Chapter 11
Slacks-Based Network DEA

Kaoru Tone and Miki Tsutsui

Abstract Traditional DEA models deal with measurements of relative efficiency of DMUs regarding multiple-inputs versus multiple-outputs. One of the drawbacks of these models is the neglect of intermediate products or linking activities. After pointing out needs for inclusion of them to DEA models, we propose a slacks-based network DEA model that can deal with intermediate products formally. Using this model we can evaluate divisional efficiencies along with the overall efficiency of decision making units (DMUs).

Keywords Data envelopment analysis • Network DEA • SBM • WSBM • Divisional efficiency • Overall efficiency

11.1 Introduction

Traditional DEA models deal with measurements of relative efficiency of DMUs regarding multiple-inputs versus multiple-outputs. One of the drawbacks of these models is the neglect of internal or linking activities. For example, many companies are comprised of several divisions that are linked as illustrated in Fig. 11.1.

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K. Tone (✉)
National Graduate Institute for Policy Studies, 7-22-1 Roppongi, Minato-ku,
Tokyo 106-8677, Japan
e-mail: tone@grips.ac.jp

M. Tsutsui
Central Research Institute of Electric Power Industry, 1-6-1 Otemachi,
Chiyoda-ku, Tokyo 100-8126, Japan

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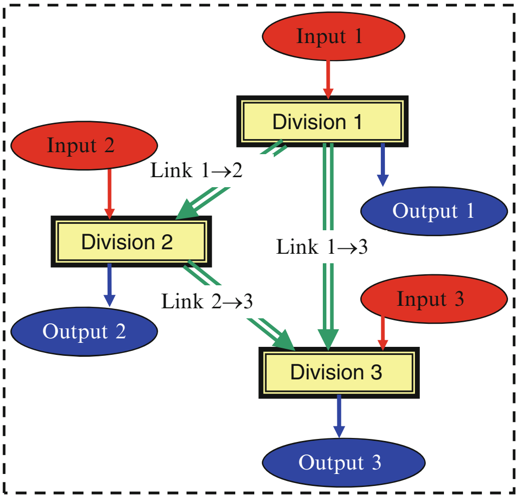


Fig. 11.1 Company with three linked divisions

In the example, the company has three divisions. Each division utilizes its own input resources for producing its own outputs. However, there are linking activities (or intermediate products) as shown by Link 1 → 2, Link 1 → 3 and Link 2 → 3. Link 1 → 2 indicates that parts of the outputs from Division 1 are utilized as inputs to Division 2. In traditional DEA models, every activity should belong to either input or output but not to both. So usually they employ multiple steps for evaluation, using intermediate products as outputs in one step and as inputs in another step. Thus, these models cannot deal with intermediate products directly in a single step.

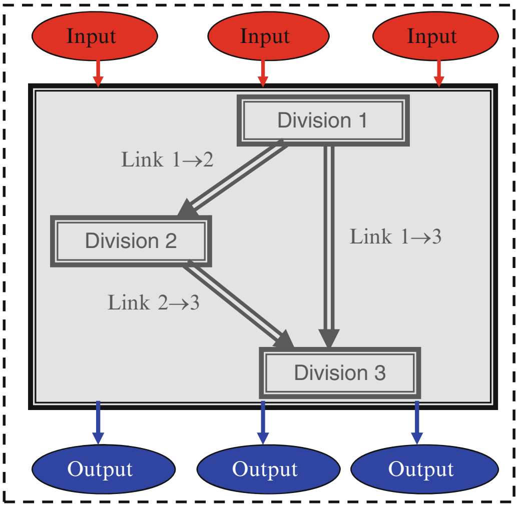
Although there may be many variants of this process flow, the existence of linking activities is an indispensable part of Network DEA models.

Within traditional DEA models there are at least two approaches for evaluating the efficiency of multi-division organizations.

11.1.1 Aggregation (Black Box)

A simple approach is to aggregate these divisions into a single company which utilizes Inputs 1, 2 and 3, and produces Outputs 1, 2 and 3 (Fig. 11.2). However, using this approach we neglect internal linking activities, and thus, we cannot evaluate the impact of division-specific inefficiencies on the overall efficiency of the company as a whole. Furthermore, this model might choose an inappropriate pair of input vs. output for evaluation and assign an unreasonable score to the concerned DMU, since DEA selects the most favorable pair for the DMU in the sense of maximizing the ratio scale (see Cooper et al. 2007, p. 25). In other words, the analysis does not fully access the underlying diagnostic value potentially available to management. This model often rouses a problem involving degree of freedom in that the number of input and output items increases relative to the

Fig. 11.2 Black box



number of DMUs. As a rule of thumb, DEA demands that the number of DMUs should be at least three times larger than the sum of numbers of inputs outputs as otherwise DEA is apt to lose discriminating power (see Cooper et al. 2007, p. 284). We will point to this in Sects. 11.5 and 11.6.

11.1.2 Separation

The second approach is to evaluate divisional efficiency individually (Fig. 11.3). In this case, we evaluate the efficiency of Division 1 of each company among the set of DMUs using Input 1 as input, and Output 1, Link 1 → 2 and Link 1 → 3 as outputs. Similarly we evaluate the efficiency of Division 2 of each company among the set of DMUs using Link 1 → 2 and Input 2 as inputs, and Link 2 → 3 and Output 2 as outputs. In this way, we can evaluate efficiency of each division of a company among the set of DMUs, and hence can find benchmarks for each division. However, this approach does not account for the continuity of links between divisions.

11.1.3 Needs for Network DEA

The above observations lead us to consider a DEA model called "Network DEA model" that accounts for divisional efficiencies as well as the overall efficiency in a unified framework. This means that we evaluate the total efficiency of DMUs as the main objective which involves divisional efficiencies as its components. Network DEA models were introduced in the innovative book (Färe and Grosskopf 1996) by

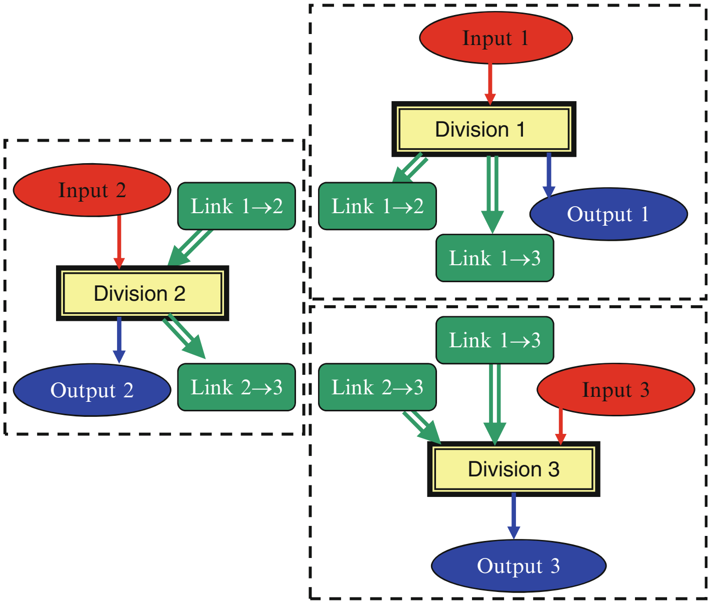


Fig. 11.3 Separation

Färe and Grosskopf (see also Färe 1991; Färe and Grosskopf 2000). They investigated the so-called "black box" for the first time. Their models were extended by several authors.

The network DEA model (Lewis and Sexton 2004) proposed by Lewis and Sexton has a multi-stage structure as an extension of the two-stage DEA model proposed in Sexton and Lewis (2003). This study solves a DEA model for each node independently. For an output-oriented model, firstly a general DEA model is solved for the upstream node at the 1st stage to obtain the optimal solution of outputs. At the next stage, a part of (or all of) optimal outputs obtained at the upstream node are applied as intermediate inputs to the next node. After solving DEA models for all nodes in turn, a final optimal output is obtained at the last node. The firm-level efficiency score is measured as the final optimal output divided by an observed output.

Prieto and Zofio (2007) applied network efficiency analysis within an input–output model initiated by Koopmans (1951). They optimized primary input allocations, intermediate products and final demand products by way of Network DEA techniques and succeeded in applying their models to input–output database of OECD countries.

Löthgren and Tambour (1999) applied Network DEA model to a sample of Swedish pharmacies with organizational objectives that necessitates a monitoring of efficiency and productivity as well as customer satisfaction. They compared the results of Network DEA models with those of traditional DEA models (see also Chen 2009; Kao 2009).

The above Network DEA models utilize the radial measure of efficiency, e.g. the CCR (Charnes et al. 1978) or the BCC (Banker et al. 1984) models as the basic DEA methodology and the production possibility set. The radial models stand on the assumption that inputs or outputs undergo proportional changes. However, this assumption needs care. For example, if we employ labor, materials and capital as inputs, some of them are substitutional and do not change proportionally.

This chapter introduces a network DEA model, that uses the slacks-based measure (SBM: Tone 2001; Pastor et al. 1999) approach for evaluating efficiency. The SBM is a non-radial method and is suitable for measuring efficiencies when inputs and outputs may change non-proportionally. This model can decompose the overall efficiency into divisional ones. Furthermore, we employ the weighted SBM model (Cooper et al. 2007; Tsutsui and Goto 2009) in order to incorporate the importance of divisions. These weights are set exogenously. We also investigate several properties of Network DEA models and show that, under the variable returns-to-scale assumption, every division has at least one efficient DMU (decision making unit) for the division, whereas under the constant returns-to-scale assumption it is possible that some division has no efficient DMUs for the division.

The remainder of this chapter unfolds as follows. In the next section, we introduce several network structures in actual business situations. Then in Sect. 11.3, we propose Network DEA (NDEA) models based on the weighted slacks-based measure (WSBM) approach. We discuss the characteristics of the divisional efficiencies in Sect. 11.4. Illustrative examples are introduced in Sect. 11.5. We extend our models in Sect. 11.6. We summarize the results and conclude the chapter in the last section. This chapter is written based on Tone and Tsutsui (2009).

11.2 Several Examples of Network Structures

We introduce network structures from actual businesses which motivated this study.

11.2.1 Electric Power Companies

Figure 11.4 exhibits typical vertically integrated electric utility companies consisting of generation, transmission and distribution divisions.

The generation division (Division 1) uses several inputs such as capital, labor and fuel (Input 1) and produces electric power. Then it becomes an intermediate input for the transmission division (Link 1–2). In the transmission division (Division 2), companies utilize capital and labor inputs (Input 2) as well as the intermediate inputs from generation division (Link 1–2). Electricity through transmission lines is sent to distribution division as intermediate output (Link 2–3) or sales to

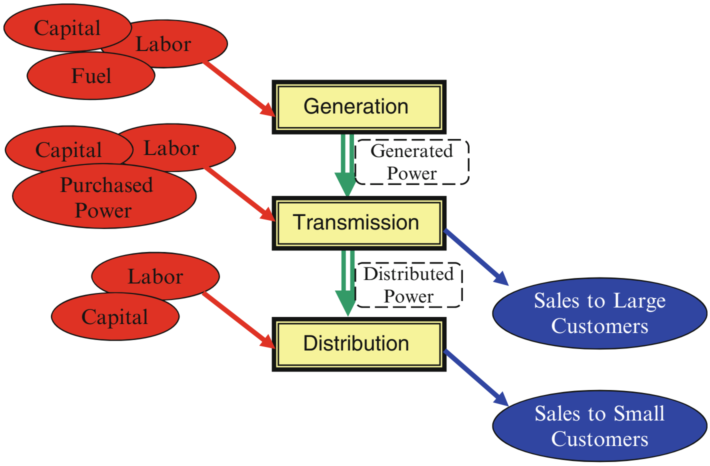


Fig. 11.4 Vertically integrated electric power companies

large customers (Output 2) that do not utilize distribution line. The distribution division (Division 3) uses capital and labor inputs (Input 3) and the intermediate input from the transmission division (Link 2–3) and provides electricity to small customers (Output 3).

11.2.2 Hospitals

Kaihara et al. (2007) private communication report the standard structure of Japanese general hospitals as depicted in Fig. 11.5. A general hospital consists of divisions, such as medical department, clinical laboratory, radiology, pharmacy and dietetic department. Each division has its own inputs; labor, materials and capital, and outputs; incomes. These divisions are connected by internal links. For example, a part of patients checked up at medical department is sent to radiology department. In order to evaluate the efficiency of general hospitals we need to account these divisions as a whole including linking activities. Thus, a network DEA model is appropriate for this purpose.

11.2.3 Broadcasting Companies

Broadcasting companies consist of two divisions; production and transmission. Using labor, materials and capital, the production division produces programs. A part of these products can be marketed to other media, while they are intermediate products to the transmission division. This division utilizes its own labor, materials and capital to send the programs to audiences. Figure 11.6 displays this

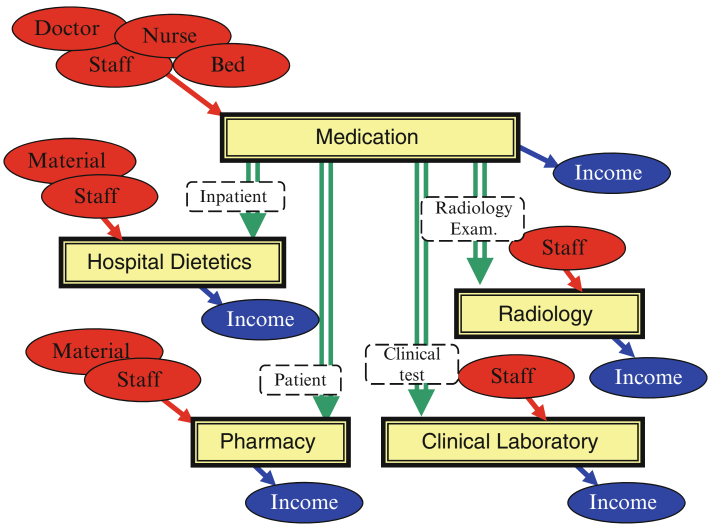


Fig. 11.5 General hospital

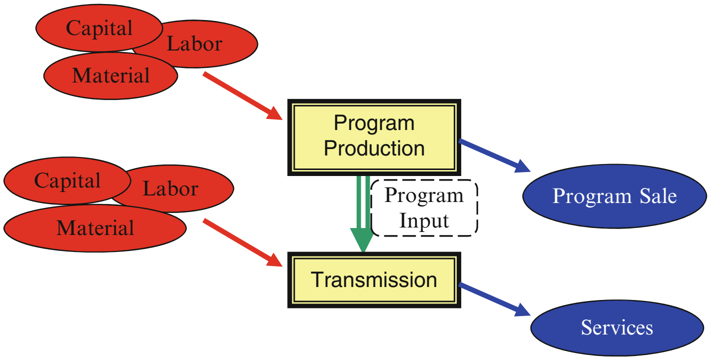


Fig. 11.6 Broadcasting companies

network structure. Product of the production division is the link (intermediate product) to the transmission division. This network structure is reported by (Asai (2007), private communication).

11.2.4 Financial Holding Companies

Seiford and Zhu (1999) pointed out that financial holding companies have two stages; profit generation and market value creation as exhibited in Fig. 11.7. Usually this process is studied in the two stage approaches; profitability and marketability. In the first stage, the profitability sector utilizes employees, assets

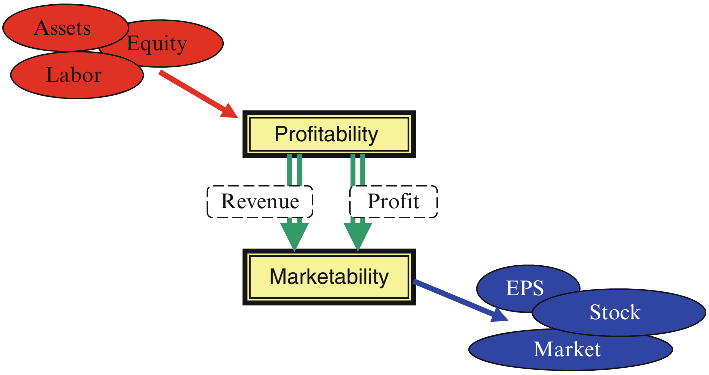


Fig. 11.7 Financial holding companies

and stockholders' equity to produce revenues and profits. The second stage measures (stock) marketability in the stock market by the revenue and profits it generates. It can be seen that revenues and profits serve as intermediate factors in the sense that they are outputs from the first stage and inputs to the second stage. The market sector produces market values, total returns to investors and earnings per share as outputs (Seiford and Zhu 1999). Thus, revenues and profits are linking activities between the two sectors and a network structure is recognized in this field.

11.3 Basic Framework of Network DEA

In this section, we introduce slacks-based Network DEA model referring to its production possibility set, efficiency and projection.

11.3.1 Notation and Production Possibility Set

We deal with n DMUs ($j = 1,\ldots, n$) consisting of K divisions ($k = 1,\ldots, K$). Let m_k and r_k be the numbers of inputs and outputs to Division k, respectively. We denote the link leading from Division k to Division h by (k,h) and the set of links by L. The observed data are $\left\{\mathbf{x}_j^k \in R_+^{m_k}\right\}(j = 1, \ldots, n;\ \ k = 1, \ldots, K)$ (input resources to DMU$_j$ at Division k), $\left\{\mathbf{y}_j^k \in R_+^{r_k}\right\}(j = 1, \ldots, n;\ \ k = 1, \ldots, K)$ (output products from DMU$_j$ at Division k) and $\left\{\mathbf{z}_j^{(k,h)} \in R_+^{t_{(k,h)}}\right\}(j = 1, \ldots, n;\ \ (k,h) \in L)$ (linking intermediate products from Division k to Division h) where $t_{(k,h)}$ is the number of items in Link (k,h).

The production possibility set $\{(\mathbf{x}^k,\mathbf{y}^k,\mathbf{z}^{(k,h)})\}$ is defined by

$$
\begin{aligned}
\mathbf{x}^k &\geq \sum\nolimits_{j=1}^{n} \mathbf{x}_j^k \lambda_j^k \quad (k = 1, \ldots, K) \\
\mathbf{y}^k &\leq \sum\nolimits_{j=1}^{n} \mathbf{y}_j^k \lambda_j^k \quad (k = 1, \ldots, K) \\
\mathbf{z}^{(k,h)} &= \sum\nolimits_{j=1}^{n} \mathbf{z}_j^{(k,h)} \lambda_j^k \left(\forall (k,h)\right) \quad \left(\text{as outputs from } k\right) \\
\mathbf{z}^{(k,h)} &= \sum\nolimits_{j=1}^{n} \mathbf{z}_j^{(k,h)} \lambda_j^h \left(\forall (k,h)\right) \quad \left(\text{as inputs to } h\right) \\
\sum\nolimits_{j=1}^{n} \lambda_j^k &= 1 \quad (\forall k), \quad \lambda_j^k \geq 0 \quad \left(\forall j,k\right)
\end{aligned}
\tag{11.1}
$$

where $\boldsymbol{\lambda}^k \in R_+^n$ is the intensity vector corresponding to Division k ($k = 1, \ldots, K$).

We notice that the above model assumes the variable returns-to-scale (VRS) for production. That is, the production frontiers are spanned by the convex hull of the existing DMUs. However, if we neglect the last constraint $\sum_{j}=1^n \lambda_j^k = 1(\forall\, k)$, we can deal with the constant returns-to-scale (CRS) case as well.

DMU $_o(o = 1, \ldots, n)$ can be represented by

$$
\begin{aligned}
\mathbf{x}_o^k &= \mathbf{X}^k \boldsymbol{\lambda}^k + \mathbf{s}_o^{k-} \quad (k = 1, \ldots, K) \\
\mathbf{y}_o^k &= \mathbf{Y}^k \boldsymbol{\lambda}^k - \mathbf{s}_o^{k+} \quad (k = 1, \ldots, K) \\
\mathbf{e}\boldsymbol{\lambda}^k &= 1 \quad (k = 1, \ldots, K) \\
\boldsymbol{\lambda}^k &\geq \mathbf{0}, \quad \mathbf{s}_o^{k-} \geq \mathbf{0}, \quad \mathbf{s}_o^{k+} \geq \mathbf{0}, \quad (\forall k)
\end{aligned}
\tag{11.2}
$$

where

$$
\begin{aligned}
\mathbf{X}^k &= \left(\mathbf{x}_1^k, \ldots, \mathbf{x}_n^k\right) \in R^{m_k \times n} \\
\mathbf{Y}^k &= \left(\mathbf{y}_1^k, \ldots, \mathbf{y}_n^k\right) \in R^{r_k \times n}.
\end{aligned}
\tag{11.3}
$$

and $\mathbf{s}_o^{k-}$ ($\mathbf{s}_o^{k+}$) are the input (output) slack vectors.

As regard to the linking constraints, we have several options of which we present two possible cases.

(a) The "free" link value case.

The linking activities are freely determined (discretionary) while keeping continuity between input and output:

$$
\mathbf{Z}^{(k,h)} \boldsymbol{\lambda}^h = \mathbf{Z}^{(k,h)} \boldsymbol{\lambda}^k. \quad (\forall (k,h)) \tag{11.4a}
$$

where

$$
\mathbf{Z}^{(k,h)} = \left(\mathbf{z}_1^{(k,h)}, \ldots, \mathbf{z}_n^{(k,h)}\right) \in R^{t_{(k,h)} \times n}. \tag{11.5}
$$

This case can serve to see if the current link flow is appropriate or not in the light of other DMUs', i.e. the link flow may increase or decrease in the optimal solution of the linear programs which we will introduce in the next section.

(b) The "fixed" link value case.
The linking activities are kept unchanged (non-discretionary):

$$\begin{aligned} \mathbf{z}_o^{(k,h)} &= \mathbf{Z}^{(k,h)}\boldsymbol{\lambda}^h \quad (\forall(k,h)) \\ \mathbf{z}_o^{(k,h)} &= \mathbf{Z}^{(k,h)}\boldsymbol{\lambda}^k. \quad (\forall(k,h)) \end{aligned} \tag{11.4b}$$

This case corresponds to the situation where the intermediate products are beyond the control of DMUs. However, if all link values are fixed, this case reduces structurally to the separation model described in Sect. 11.1.2 with non-discretionary inputs and outputs.

Throughout this chapter, we assume that all data are positive, since basically we employ the slacks-based measure (SBM) that demands positive data.

11.3.2 Efficiency

For each DMU_o, we define several efficiency scores depending on the selected orientation, input, output or non-oriented, as follows.

11.3.2.1 Input–Oriented Efficiency θ_o^*

We evaluate the input-oriented efficiency of DMU_o by solving the following linear program:

$$\theta_o^* = \min_{\lambda^k, s_o^{k-}} \sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k}\left(\sum_{i=1}^{m_k} \frac{s_{io}^{k-}}{x_{io}^k}\right)\right] \tag{11.6}$$
$$\text{subject to (11.2), and (11.4a) or (11.4b)}$$

where $\sum_{k=1}^{K} w^k = 1,\ w^k \geq 0\ (\forall\ k)$ and w^k is the relative weight of Division k which is determined corresponding to its importance, e.g. cost share and supplied exogenously.

This model is called the weighted SBM model (WSBM), an extension of the SBM. See Cooper et al. (2007) for details.

Definition 1 (Input-oriented overall efficiency)
We call θ_o^* the *overall input-efficiency* of DMU_o. If we have $\theta_o^* = 1$, the DMU_o is called *overall input-efficient*.

Definition 2 (Input-oriented divisional efficiency)
Using the optimal input slacks $\mathbf{s}_o^{k-*}$ of (11.6), we define the input-oriented divisional efficiency by

$$\theta_o^k = 1 - \frac{1}{m_k}\left(\sum\nolimits_{i=1}^{m_k} \frac{s_{io}^{k-*}}{x_{io}^k}\right) \quad (k = 1, \ldots, K). \tag{11.7}$$

θ_o^k is the divisional efficiency index which optimizes the overall efficiency θ_o^*. If we have $\theta_o^k = 1$, then the DMU_o is called *input-efficient* for the division k.

We notice that the above divisional efficiency score is not always uniquely determined,[1] although the overall efficiency is uniquely obtained as the linear program optimum. In Sect. 11.6.1, we present a scheme for deciding divisional efficiency scores uniquely.

The overall input-oriented efficiency score is the weighted arithmetic mean of the divisional scores

$$\theta_o^* = \sum\nolimits_{k=1}^{K} w^k \theta_o^k. \tag{11.8}$$

This measure is useful for comparing the total productivity of DMU_o among the concerned DMUs. It will serve not only managers but also regulatory agencies to compare DMUs in the firm-level view point.

11.3.2.2 Output-Oriented Efficiency τ_o^*

We evaluate the output-oriented efficiency of DMU_o by solving the following linear program:

$$\begin{aligned} 1/\tau_o^* = \max_{\lambda^k, s_o^{k+}} & \sum\nolimits_{k=1}^{K} w^k \left[1 + \frac{1}{r_k}\left(\sum\nolimits_{r=1}^{r_k} \frac{s_{ro}^{k+}}{y_{ro}^k}\right)\right] \\ & \text{subject to } (2), \text{and } (4\text{a}) \text{ or } (4\text{b}) \end{aligned} \tag{11.9}$$

where $\sum_{k=1}^{K} w^k = 1$, $w^k \geq 0\ (\forall\ k)$, and w^k is the relative weight of Division k which is determined corresponding to its importance.

Definition 3 (Output-oriented overall efficiency)
We call τ_o^* the *overall output-efficiency* of DMU_o. If we have $\tau_o^* = 1$, the DMU_o is called overall output-efficient.

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[1] In order to see the range in which a divisional efficiency may vary, we can solve the maximum and the minimum of θ_o^k subject to (11.2), (11.4a) or (11.4b) while keeping the overall efficiency at the optimal value θ_o^*.

Definition 4 (Output-oriented divisional efficiency)
In order to confine the score into the range [0, 1], we define the output-oriented divisional efficiency score by

$$\tau_o^k = \frac{1}{1 + \frac{1}{r_k}\left(\sum_{r=1}^{r_k} \frac{s_{ro}^{k+*}}{y_{ro}^k}\right)} \quad (k = 1, \ldots, K). \tag{11.10}$$

where $\mathbf{s}_o^{k+*}$ is the optimal output-slacks for (11.9).

The output-oriented overall efficiency score is the weighted harmonic mean of the divisional scores

$$\frac{1}{\tau_o^*} = \sum_{k=1}^{K} \frac{w^k}{\tau_o^k}. \tag{11.11}$$

11.3.2.3 Non-oriented Efficiency ρ_o^*

Accounting for both input and output slacks, we can evaluate the non-oriented efficiency of DMU$_o$ as follows:

$$\rho_o^* = \min_{\lambda^k, s_o^{k-}, s_o^{k+}} \frac{\sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k}\left(\sum_{i=1}^{m_k} \frac{s_{io}^{k-}}{x_{io}^k}\right)\right]}{\sum_{k=1}^{K} w^k \left[1 + \frac{1}{r_k}\left(\sum_{r=1}^{r_k} \frac{s_{ro}^{k+}}{y_{ro}^k}\right)\right]} \tag{11.12}$$
$$\text{subject to } (2), \text{and } (4a) \text{ or } (4b).$$

where $\sum_{k=1}^{K} w^k = 1, \ \ w^k \geq 0 \ (\forall\, k)$, and w^k is the relative weight of Division k which is determined corresponding to its importance.[2] We can solve this problem by transforming into a linear program using Charnes and Cooper transformation (see Tone 2001).

Definition 5 (Non-oriented overall efficiency)
We call ρ_o^* the *non-oriented overall efficiency* of DMU$_o$. If we have $\rho_o^* = 1$, the DMU$_o$ is called *overall efficient*.

[2] Although other forms of the objective function might be possible, we chose (11.12) for alignment with the non-oriented SBM model proposed in Tone (2001). This form serves to interpret its dual linear programming problem as the virtual profit efficiency model (see (Tone 2001)).

Definition 6 (Non-oriented divisional efficiency)
We define the non-oriented divisional efficiency score by

$$\rho_o^k = \frac{1 - \frac{1}{m_k}\left(\sum_{i=1}^{m_k} \frac{s_{io}^{k-*}}{x_{io}^k}\right)}{1 + \frac{1}{r_k}\left(\sum_{r=1}^{r_k} \frac{s_{ro}^{k+*}}{y_{ro}^k}\right)} \quad (k = 1, \ldots, K). \tag{11.13}$$

where $\mathbf{s}_o^{k-*}$ and $\mathbf{s}_o^{k+*}$ are optimal input- and output-slacks for (11.12).

The overall non-oriented efficiency score is a weighted mean of the divisional efficiency scores but is neither their arithmetic nor their harmonic mean.

We notice that the above divisional and overall efficiencies are units-invariant, i.e. they are independent of the units in which the inputs, outputs and links are measured.

Since the number of inputs and outputs may differ division by division and DEA scores are affected by the number, i.e. large number tends to give a high average score, care is needed in comparing divisional scores mutually.

Comparing the results by (11.4a) and (11.4b), we can see how the linking activities exert influence over the efficiency of each division.

11.3.3 Projection

Let an optimal solution to (11.6), (11.9) or (11.12) be ($\boldsymbol{\lambda}^{*\ k}$,$\mathbf{s}_o^{k-*}$,$\mathbf{s}_o^{k+*}$). Then we have the projection onto the frontier as follows:

$$\begin{aligned} \mathbf{x}_o^{k*} &\leftarrow \mathbf{x}_o^k - \mathbf{s}_o^{k-*} \quad (k = 1, \ldots, K) \\ \mathbf{y}_o^{k*} &\leftarrow \mathbf{y}_o^k + \mathbf{s}_o^{k+*}. \quad (k = 1, \ldots, K) \end{aligned} \tag{11.14}$$

If we employ the constraints (11.4b) for links, then the link values are unchanged (fixed). If we utilize the constraints (11.4a) (free link case), then we have the projection as follows:

$$\mathbf{z}_o^{(k,h)*} \leftarrow \mathbf{Z}^{(k,h)}\boldsymbol{\lambda}^{k*}. \quad (\forall (k,h)) \tag{11.15}$$

11.3.4 Reference Set

Using the optimal intensity vector $\boldsymbol{\lambda}^{*k}$ we have:

Definition 7 (Reference set)
We define the reference set of the division k for DMU_o by

$$\mathrm{R}_o^k = \left\{ j \middle| \lambda_j^{k*} > 0 \right\} \quad (j \in \{1, \ldots, n\}). \tag{11.16}$$

Using this notation we can express $\mathbf{x}_o^k$ and $\mathbf{y}_o^k$ as

$$\mathbf{x}_o^k = \sum_{j \in \mathrm{R}_o^k} \mathbf{x}_j^k \lambda_j^{k*} + \mathbf{s}_o^{k-*}, \quad \mathbf{y}_o^k = \sum_{j \in \mathrm{R}_o^k} \mathbf{y}_j^k \lambda_j^{k*} - \mathbf{s}_o^{k+*}. \tag{11.17}$$

11.4 Several Properties of Slacks-Based Network DEA Models

In this section we discuss several properties of the slacks-based NDEA models.

11.4.1 Overall Versus Divisional Efficiencies

We have defined the overall efficiencies corresponding to input, output and non-oriented orientations by (11.6), (11.9) and (11.12), and then the divisional efficiencies corresponding to these models are defined respectively by (11.7), (11.10) and (11.13).

Between the overall and divisional efficiencies we have:

Theorem 1 *A DMU is overall efficient if and only if it is efficient for all divisions.*

We notice that it can happen that there exists no overall efficient unit, in contrast to the traditional DEA models (see examples in Sect. 11.5.3), and furthermore that in a certain NDEA model some division may have no divisional efficient DMUs (see an example in Sect. 11.5.4).

11.4.2 Divisional Efficiency

Let us denote the sets of inputs, outputs, incoming links and outgoing links for Division k, respectively by $\mathbf{X}^k = \{\mathbf{x}_j^k\}$, $\mathbf{Y}^k = \{\mathbf{y}_j^k\}$, $\mathbf{Z}^{(pk)} = \{\mathbf{z}_j^{(pk)}\}$ and $\mathbf{Z}^{(kq)} = \{\mathbf{z}_j^{(kq)}\}$ where $j = 1, \ldots, n$. We notice that some of inputs and outputs may be vacant. However, we assume that all divisions in the model are at least indirectly connected by links.

In this section, we demonstrate that under the variable returns-to-scale (VRS) assumption every division has at least one divisionally efficient DMU. However, the constant returns-to-scale (CRS) cases are mixed. For the fixed link case under CRS, every division has at least one divisionally efficient DMU whereas for the free link case under CRS it is possible that some division has no divisionally efficient DMU.

11.4.2.1 The Variable Returns-to-Scale (VRS) Case

Under the VRS assumption, we have the following theorem:

Theorem 2 *Under the variable returns-to-scale assumption, every division has at least one divisionally efficient DMU.*

Proof We sort the n DMUs in the division k in ascending order in input values using Input i as the ith key. We further sort the resultant in descending order in output values using Output r as the $m_k + r$ th key. Then the lexicographical minimum (top) DMU has $\mathbf{s}_o^{k-} = \mathbf{0}$ and $\mathbf{s}_o^{k+} = \mathbf{0}$ for every feasible $\boldsymbol{\lambda}^k$ under the VRS assumption, even if there are tied DMUs. Thus, the division has at least one efficient DMU regardless of the orientation.

Q.E.D

11.4.2.2 The Constant Returns-to-Scale (CRS) Case

For the CRS assumption, we have two options; the free link case (11.4a) and the fixed link case (11.4b). For the later we have:

Theorem 3 *Under the constant returns-to-scale assumption with the fixed link case, every division has at least one divisionally efficient DMU.*

Proof As we noticed earlier in Sect. 11.3.1, the fixed link case reduces to the separation model with non-discretionary inputs and outputs corresponding to the fixed links. Hence we can solve this case separately division by division. Therefore, every division has at least one efficient DMU in the division.

Q.E.D.

As a consequence of the separation model, we have:

Corollary 1 *For the fixed link case, DMUs in the reference set are divisionally efficient.*

So far, we have demonstrated the existence of the divisionally efficient ($\theta_o^k = 1$) DMU for slacks-based NDEA models under the VRS assumption as well as for the CRS with the fixed link case. The remaining is the case of the CRS with the free link. In Sect. 11.5.4, we show a counter example that has no divisionally efficient DMU in this case.

11.4.3 Efficiency of the Projected DMU

We defined the projection of DMU_o by (11.14) and (11.15) (free link case).

Theorem 4 *The projected DMU is overall efficient.*

Proof We prove the theorem in the input-oriented case.

We evaluate the efficiency of the projected DMU ($\mathbf{x}_o^{k*}$,$\mathbf{y}_o^{k*}$,$\mathbf{z}_o^{k*}$) ($k = 1, \ldots, K$). Let an optimal solution be $\left(\overline{\boldsymbol{\lambda}}^{k*}, \overline{\mathbf{s}}_o^{k-*}, \overline{\mathbf{s}}_o^{k+*}\right)(k = 1, \ldots, K)$. Then we have:

$$\mathbf{x}_o^{k*} = \mathbf{X}^k\overline{\boldsymbol{\lambda}}^{k*} + \overline{\mathbf{s}}_o^{k-*}, \mathbf{y}_o^{k*} = \mathbf{Y}^k\overline{\boldsymbol{\lambda}}^{k*} - \overline{\mathbf{s}}_o^{k+*}, \mathbf{z}_o^{(k,h)*} = \mathbf{Z}^k\overline{\boldsymbol{\lambda}}^{k*}. \tag{11.18}$$

Replacing ($\mathbf{x}_o^{k*}$,$\mathbf{y}_o^{k*}$) by (11.14), we have

$$\mathbf{x}_o^k = \mathbf{X}^k\overline{\boldsymbol{\lambda}}^{k*} + \overline{\mathbf{s}}_o^{k-*} + \mathbf{s}_o^{k-*} \text{ and } \mathbf{y}_o^k = \mathbf{Y}^k\overline{\boldsymbol{\lambda}}^{k*} - \overline{\mathbf{s}}_o^{k+*} - \mathbf{s}_o^{k+*}. \tag{11.19}$$

Corresponding to this expression we have the overall efficiency:

$$\overline{\rho}_o = \min\sum_{k=1}^{K} w^k\left[1 - \frac{1}{m_k}\left(\sum_{i=1}^{m_k} \frac{s_{io}^{k-*} + \overline{s}_{io}^{k-*}}{x_{io}^k}\right)\right]. \tag{11.20}$$

If any member of $\left\{\overline{\mathbf{s}}_o^{k-*}\right\}(k = 1, \ldots, K)$ is positive, then it holds that

$$\overline{\rho}_o < \rho_o^*. \tag{11.21}$$

This contradicts the optimality of ρ_o^*. Thus, we have $\overline{\mathbf{s}}_o^{k-*} = \mathbf{0}\ \ (k = 1, \ldots, K)$. Hence, the projected DMU is overall efficient.

Similarly we can prove the theorem in the output-oriented and non-oriented models.

Q.E.D.

11.5 Illustrative Examples

We present an illustrative example of electric power companies for describing slacks-based Network DEA and compare the results with traditional approaches. Also we demonstrate an example with the free link case that has no divisionally efficient DMUs.

11.5.1 Data

As introduced in Sect. 11.2 (Fig. 11.4), the vertically integrated electric power companies consist of several divisions such as generation, transmission and distribution. For illustrative purpose, we choose ten vertically integrated power companies in the U.S. in 1994 obtained from "Form No.1" published by the Federal Energy Regulatory Commission (FERC). The inputs, outputs and links are as follows:

Table 11.1 Sample data

	Div1	Div2		Div3		Link	
DMU	Input1	Input2	Output2	Input3	Output3	Link12	Link23
A	0.838	0.277	0.879	0.962	0.337	0.894	0.362
B	1.233	0.132	0.538	0.443	0.18	0.678	0.188
C	0.321	0.045	0.911	0.482	0.198	0.836	0.207
D	1.483	0.111	0.57	0.467	0.491	0.869	0.516
E	1.592	0.208	1.086	1.073	0.372	0.693	0.407
F	0.79	0.139	0.722	0.545	0.253	0.966	0.269
G	0.451	0.075	0.509	0.366	0.241	0.647	0.257
H	0.408	0.074	0.619	0.229	0.097	0.756	0.103
I	1.864	0.061	1.023	0.691	0.38	1.191	0.402
J	1.222	0.149	0.769	0.337	0.178	0.792	0.187
Average	1.020	0.127	0.763	0.560	0.273	0.832	0.290

Generation (Div 1):

Input 1 = Labor input (number of employees)

Transmission (Div 2):

Input 2 = Labor input (number of employees)
Output 2 = Electric power sold to large customers

Distribution (Div 3):

Input 3 = Labor input (number of employees)
Output 3 = Electric power sold to small customers

Link (1–2) = Electric power generated

(Output from Generation Division and Input to Transmission Division)

Link (2–3) = Electric power distributed

(Output from Transmission Division and Input to Distribution Division)

Table 11.1 exhibits data for inputs, outputs and links of the ten DMUs; A to J. Numbers in each column of the table are obtained from the source data by dividing some standard of the column. So we do not denote the units. This has no effect on the efficiency scores, since all DEA models employed are units-invariant.

11.5.2 Results of Black Box and Separation Models

First, we solved the aggregated (black box) model explained in Sect. 11.1.1, using Inputs 1, 2 and 3, and Outputs 2 and 3 where Links were neglected (see Fig. 11.2). Throughout this section, we utilized the input-oriented SBM (slacks-based measure) under the variable returns-to-scale (VRS) assumption for evaluating

Table 11.2 SBM scores for black box and separation models

DMU	Aggregation (Black box)	Separation: Overall score[a]	Divisional score: Div1	Div2	Div3
A	1.000	0.659	0.633	0.662	0.684
B	0.531	0.657	0.260	0.763	1.000
C	1.000	0.984	1.000	1.000	0.959
D	1.000	0.719	0.297	1.000	1.000
E	1.000	0.547	0.202	1.000	0.665
F	0.681	0.844	1.000	0.635	0.792
G	1.000	0.855	0.712	1.000	0.926
H	1.000	0.893	0.787	0.890	1.000
I	1.000	0.915	1.000	1.000	0.786
J	1.000	0.640	0.263	0.672	1.000
Average	0.921	0.771	0.615	0.862	0.881

[a]Overall score indicates 0.4*Div1 + 0.2*Div2 + 0.4* Div3

efficiency (see (Cooper et al. 2007)). The column "Aggregation (Black box)" in Table 11.2 reports the results.

Next, we solved the separation model explained in Sect. 11.1.2.[3] Table 11.2 reports the results where "Overall score" indicates the weighted average 0.4 × Div1 + 0.2 × Div2 + 0.4 × Div3. The numbers 0.4, 0.2 and 0.4 are weights to Div 1, Div 2 and Div 3, respectively, which are utilized in the following Network DEA model too. This weight selection is just for illustrative purpose. No significant correlation is observed between the two efficiencies; Aggregation and Overall. This is quite natural, since we neglected the internal linking activities in the former.

The scores of the black box model tend to be higher than those of the separation model (Fig. 11.8). Actually, these two models cannot be fairly comparable, because the number of inputs is different between the two models. However, this figure clearly explains that the discriminate power of the black box model is inferior to that of the separation model. In addition, it shows that the ranks of the scores of the two models are not always corresponding, e.g. F is scored worse in the black box model, while better in the separation model.

11.5.3 Results of Slacks-Based Network DEA

We now return to the Network DEA model taking account the links inside the black box. We minimize the objective function (11.6) subject to the constraints (11.2),

[3] In solving the separation model, links were treated as ordinary (discretionary) inputs or outputs as explained in Sect. 1.2, and hence the continuity of link values between divisions were not assured. Also, the separation model takes into account the inefficiency associated with the link variables, whereas the NSBM does not.

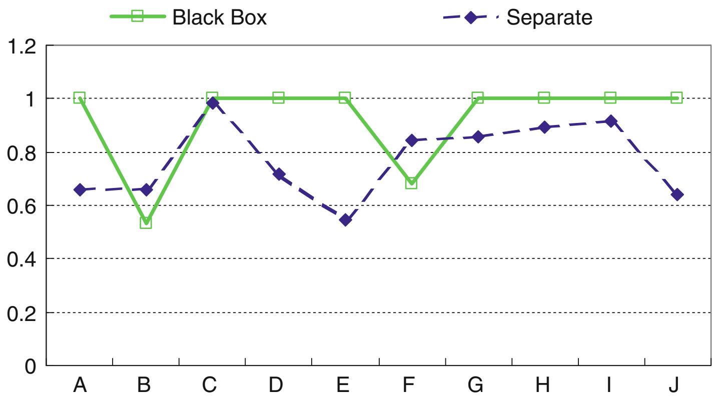

Fig. 11.8 Comparisons of scores between black box and separation models

Table 11.3 Slacks-based network DEA: fixed link case

DMU	Overall score	Divisional score			Reference			Link	
		Div1 (0.4)	Div2 (0.2)	Div3 (0.4)	Div1	Div2	Div3	Link12	Link23
A	0.478	0.633	0.339	0.393	C1,F1	C2,D2,E2,I2	D3,H3	0.894	0.362
B	0.739	0.349	1.000	1.000	C1,G1	B2	B3	0.678	0.188
C	0.968	1.000	1.000	0.919	C1	C2	B3,D3,J3	0.836	0.207
D	0.719	0.297	1.000	1.000	C1,F1	D2	D3,H3	0.869	0.516
E	0.456	0.263	1.000	0.377	C1,G1	E2	D3,H3	0.693	0.407
F	0.719	1.000	0.403	0.596	F1	C2,H2,I2	D3,H3	0.966	0.269
G	0.947	1.000	1.000	0.868	G1	G2	D3,H3	0.647	0.257
H	0.969	0.922	1.000	1.000	C1,G1	H2	H3	0.756	0.103
I	0.832	1.000	1.000	0.581	I1	I2	D3,H3	1.191	0.402
J	0.590	0.288	0.377	1.000	C1,G1	C2,G2,H2	J3	0.792	0.187
Average	0.742	0.675	0.812	0.773				0.832	0.29

and (11.4a) or (11.4b), i.e. the input-oriented network model under VRS assumption. As weights to objective function, we employ $w^1 = 0.4$ (Division 1), $w^2 = 0.2$ (Division 2) and $w^3 = 0.4$ (Division 3). This set of weights conforms to the above weights in Sect. 11.5.2. The results of the fixed link case (11.4b) are displayed in Table 11.3 while the free link case (11.4a) is exhibited in Table 11.4 where the overall efficiency (θ_o^*) together with divisional efficiencies is displayed.[4] The divisional efficiency means the individual term (11.7) in the objective function. In the "Reference" column, A1 indicates DMU A in the Division 1. This means $\lambda_A^1 > 0$ in the optimal solution. Since the constraint (11.4a) is tighter than (11.4b),

[4] We checked the uniqueness of the divisional efficiency scores as described in Footnote 1 and found no alternate optima.

Table 11.4 Slacks-based network DEA: free link case

DMU	Overall score	Divisional score			Reference			Projected link			
		Div1(0.4)	Div2(0.2)	Div3(0.4)	Div1	Div2	Div3	Link12	Link12/Data	Link23	Link23/Data
A	0.385	0.383	0.383	0.389	C1	C2,D2,E2,I2	D3,H3	0.836	0.935	0.355	0.979
B	0.433	0.260	0.341	0.652	C1	C2	D3,H3	0.836	1.233	0.207	1.101
C	0.968	1.000	1.000	0.919	C1	C2	B3,D3,J3	0.836	1.000	0.207	1.000
D	0.719	0.297	1.000	1.000	C1,F1	D2	D3,H3	0.869	1.000	0.516	1.000
E	0.456	0.263	1.000	0.377	C1,G1	E2	D3,H3	0.693	1.000	0.407	1.000
F	0.484	0.406	0.420	0.593	C1	C2,D2,G2	D3,H3	0.836	0.865	0.267	0.991
G	0.778	0.712	0.740	0.863	C1	C2,D2,G2	D3,H3	0.836	1.292	0.254	0.988
H	0.969	0.922	1.000	1.000	C1,G1	H2	H3	0.756	1.000	0.103	1.000
I	0.832	1.000	1.000	0.581	I(1)	I2	D3,H3	1.191	1.000	0.402	1.000
J	0.506	0.271	0.338	0.825	C1,G1	C2,H2	D3,H3	0.821	1.037	0.188	1.005
Average	0.653	0.551	0.722	0.720				0.851	1.036	0.291	1.006

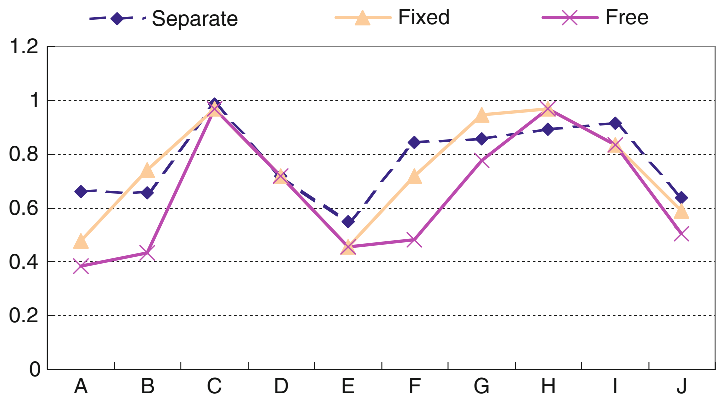

Fig. 11.9 Comparisons of scores among separate and two network models

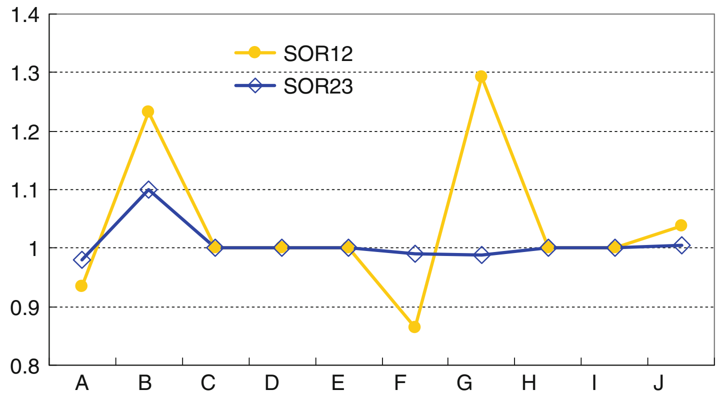

Fig. 11.10 Link effect ratio

the overall score of the former (fixed link case) is larger than that of the latter (free link case) for every DMU.

Figure 11.9 compares scores of the separate model and network models (fixed and free link cases). The trends of three models are roughly similar but exhibit sharp contrast to that of the black box model explained in Fig. 11.8. However, we can find gaps among three models, which must be caused by the difference of assumption on the links among divisions. As we mentioned, the separation model does not take account of the links, and therefore, the gap between the separation and network models implies the “linking effects”. The separation model cannot catch the full story in the case when the linking effects inside DMUs actually exist.

Concerning two network models, the scores of the fixed link case exceed or equal to those of the free case. The gap of two models explains “link effects”. Figure 11.10 shows the “Link effect ratio (LER)” of links measured as projected

Table 11.5 Data for four DMUs

	Div1	Div2		Div3		Link	
DMU	Input1	Input2	Output2	Input3	Output3	Link12	Link23
K	3	10	2	5	2	8	2
L	14	1	1	5	5	9	5
M	16	2	2	11	4	7	4
N	19	0.5	2	7	4	11	4

Table 11.6 Results of the input-oriented free link CRS model

DMU	Overall Score	Div1(0.4)	Div2(0.2)	Div3(0.4)
K	0.71	0.875	0.2	0.8
L	0.6723	0.3683	0.625	1
M	0.2986	0.2578	0.25	0.3636
N	0.5154	0.2171	1	0.5714

links in free case divided by actual links (see Table 11.4). If there exists the gap between the two network models in Fig. 11.10, the link effect ratio is not equal to unity, and if the ratio is larger than unity, the DMU should increase the link value, and vice versa.

11.5.4 Example with No Divisionally Efficient DMUs

As we have demonstrated in Theorems 2 and 3, the VRS models and the fixed link CRS model have at least one efficient DMU within every division. However, as to the free link CRS model, the proposition is not always effected. In this section, we exhibit a counter example which has no efficient DMU within a certain division. We observe 4 DMUs with the same network structure as the previous example. Table 11.5 exhibits the data. We solved this problem using the input-oriented free link NDEA model under the CRS assumption and obtained the results exhibited in Table 11.6. We found no efficient DMU in Division 1, while other divisions have an efficient DMUs; N for Division 2 and L for Division 3. This indicates that all DMUs in Division 1 need improvement. Table 11.7 reports the projection of inputs, outputs and links onto the efficient frontiers by the formulas (11.14) and (11.15). Actually, all inputs to Division 1 are reduced proportionally to their scores of Division 1. On the other hand, other divisions and links have benchmarks that remain unchanged in the projection. This occurrence of vacancy of divisionally efficient DMUs in some division is one of characteristics of this model which cannot be expected by traditional DEA models.

Table 11.7 Projection onto efficient frontiers

DMU	Div1		Div2				Div3				Link			
	Input1	Prj[a]	Input2	Prj	Output2	Prj	Input3	Prj	Output3	Prj	Link12	Prj	Link23	Prj
K	3	2.625	10	2	2	2	5	4	2	4	8	7	2	4
L	14	5.156	1	0.625	1	2.5	5	5	5	5	9	13.75	5	5
M	16	4.125	2	0.5	2	2	11	4	4	4	7	11	4	4
N	19	4.125	0.5	0.5	2	2	7	4	4	4	11	11	4	4

[a]Prj indicates projection onto efficient frontiers

11.6 Extensions

In this section, we introduce several extensions of the NDEA model.

11.6.1 *Uniqueness Issue of Divisional Efficiencies*

Although the overall efficiency is uniquely determined by the program (11.6) in the input-oriented model, slacks are not necessarily unique. Hence, the divisional efficiency in (11.7) may suffer from non-uniqueness issue.

In the model (11.6), we use w^k as the relative weight of Division k, which reflects importance of each division. Based on w^k, we can prioritize divisions. Under this priority principle, we propose the following scheme for overcoming this non-uniqueness problem. If any other priority rule exists, we can cope with it in the similar way.

For convenience sake, here we define the last division K has the top priority and those of $K-1$, $K-2$,. . ., 1 decrease in this order.

11.6.1.1 Divisional Efficiency in K

First, we solve the program (11.6) and obtain the overall efficiency θ_o^*. Then we minimize divisional efficiency in K while keeping the overall efficiency at θ_o^*.

Let us denote the divisional efficiency in K thus obtained by θ_o^{K*}.

$$\theta_o^{K*} = \min\ 1 - \frac{1}{m_K}\left(\sum\nolimits_{i=1}^{m_K} \frac{s_{io}^{K-*}}{x_{io}^K}\right) \tag{11.22}$$

subject to

$$\sum\nolimits_{k=1}^{K} w^k \left[1 - \frac{1}{m_k}\left(\sum\nolimits_{i=1}^{m_k} \frac{s_{io}^{k-}}{x_{io}^k}\right)\right] = \theta_o^* \tag{11.23}$$

and (11.2), (11.4a) or (11.4b).

11.6.1.2 Divisional Efficiency in k

We repeat this process in the descending order of priority until $k = 2$. Thus, divisional efficiency in k (θ_o^{k*}) is measured by the following program.

$$\theta_o^{k*} = \min 1 - \frac{1}{m_k}\left(\sum\nolimits_{i=1}^{m_K} \frac{s_{io}^{k-*}}{x_{io}^k}\right) \tag{11.24}$$

subject to

$$1 - \frac{1}{m_K}\left(\sum\nolimits_{i=1}^{m_K} \frac{s_{io}^{K-}}{x_{io}^{K}}\right) = \theta_o^{K*}$$
$$\vdots \tag{11.25}$$
$$1 - \frac{1}{m_k}\left(\sum\nolimits_{i=1}^{m_k} \frac{s_{io}^{k-}}{x_{io}^{k}}\right) = \theta_o^{k+1*}$$

and (11.2), (11.4a) or (11.4b), and (11.23).

Divisional efficiency in the division 1 can be obtained from θ_o^*, θ_o^{K*}, ..., θ_o^{2*}.

Through this scheme, we can obtain unique divisional efficiency scores θ_o^{k*} $(\forall k)$ for the input-oriented model. As for the output-oriented and non-oriented models, we can develop similar processes for uniqueness issues.

11.6.2 Incorporation of Link Flows in Efficiency Measurements

In the above cases, link flows do not directly concern with the objective function. They are related with efficiency scores only through link constraints (11.4a) or (11.4b). However, if we want to account their excesses (in the input-oriented case) or shortfalls (in the output-oriented case) into the objective function, we can modify the model as follows.

(i) In the input-oriented case, we consider the slacks of the link (f,k) as input to Division k and set link constraints as

$$\begin{aligned} \mathbf{z}_o^{(f,k)} &= \mathbf{Z}^{(f,k)}\boldsymbol{\lambda}^k + \mathbf{s}_o^{(f,k)-} \\ \mathbf{Z}^{(f,k)}\boldsymbol{\lambda}^f &= \mathbf{Z}^{(f,k)}\boldsymbol{\lambda}^k \\ \mathbf{s}_o^{(f,k)-} &\geq \mathbf{0} \end{aligned} \tag{11.4c}$$

The objective function is modified as:

$$\theta_o^* = \min \sum\nolimits_{k=1}^{K} w^{k-}\left[1 - \frac{1}{m_k + \sum_{f\in P_k} t_{(f,k)}}\left(\sum\nolimits_{i=1}^{m_k} \frac{s_{io}^{k-}}{x_{io}^{k}} + \sum\nolimits_{f\in P_k} \frac{s_{fo}^{(f,k)-}}{z_{fo}^{(f,k)}}\right)\right]$$
subject to (2) and (4c),
$$\tag{11.26}$$

where $\sum_{k=1}^{K} w^{k-} = 1$, $w^{k-} \geq 0$ $(\forall\ k)$ and P_k is the set of divisions having the link $(f,k) \in L$ (antecessor of Division k) and $t_{(f,k)}$ is the number of items in the link. We optimize (11.26) in terms of $\{\boldsymbol{\lambda}^k\}$, $\{\mathbf{s}_o^{k-}\}$ and $\{\mathbf{s}_o^{(f,k)-}\}$.

(ii) In the output-oriented case, we consider the slacks of the link (k,h) as output from Division k and set link constraints as;

$$\begin{aligned} \mathbf{z}_o^{(k,h)} &= \mathbf{Z}^{(k,h)}\boldsymbol{\lambda}^k - \mathbf{s}_o^{(k,h)+} \\ \mathbf{Z}^{(k,h)}\boldsymbol{\lambda}^h &= \mathbf{Z}^{(k,h)}\boldsymbol{\lambda}^k \\ \mathbf{s}_o^{(k,h)+} &\geq \mathbf{0} \end{aligned} \tag{11.4d}$$

The objective function is modified to

$$1/\tau_o^* = \max\sum_{k=1}^{K} w^k \left[1 + \frac{1}{r_k + \sum_{h\in F_k} t_{(k,h)}}\left(\sum_{r=1}^{r_k}\frac{s_{ro}^{k+}}{y_{ro}^k} + \sum_{h\in F_k}\frac{s_{ho}^{(k,h)+}}{z_{ho}^{(k,h)}}\right)\right]$$
$$\text{subject to (2) and (4d).} \tag{11.27}$$

where $\sum_{k=1}^{K} w^k = 1$, $w^k \geq 0$ ($\forall$ k) and F_k is the set of divisions having the link $(k,h) \in L$ (successor of Division k). We optimize (11.27) in terms of $\{\boldsymbol{\lambda}^k\}$, $\{\mathbf{s}_o^{k+}\}$ and $\{\mathbf{s}_o^{(k,h)+}\}$.

(iii) In the case that links are categorized into either input type (the less the better) or output type (the more the better), we can unify the above (i) and (ii) models into the non-oriented case in the similar way as the non-oriented model described in Sect. 11.3.2.3.

11.6.3 The Role of Intensity Vector λ

One of the characteristics of the NDEA is that it has an intensity vector $\boldsymbol{\lambda}^k = (\lambda_1^k, \ldots, \lambda_n^k)^T \in R^n$ ($\boldsymbol{\lambda}^k \geq \mathbf{0}$) specific to each Division k ($k = 1, \ldots, K$). We observe the role of this vector in this section.

11.6.3.1 The Identical Intensity Vector Case

In this case we assume a common intensity vector $\boldsymbol{\lambda} = \boldsymbol{\lambda}^k$ for every division k ($k = 1, \ldots, K$). Thus, DMU$_o$ can be expressed as follows:

$$\begin{aligned} \mathbf{x}_o^k &= \mathbf{X}^k\boldsymbol{\lambda} + \mathbf{s}_o^{k-} \\ \mathbf{y}_o^k &= \mathbf{Y}^k\boldsymbol{\lambda} - \mathbf{s}_o^{k+} \\ \mathbf{z}_o^{(k,h)} &= \mathbf{Z}^{(k,h)}\boldsymbol{\lambda} \\ \mathbf{e}\boldsymbol{\lambda} &= 1 \\ \boldsymbol{\lambda} \geq \mathbf{0},\ & \mathbf{s}_o^{k-} \geq \mathbf{0},\ \mathbf{s}_o^{k+} \geq \mathbf{0}. \end{aligned} \tag{11.28}$$

Now let us define matrices **X**, **Y** and **Z** as follows:

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}^1 \\ \mathbf{X}^2 \\ \vdots \\ \mathbf{X}^K \end{pmatrix} \in R_+^{(m_1+\cdots+m_K)\times n}, \mathbf{Y} = \begin{pmatrix} \mathbf{Y}^1 \\ \mathbf{Y}^2 \\ \vdots \\ \mathbf{Y}^K \end{pmatrix} \in R_+^{(r_1+\cdots+r_K)\times n}, \tag{11.29}$$

$$\mathbf{Z} = \left(\mathbf{Z}^{(k,h)} | (k,h) \in L\right) \in R_+^{\left(\sum_{(k,h)\in L} t_{(k,h)}\right)\times n}. \tag{11.30}$$

Using these notations, DMU_o can be expressed as,

$$\begin{aligned} \mathbf{x}_o &= \mathbf{X}\boldsymbol{\lambda} + \mathbf{s}_o^- \\ \mathbf{y}_o &= \mathbf{Y}\boldsymbol{\lambda} - \mathbf{s}_o^+ \\ \mathbf{z}_o &= \mathbf{Z}\boldsymbol{\lambda} \end{aligned} \tag{11.31}$$

where $\mathbf{s}_o^- = \left(s_o^{1-}, \ldots, s_o^{K-}\right)^T \in R^{m_1+\cdots+m_K}$ and $\mathbf{s}_o^+ = \left(s_o^{1+}, \ldots, s_o^{K+}\right)^T \in R^{r_1+\cdots+r_K}$.

Thus this case reduces to a traditional DEA model added by the last linking constraint. This model has $(m_1 + \cdots + m_K)$ inputs, $(s_1 + \cdots + s_K)$ outputs and $\sum_{(k,h)\,\in\, L} t_{(k,h)}$ linking constraints. In the case the sum of these numbers grows up to n (the number of DMUs), this model might lose discriminating power. As a rule of thumb, DEA demands that the number of DMUs should be at least three times larger than the sum of the number of inputs and outputs. The equality condition for the linking constraints will further narrow the feasible region and many DMUs may be judged as efficient in consequence.

11.6.3.2 Connectivity Among Divisions

In the preceding section, we have observed a special case regarding the decision variable $\boldsymbol{\lambda}$; identical. In this case, all divisions of DMU_o are evaluated by an identical set of referent DMUs, i.e. all divisions have the same benchmarks. In the NDEA models, however, benchmarks can vary division by division. Such diversified benchmarks among divisions might embrace supervisors in choosing peers to follow as a company.

These two extreme cases can be unified via the following connectivity index $\delta^{(h,k)}(\geq 0)$ $((h,k) \in L)$ as

$$\left|\lambda_j^h - \lambda_j^k\right| \leq \delta^{(h,k)} \quad (j = 1, \ldots n; (h,k) \in L) \tag{11.32}$$

The case $\delta^{(h,k)} = 0$ $(\forall\ (h,k))$ corresponds to the identical $\boldsymbol{\lambda}$, while the case $\delta^{(h,k)} = \infty$ $(\forall\ (h,k))$ corresponds to the independent $\boldsymbol{\lambda}$ setting, i.e. slacks-based NDEA models.

In our experiments with 56 Japanese (11.9) and the US (47) electric power companies, the identical **λ** case, i.e. $\delta^{(h,k)} = 0$ (∀ (h,k)), almost lost discriminating power and many companies were judged efficient, whereas $\delta^{(h,k)} = 0.01$ (∀ (h,k)) case demonstrated discrimination of efficiency and reasonable connectivity among divisions, i.e. compelling benchmarks. Appropriate setting of the connectivity index is an experimental issue. See Tsutsui (2007) for details.

11.7 Concluding Remarks

In this chapter, we have proposed a network DEA model based on the weighted SBM (WSBM) approach which accounts for the importance of each division. Thus, we can evaluate multi-divisional efficiencies and the overall efficiency in a unified framework.

The following subjects are discussed.

1. We have developed the slacks-based NDEA model under the fixed (non-discretionary) link and the free (discretionary) link assumptions. In the latter case, the optimal link values may increase or decrease from the observed ones. Comparisons of both results (fixed and free) give suggestions for improvements in the intermediate production policy. Thus, we can analyze economy and diseconomy of internal links by comparing fixed-link and free-link models.
2. We have proved that, under the VRS assumption, every division has at least one divisionally efficient DMU. This also holds for the case the fixed link under the CRS.
3. For the CRS and free link case, we have demonstrated a counter example in which a division has no divisionally efficient DMU. This may suggest improvements of the division as a whole. Also, it may reflect an unstable or unbalanced network structure in the problem of concern.

As an extension model of slacks-based NDEA to intertemporal analysis, Tone and Tsutsui (2010) proposed slacks-based dynamic DEA model, which takes account carry-over activities of DMUs such as retained earnings and facilities. These are incorporated into the model as input from the previous period and output to the next period.

Network and dynamic model is also proposed in Tone and Tsutsui (2014). Vertically, this model deals with multiple divisions connected by links of network structure within each period and, horizontally, it combines the network structure by means of carry-over activities between two succeeding periods.

Finally, we hope that these studies serve as a basis for extending theory and applications of DEA models which have been growing rapidly worldwide.

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