^*; \widetilde{z}_h^*\right)$ as:

$$\begin{aligned} \varepsilon_{ih}^{N(I)}\left(x_h^*, y_h^*; \widetilde{z}_h^*\right) &\equiv -\sum_{i=1}^{m} x_{ih} \frac{\partial F^{N(I)}(\cdot)}{\partial x_{ih}} \Big/ \sum_{r=1}^{s} y_{rh} \frac{\partial F^{N(I)}(\cdot)}{\partial y_{rh}} \\ &= \frac{\beta_h^* \sum_{i=1}^{m} v_i^* x_{ih}}{\sum_{r=1}^{s} u_r^* y_{rh}} = \frac{\beta_h^*}{\beta_h^* + \omega_I^* + \omega_{II}^*} = \frac{\beta_h^*}{\beta_h^* + \omega_I^* + \omega_{II}^*} \end{aligned} \tag{7.13}$$

Note that in (7.13), $\sum_{i=1}^{m} v_i^* x_{ih} = 1$ due to (7.3.1); and $\sum_{r=1}^{s} u_r y_{rh} = \beta_h^* + \omega_I^* + \omega_{II}^*$, due to duality, the objective function values of (7.1) and (7.3) are the same, i.e., $\beta_h^* = \sum_{r=1}^{s} u_r y_{rh} - \omega_I^* - \omega_{II}^*$.

Based on (7.13), we have now the following proposition.

Proposition 1 *The input-oriented returns to scale are increasing (IRS) (i.e.,* $\varepsilon_{ih}^{N(I)}(\cdot) > 1$*) if* $\omega_I^* + \omega_{II}^* < 0$ *in all optimal solutions, constant (CRS) (i.e.,* $\varepsilon_{ih}^{N(I)}(\cdot) = 1$*) if* $\omega_I^* + \omega_{II}^* = 0$ *in an optimal solution, and decreasing (DRS) (i.e.,* $\varepsilon_{ih}^{N(I)}(\cdot) < 1$*) if* $\omega_I^* + \omega_{II}^* > 0$ *in all optimal solutions.*

Proof The proof is similar to that of determining the RTS underlying black-box DEA model. See Banker and Thrall (1992) and Banker et al. (2004). □

We now discuss the analytical SE evaluation of a fully network efficient firm h in its sub-technologies for which the constraints (7.3.2) and (7.3.3) are of special interest. Note that the network technology constraint is the sum of its two sub-technology constraints – (7.3.2) and (7.3.3), i.e.,

$$\sum_{r=1}^{s} u_r^*\left(y_{rh}+s_r^+\right) - \sum_{i=1}^{m} v_i^*\left(\beta_h^* x_{ih}-s_i^-\right) - \omega_I^* - \omega_{II}^* = \left(\sum_{d=1}^{p} w_d^* \widetilde{z}_{dh}^* - \sum_{i=1}^{m} v_i^*\left(\beta_h^* x_{ih}-s_i^-\right) - \omega_I^*\right) + \left(\sum_{r=1}^{s} u_r^*\left(y_{rh}+s_r^+\right) - \sum_{d=1}^{p} w_d^* \widetilde{z}_{dh}^* - \omega_{II}^*\right) \tag{7.14}$$

Since $\sum_{r=1}^{s} u_r^* y_{rh} - \sum_{i=1}^{m} v_i^*\left(\beta_o^* x_{ih}-s_i^-\right) - \omega_I^* - \omega_{II}^* = 0$ for the technically efficient firm h in $T_{VRS}^{N(I)}$, h will also be efficient in $T_{VRS}^{I(I)}$ and $T_{VRS}^{II(I)}$, in which case the respective transformation functions are:

$$F^{I(I)}\left(x_h^*, \widetilde{z}_h^*\right) \equiv \sum_{d=1}^{p} w_d^* \widetilde{z}_{dh} - \sum_{i=1}^{m} v_i^*\left(\beta_h^* x_{ih}-s_i^-\right) - \omega_I^* = 0 \tag{7.15}$$

$$F^{II(I)}\left(\widetilde{z}_h^*, y_h^*\right) \equiv \sum_{r=1}^{s} u_r^*\left(y_{rh}+s_r^+\right) - \sum_{d=1}^{p} w_d^* \widetilde{z}_{dh} - \omega_{II}^* = 0 \tag{7.16}$$

Using (7.13), one can obtain the respective sub-technology specific input-oriented SEs as:

$$\varepsilon_{ih}^{I(I)}\left(x_h^*, \widetilde{z}_h^*\right) \equiv -\sum_{i=1}^{m} x_{ih}\frac{\partial F^{I(I)}(\cdot)}{\partial x_{ih}} \Big/ \sum_{d=1}^{p} \widetilde{z}_{dh}\frac{\partial F^{I(I)}(\cdot)}{\partial \widetilde{z}_{dh}} = \frac{\beta_h^* \sum_{i=1}^{m} v_i^* x_{ih}}{\sum_{d=1}^{p} w_d^* \widetilde{z}_{dh}} = \frac{\beta_h^*}{\beta_h^* + \omega_I^*} \tag{7.17}$$

$$\varepsilon_{ih}^{II(I)}\left(\widetilde{z}_h^*, y_h^*\right) \equiv -\sum_{d=1}^{p} \widetilde{z}_{ih}\frac{\partial F^{II(I)}(\cdot)}{\partial \widetilde{z}_{dh}} \Big/ \sum_{r=1}^{s} y_{rh}\frac{\partial F^{II(I)}(\cdot)}{\partial y_{ro}} = \frac{\sum_{d=1}^{p} w_d^* \widetilde{z}_{dh}}{\sum_{r=1}^{s} u_r^* y_{rh}} = \frac{\beta_h^* + \omega_I^*}{\beta_h^* + \omega_I^* + \omega_{II}^*} \tag{7.18}$$

Note that in (7.17), $\sum_{d=1}^{p} w_d^* \widetilde{z}_{dh} = \beta_h^* + \omega_I^*$. This is because $\sum_{i=1}^{m} v_i^* s_i^- = 0$ in (7.15) due to complementary slackness condition. In (7.18), $\sum_{r=1}^{s} u_r^* y_{rh} = \beta_h^* + \omega_I^* + \omega_{II}^*$. This is due to $\sum_{r=1}^{s} u_r^* s_r^+ = 0$ in (7.16) due to complementary slackness condition, and $\sum_{d=1}^{p} w_d^* \widetilde{z}_{dh} = \beta_h^* + \omega_I^*$ in (7.15).

One can now show that the SE of network firm h in $T_{VRS}^{N(I)}$ is the product of those of the two sub-technologies – $T_{VRS}^{I(I)}$ and $T_{VRS}^{II(I)}$, i.e.,

$$\begin{aligned} \varepsilon_{ih}^{I(I)}\left(x_o^*, \widetilde{z}_h^*\right) * \varepsilon_{ih}^{II(I)}\left(\widetilde{z}_h^*, y_h^*\right) &= \frac{\beta_h^*}{\beta_h^* + \omega_I^*} \cdot \frac{\beta_h^* + \omega_I^*}{\beta_h^* + \omega_I^* + \omega_{II}^*} \\ &= \frac{\beta_h^*}{\beta_h^* + \omega_I^* + \omega_{II}^*} = \varepsilon_{ih}^{N(I)}\left(x_h^*, y_h^*; \widetilde{z}_h^*\right) \end{aligned} \tag{7.19}$$

Note that DEA technologies are not differentiable at extreme efficient points due to multiple optimal solutions for ($\omega_I + \omega_{II}$). We, therefore, set up the following linear problem to find out the input-oriented right-hand SE ($\varepsilon_{ih-}^{N(I)}(\cdot)$) for firm h in $T_{VRS}^{N(I)}$ as:

$$\left[\frac{\beta_h^*}{\varepsilon_{ih-}^{N(I)}(\cdot)} - \beta_h^*\right] = \max \omega_I + \omega_{II} \tag{7.20}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{i=1}^{m} v_i\left(\beta_h^* x_{ih} - s_i^-\right) = 1, \sum_{d=1}^{p} w_d \widetilde{z}_{dh}^* - \sum_{i=1}^{m} v_i\left(\beta_h^* x_{ih} - s_i^-\right) - \omega_I = 0, \\ & \sum_{r=1}^{s} u_r\left(y_{rh} + s_r^+\right) - \sum_{d=1}^{p} w_d \widetilde{z}_{dh}^* - \omega_{II} = 0, \\ & \sum_{d=1}^{p} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \omega_I \le 0\ (\forall j \ne h), \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{p} w_d z_{dj} - \omega_{II} \le 0\ \left(\forall j \ne h\right), \\ & v_i, u_r, w_d \ge \varepsilon\ \ (\forall i, r, d), \omega_I, \omega_{II} : \text{free} \end{aligned}$$

Similarly, the input-oriented left-hand SE ($\varepsilon_{ih+}^{N(I)}(\cdot)$) can be obtained by replacing the “max” with “min” in objective of model (7.20).

Let the max of ($\omega_I + \omega_{II}$) in (7.20) be $(\omega_I + \omega_{II})^+$ in which the values of ω_I and ω_{II} are $\overline{\omega}_I$ and $\overline{\omega}_{II}$; and let the min of ($\omega_I + \omega_{II}$) in (7.20) be $(\omega_I + \omega_{II})^-$ in which the values of ω_I and ω_{II} are $\underline{\omega}_I$ and $\underline{\omega}_{II}$. Banker and Thrall (1992) used the upper and lower bounds of the intercept multiplier of the (black-box) BCC DEA model to define the left-hand (lower bound) and right-hand (upper-bound) SEs. Following Banker and Thrall (1992), we now use the SE expression (7.17) to determine the left-and right-hand SEs of firm h in $T_{VRS}^{I(I)}$ as

$$\varepsilon_{ih+}^{I(I)}(\cdot) = \frac{\beta_h^*}{\beta_h^* + \overline{\omega}_I} \quad \text{and} \quad \varepsilon_{ih-}^{I(I)}(\cdot) = \frac{\beta_h^*}{\beta_h^* + \underline{\omega}_I} \tag{7.21}$$

Similarly, one can use the SE expression (7.18) to determine the left- and right-hand SEs of firm h in $T_{VRS}^{II(I)}$ as

$$\varepsilon_{ih+}^{II(I)}(\cdot) = \frac{\beta_h^* + \overline{\omega}_I}{\beta_h^* + (\omega_I + \omega_{II})^+} \text{ and } \varepsilon_{ih-}^{II(I)}(\cdot) = \frac{\beta_h^* + \underline{\omega}_I}{\beta_h^* + (\omega_I + \omega_{II})^-} \tag{7.22}$$

While defining these sub-technology specific SEs, we have followed Banker and Thrall (1992) to consider the upper and lower bounds of (ω_I + ω_{II}) in the program (7.20), i.e., $(\overline{\omega}_I, \overline{\omega}_{II})$ and $\left(\underline{\omega}_I, \underline{\omega}_{II}\right)$ to determine the left- and right-hand SEs. However, if one considers the individual max (min) values of ω_I and ω_{II} (i.e., $\omega_I^+(\omega_I^-)$ and $\omega_{II}^+(\omega_{II}^-)$), which can be obtained by replacing max (min) (ω_I + ω_{II}) in the objective of (7.20) with max (min)ω_I and max (min)ω_{II} respectively, then our SE expressions in (7.21) and (7.22) may produce the incorrect values of left- and right-hand SEs. This is possible only when $\overline{\omega}_I\left(\underline{\omega}_I\right) \neq \omega_I^+\left(\omega_I^-\right)$ and $\overline{\omega}_{II}\left(\underline{\omega}_{II}\right) \neq \omega_{II}^+\left(\omega_{II}^-\right)$.

We have now our proposition 2.

Proposition 2

(2.1) *Assuming alternate optima in* (ω_I + ω_{II}), $T_{VRS}^{N(I)}$ *exhibits IRS* ($\varepsilon_{ih-}^{N(I)}(\cdot) > 1$) *if* $(\omega_I + \omega_{II})^- < 0$, *CRS* ($\varepsilon_{ih-}^{N(I)}(\cdot) \leq 1 \leq \varepsilon_{ih+}^{N(I)}(\cdot)$) *if* $(\omega_I + \omega_{II})^- \geq 0 \geq (\omega_I + \omega_{II})^+$ *and DRS* ($\varepsilon_{ih+}^{N(I)}(\cdot) < 1$) *if* $(\omega_I + \omega_{II})^+ > 0$.

(2.2) *Assuming alternate optima in* ω_I, $T_{VRS}^{I(I)}$ *exhibits IRS* ($\varepsilon_{ih-}^{I(I)}(\cdot) > 1$) *if* $\underline{\omega}_I < 0$, *CRS* ($\varepsilon_{ih-}^{I(I)}(\cdot) \leq 1 \leq \varepsilon_{ih+}^{I(I)}(\cdot)$) *if* $\underline{\omega}_I \geq 0 \geq \overline{\omega}_I$, *and DRS* ($\varepsilon_{ih+}^{I}(\cdot) < 1$) *if* $\overline{\omega}_I > 0$.

(2.3) *Assuming alternate optima in* (ω_I + ω_{II}), $T_{VRS}^{II(I)}$ *exhibits IRS* ($\varepsilon_{ih-}^{II(I)}(\cdot) > 1$) *if* $\underline{\omega}_{II} < 0$, *CRS* ($\varepsilon_{ih-}^{II(I)}(\cdot) \leq 1 \leq \varepsilon_{ih+}^{II(I)}(\cdot)$) *if* $\underline{\omega}_{II} \geq 0 \geq \overline{\omega}_{II}$, *and DRS* ($\varepsilon_{ih+}^{II(I)}(\cdot) < 1$) *if* $\overline{\omega}_{II} > 0$.

Proof The proof is similar to that of determining the RTS underlying black-box DEA model. See Banker and Thrall (1992) and Banker et al. (2004).

Banker et al. (1984) are the first to show that the intercept ω in the multiplier form of the (black-box) BCC DEA model can be used to estimate RTS. Several contributions exist, at the extreme points, on the evaluation of right-hand (upper bound) and left-hand SE (lower bound) measures in the black-box models. See, e.g., among others, Banker and Thrall (1992), Førsund (1996), Tone and Sahoo (2004), Tone and Sahoo (2005), Tone and Sahoo (2006), Hadjicostas and Soteriou (2006), Podinovski et al. (2009), and Sahoo et al. (2012).

7.2.3.2 Estimating SE Using Approach II

To compute the input-oriented SE of firm h in $T_{VRS}^{I(II)}$, we first set up the dual of model (7.8) as

$$TE_{ih}^{I(II)} = \max \sum_{d=1}^{p} w'_d z_{dh} - \omega'_I \tag{7.23}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} v_i x_{ih} = 1, \tag{7.23.1}$$

$$\sum_{d=1}^{p} w'_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \omega'_I \leq 0 \ (\forall j), \tag{7.23.2}$$

$$v_i, w'_d \geq 0 \ (\forall i, d); \omega'_I : \text{free} \tag{7.23.3}$$

Assume that unique optimal solutions in (7.23) exist. The duality theory suggests that the following transformation function for firm h, $F^{I(II)}(z_h, x_h) = 0$ holds, i.e.,

$$F^{I(II)}(z_h, x_h) \equiv \sum_{d=1}^{p} w'^*_d \left(z_{dh} + s_d^+\right) - \sum_{i=1}^{m} v_i^* \left(\delta_h^* x_{ih} - s_i^-\right) - \omega'^*_I = 0 \tag{7.24}$$

where s_i^- and s_d^+ are respectively the i^{th} input and d^{th} output slacks in model (7.8). Using the SE formula (7.13), one can obtain the input-oriented SE of firm h in $T_{VRS}^{I(II)}$ as

$$\varepsilon_{ih}^{I(II)}(z_h, x_h) \equiv -\sum_{i=1}^{m} x_{ih} \frac{\partial F^{I(II)}(\cdot)}{\partial x_{ih}} \Big/ \sum_{d=1}^{p} z_{dh} \frac{\partial F^{I(II)}(\cdot)}{\partial z_{dh}} = \frac{\delta_h^* \sum_{i=1}^{m} v_i^* x_{ih}}{\sum_{d=1}^{p} w'^*_d z_{dh}} = \frac{\delta_h^*}{\delta_h^* + \omega'^*_I} \tag{7.25}$$

Notice that in (7.25), $\sum_{i=1}^{m} v_i^* x_{ih} = 1$ due to (7.23.1); and $\sum_{d=1}^{p} w'^*_d z_{dh} = \delta_h^* + \omega'^*_I$, which is because, by duality, the objective function values of (7.8) and (7.23) are the same, i.e., $\delta_h^* = \sum_{d=1}^{p} w'^*_d z_{dh} - \omega'^*_I$.

One can compute the output-oriented SE of firm h in $T_{VRS}^{II(II)}$ by setting up the dual of (7.9) as

$$\left(TE_{oh}^{II(II)}\right)^{-1} = \min \sum_{d=1}^{p} w'_d z_{dh} + \omega'_{II} \tag{7.26}$$

$$\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rh} = 1, \tag{7.26.1}$$

$$\sum_{d=1}^{p} w'_d z_{dj} - \sum_{r=1}^{s} u_r y_{rj} + \omega'_{II} \geq 0 \ (\forall j), \tag{7.26.2}$$

$$u_r, w'_d \geq 0 \ (\forall r, d); \omega'_{II} : \text{free} \tag{7.26.3}$$

Assume that unique optimal solutions in the model (7.26) exist. Due to duality theory, the following transformation function for firm h, $F^{II(II)}(z_h, y_h) = 0$ holds, i.e.,

$$F^{II(II)}(z_h, y_h) \equiv \sum_{i=1}^{m} u_r^* \left(\mu_h^* y_{rh} + s_r^+\right) - \sum_{d=1}^{p} w'^*_d \left(z_{dh} - s_d^-\right) - \omega'^*_{II} = 0 \tag{7.27}$$

where s_d^- and s_r^+ are respectively the d^{th} input slack and r^{th} output slacks of the model (7.9). Using the SE formula (7.13), one can obtain the output-oriented SE of firm h in $T_{VRS}^{II(II)}$ as

$$\begin{aligned} \varepsilon_{oh}^{II(II)}(z_h, y_h) &\equiv -\sum_{d=1}^{p} z_{dh} \frac{\partial F^{II(II)}(\cdot)}{\partial z_{dh}} \Big/ \sum_{r=1}^{s} y_{rh} \frac{\partial F^{II(II)}(\cdot)}{\partial y_{rh}} \\ &= \frac{\sum_{d=1}^{p} w'^*_d z_{dh}}{\mu_h^* \sum_{r=1}^{s} u_r^* y_{rh}} = \frac{\mu_h^* - \omega'^*_{II}}{\mu_h^*} = 1 - \frac{\omega'^*_{II}}{\mu_h^*} \end{aligned} \tag{7.28}$$

Notice that in (7.28), $\sum_{r=1}^{s} u_r^* y_{rh} = 1$ due to (7.26.1); and $\sum_{d=1}^{p} w'^*_d z_{dh} = \mu_h^* - \omega'^*_{II}$, which is because, by duality, the objective function values of (7.9) and (7.26) are the same, i.e., $\mu_h^* = \sum_{d=1}^{p} w'^*_d z_{dh} + \omega'^*_{II}$.

Since in many cases ω'^*_I and ω'^*_{II} are not uniquely determined in (7.23) and (7.26) respectively, the SE estimates are not unique. There is thus a need to find out both right- and left-hand SEs.

We set up the following model to compute the input-oriented right-hand SE of firm h, $\varepsilon_{ih-}^{I(II)}(\cdot)$ in $T_{VRS}^{I(II)}$ as

$$\frac{\delta_h^*}{\varepsilon_{ih-}^{I(II)}(\cdot)} - \delta_h^* = \max \ \omega'_I \tag{7.29}$$

$$\text{s.t.}\quad \sum_{i=1}^{m} v_i\left(\delta_h^* x_{ih} - s_i^-\right) = 1, \sum_{d=1}^{p} w_d^{'}\left(z_{dh} + s_d^+\right) - \sum_{i=1}^{m} v_i\left(\delta_h^* x_{ih} - s_i^-\right) - \omega_I^{'} = 0,$$
$$\sum_{d=1}^{p} w_d^{'} z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \omega_I^{'} \le 0 \quad (\forall j \ne h),$$
$$v_i, w_d^{'} \ge 0 \ \ \forall i, d; \text{and} \ \ \omega_I^{'} : \text{free.}$$

Denote optimal solution of $\omega_I^{'}$ be ω'^{-}_I. $\varepsilon_{ih-}^{I(II)}(\cdot)$ can be computed as:

$$\varepsilon_{ih-}^{I(II)}(\cdot) = \frac{\delta_h^*}{\delta_h^* + \omega'^{-}_I} \tag{7.30}$$

Similarly, the input-oriented left-hand SE of firm h, $\varepsilon_{ih+}^{I(II)}(\cdot)$ in $T_{VRS}^{I(II)}$ can be computed by running the model (7.29) with 'min' instead of 'max'.

The output-oriented right-hand *SE* of firm h in $T_{VRS}^{II(II)}$ can be computed by setting up the following linear problem:

$$\mu_h^*\left(1 - \varepsilon_{oh-}^{II(II)}(\cdot)\right) = \max \omega_{II}^{'} \tag{7.31}$$
$$\text{s.t.}\quad \sum_{r=1}^{s} u_r\left(\mu_h^* y_{rh} + s_r^+\right) = 1, \ \sum_{d=1}^{p} w_d^{'}\left(z_{dh} - s_d^-\right) - \sum_{r=1}^{s} u_r\left(\mu_h^* y_{rh} + s_r^+\right) + \omega_{II}^{'} = 0,$$
$$\sum_{d=1}^{p} w_d^{'} z_{dj} - \sum_{r=1}^{s} u_r y_{rj} + \omega_{II}^{'} \ge 0 \quad (\forall j \ne h), \ \ u_r, w_d^{'} \ge 0 \ \forall r, d; \ \text{and} \ \omega_{II}^{'} : \text{free.}$$

Denote optimal solution of $\omega_I^{'}$ in (7.31) as ω'^{-}_{II}. The output-oriented right-hand SE of firm h in $T_{VRS}^{II(II)}$ can be computed as

$$\varepsilon_{oh-}^{II(II)}(\cdot) = 1 - \frac{\omega'^{-}_{II}}{\mu_h^*} \tag{7.32}$$

Similarly, the output-oriented left-hand SE of firm $h\varepsilon_{oh+}^{II(II)}(\cdot)$ in $T_{VRS}^{II(II)}$ can be computed by running (7.31) with 'min' instead of 'max'.

Note that unlike in Approach I, it is not possible in Approach II to decompose the network technology SE into its sub-technology specific SEs. We, however, note that Kao and Hwang (2011) develop an *ad hoc* approach to obtain network scale efficiency as the product of the sub-technology specific scale efficiencies.

7.2.4 Modeling Efficiency Against Different Efficiency Frontiers

Real-world firms suffer from profit loss due to allocative inefficiency arising from sub-optimal decision concerning the production and consumption of intermediate products connecting the sub-technologies. This profit loss has implications for revenue growth and cost control exercises. Production managers have every incentive to choose right input- and output-mix; otherwise, the opportunity cost of doing so is surprisingly high. Therefore, it is imperative to know the extent of output loss of firms suffering from such allocative inefficiencies.

In order to compute the loss in final output, one needs to model the TE of a firm against the three frontiers: the two network frontiers revealed from Approach I and Approach II, and the black-box (BB) frontier that ignore intermediate products connecting the sub-technologies. For this purpose, we specifically consider describing the network frontier under both approaches comprising of only inputs (x) and final outputs (y).

Using the model (7.1) under Approach I, we first project all the firms onto the network efficiency frontier. Let their projected input and final output vectors be $\left(\overline{x}_j, \overline{y}_j\right)$ where $\overline{x}_j = \beta_h^* x_j - s^-$ and $\overline{y}_j = y_j + s^+$ for all $j = 1, 2, \ldots, n$. We define the frontier of network technology set $(T_{VRS}^{N(I)})$, $\partial T_{VRS}^{N(I)}$, comprising of all of these projected input and final output vector as

$$\partial T_{VRS}^{N(I)} = \left\{ \left(\widehat{x}, \widehat{y}\right) \middle| \sum_{j=1}^{n} \overline{x}_j \lambda_j \leq \widehat{x}, \sum_{j=1}^{n} \overline{y}_j \lambda_j \geq \widehat{y}, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \right\} \tag{7.33}$$

Similarly, the frontier of network technology set under the Approach II $(T_{VRS}^{N(II)})$, $\partial T_{VRS}^{N(II)}$, comprising of all the projected input and final output vectors $\left(\widetilde{x}_j, \widetilde{y}_j\right)$ can be set up as

$$\partial T_{VRS}^{N(II)} = \left\{ \left(\breve{x}, \breve{y}\right) \middle| \sum_{j=1}^{n} \widetilde{x}_j \lambda_j \leq \breve{x}, \sum_{j=1}^{n} \widetilde{y}_j \lambda_j \geq \breve{y}, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \right\} \tag{7.34}$$

Here $\left(\widetilde{x}_j, \widetilde{y}_j\right) = \left(\delta_j^* x_j - s^-, \mu_j^* y_j + s^+\right)$ for $j = 1, 2, \ldots, n$ and δ_j^* and μ_j^* are obtained from the model (7.8) and model (7.9) respectively.

We then define the BB-based technology set $(T_{VRS}^{BB)})$ comprising of all the observed input and final output vectors as

$$T_{VRS}^{BB}(x, y) = \left\{ (x, y) \middle| \sum_{j=1}^{n} x_j \lambda_j \leq x, \sum_{j=1}^{n} y_j \lambda_j \geq y, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \right\} \tag{7.35}$$

Now consider evaluating the output TE of firm h (TE_{oh}) against $\partial T_{VRS}^{N(I)}$, $\partial T_{VRS}^{N(II)}$ and T_{VRS}^{BB} whose actual input and final output vector is (x_h, y_h). The respective output-oriented TEs can be obtained by setting up the following LP programs:

$$\left(TE_{oh}^{N(I)}\right)^{-1} = \max\left\{\theta_I : (x_h, \theta_I y_h) \in \partial T_{VRS}^{N(I)}\right\} \tag{7.36}$$

$$\left(TE_{oh}^{N(II)}\right)^{-1} = \max\left\{\theta_{II} : (x_h, \theta_{II} y_h) \in \partial T_{VRS}^{N(II)}\right\} \tag{7.37}$$

$$\left(TE_{oh}^{BB)}\right)^{-1} = \max\left\{\theta_{BB} : (x_h, \theta_{BB} y_h) \in T_{VRS}^{BB}\right\} \tag{7.38}$$

Since no allocative inefficiency is assumed in the construction of network technology under Approach II, one could *a priori* expect, for any given level of input, an output level in $\partial T_{VRS}^{N(II)}$ that is no less than that in $\partial T_{VRS}^{N(I)}$, which allows for inefficiencies. One could, therefore, interpret $(\theta_{II}^* - \theta_I^*)y_h$ as the output loss due to allocative inefficiency. Since the BB technology does not regard efficiencies concerning the internal operations of firm, one could expect, with any given level of input, the least output in this technology, i.e., $\theta_{BB}^* y_h \leq \theta_I^* y_h \leq \theta_{II}^* y_h$.

7.3 An Illustrative Example

Consider a simple hypothetical data set exhibited in Table 7.1. There are nine firms labeled as A, B, C, D, E, F, G, H and I. Each firm in T^I uses one input (x) to produce an intermediate product (measure) (z), which is then taken as input to T^{II} by the same firm to produce one final output (y).

7.3.1 On TE Estimates

Based on Table 7.1, Fig. 7.2 exhibits the two *independent* sub-technology frontiers in a counterclockwise orientation under the VRS specification. These frontiers are drawn by keeping z unaltered. Figure 7.3 exhibits the BB frontier involving observed x and y under an appropriate RTS specification (identified with lines: A-D-H-C), and the network production frontiers revealed from both approaches (model (7.1) under Approach I and models (7.8) and (7.9) under Approach II).

Table 7.1 Example data set

Firms	x	z	y
A	1.5	1	1
B	4	6	4
C	6	7	7
D	2	2.5	3
E	5	4	6
F	4	3.5	2
G	5.5	5	5
H	4.5	6	6
I	7	6.5	6.5

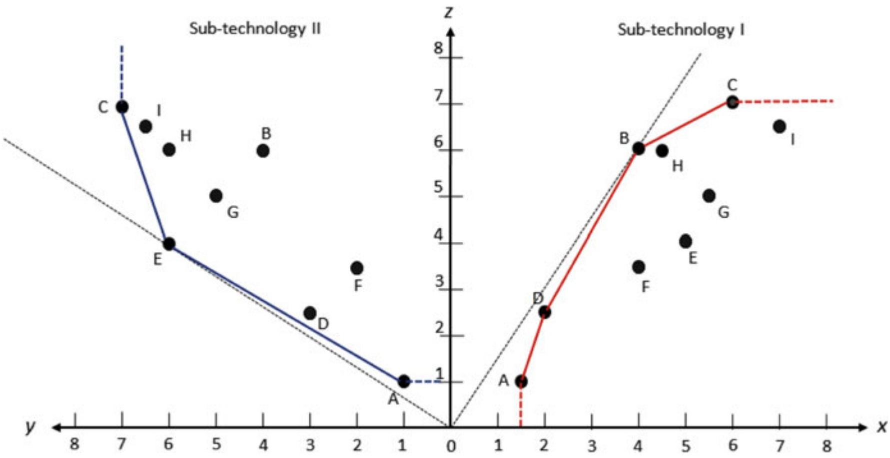

Fig. 7.2 Independent sub-technologies (Approach II)

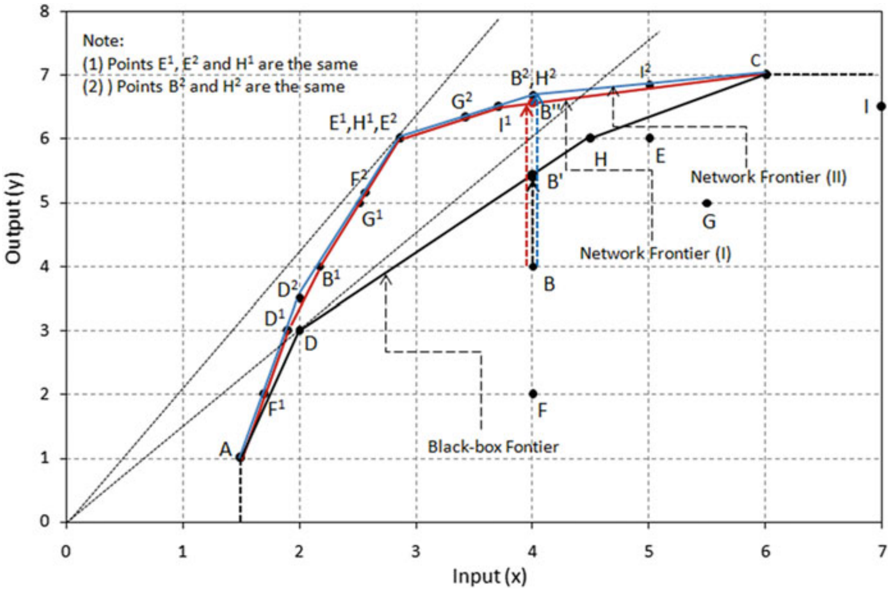

Fig. 7.3 Black-box technology vs. network technologies

Since the BB technology considers only the relation between *inputs* and *final outputs*, and makes no assumptions regarding the internal operations of firm, it provides no insights regarding the locations of inefficiency and scale economies. For example, firms – D and H that appear efficient in the BB technology turn out to be

Table 7.2 TE decomposition results

	Firms	TE_h^N	TE_h^I	TE_h^{II}	I_h^{VRS}
Approach I	A	1	1	1	1
	B	0.543	1	0.467	1.163
	C	1	1	1	1
	D	0.95	1	0.880	1.080
	E	0.571	0.571	1	1
	F	0.425	0.583	0.457	1.594
	G	0.457	0.623	0.680	1.078
	H	0.635	0.889	0.667	1.071
	I	0.531	0.612	0.846	1.024
Approach II	A	1	1	1	Not applicable
	B	0.6	1	0.6	
	C	1	1	1	
	D	0.857	1	0.857	
	E	0.571	0.571	1	
	F	0.249	0.643	0.387	
	G	0.492	0.623	0.789	
	H	0.800	0.889	0.900	
	I	0.679	0.714	0.951	

inefficient in the network technologies (identified with broken lines: A-F^1-D^1-B^1-G^1-E^1(H^1)-I^1-C under Approach I and A-D^2-F^2-E^2-G^2-B^2(H^2)-I^2-C under Approach II). Note that superscripts – 1 and 2 indicate, respectively, the projected points of the corresponding inefficient firms in both approaches. Points such as E^1, E^2 and H^1 are the same projected point for firm E (under both approaches) and for H (under Approach I). Similarly, points – B^2 and H^2 – are the same projected point for firms – B and H under Approach II. As regards the RTS, D that appears exhibiting CRS in the BB technology exhibits IRS (if projected in an input-oriented manner) in the network technologies.

We report in Table 7.2 the TE decomposition results obtained from Approach I (top part) and Approach II (bottom part), which will facilitate managerial insights regarding specific area of improvement for the network inefficient firms. The upshot of these results is summarized below.

1. Both approaches are in complete agreement in identifying the network efficient firms. The examples of such firms are A, C and E.
2. As expected, I_h^{VRS} is greater than 1 for those firms (B, D, F, G, H and I) that are technically inefficient in $T_{VRS}^{II(I)}$. Technical inefficiency arises only when the intermediate products consumed by these firms are not minimal implying that there is an overproduction of these outputs in $T_{VRS}^{I(I)}$. The results of our model (7.1) reveal that the optimal quantities of these products ($\tilde{z}^*$) are 2.8 (6), 2.2 (2.5), 1.6 (3.5), 3.4 (5), 4 (6) and 5.5 (6.5) for B, D, F, G, H and I respectively (the terms in brackets are their respective actual quantities). This is why the estimated sub-technology frontiers in Figs. 7.4 and 7.5 are different from those in Fig. 7.2.

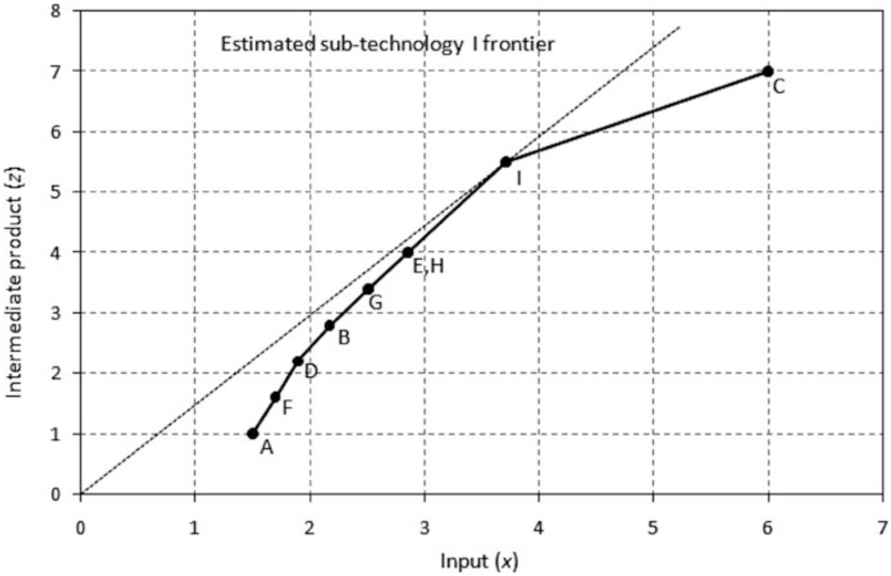

Fig. 7.4 Estimated sub-technology frontier I (Approach I)

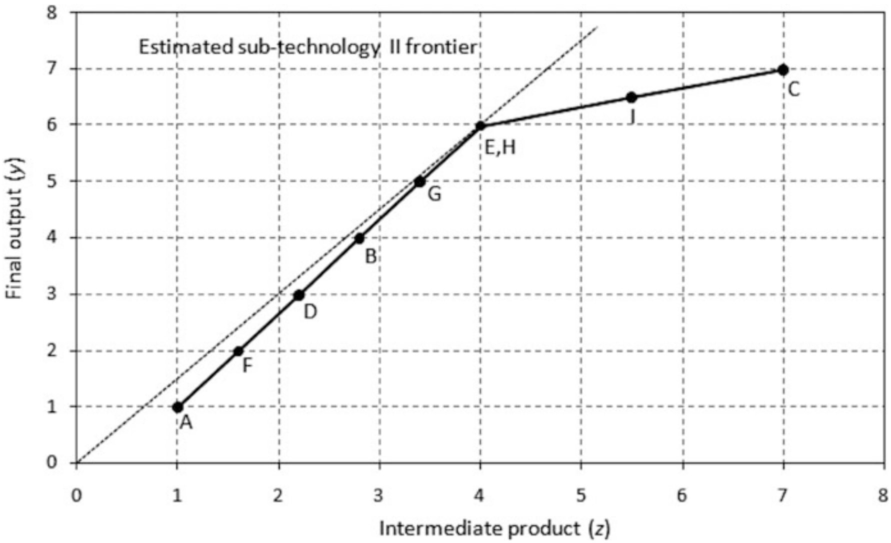

Fig. 7.5 Estimated sub-technology frontier II (Approach I)

3. The finding that the two approaches yield differential TE decomposition results for network inefficient firms is not at all strange. As expected, the decision to allow allocative inefficiency into the system in Approach I yields a frontier different from the one yielded from Approach II with no allocative inefficiency.

7.3.2 On Modeling the Output Losses

From our empirical application, one can observe that $T_{VRS}^{N(II)} \supseteq T_{VRS}^{N(I)} \supseteq T_{VRS}^{BB}$. As a result, one expect the following relationship: $TE_{oh}^{N(II)} \leq TE_{oh}^{N(I)} \leq TE_{oh}^{BB}$. Therefore, modeling of the TE of a firm against three different frontiers yields valuable information concerning whether the output loss is due to either missing of intermediate products connecting the sub-technologies, or the allocative inefficiency arising from any sub-optimal decision as to how much of intermediate products to produce and consume in the world of changing prices.

Let us consider, e.g., the TE evaluation of firm B. If B's output TE is evaluated against ∂T_{VRS}^{BB}, the projection is made on the point B′ (4, 5.4) where $TE_{oB}^{BB} = 0.741$ (=4/5.4). If it is evaluated against $\partial T_{VRS}^{N(I)}$, the projection is made on the point B″ (4, 6.563) where $TE_{oB}^{N(I)} = 0.610(=4/6.563)$; and if it is against the $\partial T_{VRS}^{N(II)}$, then the projection is made onto the point B^2 (4, 6.667) where $TE_{B}^{N(II)} = 0.600(=4/6.667)$. Since the potential output of 5.4 identified against ∂T_{VRS}^{BB} is the least as compared to 6.563 and 6.667 against $\partial T_{VRS}^{N(I)}$ and $\partial T_{VRS}^{N(II)}$, respectively, TE_{oB}^{BB} is highest. The output loss of 1.163 (= 6.563 − 5.4) against $\partial T_{VRS}^{N(I)}$ (with allocative inefficiency) is due to not accounting for the intermediate products connecting the sub-technologies in T_{VRS}^{BB}. This loss is lower as compared to the output loss of 1.267 (= 6.667 − 5.4) against $\partial T_{VRS}^{N(II)}$ with no allocative inefficiency. Therefore, the output loss of 0.104 (= 6.667 − 6.563) can be purely attributed to the allocative inefficiency associated with the possible sub-optimal decisions con-cerning the production and consumption of intermediate products connecting the sub-technologies.

We now discuss in the immediately following section the sources of input-oriented scale effects.

7.3.3 On SE Estimates

7.3.3.1 SE Estimates Using Approach I

Using Approach I we run both max and min forms of model (7.20) to compute the input-oriented left- and right-hand SEs of firms not only for the network technology but also for the sub-technologies (using formulas (7.21) and (7.22)). The SE results are reported in Table 7.3 (top part). The results reveal that $T_{VRS}^{N(I)}$ finds five firms – A, B, D, F and G operating under IRS, two firms – E and H under CRS and two firms – C and I under DRS. While the sources of increasing returns of firms in $T_{VRS}^{N(I)}$ are all located in both of the sub-technologies, the same is not the case for firms exhibiting decreasing and/or constant returns. For example, $T_{VRS}^{N(I)}$ exhibiting DRS for firm I is

Table 7.3 Upper and lower bounds of *SE* estimates

	Firms	Network $\varepsilon_{-}^{N}(\cdot)$	Network $\varepsilon_{+}^{N}(\cdot)$	Network RTS	Sub-technology I $\varepsilon_{-}^{I}(\cdot)$	Sub-technology I $\varepsilon_{+}^{I}(\cdot)$	Sub-technology I RTS	Sub-technology II $\varepsilon_{-}^{II}(\cdot)$	Sub-technology II $\varepsilon_{+}^{II}(\cdot)$	Sub-technology II RTS
Approach I	A	7.500	∞	IRS	4.500	∞	IRS	1.667	∞	IRS
	B	1.583	2.000	IRS	1.357	1.714	IRS	1.167	1.167	IRS
	C	0	0.187	DRS	0	0.562	DRS	0	0.333	DRS
	D	2.333	3.167	IRS	1.909	2.591	IRS	1.222	1.222	IRS
	E	0.278	1.389	CRS	1.250	1.250	IRS	0.222	1.111	CRS
	F	4.250	4.250	IRS	3.188	3.188	IRS	1.333	1.333	IRS
	G	1.467	1.467	IRS	1.294	1.294	IRS	1.133	1.133	IRS
	H	0.278	1.389	CRS	1.250	1.250	IRS	0.222	1.111	CRS
	I	0.125	0.333	DRS	0.443	1.182	CRS	0.282	0.282	DRS
Approach II	A	7.500	∞	IRS	4.500	∞	IRS	1.667	∞	IRS
	B	1.583	1.583	IRS	0.333	1.167	CRS	0.300	0.300	DRS
	C	0	0.143	DRS	0	0.429	DRS	0	0.333	DRS
	D	3.167	3.167	IRS	1.400	2.400	IRS	1.191	1.191	IRS
	E	0.278	1.389	CRS	1.250	1.250	IRS	0.222	1.111	CRS
	F	4.250	4.250	IRS	1.286	1.286	IRS	1.129	1.129	IRS
	G	1.467	1.467	IRS	1.200	1.200	IRS	0.263	0.263	DRS
	H	0.278	1.389	CRS	0.333	1.167	CRS	0.300	0.300	DRS
	I	0.333	0.333	DRS	0.385	0.385	DRS	0.317	0.317	DRS

due to DRS in $T_{VRS}^{II(I)}$ even though CRS prevails in $T_{VRS}^{I(I)}$. Similarly, $T_{VRS}^{N(I)}$ exhibiting CRS for firms – E and H is precisely due to CRS in $T_{VRS}^{II(I)}$ even though IRS prevails in $T_{VRS}^{I(I)}$.

7.3.3.2 SE Estimates Using Approach II

Using Approach II we run both max and min forms of the model (7.29), which is based on the optimal input TE values of the model (7.8), to compute the input-oriented left- and right-hand SEs of the firms in $T_{VRS}^{I(II)}$. Similarly, we run both max and min forms of the model (7.31), which is based on the optimal output *TE* values of the model (7.9), to compute the output-oriented left- and right-hand SEs of the firms in $T_{VRS}^{II(II)}$. However, under this approach, it is not possible to compute the input-oriented network SEs of firms using (7.29) and (7.31). Therefore, in order to compute the input-oriented left- and right-hand network SEs, we use firms' projected input–output vectors, $(\tilde{x}, \tilde{y}) = (\delta^* x - s^-, \mu^* y + s^+)$ obtained from (7.8) and (7.9), in model (7.29). The input-oriented network SE estimates are reported in Table 7.3 (bottom part). We find five firms – A, B, D, F and G operating under IRS, two firms – E and H under CRS and the remaining two firms – C and I under DRS (which call can be visualized in Fig. 7.3).

Note that since it is not possible in this approach to decompose network SE into its sub-technology specific SEs, the scale economies/diseconomies revealed from sub-technologies [(7.29) and (7.31)] may not attribute to the network scale economy/diseconomy obtained from the use of projected data of network firms $(\widetilde{x}, \widetilde{y})$ in (7.29). For example, consider firm B whose sub-technologies exhibit CRS and DRS (CRS in $T_{VRS}^{I(II)}$ and DRS in $T_{VRS}^{II(II)}$), but its network technology, $T_{VRS}^{N(II)}$ exhibits IRS. It is therefore quite improbable to argue that the sources of increasing returns in the network technology are due to CRS and DRS in the sub-technologies. Note that the very purpose of computing the input-oriented network SE of firms under Approach II is just to compare these SE estimates with those obtained under Approach I.

Notice that though the network technologies revealed from both approaches look similar (Fig. 7.3), and the (input-oriented) RTS possibilities of network firms are the same; the degrees of underlying SE estimates of some network firms are different due to differential nature (flatness/steepness) of some production facets. For example, $T_{VRS}^{N(I)}$ finds B exhibiting IRS whose value ranges from 1.583 to 2 since its SE is estimated against the vertex point B^1 connecting two facets – D^1B^1 and B^1G^1. $T_{VRS}^{I(II)}$ also finds this firm operating under the same IRS but its SE value is exact at 1.583 since it is estimated against a point on the facet D^2F^2. So are the cases with firms – D and I.

On comparison between the two approaches with regard to the sources of scale economies of firms, we find some divergent information on their RTS possibilities. Though both approaches maintain input-orientation in T_{VRS}^{I}, they yield contrasting RTS possibilities for some firms. For example, while $T_{VRS}^{I(I)}$ finds both B and H operating under IRS, and I under CRS, $T_{VRS}^{I(II)}$ finds B and H under CRS, and I under DRS. These contrasting *RTS* information are because the estimated T_{VRS}^{I} revealed from both approaches are different (see Figs. 7.2 (right) and 7.4). However, there are contrasting information on the RTS possibilities in T_{VRS}^{II} even though the estimated sub-technology II frontiers are exactly the same in both approaches (see Figs. 7.2 (left) and 7.5). This is simply due to the different orientations maintained in T_{VRS}^{II} for the measurement of efficiency and scale elasticity (i.e., the input orientation in $T_{VRS}^{II(I)}$ and the output orientation in $T_{VRS}^{II(II)}$). Note that the finding that the estimated sub-technology frontiers in T_{VRS}^{II} are the same in both approaches is just a coincidence.

Finally, the finding that firms – E and H exhibit CRS in the network technology and IRS and CRS in the sub-technology I and sub-technology II respectively reminds one that the CRS assumption maintained in the neoclassical theory for justifying the black-box structure of production technology does not necessarily allow one to infer that there are no scale benefits available in the sub-technologies. One can, therefore, argue that it is crucial for the firm's ownership to locate the sources of scale effects in their sub-technologies, which will enable the firm management to improve productivity.

However, the modeling of a firm technology by considering only the inputs consumed and the final outputs produced often yields the imprecise estimates of production function; and as a result, yields erroneous inferences concerning the RTS behavior of firms (see, e.g., D and H in Fig. 7.3). This is because the black-box characterization obscures important relations by ignoring the interdependencies that exist between the sub-technologies.

7.4 Concluding Remarks

To reveal the sources of efficiency and scale economies, two approaches are suggested, based on the premise as to whether the two-stage network technology considered in each approach allows allocative inefficiency. The first approach is developed by making use of a single network technology for the two interdependent sub-technologies. This approach allows for allocative inefficiency that may arise due to any sub-optimal decision as to how much of intermediate products to produce and consume by the sub-technology managers in the world of changing prices. In the second approach, however, the technology structure is determined by assuming its sub-technologies to be independent, implying that there is no allocative inefficiency.

Instances of real-life firms suffering from profit loss due to allocative inefficiency are not usually uncommon as they most often face uncertainty in forecasting prices in their production decisions. Therefore, production managers are given incentives to choose right output-mix and right input-mix in the world of changing market prices in order to improve upon profit. And, even if managers are not held responsible for the changing prices, management would still like to know the opportunity cost of using the sub-optimal input and output mixes. Therefore, the network production system is modeled by our two approaches to know the extent of output loss of a firm suffering from allocative inefficiencies.

The current study is limited to the estimation of TE and SE only. The potential future research subject could be the one where one could interpret a two-stage network firm as a multi-product firm producing both intermediate products and final outputs, and then, measure *economies of scope* by linking it with I_h^{VRS}, a proxy for the indication of allocative inefficiency.

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