

Chapter 14

Network DEA II

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Abstract The original DEA model by Charnes et al. (Eur. J. Oper. Res. 2:429–444, 1978) is set to analyze production as a black box, i.e., there is no information about the processes inside. Network DEA was proposed for analysis of the contents of the black box. This theory allows the researcher to model processes within the black box by formulating sub-technology DEA models. The interaction of sub-technology DEA models preserves the DEA structure, and the network model can therefore be solved using linear programming. This chapter discusses network DEA models, both static and dynamic. The discussion also explores various useful objective functions that can be applied to the models to find the optimal allocation of resources for processes within the black box that are normally invisible to DEA.

Keywords Data Envelopment Analysis (DEA) • Network • Intermediate products • Dynamic production

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14.1 Introduction

It is commonly observed that the DEA model proposed by Charnes et al. (1978) is a “black box” that receives inputs and produces outputs, but the transformation process by which this occurs is opaque to the analyst. As Tone and Tsutsui (2014) remark, “One of the drawbacks of these models is the omission of the internal structure of the DMUs.” Färe et al. (2007a) built on Shephard and Färe (1975) with a sequence of models where the interior (“black box”) of the CCR model could be analyzed. The primary device for achieving this was the use of a network. The insight they had was that when multiple DEA models are connected in a network, the network itself is a DEA model, and can be calculated using linear programming.

In this chapter we extend and update our paper (Färe et al. 2007a) with additional discussion of DEA sub-technologies, objective functions, and static and dynamic DEA models. We start with a discussion of the CCR model from an axiomatic perspective. Then we turn to objective functions, which can be either price dependent or not. In the case where objective functions are price independent, we discuss both distance functions and slack-based functions. The static network model is introduced next, starting with a generic model and extending it to three common cases. We end the chapter with dynamic DEA. The models discussed in this chapter are relatively simple, but provide the tools for construction of models describing arbitrarily complex processes.

14.2 The Black Box and Sub-technologies

Production models are frequently modeled as a black box, where inputs $x = (x_1, \dots, x_N) \in \mathbb{R}_+^N$ are transformed into outputs $y = (y_1, \dots, y_M) \in \mathbb{R}_+^M$ (Fig. 14.1).

In this chapter we go inside the black box and define sub-technologies as its smallest building blocks. First, we establish an axiomatic structure for the models, then discuss the connection of sub-technologies through a directed network.

A technology or sub-technology may be modeled as

$$T = \{(x, y) : x \text{ can produce } y\}$$

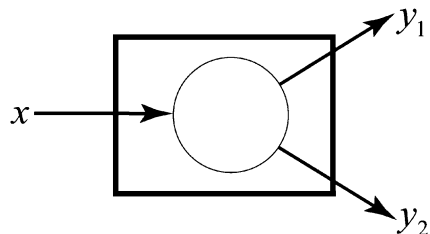


Fig. 14.1 The black box

or by its output sets,

$$P(x) = \{y : (x, y) \in T\} \quad x \in \mathbb{R}_+^N$$

or by its input sets,

$$L(y) = \{x : (x, y) \in T\},$$

and it holds that

$$y \in P(x) \Leftrightarrow (x, y) \in T \Leftrightarrow x \in L(y).$$

Suppose that we have $k = 1, \dots, K$ observations of inputs and good outputs (x^k, y^k) , then we can formulate the activity analysis or data envelopment analysis model as

$$T = \left\{ (x, y) : \begin{aligned} \sum_{k=1}^K z_k x_{kn} &\leq x_n, \quad n = 1, \dots, N, \\ \sum_{k=1}^K z_k y_{km} &\geq y_m, \quad m = 1, \dots, M, \\ z_k &\geq 0, \quad k = 1, \dots, K \end{aligned} \right\}$$

or equivalently,

$$P(x) = \left\{ y : \begin{aligned} \sum_{k=1}^K z_k x_{kn} &\leq x_n, \quad n = 1, \dots, N, \\ \sum_{k=1}^K z_k y_{km} &\geq y_m, \quad m = 1, \dots, M, \\ z_k &\geq 0, \quad k = 1, \dots, K \end{aligned} \right\}$$

i.e.,

$$L(y) = \left\{ x : \begin{aligned} \sum_{k=1}^K z_k x_{kn} &\leq x_n, \quad n = 1, \dots, N, \\ \sum_{k=1}^K z_k y_{km} &\geq y_m, \quad m = 1, \dots, M, \\ z_k &\geq 0, \quad k = 1, \dots, K \end{aligned} \right\}.$$

If the data meet the Kemeny et al. (1956) conditions, then the technology satisfies the following conditions;

- (i) $P(0) = \{0\}$; no free lunch,
- (ii) $P(x)$ is bounded for each x ; scarcity,
- (iii) T is closed.

Additional properties, from the definitions are;

- (iv) T is convex ($\Rightarrow L(y)$ and $P(x)$ are convex),
- (v) $x' \geq x \in L(y) \Rightarrow x' \in L(y)$; free disposability of inputs,
- (vi) $y' \leq y \in P(x) \Rightarrow y' \in P(x)$; free disposability of outputs,
- (vii) T is a cone; constant returns to scale (CRS).

The last condition holds, since the intensity variables z_i are only non-negative. If, in addition to non-negativity, $\sum_{k=1}^K z_k = 1$, variable returns to scale are modeled. Conditions (v) and (vi) follow from the inequalities of the inputs and outputs expressions.

$$\begin{aligned}
 (a) \quad & \sum_{k=1}^K y_{km} > 0 & m = 1, \dots, M \\
 (b) \quad & \sum_{m=1}^M y_{km} > 0 & k = 1, \dots, K \\
 (c) \quad & \sum_{k=1}^K x_{kn} > 0 & n = 1, \dots, N \\
 (d) \quad & \sum_{n=1}^N x_{kn} > 0 & k = 1, \dots, K
 \end{aligned}$$

Output condition (a) states that each output is produced by some k (DMU), and (b) requires that each activity produce some output. The input conditions (c) and (d) say that each input is used by some k and that each k uses at least one input. If $z_k \geq 0$, $k = 1, \dots, K$ and $\sum_{k=1}^K z_k \leq 1$, the technology exhibits non-increasing returns to scale.

Before we link technologies (and/or sub-technologies) together to model processes within the black box, we study optimization problems on a technology. These problems include profit and revenue maximization, cost minimization and distance function measures.

14.3 Objective Functions

Network models are frequently applied in performance measurement, including optimal resource allocations. An objective function that specifies the evaluation of measurement or allocation is required for optimization. Two types of objective functions are used; (i) those that require prices, and (ii) those that require only measures of inputs and outputs (quantity functions). These quantity functions can be either slack based or distance functions.

The distance functions have their duals among the objective functions involving prices. The most common are; the profit function dual to the directional technology function, the revenue function dual to Shephard's output distance function, and the cost function dual to Shephard's input distance function. The slack based objective functions do not have natural duals (Färe et al. 2007b).

Let $g = (g_x, g_y) \in \mathbb{R}_+^{N+M}$ be a directional vector for inputs (g_x) and outputs (g_y). This vector provides the direction in which a given input/output observation is projected onto the frontier of T . The optimization problem that defines the directional technology distance function is

$$\vec{D}_T(x, y; g) = \max \left\{ \beta : (x - \beta g_x, y + \beta g_y) \in T \right\}$$

Inputs are contracted while outputs are expanded.¹ Under "g-disposability" this function characterizes T , i.e.,

$$\vec{D}_T(x, y; g) \geq 0 \Leftrightarrow (x, y) \in T,$$

and it can be used as a measure of technical efficiency.

If we choose the directional vector to equal $g = (0, g_y)$, then we have a directional output distance function

$$\vec{D}_0(x, y; g_y) = \max \left\{ \beta : (y + \beta g_y) \in P(x) \right\},$$

which may also be a measure of technical efficiency, now output based.

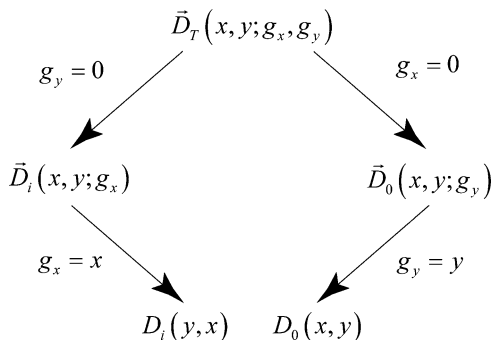
It is interesting to note how $\vec{D}_0(x, y; g_y)$ is related to Shephard's output distance function. The latter is defined as

$$D_i(x, y) = \min \{ \lambda : y/\lambda \in P(x) \}.$$

Shephard's output distance function or its reciprocal is the Farrell output oriented measure of technical efficiency.

¹This distance function was introduced by Luenberger as shortage function (see for example Luenberger 1995).

Fig. 14.2 Relation of directional distance functions



To relate the two output distance function to each other, choose $g_y = y$, then

$$\vec{D}_0(x, y; y) = 1 - 1/D_0(x, y)$$

Thus, we have shown how the directional output distance function generalizes Shephard’s (radial) output distance function. Similarly, we may choose $g = (g_x, 0)$ to obtain a directional input distance function

$$\vec{D}_i(x, y; g_x) = \max\{\beta : (x - \beta g_x) \in L(y)\},$$

and relate it to Shephard’s input distance function defined as

$$D_i(y, x) = \max\{\lambda : x/\lambda \in L(y)\}.$$

Choose $g_x = x$ and we have

$$\vec{D}_i(y, x; x) = -1 + 1/D_i(y, x).$$

Our derivations above show that the directional output distance function is the origin of the other four distance functions. This can be illustrated as (Fig. 14.2).

Finally we have (Färe and Lovell 1978)

$$D_0(x, y) = 1/D_i(y, x) \Leftrightarrow \text{CRS}.$$

The above distance functions are associated with measurement of technical efficiency. Each projects the observation in question onto a corresponding isoquant.²

²Note that an isoquant may contain the efficient subset as a proper subset and does not reflect Pareto/Koopmans efficiency.

Turning to the duals, assume input prices $w \in \mathbb{R}_+^N$ and output prices $p \in \mathbb{R}_+^M$ are known. Then we can define the profit function as

$$\begin{aligned}\Pi(p, w) &= \max\{py - wx : (x, y) \in T\} \\ &= \max\{py - wx : \vec{D}_T(x, y; g) \geq 0\}\end{aligned}$$

The last inequality holds, since the directional technology distance function characterizes T . From this it follows that (Färe and Grosskopf 2004)

$$\frac{\Pi(p, w) - (py - wx)}{pg_y + wg_x} \geq \vec{D}_T(x, y; g)$$

where the LHS is the Nerlovian profit indicator, which is the normalized difference between maximal profit $\Pi(p, w)$ and observed profit $(py - wx)$. This indicator is larger than the corresponding directional distance function (a measure of technical efficiency). By adding a measure of allocative efficiency, the inequality becomes an equality, and the Nerlovian indicator is decomposed into technical and allocative efficiency (Chambers et al. 1998).

In order to derive results similar to the Nerlovian indicator, we first define the revenue and cost functions

$$\begin{aligned}R(x, p) &= \max\{py : y \in P(x)\} \\ &= \max\{py : \vec{D}_0(x, y; g_y) \geq 0\}\end{aligned}$$

and

$$\begin{aligned}C(y, w) &= \min\{wx : x \in L(y)\} \\ &= \min\{wx : \vec{D}_i(x, y; g_x) \geq 0\}.\end{aligned}$$

From these expressions we get

$$\frac{R(x, p) - py}{pg_y} \geq \vec{D}_0(x, y; g_y)$$

and

$$\frac{wx - C(y, w)}{wg_x} \geq \vec{D}_i(x, y; g_x)$$

respectively.

Each of these inequalities can be closed by adding, as above, an allocative inefficiency component. Hence arriving at a revenue and a cost indicator with their corresponding decompositions.

Prior to a discussion of the radial distance function (Shephard 1953, 1970), we provide activity analysis/DEA models for calculating profit and the directional technology distance function. Since, under constant returns to scale, profit is zero, we choose the variable returns to scale formulation allowing for losses and profit. Maximal profit is then estimated as

$$\begin{aligned} \Pi(p, w) &= \max_{(x,y,z)} (py - wx) \\ \text{s.t. } &\sum_{k=1}^K z_k x_{kn} \leq x_n, \quad n = 1, \dots, N \\ &\sum_{k=1}^K z_k y_{km} \geq y_m, \quad m = 1, \dots, M \\ &\sum_{k=1}^K z_k = 1, \quad z_k \geq 0, \quad k = 1, \dots, K \end{aligned}$$

One may, of course, allow the price vectors p and w to vary with the observation.

To estimate the technology distance function, the researcher may choose the directional vectors or endogenize them.³ Here we consider the case of $g = (g_x, g_y)$ without any specific choice. Thus, for observation k' , we have

$$\begin{aligned} \vec{D}_T(x^{k'}, y^{k'}; g_x, g_y) &= \max \beta \\ \text{s.t. } &\sum_{k=1}^K z_k x_{kn} \leq x_{k'n} - \beta g_{x_n}, \quad n = 1, \dots, N \\ &\sum_{k=1}^K z_k y_{km} \geq y_{k'm} + \beta g_{y_m}, \quad m = 1, \dots, M \\ &\sum_{k=1}^K z_k = 1, \quad z_k \geq 0, \quad k = 1, \dots, K. \end{aligned}$$

From our two calculations and the data on $(x^{k'}, y^{k'})$, (p, w) we may calculate the Nerlovian profit indicator. Similar problems can be formulated for the revenue and cost indicators.⁴

In the discussion above, profit, revenue and cost indicators took an additive structure. The traditional Farrell (1957) decomposition of cost and revenue efficiency is multiplicative, not additive, and is based on Shephard's (1953, 1970) distance function.

³ For endogenous directions, see (Färe et al. 2013)

⁴ For an example of aggregation of these indicators, see (Färe and Grosskopf 2004).

To derive the revenue measure, note that

$$\begin{aligned} R(x, p) &= \max\{py : y \in P(x)\} \\ &= \max\{py : D_0(x, y) \leq 1\}, \end{aligned}$$

where the last equality holds because

$$D_0(x, y) \leq 1 \Leftrightarrow y \in P(x).$$

From this definition of revenue, it follows (Färe and Grosskopf 2004) that

$$\frac{R(x, p)}{py} \geq 1/D_0(x, y),$$

i.e., the ratio of maximal revenue to observed revenue is larger than the reciprocal of the output distance function.⁵ By multiplying the RHS with an allocative efficiency component, the Farrell decomposition of revenue efficiency is obtained. It is the product, not sum, of technical and allocative efficiencies.⁶

To set the stage for the slack-based measure of technical efficiency consider the Leontief production function

$$y = \min\{x_1, x_2\}$$

The isoquant for $y = 1$, is

$$\{(x_1, x_2) : \min\{x_1, x_2\} = 1\}$$

and the Koopmans efficient input set is

$$\{x_1, x_2\} = \{1, 1\}$$

Thus the isoquant and the efficient subset do not coincide. The following figure illustrates the relation between the isoquant and efficient subset (Fig. 14.3).

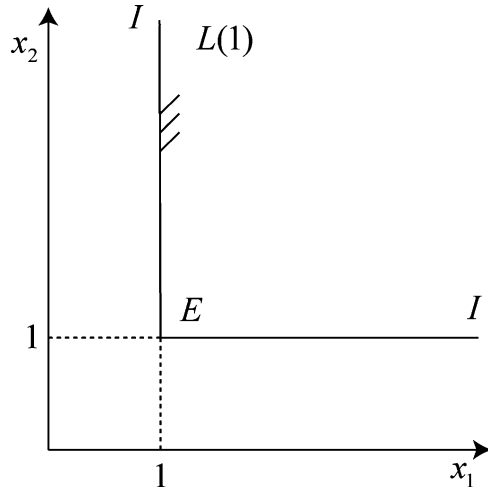
The isoquant consists of the points along II while E is the only efficient point.

The input distance functions introduced above have the property that they project an input vector onto the isoquant, and not necessarily onto the efficient point(s). Measures that project an input vector onto efficient points are the multiplicative Russell measure (Färe and Lovell 1978) and the additive slack-based measures (Färe and Grosskopf 2010; Tone 2001).

⁵ This expression is referred to as the Mahler inequality.

⁶ For the decomposition of Farrell's cost efficiency measure, see (Färe and Grosskopf 2004).

Fig. 14.3 Leontief isoquant (L,L) and efficient set (E)



The input oriented Russell measure (RM) has the following DEA formulation

$$RM_i(x^{k'}, y^{k'}) = \frac{1}{N} \min \sum_{n=1}^N \lambda_n$$

$$s.t. \quad \sum_{k=1}^K z_k x_{kn} \leq \lambda_n x_{k'n}, \quad n = 1, \dots, N$$

$$\sum_{k=1}^K z_k y_{km} \geq y_{k'm}, \quad m = 1, \dots, M$$

$$z_k \geq 0, \quad k = 1, \dots, K .$$

If an input $x_{k'n}$ is zero, we modify the measure and set $\lambda_n = 1$. This measure is one if and only if $x^{k'}$ belongs to the efficient subset of $L(y^{k'})$, where

$$Eff L(y) = \{x : x \in L(y), x' \leq x \Rightarrow x' \notin L(y)\}$$

The input oriented slack-based measure (SB) by Färe and Grosskopf (2010) is based on the input oriented directional distance function, and has the following DEA formulation

$$SB_i(x^{k'}, y^{k'}) = \max \sum_{n=1}^N \beta_n$$

$$\sum_{k=1}^K z_k y_{kn} \leq x_{k'n} - \beta_n \bullet 1_n, \quad n = 1, \dots, N$$

$$\sum_{k=1}^K z_k y_{km} \geq y_{k'm}, \quad m = 1, \dots, M$$

$$z_k \geq 0, \quad k = 1, \dots, K.$$

This measure is zero if and only if $x^{k'}$ belongs to the efficient subset of $L(y^{k'})$.⁷ Note that β_n is independent of the unit of measurement, since the direction $g_n = 1_n$ is in the same units as x_n , thus $\beta_{n_i} \forall i$ can be added.

14.4 Static Network Models

In this section we go inside the black box and model it as a network of sub-technologies. This approach has its origin in Shephard and Färe (1975), who wrote that “many production systems (technologies) may be conceptualized as the joint interaction of a finite number of production sub-technologies called activities.” The static model is useful for analyzing the allocation of intermediate products and also provides the basic structure of dynamic DEA models.

We restrict our presentation to three sub-technologies $P^1, P^2,$ and P^3 . These three sub-technologies are connected by the directed network shown in Fig. 14.4.

To complete a network of these three sub-technologies, we add a distribution process and a sink, or collection of final outputs.⁸ Inputs are denoted by $x = (x_1, \dots, x_N) \in \mathfrak{R}_+^N$, the network exogenous vector, i.e., total availability is attached to the three sub-technologies, ${}^i_0x, i = 1, 2, 3 \dots$ where 0 denotes source and i denotes use (Fig. 14.5). For example, 1_0x is the input vector from source 0 used

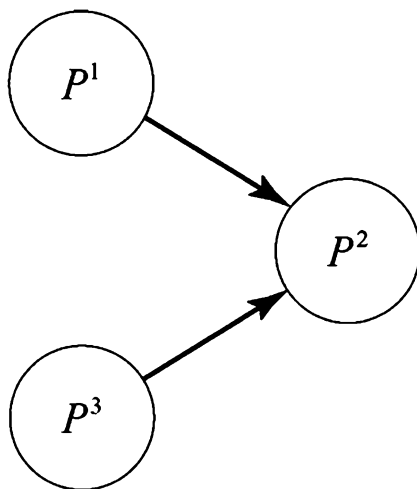
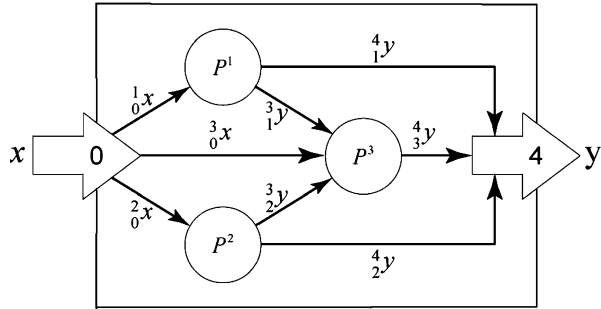


Fig. 14.4 Representation of sub-technologies

⁷ We note that the input oriented Tone (2001) measure requires all inputs to be positive. See Färe and Grosskopf (2010) for a discussion inputs to this measure.

⁸ This model is adopted from Färe and Grosskopf (1996a).

Fig. 14.5 A network technology



in activity 1. The total amounts used in the three activities cannot exceed the total amount available, i.e.,

$$x \geq \sum_{i=1}^3 x_0^i .$$

In Fig. 14.5 activity \$P^1\$ uses \$x_0^1\$ as an exogenous input and produces

$$y_1^3 + y_1^4$$

as outputs. The \$y_1^3\$ is an input into activity 3, while \$y_1^4\$ is the final output from \$P^1\$. Activity \$P^3\$ uses \$x_0^3\$ as an exogenous input and \$y_1^3, y_2^3\$ as intermediate inputs. The final product from activity \$P^3\$ is output vector \$y_3^4\$. The total network output is the sum of the final output of the three activities,

$$y_1^4 + y_2^4 + y_3^4 .$$

Note that even if an activity does not produce one of the listed outputs, that output is set at zero and the network structure remains the same. It is also noteworthy that some outputs may be both final and intermediate outputs, e.g. spare parts.

Based on the description above, a generic network model takes the form

$$P(x) = \left\{ \begin{aligned} & (y_1^4 + y_2^4 + y_3^4) : \\ & (y_1^4 + y_1^3) \in P^1(x_0^1) \\ & (y_2^4 + y_2^3) \in P^2(x_0^2) \\ & y_3^4 \in P^3(x_0^3, y_1^3 + y_2^3) \\ & x_0^1 + x_0^2 + x_0^3 \leq x \end{aligned} \right\} ,$$

where \$P^i(\bullet)\$ \$i = 1, 2, 3\$ are output sets, i.e., \$P(x) = \{y : x \text{ can produce } y\}\$. Thus the network model \$P(x)\$ is formed by the individual sub-technologies.

Next we formulate the network technology as a DEA model, i.e.,

$$\begin{aligned}
 P(x) = \{y = (&^4_1y + ^4_2y + ^4_3y) : \\
 \text{(a) } &^4_3y_m \leq \sum_{k=1}^K z_k^3 \ ^4_3y_{km}, \quad m = 1, \dots, M, \\
 \text{(b) } &\sum_{k=1}^K z_k^3 \ ^3_0x_{kn} \leq \ ^3_0x_n, \quad n = 1, \dots, N, \\
 \text{(c) } &\sum_{k=1}^K z_k^3 \ ^3_1y_{km} \leq \ ^3_1y_m, \quad m = 1, \dots, M, \\
 \text{(d) } &\sum_{k=1}^K z_k^3 \ ^3_2y_{km} \leq \ ^3_2y_m, \quad m = 1, \dots, M, \\
 \text{(e) } &z_k^3 \geq 0, \quad k = 1, \dots, K, \\
 \text{(f) } &(\ ^3_1y_m + \ ^4_1y_m) \leq \sum_{k=1}^K z_k^1 (\ ^3_1y_{km} + \ ^4_1y_{km}), \quad m = 1, \dots, M, \\
 \text{(g) } &\sum_{k=1}^K z_k^1 \ ^1_0x_{kn} \leq \ ^1_0x_n, \quad n = 1, \dots, N, \\
 \text{(h) } &z_k^1 \geq 0, \quad k = 1, \dots, K, \\
 \text{(i) } &(\ ^3_2y_m + \ ^4_2y_m) \leq \sum_{k=1}^K z_k^2 (\ ^3_2y_{km} + \ ^4_2y_{km}), \quad m = 1, \dots, M, \\
 \text{(j) } &\sum_{k=1}^K z_k^2 \ ^2_0x_{kn} \leq \ ^2_0x_n, \quad n = 1, \dots, N, \\
 \text{(k) } &z_k^2 \geq 0, \quad k = 1, \dots, K, \\
 \text{(l) } &\ ^1_0x_n + \ ^2_0x_n + \ ^3_0x_n \leq x_n, \quad n = 1, \dots, N\}.
 \end{aligned}$$

where $\ ^i_0y_{km}$ and $\ ^i_0x_{kn}$, $k = 1, \dots, K$, $m = 1, \dots, M$ and $n = 1, \dots, N$ are observed data. In the network DEA model the first sub-technology is specified by (f)–(h), the second by (i)–(k) and the third by (a)–(e). Note that each sub-technology has a vector of intensity variables (e), (h) and (k). Here they all satisfy constant returns to scale.

Färe and Grosskopf (1996b) have shown that if each sub-technology satisfies standard axioms such as free disposability of inputs and outputs, compactness of the output set and other axioms, then the network technology has the same properties as the sub-technologies from which it is constructed.

If we wish to compute, for example, output efficiency of the network DEA model, we may use the standard Farrell (1957) output measure using linear programming;

$$F_0(x^k, y^k) = \max \{ \theta : \theta y^k \in P(x^k) \},$$

where $P(x^{k'})$ is the network for observation k' . Note that in this model the ratio of outputs from P^1 and P^2 to observed output may vary. If this effect is not appropriate, we may modify the LHS of (f) and (i) to read

$$(f') \quad \theta ({}^3_1y_{k'm} + {}^4_1y_{k'm}) \leq$$

and

$$(i') \quad \mu ({}^3_2y_{k'm} + {}^4_2y_{k'm}) \leq$$

where k' is the observed data and θ, μ are non-negative scalars. When $F_0(x^{k'}, y^{k'})$ is computed under these more restrictive conditions, the optimal θ and μ are the sub-vector Farrell efficiency scores.

From standard DEA modeling practice (Cooper et al. 2004), one knows that the primal envelopment model has a dual multiplier formulation. Hence, when we use the “fixed” mix formulation, i.e., when (f') and (i') are used, we may write the network in its dual form, as in Kao (2009). To see how the network model may be specified as a traditional neoclassic production model, suppose for simplicity that ${}^4_1y = {}^4_2y = 0$ and that ${}^3_1y, {}^3_2y$, and ${}^4_3y \in \mathbb{R}_+$, so that each sub-technology produces a single output and no final products are produced by P^1 and P^2 . If we define a production function by

$$F(x) = \max\{y : y \in P(x)\},$$

then the network model can be written as follows⁹:

$$\begin{aligned} {}^4_1y &= F^3(x_0^3, F^1({}_0^1x), F^2({}_0^2x)) \\ &= F^3(x - {}_0^1x - {}_0^2x, F^1({}_0^1x), F^2({}_0^2x)), \end{aligned}$$

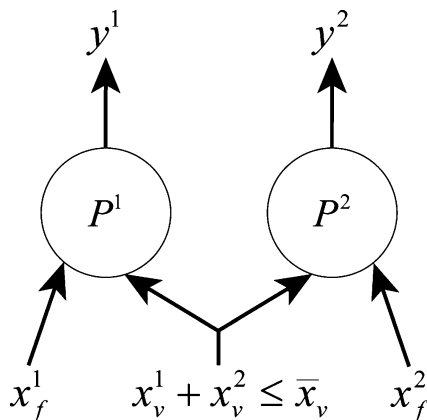
i.e., as function of functions. This example illustrates the creation of “parametric” neoclassical models of network technologies.

Next, we formulate three commonly used network models; (i) the Johansen (1972) model, (ii) the two stage model and (iii) the externality model. These three models are set up here generically, and like the network models above can be extended to any finite set of sub-technologies. We start with the Johansen model as formulated by Färe et al. (1992). Here we express it and the other models by means of diagrams (Fig. 14.6).

We have two technologies P^1 and P^2 producing two vectors of outputs y^1 and y^2 . They use fixed inputs (non-allocable) x_f^1 and x_f^2 respectively, and variable allocable

⁹ For this to exist $P(x)$ must be nonempty and compact.

Fig. 14.6 The Johansen model



inputs x_v^1 and x_v^2 . The sum of these input vectors cannot exceed a given vector \bar{x}_v . Thus the total can be allocated between the two technologies P^1 and P^2 . Such an allocation can be implemented by choosing one of the objective functions discussed in Sect. 14.2. For example, if output prices are known, one may maximize total revenue for the network, viz,

$$\begin{aligned} \max & p^1 y^1 + p^2 y^2 \\ \text{s.t.} & y^1 \in P^1, y^2 \in P^2 \end{aligned}$$

where p^1 and p^2 are two output price vectors. The solution to this problem yields the optimal output vector for each technology and the optimal allocation of the variable inputs.¹⁰ This model may be used in the discussion of merger and coalition formation (Bogetoft and Wang 2005; Färe et al. 2011).

While the Johansen model organizes the technologies in parallel, our second model organizes the technologies in a sequence (Chen et al. 2009) (Fig. 14.7).¹¹

Again we have two technologies P^1 and P^2 , now ordered sequentially with the output from P^1 being an intermediate input into P^2 to produce the final output y^2 . Each technology has its own input x^1, x^2 . It is not hard to adapt the concept of allocation from the Johansen model to introduce savings and borrowing into the model (Färe and Grosskopf 1996a).

Suppose the externality model consists of one polluter and one receptor, with the technologies P^1 and P^2 , respectively. Each technology uses an input, x^1 or x^2 , to produce desirable outputs y^1 and y^2 , and the polluter also produces an undesirable output u . The undesirable output u is an input into the receptor technology, as illustrated below (Fig. 14.8).

¹⁰ One may, of course, solve this problem using DEA.

¹¹ For a review see (Cook et al. 2010)

Fig. 14.7 A two stage model of production

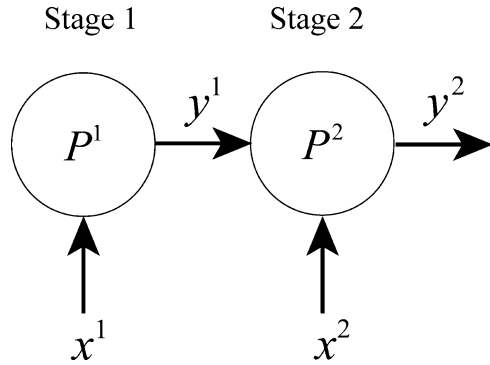
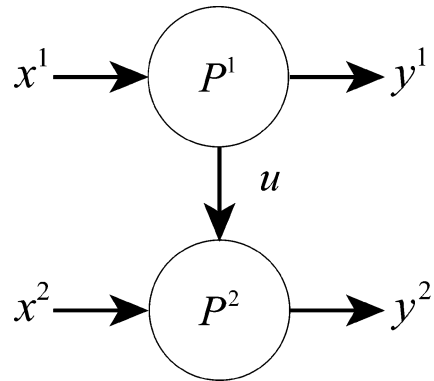


Fig. 14.8 The network model with externalities



The upstream firm (technology P^1) may be, for example, a paper mill, and the downstream firm P^2 a fishery. The upstream firm produces y^1 and u jointly and in the terminology of Shephard and Färe (1974) we say that y^1 is nulljoint with u if

$$(y^1, u) \in P^1(x^1) \text{ and } u = 0 \Rightarrow y^1 = 0,$$

i.e., if no undesirable outputs are produced, no desirable outputs can be produced. The downstream technology takes u as an additional input with negative implications, i.e.,

$$u^1 \geq u \Rightarrow P^2(x^2, u^1) \leq P^2(x^2, u).$$

This means that more desirable input does not increase production, but may decrease it. This type of network models have also been applied in the studies of property rights (see Färe and Grosskopf 2004, for a DEA formulation of such a problem).

14.5 Dynamic Network Models¹²

“Late in 1974, Dr Thomas Varley, Office of Naval Research, asked whether I could formulate a production function for shipbuilding. In thinking about this matter it became apparent that the usual steady state (static) production function could at best provide a faint model of this production technology. Imagine that you visit a shipyard. Day by day a tremendous amount of production activity of great variety is carried on, yet no ships are turned out. This goes on for a long time. Eventually a ship emerges. What was being produced day by day all during this time? It is clear that the daily, weekly, monthly outputs of the system were intermediate products. The shipbuilding production system, like construction, is a dynamically evolving process.”(Shephard and Färe 1980, page V)

With this description of a production process as a motivation for the dynamics, let us start by formulating a dynamic model as a network. Assume there are three time periods $t-1, t, t+1$, and that each has its production technology $P^\tau, \tau = t-1, t, t+1$. A dynamic model has the property that a decision in one time period impacts on later time periods. For example, if I save now, then my possible consumption may increase later. Therefore we introduce intermediate products, i.e., those products that are held over between time periods, ${}^\tau y \in \mathfrak{R}_+^M$. If $\tau = 1$, then ${}^t y$ is the intermediate vector of outputs produced at time t and entering the production process at $t+1$, i.e., an input at P^{t+1} . Figure 14.9 illustrates this setup.

Each of the production sub-technologies P^τ uses exogenous inputs x^τ to produce the final output ${}^\tau y$ and intermediate inputs ${}^i y$. To complete the network model (see the figure below) we add initial conditions (distribution process in the static model) and transversality conditions (sink in the static model).

The initial condition is given by ${}^i \bar{y}$ and may be thought of as the stocks of “capital” initially available. The transversality condition could include the number of periods, say $t+1 = T$, the state of the system at $T, {}^i_{t+1} \bar{y}$, and the final output vector

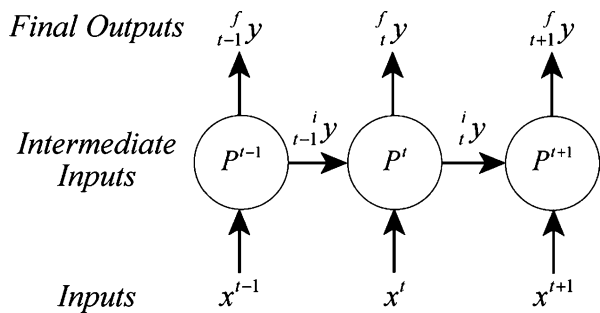
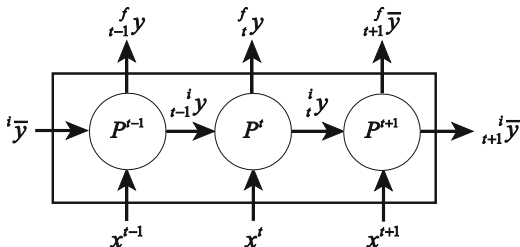


Fig. 14.9 Dynamic sub-technologies

¹²This section is to a large extent adapted from Färe, Grosskopf and Whittaker (2007) For a recent survey of nonparametric dynamic efficiency see (Fallah-Fini et al. 2013).

Fig. 14.10 A dynamic network model



from the last period, $f_{t+1}^f \bar{y}$. The chosen conditions are specific to the research problem to be analyzed.

Thus far we have not introduced discounting. However, if outputs are given in value terms, a discount factor of δ^t , $0 \leq \delta^t \leq 1$, may be introduced to account for the lesser of future income compared to present income. For example, $\delta_{t,y}^{f_t y}$ are the discounted values of the final output in period t (Fig. 14.10).

The dynamic network DEA model consists of the interaction of a finite number of static models. Therefore let us start by studying the sub-technology P^t from above. This technology uses inputs x^t and intermediate inputs $i_{t-1}y$ to produce outputs $f_t y + i_t y$, where $f_t y$ is the final output and $i_t y$ is the intermediate output that is used as an input in the next period. Thus we may write

$$\begin{aligned}
 P^t(x^t, i_{t-1}y) &= \left\{ (f_t y + i_t y) : \right. \\
 (f_t y_m + i_t y_m) &\leq \sum_{k=1}^{K^t} z_k^t (f_t y_{km} + i_t y_{km}) & m = 1, \dots, M, \\
 \sum_{k=1}^{K^t} z_k^{t-1} i_t y_{km} &\leq i_{t-1} y_{k'm}, & m = 1, \dots, M, \\
 \sum_{k=1}^{K^t} z_k^t x_{kn}^t &\leq x_{k'n}^t, & n = 1, \dots, N, \\
 z_k^t &\geq 0, & k = 1, \dots, K^t \}
 \end{aligned}$$

where $f_t y_{km}$, $i_t y_{km}$, $i_{t-1} y_{k'm}$, and x_{kn}^t are observed inputs and outputs. We allow the number of observations to differ between periods, hence the notation K^t . The output vector $(f_t y + i_t y)$ is specified so that one can include M^t for all t , i.e., $M = M^{t-1} + M^t + \dots$ by including appropriate zeroes.

Recall from the static model, that if the sub-technologies have the properties (i)–(vii), then the whole network model also has those properties. This observation also applies to the dynamic model below:

$$\begin{aligned}
P^t(x^{t-1}, x^t, x^{t+1}, i\bar{y}) = & \left\{ \left(\begin{array}{c} f \\ t-1 \end{array} y^f_t y, \left(\begin{array}{c} f \\ t+1 \end{array} y + \begin{array}{c} i \\ t+1 \end{array} y \right) \right) : \\
\left(\begin{array}{c} f \\ t-1 \end{array} y_m + \begin{array}{c} i \\ t-1 \end{array} y_m \right) \leq & \sum_{k=1}^{K^{t-1}} z_k^{t-1} \left(\begin{array}{c} f \\ t-1 \end{array} y_{km} + \begin{array}{c} i \\ t-1 \end{array} y_{km} \right) \quad m = 1, \dots, M, \\
\sum_{k=1}^{K^{t-1}} z_k^{t-1} i y_{km} \leq & i \bar{y}_{km}, \quad m = 1, \dots, M, \\
\sum_{k=1}^{K^{t-1}} z_k^{t-1} x_{kn}^{t-1} \leq & x_n^{t-1}, \quad n = 1, \dots, N, \\
z_k^{t-1} \geq 0, & \quad k = 1, \dots, K^{t-1}, \\
\left(\begin{array}{c} f \\ t \end{array} y_m + \begin{array}{c} i \\ t \end{array} y_m \right) \leq & \sum_{k=1}^{K^t} z_k^t \left(\begin{array}{c} f \\ t \end{array} y_{km} + \begin{array}{c} i \\ t \end{array} y_{km} \right) \quad m = 1, \dots, M, \\
\sum_{k=1}^{K^t} z_k^t \begin{array}{c} i \\ t-1 \end{array} y_{km} \leq & \begin{array}{c} i \\ t-1 \end{array} \bar{y}_m, \quad m = 1, \dots, M, \\
\sum_{k=1}^{K^t} z_k^t x_{kn}^t \leq & x_n^t, \quad n = 1, \dots, N, \\
z_k^t \geq 0, & \quad k = 1, \dots, K^t, \\
\left(\begin{array}{c} f \\ t+1 \end{array} y_m + \begin{array}{c} i \\ t+1 \end{array} y_m \right) \leq & \sum_{k=1}^{K^{t+1}} z_k^{t+1} \left(\begin{array}{c} f \\ t+1 \end{array} y_{km} + \begin{array}{c} i \\ t+1 \end{array} y_{km} \right) \quad m = 1, \dots, M, \\
\sum_{k=1}^{K^{t+1}} z_k^{t+1} \begin{array}{c} i \\ t \end{array} y_{km} \leq & \begin{array}{c} i \\ t \end{array} \bar{y}_{km}, \quad m = 1, \dots, M, \\
\sum_{k=1}^{K^{t+1}} z_k^{t+1} x_{kn}^{t+1} \leq & x_n^{t+1}, \quad n = 1, \dots, N, \\
z_k^{t+1} \geq 0, & \quad k = 1, \dots, K^{t+1} \},
\end{aligned}$$

Note that each sub-technology has its own intensity vector z^τ , $\tau = t - 1$, t , $t + 1$, and that the interaction between time periods comes through the intermediate outputs.

Färe and Grosskopf (1997) use this model to study the inefficiency of APEC countries due to dynamic misallocation of resources. They used the sum of Shephard (1970) sub-technology distance functions as their optimization criterion. Nemota and Goto (2003) applied the dynamic network model to study Japanese electricity production over time. They used cost minimization for the optimization criterion. Jaenicke (2000) applied the dynamic model in the analysis of the yield effects of crop rotation. Nemota and Goto (1999) applied the dual linear programming problem formulation to the cost minimization and derived the fundamental

equation (Hamilton-Jacobi-Bellman) of dynamic programming. We also refer the reader to the work of J.K. Sengupta. A search for “J.K. Sengupta dynamic models” using any one of the major internet search engines produces a large number of dynamic non-parametric models that he has analyzed.

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