Chapter 14
A Radial Framework for Estimating the Efficiency and Returns to Scale of a Multi-component Production System in DEA

Jingjing Ding, Chenpeng Feng, and Huaqing Wu

Abstract This chapter provides radial measurements of efficiency for the production process possessing multi-components under different production technologies. Our approach is based on the construction of various empirical production possibility sets. Then we propose a procedure that is unaffected affected by multiple optima for estimating returns to scale. The theoretical connections between the traditional black box and the proposed multi-component approach are established, which ascertains consistency in estimating the efficiency and returns to scale. Moreover, we introduce two homogeneity conditions, which clarify the difference between our approach and the existing one, and are important for evaluating performance in multi-component setting. Finally, an empirical study of the pollution treatment processes in China is presented, and compared to the results from black-box approach. Many insightful findings related to the operations of the pollution treatment processes in China are secured.

Keywords Data envelopment analysis • Efficiency • Returns to scale • Multi-component • Pollution treatment process

J. Ding (✉) • C. Feng
School of Management, Hefei University of Technology, No. 193 Tunxi Road, Hefei, Anhui Province 230009, P.R. China
e-mail: jingding@hfut.edu.cn; cpfeng@hfut.edu.cn

H. Wu
School of Economics, Hefei University of Technology, No. 193 Tunxi Road, Hefei, Anhui Province 230009, P.R. China
e-mail: wuhuaqing@hfut.edu.cn

S.-N. Hwang et al. (eds.), *Handbook of Operations Analytics Using Data Envelopment Analysis*, International Series in Operations Research & Management Science 239, DOI 10.1007/978-1-4899-7705-2_14

14.1 Introduction

We consider the estimation of the efficiency and returns to scale (RTS) for a production system which can be modeled as having multi-components based on data envelopment analysis (DEA). There are many production systems bearing this situation. For example, Beasley (1995) studied the performances of universities, each of which had two components: research and teaching. Cook et al. (2000) modeled a banking production system as having two components: service and sales. We are mainly concerned with radial measurements, and the theoretical connection with the existing black-box approach.

DEA is a nonparametric technique for measuring the relative efficiencies of a set of peer decision-making units (DMUs) involving multiple inputs and outputs. Charnes et al. (1978) first introduced it. In this pioneer paper, the authors constructed a nonlinear programming model to evaluate the efficiency of activity conducted by a non-profit organization. The model is known as the CCR model in the literature. As is known, the CCR model captures both technical and scale inefficiencies. Banker et al. (1984) proposed a new model (BCC model) which extended the CCR model by separating technical efficiency and scale efficiency. Recently, DEA has been extended to many areas in management science and operational research field.

At the early stage of development, DEA treats a DMU under evaluation as a black box. Thus, it cannot provide users with specific information concerning the sources of inefficiency within an organization. Färe and Grosskopf (2000) introduced a network DEA technique, which opened the black box, and explicitly modeled the internal mechanism of a DMU. Lewis and Sexton (2004) also published a research paper in this direction. Färe and Grosskopf (2000) and Lewis and Sexton (2004) proposed radial measurements of efficiency in network DEA literature. By contrast, Tone and Tsutsui (2009) extended radial measurements in network DEA to non-radial measurements of efficiency by introducing slack-based network DEA model. Kao and Hwang (2008) and Kao (2009a, b) proposed models for evaluating DMUs with serial network structure, parallel network structure and the mixture of the above two structures. DMUs with a two-stage production process have been extensively studied both from a theoretical and from a practical perspective. Included among these studies are Liang et al. (2008) and Chen et al. (2006, 2009a, b, 2010). We refer the reader to review papers, such as Cook et al. (2010) and Castelli et al. (2010) for more references.

The value of returns to scale (RTS) measures the percentage change in output from a given percentage change in inputs in economic theory. Unlike main researches in economic literature, which are concerned about production processes with a single output, extensions to the situations of multiple outputs are spurred by Banker et al. (1984). Since then, RTS has been studied extensively. Banker et al. (2004) published an excellent review on different methods used to handle RTS. According to the paper, there are two approaches followed in the literature to study RTS. The first approach is proposed by Färe et al. (1985, 1994) and the other one is devised by Banker et al. (1984). In this paper, we follow the first approach, which has the advantage of being unaffected by the possible existence of multiple optima.

The existing papers concerning RTS are mainly based on the black-box assumption. However, very few of these papers deal with RTS, when the black-box assumption is dropped. Research papers with RTS consideration include Chen et al. (2009a), Tsai and Molinero (2002). Those two papers both follow the framework proposed by Banker et al. (1984), and could suffer from the existence of multiple optima.

Our current paper studies a production process with a multi-component structure. Before moving on, we firstly differentiate two cases of production processes having a multi-component according to data availability. The first case has the data on how the shared inputs and shared outputs are split among sub-decision making units (SDMUs). The second case does not have data on how the shared inputs/outputs are split among SDMUs. Beasley (1995) and Cook et al. (2000) investigated models for evaluating performance in the second case, but did not study the RTS of the productions. In addition, how to extend their models to treat RTS is not clear. The difficulties are twofold in multi-component setting: (1) the nonlinearity of the proposed models and (2) the impact of potential multiple optima on testing RTS by following Banker's approach. Our work focuses on production processes with multi-components of (1). In doing so, we avoid the problem of nonlinearity, to center on investigating RTS.

The contributions of our work mainly lie in three aspects. Firstly, we propose radial measurements for efficiency evaluation and a procedure to determine the RTS of a DMU that is unaffected by possible multiple optima. Secondly, we establish theoretical connection between the black-box approach by Färe et al. (1985, 1994) and our multi-component approach, which helps to connect the black-box approach with the network approach, and ensures consistency between both approaches in dealing with RTS. In addition, two homogeneity conditions are proposed and are important for evaluating performance in multi-component setting. They are not pointed out before in the literature. Thirdly, in this work, we use the proposed method to study the efficiency and RTS of pollution treatment processes in China based on real data. We model the processes as having two components, which is different from the traditional approach, and secure various insightful findings related to the operations of the pollution treatment processes in China.

The paper unfolds as follows: Section 14.2 proposes a radial evaluation model under variable returns to scale assumption (14.2.1), and establishes the theoretical connection of the proposed model to the black-box model (14.2.2). Section 14.3 provides a procedure for determining the RTS of a DMU. Section 14.4 establishes the theoretical connection of the proposed approach for estimating RTS to Färe et al. (1985, 1994). In Sect. 14.5, we apply the prospective method to study the performance of pollution treatment processes in China. Section 14.6 concludes the paper.

14.2 Radial Performance Measurement for a Multi-component System

A production unit (denoted as a DMU) with multi-component structure studied in this paper is depicted in Fig. 14.1. The DMU consists of two sub-decision-making units (SDMUs) without loss of generality. It is assumed that some inputs of DMU are shared by $SDMU_1$ and $SDMU_2$, and some outputs are the results of $SDMU_1$ and $SDMU_2$. In addition to shared inputs and outputs, there are inputs and outputs of the DMU dedicated to, or are the results of, $SDMU_1$ or $SDMU_2$ exclusively. We assume to deal with n DMUs in this paper. In the sequel, when referring to a specific DMU, we denote it by a subscript j, that is, DMU_j, $SDMU_{1j}$, and $SDMU_{2j}$ ($j = 1,\ldots,n$).

The variables in Fig. 14.1 are defined as follows: $X_1 = \left(x_1^1, \ldots, x_m^1\right)$ indicates m inputs dedicated to $SDMU_1$; $X_2 = \left(x_1^2, \ldots, x_h^2\right)$ indicates h inputs dedicated to $SDMU_2$; $X_s = \left(x_1^s, \ldots, x_l^s\right)$ indicates l inputs shared by $SDMU_1$ and $SDMU_2$; $Y_1 = \left(y_1^1, \ldots, y_s^1\right)$ indicates s outputs produced exclusively by $SDMU_1$; $Y_2 = \left(y_1^2, \ldots, y_q^2\right)$ indicates q outputs produced exclusively by $SDMU_2$; $Y_s = \left(y_1^s, \ldots, y_u^s\right)$ indicates u outputs produced together by $SDMU_1$ and $SDMU_2$. When referring to the specific data of DMU_j, we shall use a secondary index j. For instance, the m inputs dedicated to $SDMU_{1j}$, the $SDMU_1$ of DMU_j, are denoted as $X_{1j} = \left(x_{1j}^1, \ldots, x_{mj}^1\right)$.

We differentiate two cases of production processes with multi-component structure according to data availability. In the first case, the data on how the shared inputs and shared outputs are split between $SDMU_1$ and $SDMU_2$ are available. In this case, we use $X_{s1} = \left(x_1^{s1}, \ldots, x_l^{s1}\right), X_{s2} = \left(x_1^{s2}, \ldots, x_l^{s2}\right)$, and $Y_{s1} = \left(y_1^{s1}, \ldots, y_u^{s1}\right)$, $Y_{s2} = \left(y_1^{s2}, \ldots, y_u^{s2}\right)$ to denote the observational data fulfilling $X_s = X_{s1} + X_{s2}$ and $Y_s = Y_{s1} + Y_{s2}$. Note that these are component wise additions indicating $X_s(i) = X_{s1}(i) + X_{s2}(i)$, $i,\ldots,l$, and $Y_s(j) = Y_{s1}(j) + Y_{s2}(j)$, $j = 1,\ldots,u$. In the second case, it is not known how the shared inputs/outputs are split. *We deal with the former case in this paper.*

To be specific, we take pollution treatment processes in China as an example. If we are going to investigate the performances of pollution treatment processes in all provinces, provinces are naturally modeled as DMUs. When the black box of a

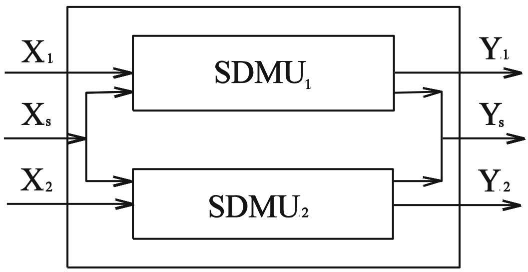


Fig. 14.1 Structure of multi-component system

DMU is opened, it can be found that cities can be further classified into two SDMUs: capital city and non-capital cities. The capital city is the political, economic and cultural center of a province. Thus, the environment beyond the control of the management of the pollution treatment process in capital city and non-capital cities is arguably different. This makes sense: For example, a capital city often consumes more inputs such as capital inputs: pollution treatment facilities. As will be shown in this paper, the average capital city consumes approximately more than one fifth of the total inputs, but produces less than one fifth of the total outputs. In this case, we might reasonably claim that the capital city consumes more inputs as compared with noncapital cities.

14.2.1 Basic Model

Let us begin with the construction of production possibility set (PPS) of each SDMU. Based on the PPS of SDMUs, the PPS of a DMU is derived. We assume first variable returns of scale for all SDMUs. Note that the PPS considered is similar to that in Tsai and Molinero (2002).

The PPS of $SDMU_1$:

$$T_1^{VRS} = \left\{ \left(X^1, Y^1\right) \left| \begin{array}{l} \sum_{j=1}^{n} \lambda_j^1 x_{ij}^1 \leq x_i^1, i = 1, \ldots, m, \sum_{j=1}^{n} \lambda_j^1 y_{rj}^1 \geq y_r^1, r = 1, \ldots, s \\ \sum_{j=1}^{n} \lambda_j^1 x_{ij}^{s1} \leq x_i^{s1}, i = 1, \ldots, l, \sum_{j=1}^{n} \lambda_j^1 y_{rj}^{s1} \geq y_r^{s1}, r = 1, \ldots, u \\ \sum_{j=1}^{n} \lambda_j^1 = 1, \lambda_j^1 \geq 0 \end{array} \right. \right\} \tag{14.1}$$

where $\left(X^1, Y^1\right) = \left(x_1^1, \ldots, x_m^1, x_1^{s1}, \ldots, x_l^{s1}, y_1^1, \ldots, y_s^1, y_1^{s1}, \ldots, y_u^{s1}\right)$.

The PPS of $SDMU_2$:

$$T_2^{VRS} = \left\{ \left(X^2, Y^2\right) \left| \begin{array}{l} \sum_{j=1}^{n} \lambda_j^2 x_{ij}^2 \leq x_i^2, i = 1, \ldots, h, \sum_{j=1}^{n} \lambda_j^2 y_{rj}^2 \geq y_r^2, r = 1, \ldots, q \\ \sum_{j=1}^{n} \lambda_j^2 x_{ij}^{s2} \leq x_i^{s2}, i = 1, \ldots, l, \sum_{j=1}^{n} \lambda_j^2 y_{rj}^{s2} \geq y_r^{s2}, r = 1, \ldots, u \\ \sum_{j=1}^{n} \lambda_j^2 = 1, \lambda_j^2 \geq 0 \end{array} \right. \right\} \tag{14.2}$$

where $\left(X^2, Y^2\right) = \left(x_1^2, \ldots, x_h^2, x_1^{s2}, \ldots, x_l^{s2}, y_1^2, \ldots, y_q^2, y_1^{s2}, \ldots, y_u^{s2}\right)$.

The PPS of DMU:

$$T^{VRS}=\left\{(X,Y)\left|\begin{array}{l}\sum_{j=1}^{n}\lambda_j^1x_{ij}^1\le x_i^1, i=1,\ldots,m, \sum_{j=1}^{n}\lambda_j^1x_{ij}^{s1}+\sum_{j=1}^{n}\lambda_j^2x_{ij}^{s2}\le x_i^s, i=1,\ldots,l\\ \sum_{j=1}^{n}\lambda_j^2x_{ij}^2\le x_i^2, i=1,\ldots,h, \sum_{j=1}^{n}\lambda_j^1y_{rj}^1\ge y_r^1, r=1\ldots s\\ \sum_{j=1}^{n}\lambda_j^1y_{rj}^{s1}+\sum_{j=1}^{n}\lambda_j^2y_{rj}^{s2}\ge y_r^s, r=1,\ldots,u, \sum_{j=1}^{n}\lambda_j^2y_{rj}^2\ge y_r^2, r=1,\ldots,q\\ \sum_{j=1}^{n}\lambda_j^1=1, \sum_{j=1}^{n}\lambda_j^2=1, \lambda_j^1,\lambda_j^2\ge 0\end{array}\right.\right\} \tag{14.3}$$

where $(X,Y)=\left(x_1^1,\ldots,x_m^1,x_1^s,\ldots,x_l^s,x_1^2,\ldots,x_h^2,y_1^1,\ldots,y_s^1,y_1^s,\ldots,y_u^s,y_1^2,\ldots,y_q^2\right)$.

It should be noted that the PPS of DMU is the addition of the PPS's of $SDMU_1$ and $SDMU_2$. We assume that if $SDMU_1$ (X^1, Y^1) and $SDMU_2$ (X^2, Y^2) are possible, then one can set up a DMU consisting of a $SDMU_1$ and a $SDMU_2$. Most importantly, the two SDMUs do not interfere with each other and carry out (X^1, Y^1) and (X^2, Y^2) independently. The result is then that DMU built in this way consumes (X^1+X^2), and produces (Y^1+Y^2).

The performance of a DMU can be measured under two different situations: first, price information is given, and second, prices are not available. In the latter situation, Shephard's input distance function is a frequently used measurement (Shephard's 1970). Suppose $L(Y)$ is the input requirement set derived from T^{VRS}. Shephard's input distance function is given below.

$$D(X,Y)=\max\{\lambda : X/\lambda\in L(Y), \lambda\in R\} \tag{14.4}$$

Clearly, $D(X,Y)$ is greater than or equal to 1, if $X\in L(Y)$, with $D(X,Y)=1$, if and only if it is impossible to improve input vector X proportionately without worsening the output vector. Let $\theta=1/\lambda$. It follows that

$$[D(X,Y)]^{-1}=\min\{\theta : \theta X\in L(Y)\} \tag{14.5}$$

According to (14.3) and (14.5), the performance of DMU_0 with multi-components can be estimated by the following linear programming model.

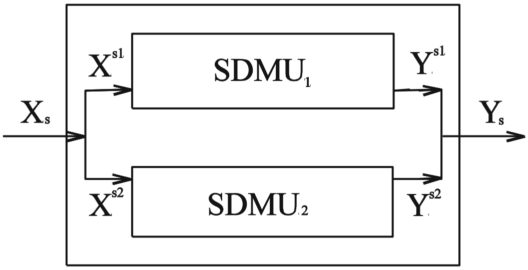
$$
\begin{aligned}
&\theta_T^* = \min\ \theta \\
&\text{s.t.}\ \sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^k x_{ij}^{sk} \le \theta x_{io}^{s} && i = 1, \ldots, l \text{ (shared inputs)} \\
&\sum_{j=1}^{n}\lambda_j^1 x_{ij}^{1} \le \theta x_{io}^{1} && i = 1, \ldots, m \text{ (inputs dedicated to SDMU}_1\text{)} \\
&\sum_{j=1}^{n}\lambda_j^2 x_{ij}^{2} \le \theta x_{io}^{2} && i = 1, \ldots, h \text{ (inputs dedicated to SDMU}_2\text{)} \\
&\sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^k y_{rj}^{sk} \ge y_{ro}^{s} && r = 1, \ldots, u \text{ (shared outputs)} \\
&\sum_{j=1}^{n}\lambda_j^1 y_{rj}^{1} \ge y_{ro}^{1} && r = 1, \ldots, s \text{ (outputs produced by SDMU}_1\text{)} \\
&\sum_{j=1}^{n}\lambda_j^2 y_{rj}^{2} \ge y_{ro}^{2} && r = 1, \ldots, q \text{ (outputs produced by SDMU}_2\text{)} \\
&\sum_{j=1}^{n}\lambda_j^k = 1 && k = 1, 2 \\
&\lambda_j^k \ge 0, k = 1, 2, && j = 1, \ldots, n.
\end{aligned}
\tag{14.6}
$$

where the decision variables are λ_j^k $(j = 1, \ldots, n;\ k = 1, 2)$ and θ. It should be noted that x_{ij}^k, x_{ij}^{sk}, y_{rj}^k and y_{rj}^{sk} are observational data that correspond to the types of inputs and outputs labeled in (14.6).

14.2.2 Theoretical Connection with Black-Box Approach

In this section, we formally derive the black-box equivalent PPS that corresponds to T^{VRS}, which can give an insight into model (14.6). Before moving on, we assume that the structure depicted in Fig. 14.1 consumes all inputs shared by SDMU_1 and SDMU_2, and all the outputs of DMU are the results of SDMU_1 and SDMU_2. We adopt the convention that DMU consumes m inputs $X_j = (x_{1j}, \ldots, x_{mj})$ and produces s outputs $Y_j = \left(y_{1j}, \ldots, y_{sj}\right)$. Thus, based on the notations provided above for DMUs with multi-component structure, the assumption here implies that $X_j^{sk} = \left(x_{1j}^{sk}, \ldots, x_{mj}^{sk}\right)$ and $Y_j^{sk} = \left(y_{1j}^{sk}, \ldots, y_{sj}^{sk}\right)$ with $X_j^{s1} + X_j^{s2} = X_j^s = X_j$, and $Y_j^{s1} + Y_j^{s2} = Y_j^s = Y_j$. Later in the paper, the s in the superscript is deleted for

Fig. 14.2 Structure of DMU with all shared inputs and outputs



simplicity. In cases where inputs or outputs are not entirely shared by SDMU$_1$ and SDMU$_2$ (See Fig. 14.1), the values of those inputs/outputs dedicated to SDMU$_1$ (SDMU$_2$) are zeros for SDMU$_2$ (SDMU$_1$). Therefore, the structure of the DMU in Fig. 14.1 reduces to structure provided in Fig. 14.2.

In light of the structure depicted in Fig. 14.2, T_1^{VRS}, T_2^{VRS} and T^{VRS} in the previous section are rewritten as follows:

$$T_1^{VRS} = \left\{ \left(X^1, Y^1\right) \middle| \sum_{j=1}^{n} \lambda_j^1 x_{ij}^1 \le x_i^1, i = 1, \ldots, m, \right.$$
$$\left. \sum_{j=1}^{n} \lambda_j^1 y_{rj}^1 \ge y_r^1, r = 1, \ldots, s, \sum_{j=1}^{n} \lambda_j^1 = 1, \lambda_j^1 \ge 0 \right\} \tag{14.7}$$

where $\left(X^1, Y^1\right) = \left(x_1^1, \ldots, x_m^1, y_1^1, \ldots, y_s^1\right)$.

$$T_2^{VRS} = \left\{ \left(X^2, Y^2\right) \middle| \sum_{j=1}^{n} \lambda_j^2 x_{ij}^2 \le x_i^2, i = 1, \ldots, m, \right.$$
$$\left. \sum_{j=1}^{n} \lambda_j^2 y_{rj}^2 \ge y_r^2, r = 1, \ldots, s, \sum_{j=1}^{n} \lambda_j^2 = 1, \lambda_j^2 \ge 0 \right\} \tag{14.8}$$

where $\left(X^2, Y^2\right) = \left(x_1^2, \ldots, x_m^2, y_1^2, \ldots, y_s^2\right)$.

$$T^{VRS} = \left\{ (X, Y) \middle| \begin{array}{l} \sum_{k=1}^{2} \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le x_i, i = 1, \ldots, m, \sum_{k=1}^{2} \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_r, r = 1, \ldots, s \\ \sum_{j=1}^{n} \lambda_j^k = 1, \lambda_j^k \ge 0, k = 1, 2 \end{array} \right\} \tag{14.9}$$

where $(X, Y) = (x_1, \ldots, x_m, y_1, \ldots, y_s)$.

We proceed to give a result on the convexity of T^{VRS} that is necessary for the exposition of this paper.

Property 1 T^{VRS} is convex set.

Proof Suppose (X_1, Y_1) and (X_2, Y_2) belong to T^{VRS}. By definition, there are sets of nonnegative multipliers $\lambda_j^{k1*}, \lambda_j^{k2*}$ with $\sum_{j=1}^{n} \lambda_j^{k1*} = 1$ and $\sum_{j=1}^{n} \lambda_j^{k2*} = 1$ such that

$$\sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^{k1*} y_{rj}^k \geq y_r^1, r = 1, \ldots, s, \sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^{k1*} x_{ij}^k \leq x_i^1, i = 1, \ldots, m,$$

$$\sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^{k2*} y_{rj}^k \geq y_r^2, r = 1, \ldots, s, \sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^{k2*} x_{ij}^k \leq x_i^2, i = 1, \ldots, m.$$

For any convex pair α, β, we have $\sum_{k=1}^{2}\sum_{j=1}^{n}\left(\alpha\lambda_j^{k1*} + \beta\lambda_j^{k2*}\right) y_{rj}^k \geq \alpha y_r^1 + \beta y_r^2, r = 1, \ldots, s,$ $\sum_{k=1}^{2}\sum_{j=1}^{n}\left(\alpha\lambda_j^{k1*} + \beta\lambda_j^{k2*}\right) x_{ij}^k \leq \alpha x_i^1 + \beta x_i^2, i = 1, \ldots, m,$ and $\sum_{j=1}^{n}\left(\alpha\lambda_j^{k1*} + \beta\lambda_j^{k2*}\right) = 1.$ This ensures that $\alpha(X_1, Y_1) + \beta(X_2, Y_2) = (\alpha X_1 + \beta X_2, \alpha Y_1 + \beta Y_2) \in T^{VRS}$. □

Assumption 1 Assume there are n DMUs, each of which consists of two production units $\mathrm{SDMU_{1j}}$, $\mathrm{SDMU_{2j}}$, $j = 1,\ldots,n$ using the production technology characterized by T_1^{VRS} and T_2^{VRS} respectively. Let there be an extended data set (**EDS**) of n^2 distinct DMUs, each of which comprises $\mathrm{SDMU_{1j}}$ and $\mathrm{SDMU_{2k}}$ with $j, k \in \{1, \ldots, n\}$.

Let (x_{ij}, y_{rj}) denote the input and output bundle of $\mathrm{DMU_j}$ in **EDS**. Define T_b^{VRS}, T_b^{CRS}, and T_b^{NIRS} as below, where the superscripts CRS and NIRS, respectively, stand for constant returns to scale and non-increasing returns to scale:

$$T_b^{VRS} = \left\{ (X,Y) \middle| \sum_{j=1}^{n^2}\lambda_j x_{ij} \leq x_i, i = 1,\ldots,m, \sum_{k=1}^{n^2}\lambda_j y_{rj} \geq y_r, r = 1,\ldots,s, \sum_{j=1}^{n^2}\lambda_j = 1, \lambda_j \geq 0 \right\}$$

$$T_b^{CRS} = \left\{ (X,Y) \middle| \sum_{j=1}^{n^2}\lambda_j x_{ij} \leq x_i, i = 1,\ldots,m, \sum_{k=1}^{n^2}\lambda_j y_{rj} \geq y_r, r = 1,\ldots,s, \lambda_j \geq 0 \right\}$$

$$T_b^{NIRS} = \left\{ (X, Y) | \sum_{j=1}^{n^2} \lambda_j x_{ij} \le x_i, i = 1, \ldots, m, \sum_{k=1}^{n^2} \lambda_j y_{rj} \ge y_r, r = 1, \ldots, s, \right.$$
$$\left. \sum_{j=1}^{n^2} \lambda_j \le 1, \lambda_j \ge 0 \right\}$$

where $(X, Y) = (x_1, \ldots, x_m, y_1, \ldots, y_s)$.

We now establish that the PPS of the general multi-component system with two different SDMUs can be recovered by DMUs in **EDS** through the black-box approach. The connections between the multi-component PPS's and the above mentioned black-box PPS's are summarized in Theorem 1.

Theorem 1 $T_b^{VRS} = T^{VRS}$, $T_b^{CRS} = T^{CRS}$, *and* $T_b^{NIRS} = T^{NIRS}$, *where*

$$T^{CRS} = \left\{ (X, Y) \left| \begin{array}{l} \sum_{k=1}^{2} \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le x_i, i = 1, \ldots, m, \sum_{k=1}^{2} \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_r, r = 1, \ldots, s \\ \sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2, \lambda_j^k \ge 0, k = 1, 2 \end{array} \right. \right\}$$

and

$$T^{NIRS} = \left\{ (X, Y) \left| \begin{array}{l} \sum_{k=1}^{2} \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le x_i, i = 1, \ldots, m, \sum_{k=1}^{2} \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_r, r = 1, \ldots, s \\ \sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2 \le 1, \lambda_j^k \ge 0, k = 1, 2 \end{array} \right. \right\}$$

Proof See Appendix. □

Let us close this section by pointing out the difference between T^{CRS} and $\overline{T}^{CRS}$, which is defined by

$$\overline{T}^{CRS} = \left\{ (X, Y) | \sum_{k=1}^{2} \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le x_i, i = 1, \ldots, m, \sum_{k=1}^{2} \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_r, r = 1, \ldots, s, \right.$$
$$\left. \lambda_j^k \ge 0, k = 1, 2 \right\}.$$

Researchers in the literature tend to define $\overline{T}^{CRS}$ as the CRS PPS for the production system in Fig. 14.1. Tsai and Molinero (2002) is a case in point. Obviously, the production frontier determined by T^{CRS} is dominated by the one defined by $\overline{T}^{CRS}$. In Fig. 14.3, we use a set of two DMUs with one input and one output for illustration.

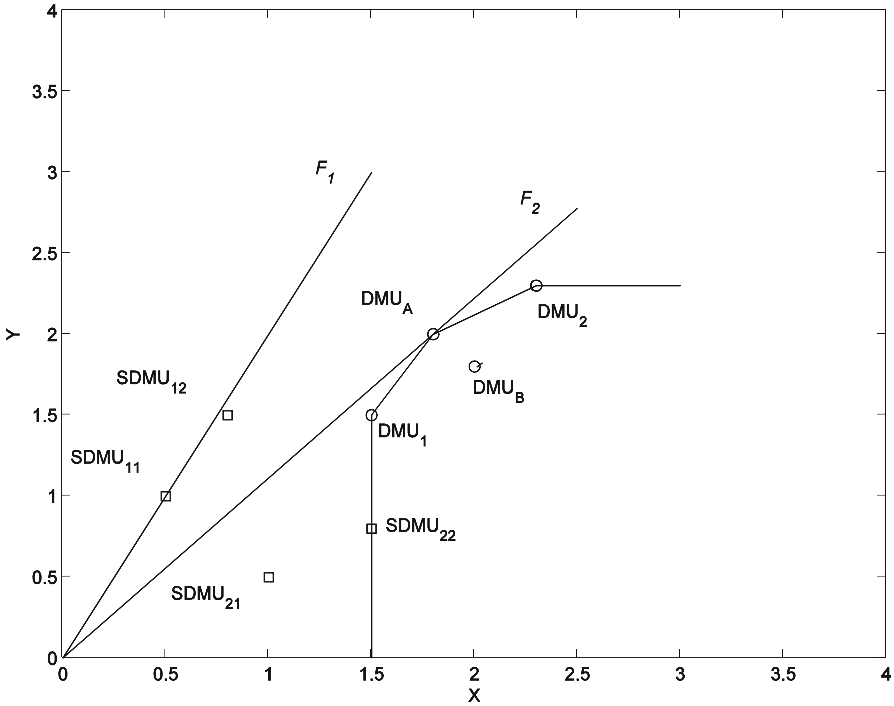


Fig. 14.3 Graphical illustration of T^{CRS} and $\overline{T}^{CRS}$

Here, DMU_1 and DMU_2 comprise of (SDMU_{11}, SDMU_{21}) and (SDMU_{12}, SDMU_{22}) respectively. DMU_A and DMU_B are generated by combining respectively SDMU_{11} and SDMU_{22}, SDMU_{21} and SDMU_{22}. In light of Theorem 1, T^{CRS} is the conic hull constructed by DMU_1, DMU_2, DMU_A and $\mathrm{DMU}_\mathrm{B.}$ This is the region to the right of frontier F_2. PPS provided by $\overline{T}^{CRS}$ is the region to the right of frontier F_1.

Figure 14.3 shows that the production frontier of $\overline{T}^{CRS}$ is determined by SDMU_{11}. Apparently, the production process of SDMU is arguably different from that of DMU. Therefore, the use of SDMU as a benchmarking point for DMU is not appropriate. To highlight the difference between SDMU and DMU, criteria for homogeneity are essential. The homogeneity in this context refers to the characteristic of the efficient frontier that a benchmarking point on the frontier constructed for evaluating the performance of a DMU should be comparable to the DMU in terms of the internal production process. Two homogeneity conditions for the construction of a virtual DMU, i.e., weak condition and strong condition, are introduced below:

(1) Weak homogeneity condition: $t_1 = 0$ if and only if $t_2 = 0$.
(2) Strong homogeneity condition: $t_1 = t_2$.

Clearly, if a virtual DMU built by SDMU_1 and SDMU_2 satisfies the strong homogeneity condition, the weak homogeneity condition is automatically satisfied. However, the opposite is not true. Comparing the definition of T^{CRS} with that of $\overline{T}^{CRS}$, the difference is the distinct requirements of the sum of the levels of elementary activities involved (i.e., $t_1 = \sum_{j=1}^{n} \lambda_j^1, t_2 = \sum_{j=1}^{n} \lambda_j^2$). Specifically, T^{CRS} requires $t_1 = t_2$, while $\overline{T}^{CRS}$ does not. $\overline{T}^{CRS}$ is claimed to violate the strong homogeneity condition.

This small example shows that SDMU_{11} is chosen as a benchmarking point, as can be seen from Fig. 14.3 where F_1 is completely specified by SDMU_{11}. If we do not set conditions for choosing a benchmarking point, the frontier is arguably too ideal. The main consequence is the potential under estimation of the efficiency of a DMU, since an improper benchmarking point is chosen. This specification of conditions is comparable to the modeling consideration in the evaluation considering environment constraints. One might expect that the performance of a DMU be evaluated by comparing it to the DMUs possessing similar environment characteristics (See, for example, Ruggiero (1998)).

14.3 Procedure for Estimating the Returns to Scale

In economic theory, the value of RTS measures the percentage change in output from a given percentage change in inputs. Let $y = f(x)$ denote a production function for a single-output technology. The production function is said to have IRS if $f(ax) > af(x)$, for any $a > 1$. The production function exhibits DRS if $f(ax) > af(x)$, for any $a \in [0, 1)$. If $f(ax) = af(x)$ for all scalars $a \geq 0$, the production function exhibits CRS. Banker et al. (1984), who introduced the concept of Most Productive Scale Size (MPSS) into the DEA literature, spurred extensions to the situations of multiple inputs and outputs. For a technically efficient DMU_0 with input and output bundle (X_0, Y_0) to be MPSS, the following optimization model should achieve a value of one. Note that the subscript 0 is usually used to indicate the DMU under evaluation in the literature. In the sequel, we shall frequently refer to DMU_0 when a specific DMU is discussed.

$$
\begin{aligned}
\max\ & \frac{\beta}{\alpha} \\
\text{s.t.}\ & (\alpha X_0, \beta Y_0) \in T \\
& \alpha, \beta \geq 0.
\end{aligned}
\tag{14.10}
$$

where T is the empirical production possibility set. If the optimal value is larger than 1, it means that either the current input level can be reduced with a less percentage of losses in outputs, or it can be increased with a larger percentage of gains in outputs. Therefore, DMU_0 can benefit from the adjustment of input levels.

By analogy, the following model is proposed for testing whether DMU_0 with multi-component structure is MPSS, where T in (14.10) is substituted by T^{VRS}.

$$
\begin{aligned}
&\max \frac{\phi}{\theta} \\
&\text{s.t.} \sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^k x_{ij}^{sk} + s^{s-} = \theta x_{io}^{s} \qquad i = 1, \ldots, l \text{ (shared inputs)} \\
&\qquad \sum_{j=1}^{n}\lambda_j^1 x_{ij}^{1} + s_i^{1-} = \theta x_{io}^{1} \qquad i = 1, \ldots, m \text{ (inputs dedicated to SDMU}_1\text{)} \\
&\qquad \sum_{j=1}^{n}\lambda_j^2 x_{ij}^{2} + s_i^{2-} = \theta x_{io}^{2} \qquad i = 1, \ldots, h \text{ (inputs dedicated to SDMU}_2\text{)} \\
&\qquad \sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^k y_{rj}^{sk} - s_r^{s+} = \phi y_{ro}^{s} \qquad r = 1, \ldots, u \text{ (shared outputs)} \\
&\qquad \sum_{j=1}^{n}\lambda_j^1 y_{rj}^{1} - s_r^{1+} = \phi y_{ro}^{1} \qquad r = 1, \ldots, s \text{ (outputs produced by SDMU}_1\text{)} \\
&\qquad \sum_{j=1}^{n}\lambda_j^2 y_{rj}^{2} - s_r^{2+} = \phi y_{ro}^{2} \qquad r = 1, \ldots, q \text{ (outputs produced by SDMU}_2\text{)} \\
&\qquad \sum_{j=1}^{n}\lambda_j^k = 1 \qquad k = 1, 2, \\
&\qquad \lambda_j^k \geq 0, k = 1, 2, j = 1, \ldots, n.
\end{aligned}
\tag{14.11}
$$

Cooper et al. (1996) proposed an approach to transform the above non-linear model to an equivalent linear model. Firstly, let us divide both sides of the constraints by ϕ. The resulting model is given in (14.12). Secondly, by letting $\theta/\phi = t$, $s_r^{s+}/\phi = \overline{s}_r^{s+}$, $s_r^{1+}/\phi = \overline{s}_r^{1+}$, $s_r^{2+}/\phi = \overline{s}_r^{2+}$, $s_i^{s-}/\phi = \overline{s}_i^{s-}$, $s_i^{1-}/\phi = \overline{s}_i^{1-}$, $s_i^{2-}/\phi = \overline{s}_i^{2-}$ and $\lambda_j^k/\phi = \overline{\lambda}_j^k$, we can obtain model (14.13). Since ϕ in (14.13) is a free variable, it is safe to delete it. Finally, model (14.13) can be further reduced to an equivalent model (14.14). Note that we call two optimization problems equivalent if from a solution of one, a solution of the other is readily found, and vice versa.

$$
\begin{aligned}
&\max \frac{\phi}{\theta} \\
&\text{s.t.} \sum_{k=1}^{2}\sum_{j=1}^{n}\frac{\lambda_j^k}{\phi}x_{ij}^{sk}+\frac{s_i^{s-}}{\phi}=\frac{\theta}{\phi}x_{io}^{s}\; i=1,\ldots,l\,(\text{shared inputs}) \\
&\quad \sum_{j=1}^{n}\frac{\lambda_j^1}{\phi}x_{ij}^{1}+\frac{s_i^{1-}}{\phi}=\frac{\theta}{\phi}x_{io}^{1} \quad i=1,\ldots,m\,(\text{inputs dedicated to SDMU}_1) \\
&\quad \sum_{j=1}^{n}\frac{\lambda_j^2}{\phi}x_{ij}^{2}+\frac{s_i^{2-}}{\phi}=\frac{\theta}{\phi}x_{io}^{2} \quad i=1,\ldots,h\,(\text{inputs dedicated to SDMU}_2) \\
&\quad \sum_{k=1}^{2}\sum_{j=1}^{n}\frac{\lambda_j^k}{\phi}y_{rj}^{sk}-\frac{s_r^{s+}}{\phi}=y_{ro}^{s} \quad r=1,\ldots,u\,(\text{shared outputs}) \\
&\quad \sum_{j=1}^{n}\frac{\lambda_j^1}{\phi}y_{rj}^{1}-\frac{s_r^{1+}}{\phi}=y_{ro}^{1} \quad r=1,\ldots,s\,(\text{outputs produced by SDMU}_1) \\
&\quad \sum_{j=1}^{n}\frac{\lambda_j^2}{\phi}y_{rj}^{2}-\frac{s_r^{2+}}{\phi}=y_{ro}^{2} \quad r=1,\ldots,q\,(\text{outputs produced by SDMU}_2) \\
&\quad \sum_{j=1}^{n}\frac{\lambda_j^k}{\phi}=\frac{1}{\phi} \quad k=1,2 \\
&\quad \lambda_j^k,\phi\geq 0, k=1,2, j=1,\ldots,n.
\end{aligned}
\tag{14.12}
$$

$$
\begin{aligned}
&\max \frac{1}{t} \\
&\text{s.t.} \sum_{k=1}^{2}\sum_{j=1}^{n}\overline{\lambda}_j^k x_{ij}^{sk}+\overline{s}_i^{s-}=tx_{io}^{s} \quad i=1,\ldots,l\,(\text{shared inputs}) \\
&\quad \sum_{j=1}^{n}\overline{\lambda}_j^1 x_{ij}^{1}+\overline{s}_i^{1-}=tx_{io}^{1} \quad i=1,\ldots,m\,(\text{inputs dedicated to SDMU}_1) \\
&\quad \sum_{j=1}^{n}\overline{\lambda}_j^2 x_{ij}^{2}+\overline{s}_i^{2-}=tx_{io}^{2} \quad i=1,\ldots,h\,(\text{inputs dedicated to SDMU}_2) \\
&\quad \sum_{k=1}^{2}\sum_{j=1}^{n}\overline{\lambda}_j^k y_{rj}^{sk}-\overline{s}_r^{s+}=y_{ro}^{s} \quad r=1,\ldots,u\,(\text{shared outputs}) \\
&\quad \sum_{j=1}^{n}\overline{\lambda}_j^1 y_{rj}^{1}-\overline{s}_r^{1+}=y_{ro}^{1} \quad r=1,\ldots,s\,(\text{outputs produced by SDMU}_1) \\
&\quad \sum_{j=1}^{n}\overline{\lambda}_j^2 y_{rj}^{2}-\overline{s}_r^{2+}=y_{ro}^{2} \quad r=1,\ldots,q\,(\text{outputs produced by SDMU}_2) \\
&\quad \sum_{j=1}^{n}\overline{\lambda}_j^1=\sum_{j=1}^{n}\overline{\lambda}_j^2=\frac{1}{\phi} \\
&\quad \lambda_j^k\geq 0, k=1,2, j=1,\ldots,n.
\end{aligned}
\tag{14.13}
$$

$$
\begin{aligned}
& t^* = \min\ t \\
\text{s.t.}\ & \sum_{k=1}^{2}\sum_{j=1}^{n}\overline{\lambda}_j^k x_{ij}^{sk} + \overline{s}_i^{s-} = t x_{io}^s \qquad i = 1, \ldots, l \text{ (shared inputs)} \\
& \sum_{j=1}^{n}\overline{\lambda}_j^1 x_{ij}^1 + \overline{s}_i^{1-} = t x_{io}^1 \qquad i = 1, \ldots, m \text{ (inputs dedicated to SDMU}_1) \\
& \sum_{j=1}^{n}\overline{\lambda}_j^2 x_{ij}^2 + \overline{s}_i^{2-} = t x_{io}^2 \qquad i = 1, \ldots, h \text{ (inputs dedicated to SDMU}_2) \\
& \sum_{k=1}^{2}\sum_{j=1}^{n}\overline{\lambda}_j^k y_{rj}^{sk} - \overline{s}_r^{s+} = y_{ro}^s \qquad r = 1, \ldots, u \text{ (shared outputs)} \\
& \sum_{j=1}^{n}\overline{\lambda}_j^1 y_{rj}^1 - \overline{s}_r^{1+} = y_{ro}^1 \qquad r = 1, \ldots, s \text{ (outputs produced by SDMU}_1) \\
& \sum_{j=1}^{n}\overline{\lambda}_j^2 y_{rj}^2 - \overline{s}_r^{2+} = y_{ro}^2 \qquad r = 1, \ldots, q \text{ (outputs produced by SDMU}_2) \\
& \sum_{j=1}^{n}\overline{\lambda}_j^1 = \sum_{j=1}^{n}\overline{\lambda}_j^2 \\
& \lambda_j^k \geq 0, k = 1, 2, j = 1, \ldots, n.
\end{aligned}
\tag{14.14}
$$

Assume that $t^*, \overline{\lambda}_j^{k*}$ are the optimal solution to model (14.14). It follows that $\phi^* = 1/\sum_{j=1}^{n}\lambda_j^{2*}$ and $\theta^* = t^*\phi^* = t^*/\sum_{j=1}^{n}\overline{\lambda}_j^{1*} = t^*/\sum_{j=1}^{n}\overline{\lambda}_j^{2*}$. Apparently, Proposition 1 holds.

Proposition 1 *If* $t^* = 1$, *then DMU is MPSS, and constant returns to scale prevails at DMU; Otherwise, the unit is not MPSS.*

RTS generally has an unambiguous meaning only if DMU_0 is on the efficiency frontier. For any inefficient DMU0 to become efficient, based on the optimal solutions of model (14.6), it can be projected onto the efficient frontier by formulas as follows:

(1) $\overline{y}_{ro}^s = y_{ro}^s + s_r^{+s*}, \overline{y}_{ro}^1 = y_{ro}^1 + s_r^{+1*}, \overline{y}_{ro}^2 = y_{ro}^2 + s_r^{+2*}$.
(2) $\overline{x}_{io}^s = t^* x_{io}^s - s_i^{-s*}, \overline{x}_{io}^1 = t^* x_{io}^1 - s_i^{-1*}, \overline{x}_{io}^2 = t^* x_{io}^2 - s_i^{-2*}$.

For those who are interested in the projection operation and the concept of efficient frontier, we recommend Cooper et al. (2004). A full treatment of the topics is beyond the scope of this paper. Before proceeding to discuss how to determine RTS of a DMU, we now introduce the scale efficiency of a production unit in Definition 1.

Definition 1 Scale efficiency: $\theta_S^* = t^*/\theta_T^*$.

Scale efficiency reflects the RTS characteristic of DMU_0. It should be noted that if DMU_0 is not an efficient unit, the scale efficiency actually reflects the RTS characteristic of the corresponding projection on the efficient frontier by formulas (14.1) and (14.2). Let us denote it as DMU_o^* for the convenience of reference.

Obviously, it can be seen that $\theta_S^* \leq 1$, since the feasible set of model (14.6) is a subset of the feasible set of model (14.14). If $\theta_S^* = 1$, DMU_o^* should achieve an efficiency rating of 1 by model (14.14). If not, it contradicts that $\theta_S^* = 1$, i.e., $t^* = \theta_T^*$. Therefore, by Proposition 1, DMU_0^* is MPSS. In other words, DMU_0 exhibits or is projected onto a region of the efficient frontier exhibits constant returns to scale.

If $\theta_S^* < 1$, or equivalently, the optimal objective function (ϕ/θ) of model (14.11) is larger than 1, the current input–output data of DMU_0^* can be improved in productivity by adjusting the scale of it. This is because the percentage by which the outputs gain equiproportionate increase due to the adjustment of the scale will outweigh the percentage by which the inputs increase equiproportionate, or the input equiproportionate reduction will outweigh the output equiproportionate reduction. To sum up, if $\theta_S^* < 1$, DMU_0 is currently not located in CRS region of the frontier or not projected onto a region of the frontier that exhibits CRS.

Below we provide Proposition 2 to shed light on how to determine whether IRS or DRS prevail at DMU_0 with the aid of model (14.15).

Proposition 2. (Conditions for the Determination of RTS (Multi-component))

(1) *If* $\theta_S^* = 1$, *then DMU0 exhibits or is projected onto a region of the efficient frontier exhibits constant returns to scale.*
(2) *If* $\theta_S^* < 1$ *and the optimal values of models (14.14) and (14.15) below coincide, then* DMU_0 *exhibits or is projected onto a region of the efficient frontier that exhibits increasing returns to scale.*
(3) *If* $\theta_S^* < 1$ *and the optimal values of models (14.6) and (14.15) below coincide,* DMU_0 *exhibits or is projected onto a region of the efficient frontier that exhibits decreasing returns to scale.*

A short proof of the proposition is in order. We consider the condition (2): $\theta_S^* < 1$ and the optimal values of models (14.14) and (14.15) coincide. The condition (3) can be established similarly.

Let $\overline{\lambda}_j^{1*}$ and $\overline{\lambda}_j^{2*}$ be the optimal solutions of models (14.14) and (14.15). It is clear that $\sum_{j=1}^{n} \overline{\lambda}_j^{1*} = \sum_{j=1}^{n} \overline{\lambda}_j^{2*} < 1$. DMU_0^* can make improvement through output augmentation since $\phi^* = 1/\sum_{j=1}^{n} \overline{\lambda}_j^{1*} > 1$. As DMU_0^* is technically efficient, the only way that it can increase the output level is by increasing the level of inputs. As the percentage by which the outputs increase outweighs the percentage by which the inputs increase, DMU_0 is currently located in the region that shows increasing returns to scale.

We have to show now it is impossible to lower its output level, and at the same time improve the productivity, i.e., achieve MPSS, since we have not checked if model (14.15) can achieve an value less than that of model (14.6) (i.e., θ_T^*) if $\sum_{j=1}^{n}\lambda_j^1 = \sum_{j=1}^{n}\lambda_j^2 \leq 1$ is replaced by $\sum_{j=1}^{n}\lambda_j^1 = \sum_{j=1}^{n}\lambda_j^2 \geq 1$. It should be noted that an optimal value less than θ_T^* in this context indicates DMU_0^* can gain benefits by lowering its input level. If this were true, the RTS of DMU_0^* will have an ambiguous meaning, since it can gain positive change in productivity by either lowering or augmenting its input level.

We claim impossibility by contradiction. Suppose $\overline{\lambda}_{1j}^{1*}, \overline{\lambda}_{1j}^{2*}, t_1^*$ and $\overline{\lambda}_{2j}^{1*}, \overline{\lambda}_{2j}^{2*}, t_2^*$ are the respective optimal solutions of model (14.15) and the model similar to model (14.15) except that $\sum_{j=1}^{n}\lambda_{1j}^{1*} = \sum_{j=1}^{n}\lambda_{1j}^{2*} < 1$ is replaced by $\sum_{j=1}^{n}\lambda_{2j}^{1*} = \sum_{j=1}^{n}\lambda_{2j}^{2*} > 1$. In addition, $t_1^* = t^* \leq t_2^* < \theta_T^*$ (i.e., $\theta_S^* < 1$). Thus, there exists a convex combination of the two solutions with $t^* = at_1^* + (1-a)t_2^* < \theta_T^*$, and $\sum_{j=1}^{n}\left(a\lambda_{1j}^{1*} + (1-a)\lambda_{2j}^{1*}\right) = \sum_{j=1}^{n}\left(a\lambda_{1j}^{2*} + (1-a)\lambda_{2j}^{2*}\right) = 1$, which contradicts the premise that θ_T^* is the optimal value of model (14.6). Thus, impossibility holds and condition (2) has an unambiguous meaning.

$$
\begin{aligned}
& t_{nirs}^* = \min\ t \\
& \text{s.t.}\ \sum_{k=1}^{2}\sum_{j=1}^{n}\overline{\lambda}_j^k x_{ij}^{sk} + \overline{s}_i^{s-} = tx_{io}^s \qquad i = 1, \ldots, l \ (\text{shared inputs}) \\
& \qquad \sum_{j=1}^{n}\overline{\lambda}_j^1 x_{ij}^1 + \overline{s}_i^{1-} = tx_{io}^1 \qquad i = 1, \ldots, m \ (\text{inputs dedicated to } \mathrm{SDMU}_1) \\
& \qquad \sum_{j=1}^{n}\overline{\lambda}_j^2 x_{ij}^2 + \overline{s}_i^{2-} = tx_{io}^2 \qquad i = 1, \ldots, h \ (\text{inputs dedicated to } \mathrm{SDMU}_2) \\
& \qquad \sum_{k=1}^{2}\sum_{j=1}^{n}\overline{\lambda}_j^k y_{rj}^{sk} - \overline{s}_r^{s+} = y_{ro}^s \qquad r = 1, \ldots, u \ (\text{shared outputs}) \\
& \qquad \sum_{j=1}^{n}\overline{\lambda}_j^1 y_{rj}^1 - \overline{s}_r^{1+} = y_{ro}^1 \qquad r = 1, \ldots, s \ (\text{outputs produced by } \mathrm{SDMU}_1) \\
& \qquad \sum_{j=1}^{n}\overline{\lambda}_j^2 y_{rj}^2 - \overline{s}_r^{2+} = y_{ro}^2 \qquad r = 1, \ldots, q \ (\text{outputs produced by } \mathrm{SDMU}_2) \\
& \qquad \sum_{j=1}^{n}\overline{\lambda}_j^1 = \sum_{j=1}^{n}\overline{\lambda}_j^2 \leq 1 \\
& \qquad \lambda_j^k \geq 0, k = 1, 2, j = 1, \ldots, n.
\end{aligned}
\tag{14.15}
$$

14.4 Theoretical Connection Between Black Box Approach and Multi-component Approach

In this section, we establish the equivalence between the method proposed in the previous section and the traditional black approach provided by Färe et al. (1985, 1994). This further ensures consistency in transition from black box to multi-component setting.

The efficiency measurements based on CRS, VRS, and NIRS respectively are provided as follows:

1. Efficiency index based on CRS;

$$
\begin{aligned}
\theta_b^{crs} &= \min\ \theta \\
\text{s.t.}\ &\sum_{j=1}^{n^2} \lambda_j y_{rj} \geq y_{ro} \qquad r = 1, \ldots, s. \\
&\sum_{j=1}^{n^2} \lambda_j x_{ij} \leq \theta x_{io} \qquad i = 1, \ldots, m. \\
&\lambda_j \geq 0, j = 1, \ldots, n^2.
\end{aligned} \tag{14.16}
$$

2. Efficiency index based on VRS;

$$
\begin{aligned}
\theta_b^{vrs} &= \min\ \theta \\
\text{s.t.}\ &\sum_{j=1}^{n^2} \lambda_j y_{rj} \geq y_{ro} \qquad r = 1, \ldots, s. \\
&\sum_{j=1}^{n^2} \lambda_j x_{ij} \leq \theta x_{io} \qquad i = 1, \ldots, m. \\
&\sum_{j=1}^{n^2} \lambda_j = 1 \\
&\lambda_j \geq 0, j = 1, \ldots, n^2.
\end{aligned} \tag{14.17}
$$

3. Efficiency index based on NIRS;

$$
\begin{aligned}
\theta_b^{nirs} &= \min\ \theta \\
\text{s.t.}\ &\sum_{j=1}^{n^2} \lambda_j y_{rj} \geq y_{ro} \qquad r = 1, \ldots, s. \\
&\sum_{j=1}^{n^2} \lambda_j x_{ij} \leq \theta x_{io} \quad i = 1, \ldots, m. \\
&\sum_{j=1}^{n^2} \lambda_j \leq 1 \\
&\lambda_j \geq 0, j = 1, \ldots, n^2.
\end{aligned} \tag{14.18}
$$

Färe et al. (1985, 1994) provided the following proposition for determining RTS.

Proposition 3 (Conditions for Determination of RTS (Black Box))

(1) *DMU_0 exhibits or is projected onto a region of the efficient frontier that exhibits constant returns to scale, if* $\theta_b^{crs} = \theta_b^{vrs} = \theta_b^{nirs}$.
(2) *DMU_0 exhibits or is projected onto a region of the efficient frontier that exhibits increasing returns to scale, if* $\theta_b^{crs} = \theta_b^{nirs} < \theta_b^{vrs}$.
(3) *DMU_0 exhibits or is projected onto a region of the efficient frontier that exhibits decreasing returns to scale, if* $\theta_b^{crs} < \theta_b^{nirs} = \theta_b^{vrs}$.

Formally, the following theorem holds.

Theorem 2 *Proposition 2 is equivalent to Proposition 3.*

Proof In light of Theorem 1, we can derive that $\theta_b^{vrs} = \theta_T^*, \theta_b^{crs} = t^*$ and $\theta_b^{crs} = t_{nirs}^*$, since the corresponding PPS's are equal. Since $\theta_s^* = 1$ indicates $t^* = \theta_T^* = t_{nirs}^*$, it follows that the first condition of Proposition 3 is equivalent to the first condition of Proposition 2. By the same reasoning, condition 2 of the propositions is equivalent as well as their conditions 3. Thus, Proposition 3 is equivalent to Proposition 2. □

14.5 Application

In this section, data extracted from *Environmental Statistics 2009* are used for illustration. We analyze the performances (efficiency and RTS) of the pollution treatment processes for waste water and waste air in China. Provinces are deemed as DMUs, each of which consists of two SDMUs, namely, capital city and non-capital cities. The pollution treatment process is depicted in Fig. 14.4.

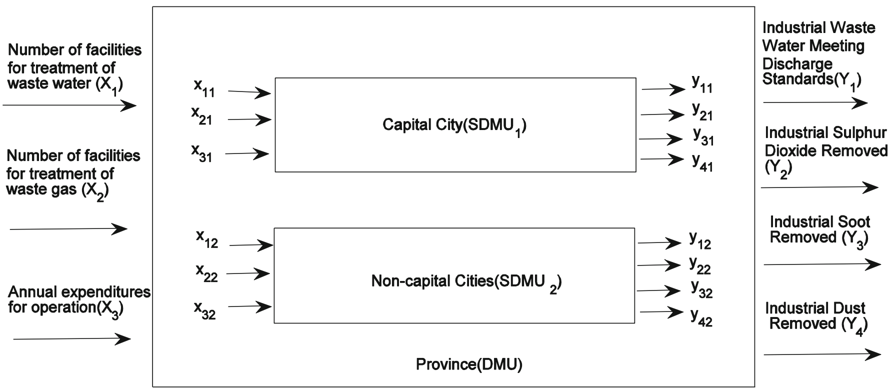


Fig. 14.4 Treatment process for wastewater and gas

The inputs involved in this application are three indicators: (1) number of facilities for treatment of wastewater in set (X_1); (2) number of facilities for treatment of waste gas in set (X_2); (3) annual expenditures in 10,000 Yuan (X_3). The outputs include (1) the industrial wastewater meeting discharge standards in 10,000 t (Y_1), (2) industrial sulphur dioxide removed in 10,000 t (Y_2), (3) industrial soot removed in 10,000 t (Y_3), and (4) industrial dust removed in 10,000 t (Y_4).

The inputs (X_1, X_2, X_3) are shared by capital city (SDMU_1) and non-capital cities (SDMU_2), and the outputs are the results of SDMU_1 and SDMU_2 fulfilling $X_i = \sum_{k=1}^{2} x_{ik}$ and $Y_r = \sum_{k=1}^{2} y_{rk}$.

Table 14.1 provides the input/output data by DMU (province), and Table 14.2 provides data on inputs/outputs by SDMU_1 (capital city). Table 14.3 presents the descriptive statistics of the data on inputs/outputs. In light of Table 14.3, capital city consumes relative more inputs and produces comparatively less outputs. An average capital city consumes inputs 19 %, 21 % and 26 % of the means of X_1, X_2 and X_3 respectively. However, the amounts produced account for 20 %, 17 %, 17 % and 14 % respectively of the means of Y_1, Y_2, Y_3 and Y_4 by an average capital city. Thus, roughly speaking, the average capital city consumes approximately more than one fifth of the total inputs, but produces less than one fifth of the total outputs. In this case, we might reasonably claim that the capital city consumes more inputs as compared with the noncapital cities. In the sequel, we will present the computational results associated with efficiency and returns to scale.

14.5.1 *Efficiency*

The efficiencies of DMUs by using the black-box approach and the proposed multi-component approach are presented in Table 14.4. From the black-box approach, the results of $\overline{\theta}_o^{crs}$ (CCR model), $\overline{\theta}_o^{nirs}$ and $\overline{\theta}_o^{vrs}$ (BCC model) are reported in columns 2–4. Column 5 presents results by Kao's parallel model which, in fact, are based on the $\overline{T}^{CRS}$ (see Kao (2009b)). Using the multi-component approach the results of $t^*, \theta^*_{nirs}, \theta^*_T$ by models (14.14), (14.15) and (14.6) are presented in columns 6–8.

Now we focus on the results of $\overline{\theta}_o^{vrs}$ and θ^*_T, both of which are based on the VRS assumption. Note that $\overline{\theta}_o^{vrs}$ is the result of the black-box approach without considering the internal mechanism of a DMU, and θ^*_T is the result of multi-component approach. The difference between the two efficiency indexes can be attributed to the level of information requirements. Obviously, if more information is available, we are able to refine the results from the black-box approach. Overall, notice from Table 14.4 that the mean of θ^*_T is approximately 87.6 % of the mean of $\overline{\theta}_o^{vrs}$, with a standard deviation of 0.103. Their distributions are provided in Fig. 14.5. The distribution of θ^*_T is more bell-shaped, while the distribution of $\overline{\theta}_o^{vrs}$ is obviously skewed to the left.

Table 14.1 Number of facilities, annual expenditures for operation and treatment of wastewater and waste gas by province

Province	X_1 (set)	X_2 (set)	X_3 (10,000 Yuan)	Y_1 (10,000 t)	Y_2 (10,000 t)	Y_3 (10,000 t)	Y_4 (10,000 t)
Beijing	514.00	2547.00	115,631.00	8221.00	11.20	199.90	108.40
Tianjin	875.00	3052.00	230,766.70	20,413.00	22.10	307.90	119.30
Hebei	5822.00	12,998.00	837,126.40	115,699.00	92.80	2158.30	741.80
Shanxi	2797.00	9216.00	588,709.40	35,229.00	108.70	2102.50	306.70
Inner Mongolia	836.00	4406.00	343,880.80	24,092.00	141.10	2358.20	236.20
Liaoning	1822.00	10,247.00	427,415.30	73,544.00	89.10	1520.40	675.30
Jilin	629.00	2983.00	118,217.20	33,443.00	8.90	885.50	293.50
Heilongjiang	990.00	4396.00	320,333.30	33,768.00	4.10	1133.30	107.00
Shanghai	1790.00	3839.00	394,044.20	41,364.00	24.00	524.40	103.40
Jiangsu	6469.00	11,365.00	994,868.20	253,940.00	177.20	2117.30	295.30
Zhejiang	7630.00	13,600.00	683,111.40	182,094.00	128.60	1297.20	475.70
Anhui	1795.00	4669.00	307,930.90	64,438.00	133.40	1097.20	321.00
Fujian	4196.00	6833.00	244,448.40	137,825.00	29.70	485.80	226.90
Jiangxi	1767.00	3786.00	202,398.00	63,863.00	130.60	1058.90	497.50
Shandong	4590.00	11,453.00	1,071,702.00	174,953.00	222.20	2683.20	614.00
Henan	3211.00	8225.00	1,113,638.00	126,308.00	134.60	2382.30	585.00
Hubei	2050.00	4921.00	378,840.30	87,753.00	71.80	781.30	362.10
Hunan	3149.00	4884.00	270,813.90	85,057.00	70.40	816.10	295.90
Guangdong	9968.00	13,876.00	1,031,626.00	191,413.00	108.80	1108.60	609.30
Guangxi	2552.00	5687.00	257,770.60	176,290.00	56.80	503.90	248.60
Hainan	293.00	401.00	53,475.40	5674.00	4.00	68.60	9.60
Chongqing	1550.00	2978.00	389,378.20	62,648.00	65.30	279.80	41.70
Sichuan	4757.00	6811.00	373,030.70	103,191.00	61.70	1006.90	239.30
Guizhou	1798.00	2708.00	185,327.00	8386.00	80.20	1015.10	143.90
Yunnan	2032.00	5267.00	283,327.80	30,574.00	114.40	817.60	373.70

(continued)

Table 14.1 (continued)

Province	X_1 (set)	X_2 (set)	X_3 (10,000 Yuan)	Y_1 (10,000 t)	Y_2 (10,000 t)	Y_3 (10,000 t)	Y_4 (10,000 t)
Tibet	13.00	39.00	563.50	274.00	0.00	0.10	0.00
Shananxi	2780.00	4036.00	203,039.20	47,132.00	36.40	440.60	190.70
Gansu	747.00	2734.00	604,078.40	9670.00	139.10	327.50	84.00
Qinghai	148.00	771.00	40,161.50	3767.00	1.10	139.10	66.80
Ningxia	380.00	1259.00	91,000.20	17,884.00	13.30	495.70	37.70
Xinjiang	775.00	4177.00	106,242.50	15,078.00	4.60	429.80	61.10

Table 14.2 Number of facilities, annual expenditures for operation and treatment of wastewater and waste gas by capital city

Capital city	X_1 (set)	X_2 (set)	X_3 (10,000 Yuan)	Y_1 (10,000 t)	Y_2 (10,000 t)	Y_3 (10,000 t)	Y_4 (10,000 t)
Beijing	514.00	2547.00	115,631.00	8221.00	11.20	199.90	108.40
Tianjin	875.00	3052.00	230,766.70	20,413.00	22.10	307.90	119.30
Shijiazhuang	547.00	1434.00	80,319.80	20,795.00	19.60	278.20	29.30
Taiyuan	310.00	1234.00	133,666.90	1994.00	19.00	529.20	87.40
Hohhot	74.00	550.00	42,033.10	2802.00	14.40	320.30	8.60
Shenyang	352.00	2181.00	25,760.90	6706.00	3.00	91.10	1.60
Changchun	103.00	671.00	16,906.40	5181.00	0.60	153.90	85.00
Harbin	156.00	863.00	32,357.70	3367.00	2.10	180.40	16.60
Shanghai	1790.00	3839.00	394,044.20	41,364.00	24.00	524.40	103.40
Nanjing	701.00	1123.00	152,446.10	36,606.00	42.00	271.80	78.00
Hangzhou	1051.00	1672.00	144,510.20	63,889.00	8.60	120.20	78.80
Hefei	170.00	389.00	14,842.70	2011.00	0.50	89.10	13.10
Fuzhou	367.00	537.00	58,322.90	5429.00	6.30	129.20	1.60
Nanchang	198.00	499.00	28,708.40	9471.00	2.30	47.90	27.80
Jinan	208.00	765.00	125,283.30	4693.00	12.60	157.60	68.60
Zhengzhou	388.00	1408.00	66,162.40	12,696.00	6.60	295.80	55.70
Wuhan	248.00	551.00	69,037.50	23,603.00	10.00	249.90	35.20
Changsha	289.00	315.00	8248.70	3679.00	2.60	50.10	20.10
Guangzhou	1161.00	2069.00	176,626.00	33,045.00	15.60	209.10	5.90
Nanning	367.00	840.00	28,386.60	12,696.00	2.00	38.60	34.10
Haikou	35.00	19.00	2984.30	481.00	0.00	0.00	0.00
Chongqing	1550.00	2978.00	389,378.20	62,648.00	65.30	279.80	41.70
Chengdu	1759.00	1545.00	67,992.50	20,421.00	5.60	216.70	13.50
Guiyang	385.00	753.00	51,893.60	2244.00	24.10	95.50	32.20
Kunming	546.00	1414.00	102,620.40	4405.00	63.20	125.20	41.00

(continued)

Table 14.2 (continued)

Capital city	X_1 (set)	X_2 (set)	X_3 (10,000 Yuan)	Y_1 (10,000 t)	Y_2 (10,000 t)	Y_3 (10,000 t)	Y_4 (10,000 t)
Lhasa	11.00	11.00	355.00	267.00	0.00	0.00	0.00
Xi'an	414.00	920.00	18,480.40	17,862.00	2.90	65.90	14.60
Lanzhou	73.00	512.00	493,456.40	2311.00	2.00	103.70	11.10
Xinning	123.00	420.00	32,555.10	3443.00	1.10	115.80	38.40
Yinchuan	106.00	242.00	18,244.00	4923.00	1.40	12.70	6.90
Urumqi	103.00	474.00	27,511.10	4564.00	0.40	39.80	10.40

Table 14.3 Descriptive statistics on input and output variables

Variables	Mean (province)	Std. dev. (province)	Mean (capital city)	Std. dev. (capital city)	Mean (non-capital city)	Std. dev. (non-capital city)
X_1 (set)	2539.5	2368.8	483.03	494.6	2056.5	2188.7
X_2 (set)	5618.2	3920.8	1155.7	949.74	4462.5	3801.1
X_3 (10,000 Yuan)	395,580	319,360	101,600	122,600	293,980	302,330
Y_1 (10,000 t)	72,064	67,491	14,265	17,043	57,799	61,115
Y_2 (10,000 t)	73.748	59.378	12.616	16.88	61.132	57.118
Y_3 (10,000 t)	985.26	758.18	170.96	134.95	814.3	711.99
Y_4 (10,000 t)	273.27	210.9	38.332	35.447	234.94	212.8

Furthermore, according to Fig. 14.5, 15 provinces are classified as efficient by the BCC. It can be seen the discrimination power of BCC model in this application is too weak. By contrast, 12 of them are degraded in efficiencies by the multi-component approach. They are Hebei, Liaoning, Zhejiang, Jiangxi, Shandong, Henan, Guangdong, Guangxi, Tibet, Gansu, Qinghai, and Ningxia. Seven of them are given efficiency scores lower than 0.9.

Finally, we point out that the efficiency scores based on T^{CRS} are almost the same as those based on $\overline{T}^{CRS}$. Though the differences of θ^*_{Kao} and t^* are negligible, we can find that the efficiencies of some DMUs such as Zhejiang and Hunan are adjusted slightly.

14.5.2 Returns to Scale

The RTS of provinces can be determined by Proposition 3 (black box), and Proposition 2 (multi-component). The results are presented in Table 14.5.

Table 14.5 shows that approximately half of the provinces which are classified by the black-box approach as CRS and IRS are reclassified as DRS or CRS by the multi-component approach. Those classified as DRS by the black-box approach remain the same by the both approaches. We concentrate here on the results of the multi-component approach. In summary, six provinces show IRS, five provinces show CRS and the rest show DRS. Among those that show CRS, Inner Mongolia and Jilin have the MPSS because the optimal value in Model (14.10) that corresponds to t^* in Table 14.4 equals one. We proceed to rearrange the results by the multi-component approach according to the administrative regions of China. The results are provided in Table 14.6.

From Table 14.6, the developed provinces are more likely to show DRS. In particular, East China shows DRS entirely. Another obvious finding is that the provinces that show IRS are mainly located in the west of China, which is less developed area of China.

Table 14.4 Results of various models

Provinces	θ_b^{crs}	θ_b^{nirs}	θ_b^{vrs}	θ_{Kao}	t*	θ^*_{nirs}	θ^*_T
Beijing	0.4944	0.4944	0.5062	0.373	0.373	0.373	0.373
Tianjin	0.4681	0.4681	0.4748	0.3937	0.3937	0.4003	0.4003
Hebei	0.5428	1	1	0.4718	0.4718	0.8754	0.8754
Shanxi	0.5161	0.5654	0.5654	0.4952	0.4952	0.536	0.536
Inner Mongolia	1	1	1	1	1	1	1
Liaoning	0.902	1	1	0.7609	0.7609	0.8761	0.8761
Jilin	1	1	1	1	1	1	1
Heilongjiang	0.7037	0.7037	0.7049	0.5528	0.5528	0.6043	0.6043
Shanghai	0.5177	0.5177	0.519	0.3241	0.3241	0.3756	0.3756
Jiangsu	0.9221	1	1	0.6711	0.6711	1	1
Zhejiang	0.5352	1	1	0.5034	0.5036	0.7536	0.7536
Anhui	0.9099	0.9405	0.9405	0.7872	0.7872	0.7941	0.7941
Fujian	0.8612	0.8612	0.8614	0.7703	0.7703	0.8023	0.8023
Jiangxi	1	1	1	0.9605	0.9618	0.9709	0.9709
Shandong	0.8142	1	1	0.6254	0.6254	0.9473	0.9473
Henan	0.9133	1	1	0.7209	0.7209	0.9412	0.9412
Hubei	0.8199	0.8199	0.8201	0.6874	0.6874	0.7555	0.7555
Hunan	0.774	0.774	0.7742	0.7004	0.7006	0.7006	0.7006
Guangdong	0.5486	1	1	0.4723	0.4723	0.7266	0.7266
Guangxi	1	1	1	0.9612	0.9614	0.9708	0.9708
Hainan	0.6608	0.6608	0.7323	0.4066	0.4066	0.4066	0.4243
Chongqing	0.9163	0.9163	0.9203	0.6495	0.6495	0.7123	0.7123
Sichuan	0.6593	0.6593	0.6596	0.615	0.6154	0.6179	0.6179
Guizhou	0.8909	0.8909	0.8961	0.8397	0.8397	0.8407	0.8407
Yunnan	0.6877	0.6877	0.6882	0.5962	0.5965	0.5965	0.597
Tibet	0.711	0.711	1	0.5031	0.5031	0.5031	0.8519
Shananxi	0.5248	0.5248	0.5274	0.4946	0.495	0.495	0.4953
Gansu	1	1	1	0.9309	0.9346	0.9346	0.9351
Qinghai	0.9673	0.9673	1	0.6992	0.6992	0.6992	0.706
Ningxia	1	1	1	0.889	0.8903	0.8903	0.8903
Xinjiang	0.5407	0.5407	0.5434	0.4591	0.4591	0.5198	0.5198

14.6 Summary and Conclusion

This paper studies the efficiency evaluation and RTS estimation in the situation where a DMU has multi-component structure. Radial measurements for efficiency evaluation and a procedure to determine the RTS of a DMU that is unaffected by possible multiple optima are provided. In doing so, we emphasize the theoretical connections between the black-box approach, which has been extensively studied in the literature, and the proposed methods. The strong relationship as is given by theorem 1 ensures a consistent transition from the black-box approach to the multi-component approach.

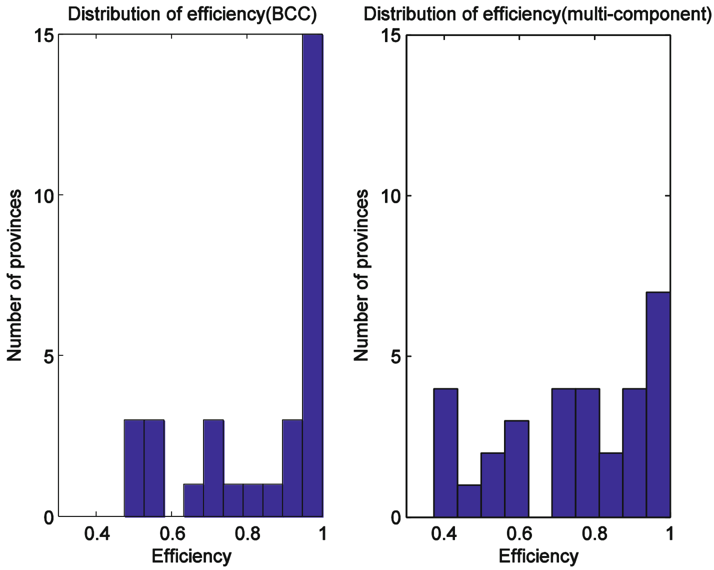


Fig. 14.5 Distribution of efficiency scores

In the application section, we use the proposed method to study the efficiencies and RTS of pollution treatment processes in China. The results show that the multi-component approach has strong discrimination power: the efficiency scores obtained are distributed in a bell-shaped manner, contrast this to the weak discrimination power as evidenced by the black-box approach with the distribution of efficiency scores skewed to the left. It is also found that six provinces show IRS, five provinces show CRS, and the rest show DRS. Among those that show CRS, Inner Mongolia and Jilin have the MPSS. Furthermore, the developed provinces are more likely to show DRS. In particular, East China shows DRS entirely. In contrast, the provinces that show IRS are mainly located in the west, which is a less developed area of China.

Acknowledgments The research is supported by National Natural Science Funds of China (Nos. 71301155; 71471053) and the Fundamental Research Funds for the Central Universities (No. JZ2015HGBZ0481).

Appendix

Proof of Theorem 1 Before we prove theorem 1, we establish Lemma 1.

Table 14.5 Results of various models

Provinces	RTS (black box)	RTS (multi-component)
Beijing	IRS	CRS
Tianjin	IRS	DRS
Hebei	DRS	DRS
Shanxi	DRS	DRS
Inner Mongolia	CRS	CRS
Liaoning	DRS	DRS
Jilin	CRS	CRS
Heilongjiang	IRS	DRS
Shanghai	IRS	DRS
Jiangsu	DRS	DRS
Zhejiang	DRS	DRS
Anhui	DRS	DRS
Fujian	IRS	DRS
Jiangxi	CRS	DRS
Shandong	DRS	DRS
Henan	DRS	DRS
Hubei	IRS	DRS
Hunan	IRS	CRS
Guangdong	DRS	DRS
Guangxi	CRS	DRS
Hainan	IRS	IRS
Chongqing	IRS	DRS
Sichuan	IRS	DRS
Guizhou	IRS	DRS
Yunnan	IRS	IRS
Tibet	IRS	IRS
Shananxi	IRS	IRS
Gansu	CRS	IRS
Qinghai	IRS	IRS
Ningxia	CRS	CRS
Xinjiang	IRS	DRS

Lemma A1 *Define* $\hat{T}_b^{VRS}, \hat{T}^{VRS}$ *as follows:*

$$\hat{T}_b^{VRS} = \left\{ (X, Y) \middle| \sum_{j=1}^{n^2} \lambda_j x_{ij} = x_i, i = 1, \ldots, m, \right.$$
$$\left. \sum_{j=1}^{n^2} \lambda_j y_{rj} = y_r, r = 1, \ldots, s, \sum_{j=1}^{n^2} \lambda_j = 1, \lambda_j \geq 0 \right\}$$

and

Table 14.6 RTS by administrative regions

Region	Provinces		
	IRS	CRS	DRS
North China		Beijing, Inner Mongolia	Tianjin, Hebei, Shanxi
Northeast		Jilin	Liaoning, Heilongjiang
East China			Shanghai, Jiangsu, Zhejiang, Anhui, Fujian, Jiangxi, Shandong
South-central China	Hainan	Hunan	Henan, Hubei, Guangdong, Guangxi
Southwest	Yunnan, Tibet		Chongqing, Sichuan, Guizhou
Northwest	Shaanxi, Gansu, Qinghai	Ningxia	Xinjiang
Total	6	5	20

$\hat{T}^{VRS} = \left\{ (X,Y) \middle| \sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^k x_{ij}^k = x_i, i = 1, \ldots, m, \sum_{k=1}^{2}\sum_{j=1}^{n}\lambda_j^k y_{rj}^k = y_r, r = 1, \ldots, s, \right.$
$\left. \sum_{j=1}^{n}\lambda_j^k = 1, \lambda_j^k \geq 0 \right\}$. *Then* $\hat{T}_b^{VRS} = \hat{T}^{VRS}$.

Proof (1) $\hat{T}_b^{VRS} \subseteq \hat{T}^{VRS}$;

Let $\mathrm{DMU_j}$ be some DMU in **EDS**, and $(x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{rj})$ be its input–output bundle. Suppose it is made of $\mathrm{SDMU_{1k}}$, and $\mathrm{SDMU_{2m}}$, where $k, m \in \{1, \ldots, n\}$. Obviously, $\left(x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{rj}\right) \in \hat{T}_b^{VRS}$, since it can be decomposed into input–output bundle of $\mathrm{SDMU_{1k}}$, and that of $\mathrm{SDMU_{2m}}$. To put it another way, if we set a multiplier corresponding to $\mathrm{SDMU_{1k}}$ and $\mathrm{SDMU_{2m}}$ equal to 1 and other multipliers equal to zero, we can see that $(x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{rj})$ satisfies the condition to be an element of $\hat{T}^{VRS}$. Therefore $\hat{T}_b^{VRS} \subseteq \hat{T}^{VRS}$ holds.

(2) $\hat{T}_b^{VRS} \supseteq \hat{T}^{VRS}$;

For any $(X,Y) \in \hat{T}^{VRS}$, there exist two sets of convex multipliers $(\lambda_1^1, \ldots, \lambda_n^1)$ and $(\lambda_1^2, \ldots, \lambda_n^2)\left(\lambda_j^1, \lambda_j^2 \geq 0, \sum_{j=1}^{n}\lambda_j^1 = 1, \sum_{j=1}^{n}\lambda_j^2 = 1\right)$ such that

$$
\begin{aligned}
x_i &= \sum_{j=1}^{n}\lambda_j^1 x_{ij}^1 + \sum_{j=1}^{n}\lambda_j^2 x_{ij}^2 (i = 1, \ldots, m), \\
y_r &= \sum_{j=1}^{n}\lambda_j^1 y_{rj}^1 + \sum_{j=1}^{n}\lambda_j^2 y_{rj}^2 (r = 1, \ldots, s).
\end{aligned}
\tag{14.19}
$$

We need to show that there always exists a convex multiplier $\sum_{j=1}^{n^2} \lambda_j = 1, \lambda_j \geq 0$, such that $x_i = \sum_{j=1}^{n^2} \lambda_j x_{ij}, y_r = \sum_{j=1}^{n^2} \lambda_j y_{rj}$, where $(x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{rj})$ is the input–output bundle of DMU_j in **EDS**. In other words, there is a convex multiplier such that the following equations hold:

$$x_i = \sum_{j=1}^{n} \lambda_j \left(x_{i1}^1 + x_{ij}^2 \right) + \sum_{j=n+1}^{2n} \lambda_j \left(x_{i2}^1 + x_{i(j-n)}^2 \right) +, \ldots, + \sum_{j=n^2-n+1}^{n^2} \lambda_j \left(x_{in}^1 + x_{i(j-n^2-n)}^2 \right)$$
$$y_r = \sum_{j=1}^{n} \lambda_j \left(y_{r1}^1 + y_{rj}^2 \right) + \sum_{j=n+1}^{2n} \lambda_j \left(y_{r2}^1 + y_{r(j-n)}^2 \right) +, \ldots, + \sum_{j=n^2-n+1}^{n^2} \lambda_j \left(y_{rn}^1 + y_{r(j-n^2-n)}^2 \right) \tag{14.20}$$

where $(x_{1j}^1, \ldots, x_{mj}^1, y_{1j}^1, \ldots, y_{sj}^1)$ and $(x_{1j}^2, \ldots, x_{mj}^2, y_{1j}^2, \ldots, y_{sj}^2)$, $j = 1, \ldots, n$, are the respective input bundle and output bundle of $\text{SDMU}_{1\text{j}}$, and $\text{SDMU}_{2\text{j}}$. That is to say, $\sum_{j=1}^{n^2} \lambda_j = 1, \lambda_j \geq 0$ must satisfy the following conditions:

$$\lambda_j^1 = \sum_{k=(j-1)n+1}^{(j-1)n+n} \lambda_k,\ \lambda_j^2 = \sum_{k=1}^{n} \lambda_{n(j-1)+k},\ j = 1, \ldots, n \tag{14.21}$$

To facilitate understanding, we organize the conditions as matrix products.

$$\begin{bmatrix} \lambda_1 & \lambda_{n+1} & \ldots & \lambda_{n^2-n+1} \\ \lambda_2 & \lambda_{n+2} & \ldots & \lambda_{n^2-n} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_n & \lambda_{n+n} & \ldots & \lambda_{n^2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \ldots \\ \lambda_n^2 \end{bmatrix} \tag{14.22}$$

$$\begin{bmatrix} \lambda_1 & \lambda_{n+1} & \ldots & \lambda_{n^2-n+1} \\ \lambda_2 & \lambda_{n+2} & \ldots & \lambda_{n^2-n} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_n & \lambda_{n+n} & \ldots & \lambda_{n^2} \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^1 \\ \lambda_2^1 \\ \ldots \\ \lambda_n^1 \end{bmatrix} \tag{14.23}$$

The above illustration indicates that the row j of the matrix is summed to λ_j^2, and the column j the matrix is summed to λ_j^1. Let us now combine (14.22) and (14.23) into the following equations where A is $2n$ by n^2.

$$\mathbf{A}\lambda = \begin{bmatrix} \overbrace{11,\ldots,1}^{n} & \overbrace{00,\ldots,0}^{n} & \overbrace{00,\ldots,0}^{n} & \ldots & \overbrace{00,\ldots,0}^{n} \\ 00,\ldots,0 & 11,\ldots,1 & 00,\ldots,0 & \ldots & 00,\ldots,0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 00,\ldots,0 & 00,\ldots,0 & 00,\ldots,0 & \ldots & 11,\ldots,1 \\ 10,\ldots,0 & 10,\ldots,0 & 10,\ldots,0 & \ldots & 10,\ldots,0 \\ 01,\ldots,0 & 01,\ldots,0 & 01,\ldots,0 & \ldots & 01,\ldots,0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 00,\ldots,1 & 00,\ldots,1 & 00,\ldots,1 & \ldots & 00,\ldots,1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdots \\ \lambda_{n^2} \end{bmatrix} = \begin{bmatrix} \lambda_1^1 \\ \lambda_2^1 \\ \cdots \\ \lambda_n^1 \\ \lambda_1^2 \\ \lambda_2^2 \\ \cdots \\ \lambda_n^2 \end{bmatrix} = \mathbf{\Gamma} \quad (14.24)$$

We are going to prove (14.24) always has a nonnegative solution $\lambda_1^*, \ldots, \lambda_{n^2}^*$. Note that $\sum_{j=1}^{n^2} \lambda_j^* = 1$ automatically holds provided $\sum_{j=1}^{n} \lambda_j^1 = 1$ and $\sum_{j=1}^{n} \lambda_j^2 = 1$. Our problem reduces to the existence of nonnegative solution to (14.24). We claim the nonnegative solution always exists, by way of contradiction. Before moving on, we reduce (14.24) to (14.25).

$$\overline{\mathbf{A}}\lambda = \begin{bmatrix} \overbrace{00,\ldots,0}^{n} & \overbrace{11,\ldots,1}^{n} & \overbrace{00,\ldots,0}^{n} & \ldots & \overbrace{00,\ldots,0}^{n} \\ 00,\ldots,0 & 00,\ldots,0 & 11,\ldots,1 & \ldots & 00,\ldots,0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 00,\ldots,0 & 00,\ldots,0 & 00,\ldots,0 & \ldots & 11,\ldots,1 \\ 10,\ldots,0 & 10,\ldots,0 & 10,\ldots,0 & \ldots & 10,\ldots,0 \\ 01,\ldots,0 & 01,\ldots,0 & 01,\ldots,0 & \ldots & 01,\ldots,0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 00,\ldots,1 & 00,\ldots,1 & 00,\ldots,1 & \ldots & 00,\ldots,1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdots \\ \lambda_{n^2} \end{bmatrix} = \begin{bmatrix} \lambda_2^1 \\ \lambda_3^1 \\ \cdots \\ \lambda_n^1 \\ \lambda_1^2 \\ \lambda_2^2 \\ \cdots \\ \lambda_n^2 \end{bmatrix} = \overline{\mathbf{\Gamma}} \quad (14.25)$$

Note that we have eliminated the first row of $\mathbf{A}$ and the first element of $\mathbf{\Gamma}$ by elementary row operation. Assume, now, that $\overline{A}\lambda = \overline{\Gamma}$ doesn't have a nonnegative solution, i.e., $\overline{\mathbf{\Gamma}}$ doesn't belong to the conic hull constructed by the column vectors of $\bar{A}$. By Farkas lemma, there exists $\mathbf{x} \in R^{2n-1}$, such that

(1) $\mathbf{x}^{\mathbf{T}}\overline{\Gamma} > 0$;
(2) $\mathbf{x}^{\mathbf{T}}\overline{\mathbf{A}}(i) \leq 0, \overline{\mathbf{A}}(i)$ denotes the i th column of $\bar{\mathbf{A}}$, $i = 1, \ldots, n^2$.

By (2), it follows that

(1) $\mathbf{x}(i) \leq 0,\ i = n, \ldots, 2n-1,\ \big(\mathbf{x}(i)$ denotes the ith component of vector $\mathbf{x}$);
(2) For any $k = 1, \ldots, n-1$, we have $x(k) + x(i) \leq 0,\ i = n, \ldots, 2n-1$, i.e., $x(k) \leq \min_{j=n,\ldots 2n-1} -x(j)$.

Combining the previous two conditions, we obtain

$$
\begin{aligned}
x^T\overline{\Gamma} &= \sum_{k=1}^{n-1} x(k)\lambda_{k+1}^1 + \sum_{j=n}^{2n-1} x(j)\lambda_j^2 \le \left(\min_{j=n,\dots,2n-1} -x(j)\right)\sum_{k=1}^{n-1}\lambda_{k+1}^1 + \sum_{j=n}^{2n-1} x(j)\lambda_j^2 \\
&= \left(-\max_{j=n,\dots,2n-1} x(j)\right)\sum_{k=1}^{n-1}\lambda_{k+1}^1 + \sum_{j=n}^{2n-1} x(j)\lambda_j^2 \\
&\le \left(-\max_{j=n,\dots,2n-1} x(j)\right)\sum_{k=1}^{n-1}\lambda_{k+1}^1 + \max_{j=n,\dots,2n-1} x(j) \\
&= \left(\max_{j=n,\dots,2n-1} x(j)\right)\left(1-\sum_{k=1}^{n-1}\lambda_{k+1}^1\right) \le 0
\end{aligned}
\tag{14.26}
$$

To see why the last relation holds, note that $\sum_{j=1}^{n} \lambda_j^1 = 1$ and $\mathbf{x}(i) \le 0,\ i = n, \dots, 2n-1$. So it follows that $1 - \sum_{k=1}^{n-1} \lambda_{k+1}^1 = \lambda_1^1 \ge 0$, and $\max_{j=n,\dots,2n-1} x(j) \le 0$. Therefore, the product of the two parts is less than or equal to zero.

This contradicts $\mathbf{x}^T\overline{\mathbf{\Gamma}} > 0$. Therefore, $\overline{\mathbf{\Gamma}}$ belongs to the conic hull constructed by the column vectors of $\bar{\mathbf{A}}$, i.e., there is $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n^2}) \ge 0$ such that $\overline{\mathbf{A}}\lambda = \overline{\mathbf{\Gamma}}$, which also means that $\mathbf{A}\lambda = \mathbf{\Gamma}$. By our construction, we know that there exists $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n^2}) \ge 0$ such that (14.22) and (14.23) hold. In turn, this establishes that $(X, Y) \in \hat{T}_b^{VRS}$. □

Proof of Theorem 1 Let $(x_{1j}, \dots, x_{mj}, y_{1j}, \dots, y_{rj})$ be an arbitrary point in T_b^{VRS}. We first prove that $T_b^{VRS} \subseteq T^{VRS}$. By definition, there exists one point $\left(\bar{x}_{1j}, \dots, \bar{x}_{mj}, \bar{y}_{1j}, \dots, \bar{y}_{rj}\right)$ in $\hat{T}_b^{VRS}$ such that $x_{ij} \ge \bar{x}_{ij}$ and $y_{rj} \le \bar{y}_{rj}$. In light of Lemma 1, $\left(\bar{x}_{1j}, \dots, \bar{x}_{mj}, \bar{y}_{1j}, \dots, \bar{y}_{rj}\right)$ also belongs to $\hat{T}^{VRS}$. Therefore $\left(x_{1j}, \dots, x_{mj}, y_{1j}, \dots, y_{rj}\right) \in T^{VRS}$, since there is a point in T^{VRS} such that $x_{ij} \ge \bar{x}_{ij}$ and $y_{rj} \le \bar{y}_{rj}$ hold. By analogy, we can prove $T_b^{VRS} \supseteq T^{VRS}$. Therefore, $T_b^{VRS} = T^{VRS}$ holds.

By substituting the convex condition in the definition of T^{VRS} and T_b^{VRS} for $\sum_{j=1}^{n} \lambda_j^k = t\ (k = 1, 2)$ and $\sum_{j=1}^{n^2} \lambda_j = t\ (t \ge 0)$ respectively, it follows that $T^{VRS}(t) = T_b^{VRS}(t)$, since they are obtained by scaling up or down T^{VRS} and T_b^{VRS} by the same factor t. Given the fact that $T_b^{CRS} = \bigcup_{t\in[0,\infty)} T_b^{VRS}(t)$, $T_b^{NIRS} = \bigcup_{t\in[0,1]} T_b^{VRS}(t)$, and $T^{CRS} = \bigcup_{t\in[0,\infty)} T^{VRS}(t)$, $T^{NIRS} = \bigcup_{t\in[0,1]} T^{VRS}(t)$, it follows $T_b^{CRS} = T^{CRS}$ and $T_b^{NIRS} = T^{NIRS}$. □

References

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for the estimation of technical and scale inefficiencies in data envelopment analysis. *Management Science, 30*(9), 1078–1092.

Banker, R. D., Cooper, W. D., Seiford, L. M., Thrall, R. M., & Zhu, J. (2004). Returns to scale in different DEA models. *European Journal of Operational Research, 154*(2), 345–362.

Beasley, J. E. (1995). Determining teaching and research efficiencies. *Journal of the Operational Society, 46*(4), 441–452.

Castelli, L., Pesenti, R., & Ukovich, W. (2010). A classification of DEA models when the internal structure of the decision making units is considered. *Annals of Operations Research, 173*(1), 207–235.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research, 2*(6), 429–444.

Chen, Y., Cook, W. D., Li, N., & Zhu, J. (2009a). Additive efficiency decomposition in two-stage DEA. *European Journal of Operational Research, 196*(3), 1170–1176.

Chen, Y., Cook, W. D., & Zhu, J. (2010). Deriving the DEA frontier for two-stage processes. *European Journal of Operational Research, 202*(1), 138–142.

Chen, Y., Liang, L., Yang, F., & Zhu, J. (2006). Evaluation of information technology investment: A data envelopment analysis approach. *Computers & Operations Research, 33*(5), 1368–1379.

Chen, Y., Liang, L., & Zhu, J. (2009b). Equivalence in two-stage DEA approaches. *European Journal of Operational Research, 193*(2), 600–604.

Cook, W. D., Hababou, M., & Tuenter, H. J. H. (2000). Multicomponent efficiency measurement and shared inputs in data envelopment analysis: An application to sales and service performance in bank branches. *Journal of Productivity Analysis, 14*(3), 209–224.

Cook, W. D., Liang, L., & Zhu, J. (2010). Measuring performance of two-stage network structures by DEA: A review and future perspective. *Omega, 38*(6), 423–430.

Cooper, W. W., Seiford, L. M., & Zhu, J. (2004). *Handbook on data envelopment analysis*. Boston: Kluwer Academic.

Cooper, W. W., Thompson, R. G., & Thrall, R. M. (1996). Extensions and new developments in DEA. *Annals of Operations Research, 66*(2), 1–45.

Färe, R., & Grosskopf, S. (2000). Network DEA. *Socio-Economic Planning Sciences, 34*(1), 25–49.

Färe, R., Grosskopf, S., & Lovell, C. A. K. (1985). *The measurement of efficiency of production*. Boston: Kluwer Nijhoff.

Färe, R., Grosskopf, S., & Lovell, C. A. K. (1994). *Production frontiers*. Cambridge: Cambridge University Press.

Kao, C. (2009a). Efficiency decomposition in network data envelopment analysis: A relational model. *European Journal of Operational Research, 192*(3), 949–962.

Kao, C. (2009b). Efficiency measurement for parallel production systems. *European Journal of Operational Research, 196*(3), 1107–1112.

Kao, C., & Hwang, S.-N. (2008). Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan. *European Journal of Operational Research, 185*(1), 418–429.

Lewis, H. F., & Sexton, T. R. (2004). Network DEA: Efficiency analysis of organizations with complex internal structure. *Computers & Operations Research, 31*(9), 1365–1410.

Liang, L., Cook, W. W., & Zhu, J. (2008). DEA models for two-stage processes: Game approach and efficiency decomposition. *Naval Research Logistics, 55*(7), 643–653.

Ruggiero, J. (1998). Non-discretionary inputs in data envelopment analysis. *European Journal of Operational Research, 111*(3), 461–469.

Shephard, R. W. (1970). The theory of cost and production functions. Princeton: Princeton University Press.

Tone, K., & Tsutsui, M. (2009). Network DEA: A slacks-based measure approach. *European Journal of Operational Research, 197*(1), 243–252.

Tsai, P. F., & Molinero, C. M. (2002). A variable returns to scale data envelopment analysis model for the joint determination of efficiencies with an example of the UK health service. *European Journal of Operational Research, 141*(1), 21–38.