4 Forces and stresses

Previously in this book, we discussed the kinematic structure of a flow but made no reference to the external action that is necessary to establish a flow or to the physical mechanism that is necessary to sustain the motion of the fluid. To address these issues, in this chapter we turn our attention to the hydrodynamic forces developing in a fluid as a result of the motion and introduce constitutive equations relating the stresses developing at the surface of infinitesimal fluid parcels to the parcel motion and deformation. The constitutive equations will then be incorporated into an integrated theoretical framework based on Newton's law of motion that will allow us to compute the structure of a steady flow and the evolution of an unsteady flow from a specified initial configuration.

4.1 Forces acting in a fluid

Two types of forces are exerted on any coherent piece of a material: a homogeneous force acting on its volume, and a surface force acting on its boundaries.

4.1.1 Body force

A fluid parcel, like any other piece of material, is subject to a force mediated by an ambient gravitational, electrical, electromagnetic, or any other external force field acting on its volume. Electrical and electromagnetic forces arise when the fluid is electrically charged or contains molecules or small particles of a polarized material.

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Under the influence of such fields, the molecules residing inside a fluid parcel are acted upon individually and independently by a force that may be constant or vary with position inside the parcel. The sum of the forces exerted on the individual molecules amounts to a net body force that is proportional to the number of molecules residing inside the parcel, and thus to the parcel volume.

Gravitational body force

Let $\delta\mathbf{F}_p$ be the gravitational force exerted on a small fluid parcel with volume δV_p, density ρ, and mass $\delta m_p = \rho\,\delta V_p$. By definition,

$$\delta\mathbf{F}_p = \mathbf{g}\,\rho\,\delta V_p, \qquad (4.1.1)$$

where $\mathbf{g}$ is the acceleration of gravity. The right-hand side of (4.1.1) has units of acceleration multiplied by mass, which amounts to force.

One distinguishing feature of the body force due to gravity is that it is considered to be independent of molecular motions. This means that a certain mass of fluid weighs the same, independent of whether the fluid is stationary or flows.

4.1.2 Surface force

A different type of force arises at the surface of a fluid parcel and at the boundaries of a flow, such as the surface of a bubble rising through an ambient liquid or the windshield of a moving vehicle. More generally, a surface force can be defined on any fictitious surface that is drawn inside the bulk of a fluid or over its boundary.

Understanding the physical origin of the surface force requires consideration of molecular motions and necessitates a distinction between gases and liquids. A key idea is the equivalence between local hydrodynamic force and rate of exchange of momentum between adjacent fluid layers due to molecular excursions.

Gases

To understand the origin of surface forces developing in a gas, we draw a surface in the interior of the gas and consider the momentum of the molecules that cross the surface from either side. A key realization is that the momentum normal to the surface is responsible for a normal force.

If the molecules move with different average tangential velocities on either side of the surface, the net transport of tangential momentum is responsible for a tangential surface force necessary to accelerate or decelerate the molecules. In the present context, the average velocity of a molecule can be identified with the velocity of the fluid at the location where a molecule last underwent a collision with one of its peers. The effective force field due to the tangential surface force slows down fast-moving molecules as they approach regions of slower-moving fluid.

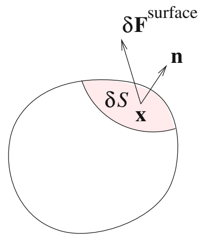


Figure 4.2.1 Illustration of a small section on the surface of a fluid parcel, δS, introduced to define the hydrodynamic traction exerted on the parcel.

Liquids

The physical origin of surface forces developing in a liquid is somewhat different. The molecules of a liquid perform oscillatory motion around a mean position with an amplitude that is determined by their distance from closely spaced neighbors. Occasional excursions into vacant spots are responsible for momentum transport attributed to the action of a surface force.

PROBLEM

4.1.1 *Friction*

The friction on a body sliding over a horizontal surface imparts to the body a tangential surface force that depends on the body weight. Does this frictional force also depend on the contact area?

4.2 Traction and the stress tensor

Consider a small surface with area δS centered at a point, $\mathbf{x} = (x, y, z)$, in a stationary or moving fluid, as illustrated in Figure 4.2.1. The designated outer side of the surface is indicated by the direction of the unit vector normal to the surface at the point $\mathbf{x}$, denoted by $\mathbf{n} = (n_x, n_y, n_z)$. According to our discussion in Section 4.1, a body of fluid whose instantaneous boundary includes the small surface under consideration experiences a surface force, $\delta\mathbf{F}^{\text{surface}}$, that may point in any direction; that is, it may have a component normal to the surface and a component tangential to the surface.

Traction

The ratio between the surface force, $\delta\mathbf{F}^{\text{surface}}$, and the area of the surface, δS, is the average stress exerted on the small surface. As the surface area δS becomes infinitesimal, the average stress tends to a limit defined as the traction exerted on an infinitesimal surface and is denoted by $\mathbf{f}$. Thus, by definition,

$$\mathbf{f} \equiv \frac{\delta\mathbf{F}^{\text{surface}}}{\delta S} \tag{4.2.1}$$

in the limit at δS becomes infinitesimal. The three scalar components of the traction have units of force per area, which amounts to stress.

Force in terms of traction

Rearranging equation (4.2.1), we obtain an expression for the surface force exerted on an infinitesimal surface in terms of the traction,

$$\delta \mathbf{F}^{\text{surface}} = \mathbf{f}\, \delta S. \tag{4.2.2}$$

Integrating the traction over a specified surface area, such as the boundary of a fluid parcel, we obtain a resultant surface force.

Dependence on position and orientation

It is clear from relation (4.2.1) that the traction is defined only when the location and orientation of an infinitesimal surface upon which the traction is exerted are specified, respectively, in terms of the coordinates of the center-point, $\mathbf{x}$, and orientation of the unit normal vector, $\mathbf{n}$. This requirement is signified by writing

$$\mathbf{f}(\mathbf{x}, \mathbf{n}), \tag{4.2.3}$$

where the parentheses enclose the arguments of the three scalar components of the traction. If a flow is unsteady, or the position or orientation of the surface change in time, time, t, should be added to the arguments on the right-hand side of (4.2.3).

The stress tensor

The traction exerted on a small surface that is perpendicular to the x, y, or z axis, is denoted by

$$\begin{aligned} \mathbf{f}^{(x)} &= [\, f_x^{(x)}, f_y^{(x)}, f_z^{(x)}\,], \\ \mathbf{f}^{(y)} &= [\, f_x^{(y)}, f_y^{(y)}, f_z^{(y)}\,], \\ \mathbf{f}^{(z)} &= [\, f_x^{(z)}, f_y^{(z)}, f_z^{(z)}\,], \end{aligned} \tag{4.2.4}$$

respectively, where the unit normal vector, $\mathbf{n}$ points in the positive directions of these axes, as depicted in Figure 4.2.2. Stacking these vectors on top of one another in a particular order, we obtain the 3×3 stress tensor

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} f_x^{(x)} & f_y^{(x)} & f_z^{(x)} \\ f_x^{(y)} & f_y^{(y)} & f_z^{(y)} \\ f_x^{(z)} & f_y^{(z)} & f_z^{(z)} \end{bmatrix} \tag{4.2.5}$$

in a three-dimensional flow, and a corresponding 2×2 tensor in a two-dimensional flow.

Next, we introduce the standard two-index notation for the components of the stress tensor,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \tag{4.2.6}$$

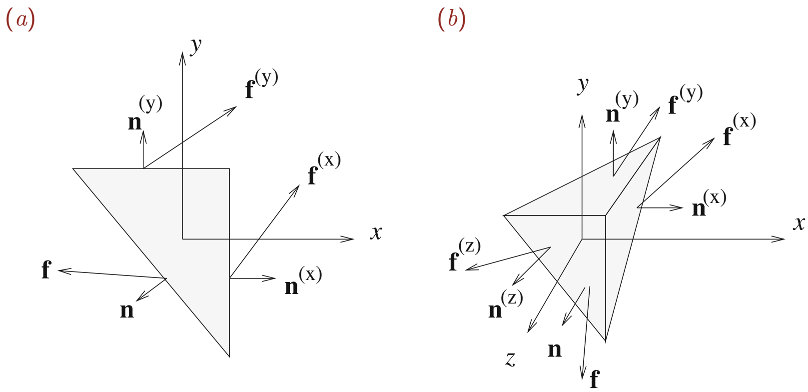


Figure 4.2.2 Illustration of (a) a triangular fluid parcel in a two-dimensional flow and (b) a tetrahedral fluid parcel in a three-dimensional fluid. These parcels are used as devices for computing the traction exerted on an arbitrary surface in terms of (a) the unit vector normal to the surface, and (b) the stress tensor.

and find that, by definition,

$$\sigma_{ij} \equiv f_j^{(i)} \tag{4.2.7}$$

for $i, j = x, y, z$. The first index of σ_{ij} indicates the component of the normal vector on the infinitesimal surface upon which the traction is exerted. The second index indicates the component of the corresponding traction.

We will see that, in the absence of an externally induced torque, the stress tensor is symmetric,

$$\sigma_{ij} = \sigma_{ji}. \tag{4.2.8}$$

For example, $\sigma_{xy} = \sigma_{yx}$.

In the case of two-dimensional flow in the xy plane, the stresses are encapsulated in a 2×2 stress tensor,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}. \tag{4.2.9}$$

The five omitted components involving the subscript z are either constant or zero.

Traction in terms of the stress tensor

We will demonstrate that the dependence of the traction on the position vector, $\mathbf{x}$, and normal vector, $\mathbf{n}$, displayed symbolically in (4.2.3), can be decoupled in a simple fashion, yielding

$$\mathbf{f}(\mathbf{x}, \mathbf{n}) = \mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{x}). \tag{4.2.10}$$

Specifically,

$$\big[\, f_x, f_y, f_z \,\big] = \big[\, n_x, n_y, n_z \,\big] \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}. \tag{4.2.11}$$

In index notation,

$$f_j(\mathbf{x}, \mathbf{n}) = n_i\,\sigma_{ij} = n_x\,\sigma_{xj} + n_y\,\sigma_{yj} + n_z\,\sigma_{zj}, \tag{4.2.12}$$

where summation is implied over the repeated index, i, in the middle expression of (4.2.12), while the index j is free to vary over x, y, or z.

An important consequence of (4.2.10) is that, if the nine components of the stress tensor are known at a point, then the traction exerted on any infinitesimal surface centered at that point can be evaluated in terms of the unit normal vector, $\mathbf{n}$, merely by carrying out a vector-matrix multiplication.

To confirm that expression (4.2.10) is consistent with the foregoing definitions, we choose $\mathbf{n} = (1, 0, 0)$ and carry out the vector-matrix multiplication on the right-hand side of (4.2.12) to find that $\mathbf{f} = \mathbf{f}^{(x)}$, as required. Working in a similar fashion with $\mathbf{n} = (0, 1, 0)$ and $\mathbf{n} = (0, 0, 1)$, we obtain $\mathbf{f} = \mathbf{f}^{(y)}$ and $\mathbf{f} = \mathbf{f}^{(z)}$, as required.

It remains to show that (4.2.10) holds true for general orientations of the unit normal vector, $\mathbf{n}$. For simplicity, we present the proof for two-dimensional flow in the xy plane with reference to the 2×2 stress tensor defined in (4.2.9).

Force balance on a small triangular parcel

Consider a small area of fluid enclosed by an infinitesimal triangle with two sides perpendicular to the x and y axes, as shown in Figure 4.2.2(a). Newton's second law of motion requires that the rate of change of the momentum of the fluid enclosed by the triangle is balanced by the forces exerted on the triangle. The forces include the body force and the surface force associated with the traction exerted on the three sides.

The momentum of the parcel and the body force exerted on the parcel are both proportional to the area of the triangle, $\frac{1}{2}\,\Delta x\,\Delta y$. The surface force exerted on the vertical side is equal to $\mathbf{f}^{(x)}\Delta y$, the surface force exerted on the horizontal side is equal to $\mathbf{f}^{(y)}\Delta x$, and the surface force exerted on the slanted side is equal to $\mathbf{f}\Delta\ell$, where $\Delta\ell$ is the length of the slanted side, $\Delta\ell = (\Delta x^2 + \Delta y^2)^{1/2}$.

In the limit as Δx and Δy tend to zero, the fluid momentum and the body force become negligible compared to the surface force exerted on the sides, and the sum of the three surface forces must balance to zero. Setting the x and y components of the sum to zero, we obtain

$$f_x\,\Delta\ell + \sigma_{xx}\,\Delta y + \sigma_{yx}\,\Delta x = 0, \qquad f_y\,\Delta\ell + \sigma_{xy}\,\Delta y + \sigma_{yy}\,\Delta x = 0. \tag{4.2.13}$$

Using elementary trigonometry, we find that the x and y components of the outward unit

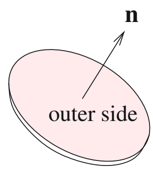


Figure 4.2.3 Illustration of a thin fluid layer with a designated inner and outer side. The outer side is indicated by the direction of the unit normal vector, $\mathbf{n}$.

vector normal to the slanted side of the triangle are given by

$$n_x = -\frac{\Delta y}{\Delta \ell}, \qquad n_y = -\frac{\Delta x}{\Delta \ell}. \tag{4.2.14}$$

Combining equations (4.2.13) and (4.2.14), we find that

$$f_x = n_x\, \sigma_{xx} + n_y\, \sigma_{yx}, \qquad f_y = n_x\, \sigma_{xy} + n_y\, \sigma_{yy}, \tag{4.2.15}$$

which are precisely the x and y components of (4.2.10).

To carry out an analogous proof for three-dimensional flow, we consider the forces exerted on the sides and over the volume of a tetrahedral fluid parcel, as illustrated in Figure 4.2.2(b), and work in similar ways (Problem 4.2.1).

4.2.1 Traction on either side of a fluid surface

Next, we consider a thin fluid layer with a designated outer side indicated by the direction of the unit normal vector, $\mathbf{n}$, and a designated inner side indicated by the direction of the opposite normal vector, $\mathbf{n}^{\text{inner}} = -\mathbf{n}$, as illustrated in Figure 4.2.3. Balancing the rate of change of momentum of the fluid residing inside the thin layer with the forces exerted on the layer, and repeating the preceding arguments on the insignificance of the fluid momentum and body force compared to the surface force, we derive the force balance equation

$$\mathbf{f}^{\text{outer}} + \mathbf{f}^{\text{inner}} = \mathbf{0}, \tag{4.2.16}$$

which is a statement of Newton's third law of action and reaction, stating that the force exerted on one body by a second body is equal in magnitude and opposite in direction to the force exerted by the second body on the first body.

It is reassuring to confirm that expression (4.2.10) is consistent with the physical law expressed by (4.2.16). Substituting the former into the latter, we obtain

$$\mathbf{n} \cdot \boldsymbol{\sigma} + \mathbf{n}^{\text{inner}} \cdot \boldsymbol{\sigma} = \mathbf{0}, \tag{4.2.17}$$

which is true in light of the definition $\mathbf{n}^{\text{inner}} = -\mathbf{n}$. More generally, for (4.2.16) to be true, it must be that

$$\mathbf{f}(\mathbf{x}, -\mathbf{n}) = -\mathbf{f}(\mathbf{x}, \mathbf{n}), \tag{4.2.18}$$

which is clearly satisfied by the right-hand side of (4.2.10).

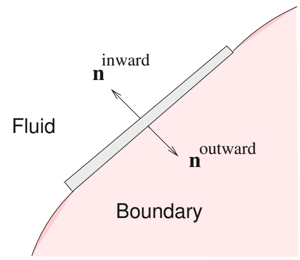


Figure 4.2.4 Illustration of a thin fluid layer adjacent to a boundary used to define the hydrodynamic force exerted on the boundary.

4.2.2 Traction on a boundary

Now we consider a small fluid surface residing at the boundary of a flow. The outer side of the fluid surface is indicated by the unit normal vector $\mathbf{n}^{\text{outward}}$ pointing into the boundary, as illustrated in Figure 4.2.4. Newton's third law of action and reaction requires that the traction exerted on the surface should balance the traction exerted by the fluid on the boundary, $\mathbf{f}^{\text{boundary}}$, so that

$$\mathbf{f}^{\text{boundary}} + \mathbf{n}^{\text{outward}} \cdot \boldsymbol{\sigma} = \mathbf{0}. \tag{4.2.19}$$

In terms of the inward unit normal vector pointing into the fluid, $\mathbf{n}^{\text{inward}} = -\mathbf{n}^{\text{outward}}$, we obtain

$$\mathbf{f}^{\text{boundary}} = \mathbf{n}^{\text{inward}} \cdot \boldsymbol{\sigma}. \tag{4.2.20}$$

Expression (4.2.20) allows us to compute the traction exerted on a boundary in terms of the stress tensor evaluated at the boundary.

4.2.3 Symmetry of the stress tensor

The torque with respect to a specified point, $\mathbf{x}_0$, due to a force, $\mathbf{F}$, applied at a point, $\mathbf{x}$, is defined by the outer vector product

$$\mathbf{T}(\mathbf{x}_0) \equiv (\mathbf{x} - \mathbf{x}_0) \times \mathbf{F}. \tag{4.2.21}$$

A fundamental law of mechanics originating from Newton's second law of motion requires that the rate of change of angular momentum of a fluid parcel should be balanced by the torque exerted on the fluid parcel, including the torque due to the body force and the torque due to the surface force.

Applying this law for a rectangular fluid parcel whose sides are parallel to the x, y, and z, axes, we find, that, in the absence of a body force inducing a torque, the tangential component of the traction in the jth direction exerted on the side that is perpendicular to the ith axis must be equal to the tangential component of the traction in the ith direction exerted on the side that is perpendicular to the jth axis, otherwise an imbalance will arise (Problem 4.2.2). Thus,

$$f_j^{(i)} = f_i^{(j)}, \tag{4.2.22}$$

stating that the stress tensor is symmetric,

$$\sigma_{ij} = \sigma_{ji}. \tag{4.2.23}$$

The diagonal components of the stress tensor can be arbitrary.

It is important to emphasize that the stress tensor is symmetric only in the absence of an externally induced torque, that is, in the absence of an external force field causing point particles to spin. This condition is tacitly assumed in the remainder of this book.

Problems

4.2.1 *Traction in three-dimensional flow*

Prove expression (4.2.10) for three-dimensional flow. *Hint:* Perform a force balance over the polyhedral volume depicted in Figure 4.2.2(b).

4.2.2 *Symmetry of the stress tensor*

Demonstrate the symmetry of the stress tensor for two-dimensional flow in the absence of an externally induced torque.

4.3 Traction jump across a fluid interface

Equation (4.2.16) states that the traction exerted on one side of a surface drawn inside a fluid is equal in magnitude and opposite in direction to that exerted on the other side. To derive this relation, we performed a force balance over a thin fluid layer centered at the surface, considering the force exerted along the edges infinitesimal. If the fluid residing inside this layer is homogeneous, the edge force scales with the layer thickness and is negligible indeed compared to the surface force exerted on the two sides.

However, if a thin layer straddles the interface between two different fluids instead of a regular surface residing inside a homogeneous fluid, differences in the magnitude of intermolecular forces on either side of the layer generate an effective edge force that does not scale with the layer thickness.

4.3.1 Interfacial tension

The interfacial edge force can be expressed in terms of the interfacial tension, also called the surface tension, γ, defined as the *tangential force per differential arc length* exerted around the edge of a section of an interface. In fact, the surface tension is the integrated normal stress exerted in a plane that is normal to the interface over a thin interfacial layer where the

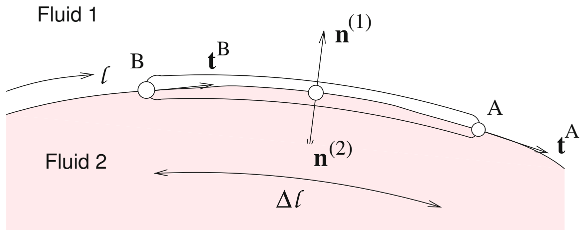


Figure 4.3.1 Illustration of forces exerted on a thin fluid layer centered at a two-dimensional interface, including the hydrodynamic force due to the fluid stresses and the force due to the surface tension.

physical properties of the medium undergo a rapid transition. Interfaces with membrane-like constitution exhibit tangential as well as normal interfacial tensions and possibly bending moments. In the most general case, an interface behaves line a thin shell, such as a dome or an egg shell.

Simple interfaces and surfactants

Consider a simple interface separating two immiscible liquids or a gas from a liquid. The surface tension pulls the interfacial layer in a direction that is tangential to the interface and normal to the edges. The magnitude of the surface tension depends on the local temperature and on the molecular constitution of the interface determined by the concentration of surface active substances residing over the interface, called surfactants, as will be discussed in Section 4.10. The higher the temperature or the concentration of a surfactant, the lower the surface tension.

Surfactants are often added to liquids to lower the surface tension and achieve a desired effect. A dish or laundry detergent is a common household surfactant used to lower the strength of the forces anchoring particles to a soiled surface. In engineering applications, surfactants are used to disperse oil spills.

4.3.2 Force balance at a two-dimensional interface

To illustrate the action of the surface tension, we consider a small section of a two-dimensional interface with length $\Delta\ell$, as shown in Figure 4.3.1. Surface tension pulls the layer forward and backward at the two edges in directions that are tangential to the interface at the two end points, A and B.

Balancing the surface force exerted on the upper and lower sides due to the stresses in each fluid and the edge forces, we obtain the vectorial equilibrium condition

$$\left[\, \mathbf{n}^{(1)} \cdot \boldsymbol{\sigma}^{(1)} \,\right] \Delta\ell + \left[\, \mathbf{n}^{(2)} \cdot \boldsymbol{\sigma}^{(2)} \,\right] \Delta\ell + \gamma^{\mathrm{A}}\, \mathbf{t}^{\mathrm{A}} - \gamma^{\mathrm{B}}\, \mathbf{t}^{\mathrm{B}} = \mathbf{0}, \tag{4.3.1}$$

where the unit normal vector $\mathbf{n}^{(1)}$ points into the fluid labeled 1 by convention, while the unit normal vector $\mathbf{n}^{(2)} = -\mathbf{n}^{(1)}$ points into the fluid labeled 2.

Now we express the second normal vector in terms of the first normal vector and rearrange the resulting expression to obtain

$$\mathbf{n}^{(1)} \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) = -\frac{\gamma^{A}\,\mathbf{t}^{A} - \gamma^{B}\,\mathbf{t}^{B}}{\Delta \ell}. \tag{4.3.2}$$

If the surface tension is uniform across the length of the interfacial element under consideration, then $\gamma^{B} = \gamma^{A}$. If, in addition, the interface is flat, $\mathbf{t}^{B} = \mathbf{t}^{A}$, the vectorial difference $\mathbf{t}^{B} - \mathbf{t}^{A}$ is zero and the right-hand side of (4.3.2) vanishes. Equation (4.3.2) then requires that the traction is continuous across the interface in the absence of a net contribution due to the surface tension.

As the length of the interfacial segment, $\Delta \ell$, tends to zero, the fraction on the right-hand side of (4.3.2) tends to the derivative of the product $\gamma \mathbf{t}$ with respect to arc length, ℓ, measured in the direction of the tangent vector $\mathbf{t}$ from an arbitrary origin, yielding

$$\mathbf{n}^{(1)} \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) = -\frac{\mathrm{d}(\gamma \mathbf{t})}{\mathrm{d}\ell}. \tag{4.3.3}$$

Expanding the derivative of the product on the right-hand side, we find that

$$\mathbf{n}^{(1)} \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) = -\gamma\,\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\ell} - \frac{\mathrm{d}\gamma}{\mathrm{d}\ell}\,\mathbf{t}. \tag{4.3.4}$$

The second term on the right-hand side contributes a traction discontinuity tangential to the interface, known as the *Marangoni traction.* If the surface tension is uniform over the interface, the Marangoni traction does not appear.

Curvature

To interpret the first term on the right-hand side of (4.3.4), we consider the difference between the two nearly equal tangential vectors $\mathbf{t}^{A}$ and $\mathbf{t}^{B}$. As the arc length, $\Delta \ell$, tends to zero, the difference between these vectors tends to a new vector that is directed normal to the interface. More precisely, in this limit, the ratio $(\mathbf{t}^{A} - \mathbf{t}^{B})/\Delta \ell$ tends to the derivative

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\ell} = -\kappa\,\mathbf{n}^{(1)}, \tag{4.3.5}$$

where κ is the positive or negative curvature of the interface; for the shape shown in Figure 4.3.1, the curvature is positive, $\kappa > 0$.

To understand why the derivative $\mathrm{d}\mathbf{t}/\mathrm{d}\ell$ is normal to the interface, we approximate the derivative with the ratio $(\mathbf{t}^{A} - \mathbf{t}^{B})/\Delta \ell$, and rearrange to obtain

$$\mathbf{t}^{A} \simeq \mathbf{t}^{B} - \kappa\,\mathbf{n}^{(1)}. \tag{4.3.6}$$

The second term on the right-hand side inclines $\mathbf{t}^{B}$ against $\mathbf{n}^{(1)}$ to generate $\mathbf{t}^{A}$, as shown in Figure 4.3.1.

Taking the inner product of both sides of equation (4.3.5) with the unit normal vector $\mathbf{n}^{(1)}$, we derive an expression for the curvature,

$$\kappa = -\mathbf{n}^{(1)} \cdot \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\ell}, \tag{4.3.7}$$

which can be restated as

$$\kappa = -\frac{\mathrm{d}x}{\mathrm{d}\ell}\,\mathbf{n}^{(1)} \cdot \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}x} = -\frac{\mathrm{d}y}{\mathrm{d}\ell}\,\mathbf{n}^{(1)} \cdot \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}y}. \tag{4.3.8}$$

By definition, $\kappa = 1/R$, where R is the positive or negative radius of curvature of the interface. For the shape shown in Figure 4.3.1, the radius of curvature is positive, $R > 0$.

Conversely, the derivative of the unit normal vector is parallel to the unit tangential vector,

$$\frac{\mathrm{d}\mathbf{n}^{(1)}}{\mathrm{d}\ell} = \kappa\,\mathbf{t}, \tag{4.3.9}$$

yielding

$$\kappa = \mathbf{t} \cdot \frac{\mathrm{d}\mathbf{n}^{(1)}}{\mathrm{d}\ell}. \tag{4.3.10}$$

Equations (4.3.5) and (4.3.9) comprise the *Frenet–Serret relations* in differential geometry.

Laplace pressure

Substituting (4.3.5) into (4.3.4) and rearranging, we derive the final expression for the jump in the interfacial traction across a two-dimensional interface,

$$\Delta\mathbf{f} \equiv \mathbf{n}^{(1)} \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) = \gamma\kappa\,\mathbf{n}^{(1)} - \frac{\mathrm{d}\gamma}{\mathrm{d}\ell}\,\mathbf{t}. \tag{4.3.11}$$

The first term on the right-hand side of (4.3.11) contributes a traction discontinuity normal to the interface, known as the *Laplace pressure*; however, bear in mind that the term *pressure* is appropriate only in the absence of fluid motion on either side of the interface. If either the curvature of the interface or the surface tension vanishes, the Laplace pressure is zero and the normal stress is continuous across the interface.

It is important to bear in mind that the interfacial tension is independent of the radius of curvature of the interface, except when the radius of curvature is so small that it becomes comparable to the molecular size. In mainstream engineering applications, the surface tension is regarded as a genuine physical constant.

Local coordinates

Consider the jump in traction across the curved interface depicted in Figure 4.3.2(a). The origin of the Cartesian axes lies at a point on the interface, the x axis is tangential to the

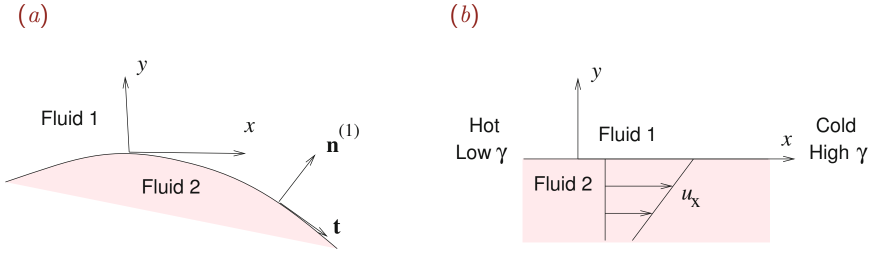


Figure 4.3.2 (*a*) A local coordinate system with the x axis tangential to a two-dimensional interface at a point is used to evaluate the jump in the traction across the interface. (*b*) A differentially heated interface drives a thermocapillary flow.

interface, and the y axis is normal to the interface pointing into the fluid labeled 1. At the origin of the Cartesian axes, the components of the unit normal vector, $\mathbf{n}^{(1)}$, are

$$n_x^{(1)} = 0, \qquad n_y^{(1)} = 1, \tag{4.3.12}$$

and the jump in the interfacial traction is given by

$$\Delta\mathbf{f} \equiv \mathbf{n}^{(1)} \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) = (\sigma_{yx}^{(1)} - \sigma_{yx}^{(2)})\,\mathbf{e}_x + (\sigma_{yy}^{(1)} - \sigma_{yy}^{(2)})\,\mathbf{e}_y, \tag{4.3.13}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors parallel to the x or y axis.

Applying equation (4.3.11) at the origin and setting $\mathbf{t} = \mathbf{e}_x$ and $\mathbf{n}^{(1)} = \mathbf{e}_y$, we obtain

$$\Delta\mathbf{f} = \gamma\kappa\,\mathbf{e}_y - \frac{\mathrm{d}\gamma}{\mathrm{d}\ell}\,\mathbf{e}_x. \tag{4.3.14}$$

Comparing this equation with (4.3.13), we derive an expression for the jump in the shear stress,

$$\sigma_{yx}^{(1)} - \sigma_{yx}^{(2)} = -\frac{\mathrm{d}\gamma}{\mathrm{d}\ell}, \tag{4.3.15}$$

and another expression for the jump in the normal stress,

$$\sigma_{yy}^{(1)} - \sigma_{yy}^{(2)} = \gamma\,\kappa. \tag{4.3.16}$$

For the configuration shown in Figure 4.3.2(*a*), the curvature κ is positive

A heated liquid layer

As an application, we consider a liquid layer that is hot at the left end and cold at the right end, as illustrated in Figure 4.3.2(*b*). In this case, $\mathrm{d}T/\mathrm{d}x < 0$ and therefore $\mathrm{d}\gamma/\mathrm{d}x > 0$, where T is the temperature. Equation (4.3.15) yields

$$\sigma_{yx}^{(1)} - \sigma_{yx}^{(2)} = -\frac{\mathrm{d}\gamma}{\mathrm{d}x} < 0. \tag{4.3.17}$$

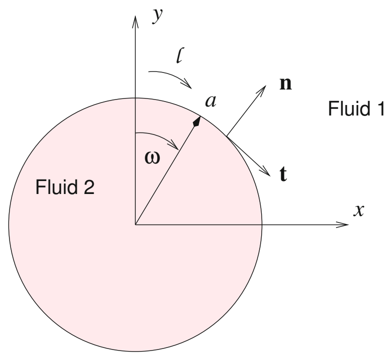


Figure 4.3.3 Illustration of a circular interface of radius a enclosing a fluid labeled 2, showing the unit tangent vector, $\mathbf{t}$, and the outward unit normal vector, $\mathbf{n}$.

Because the shear stress is insignificant in the gas above the layer, we can approximate

$$\sigma_{yx}^{(1)} \simeq 0. \qquad (4.3.18)$$

For a Newtonian fluid,

$$\sigma_{yx}^{(2)} = \mu_2 \frac{\mathrm{d}u_x}{\mathrm{d}y} > 0, \qquad (4.3.19)$$

where μ_2 is the viscosity of the lower fluid, as discussed in Section 4.5. The positive sign of the slope $\mathrm{d}u_x/\mathrm{d}y$ is consistent with the velocity profile drawn in Figure 4.3.2(b). Physically, the high surface tension at the cold end pulls the fluid against the low surface tension at the hot end to drive a surface-tension induced flow.

A flow induced by temperature differences causing variations in surface tension is called a *thermocapillary flow.*

Curvature of a circle

We have seen that the curvature of an interface determines the jump in the normal component of the traction due to surface tension. To gain experience on the computation of the curvature, we consider a circle of radius a centered at the origin, described in parametric form by the equations

$$x = a \sin\omega, \qquad y = a\cos\omega, \qquad (4.3.20)$$

where ω is the polar angle measured around the center of the circle in the clockwise direction, varying from 0 to 2π, as shown in Figure 4.3.3. The components of the unit tangent vector pointing in the clockwise direction are

$$t_x = \frac{\mathrm{d}x}{\mathrm{d}\ell}, \qquad t_y = \frac{\mathrm{d}y}{\mathrm{d}\ell}, \qquad (4.3.21)$$

where $\mathrm{d}\ell = (\mathrm{d}x^2 + \mathrm{d}y^2)^{1/2}$ is the differential arc length measured in the clockwise direction. Using the parametric representation, we find that

$$\mathrm{d}x = a\,\cos\omega\,\mathrm{d}\omega, \qquad \mathrm{d}y = -a\,\sin\theta\,\mathrm{d}\omega, \qquad \mathrm{d}\ell = a\,\mathrm{d}\omega, \tag{4.3.22}$$

and thus

$$t_x = \cos\omega, \qquad t_y = -\sin\omega. \tag{4.3.23}$$

Based on these formulas, we compute

$$\frac{\mathrm{d}t_x}{\mathrm{d}\ell} = \frac{\mathrm{d}(\cos\omega)}{\mathrm{d}(a\omega)} = -\frac{1}{a}\,\sin\omega, \qquad \frac{\mathrm{d}t_y}{\mathrm{d}\ell} = \frac{\mathrm{d}(-\sin\theta)}{\mathrm{d}(a\theta)} = -\frac{1}{a}\,\cos\omega. \tag{4.3.24}$$

In unified vector form,

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\ell} = -\frac{1}{a}\,\mathbf{n}, \tag{4.3.25}$$

where $\mathbf{n} = (\sin\omega, \quad \cos\omega)$ is the unit vector normal to the circle pointing outward, as shown in Figure 4.3.3. Comparing (4.3.25) with (4.3.5), we confirm that the curvature of the circle is equal to the inverse of its radius, $\kappa = 1/a$.

Formulas for the curvature

The shape of a two-dimensional interface that does not turn upon itself but has a monotonic shape can be described by a single-valued function,

$$y = f(x). \tag{4.3.26}$$

Using elementary geometry, we find that

$$\mathbf{n}^{(1)} = \frac{1}{(1+f'^2)^{1/2}}\,(-f'\,\mathbf{e}_x + \mathbf{e}_y\,), \qquad \mathbf{t} = \frac{1}{(1+f'^2)^{1/2}}\,(\mathbf{e}_x + f'\,\mathbf{e}_y),$$
$$\frac{\mathrm{d}\ell}{\mathrm{d}x} = \sqrt{1+f'^2}, \tag{4.3.27}$$

where a prime denotes a derivative with respect to x. Substituting these expressions into the formula (4.3.7) for the curvature, and simplifying, we derive the expressions

$$\kappa = -\frac{f''}{(1+f'^2)^{3/2}} = \frac{1}{f'}\Big(\frac{1}{\sqrt{1+f'^2}}\Big)' = -\Big(\frac{f'}{\sqrt{1+f'^2}}\Big)'. \tag{4.3.28}$$

The slope angle of the interface, θ, is defined by the equation

$$\tan\theta = f'. \tag{4.3.29}$$

We note that the fraction in the second expression of (4.3.28) is equal to $|\cos\theta|$ and derive the alternative expression

$$\kappa = \frac{1}{f'}\,\frac{\mathrm{d}|\cos\theta|}{\mathrm{d}x}. \tag{4.3.30}$$

The curvature of an interface that is described parametrically by the functions

$$x = X(\xi), \qquad y = Y(\xi) \tag{4.3.31}$$

is given by

$$\kappa = \frac{X_{\xi\xi}\, Y_\xi - Y_{\xi\xi}\, X_\xi}{(X_\xi^2 + Y_\xi^2)^{3/2}}, \tag{4.3.32}$$

where a subscript denotes a derivative with respect to the parametric variable, ξ. Formula (4.3.28) arises by setting $\xi = x$.

Problems

4.3.1 *Curvature of an ellipse*

Consider a horizontal ellipse centered at the origin of the xy plane, described in parametric form by the equations $x = a \cos \eta$ and $y = b \sin \eta$, where η is the natural parameter of the ellipse varying between 0 and 2π, and a, b are the ellipse semi-axes. Derive an expression for the curvature of the ellipse in terms of a, b, and η. Confirm that, as b tends to a, the curvature of the ellipse reduces to that of a circle.

4.3.2 💻 *Computation of the curvature*

A line in the xy plane can be described by a set of $N + 1$ marker points with coordinates (x_i, y_i) for $i = 1, \ldots, N + 1$. An approximation to the components of the tangent vector at the ith point is provided by the central-difference formulas

$$t_x^{(i)} = \frac{x_{i+1} - x_{i-1}}{\Delta \ell_i}, \qquad t_y^{(i)} = \frac{y_{i+1} - y_{i-1}}{\Delta \ell_i}, \tag{4.3.33}$$

where

$$\Delta \ell_i = \big[\, (x_{i+1} - x_{i-1})^2 + (y_{i+1} - y_{i-1})^2 \,\big]^{1/2}. \tag{4.3.34}$$

The derivatives of the components of the tangent vector with respect to arc length can be approximated with the corresponding formulas

$$\frac{\mathrm{d}t_x^{(i)}}{\mathrm{d}\ell} = \frac{t_x^{(i+1)} - t_x^{(i-1)}}{\Delta \ell_i}, \qquad \frac{\mathrm{d}t_y^{(i)}}{\mathrm{d}\ell} = \frac{t_y^{(i+1)} - t_y^{(i-1)}}{\Delta \ell_i}. \tag{4.3.35}$$

The components of the outward normal vector at the marker points are given by

$$n_x^{(i)} = t_y^{(i)}, \qquad n_y^{(i)} = -t_x^{(i)}. \tag{4.3.36}$$

Write a computer program that reads or generates the coordinates of a set of marker points, computes the right-hand sides of (4.3.33)–(4.3.36), and then evaluates the curvature

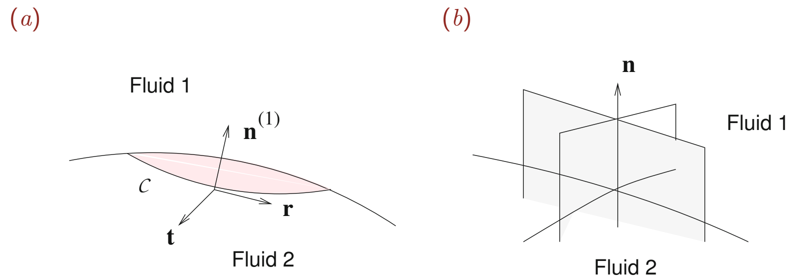


Figure 4.4.1 (a) Illustration of of a thin fluid layer straddling a three-dimensional interface. The surface tension pulls the interfacial patch in the direction of the unit tangent vector, $\mathbf{t}$. (b) The mean curvature of a three-dimensional surface is equal to the average of two directional curvatures in two perpendicular planes containing the normal vector.

at the marker points from the expression $\kappa = -\mathbf{n} \cdot \mathrm{d}\mathbf{t}/\mathrm{d}\ell$. Perform a series of computations with marker points distributed evenly along a circle, and compare the numerically computed with the exact curvature.

4.3.3 *Motion induced by curvature*

Interfaces exhibit a variety of motions under the influence of surface tension. In a simplified model, point particles distributed along a two-dimensional interface move normal to the interface with velocity that is proportional to the local curvature. If $\mathbf{X}^{(i)}$ is the position of the ith marker point, then the motion of the marker point is described by the vectorial differential equation

$$\frac{\mathrm{d}\mathbf{X}^{(i)}}{\mathrm{d}t} = \kappa\,[\mathbf{n}^{(1)}]_i, \tag{4.3.37}$$

where t stands for time, $\mathbf{n}^{(1)}$ is the outward normal vector, and κ is the curvature.

Write a computer program that computes the motion of marker points distributed along an interface using the finite-difference approximations discussed in Problem 4.3.2 and the modified Euler method for integrating in time the differential equations (4.3.37). Run the program to compute the evolution of marker points distributed along a circle or an ellipse with axes ratio equal to two. Discuss the nature of the motion in each case.

4.4 Force balance at a three-dimensional interface

In Section 4.3, we derived a force balance at a two-dimensional interface and discussed the computation of the curvature. To derive the counterpart of the force balance equation (4.3.11) for a three-dimensional interface, we consider a thin material patch straddling the interface, as illustrated in Figure 4.4.1(a).

Let $\mathbf{n}^{(1)}$ be the unit vector normal to the interface pointing into fluid labeled 1, and $\mathbf{r}$ be the unit vector tangential to the edge of the patch, $\mathcal{C}$. Surface tension pulls the layer in the direction of the unit vector $\mathbf{t}$ that is tangential to the interface and normal to both $\mathbf{n}^{(1)}$ and $\mathbf{r}$. Recalling the geometrical interpretation of the outer vector product, discussed in Section 2.3, we write

$$\mathbf{t} = \mathbf{r} \times \mathbf{n}^{(1)}. \tag{4.4.1}$$

All vectors involved in this equation are unit vectors, that is, they have unit length.

Next, we balance the surface force due to the fluid stresses on either side of the interface and the edge force due to the surface tension, writing

$$(\, \mathbf{n}^{(1)} \cdot \boldsymbol{\sigma}^{(1)}\,)\,\Delta S + (\, \mathbf{n}^{(2)} \cdot \boldsymbol{\sigma}^{(2)}\,)\,\Delta S + \oint_{\mathcal{C}} \gamma\,\mathbf{t}\,\mathrm{d}\ell = \mathbf{0}, \tag{4.4.2}$$

where ΔS is the surface area of the patch, ℓ is the arc length around the edge of the patch, $\mathcal{C}$, and $\mathbf{n}^{(2)}$ is the unit vector normal to the interface pointing into the fluid labeled 2. Equation (4.4.2) is the three-dimensional counterpart of equation (4.3.1). Setting $\mathbf{n}^{(2)} = -\mathbf{n}^{(1)}$ and rearranging, we obtain

$$\Delta\mathbf{f} \equiv \mathbf{n}^{(1)} \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) = -\frac{1}{\Delta S} \oint_{\mathcal{C}} \gamma\,\mathbf{t}\,\mathrm{d}\ell. \tag{4.4.3}$$

In the limit as the loop $\mathcal{C}$ shrinks to a point and ΔS tends to zero, equation (4.4.3) provides us with the expression

$$\Delta\mathbf{f} = \gamma\,2\,\kappa_m\,\mathbf{n}^{(1)} - \frac{\partial\gamma}{\partial\ell}\,\boldsymbol{\tau}, \tag{4.4.4}$$

subject to the following definitions:

- κ_m is the mean curvature of the interface defined in terms of the surface divergence of the unit normal vector, as discussed in Section 4.4.1.
- $\boldsymbol{\tau}$ is the unit vector tangent to the interface pointing in the direction where the surface tension changes most rapidly.
- ℓ is the arc length measured in the direction of the tangential vector $\boldsymbol{\tau}$, and $\partial\gamma/\partial\ell$ is the corresponding maximum rate of change of the surface tension with respect to arc length.

The first term on the right-hand side of (4.4.4) expresses a discontinuity in the normal direction identified as the *Laplace pressure*, whereas the second term expresses a discontinuity in the tangential direction identified as the *Marangoni traction.*

Tangential coordinates

If the x axis is chosen to be normal to the interface at a point, and correspondingly the yz plane is tangential to the interface at that point, then

$$\Delta\mathbf{f} = \gamma\,2\kappa_m\,\mathbf{e}_x - \frac{\partial\gamma}{\partial y}\,\mathbf{e}_y - \frac{\partial\gamma}{\partial z}\,\mathbf{e}_z, \tag{4.4.5}$$

where $\mathbf{e}_y$ and $\mathbf{e}_z$ are tangential unit vectors along the y and z axes.

4.4.1 Mean curvature

If the yz plane is tangential to an interface at a point, then the mean curvature at that point is given by the surface divergence of the normal vector,

$$2\kappa_m \equiv \boldsymbol{\nabla}_s \cdot \mathbf{n} = \frac{\partial n_y}{\partial y} + \frac{\partial n_z}{\partial z}. \tag{4.4.6}$$

Requiring that

$$n_x^2 + n_y^2 + n_z^2 = 1, \tag{4.4.7}$$

and then

$$n_x \frac{\partial n_x}{\partial x} + n_y \frac{\partial n_x}{\partial y} + n_z \frac{\partial n_x}{\partial z} = 0, \tag{4.4.8}$$

and setting $n_x =$, $n_y = 0$, and $n_z = 0$, we find that $\partial n_x/\partial x = 0$, which shows that the normal derivative of the normal component of the normal vector is zero. This property allows us to write the more general expression

$$2\kappa_m \equiv \boldsymbol{\nabla} \cdot \mathbf{n} = \frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} + \frac{\partial n_z}{\partial z}, \tag{4.4.9}$$

with reference to an arbitrary system of Cartesian coordinates whose axes are not necessarily tangential or normal to the interface.

Mean curvature of a surface described as $F(x, y, z) = 0$

A three-dimensional interface can be described implicitly by an equation of the form

$$F(x, y, z) = 0. \tag{4.4.10}$$

Given two of the three coordinates, x, y, or z, this equation can be used to compute the third coordinate by analytical or numerical methods. The unit vector normal to the interface is given by

$$\mathbf{n} = \frac{1}{|\boldsymbol{\nabla} F|}\, \boldsymbol{\nabla} F \tag{4.4.11}$$

and the mean curvature is given by

$$2\kappa_m = \boldsymbol{\nabla} \cdot \Big(\frac{1}{|\boldsymbol{\nabla} F|}\, \boldsymbol{\nabla} F \Big) = \frac{1}{|\boldsymbol{\nabla} F|}\, \nabla^2 F - \frac{1}{|\boldsymbol{\nabla} F|^3}\, \boldsymbol{\nabla} F \cdot (\boldsymbol{\nabla}\boldsymbol{\nabla} F) \cdot \boldsymbol{\nabla} F, \tag{4.4.12}$$

where $\boldsymbol{\nabla}\boldsymbol{\nabla} F \equiv \boldsymbol{\Phi}$ is a symmetric matrix encapsulating the second partial derivatives,

$$\Phi_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}. \tag{4.4.13}$$

Mean curvature of a surface described as $z = f(x, y)$

For an interface that is described explicitly by a function

$$z = f(x, y), \tag{4.4.14}$$

we set $F(x, y, z) = z - f(x, y)$ and derive the formula

$$2\kappa_m = -\frac{(1 + f_y^2)\, f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2)\, f_{yy}}{(1 + f_x^2 + f_y^2)^{3/2}}, \tag{4.4.15}$$

where the subscript x denotes a derivative with respect to x and the subscript y denotes a derivative with respect to y.

For a nearly flat interface, the partial derivatives are small compared to unity, yielding

$$2\kappa_m \simeq -(f_{xx} + f_{yy}). \tag{4.4.16}$$

The term inside the parentheses is the Laplacian of $f(x, y)$.

Spherical polar coordinates

The unit normal vector for an interface that is described in spherical polar coordinates, (r, θ, φ), as

$$r = f(\theta, \varphi), \tag{4.4.17}$$

is computed from (4.4.11) with

$$\boldsymbol{\nabla} F = \mathbf{e}_r - \frac{f_\theta}{r}\, \mathbf{e}_\theta - \frac{f_\varphi}{r \sin\theta}\, \mathbf{e}_\varphi, \tag{4.4.18}$$

where a subscript after f indicates a corresponding partial derivative.

For a nearly spherical interface of radius r, the mean curvature can be approximated with the linearized expression

$$2\kappa_m \simeq \frac{2}{r} - \frac{\cot\theta}{r^2}\, f_\theta - \frac{1}{r^2}\, f_{\theta\theta} - \frac{1}{r^2 \sin^2\theta}\, f_{\varphi\varphi} \tag{4.4.19}$$

involving first and second partial derivatives.

4.4.2 Directional curvatures

To compute the mean curvature of a three-dimensional interface, we consider the traces of the interface in two conjugate orthogonal planes that are normal to the interface at a point, and thus contain the normal vector, as depicted in Figure 4.4.1(b).

If κ_1 and κ_2 are the curvatures of the two traces at that point, computed using formula (4.3.5) with the x and y axes residing in each of the two planes, then the mean curvature of the interface is given by

$$\kappa_m = \tfrac{1}{2}\,(\kappa_1 + \kappa_2). \tag{4.4.20}$$

A theorem due to Euler reassures us that the mean value of the conjugate directional curvatures is independent of the orientation of the two planes, provided that the planes remain mutually orthogonal.

Principal curvatures

There is a particular orientation of the normal plane corresponding to maximum directional curvature, $\kappa_{\max}$, and a conjugate orthogonal orientation corresponding to minimum directional curvature, $\kappa_{\min}$. These are the principal curvatures of the interface at the chosen point. The mean curvature is

$$\kappa_m = \frac{1}{2}\,(\kappa_{\max} + \kappa_{\min}). \tag{4.4.21}$$

In the case of a sphere, the principal curvatures and the mean curvature are equal.

Euler's theorem states that the curvature in an arbitrary direction is related to the principal curvatures by

$$\kappa = \kappa_{\max}\,\cos^2\alpha + \kappa_{\min}\,\sin^2\alpha, \tag{4.4.22}$$

where α is the angle subtended between (a) the tangential vector pointing in a chosen direction, and (b) the tangential vector pointing in the direction of maximum curvature.

4.4.3 Axisymmetric interfaces

Next, we consider the geometrical properties of an axisymmetric interface, as shown in Figure 4.4.2. The mean curvature is the average of the two principal curvatures: one is the curvature of the trace of the interface in the σx (azimuthal) plane, corresponding to a certain value of the azimuthal angle, φ, denoted by κ_1, and the second is the curvature of the trace of the interface in the conjugate orthogonal plane, denoted by κ_2.

Description as $\sigma = w(x)$

The shape of an interface in an azimuthal plane can be described by a function,

$$\sigma = w(x), \tag{4.4.23}$$

as shown in Figure 4.4.2(a). The unit normal vector is

$$\mathbf{n} = \frac{1}{\sqrt{1+w'^2}}\,(\,\mathbf{e}_\sigma - w'\,\mathbf{e}_x\,), \tag{4.4.24}$$

where a prime denotes a derivative with respect to x. The mean curvature is given by the divergence of the normal vector,

$$2\kappa_m = \frac{\partial n_x}{\partial x} + \frac{1}{\sigma}\,\frac{\partial(\sigma n_\sigma)}{\partial \sigma}. \tag{4.4.25}$$

Making substitutions, we obtain

$$2\kappa_m = -\Big(\frac{w'}{\sqrt{1+w'^2}}\Big)' + \frac{1}{\sigma}\,\frac{\partial}{\partial\sigma}\Big(\frac{\sigma}{\sqrt{1+w'^2}}\Big). \tag{4.4.26}$$

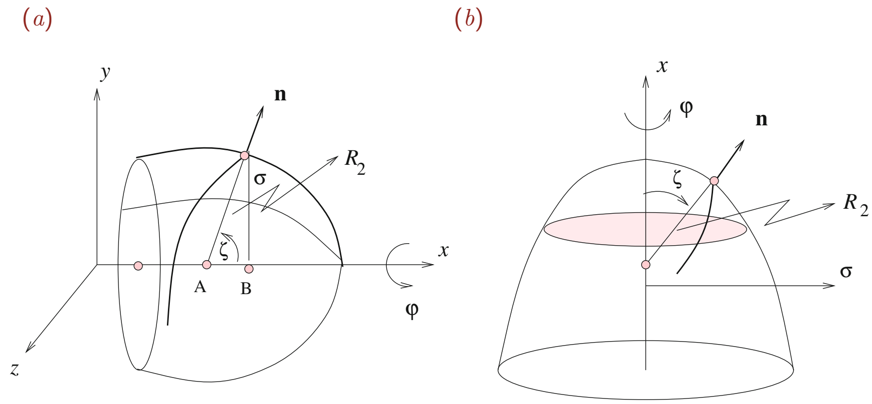


Figure 4.4.2 Illustration of an axisymmetric interface. (a) The second principal curvature at a point in the xy plane is the curvature of the line representing the trace of the interface in a plane that is normal to the interface and normal to the xy plane, drawn as a heavy line. (b) An axisymmetric interface can be described by a function $x = f(\sigma)$. The second principal curvature is the curvature of the line drawn with a heavy line.

Carrying out the differentiations, we find that

$$2\,\kappa_m = -\frac{w''}{(1+w'^2)^{3/2}} + \frac{1}{w}\,\frac{1}{\sqrt{1+w'^2}} \tag{4.4.27}$$

or

$$2\,\kappa_m = \frac{1}{ww'}\Big(\frac{w}{\sqrt{1+w'^2}}\Big)'. \tag{4.4.28}$$

The first term on the right-hand side of (4.4.27) is the principal curvature in an azimuthal plane. The second term is the second principal curvature,

$$\kappa_2 = \frac{1}{R_2}, \qquad R_2 = \sigma\,\sqrt{1+w'^2} = \frac{\sigma}{\sin\zeta} = \frac{\sigma}{n_\sigma}, \tag{4.4.29}$$

where R_2 is the second principal radius of curvature and the angle ζ is defined in Figure 4.4.2(a). We have found that R_2 is the signed distance between (a) the point where the curvature is evaluated and (b) the intersection of the extension of the normal vector with the x axis. If n_σ is negative, R_2 is also negative.

In the case of a sphere, points A and B in Figure 4.4.2(a) coincide with the center of the sphere, and both principal curvatures are equal to the radius of the sphere.

Description as $x = f(\sigma)$

Alternatively, the shape of an interface can be described by a function

$$x = f(\sigma), \tag{4.4.30}$$

as shown in Figure 4.4.2(*b*). The unit normal vector is given by

$$\mathbf{n} = \frac{1}{\sqrt{1+f'^2}} \, (\, \mathbf{e}_x - f' \, \mathbf{e}_\sigma \,), \tag{4.4.31}$$

where a prime denotes a derivative with respect to σ. The mean curvature is given by the divergence of the normal vector,

$$2\kappa_m = \frac{\partial n_x}{\partial x} + \frac{1}{\sigma} \frac{\partial(\sigma \, n_\sigma)}{\partial \sigma}. \tag{4.4.32}$$

Making substitutions, we obtain

$$2\kappa_m = -\frac{1}{\sigma} \Big(\frac{\sigma f'}{\sqrt{1+f'^2}} \Big)'. \tag{4.4.33}$$

Carrying out the differentiations, we find that

$$2\kappa_m = -\frac{f''}{(1+f'^2)^{3/2}} - \frac{1}{\sigma} \frac{f'}{\sqrt{1+f'^2}}. \tag{4.4.34}$$

The first term on the right-hand side is the principal curvature in a meridional plane,

$$\kappa_1 = -\frac{f''}{(1+f'^2)^{3/2}} = \frac{1}{f'} \Big(\frac{1}{\sqrt{1+f'^2}} \Big)'. \tag{4.4.35}$$

The second term is the second principal curvature,

$$\kappa_2 = \frac{1}{R_2}, \qquad R_2 = -\frac{\sigma}{f'} \sqrt{1+f'^2} = \frac{\sigma}{\sin\zeta} = \frac{\sigma}{n_\sigma}, \tag{4.4.36}$$

where R_2 is the second principal radius of curvature and the angle ζ is defined in Figure 4.4.2(*b*).

Problems

4.4.1 *Mean curvature*

(*a*) Compute the mean curvature of a periodic surface described by the equation

$$z = a \sin(kx) + b \sin(ly), \tag{4.4.37}$$

where a, b, k, and l are given constants. State the units of each constant.

(*b*) Based on formula (4.4.20) and its accompanying interpretation discussed in the text, show that the mean curvature of a sphere of radius a is equal to $\kappa_m = 1/a$, whereas the mean curvature of a circular cylinder of radius a is equal to $\kappa_m = 1/(2a)$.

(*c*) The sphere and the circular cylinder are two shapes with constant mean curvature. Describe and discuss one additional shape.

4.4.2 *Jump in traction in local coordinates*

Derive the counterparts of equations (4.3.15) and (4.3.16) for a three-dimensional interface.

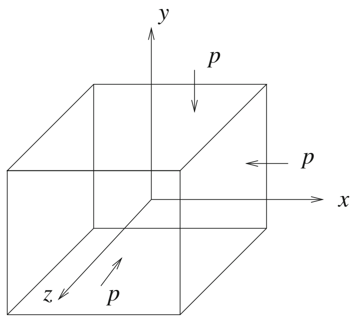


Figure 4.5.1 The traction exerted at the three sides of a cubical parcel of a stationary fluid has only a normal component defined in terms of the thermodynamic pressure, p.

4.5 Stresses in a fluid at rest

If a fluid does not exhibit macroscopic motion as seen by a stationary observer, that is, the observable fluid velocity vanishes, the molecules are in a state of dynamic equilibrium determined by the physical conditions prevailing in their immediate environment.

Consider a small cubic fluid parcel with all six faces perpendicular to the x, y, or z axis, as illustrated in Figure 4.5.1. In the absence of macroscopic fluid motion, the traction exerted on the sides that are perpendicular to the x axis must be directed normal to these side. To demonstrate this by *reduction ad absurdum*, we note that, if this were not true, a tangential component pointing in a physically indeterminate direction would arise. In the notation of Section 4.2,

$$f_x^{(x)} \neq 0, \qquad f_y^{(x)} = 0, \qquad f_z^{(x)} = 0, \tag{4.5.1}$$

where $\mathbf{f}^{(x)}$ is the traction exerted on the right side of a face that is perpendicular to the x axis.

Similar arguments can be made to show that the traction exerted on the sides that are perpendicular to the y or z axis are directed normal to these sides,

$$f_x^{(y)} = 0, \qquad f_y^{(y)} \neq 0, \qquad f_z^{(y)} = 0, \tag{4.5.2}$$

and

$$f_x^{(z)} = 0, \qquad f_y^{(z)} = 0, \qquad f_z^{(z)} \neq 0. \tag{4.5.3}$$

If the size of the cubic parcel is infinitesimal, the fluid residing inside the parcel is perfectly or nearly homogeneous and the non-vanishing components of the three tractions, $f_x^{(x)}$, $f_y^{(y)}$, and $f_z^{(z)}$, must be identical. By definition, the common value of these normal components is the negative of the pressure, p,

$$f_x^{(x)} = f_y^{(y)} = f_z^{(z)} \equiv -p. \tag{4.5.4}$$

We conclude that, in hydrostatics, the stress tensor introduced in equation (4.2.6) in terms of three tractions is defined exclusively in terms of the pressure, and is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.5.5}$$

In compact notation,

$$\boldsymbol{\sigma} = -p\,\mathbf{I}, \tag{4.5.6}$$

where $\mathbf{I}$ is the unit or identity matrix shown on the right-hand side of (4.5.5).

Traction on a surface

As an application, we use expression (4.5.6) to evaluate the traction exerted on a surface that resides inside or at the boundary a stationary fluid. Substituting (4.5.6) into formula (4.2.10), we find that

$$\mathbf{f}(\mathbf{x}, \mathbf{n}) = \mathbf{n} \cdot (-p\,\mathbf{I}) = -p\,\mathbf{n} \cdot \mathbf{I}, \tag{4.5.7}$$

and then

$$\mathbf{f}(\mathbf{x}, \mathbf{n}) = -p\,\mathbf{n}. \tag{4.5.8}$$

The last equation results from the identity $\mathbf{n} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{n} = \mathbf{n}$. Equation (4.6.1) shows that the traction exerted on a surface in hydrostatics is directed normal to the surface and points against the surface, while the tangential component is identically zero.

Traction on a boundary

Substituting (4.2.20) into expression (4.5.6) to evaluate the traction on a boundary that is immersed in, or confines a stationary fluid, we obtain

$$\mathbf{f}^{\text{boundary}} = \mathbf{n}^{\text{inward}} \cdot (-p\,\mathbf{I}) = -p\,\mathbf{n}^{\text{inward}} \cdot \mathbf{I}, \tag{4.5.9}$$

and then

$$\mathbf{f}^{\text{boundary}} = -p\,\mathbf{n}^{\text{inward}}. \tag{4.5.10}$$

The last equality results from the identity $\mathbf{n}^{\text{inward}} \cdot \mathbf{I} = \mathbf{n}^{\text{inward}}$. Thus, the traction exerted on a fluid boundary in hydrostatics is directed normal to the boundary and points against the boundary, while the tangential component of the traction is identically zero.

4.5.1 Pressure from molecular motions

The hydrostatic pressure distribution established in a fluid at rest cannot be computed working exclusively in the context of fluid mechanics. Additional information concerning the relationship between the density and the pressure for a particular fluid under consideration is required.

Gases

Molecular thermodynamics states that the pressure of a small gas parcel is determined by (a) the number of molecules residing inside the parcel expressed by the local fluid density, ρ, (b) the kinetic energy of the molecules determined by the absolute temperature, T, and (c) the nature and intensity of the intermolecular forces due to an intermolecular potential.

For an ideal gas, intermolecular forces are negligible and the pressure derives from the density and temperature in terms of the ideal gas law,

$$p = \frac{RT}{M}\,\rho, \tag{4.5.11}$$

where M is the molecular mass, defined as the mass of one mole comprised of a collection of N_A molecules;

$$N_\mathrm{A} = 6.022 \times 10^{26} \tag{4.5.12}$$

is the Avogadro number;

$$R = 8.314 \times 10^3 \mathrm{kg\,m^2/(sec^2 \cdot kmole \cdot K)} \tag{4.5.13}$$

is the ideal gas constant; T is Kelvin's absolute temperature, which is equal to the Celsius centigrade temperature reduced by 273 units. The gram-molecular mass of an element is equal to the atomic weight of the element listed in the periodic table, expressed in grams.

Liquids

Because liquids are nearly incompressible, the pressure can be regarded a function of the density alone, independent of pressure. The computation of the hydrostatic pressure distribution in gases and liquids will be discussed in detail in Chapter 5.

4.5.2 Jump in pressure across an interface in hydrostatics

Equations (4.3.11) and (4.4.4) provide us with expressions for the jump in the traction across a two- or three-dimensional interface. If the fluids on either side of the interface are stationary, the corresponding stress tensors are given by (4.5.6) in term of the pressure and the jump in the traction is given by

$$\Delta\mathbf{f} \equiv \mathbf{n}^{(1)} \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) = \mathbf{n}^{(1)} \cdot [\,-p^{(1)}\,\mathbf{I} - (-p^{(2)}\,\mathbf{I})\,], \tag{4.5.14}$$

where the superscript 1 or 2 denotes the choice of fluid. Simplifying, we obtain

$$\Delta\mathbf{f} = (p^{(2)} - p^{(1)})\,\mathbf{n}^{(1)}. \tag{4.5.15}$$

Because the jump in traction is normal to the interface, surface tension variations are not accepted in hydrostatics.

Comparing the right-hand side of (4.5.15) with the right-hand side of the force equilibrium equation (4.3.11) for a two-dimensional interface with uniform surface tension, γ, we find that

$$p^{(2)} - p^{(1)} = \gamma\,\kappa, \tag{4.5.16}$$

where κ is the curvature of the interface. We have shown that the jump in the pressure across a two-dimensional interface in hydrostatics is equal to the product of the surface tension and the curvature of the interface.

Working in a similar fashion for a three-dimensional interface, we refer to (4.4.4) and find that

$$p^{(2)} - p^{(1)} = \gamma\,2\,\kappa_m, \tag{4.5.17}$$

where κ_m is the mean curvature of the interface. We have found that the jump in the pressure across a three-dimensional interface in hydrostatics is equal to the product of the surface tension and twice the mean curvature of the interface.

Laplace's law

As an application, we compute the jump in pressure across a spherical interface of radius a representing the surface of a liquid drop or bubble. Designating the outer fluid as fluid 1 and the inner fluid as fluid 2, we find that the mean curvature is $\kappa_m = 1/a$. Consequently, the pressure jump across the spherical interface is given by Laplace's law,

$$p^{(2)} - p^{(1)} = 2\,\frac{\gamma}{a}. \tag{4.5.18}$$

The pressure inside a drop or bubble is higher than the ambient pressure due to the interfacial tension by $2\gamma/a$.

Problems

4.5.1 *Jump in pressure across a circular interface*

Derive an expression for the jump in pressure across a circular interface of radius a representing the trace of a cylindrical thread in the xy plane.

4.5.2 *Curvature of a soap film*

Explain why the mean curvature of a thin soap film attached to a wire frame must be zero.

4.6 Constitutive equations

In the absence of macroscopically observable fluid motion, the traction exerted on a specified side of a small fluid surface indicated by the unit normal vector, $\mathbf{n}$, is given by equation (4.6.1),

$$\mathbf{f}(\mathbf{x}, \mathbf{n}) = -p\,\mathbf{n}, \tag{4.6.1}$$

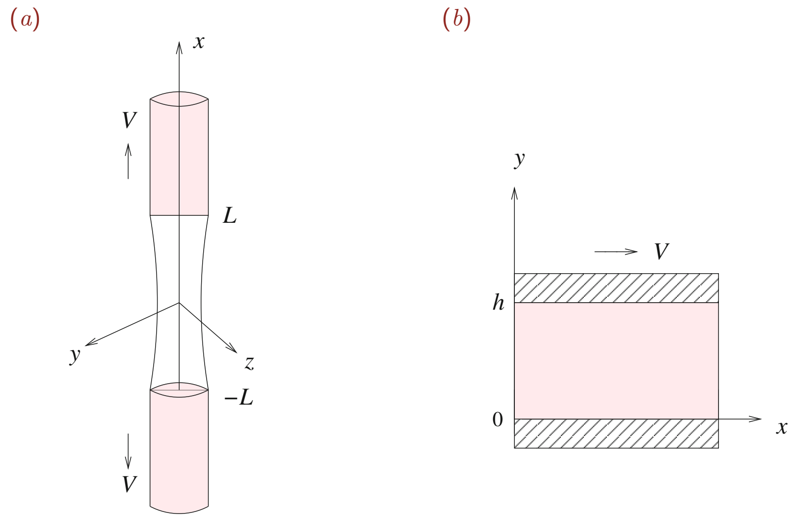


Figure 4.6.1 (a) Stretching of a liquid bridge between two coaxial cylinders that are pulled apart along their axes with velocity V. (b) Shear flow in a two-dimensional channel confined between two parallel plates located at $y = 0$ and h; the motion is due to the parallel translation of the upper plate with velocity V.

in terms of the pressure, p. In the presence of macroscopic fluid motion, this equation is modified in two ways. First, the normal component of the traction is accompanied by a new contribution that depends on the physical properties of the fluid and the nature of the fluid motion. Second, a tangential component of the traction is established.

To understand how these new contributions arise from a physical point of view, it is helpful to consider the tractions developing in two complementary flows: (a) an extensional flow where the fluid stretches and elongates, and (b) a channel flow where the fluid is sheared due to boundary motion, as shown in Figure 4.6.1.

Stretching of a thread

In one experiment, a thread of liquid is suspended between two rods forming an axisymmetric bridge, and the rods are pulled apart with velocity V extending the thread, as illustrated in Figure 4.6.1(a). A force is required to pull the rods apart and thus overcome the normal component of the hydrodynamic traction imparted by the fluid to the tips of the rods, so that

$$f_x^{(x)} \neq -p, \tag{4.6.2}$$

where p is the pressure discussed in Section 4.5 in the context of hydrostatics, Our intuition suggests that the faster the rods are pulled apart, the higher the magnitude of the normal component of the traction. The greater the distance between the rods, the lower the magnitude of the normal component of the traction.

For most common fluids, a linear relationship exists between the traction, the velocity of the rods, and the inverse of their distance, so that

$$f_x^{(x)} = -p + 2\,\mu_{\rm ext}\,\frac{V}{L}, \tag{4.6.3}$$

where L is half the instantaneous distance between the rods and $\mu_{\rm ext}$ is a physical constant associated with the fluid called the extensional viscosity of the fluid.

Shearing of a layer

In another experiment, a fluid is placed in a channel confined between two parallel plates. The upper plate translates in the direction of the x axis parallel to itself with constant velocity V, while the lower plate is held stationary, as depicted in Figure 4.6.1(b). A force in the x direction must be exerted on the upper plate to balance the tangential component of the traction developing due to the fluid motion, so that

$$f_x^{(y)} \neq 0. \tag{4.6.4}$$

The faster the velocity of the translating plate, the higher the magnitude of the traction; the greater the distance between the two plates, the lower the magnitude of the traction.

For most common fluids, a linear relationship exists between the traction, the velocity of the moving plate, and the inverse of the distance between the plates, h,

$$f_x^{(y)} = \mu_{\rm shear}\,\frac{V}{h}, \tag{4.6.5}$$

where $\mu_{\rm shear}$ is a physical constant associated with the fluid called the shear viscosity of the fluid.

4.6.1 Simple fluids

We have demonstrated by example that stresses develop in a fluid as a result of the motion. To proceed further, we consider the tractions developing at the surface of a small fluid parcel in motion and argue the following properties characterizing a simple fluid:

- If a fluid parcel translates, rotates, or translates and rotates as a rigid body, tractions do not develop at the parcel surface.
- Tractions develop only when a parcel deforms.
- The distribution of traction over the parcel surface at any particular time instant depends only on the type and rate of deformation of the parcel at that particular time instant.

Our analysis of kinematics in Chapter 2 has revealed that a small spherical fluid parcel in motion deforms to obtain an ellipsoidal shape whose axes are parallel to the three eigenvectors of the rate-of-deformation tensor given in Table 2.1.1. The directional rates of deformation are equal to the corresponding eigenvalues.

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -p + 2\mu \dfrac{\partial u_x}{\partial x} & \mu \,(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y}) & \mu \,(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z}) \\ \mu \,(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}) & -p + 2\mu \dfrac{\partial u_y}{\partial y} & \mu \,(\dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z}) \\ \mu \,(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}) & \mu \,(\dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y}) & -p + 2\mu \dfrac{\partial u_z}{\partial z} \end{bmatrix}$$

Table 4.6.1 Components of the stress tensor for an incompressible Newtonian fluid in Cartesian coordinates.

Temperature (°C)	*Water* (cp)	*Air* (cp)
20	1.002	0.0181
40	0.653	0.0191
80	0.355	0.0209

Table 4.6.2 The viscosity of water and air at three temperatures; cp stands for centipoise, which is one hundredth of the viscosity unit poise defined as 1 g/(cm sec). Thus, cp$\equiv 10^{-2}$g/(cm sec).

With these observations as a point of departure, we proceed to relate the stress tensor to the physical properties of the fluid and to the structure of the velocity field by a constitutive equation.

4.6.2 Incompressible Newtonian fluids

The constitutive equation for an incompressible Newtonian fluid reads

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + \mu\, 2\,\mathbf{E}, \tag{4.6.6}$$

where p is the pressure, the coefficient μ is the fluid viscosity, sometimes also called the dynamic viscosity, and $\mathbf{E}$ is the rate-of-deformation tensor given in Table 2.1.1. Note that the Newtonian constitutive relation respects the symmetry of the stress tensor discussed at the end of Section 4.2. Explicitly, the components of the stress tensor are given by the matrix equation shown in Table 4.6.1. The viscosity of water and air at three temperatures is given in Table 4.6.2.

In the absence of flow, we recover the hydrostatic stress tensor defined in equation (4.5.6), involving the hydrostatic pressure alone.

Unidirectional shear flow

As an example, we consider flow in a two-dimensional channel confined between two parallel plane walls. The motion of the fluid is generated by the translation of the upper wall, as shown in Figure 4.6.1(b). Physical intuition suggests that, at low and moderate velocities, the fluid will translate along the x axis with a position-dependent velocity u_x varying along the y axis; to signify this dependence, we write $u_x(y)$. Using equation (2.1.28), we find that the rate-of-deformation tensor is given by

$$\mathbf{E} = \frac{1}{2} \begin{bmatrix} 0 & \dfrac{du_x}{dy} \\ \dfrac{du_x}{dy} & 0 \end{bmatrix}. \tag{4.6.7}$$

Substituting this expression into the right-hand side of (4.6.6), we obtain the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} -p & \mu \dfrac{du_x}{dy} \\ \mu \dfrac{du_x}{dy} & -p \end{bmatrix}. \tag{4.6.8}$$

The off-diagonal components involving the local slope of the velocity profile, du_x/dy arise as a result of the fluid motion.

The x component of the traction exerted on a fluid surface that is perpendicular to the y axis, identified as the shear stress, is

$$f_x^{(y)} = \sigma_{yx} = \mu \frac{du_x}{dy}. \tag{4.6.9}$$

Physically, this traction can be attributed to the friction experienced by adjacent fluid layers as they slide over one another with gradually varying velocities.

4.6.3 Viscosity

Strictly speaking, the viscosity of a Newtonian fluid is a proportionality coefficient relating the stress tensor to the rate-of-deformation tensor, as shown in equation (4.6.6). However, it is reassuring to know that this mathematical definition, established by phenomenological observation, has a firm physical foundation. In fact, the viscosity is a genuine physical property dependent on the local physical conditions, including the temperature.

As the temperature increases, the viscosity of liquids decreases whereas the viscosity of gases increases, as shown in Table 4.6.2. This dichotomy is a reflection of the different physical mechanisms that are responsible for the development of stresses in these two complementary classes of fluids.

In the case of liquids, the viscosity is due to occasional molecular excursions from a mean position to neighboring empty sites. In the cases of gases, the viscosity is due to the relentless molecular excursions from regions of high velocity to regions of low velocity in the course of random motion due to thermal fluctuations.

4.6.4 Viscosity of a gas

To demonstrate the relation between molecular and macroscopic fluid motion, we consider a gas in unidirectional shear flow and derive an expression for the viscosity in terms of molecular properties. In the simplest kinetic theory, the molecules are modeled as rigid spheres moving with the local fluid velocity defined in Section 1.4, and with a randomly fluctuating velocity. The square of the average magnitude of the fluctuating component is

$$\overline{v}^2 = \frac{8}{\pi}\,\frac{k_B T}{M}, \tag{4.6.10}$$

where k_B is the Boltzmann's constant, T is the absolute temperature, and M is the molecular mass. In the course of the random motion, two molecules occasionally collide after having traveled an average distance equal to the mean free path, λ.

Consider a macroscopically stationary gas with vanishing fluid velocity. The number of molecules crossing an infinitesimal surface area during an infinitesimal time interval as a result of the fluctuating motion is denoted by n_{crossing}. Using principles of statistical mechanics, we find that n_{crossing} is proportional to (a) the number of molecules per infinitesimal volume, defined as the number density n, and (b) the average magnitude of the fluctuating velocity, $\overline{v}$. It can be shown that

$$n_{\text{crossing}} = \tfrac{1}{4}\, n\, \overline{v}. \tag{4.6.11}$$

The units of n_{crossing} are number of particles over time and length squared.

A molecule crossing a surface at a particular instant has collided with another molecule above or below the surface at an average distance a. Using principles of statistical mechanics, we find that

$$a = \tfrac{2}{3}\,\lambda, \tag{4.6.12}$$

where λ is the mean free path. Relations (4.6.11) and (4.6.12) have been derived taking into consideration that, since the molecules move randomly in all directions, only one component of the velocity brings them toward the crossing surface under consideration.

Relations (4.6.10)–(4.6.12) also hold true when the fluid exhibits macroscopic motion, provided that the molecular velocity is computed relative to the average velocity at the position where a molecule last underwent a collision.

Momentum transport

Shown in Figure 4.6.2 is a schematic illustration of the instantaneous distribution of molecules in a gas undergoing unidirectional shear flow along the x axis. Without loss of generality, we have assumed that the fluid velocity increases in the positive direction of the y axis. In the course of the motion, gas molecules cross a horizontal plane corresponding to a certain value of y, drawn with the heavy horizontal line, from either side. Because the x velocity of molecules crossing from above is higher than the x velocity of molecules crossing from below, x momentum is transferred in the negative direction of the y axis. The rate of transport of x momentum across a surface that is perpendicular to the y axis amounts to a hydrodynamic traction, $f_x^{(y)}$.

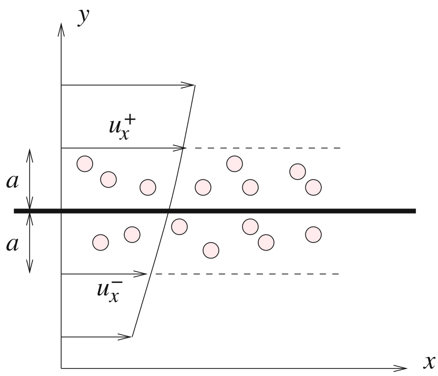


Figure 4.6.2 A molecular model of a gas in shear flow is used to derive an expression for the viscosity in terms of molecular properties, as shown in equation (4.6.17).

The rate of momentum transport defined in the last paragraph can be quantified by setting

$$f_x^{(y)} = -M\,(n_{\text{crossing}}^- u_x^- - n_{\text{crossing}}^+ u_x^+), \tag{4.6.13}$$

where u_x is the fluid velocity and the superscripts + and − indicate that the superscripted variable is evaluated at a distance equal to a above or below the transport surface. Effectively, the collection of molecules crossing the y plane during an infinitesimal period of time are represented by model molecules distinguished by the following two important properties:

- The model molecules last underwent a collision at a distance a above or below the y plane.
- The model molecules move with an average velocity that is equal to the local fluid velocity evaluated at the position of the last collision.

Because the flow is unidirectional, the mean fluid velocity normal to a horizontal plane is zero and the number of molecules crossing the y plane from either side during an infinitesimal time period are equal,

$$n_{\text{crossing}}^- = n_{\text{crossing}}^+. \tag{4.6.14}$$

Combining equations (4.6.11)–(4.6.13), we obtain

$$f_x^{(y)} = M\,\frac{1}{4}\,n\,\overline{v}\,\frac{u_x^+ - u_x^-}{2a}\,\frac{4}{3}\,\lambda. \tag{4.6.15}$$

Since a is small compared to the macroscopic length scale of the shear flow, the fraction on the right-hand side can be approximated with a derivative, yielding the final result

$$f_x^{(y)} = \frac{1}{3}\,nM\overline{v}\lambda\,\frac{\mathrm{d}u_x}{\mathrm{d}y}. \tag{4.6.16}$$

Now comparing equations (4.6.16) and (4.6.9), we derive an expression for the viscosity of a gas in terms of the number density, n, the molecular mass, M, the magnitude of the fluctuating velocity component, $\overline{v}$, and the mean free path, λ,

$$\mu = \tfrac{1}{3}\, n M \overline{v} \lambda. \tag{4.6.17}$$

The units of the four terms on the right-hand side following the numerical fraction $\frac{1}{3}$ are as follows:

$$\frac{\text{Particle}}{\text{Volume}} \times \frac{\text{Mass}}{\text{Particle}} \times \frac{\text{Length}}{\text{Time}} \times \text{Length} = \frac{\text{Mass}}{\text{Length Time}}, \tag{4.6.18}$$

as required.

We have derived the Newtonian constitutive equation from molecular considerations and obtained a prediction for the viscosity of a gas in terms of molecular properties.

4.6.5 Ideal fluids

If the viscosity of a fluid vanishes, the fluid is frictionless and is called ideal. The stress tensor in an ideal fluid is given by a simplified version of (4.6.6),

$$\boldsymbol{\sigma} = -p\,\mathbf{I}. \tag{4.6.19}$$

However, in practice, no fluid is ideal and the absence of viscosity should be interpreted strictly as insignificance of hydrodynamic forces or stresses associated with the fluid viscosity. Viscous stresses are always important near solid boundaries, as discussed in Chapter 10. The formal requirement for viscous stresses to be negligible will be discussed in Chapter 6 with reference to the Reynolds number.

4.6.6 Significance of the pressure in an incompressible fluid

The physical interpretation of the pressure in the Newtonian constitutive equation (4.6.6), or any other constitutive equation, is not entirely clear. Strictly speaking, the pressure is a mathematical entity defined in terms of the trace of the stress tensor,

$$p = -\tfrac{1}{3}\,\text{trace}(\boldsymbol{\sigma}) = -\tfrac{1}{3}\,(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}). \tag{4.6.20}$$

All we can say with confidence is that, as a fluid becomes quiescent, the dynamic pressure reduces to the hydrostatic pressure computed from consideration of random molecular motions.

In the case of compressible gases, an equation relating the pressure to the density to the temperature can be derived working in the framework of equilibrium thermodynamics, as discussed in Section 4.7

Problems

4.6.1 *Flow in a channel.*

Consider steady unidirectional flow in a channel due to the translation of the upper wall, as depicted in Figure 4.6.1(b).

(a) Perform a force balance over a rectangular fluid layer confined between two y levels to show that, if the pressure is uniform, the shear stress $f_x^{(y)}$ must be independent of y.

(b) Having established that $f_x^{(y)}$ is constant, solve the first-order differential equation (4.6.9) for u_x in terms of y subject to the no-slip boundary conditions $u_x(y=0)=0$ and $u_x(y=h)=V$, and evaluate $f_x^{(y)}$ in terms of μ, V, and the channel width, h.

4.6.2 *Extensional flow*

(a) Consider a two-dimensional extensional flow in the xy plane with velocity components

$$u_x = \xi x, \qquad u_y = -\xi y, \tag{4.6.21}$$

where ξ is the rate of extension with units of inverse time. The corresponding pressure field is uniform throughout the domain of flow. Confirm that the fluid is incompressible, sketch the streamline pattern, and evaluate the stress tensor.

(b) Repeat (a) for axisymmetric extensional flow with Cartesian velocity components

$$u_x = \xi x, \qquad u_y = -\tfrac{1}{2}\,\xi y, \qquad u_z = -\tfrac{1}{2}\,\xi. \tag{4.6.22}$$

(c) The axisymmetric extensional flow discussed in (b) describes the motion inside the thread illustrated in Figure 4.6.1(a). Assuming that the fluid is Newtonian, compute the force necessary to pull the rods apart with velocity V in terms of the half-length of the thread, L, the fluid viscosity, μ, and the cross-sectional area of the rods, A.

4.7 Pressure in compressible fluids

Consider a small fluid parcel of a compressible gas with volume V. To decrease the volume of the parcel by a differential amount, $\mathrm{d}V$, we may apply an external pressure, p, by way of an ideal *frictionless* piston. The differential work required to carry out this reduction is

$$\delta W = -p\,\mathrm{d}V = -p\,n\,\mathrm{d}v, \tag{4.7.1}$$

where n is the number of moles contained in the parcel and v is the specific volume defined as the volume occupied by one mole of gas; by definition,

$$V = nv. \tag{4.7.2}$$

In our experiment, $\mathrm{d}V$ and $\mathrm{d}v$ are both negative due to compression, while δW is positive. In the case of expansion, $\mathrm{d}V$ and $\mathrm{d}v$ would be both positive, while δW would be negative, indicating that energy would be released instead of supplied.

The notation δW emphasizes that the differential work can be computed only after the frictional properties of the piston have been specified. In formal thermodynamics, we say that δW is an *inexact differential.*

Part of the work in compression or expansion is spent to increase the temperature of the parcel, T, by a differential amount, $\mathrm{d}T$, and therefore the internal (thermal) energy of the parcel, U, by a differential amount, $\mathrm{d}U$. The remainder of the work escapes as a (negative) process-dependent heat loss, δQ.

Energy conservation for a closed system in the absence of significant kinetic or potential energy requires that

$$\mathrm{d}U = \delta W + \delta Q, \tag{4.7.3}$$

which can be rearranged as

$$\delta W = \mathrm{d}U - \delta Q. \tag{4.7.4}$$

A certain change in internal energy, $\mathrm{d}U$, can be achieved by different combinations of δW and δQ satisfying this equation.

Reversible process

To quantify the heat loss in the case of a *reversible process*, we write

$$\delta Q_{\mathrm{rev}} = T\,\mathrm{d}S, \tag{4.7.5}$$

and obtain

$$\delta W_{\mathrm{rev}} = \mathrm{d}U - T\,\mathrm{d}S, \tag{4.7.6}$$

where S is the entropy. Dividing this equation by the number of moles of the gas, n, we obtain

$$\delta w_{\mathrm{rev}} = \mathrm{d}u - T\,\mathrm{d}s, \tag{4.7.7}$$

where

$$w = \frac{W}{n}, \qquad u = \frac{U}{n}, \qquad s = \frac{S}{n} \tag{4.7.8}$$

are the specific work, specific internal energy, and specific entropy.

Now substituting into (4.7.7) the expression $\delta w_{\mathrm{rev}} = -p\,\mathrm{d}v$ and rearranging, we derive a process-independent differential relation in the absence of inexact differentials,

$$\mathrm{d}u = T\,\mathrm{d}s - p\,\mathrm{d}v. \tag{4.7.9}$$

If the specific volume of a gas is made to change by the same small amount, $\mathrm{d}v$, according to two different processes, the corresponding changes in the specific internal energy, $\mathrm{d}u$, and specific entropy, $\mathrm{d}s$, will be such that equation (4.7.9) is satisfied in both cases.

Ideal gas

The specific internal energy of an arbitrary gas depends on the temperature, T, and specific volume, v. Due of the absence of intermolecular forces, the specific internal energy of an ideal gas depends on the temperature alone. The change in the specific internal energy is given by

$$\mathrm{d}u = c_v \, \mathrm{d}T, \tag{4.7.10}$$

where c_v is the specific heat capacity under constant volume. Substituting this equation into the balance equation (4.7.9) and rearranging, we find that

$$\mathrm{d}s = \frac{c_v}{T} \, \mathrm{d}T + \frac{p}{T} \, \mathrm{d}v, \tag{4.7.11}$$

which shows that

$$\frac{c_v}{T} = \left(\frac{\partial s}{\partial T}\right)_v, \tag{4.7.12}$$

that is, the ratio c_v/T is the partial derivative of the specific entropy with respect to the temperature under constant volume. For a mono-atomic ideal gas, $c_v = \frac{3}{2} R$. For a diatomic ideal gas, $c_v = \frac{5}{2} R$.

Change of entropy

Now solving the equation of state for an ideal gas, $pv = RT$, for the pressure, and substituting the result into (4.7.1), we find that

$$\delta w = -p \, \mathrm{d}v = -RT \, \frac{\mathrm{d}v}{v}. \tag{4.7.13}$$

Substituting this expression into the balance equation(4.7.11), we obtain

$$-R \, \frac{\mathrm{d}v}{v} = c_v \, \frac{\mathrm{d}T}{T} - \mathrm{d}s. \tag{4.7.14}$$

Rearranging, we derive an expression for the change in the specific entropy,

$$\mathrm{d}s = c_v \, \frac{\mathrm{d}T}{T} + R \, \frac{\mathrm{d}v}{v}, \tag{4.7.15}$$

applicable for an ideal gas.

Next, we treat c_v as a constant and integrate (4.7.15) between two states labeled A and B to derive an expression for the difference in entropy,

$$\Delta s \equiv s_\mathrm{B} - s_\mathrm{A} = c_v \, \ln \frac{T_\mathrm{B}}{T_\mathrm{A}} + R \, \ln \frac{v_\mathrm{B}}{v_\mathrm{A}}, \tag{4.7.16}$$

which can be rearranged into

$$\Delta s \equiv s_\mathrm{B} - s_\mathrm{A} = \ln \left[\left(\frac{T_\mathrm{B}}{T_\mathrm{A}}\right)^{c_v} \left(\frac{v_\mathrm{B}}{v_\mathrm{A}}\right)^{R} \right]. \tag{4.7.17}$$

For example, in the case of constant pressure, $p_B = p_A$, we use the equation of state to write $v_B/v_A = T_B/T_A$ and

$$\Delta s = c_p \ln \frac{T_B}{T_A}, \tag{4.7.18}$$

where $c_p = c_v + R$ is the specific heat capacity under constant pressure.

Isentropic compression or expansion

In the case of a constant entropy (isentropic) reversible process, $s_B = s_A$, the general equation (4.7.17) for an ideal gas yields

$$\left(\frac{T_B}{T_A}\right)^{c_v} \left(\frac{v_B}{v_A}\right)^{R} = 1, \tag{4.7.19}$$

and thus

$$T^{c_v} v^R = A, \tag{4.7.20}$$

where A is a constant. This equation provides us with a relation between the specific volume, v, and the temperature, T. Introducing the density, $\rho = M/v$, we write

$$T^{c_v} = B\rho^R, \tag{4.7.21}$$

where M is the molecular weight and $B \equiv A/M^R$ is a new constant. Using the ideal gas law expressed in the form

$$T = \frac{Mp}{R\rho}, \tag{4.7.22}$$

we eliminate the temperature and thus obtain a relation between the density and the pressure in an ideal compressible gas in isentropic transition,

$$p = D\rho^k, \tag{4.7.23}$$

where D is a new constant and

$$k \equiv \frac{c_v + R}{c_v} = \frac{c_p}{c_v} \tag{4.7.24}$$

is the heat capacity ratio.

Speed of sound

The square of the speed of sound, c, is given by the formula

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s. \tag{4.7.25}$$

The right-hand side is the derivative of the pressure with respect to the density at constant entropy. Using (4.7.23), we find that

$$c^2 = D\,k\,\rho^{k-1} = k\,\frac{p}{\rho}, \tag{4.7.26}$$

and then

$$c^2 = k\,\frac{RT}{M}. \tag{4.7.27}$$

The higher the temperature, the faster the speed of sound, in agreement with physical intuition.

Problem

4.7.1 *Speed of sound in the atmosphere*

Use equation (4.7.27) to predict the speed of sound in the atmosphere regarded as an ideal gas with $k = 1.4$ and molecular mass $M = 28.97$ kg/kmole at 25° C.

4.8 Simple non-Newtonian fluids

The Newtonian constitutive law for an incompressible fluid, expressed by equation (4.6.6), describes the stresses developing in a fluid consisting of small molecules. Fluids containing or consisting of macromolecules, such as polymeric solutions and melts, and fluids containing suspended rigid or deformable particles, exhibit a more complicated behavior described by more involved constitutive equations. Examples include pastes, bubbly liquids, and biological fluids, such as blood.

To derive a constitutive equation for a non-Newtonian fluid, we consider the motion of a small fluid parcel and seek to establish a relation between the instantaneous traction exerted on the parcel surface, expressed in terms of the stress tensor, $\boldsymbol{\sigma}$, and the entire history of the parcel deformation. In the simplest class of materials, the traction depends only on the instantaneous rate of parcel deformation expressed by the rate-of-deformation tensor, $\mathbf{E}$.

A distinguishing feature of a non-Newtonian fluid is that the relation between the stress tensor, $\boldsymbol{\sigma}$, and the rate-of-deformation tensor, $\mathbf{E}$, is nonlinear. In contrast, the corresponding relation for a Newtonian fluid is linear.

4.8.1 Unidirectional shear flow

In the case of two-dimensional unidirectional shear flow along the x axis, the Newtonian shear stress, given by

$$\sigma_{yx} = \mu\,\frac{\mathrm{d}u_x}{\mathrm{d}y}, \tag{4.8.1}$$

can be generalized by allowing the viscosity to depend on the magnitude of the shear rate, $|\mathrm{d}u_x/\mathrm{d}y|$, where the vertical bars indicate the absolute value. If the viscosity decreases as the shear rate increases, the fluid is shear-thinning or pseudo-plastic. If the viscosity increases as the shear rate increases, the fluid is shear-thickening or dilatant.

Physically, the dependence of the viscosity on the shear rate is attributed to changes in the configuration of molecules, changes in the shape and relative position of particles suspended in a fluid, and to the spontaneous formation of internal microstructure due to intermolecular force fields and other particle interactions.

Power-law fluids

The shear stress developing in a certain class of non-Newtonian fluids in unidirectional shear flow can be described by the *Ostwald-de Waele model.* In this model, the viscosity is proportional to the magnitude of the shear rate raised to a certain power,

$$\mu = \mu_0 \left| \frac{\mathrm{d}u_x}{\mathrm{d}y} \right|^{n-1}, \tag{4.8.2}$$

where μ_0 is a reference viscosity and n is the power-law exponent. When $n = 1$, we obtain a Newtonian fluid with viscosity μ_0; when $n < 1$, we obtain a shear-thinning fluid; when $n > 1$, we obtain a shear-thickening fluid.

Substituting (4.8.2) into expression (4.6.9), we derive an expression for the shear stress,

$$f_x^{(y)} = \sigma_{yx} = \mu_0 \left| \frac{\mathrm{d}u_x}{\mathrm{d}y} \right|^{n-1} \frac{\mathrm{d}u_x}{\mathrm{d}y}. \tag{4.8.3}$$

When $n = 1$, we recover the Newtonian shear stress.

4.8.2 Channel flow

As an application, we consider flow in a channel due to the translation of the upper wall with velocity V, as illustrated in Figure 4.6.1(b). Performing a force balance over a rectangular fluid layer, we find that, if the pressure is uniform, the shear stress, σ_{yx}, is independent of y and the right-hand side of (4.8.3) is constant (Problem 4.5.1).

The fluid velocity at the upper wall located at $y = h$ is equal to the wall velocity, V, while the fluid velocity at the stationary lower wall located at $y = 0$ is zero. Integrating equation $\mathrm{d}u_x/\mathrm{d}y = c$, where c is the constant shear rate, and using the aforementioned boundary conditions, we derive a linear velocity profile with shear rate

$$\frac{\mathrm{d}u_x}{\mathrm{d}y} = \frac{V}{h}, \tag{4.8.4}$$

independent of the value of the power-law exponent, n.

Although the velocity profile is linear for any value of n, the magnitude of the shear stress depends on n, as shown in equation (4.8.3). This distinction emphasizes that the

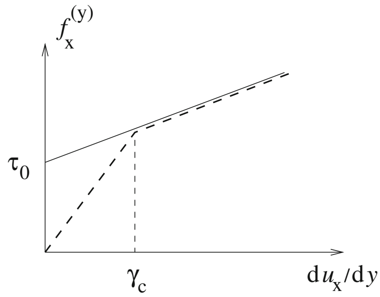


Figure 4.8.1 Rheological response of a Bingham plastic showing a yield-stress behavior in unidirectional flow.

kinematic appearance of a flow does not necessarily reflect the magnitude of the stresses developing in the fluid. Two flows that are kinematically identical may support different stress fields.

4.8.3 Yield-stress fluids

A class of heterogeneous fluids, called Bingham plastics, flow only when the shear stress established due to the motion exceeds a certain threshold. Examples include pastes and concentrated suspensions of fine particles. An idealized constitutive equation between stress and shear rate for this class of materials is

$$\frac{du_x}{dy} = 0 \qquad \text{if} \qquad |\sigma_{yx}| < \tau_0 \tag{4.8.5}$$

and

$$f_x^{(y)} = \sigma_{yx} = \tau_0 + \mu\,\frac{du_x}{dy} \qquad \text{if} \qquad |\sigma_{yx}| > \tau_0, \tag{4.8.6}$$

where μ is the viscosity and τ_0 is the yield stress. The relation between the shear stress and the shear rate is represented by the solid line in Figure 4.8.1.

As an application, we consider the familiar unidirectional flow in a channel confined between two parallel walls, generated by exerting on the upper wall a force, F, parallel to the x axis. If the fluid is a Bingham plastic whose rheological behavior is described by equations (4.8.5) and (4.8.6), a shear flow across the entire cross-section of the channel will be established only if the externally imposed force F over a certain length of the channel, L, counteracting the shear stress, $\sigma_{yx} = F/L$, is greater than the yield-stress threshold, τ_0.

Assuming that this occurs, we treat σ_{yx} as a constant, solve equation (4.8.6) for du_x/dy, and then integrate with respect to y subject to the boundary condition $u_x(y = 0) = 0$ to obtain a linear velocity profile,

$$u_x = \frac{y}{\mu}\,(\frac{F}{L} - \tau_0). \tag{4.8.7}$$

The velocity at the upper wall is

$$u_x(y = h) = \frac{h}{\mu}\,(\frac{F}{L} - \tau_0) = V. \tag{4.8.8}$$

In practice, equation (4.8.8) allows us to estimate the values of the physical constants μ and τ_0 from laboratory observations.

Problem

4.8.1 *Yield-stress fluid*

The relation between the shear stress and shear rate for a class of yield-stress fluids in unidirectional flow is described by the broken line in Figure 4.8.1, where γ_c is the critical shear rate.

(a) State the equations describing this rheological behavior.

(b) Compute the shear stress established in a channel with parallel walls, where the upper wall translates with velocity V while the lower wall is held stationary.

4.9 Stresses in polar coordinates

We have discussed tractions and stresses in Cartesian coordinates. In practice, it is often convenient to work in cylindrical, spherical, or plane polar coordinates, with the benefit of reduced algebraic manipulations and ease in the implementation of boundary conditions. In this section, we define the components of the stress tensor in polar coordinates and relate them to the pressure and to the corresponding components of the rate-of-deformation tensor using the constitutive equation for an incompressible Newtonian fluid.

4.9.1 Cylindrical polar coordinates

Consider the cylindrical polar coordinates, (x, σ, φ), depicted in Figure 4.9.1(a). The traction exerted on a small surface that is perpendicular to the x axis, $\mathbf{f}^{(x)}$, acting on the side that faces the positive direction of the x axis, can be resolved into its cylindrical polar components as

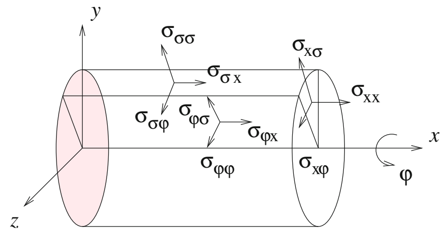
$$\mathbf{f}^{(x)} = f_x^{(x)}\,\mathbf{e}_x + f_\sigma^{(x)}\,\mathbf{e}_\sigma + f_\varphi^{(x)}\,\mathbf{e}_\varphi, \tag{4.9.1}$$

where $\mathbf{e}_x$, $\mathbf{e}_\sigma$, and $\mathbf{e}_\varphi$ are unit vectors pointing, respectively, in the axial, radial, and azimuthal direction. Note that the orientation of $\mathbf{e}_x$ is constant, whereas the orientations of $\mathbf{e}_\sigma$ and $\mathbf{e}_\varphi$ change with position in the flow.

The traction exerted on a small surface that is perpendicular to the distance from the x axis, $\mathbf{f}^{(\sigma)}$, and is thus parallel to the axial and azimuthal directions at a designated center of the surface, can be resolved into corresponding components as

$$\mathbf{f}^{(\sigma)} = f_x^{(\sigma)}\,\mathbf{e}_x + f_\sigma^{(\sigma)}\,\mathbf{e}_\sigma + f_\varphi^{(\sigma)}\,\mathbf{e}_\varphi. \tag{4.9.2}$$

(a)



(b)

$$\sigma_{xx} = -p + 2\mu \frac{\partial u_x}{\partial x}, \qquad \sigma_{x\sigma} = \sigma_{\sigma x} = \mu \left(\frac{\partial u_x}{\partial \sigma} + \frac{\partial u_\sigma}{\partial x} \right)$$

$$\sigma_{x\varphi} = \sigma_{\varphi x} = \mu \left(\frac{\partial u_\varphi}{\partial x} + \frac{1}{\sigma} \frac{\partial u_x}{\partial \varphi} \right), \qquad \sigma_{\sigma\sigma} = -p + 2\mu \frac{\partial u_\sigma}{\partial \sigma}$$

$$\sigma_{\sigma\varphi} = \sigma_{\varphi\sigma} = \mu \left(\sigma \frac{\partial}{\partial \sigma} \left(\frac{u_\varphi}{\sigma} \right) + \frac{1}{\sigma} \frac{\partial u_\sigma}{\partial \varphi} \right), \qquad \sigma_{\varphi\varphi} = -p + 2\,\mu \left(\frac{1}{\sigma} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\sigma}{\sigma} \right)$$

Table 4.9.1 (a) Physical depiction of the components of the stress tensor in cylindrical polar coordinates and (b) expressions for the components of the stress tensor in a Newtonian fluid. Note that the stress tensor remains symmetric in these coordinates.

The traction exerted on a small surface that is normal to the azimuthal direction, $\mathbf{f}^{(\varphi)}$, can be resolved as

$$\mathbf{f}^{(\varphi)} = f_x^{(\varphi)}\, \mathbf{e}_x + f_\sigma^{(\varphi)}\, \mathbf{e}_\sigma + f_\varphi^{(\varphi)}\, \mathbf{e}_\varphi. \tag{4.9.3}$$

Stacking the coefficients of the unit vectors on the right-hand sides of (4.9.1)–(4.9.3) on top of one another in a particular order, we obtain the cylindrical polar components of the stress tensor,

$$(\sigma_{\alpha\beta}) = \begin{pmatrix} f_x^{(x)} & f_\sigma^{(x)} & f_\varphi^{(x)} \\ f_x^{(\sigma)} & f_\sigma^{(\sigma)} & f_\varphi^{(\sigma)} \\ f_x^{(\varphi)} & f_\sigma^{(\varphi)} & f_\varphi^{(\varphi)} \end{pmatrix}, \tag{4.9.4}$$

where Greek indices stand for x, σ, or φ. We have used large parentheses instead of the square brackets to indicate that the matrix shown in (4.9.4) should not be misinterpreted as a Cartesian tensor.

Now we define

$$\sigma_{\alpha\beta} \equiv f_\beta^{(\alpha)}, \tag{4.9.5}$$

where Greek indices stand for x, σ, or φ. With the convention expressed by (4.9.5), the cylindrical polar components of the stress tensor are collected in the matrix

$$(\sigma_{\alpha\beta}) = \begin{pmatrix} \sigma_{xx} & \sigma_{x\sigma} & \sigma_{x\varphi} \\ \sigma_{\sigma x} & \sigma_{\sigma\sigma} & \sigma_{\sigma\varphi} \\ \sigma_{\varphi x} & \sigma_{\varphi\sigma} & \sigma_{\varphi\varphi} \end{pmatrix}. \tag{4.9.6}$$

This matrix should not be misinterpreted as a Cartesian tensor.

Newtonian fluids

The stress components in an incompressible Newtonian fluid derive from the constitutive equation (4.6.6) as shown in Table 4.9.1(a). To derive these relations, we may write

$$\mathbf{f}^{(x)} = \sigma_{xx}\,\mathbf{e}_x + \sigma_{xy}\,\mathbf{e}_y + \sigma_{xz}\,\mathbf{e}_z, \tag{4.9.7}$$

for the axial component,

$$\begin{aligned} \mathbf{f}^{(\sigma)} &= \mathbf{f}^{(y)}\,\cos\varphi + \mathbf{f}^{(z)}\,\sin\varphi \\ &= (\sigma_{yx}\,\mathbf{e}_x + \sigma_{yy}\,\mathbf{e}_y + \sigma_{yz}\,\mathbf{e}_z)\,\cos\varphi + (\sigma_{zx}\,\mathbf{e}_x + \sigma_{zy}\,\mathbf{e}_y + \sigma_{zz}\,\mathbf{e}_z)\,\sin\varphi, \end{aligned} \tag{4.9.8}$$

for the radial component, and

$$\begin{aligned} \mathbf{f}^{(\varphi)} &= -\mathbf{f}^{(y)}\,\sin\varphi + \mathbf{f}^{(z)}\,\cos\varphi \\ &= -(\sigma_{yx}\,\mathbf{e}_x + \sigma_{yy}\,\mathbf{e}_y + \sigma_{yz}\,\mathbf{e}_z)\,\sin\varphi + (\sigma_{zx}\,\mathbf{e}_x + \sigma_{zy}\,\mathbf{e}_y + \sigma_{zz}\,\mathbf{e}_z)\,\cos\varphi, \end{aligned} \tag{4.9.9}$$

for the azimuthal component. Substituting

$$\mathbf{e}_y = \mathbf{e}_\sigma\,\cos\varphi - \mathbf{e}_\varphi\,\sin\varphi, \qquad \mathbf{e}_z = \mathbf{e}_\sigma\,\sin\varphi + \mathbf{e}_\varphi\,\cos\varphi, \tag{4.9.10}$$

and consolidating terms multiplying the unit cylindrical polar vectors in (4.9.7), we obtain

$$\sigma_{x\sigma} = \sigma_{xy}\,\cos\varphi + \sigma_{xz}\,\sin\varphi = \mu\left(\cos\varphi\,\frac{\partial u_x}{\partial y} + \sin\varphi\,\frac{\partial u_x}{\partial z}\right) + \mu\,\frac{\partial(u_y\cos\varphi + u_z\sin\varphi)}{\partial x} \tag{4.9.11}$$

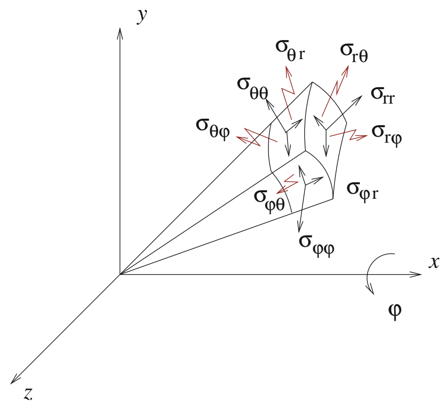
and

$$\sigma_{x\varphi} = -\sigma_{xy}\,\sin\varphi + \sigma_{xz}\,\cos\varphi = \mu\left(-\sin\varphi\,\frac{\partial u_x}{\partial y} + \cos\varphi\,\frac{\partial u_x}{\partial z}\right) + \mu\,\frac{\partial(-u_y\sin\varphi + u_z\cos\varphi)}{\partial x}, \tag{4.9.12}$$

which reproduce the second and third relations in Table 4.9.1(b). The rest of the relations can be derived working in a similar fashion. More expedient methods of deriving these relations are available.[1]

[1] Pozrikidis, C. (2011) *Introduction to Theoretical and Computational Fluid Dynamics.* Second Edition, Oxford University Press.

(a)



(b)

$$\sigma_{rr} = -p + 2\,\mu\,\frac{\partial u_r}{\partial r}, \qquad \sigma_{r\theta} = \sigma_{\theta r} = \mu\left(r\,\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

$$\sigma_{r\varphi} = \sigma_{\varphi r} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + r\,\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right), \qquad \sigma_{\theta\theta} = -p + 2\,\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)$$

$$\sigma_{\theta\varphi} = \sigma_{\varphi\theta} = \mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi}\right)$$

$$\sigma_{\varphi\varphi} = -p + \mu\,\frac{2}{r\sin\theta}\left(\frac{\partial u_\varphi}{\partial \varphi} + u_r\sin\theta + u_\theta\cos\theta\right)$$

Table 4.9.2 (a) Physical depiction of the components of the stress tensor in spherical polar coordinates and (b) components of the stress tensor in a Newtonian fluid. Note that the stress tensor remains symmetric in these coordinates.

4.9.2 Spherical polar coordinates

Consider a system of spherical polar coordinates, (r, θ, φ), defined in Table 4.9.2(a). The traction exerted on a small surface that is normal to the distance from the origin, acting on the side of the surface that faces away from the origin, $\mathbf{f}^{(r)}$, can be resolved into its spherical polar components as

$$\mathbf{f}^{(r)} = f_r^{(r)}\,\mathbf{e}_r + f_\theta^{(r)}\,\mathbf{e}_\theta + f_\varphi^{(r)}\,\mathbf{e}_\varphi, \tag{4.9.13}$$

where $\mathbf{e}_x$, $\mathbf{e}_\theta$, and $\mathbf{e}_\varphi$ are unit vectors pointing in the radial, meridional, and azimuthal directions.

The traction exerted on a small surface that is normal to the meridional direction corresponding to the angle θ, and is thus parallel to the radial and azimuthal directions,

$\mathbf{f}^{(\theta)}$, can be resolved as

$$\mathbf{f}^{(\theta)} = f_r^{(\theta)}\,\mathbf{e}_r + f_\theta^{(\theta)}\,\mathbf{e}_\theta + f_\varphi^{(\theta)}\,\mathbf{e}_\varphi. \tag{4.9.14}$$

The traction exerted on a small surface that is normal to the azimuthal direction, and is thus parallel to the radial and meridional directions, $\mathbf{f}^{(\varphi)}$, can be resolved as

$$\mathbf{f}^{(\varphi)} = f_r^{(\varphi)}\,\mathbf{e}_r + f_\theta^{(\varphi)}\,\mathbf{e}_\theta + f_\varphi^{(\varphi)}\,\mathbf{e}_\varphi. \tag{4.9.15}$$

Stacking the coefficients of the unit vectors on the right-hand sides of (4.9.13)–(4.9.15) on top of one another in a particular order, we obtain a matrix containing the spherical polar components of the stress tensor,

$$(\sigma_{\alpha\beta}) = \begin{pmatrix} f_r^{(r)} & f_\theta^{(r)} & f_\varphi^{(r)} \\ f_r^{(\theta)} & f_\theta^{(\theta)} & f_\varphi^{(\theta)} \\ f_r^{(\varphi)} & f_\theta^{(\varphi)} & f_\varphi^{(\varphi)} \end{pmatrix}, \tag{4.9.16}$$

where Greek indices stand for x, σ, or φ.

Now we introduce the standard two-index notation for the stress tensor, writing

$$\sigma_{\alpha\beta} \equiv f_\beta^{(\alpha)}, \tag{4.9.17}$$

where Greek indices stand for r, θ, or φ. With the convention expressed by (4.9.17), the matrix containing the spherical polar components of the stress tensor is given by

$$(\sigma_{\alpha\beta}) = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\varphi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\varphi} \\ \sigma_{\varphi r} & \sigma_{\varphi\theta} & \sigma_{\varphi\varphi} \end{pmatrix}. \tag{4.9.18}$$

This matrix should not be misinterpreted as a Cartesian tensor.

Newtonian fluids

The stress components for an incompressible Newtonian fluid derive from the constitutive equation (4.6.6) as shown in Table 4.9.2(b).

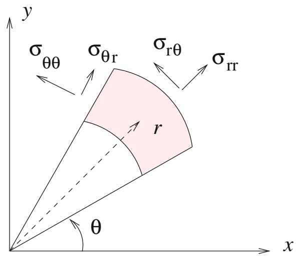
4.9.3 Plane polar coordinates

Consider a two-dimensional flow in the xy plane and refer to the plane polar coordinates, (r, θ), depicted in Table 4.9.3(a). The traction exerted on a small segment that is normal to the distance from the origin, acting on the side facing away from the origin, $\mathbf{f}^{(r)}$, can be resolved into its plane polar components as

$$\mathbf{f}^{(r)} = f_r^{(r)}\,\mathbf{e}_r + f_\theta^{(r)}\,\mathbf{e}_\theta, \tag{4.9.19}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors pointing in the radial and polar direction.

(*a*)



(*b*)

$$\sigma_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r}, \qquad \sigma_{r\theta} = \sigma_{\theta r} = \mu \left(r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right)$$

$$\sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right)$$

Table 4.9.3 (*a*) Components of the stress tensor in plane polar coordinates (r, θ) and (b) components of the stress tensor in a Newtonian fluid. Note that the stress tensor remains symmetric in these coordinates.

The traction exerted on a small surface that is normal to the direction of the polar angle θ, and is thus parallel to the distance from the origin, $\mathbf{f}^{(\theta)}$, can be resolved as

$$\mathbf{f}^{(\theta)} = f_r^{(\theta)}\, \mathbf{e}_r + f_\theta^{(\theta)}\, \mathbf{e}_\theta. \tag{4.9.20}$$

Stacking the coefficients of the unit vectors on the right-hand sides of (4.9.19) and (4.9.20) on top of one another in a particular order, we obtain the plane polar components of the stress tensor,

$$\begin{pmatrix} f_r^{(r)} & f_\theta^{(r)} \\ f_r^{(\theta)} & f_\theta^{(\theta)} \end{pmatrix}. \tag{4.9.21}$$

Next, we introduce the familiar two-index notation,

$$\sigma_{\alpha\beta} \equiv f_\beta^{(\alpha)}, \tag{4.9.22}$$

where Greek indices stand for r or θ. With the convention expressed by (4.9.22), the matrix containing the plane polar components of the stress tensor is given by

$$(\sigma_{\alpha\beta}) = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{\theta r} & \sigma_{\theta\theta} \end{pmatrix}. \tag{4.9.23}$$

This matrix should not be confused with a Cartesian tensor.

Newtonian fluids

The stress components for an incompressible Newtonian fluid derive from the constitutive equation (4.6.6) as shown in Table 4.9.3(b).

Problem

4.9.1 *Plane polar coordinates*

Work as discussed in the text for the cylindrical polar coordinates to derive the Newtonian constitutive equations in plane polar coordinates shown in Table 4.9.3.

4.10 Boundary conditions for the tangential velocity

In Section 2.10.1, we discussed the no-penetration boundary condition over impermeable boundaries and interfaces between immiscible fluids, involving the normal component of the fluid velocity. Viscous fluids obey an additional boundary condition concerning the tangential component of the fluid velocity.

4.10.1 No-slip boundary condition

Under most conditions, the vast majority of fluids satisfy the no-slip boundary condition requiring that:

- The tangential component of the fluid velocity over a solid boundary is equal to the tangential component of the boundary velocity.

- The tangential component of the fluid velocity is continuous across an interface between two immiscible fluids.

The no-slip boundary condition has been confirmed in the overwhelming majority of applications and is the standard choice in mainstream fluid dynamics. Combined with the no-slip condition, the no-penetration condition requires that the fluid velocity is equal to the local velocity of an impermeable solid boundary and continuous across an interface.

Physical origin

The physical origin of the no-slip boundary condition over a solid surface has not been established with absolute certainty. One theory argues that the molecules of a fluid next to a solid surface are adsorbed onto the surface for a short period of time, only to be desorbed and ejected into the fluid. This relentless process slows down the fluid and effectively renders the tangential component of the fluid velocity equal to the corresponding component of the boundary velocity. Another theory argues that the true boundary condition is the condition of vanishing shear stress, and the no-slip boundary condition arises due to microscopic boundary roughness. Thus, a perfectly smooth boundary would allow the fluid to slip.

4.10.2 Slip boundary condition

Exceptions to the no-slip boundary condition arise in the case of rarefied gas flow, high-pressure flow of polymer melts, flow near a three-phase contact line where a solid meets two liquids or a gas and a liquid, and flow past interfaces consisting of dual or multiple molecular layers that may exhibit relative motion, yielding a discontinuous macroscopic velocity. The no-slip boundary condition is sometimes relaxed in numerical simulations to prevent singularities stemming from excessive idealization. One example is the development of an infinite force on a sharp plate scraping fluid off a flat surface.

Consider steady unidirectional flow in a channel with parallel walls driven by the parallel translation of the upper wall along the x axis with velocity V, as illustrated in Figure 4.6.1(b). In this case, we may specify that the fluid slips over the lower wall such that the slip velocity, $u_x(y=0)$, is related to the wall shear stress by

$$u_x(y=0) = \frac{L}{\mu\beta}\,\sigma_{yx}(y=0) = \frac{L}{\beta}\left(\frac{\partial u_x}{\partial y}\right)_{y=0}, \tag{4.10.1}$$

where the constant β is the slip coefficient and L is a reference length identified, for example, with the channel width. As β tends infinity, the slip velocity tends to zero and the no-slip boundary condition prevails. The slip length is defined as $\ell = L/\beta$.

Rarefied gases

In the case of a rarefied gas, the slip coefficient, β, and slip length, ℓ, can be rigorously related to the molecular mean free path, λ, by the Maxwell relation

$$\frac{\lambda}{\ell} = \beta\,\mathrm{K}_n = \frac{\sigma}{2-\sigma}, \tag{4.10.2}$$

where $\mathrm{K}_n \equiv \lambda/L$ is the Knudsen number and σ is the tangential momentum accommodation coefficient (TMAC) expressing the fraction of molecules that undergo diffusive instead of specular reflection. In the limit $\sigma \to 2$ we recover the no-slip boundary condition, $\beta \to \infty$. In the limit $\sigma \to 0$ we recover the perfect-slip boundary condition, $\beta \to 0$.

Problem

4.10.1 *Flow in a channel with slip*

In the case of shear-driven channel flow illustrated in Figure 4.6.1(b), the pressure is uniform and the shear stress σ_{yx} is constant, independent of y. Assuming that the slip condition applies at the upper and lower walls, derive expressions for the shear shear and velocity profile of a Newtonian fluid in terms of V, h, μ, and the slip length, ℓ.

4.11 Wall stresses in Newtonian fluids

Combining the no-slip boundary condition discussed in Section 4.10.1 with the no-penetration boundary condition discussed in Section 2.10.1, we derive remarkably simple expressions for

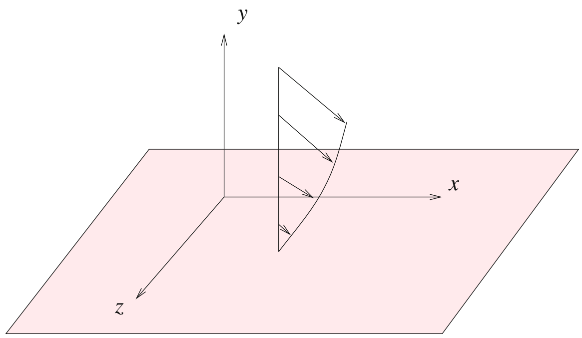


Figure 4.11.1 Flow over a plane wall subject to the no-slip boundary condition. The wall shear stress is proportional to the slope of the velocity profile with respect to distance normal to the wall, in this case y. The normal stress is equal to the negative of the pressure.

the Newtonian traction exerted on a solid surface, called the *wall stress*, amenable to a simple physical interpretation.

Consider a viscous flow above a stationary flat solid surface located at $y = 0$, as illustrated in Figure 4.11.1. The no-slip boundary condition requires that the tangential components of the velocity, and thus their derivatives with respect to z and x, are identically zero over the surface,

$$\frac{\partial u_x}{\partial x} = 0, \qquad \frac{\partial u_z}{\partial x} = 0, \qquad \frac{\partial u_x}{\partial z} = 0, \qquad \frac{\partial u_z}{\partial z} = 0, \tag{4.11.1}$$

where all partial derivatives are evaluated at $y = 0$. The no-penetration boundary condition requires that the normal component of the velocity, and thus its derivatives with respect to z and x, also vanish over the surface,

$$\frac{\partial u_y}{\partial x} = 0, \qquad \frac{\partial u_y}{\partial z} = 0, \tag{4.11.2}$$

where all partial derivatives are evaluated at $y = 0$. Thus, six of the nine components of the velocity-gradient tensor vanish over the surface.

Shear stress

The two components of the Newtonian shear stress exerted on the surface are given by

$$\sigma_{yx} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \qquad \sigma_{yz} = \mu \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right) \tag{4.11.3}$$

evaluated at $y = 0$. Using (4.11.2) to simplify (4.11.3), we find that

$$\sigma_{yx} = \mu \frac{\partial u_x}{\partial y}, \qquad \sigma_{yz} = \mu \frac{\partial u_z}{\partial y} \tag{4.11.4}$$

evaluated at $y = 0$. Equations (4.11.4) reveal that the wall shear stress is equal to the slope of the tangential velocity with respect to distance normal to the wall multiplied by the fluid viscosity in any flow, not just in a unidirectional flow.

Normal stress

The Newtonian normal stress exerted on a solid surface is given by

$$\sigma_{yy} = -p + 2\,\mu\,\frac{\partial u_y}{\partial y} \tag{4.11.5}$$

evaluated at $y = 0$, where p is the pressure. Since the fluid has been assumed incompressible, we may use the continuity equation (2.9.2) to write

$$\frac{\partial u_y}{\partial y} = -\frac{\partial u_x}{\partial x} - \frac{\partial u_z}{\partial z} \tag{4.11.6}$$

evaluated at $y = 0$, and then invoke the first and fourth equations in (4.11.1) to find that $\partial u_y/\partial y = 0$. Expression (4.11.5) thus simplifies to

$$\sigma_{yy} = -p, \tag{4.11.7}$$

which shows that the normal stress exerted on a solid surface is equal to the negative of the pressure.

Generalization

The results displayed in equations (4.11.4) and (4.11.7) apply even when a surface translates with a constant or time-dependent velocity. Moreover, these results apply when a surface is curved, provided that the zx plane is tangential to the surface and the y axis is normal to the surface at the position when the shear and normal stress are evaluated.

Problem

4.11.1 *Vorticity at a no-slip surface*

Show that the component of the vorticity vector normal to an impermeable wall vanishes, and thus the vortex lines are tangential to the surface. *Hint:* Use the second and third of equations (4.11.1).

4.12 Interfacial surfactant transport

An impure interface between two immiscible fluids is sometimes occupied by a molecular layer of surfactants affecting the local surface tension, γ. Dividing the number of surfactant molecules residing inside an infinitesimal surface patch centered at a point by the surface area of the patch, we obtain the surface concentration of the surfactant, Γ. The higher the surfactant concentration of the surfactant, the lower the surface tension.

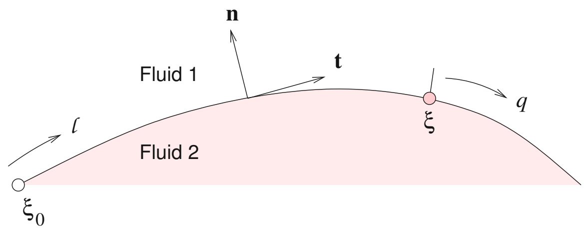


Figure 4.12.1 Point particles along a two-dimensional interface are identified by a parameter ξ that increases monotonically but in an otherwise arbitrary fashion along the interface.

Surface equation of state

For small surfactant concentrations, a linear surface equation of state can be assumed relating γ to Γ according to Gibbs' law,

$$\gamma = \frac{\gamma_0}{1-\beta}\,(1-\beta\,\frac{\Gamma}{\Gamma_0}), \tag{4.12.1}$$

where Γ_0 is a reference surfactant concentration and γ_0 is the corresponding surface tension. The dimensionless physical constant

$$\beta = RT\,\frac{\Gamma_0}{\gamma_c} \tag{4.12.2}$$

expresses the sensitivity of the surface tension to the surfactant concentration, where R is the ideal gas constant, T is the absolute temperature, and γ_c is the surface tension of a clean interface that is devoid of surfactants. More involved surface equations of state for moderate and large surfactant concentrations near saturation are available.

Interfacial convection-diffusion

The molecules of an insoluble surfactant are convected and diffuse over the interface, but do not enter the bulk of the fluid. Our objective in this section is to derive an evolution equation for the surface surfactant concentration determining the local surface tension and thus the jump in the traction across an interface.

4.12.1 Two-dimensional interfaces

Consider a chain of material point particles distributed along the inner or outer side of a two-dimensional interface. We begin by labeling the point particles using a parameter, ξ, so that their position can be described in parametric form as $\mathbf{X}(\xi)$ at any time, as shown in Figure 4.12.1.

Let ℓ be the arc length measured along the interface from an arbitrary point particle labeled ξ_0. The number of surfactant molecules residing inside a material test section of the interface confined between ξ_0 and ξ is

$$n(\xi,t) = \int_{\ell(\xi_0,t)}^{\ell(\xi,t)} \Gamma(\xi',t)\,\mathrm{d}\ell(\xi') = \int_{\xi_0}^{\xi} \Gamma(\xi',t)\,\frac{\partial \ell}{\partial \xi'}\,\mathrm{d}\xi', \tag{4.12.3}$$

where ξ' is an integration variable. Conservation of the total number of surfactant molecules inside the test section requires that

$$\frac{\partial n}{\partial t} = q(\xi_0) - q(\xi), \tag{4.12.4}$$

where $q(\xi)$ is the flux of surfactant molecules along the interface due to diffusion, and the time derivative is taken keeping ξ fixed. Substituting the expression for n from the last integral in (4.12.3), and transferring the derivative inside the integral as a material derivative, we obtain

$$\int_{\xi_0}^{\xi} \frac{\mathrm{D}}{\mathrm{D}t}\Big(\Gamma(\xi',t)\,\frac{\partial \ell}{\partial \xi'}\Big)\,\mathrm{d}\xi' = q(\xi_0) - q(\xi), \tag{4.12.5}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative. Now taking the limit as ξ tends to ξ_0, we derive a differential equation,

$$\frac{\mathrm{D}}{\mathrm{D}t}\Big(\Gamma\,\frac{\partial \ell}{\partial \xi}\Big) = -\frac{\partial q}{\partial \xi}. \tag{4.12.6}$$

Expanding the material derivative of the product on the left-hand side, we obtain

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t}\,\frac{\partial \ell}{\partial \xi} + \Gamma\,\frac{\mathrm{D}}{\mathrm{D}t}\Big(\frac{\partial \ell}{\partial \xi}\Big) = -\frac{\partial q}{\partial \xi}. \tag{4.12.7}$$

Interfacial stretching

Next, we use the Pythagorean theorem to write

$$\frac{\partial \ell}{\partial \xi} = \Big[\Big(\frac{\partial X}{\partial \xi}\Big)^2 + \Big(\frac{\partial Y}{\partial \xi}\Big)^2\Big]^{1/2}, \tag{4.12.8}$$

and compute the material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t}\Big(\frac{\partial \ell}{\partial \xi}\Big) = \frac{1}{2}\,\frac{1}{\partial \ell/\partial \xi}\,\Big[2\,\frac{\partial X}{\partial \xi}\,\frac{\mathrm{D}}{\mathrm{D}t}\Big(\frac{\partial X}{\partial \xi}\Big) + 2\,\frac{\partial Y}{\partial \xi}\,\frac{\mathrm{D}}{\mathrm{D}t}\Big(\frac{\partial Y}{\partial \xi}\Big)\Big]. \tag{4.12.9}$$

Interchanging the material derivative with the ξ derivative, and setting

$$\frac{\mathrm{D}X}{\mathrm{D}t} = u_x \qquad \frac{\mathrm{D}Y}{\mathrm{D}t} = u_y, \tag{4.12.10}$$

we find that

$$\frac{\mathrm{D}}{\mathrm{D}t}\Big(\frac{\partial \ell}{\partial \xi}\Big) = \frac{1}{\partial \ell/\partial \xi}\,\Big(\frac{\partial X}{\partial \xi}\,\frac{\partial u_x}{\partial \xi} + \frac{\partial Y}{\partial \xi}\,\frac{\partial u_y}{\partial \xi}\Big), \tag{4.12.11}$$

where $\mathbf{u} = (u_x, u_y)$ is the fluid velocity. Rearranging, we obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\Big(\frac{\partial \ell}{\partial \xi}\Big) = \frac{1}{\partial \ell/\partial \xi}\,\frac{\partial \mathbf{X}}{\partial \xi}\cdot\frac{\partial \mathbf{u}}{\partial \xi} = \frac{\partial \ell}{\partial \xi}\,\frac{\partial \mathbf{X}}{\partial \ell}\cdot\frac{\partial \mathbf{u}}{\partial \ell} = \frac{\partial \ell}{\partial \xi}\,\mathbf{t}\cdot\frac{\partial \mathbf{u}}{\partial \ell}, \tag{4.12.12}$$

where $\mathbf{t} = \partial \mathbf{X}/\partial l$ is the unit tangent vector shown in Figure 4.12.1.

Evolution equation

Substituting the last expression into (4.12.7), we obtain

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} + \Gamma\,\mathbf{t}\cdot\frac{\partial \mathbf{u}}{\partial \ell} = -\frac{\partial q}{\partial \ell}, \tag{4.12.13}$$

which is the targeted evolution equation for the surfactant concentration.

Fick's law

The diffusive flux can be described by Fick's law,

$$q = -D_s\,\frac{\partial \Gamma}{\partial \ell}, \tag{4.12.14}$$

where D_s is the surfactant surface diffusivity. Substituting this expression into (4.12.13), we derive a convection–diffusion equation,

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} + \Gamma\,\mathbf{t}\cdot\frac{\partial \mathbf{u}}{\partial \ell} = \frac{\partial}{\partial l}\Big(D_s\,\frac{\partial \Gamma}{\partial \ell}\Big). \tag{4.12.15}$$

In practice, the surfactant diffusivity is typically small.

Stretching and expansion

It is illuminating to resolve the velocity into a tangential and a normal component,

$$\mathbf{u} = u_t\,\mathbf{t} + u_n\,\mathbf{n}. \tag{4.12.16}$$

Noting that $\mathbf{t}\cdot\mathbf{n} = 0$, $\mathbf{t}\cdot\mathbf{t} = 1$, and $\mathbf{n}\cdot\mathbf{n} = 1$, and using the Frenet relations (4.3.5) and (4.3.9),

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\ell} = -\kappa\,\mathbf{n}, \qquad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\ell} = \kappa\,\mathbf{t}, \tag{4.12.17}$$

we compute

$$\mathbf{t}\cdot\frac{\partial \mathbf{u}}{\partial \ell} = \frac{\partial u_t}{\partial \ell} + u_n\,\mathbf{t}\cdot\frac{\partial \mathbf{n}}{\partial \ell} = \frac{\partial u_t}{\partial \ell} + \kappa\,u_n, \tag{4.12.18}$$

where κ is the interfacial curvature. Substituting this expression into (4.12.15), we obtain the evolution equation

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} + \Gamma\,\Big(\frac{\partial u_t}{\partial \ell} + \kappa\,u_n\Big) = \frac{\partial}{\partial \ell}\Big(D_s\,\frac{\partial \Gamma}{\partial \ell}\Big). \tag{4.12.19}$$

The first and second terms inside the parentheses on the left-hand side express, respectively, the rate of change of the surfactant concentration due to interfacial stretching, and the rate of change of the surfactant concentration due to interfacial expansion.

Stretching of a flat interface

Consider a flat interface situated along the x axis, stretched uniformly under the influence of a tangential velocity field, $u_x(x)$. Identifying ℓ with x and setting $\kappa = 0$, we find that the transport equation (4.12.19) reduces to

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} + \Gamma \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}\Big(D_s \frac{\partial \Gamma}{\partial x}\Big). \tag{4.12.20}$$

The material derivative can be resolved into Eulerian derivatives with respect to x and t, yielding

$$\frac{\partial \Gamma}{\partial t} + u_x \frac{\partial \Gamma}{\partial x} + \Gamma \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}\Big(D_s \frac{\partial \Gamma}{\partial x}\Big) \tag{4.12.21}$$

or

$$\frac{\partial \Gamma}{\partial t} + \frac{\partial (u_x \Gamma)}{\partial x} = \frac{\partial}{\partial x}\Big(D_s \frac{\partial \Gamma}{\partial x}\Big). \tag{4.12.22}$$

In fact, this equation could have been derived directly by performing a surfactant molecular balance over a differential control volume along the x axis, taking into consideration the convective and diffusive flux.

In the case of a uniformly stretched interface, $u_x = kx$, where k is a constant identified as the rate of extension. If the surfactant concentration is uniform at the initial instant, it will remain uniform at any time, governed by the linear equation

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} + k\,\Gamma = 0. \tag{4.12.23}$$

The solution reveals that the surfactant concentration decreases exponentially due to dilution,

$$\Gamma(t) = \Gamma(t = 0)\,\exp(-kt). \tag{4.12.24}$$

Expansion of a circular interface

Now consider a cylindrical interface with circular cross-section of radius a centered at the origin, expanding under the influence of a uniform radial velocity, $u_r(t)$, in the absence of circumferential motion. In plane polar coordinates, (r, θ), the transport equation (4.12.19) with constant diffusivity becomes

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} + \Gamma \frac{u_r}{a} = \frac{D_s}{a^2} \frac{\partial^2 \Gamma}{\partial \theta^2}. \tag{4.12.25}$$

In this case, the material derivative is the partial derivative with respect to time, yielding

$$\frac{\partial \Gamma}{\partial t} + \Gamma\, \frac{u_r}{a} = \frac{D_s}{a^2}\, \frac{\partial^2 \Gamma}{\partial \theta^2}. \tag{4.12.26}$$

If the surfactant concentration is uniform at the initial instant, it will remain uniform at any time, governed by the linear equation

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} + \Gamma\, \frac{u_r}{a} = 0. \tag{4.12.27}$$

In the case of expansion, $u_r > 0$, the surfactant concentration decreases exponentially due to dilution.

Interfacial markers

The material derivative expresses the rate of change of the surfactant concentration following the motion of *material point particles* residing on a selected side of an interface. In numerical practice, it may be expedient to follow the motion of interfacial marker points that move with the normal component of the fluid velocity and with an arbitrary tangential velocity, v_t. If $v_t = 0$, the marker points move normal to the interface at any instant. The velocity of a marker point is

$$\mathbf{v} = u_n\, \mathbf{n} + v_t\, \mathbf{t}. \tag{4.12.28}$$

By definition,

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \frac{\mathrm{d}\Gamma}{\mathrm{d}t} + (u_t - v_t)\, \frac{\partial \Gamma}{\partial \ell}, \tag{4.12.29}$$

where $\mathrm{d}/\mathrm{d}t$ is the rate of change of the surfactant concentration following the marker points. Substituting this expression into (4.12.19), we find that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} + (u_t - v_t)\, \frac{\partial \Gamma}{\partial \ell} + \Gamma\, (\, \frac{\partial u_t}{\partial \ell} + \kappa\, u_n\,) = \frac{\partial}{\partial \ell}\Big(D_s\, \frac{\partial \Gamma}{\partial \ell}\Big), \tag{4.12.30}$$

which can be restated as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} + \frac{\partial (u_t \Gamma)}{\partial \ell} - v_t\, \frac{\partial \Gamma}{\partial \ell} + \Gamma\, \kappa\, u_n = \frac{\partial}{\partial l}\Big(D_s\, \frac{\partial \Gamma}{\partial \ell}\Big). \tag{4.12.31}$$

The second term on the left-hand side is the derivative of the interfacial convective flux.

4.12.2 Axisymmetric interfaces

Next, we consider a chain of material point particles distributed along the inner or outer side of the trace of an axisymmetric interface in an azimuthal plane, and label the point particles using a parameter, ξ, so that their position in the chosen azimuthal plane is described in parametric form at any instant as $\mathbf{X}(\xi)$, as shown in Figure 4.12.2.

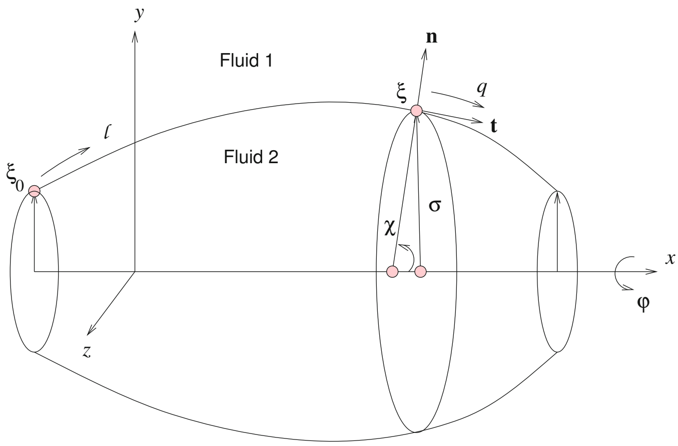


Figure 4.12.2 Point particles along the trace of an axisymmetric interface in an azimuthal plane are identified by a parameter ξ. The angle χ is subtended between the x axis and the straight line defined by the extension of the normal vector.

Let ℓ be the arc length measured along the trace of the interface from an arbitrary point particle labeled ξ_0. To derive an evolution equation for the surface surfactant concentration, we introduce cylindrical polar coordinates, (x, σ, φ), and express the number of surfactant molecules inside a ring-shaped material section of the interface confined between ξ_0 and ξ as

$$n(\xi, t) = 2\pi \int_{\ell(\xi_0,t)}^{\ell(\xi,t)} \Gamma(\xi', t)\, \sigma(\xi')\, \mathrm{d}\ell(\xi') = 2\pi \int_{\xi_0}^{\xi} \Gamma(\xi', t)\, \frac{\partial \ell}{\partial \xi'}\, \sigma(\xi')\, \mathrm{d}\xi'. \tag{4.12.32}$$

Conservation of the total number of surfactant molecules inside the test section requires that

$$\frac{\partial n}{\partial t} = 2\pi \big(\sigma_0\, q(\xi_0) - \sigma q(\xi) \big), \tag{4.12.33}$$

where q is the flux of surfactant molecules along the interface by diffusion, and the time derivative is taken keeping ξ fixed.

The counterpart of the balance equation (4.12.6) is

$$\frac{\mathrm{D}}{\mathrm{D}t} \Big(\Gamma(\xi, t)\, \sigma(\xi, t)\, \frac{\partial \ell}{\partial \xi} \Big) = -\frac{\partial (\sigma q)}{\partial \xi}, \tag{4.12.34}$$

and the counterpart of equation (4.12.13) is

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} + \Gamma \left(\mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial \ell} + \frac{u_\sigma}{\sigma} \right) = -\frac{1}{\sigma} \frac{\partial (\sigma q)}{\partial \ell}. \tag{4.12.35}$$

In deriving this equation, we have set $\mathrm{D}\sigma/\mathrm{D}t = u_\sigma$.

In terms of the normal and tangential velocities, u_n and u_t,

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} + \Gamma\,\Big(\frac{\partial u_t}{\partial \ell} + \kappa\, u_n + \frac{u_\sigma}{\sigma} \Big) = -\frac{1}{\sigma}\,\frac{\partial(\sigma q)}{\partial \ell}, \tag{4.12.36}$$

where κ is the curvature of the interface in a meridional plane. Substituting

$$u_\sigma = u_n\,\sin\chi - u_t\,\cos\chi, \tag{4.12.37}$$

we find that

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} + \Gamma\,\Big(\frac{\partial u_t}{\partial \ell} - \frac{\cos\chi}{\sigma}\,u_t + (\kappa + \frac{\sin\chi}{\sigma}\,)\,u_n \Big) = -\frac{1}{\sigma}\,\frac{\partial(\sigma q)}{\partial \ell}, \tag{4.12.38}$$

where the angle χ is defined in Figure 4.12.2. The sum of the terms inside the inner parentheses on the left-hand side is twice the mean curvature of the interface, $2\kappa_m$. The first two terms inside the outer parentheses can be consolidated to yield the final form

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} + \Gamma\,\Big(\frac{1}{\sigma}\,\frac{\partial(\sigma u_t)}{\partial \ell} + 2\,\kappa_m\,u_n \Big) = -\frac{1}{\sigma}\,\frac{\partial(\sigma q)}{\partial \ell}. \tag{4.12.39}$$

The first term inside the parentheses on the left-hand side expresses the rate of change of the surface area of an axisymmetric material ring.

Marker points

An evolution equation for interfacial marker points can be derived working as in Section 4.12.1 for a two-dimensional interface. The result is

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} + (u_t - v_t)\,\frac{\partial\Gamma}{\mathrm{d}\ell} + \Gamma\,\Big(\frac{1}{\sigma}\,\frac{\partial(\sigma u_t)}{\partial \ell} + 2\,\kappa_m\,u_n \Big) = -\frac{1}{\sigma}\,\frac{\partial(\sigma q)}{\partial \ell}, \tag{4.12.40}$$

which can be restated as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} + \frac{1}{\sigma}\,\frac{\partial(\sigma u_t \Gamma)}{\partial \ell} - v_t\,\frac{\partial\Gamma}{\partial \ell} + \Gamma\,2\,\kappa_m\,u_n = -\frac{1}{\sigma}\,\frac{\partial(\sigma q)}{\partial \ell}. \tag{4.12.41}$$

When the marker points move normal to the interface, $v_t = 0$, the third term on the right-hand side of (4.12.41) does not appear.

4.12.3 Three-dimensional interfaces

The equations derived previously in this section for two-dimensional and axisymmetric interfaces can be generalized to three-dimensional interfaces.[2] The normal component of the marker-point velocity over a three-dimensional interface must be equal to the normal component of the fluid velocity, but the tangential component can be arbitrary. The general form of marker point velocity is

$$\mathbf{v} = u_n\,\mathbf{n} + \mathbf{v}^{\text{tangential}}, \tag{4.12.42}$$

[2] Yon, S. & Pozrikidis, C. (1998) A finite-volume/boundary-element method for interfacial flow in the presence of surfactants, with applications to shear flow past a viscous drop, *Computers & Fluids* **27**, 879–902.

where $\mathbf{v}^{\text{tangential}}$ is an arbitrary tangential component. When $\mathbf{v}^{\text{tangential}} = \mathbf{0}$, the marker points move with the fluid velocity normal to the interface alone. When $\mathbf{v}^{\text{tangential}} = \mathbf{u} - u_n\,\mathbf{n}$, the marker points are point particles moving with the fluid velocity.

Adopting Fick's law for the surface diffusion, we find that the evolution of the concentration of an immiscible surfactant following the motion of interfacial marker points takes the form

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} + \boldsymbol{\nabla}_s \cdot (\mathbf{u}_s \Gamma) - \mathbf{v}^{\text{tangential}} \cdot \boldsymbol{\nabla}_s \Gamma + \Gamma\, 2\, \kappa_m\, \mathbf{u} \cdot \mathbf{n} = \boldsymbol{\nabla}_s \cdot (D_s \boldsymbol{\nabla}_s \Gamma). \tag{4.12.43}$$

To define the various terms, we introduce the tangential projection matrix, $\mathbf{P} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$, with elements

$$P_{ij} = \delta_{ij} - n_i n_j, \tag{4.12.44}$$

where δ_{ij} is Kronecker's delta representing the identity matrix and the symbol $\otimes$ denotes the tensor product. Subject to this definition,

$$\mathbf{u}_s = \mathbf{P} \cdot \mathbf{u} \tag{4.12.45}$$

is the component of the fluid velocity tangential to the interface and

$$\boldsymbol{\nabla}_s \equiv \mathbf{P} \cdot \boldsymbol{\nabla} \tag{4.12.46}$$

is the surface gradient.

In the case of two-dimensional flow, depicted in Figure 4.12.1, or axisymmetric flow, depicted in Figure 4.12.2, equation (4.12.43) reduces to (4.12.41) or (4.12.31) by setting $\mathbf{v}^{\text{tangential}} = v_t\, \mathbf{t}$.

Problems

4.12.1 *Expansing spherical interface*

Derive an evolution equation for the surface concentration of a surfactant over a uniformly expanding spherical interface.

4.12.2 *Transport on a flat interface*

Simplify equation (4.12.43) for a flat interface in the xy plane.