Chapter 13
Manufacturer-to-Retailer Versus Manufacturer-to-Consumer Rebates in a Supply Chain

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1 Introduction

Rebates are widely used as promotional tools. In this paper we investigate the effects of two kinds of rebates (from the manufacturer) on supply chains: retailer rebates and consumer rebates. Retailer rebates, also known as channel rebates, are payments from the manufacturer to the retailer based on the sales performance of the retailer. Taylor (2002) cites several examples of the use of retailer rebates, in industries that range from software to printers, from network hardware switching to automotive. Consumer rebates, which are no less widespread than retailer rebates, are payments from the manufacturer to the consumer upon the consumer's purchase of the manufacturer's product. Most everybody is familiar through personal experience with the use of consumer rebates in consumer electronics, automotive and food products industries. The magnitude of rebate offers can reach surprisingly large numbers: A *New York Times* article reports that \$10 billion worth of consumer rebates were offered in 2002 (Millman 2003).[1] Although some consumers do not claim their rebates (especially when the rebate size is small), the number of claims for consumer rebates is not negligible either: In 1998 Young America Inc. was reported to mail out 30 million rebate checks a year on behalf of companies like PepsiCo Inc., Nestle SA and OfficeMax (Bulkeley 1998). More recent statistics also suggest that the rebate

[1] In some cases, retailers themselves offer rebates to consumers. It is possible that the amount \$10 billion quoted in the article includes the rebates offered by the retailers themselves.

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activity remained strong in recent years. For example, according to a phone survey conducted by Consumer Reports National Research Center in 2009, 70 % of consumers reported having claimed a rebate within the past 12 months.[2] Similarly, high rebate activity was reported in a survey conducted by Parago, a firm that runs rebate and reward programs for its clients. The company's 2010 survey found that 47 % of consumers had submitted a rebate within the past 12 months.[3]

It is likely that rebates will remain on the scene as online shopping becomes more popular and smart phones start to play a larger role in consumers' purchases. In fact, online shopping enables instantly redeemable rebates, which are more attractive to customers. For example, Parago's shopper behavior study for 2013 found that 83 % of customers agree that "when shopping online, a discount via rebate is attractive." Similarly, 80 % of customers agreed that "the ability to submit a rebate via a smart phone is attractive," and 75 % of customers said that they wanted to scan a barcode in-store for rebates on their phone.[4]

For both retailer and consumer rebates, there do exist different implementations. Retailer rebates can be paid for each unit the retailer sells to the end customer or only for units sold in excess of a target number (Taylor 2002). Here we focus on the former type. In our model, the manufacturer uses consumer rebates for the sole purpose of selling more to the retailer. Thus, we do not address the role they may have early in a product's life cycle to learn more about demand or later to increase demand for unintended excess inventories. Consumer rebates can be in the form of mail-in rebates or coupons. Moreover, there are different kinds of coupons; some can be instantly redeemed at the time of purchase and some can be used only the next time a product is purchased. Of course, the specifics of the rebate offer have an influence on how attractive consumers find the rebate and how many customers will redeem the rebate. Here we use a stylized model of consumer rebates. We assume that (all) consumers treat a rebate of \$1 as being equivalent to a price discount of \$$\alpha$ and will redeem their rebates with probability β, where $0 < \alpha \leq 1$ and $0 < \beta \leq 1$. Thus, if a consumer rebate of x is offered on a product with price p, then the *effective retail price* is $p - \alpha x$ and if y customers buy the product, then the expected number of claims will be βy. Note that consumers are homogeneous in regard to the parameter α and we do not explicitly model a customer's decision of whether to claim a rebate or not. We shall see that modeling consumer rebates at this aggregate level allows us to identify the roles of the *claim rate* β and the *effective fraction* α in splitting the supply chain profit between the retailer and the manufacturer.

While the values of both α and β are likely to depend on many factors, we expect that they will be similar for products in a given category. For example, according to a survey of AC Nielsen's Homescan Consumer Panel, 27.7 % of households that

[2] Source: http://www.consumerreports.org/cro/magazine-archive/september-2009/personal-finance/rebates/overview/rebates-ov.htm. La28/28/2013.

[3] Source: http://www.parago.com/2011/02/11/parago-announces-surging-rebate-activity-in-2010/. La28/28/2013.

[4] Source: http://www.slideshare.net/TheresaWabler/letsmakeadeal-21181087. La28/28/2013.

reported buying computer products said mail-in rebates were very important when they bought PCs, monitors, printers and peripherals; 35.7 % said they were somewhat influenced by rebates (Ricadela and Koenig 1998). The same article reports, however, that consumers are less influenced by rebates when purchasing software. This example suggests that the value of α depends to a large degree on the product category. The claim rate, on the other hand, is likely to depend on the size of the rebate itself. For example, an educational software vendor reports that 8–10 % of its customers claim $10 rebates, and the claim rate increases to 20 % for $20 rebates (Bulkeley 1998). Likewise, according to an estimate reported in the *US News & World Report*, "for pricey items with rebates worth $50, the redemption rate is below 50 %. On smaller items with rebates under $10, redemption rates are likely to be in the single digits" (Palmer 2008). Nevertheless, the rebate sizes tend to be similar within a product category and, hence, the product category seems to be a more important determinant of the claim rate than the size of the rebate. For example, in contrast to the software vendor who faced claim rates in the 10–20 % range, the now-defunct PC seller eMachines had a mail-in rebate program, which had seen a 70–90 % claim rate prior to its cancellation (Olenick 2002). In the case of new automotive purchases, where the rebates are even larger, the usual practice is for the rebate to be instantaneously redeemable at the time of purchase, which suggests that $\alpha = \beta = 1$. In summary, while consumer response to rebate offers may vary in the size of the rebate, much of this variation may be accounted for by the product category.

In order to compare and contrast the effects of the two rebate types on the supply chain, we consider a single-retailer, single-manufacturer supply chain selling a single product, and we analyze the equilibrium outcome under each rebate policy. (The decision of what rebate type to use is not endogenous to our model; instead, we analyze and compare the equilibria under each rebate type.) We assume that the wholesale price for the product is exogenously fixed. This assumption is mainly for tractability, but it is also an approximation of an environment where rebate offers constitute a further stage of decision making in a supply chain with a well-established wholesale price. The consumer demand for the product is stochastic and depends on the effective retail price. In the case of a retailer rebate, the effective retail price is simply the retail price, whereas in the case of a consumer rebate, the effective retail price is the retail price minus the effective fraction of the consumer rebate. We assume that the expected demand for the product is a function of the effective retail price, and the realized demand is a multiplicative random perturbation of that expected demand. The assumption of a multiplicative model is not without consequence; it implies that the coefficient of variation of demand is constant with respect to price.

Under either rebate policy, before the start of the single-period selling season, the retailer must determine the retail price, and the manufacturer needs to choose the size of the rebate (or rebates, if both rebate types are used in the supply chain) simultaneously. This simultaneous determination of the rebates and the retail price can be seen as approximating a negotiation process between the manufacturer and the retailer in setting the terms of a rebate offer. Once the price and rebate(s) are announced, the retailer decides how many units of the product to purchase. The manufacturer builds that amount and delivers it to the retailer by the beginning of the selling season. At the end of the selling season, all unmet demands become lost sales, and leftover inventory is salvaged. This model would be particularly applicable to high-tech products where the short life cycle of the product can be modeled as covering a single season with a single ordering and pricing opportunity. The more replenishments take place during the life cycle of the product and the more price adjustments made, the more approximate our model becomes.

Of course, both retailer and consumer rebates provide the retailer with an incentive to stock more. However, the two rebates differ in how they achieve this result: Retailer rebates do so by increasing the retailer's margin on every unit sold, whereas consumer rebates do so by boosting the demand for the product. We find that, as expected (in equilibrium), when retailer rebates are present, the retailer will reduce the retail price (by an amount less than the rebate itself) to increase the sales volume of an item and collect a larger sum from the manufacturer in rebates, thereby passing on to the consumer some of the benefits it receives. On the other hand, a consumer rebate will induce the retailer to increase the retail price (by an amount less than the effective rebate) to take advantage of the boost in demand that arises from a consumer rebate, thereby sharing in some of the benefits offered to consumers. We show that the total supply chain profit always improves under retailer rebates, compared to no rebates. The same is true for consumer rebates, provided that the effective fraction (α) is larger than the claim rate (β). However, if $\alpha < \beta$, then total supply chain profit may suffer. We provide numerical examples to demonstrate that neither the retailer nor the manufacturer always prefers one particular kind of rebate to the other. In addition, our numerical examples suggest that, contrary to popular belief, it is possible for both firms to prefer consumer rebates even when all such rebates are redeemed.

In comparing the two rebate types, we find that the split of supply chain profits under consumer rebates depends critically on α and β. In particular, we obtain the following results:

- Under the consumer rebate equilibrium, the retailer's share of the supply chain profit will be $\frac{\alpha}{\alpha+\beta}$, and the manufacturer's $\frac{\beta}{\alpha+\beta}$. In other words, the profit will be divided so that the ratio of the retailer profit to the manufacturer profit will be α/β.
- The higher α is with respect to β (i.e., the higher consumers value the rebate relative to the rate at which consumers redeem them), the more attractive the consumer rebate becomes from the overall supply chain's perspective. Therefore, one can conclude that, everything else being equal, the more attractive the consumer rebate from the overall supply chain's perspective, the larger the retailer's share of the supply chain profit will be in equilibrium.
- Note that the retailer's share is increasing in α and decreasing in β, and the opposite is true for the manufacturer. Nevertheless, as we demonstrate through a numerical example, this does not mean that the retailer and the manufacturer are

at odds in terms of what α and β they prefer. It turns out that, under a consumer rebate equilibrium, both firms can prefer α to be larger and β to be smaller; even though the manufacturer's share of supply chain profits is smaller, the manufacturer gets more, because the increase in the supply chain profits more than compensates for the decrease in the share it gets.

In the next section, we review the related literature and compare our model to those in earlier research. Section 3 describes our model and discusses our results for the case where both rebate types are used simultaneously. In Sects. 4 and 5, we discuss our results when retailer rebates and consumer rebates are used in isolation. We provide a number of numerical examples in Sect. 6 to demonstrate some interesting equilibrium outcomes. We conclude in Sect. 7. All proofs are provided in the appendix.

2 Literature Review

The marketing and economics literature has investigated the use of consumer rebates. For example, Gerstner and Hess (1991, 1995) use a demand model where the consumer population consists of two segments; the size and reservation price of each segment is deterministic and known. The higher-end segment has a cost associated with redeeming a consumer rebate, reflecting the higher disutility price-insensitive customers have for claiming rebates. The supply chain is assumed to be serving only the higher-end segment in status quo. They examine how retailer rebates (called push price promotions) and consumer rebates (called pull price promotions) can be used to induce the retailer to serve the lower-end segment as well as the higher-end one, and how such promotions affect manufacturer and supply chain profits. Narasimhan (1984) offers a price discrimination argument to explain the use of consumer rebates. He considers a model where the firm offering the rebate is selling directly to the end consumer. In his model, a consumer need not redeem a rebate every time she purchases a product. He models the consumer's decision of how many rebates to use as a utility maximization problem, and shows that the more price-sensitive a customer, the more she engages in consumer rebates. Therefore, rebates result in the firm selling at a lower price to consumers who are more price sensitive. In this sense, the consumer rebate acts as a price discrimination device. Our model is less general than this stream of research because we do not model how individual consumers respond differently to rebate offers. Instead, we model the effect of rebates at the aggregate demand level, through the effective fraction parameter α and the claim rate parameter β. Our model is more general in the sense that we incorporate demand uncertainty and retail stock level decisions.

There is also a stream of research in marketing that considers the use of trade promotions; i.e., a discount in wholesale price offered by the manufacturer in order to induce the retailer to lower the retail price. Since the typical assumption of this research stream is that all demand is met (i.e., sales equals demand), such a discount in wholesale price is equivalent to a retailer rebate. Most of the model-based work in this research stream involves multiple competing manufacturers, and the emergence of trade promotions is explained through the equilibrium of the game among these multiple manufacturers. In this setting, the manufacturer is assumed to be selling directly to the end consumers, and the role of the retailer is ignored. See, for example, Raju et al. (1990), Lal (1990) and Rao (1991). Our model has only a single manufacturer, but we add explicit consideration of a retailer, demand uncertainty, and the retailer's decision of the stock level.

There is another marketing research stream on trade promotions that considers manufacturers selling through a retailer. For example, Lal et al. (1996) consider an infinite horizon model where two identical manufacturers sell through a single retailer. Their customer population consists of three customers: one switcher and two loyals. In this model, trade promotions exist because the manufacturers compete for the switcher. Dreze and Bell (2003) consider a single-retailer, single-manufacturer setting where customer demand is a deterministic function of price. They compare the effects of two different contractual arrangements for trade promotions: off-invoice deals that correspond to a wholesale price discount and scan-back deals that correspond to retailer rebates. In this model, even though demand is deterministic, the retailer may choose to carry inventories to take advantage of a temporary promotional offer from the manufacturer. In our model, the reason a retailer chooses to carry inventories is due to demand uncertainty. We also emphasize how the rebates affect supply chain profits and the shares that the two firms get.

There is earlier work in the operations management literature that considers the role played by retailer rebates in the presence of operational concerns like inventory costs. Taylor (2002) considers retailer rebates in a model where demand is stochastic, but the retail price is exogenously given. He shows that retailer rebates paid for units sold beyond a target level can be used to achieve supply chain coordination. He also analyzes a model where the retailer can exert sales effort to influence demand. In this case, retailer rebates can still achieve coordination, but a returns policy should also be implemented. Using a more general model, Krishnan et al. (2004) focus on the use of retailer rebates in the presence of retailer efforts. Their main focus is finding coordinating contracts. Unlike these two, we do not model the retailer's sales effort; however, we consider a model with price-dependent stochastic demand, and retail price is endogenous to our model in that the retailer decides what price to charge. We do not seek to establish channel-coordinating mechanisms, but we do show that retailer rebates improve supply chain profits. We also compare the supply chain profit under retailer rebates with that under consumer rebates.

There is an extensive operations management literature on the price setting newsvendor problem, in which a retailer faces a single-period inventory and pricing problem with stochastic, price-dependent demand. See, for example, Petruzzi and Dada (1999) for a review with extensions. Our analysis benefits from Petruzzi and Dada (1999); in particular, Lemma 4(a) in the appendix is due to them. In their multiplicative model, they assume that demand is given by $ap^{-b}\epsilon$, where ϵ is a random variable. In this demand model, the price elasticity of expected demand is constant. Our assumptions do not cover this specific model, but we do allow the (absolute) price elasticity of expected demand to be increasing in price, thereby complementing some of the existing structural results on the price setting newsvendor problem. Kalyanam (1996) finds empirical support for both constant and increasing price elasticity of demand. In this chapter, we use an inverse demand representation to write the retailer's and manufacturer's expected profit functions, which facilitates our analysis. (See the next section.) Aydin and Porteus (2008) study an inventory and pricing problem where a retailer sets the prices and inventory levels for an assortment of substitutable products, and they take advantage of a similar representation.

A closely related paper is by Chen et al. (2007), who consider the question of consumer rebates from an operations management perspective. As in our model, they consider a single-retailer, single-manufacturer supply chain where one-shot inventory and pricing decisions are made to satisfy price-dependent uncertain customer demand. Their consumer rebate is an exogenously fixed fraction of the wholesale price and the decision making is sequential: the manufacturer chooses the wholesale price first, and the retailer chooses the retail price second. Our wholesale price is exogenous but our consumer rebate is a decision variable. We add consideration of retailer rebates and our assumptions allow us to show how the claim rate and the effective fraction parameters affect the split of supply chain profits between the retailer and the manufacturer.

Since the initial publication of this chapter, several further contributions have been made to the literature on the role of consumer rebates in supply chain management. In a set of recent papers, Demirag and colleagues also compare two avenues available to manufacturers: offering rebates to consumers or offering incentives to retailers. Demirag et al. (2010) compare manufacturer-to-consumer rebates with manufacturer-to-retailer incentives, which take the form of a lump sum payment (in contrast to the manufacturer-to-retailer rebate in our model, which is a per-unit payment). The retailer uses this incentive to offer discounts to select customers, thus effectively achieving price discrimination among customers. By studying several scenarios (including both stochastic and deterministic demand models), the paper investigates which of the two schemes the manufacturer prefers. Demirag et al. (2011b) extend this work to the case with two manufacturers and two competing retailers. In a different vein, Demirag et al. (2011a) start with a model similar to ours, but they assume that the retailer is risk averse. They show that the rebate scheme preferred by the manufacturer does depend on the degree of retailer's risk aversion.

Another set of recent papers focuses on rebates paid to consumers only (whereas we study rebates paid to the retailer as well), but they allow consumer rebates to come from either the manufacturer or the retailer (whereas we allow consumer rebates to come from the manufacturer only). These papers model interactions in a two-stage supply chain using the Stackelberg equilibrium, where the manufacturer moves first, followed by the retailer. Cho et al. (2009) use a deterministic demand model, and they pay special attention to how the equilibrium depends on the fixed cost of adopting a rebate initiative. Arcelus et al. (2012) and Geng and Mallik (2011) adopt a newsvendor setting to compare retailer-driven versus manufacturer-driven rebates. Both allow the redemption rate to be a function of the rebate size—this is a dependence we do not model. Arcelus et al. (2012) treat the wholesale price as endogenous, and they find conditions under which it is best for only the retailer to offer the rebate. Geng and Mallik (2011) treat the wholesale price as exogenous, and they show that the average effective price paid by consumers is higher in the presence of rebates.

Focusing on manufacturer-to-consumer rebates only, a few recent papers study how rebates play out in the presence of supply chain initiatives that restrict the retail price. For instance, Yang et al. (2010) study how manufacturer-suggested retail prices (MSRP) interact with rebates. In a similar vein, Khouja and Zhou (2010) study a supply chain where the manufacturer implements incentives that curb the retail price. They use a model where consumers are heterogeneous in the value they derive from a rebate. Their main result is that rebates are good for the supply chain as a whole, owing to the limits on retail price.

3 Consumer and Retailer Rebates Together

In this section, we describe our model when the manufacturer uses both retailer and consumer rebates, and we derive some preliminary results. The use of both rebates at the same time is quite common in the automotive industry, where retailer rebates are usually called dealer incentives and the consumer rebates are offered in the form of cashback allowances. In the following sections, we will focus on the cases where each rebate type is used in isolation, and the results developed in this section will apply to those special cases. Let r_{R} denote the retailer rebate and r_{C} the consumer rebate, each paid to their respective recipients for every unit the customer buys. Also, let p be the retail price of the product.

Let us first describe the demand model. First, the higher the consumer rebate the larger the stochastic demand will be. Therefore, the demand should be a function of r_{C} as well as p. Let $D(p, r_{\mathrm{C}})$ denote the stochastic demand for the product. We assume that consumers treat a \$1 rebate as the equivalent of an \$$\alpha$ price discount; i.e., consumers act as if the unit retail price they are paying is $p - \alpha r_{\mathrm{C}}$. We will impose the following assumptions on the demand model:

(A1) $D(p, r_{\mathrm{C}}) = f(p - \alpha r_{\mathrm{C}})\epsilon$,
(A2) ϵ is a strictly positive random variable with a strictly increasing failure rate (IFR),
(A3) $f(\cdot)$ is strictly decreasing, and $f(x) \to 0$ as $x \to \infty$, and
(A4) $\frac{f'(\cdot)}{f(\cdot)}$ is non-increasing.

The first assumption implies that the expected demand is a function of the retail price minus the effective consumer rebate (i.e., the price after rebate). $(A1)$ and $(A2)$ implicitly assume that ϵ is independent of price and any rebate. Thus, $(A1)$ implies that the coefficient of variation of demand for the product does not change with price. The requirement in ($A2$) that ϵ be IFR is not very restrictive as many probability density functions, including the normal and Weibull with shape parameter greater than one, satisfy this assumption. (For more on IFR distributions, see Barlow and Proschan 1965.) ($A3$) is a natural assumption that means the expected demand is decreasing in price. This assumption is violated only for very few luxury items. ($A4$) implies that the magnitude of the expected demand's elasticity to price is increasing in p; i.e., as price gets larger the percentage change in demand in response to a percentage change in price gets larger. ($A4$) is satisfied by many commonly used forms of price dependency. For example, it is easy to check that ($A4$) will be satisfied when expected demand is exponentially decreasing in price; i.e., $f(x) = e^{-ap}$, or when expected demand is linearly decreasing in price; i.e., $f(x) = a - bx$, or when expected demand is given by the logit demand model; i.e., $f(x) = \dfrac{exp(u_1 - x)}{exp(u_0) + exp(u_1 - x)}$.

We define the following notation:

w : unit wholesale price charged by the manufacturer

c : unit production cost

v : unit salvage value

$\Phi(x, p - \alpha r_C)$: cumulative distribution function (cdf) of $D(p, r_C)$

$\phi(x, p - \alpha r_C)$: probability density function (pdf) of $D(p, r_C)$

$\Phi_\epsilon(\cdot)$: cdf of ϵ

$\phi_\epsilon(\cdot)$: pdf of ϵ

We assume that $\Phi(x, p - \alpha r_C)$ is twice-continuously differentiable in both its arguments. Throughout the remainder of the paper, given a function g of vector x, we use $\nabla_i g(\tilde{x})$ to denote the partial derivative of $g(x)$ with respect to the ith component of x evaluated at $x = \tilde{x}$. Similarly, $\nabla^2_{ij} g(\tilde{x})$ and $\nabla^2_{ii} g(\tilde{x})$ denote the cross-partial and second partial of $g(x)$ at $\tilde{x}$, respectively.

Before the selling season starts, the retailer determines p, and, simultaneously, the manufacturer chooses r_R and r_C. The assumption of simultaneous decision making implies that one party in the supply chain is not particularly more powerful than the other, so one party cannot impose its respective decision on the other. We assume that all the parameters and distributions are known by both the retailer and the manufacturer.

Once the price and rebates are announced, the retailer chooses the stock level and the manufacturer then builds that amount, which is delivered to the retailer by the beginning of the selling season. After the selling season is over, the retailer will salvage the leftover inventory at unit salvage value of v. We assume that $w > c > v$ for the problem to make economic sense. In the presence of retailer rebates, there is the possibility that the retailer could misreport the amount of sales to collect larger rebates from the manufacturer. For example, the retailer could dump all the leftover inventory and claim that it had been sold. While the existence of a salvage value alleviates this moral hazard problem, a complete avoidance of such misreporting of sales requires some form of possibly costly monitoring of retail sales. We return to this issue in Sect. 7.

The retailer's profit function is given by

$$\Pi_R(p,y,r_{\mathrm{C}},r_{\mathrm{R}}) = (p+r_{\mathrm{R}})\left[\int_0^y x\phi(x,p-\alpha r_{\mathrm{C}})dx + y(1-\Phi(y,p-\alpha r_{\mathrm{C}}))\right] + v\int_0^y (y-x)\phi(x,p-\alpha r_{\mathrm{C}})dx - wy. \tag{13.1}$$

Note that the optimal stock level for the product, $y^*(p,r_{\mathrm{C}},r_{\mathrm{R}})$, is given for each given retail price p, wholesale price w and rebates r_{R} and r_{C} as the critical fractile solution:

$$\Phi(y^*(p,r_{\mathrm{C}},r_{\mathrm{R}}),p-\alpha r_{\mathrm{C}}) = \frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v} \tag{13.2}$$

It is important to note how the two different kinds of rebates affect $y^*(p,r_{\mathrm{C}},r_{\mathrm{R}})$: the stock level chosen depends on the retailer rebate since the critical fractile itself is a function of the retailer rebate, whereas the consumer rebate affects the stock level through its impact on the demand distribution.

The retailer's profit function can be rewritten as the following induced profit function, obtained by substituting for $\Phi(y^*(p,r_{\mathrm{C}},r_{\mathrm{R}}),p-\alpha r_{\mathrm{C}})$ in (13.1) (see, for example, Porteus 2002):

$$\Pi_R(p,r_{\mathrm{C}},r_{\mathrm{R}}) = (p+r_{\mathrm{R}}-v)\int_0^{y^*(p,r_{\mathrm{C}},r_{\mathrm{R}})} x\phi(x,p-\alpha r_{\mathrm{C}})dx, \tag{13.3}$$

where $y^*(p,r_{\mathrm{C}},r_{\mathrm{R}})$ is as defined by (13.2). Define the inverse demand function $z(p,r_{\mathrm{C}},\xi)$ as

$$\Phi(z(p,r_{\mathrm{C}},\xi),p-\alpha r_{\mathrm{C}}) = \xi. \tag{13.4}$$

With this definition, $z(p,r_{\mathrm{C}},\xi)$ is the demand that corresponds to the ξ fractile of Φ, given the retail price p and consumer rebate r_{C}. We use this representation as it provides a more convenient way of dealing with the pricing problems to be solved. Using the inverse demand function, we can rewrite (13.2) as $y^*(p,r_{\mathrm{C}},r_{\mathrm{R}}) = z(p,r_{\mathrm{C}},\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v})$. Also, we can rewrite the retailer's induced profit function in (13.3) as

$$\Pi_R(p, r_C, r_R) = (p + r_R - v) \int_0^{\frac{p+r_R-w}{p+r_R-v}} z(p, r_C, \xi) d\xi. \tag{13.5}$$

The following proposition states our structural result on $\Pi_R(p, r_C, r_R)$.

Proposition 1 *Suppose (A1) through (A4) hold. Then, given r_C and r_R, there is a unique $p > w - r_R$ that optimizes the retailer's profit, and this unique p satisfies the first order condition (FOC) for $\Pi_R(p, r_C, r_R)$.*

The manufacturer's profit function is given by

$$\begin{aligned} \Pi_M(p, r_C, r_R) &= (w - c) y^*(p, r_C, r_R) \\ &\quad - (\beta r_C + r_R) \left[\int_0^{\frac{p+r_R-w}{p+r_R-v}} z(p, r_C, \xi) d\xi + \frac{w - v}{p + r_R - v} y^*(p, r_C, r_R) \right]. \end{aligned} \tag{13.6}$$

The first term in (13.6) is the profit margin of the manufacturer multiplied by the number of units ordered by the retailer. The term in brackets is the expected sales. Note that the rebate the manufacturer pays per unit sold is the retailer rebate r_R, plus a fraction β of the consumer rebate r_C (since a fraction β of consumers claim their rebate). Therefore, the expected total rebate payment made by the manufacturer is $\beta\, r_C + r_R$ multiplied by the expected sales. The following proposition states some structural results on $\Pi_M(p, r_C, r_R)$:

Proposition 2 *Suppose (A1) through (A4) hold. Then, given p:*

(a) *Suppose r_R is fixed so that $p + r_R > v$. Then, either the manufacturer's profit is optimized at $r_C = 0$, or there exists a unique r_C that satisfies the FOC for $\Pi_M(p, r_C, r_R)$ and such r_C optimizes the manufacturer's profit.*
(b) *Suppose r_C and p are fixed. Then, either the manufacturer's profit is optimized at $r_R = 0$, or there exists a unique r_R that satisfies the FOC for $\Pi_M(p, r_C, r_R)$ and such r_R optimizes the manufacturer's profit.*
(c) *At any r_C and r_R such that $\nabla_2 \Pi_M(p, r_C, r_R) = \nabla_3 \Pi_M(p, r_C, r_R) = 0$, we have $\nabla_2 y^*(p, r_C, r_R)/\beta > \nabla_3 y^*(p, r_C, r_R)$.*

Parts (a) and (b) of the proposition establish that the manufacturer's profit is well-behaved in the rebates. We cannot rule out the possibility that the manufacturer's profit will be decreasing in the retailer or the consumer rebate. Therefore, the manufacturer's optimal solution may involve a zero rebate. To understand part (c), note that increasing the retailer rebate by \$1 costs the manufacturer \$1 for every unit sold to consumers, while increasing the consumer rebate by \$1 costs only \$$\beta$ for every unit sold to consumers. Thus, part (c) says that, given a pair of rebates that is a candidate for the manufacturer's optimal solution, the marginal increase in units sold to the retailer, per manufacturer's effective (at-risk) cost of a rebate-dollar, is higher for consumer rebates than retailer rebates.

Let $\Pi_{SC}(p, r_{\mathrm{C}}, r_{\mathrm{R}})$ be the profit of the supply chain for a given retail price p, consumer rebate r_{C} and retailer rebate r_{R}. Note that $\Pi_{SC}(p, r_{\mathrm{C}}, r_{\mathrm{R}}) = \Pi_R(p, r_{\mathrm{C}}, r_{\mathrm{R}}) + \Pi_M(p, r_{\mathrm{C}}, r_{\mathrm{R}})$ where $\Pi_R(p, r_{\mathrm{C}}, r_{\mathrm{R}})$ and $\Pi_M(p, r_{\mathrm{C}}, r_{\mathrm{R}})$ are as defined by (13.5) and (13.6), respectively. The following proposition states how the supply chain profit will be split between the two parties under an equilibrium solution.

Proposition 3 *Suppose (A1) through (A4) hold. Furthermore, suppose that a pure-strategy Nash equilibrium exists for the game between the retailer and the manufacturer. Let* $\tilde{p}$ *be an equilibrium retail price, and* $\tilde{r}_{\mathrm{R}}$ *and* $\tilde{r}_{\mathrm{C}}$ *the corresponding equilibrium rebates. The stock level that arises under this equilibrium is given by* $y^*(\tilde{p}, \tilde{r}_{\mathrm{C}}, \tilde{r}_{\mathrm{R}})$ *where* y^* *is given in* (13.2) *. Under this equilibrium, if* $\tilde{r}_{\mathrm{C}} > 0$*, then* $\frac{\Pi_R(\tilde{p}, \tilde{r}_{\mathrm{C}}, \tilde{r}_{\mathrm{R}})}{\Pi_M(\tilde{p}, \tilde{r}_{\mathrm{C}}, \tilde{r}_{\mathrm{R}})} = \frac{\alpha}{\beta}$*.*

As we will see later on, this particular division of the supply chain profit under an equilibrium solution is due to the use of the consumer rebate, and, as stated in the proposition, will be true whenever the equilibrium consumer rebate is (strictly) positive. The key assumption that leads to this interesting result is that the demand uncertainty is multiplicative. We will discuss the rationale behind this result in detail when we discuss the use of consumer rebates in isolation. Also, this constant-split property allows us to conclude that, even when multiple Nash equilibria (with strictly positive consumer rebates) exist, there is one equilibrium that is preferred by both parties to all other equilibria, and the equilibrium preferred by both parties is the one under which the supply chain profit is at its highest among all other equilibria. If one could argue that our model captured the first order issues addressed in the automotive industry, where it is plausible to assume that both α and β are equal to one (due to the large sums involved in cashback allowances), one could say that the rebates would lead to dividing the channel profits evenly between the manufacturers and the dealers.

In the next two sections, we will consider the cases that arise when either only retailer rebates or only consumer rebates are used.

4 Retailer Rebate Only

The *retailer rebate game* is the game between the retailer and the manufacturer in the previous section with the restriction that $r_{\mathrm{C}} = 0$. We will continue to use the same notation as before, replacing r_{C} with zero where necessary. The structural results on the profit functions of the manufacturer and the retailer (adapted for $r_{\mathrm{C}} = 0$) will carry over directly from the previous section. In addition, the following proposition states how the optimal decision of one player changes with the decision of the other one.

Proposition 4 *Suppose (A1) through (A4) hold and* $r_{\mathrm{C}} = 0$. *Let* $p^*(r_{\mathrm{R}})$ *be the optimal price chosen by the retailer as a response to a given* r_{R} *and* $r_{\mathrm{R}}^*(p)$ *the optimal retailer rebate chosen by the manufacturer as a response to a given* p. *Then:*

(a) $-1 \leq \dfrac{dp^*(r_{\mathrm{R}})}{dr_{\mathrm{R}}} < 0.$

(b) $-1 < \dfrac{dr_{\mathrm{R}}^*(p)}{dp} \leq 0.$

(c) *There exists a unique Nash equilibrium for the retailer rebate game.*

The first part of the proposition above implies that when the manufacturer offers an additional \$1 rebate to the retailer for every unit sold, the retailer will decrease the selling price of the product, but the price discount will be less than \$1. Therefore, the retailer rebate results in some savings being passed on to the customer. Likewise, when the retailer reduces the price of the product by \$1, the manufacturer will increase the rebate paid to the retailer, but by less than \$1. The following proposition summarizes our results in this setting.

Proposition 5 *Suppose (A1) through (A4) hold and* $r_{\mathrm{C}} = 0$. *Let* p_{o} *be the retail price and* y_{o} *the stock level chosen by the retailer when* $r_{\mathrm{R}} = 0$. *Let* $\tilde{p}$ *be the equilibrium retail price, and* $\tilde{r}_{\mathrm{R}}$ *the equilibrium rebate that will arise under the retailer rebate game. The stock level that arises under this equilibrium is given by* $y^*(\tilde{p}, 0, \tilde{r}_{\mathrm{R}})$ *where* y^* *is as defined by* (13.2) *. Then:*

(a) $p_o - \tilde{r}_{\mathrm{R}} \leq \tilde{p} \leq p_o$,

(b) $y_o \leq y^*(\tilde{p}, 0, \tilde{r}_{\mathrm{R}})$ *and*

(c) *If* $0 < \tilde{r}_{\mathrm{R}} \leq w - c$, *then* $\Pi_{SC}(\tilde{p}, 0, \tilde{r}_{\mathrm{R}}) > \Pi_{SC}(p_o, 0, 0)$.

The first two parts of the proposition state that, as expected, the retail price will decrease and the stock level will increase when retailer rebates are used. We should note that, in parts (a) and (b) of the proposition, the inequalities are not strict, since the equilibrium may turn out to be the no-rebate case; i.e., $\tilde{r}_{\mathrm{R}}$ may be zero. It is interesting to note here how the role played by retailer rebates under endogenous retail pricing differs from that under an exogenously-fixed retail price. When the retail price is exogenous, the rebate helps the manufacturer by increasing the retailer's margin on every unit sold, thereby increasing the quantity ordered by the retailer. On the other hand, when the retail price is endogenous, the rebate serves a dual purpose for the manufacturer: As before, the rebate increases the order quantity of the retailer by increasing the retailer's margin on every unit sold, but, in addition, the rebate causes a decrease in the retail price (as stated in part (a) of the proposition), thereby increasing the customer demand, which causes a further increase in retailer's order quantity.

The last part of the proposition states that if the equilibrium rebate is (strictly) positive and below the manufacturer's unit profit margin (which would be expected to be the case in practice), then the supply chain will be strictly better off as a result of the use of the retailer rebate. This result is not surprising. Intuitively speaking, the higher the retailer rebate, the closer the supply chain becomes to one that is owned by a single decision maker, since increasing the retailer rebate brings the retailer's underage cost closer to the integrated supply chain's underage cost. Therefore, the higher the retailer rebate, the closer the performance of the supply chain becomes to that of the integrated one. We should note that the constant-split property does not hold when only retailer rebates are used.

Next, we discuss the case in which only consumer rebates are used.

5 Consumer Rebate Only

The *consumer rebate game* is the game between the retailer and the manufacturer in Sect. 3 with the restriction that $r_R = 0$. The structural results on the retailer's and manufacturer's profit functions stated in Sect. 3 (adapted for $r_R = 0$) carry over. The following proposition states how the optimal price chosen by the retailer responds to a change in the consumer rebate.

Proposition 6 *Suppose (A1) through (A4) hold and* $r_R = 0$. *Let* $p^*(r_C)$ *be the optimal price chosen by the retailer as a response to a given* r_C. *Then:*

(a) $0 < \dfrac{dp^*(r_C)}{dr_C} < \alpha.$
(b) *There exists a Nash equilibrium for the consumer rebate game.*

The proposition states that when the manufacturer offers an additional \$1 rebate to the consumer, the retailer will take advantage of this offer, and will increase the retail price, but the increase will be less than α. This means that, as is commonly thought, a consumer rebate will bring about a price increase, however the effective retail price paid by the consumer will still be less than the price that would be paid if the rebate did not exist. Unfortunately, a result on how the optimal consumer rebate responds to price eludes us. In the absence of such a result, we are not able to claim that the Nash equilibrium under the consumer rebate game will be unique. The following proposition summarizes our results for this game.

Proposition 7 *Suppose (A1) through (A4) hold and* $r_R = 0$. *Let* p_o *denote the price and* y_o *the stock level chosen by the retailer when* $r_C = 0$. *Let* $\tilde{p}$ *be an equilibrium retail price under the consumer rebate game, and* $\tilde{r}_C$ *the corresponding equilibrium rebate. Suppose that* $\tilde{r}_C > 0$. *The stock level that arises under this equilibrium is given by* $y^*(\tilde{p}, \tilde{r}_C, 0)$ *where* y^* *is as defined by* (13.2) . *Then:*

(a) $p_o \leq \tilde{p} \leq p_o + \alpha\tilde{r}_C$,
(b) $y_o \leq y^*(\tilde{p}, \tilde{r}_C, 0)$,
(c) *If* $\alpha \geq \beta$ *and* $\tilde{r}_C \leq w - c$, *then* $\Pi_{SC}(\tilde{p}, \tilde{r}_C, 0) \geq \Pi_{SC}(p_o, 0, 0)$ *and*
(d) $\dfrac{\Pi_R(\tilde{p}, \tilde{r}_C, 0)}{\Pi_M(\tilde{p}, \tilde{r}_C, 0)} = \dfrac{\alpha}{\beta}$

The first two parts of the proposition state the intuitive results that the retail price and the stock level will increase when consumer rebates are used. However, the increase in retail price will not be larger than the effective fraction of the consumer rebate, so consumers are still better off as a result of the rebate. The third part of the proposition states that, if α is larger than β, the supply chain profit will improve as a result of the consumer rebate (provided that the rebate is less than the manufacturer's profit margin, which we would expect to be the case). This result is expected: Essentially, when the cost of a \$1 rebate, modeled by β, is less than the effective fraction of the \$1 rebate, modeled by α, the supply chain is able to achieve the demand impact of an α-dollar price discount at a cost of $\beta < \alpha$ dollars. Also, we see from the last part of the proposition that the constant-split property of supply chain profit continues to hold when consumer rebates are used in isolation. Due to this constant-split property, we conclude that, even when multiple Nash equilibria exist, the equilibrium under which the supply chain profit is at its highest (among all other equilibria) is the one preferred by both parties. Furthermore, the constant-split property shows that, in an equilibrium solution, neither party is able to extract the entire supply chain profits. (Unless α or β is zero, which are not likely to be the case. Here, we assume that both α and β are strictly positive, and we do not cover the cases that arise when one or the other is zero.)

An interesting consequence of the constant-split property is that if the retailer's share of the supply chain profit under consumer rebates is larger than the manufacturer's, then it must be that $\alpha > \beta$ for the product in question, and, hence, by Proposition 7(c), the use of consumer rebates must have improved total supply chain profits.

From part (d) of Proposition 7, we observe that the manufacturer's share of the supply chain profit under consumer rebate equilibrium is $\frac{\beta}{\alpha+\beta}$. However, this observation does not imply that the manufacturer would necessarily like to design rebates so that β is high or α is low. In fact, in many numerical examples, we observed the opposite to be true. One such example is depicted in Fig. 13.1. In this example, with β fixed at 0.9, the manufacturer prefers a large α to a small one, since the manufacturer prefers getting a smaller share of the large supply chain profit

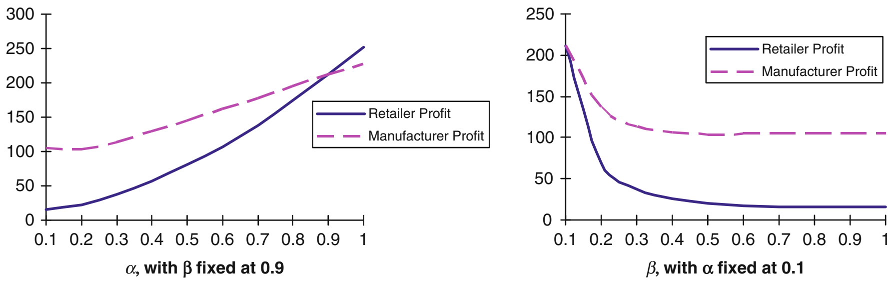

Fig. 13.1 Equilibrium retailer and manufacturer profits as a function of α (*left*) and β (*right*)

achieved under a large α value. Likewise, with α fixed at 0.1, the manufacturer prefers a small β to a large one. Note that the manufacturer's profit is not necessarily monotonic in α or β, which can be confirmed with careful scrutiny of the graphs. Another conclusion that applies to this example is that there is no conflict between the retailer and the manufacturer in terms of the attributes of a rebate: To the extent possible, both parties would like a rebate with a high customer valuation α and a small redemption rate β. We observed this to be the case in many other numerical examples. We return to this point in Sect. 7.

It is worthwhile to discuss the rationale behind the constant-split property. We will do so through a marginal analysis discussion. For the sake of the following discussion, define $\gamma(p) := -\frac{f'(p)}{f(p)}$; i.e., $\gamma(p)$ is a positive number representing the fractional decrease in expected demand in response to a marginal increase in price p. Under the consumer rebate equilibrium, the retail price must satisfy the FOC for the retailer. Hence, by part (a) of Lemma 6, the retail price p must satisfy

$$\int_0^{\frac{p+r_R-w}{p+r_R-v}} z(p,r_C,\xi)d\xi + \frac{w-v}{p+r_R-v} y^*(p,r_C,r_R) = \gamma(p+r_R-v) \int_0^{\frac{p+r_R-w}{p+r_R-v}} z(p,r_C,\xi)d\xi \tag{13.7}$$

The left-hand side of (13.7) is the expected sales of the product; a \$1 price increase means the retailer will make \$1 more on every unit sold, so the retailer's profit will increase by an amount equal to the expected sales. The right-hand side of (13.7) is γ times the (expected) profit of the retailer; a \$1 price increase will lead to a demand reduction, which will cause the retailer to lose some profit, and this loss turns out to be equal to γ times the profit of the retailer. (This is a consequence of the multiplicative demand model.) Therefore, as the FOC given by (13.7) implies, the optimal price chosen by the retailer must set the expected sales volume equal to γ times the retailer's profit.

Likewise, under the consumer rebate equilibrium, the consumer rebate must satisfy the FOC for the manufacturer. Hence, by part (b) of Lemma 6, the consumer rebate r_C must satisfy

$$\begin{aligned}&\beta\left(\frac{w-v}{p+r_R-v}y^*(p,r_C,r_R) + \int_0^{\frac{p+r_R-w}{p+r_R-v}} z(p,r_C,\xi)d\xi\right)\\&=\gamma\alpha\left[(w-c)y^*(p,r_C,r_R) - (\beta r_C + r_R)\left(\frac{w-v}{p+r_R-v}y^*(p,r_C,r_R) + \int_0^{\frac{p+r_R-w}{p+r_R-v}} z(p,r_C,\xi)d\xi\right)\right].\end{aligned} \tag{13.8}$$

The left-hand side of (13.8) is β times the expected sales of the product; a \$1 rebate increase means the manufacturer will pay β dollars more per each unit sold, so the manufacturer's profit will decrease by an amount equal to β times the expected sales. The right-hand side of (13.8) is $\gamma\ \alpha$ times the profit of the manufacturer; a \$1 rebate increase will lead to a demand increase, which will cause the manufacturer to gain some profit, and this gain turns out to be equal to $\gamma\ \alpha$ times the profit of the manufacturer. (Once again, this is a consequence of the multiplicative demand model.) Therefore, the optimal price chosen by the manufacturer must set β times the expected sales volume equal to $\gamma\ \alpha$ times the manufacturer's profit.

In summary, both parties are using the (expected) sales volume as a benchmark; one is trying to set its profit equal to the sales volume multiplied by $\frac{1}{\gamma}$, and the other is trying to set its profit equal to $\frac{\beta}{\gamma\alpha}$ times the sales volume. Since both parties will be seeing the same sales volume in equilibrium, the last part of the proposition follows.

In the next section, we provide some numerical examples to compare the effects of the retailer and consumer rebates on the profits of the supply chain partners.

6 Numerical Examples

One natural question to ask is which rebate type each player in the supply chain prefers. Unfortunately, there is no clear-cut answer to this question. In particular, as one would expect, the values of α and β have a significant impact on the equilibrium that arises under consumer rebates, and, therefore, whether a party prefers consumer rebates to retailer rebates depends very much on the values of α and β. Consider the equilibrium results depicted in Table 13.1. These equilibria are obtained under the assumption that $f(\cdot)$ is given by the logit demand function; i.e., $f(x) = \dfrac{exp(u_1 - x)}{exp(u_0) + exp(u_1 - x)}$, and ϵ is distributed uniformly between 50 and 250. The other parameter values were as follows: $w = 18.55$,

Table 13.1 Equilibria under different rebate scenarios

Rebate type	α	β	Retailer rebate	Consumer rebate	Price	Manufacturer profit	Retailer profit	Expected demand
No rebate	–	–	–	–	20.55	417.95	49.67	62.37
Retailer	–	–	3.44	–	19.32	689.79	204.72	106.33
Consumer	1	0.8	–	11.58	29.74	694.90	868.46	132.89
Consumer	1	1	–	8.94	27.37	621.60	621.17	128.35
Consumer	0.4	1	–	7.18	22.33	430.05	172.01	101.94
Consumer + retailer	1	0.8	0	11.58	29.74	694.90	868.46	132.89
Consumer + retailer	1	1	0	8.94	27.37	621.60	621.17	128.35
Consumer + retailer	0.4	1	0	7.18	22.33	430.05	172.01	101.94

$c = 4.08, v = 0, u_1 = 22.91, u_0 = 2.70$. (This is one of many randomly-generated numerical examples we tested.) For the three combinations of parameters considered for consumer rebates, there was only a single equilibrium to the "both rebate types" game and it specified zero retailer rebate.[5] Thus, the prices and profits are the same as those given in the table under consumer rebates only. When $\alpha = 1$ and $\beta = 0.8$, both the retailer and the manufacturer prefer consumer rebates to retailer rebates. However, if β increases to 1 while keeping α fixed at 1, the manufacturer will now suffer from the increased claim rate of rebates, and, therefore, will now prefer retailer rebates to consumer rebates, while the retailer's preference is not affected by the change in β. On the other hand, if α decreases to 0.4 while keeping β fixed at 1, consumer rebates will now have a smaller impact on consumer demand, and, hence, the retailer will now prefer retailer rebates to consumer rebates. Therefore, neither party always prefers one rebate type to another.

6.1 A Form of Prisoner's Dilemma in Choosing What Rebate(s) to Offer

Note from Table 13.1 that for $\alpha = 0.4$ and $\beta = 1$, a supply chain in which both rebate types are allowed will settle in the same equilibrium as a supply chain in which only consumer rebates are allowed. Notice that there is a form of prisoners' dilemma here: The retailer rebate game equilibrium, even though it is preferred by both parties, is not an equilibrium in this game with both types allowed. This leads to an interesting observation: When the supply chain plays the game where both types of rebates are allowed, the supply chain ends up using only consumer rebates in equilibrium, an outcome that hurts both parties when compared to what they could achieve if only retailer rebates are allowed. The policy implication of this observation is that there are environments in which both the retailer and the manufacturer will agree in advance, before prices and rebates are set, to not allow the use of consumer rebates.

6.2 Both Parties May Prefer Consumer Rebates Even When All Consumers Claim Them

There exist cases where both parties prefer to use the consumer rebates to stimulate customer demand. For example, when $w = 10, c = 4, v = 0, u_1 = 30, u_0 = 20$, and $\alpha = \beta = 1$, both parties prefer consumer rebates. (Under retailer rebates only, the equilibrium profits are 122.45 for the manufacturer and 32.92 for the retailer. Under

[5] Under consumer rebate equilibria, the manufacturer expected profit to retailer expected profit ratios are not precisely β: α, since our searches were over fine grids that were nevertheless discrete.

consumer rebates only, the equilibrium profits are 151.70 for both the manufacturer and the retailer.) Under the consumer rebate equilibrium, the retail price is 13.28 and the consumer rebate is 3.77, which yield an effective price of 9.51, less than the wholesale price of 10. Note that this is an environment where all consumers claim their rebates, i.e., β is one; nevertheless, both parties prefer consumer rebates to retailer rebates. Moreover, in this supply chain, even when both types of rebates are allowed, it turns out that retailer rebates are not offered in equilibrium. The policy implication is that, contrary to popular belief, there exist environments in which supply chains prefer consumer rebates even when all consumers claim them.

A variant of this result can be seen in Table 13.1, where the supply chain profits are higher under consumer rebates than retailer rebates when $\alpha = \beta = 1$. In this case, because the wholesale price is fixed so much higher than cost, the retail price and consumer rebate are both high, leading to an effective price lower than under retailer rebates, but with a much higher margin to the retailer on units sold with still a good margin to the manufacturer on an increased level of sales. The manufacturer gets slightly lower profits but the retailer gets dramatically more.

6.3 Retailer May Choose to Sell at a Loss to Make Money on Rebates

Rebates can play an interesting role in the supply chain when the exogenously-fixed wholesale price is high. For example, if $w = 20, c = 5, v = 0, u_1 = 40, u_0 = 20$, and $\alpha = \beta = 1$, then the retailer rebate game equilibrium has a retail price of 19.24, which is lower than the wholesale price, and the retailer rebate is 4.53. Thus, in this example, the wholesale price is so high that the retailer sells the product at a loss to stimulate customer demand, and makes money only on rebates collected from the manufacturer rather than directly from consumers.

6.4 The Effect of Wholesale Price

To further examine the effect of wholesale price on the equilibrium, consider the case where only consumer rebates are allowed. Figure 13.2 shows the effect of w on the rebate size in equilibrium as well as on the manufacturer's and retailer's profits. In this example, $w = 20, c = 5, v = 0, u_1 = 40, u_0 = 20$, and $\alpha = \beta = 1$.

Observe from the figure that there is a threshold for the wholesale price such that only if the wholesale price exceeds this threshold will the manufacturer offer a strictly positive consumer rebate. This is intuitive: As the wholesale price gets larger, the manufacturer's profit margin per unit gets larger as well, and the manufacturer becomes more willing to pay a rebate to drive the retailer's stock level up. In addition, the figure suggests that the manufacturer's profit is at its

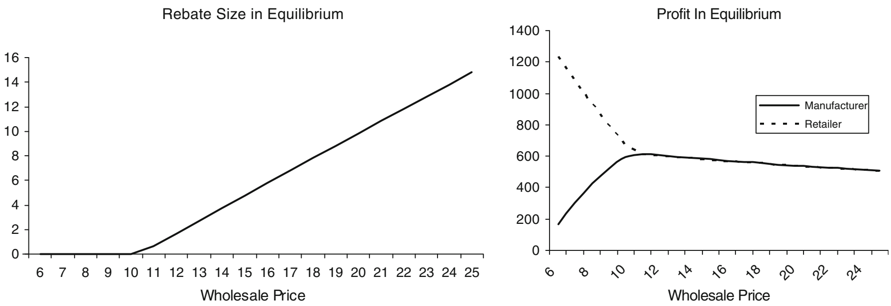

Fig. 13.2 Equilibrium rebate size (*left*) and profits (*right*) as a function of the wholesale price when only consumer rebates are allowed

highest at the threshold wholesale price. Therefore, if the manufacturer were to choose the wholesale price first, followed by a game where the consumer rebate and retail price are chosen simultaneously, then it would be optimal for the manufacturer to set the wholesale price equal to its threshold value, which would lead to a zero rebate in equilibrium. We have observed the same behavior in a number of numerical examples, but further analysis is needed to determine if this result is true in general.

7 Conclusion

We considered a supply chain where the retailer faces stochastic, effective-price-dependent demand and the manufacturer builds to order. We established some properties of the equilibrium that would arise when the manufacturer offers retailer and/or consumer rebates. We showed that supply chain profits are improved by the use of retailer rebates. On the other hand, consumer rebates may reduce the supply chain profit, but they will lead to an improvement whenever the effective fraction, α, is larger than the fraction of customers who claim their rebate, β. Furthermore, we showed that these two parameters have further significance: Under the equilibrium of the consumer rebate game, the ratio of (expected) retailer profits to (expected) manufacturer profits equals the ratio α/β. We discussed some interesting consequences of this property. We provided numerical examples to demonstrate that neither the retailer nor the manufacturer always prefers one particular kind of rebate to the other. In addition, our numerical examples suggest that, contrary to popular belief, it is possible for both firms to prefer consumer rebates even when all such rebates are redeemed.

In our model, we examined how the two rebate types differ from each other through their effects on the pricing and inventory decisions for a product. When the product's price is fixed, but the retailer is able to exert some type of hidden effort to sell the product; e.g., putting up in-store displays or advertising in local media, the effects of retailer and consumer rebates are likely to differ again and are worthy of study. Another extension worthy of study is to address the moral hazard problem of misreporting retailer sales. One approach is to add buy-backs to the model (the manufacturer buys back unsold inventory at the end of the season at a set price), which could reduce the retailer's incentive to misreport sales. It would also be interesting to add a verification cost (of sold units) to the model.

We give a partial answer to the question of why consumer rebates are offered. Our numerical examples illustrate the existence of cases where the manufacturer will prefer offering consumer rebates to offering a retailer rebate. Consumer rebates help the manufacturer by increasing the stock level at the retailer, and our results suggest that they may be useful even when all customers claim them. Therefore, perhaps it is not too surprising that some firms choose to offer instantly redeemable rebates to online shoppers even though such rebates have high redemption rates. Bulkeley (1998) cites some alternative explanations for the use of consumer rebates. For example, consumer rebates may be seen as temporary price reductions, used in order to learn more about the customer population's price elasticity. Alternatively, in high-tech products, consumer rebates can be used to offer price discounts to consumers on older-generation products, which would eliminate the need for offering price protection to the retailer. Analysis of such uses for consumer rebates is left for future research. Hopefully, some of the structural results in this paper could prove useful for researchers who would like to further analyze the question of why consumer rebates are used. Another line of extension for this research is using more elaborate models for the redemption of consumer rebates, such as having heterogeneous consumer types, with differing values of α and β. A utility-based model that describes the customer's attitude towards redeeming a rebate would contribute to our understanding of the use of consumer rebates.

It is possible that some retailers will force manufacturers to move away from mail-in consumer rebates in the future. For example, in 2005 BestBuy announced that it would no longer stock products tied to mail-in rebates and it intended to implement this policy in the span of a few years (Menzies 2005). Indeed, according to an article published in the *US News & World Report* in 2008, BestBuy phased out mail-in rebates between 2005 and 2007 (Palmer 2008). BestBuy's stated reason was that mail-in rebates were cumbersome for the consumers. To the extent that our model captures the BestBuy environment (the major violation is likely to be that the wholesale price is not exogenous), it may be that BestBuy preferred the retailer rebate regime, although in our numerical examples where that happens, the manufacturer also prefers the retailer rebate regime, so would not resist dropping consumer rebates and instituting retailer rebates. Another explanation is that BestBuy was lobbying for having the consumer rebates instantaneously redeemable at the time of consumer purchase, as is done in the automotive industry. This might have the effect of increasing both the customer valuation α and redemption rate β to 1, which would make rebates the equivalent to price discounts offered directly by the manufacturer to consumers. Depending on what the values of α and β were prior to the cancelation of the mail-in rebates, such a change might have improved the total supply chain profit as well as been appreciated by consumers. There are other explanations for BestBuy's position that are not covered by our model, such as that it helped BestBuy in its competition with other retailers. In any event, BestBuy could have been acting in its self interest, while claiming that its motivation was as a consumer advocate.

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Appendix

For the purposes of the appendix, let $h(\cdot) = \frac{\phi_\epsilon(\cdot)}{1-\Phi_\epsilon(\cdot)}$ denote the failure rate of Φ_ϵ. Throughout the appendix, we will use the following short-hand notation by dropping the functional arguments: $f = f(p - \alpha r_\mathrm{C})$, $z = z(p, r_\mathrm{C}, \xi)$, $y^* = y^*(p, r_\mathrm{C}, r_\mathrm{R})$ and $h = h\left(\frac{y^*(p, r_\mathrm{C}, r_\mathrm{R})}{f(p-\alpha r_\mathrm{C})}\right)$. In addition, define $\gamma := -\frac{f'}{f}$ and $\theta := \frac{f''}{f}$. Hence, by $(A3)$, $\gamma > 0$, and, by $(A4)$, $\gamma' = \gamma^2 - \theta \geq 0$. We first state and prove some lemmas that will be useful in the proofs of the propositions.

Lemma 1 *Suppose (A1) holds. For z(p,r_C,ξ) implicitly defined by* (13.4) *, we have:*

(a) $\nabla_1 z = -\gamma z$,
(b) $\nabla_2 z = \alpha\gamma z$,
(c) $\nabla^2_{11} z = \theta z$
(d) $\nabla^2_{22} z = \alpha^2\theta z$,
(e) $\nabla^2_{12} z = -\alpha\theta z$.

Proof of Lemma 1 By virtue of $(A1)$, we can rewrite (13.4) as $\Phi_\epsilon\left(\frac{z}{f}\right) = \xi$. Now, implicit differentiation of this identity with respect to p yields the following:

$$\nabla_1 zf - zf' = 0.$$

The first part of the lemma follows from the above equality recalling the definition of $\gamma := -\frac{f'}{f}$. The proof of the second part follows the same logic. The third part can be obtained directly by partial differentiation of the expression for $\nabla_1 z$. Likewise, the fourth and fifth parts are obtained by partial differentiation of the expression for $\nabla_2 z$. $\square$

Lemma 2 *Suppose (A1) through (A4) hold. For $y^*(p, r_C, r_R)$ implicitly defined by* (13.2)*, we have:*

(a) $\nabla_1 y^* = -\gamma y^* + \dfrac{f}{(p + r_R - v)h}$,

(b) $\nabla_2 y^* = \alpha\gamma y^* > 0$,

(c) $\nabla_3 y^* = \dfrac{f}{(p + r_R - v)h} > 0$,

(d) $\nabla_1 y^* = -\dfrac{1}{\alpha}\nabla_2 y^* + \nabla_3 y^*$,

(e) $\nabla^2_{22} y^* = \alpha^2\theta y^*$,

(f) $\nabla^2_{33} y^* = -\dfrac{f}{(p + r_R - v)^2 h} - \dfrac{f h'}{(p + r_R - v)^2 h^3} < 0$,

(g) $\nabla^2_{23} y^* = \alpha\gamma \dfrac{f}{(p + r_R - v)h} > 0$,

(h) $\nabla^2_{13} y^* = -\dfrac{1}{\alpha}\nabla^2_{23} y^* + \nabla^2_{33} y^* < 0$.

Proof of Lemma 2 *Proofs of (a) through (d)* Due to ($A1$), we can rewrite (13.2) as

$$\Phi_\epsilon\left(\frac{y^*}{f}\right) = \frac{p + r_R - w}{p + r_R - v} \tag{13.9}$$

Now, implicit differentiation of (13.9) with respect to p yields

$$\frac{\nabla_1 y^* f - f' y^*}{f^2}\phi_\epsilon\left(\frac{y^*}{f}\right) = \frac{w - v}{(p + r_R - v)^2}.$$

Recalling the definition of $h(\cdot) = \dfrac{\phi_\epsilon(\cdot)}{1 - \Phi_\epsilon(\cdot)}$ and noting that $1 - \Phi_\epsilon\left(\dfrac{y^*}{f}\right) = \dfrac{w - v}{p + r_R - v}$ [this follows from (13.9)], we can leave $\nabla_1 y^*$ alone in the above expression to obtain part (a) of the lemma. The proofs of parts (b) and (c) follow the same line of argument. Part (d) of the lemma follows directly from parts (a) through (c).

Proofs of (e) through (g) These follow from partial differentiation of the expressions obtained in parts (a) through (c). To see why $\nabla^2_{33} y^* < 0$, recall that $h(\cdot)$ is the failure rate and it is an increasing function by ($A2$). To see why $\nabla^2_{23} y^* > 0$, recall that $\gamma > 0$ by ($A3$).

Proof of (h) This follows from part (d) of the lemma. □

Lemma 3 *Given* r_{C} *and* r_{R}, *if* $\tilde{p}$ *satisfies* $\nabla_1 \Pi_R(\tilde{p}, r_{\mathrm{C}}, r_{\mathrm{R}}) = 0$, *then* $\nabla_1 y^*(\tilde{p}, r_{\mathrm{C}}, r_{\mathrm{R}}) < 0$.

Proof of Lemma 3 Omitted. See Aydin and Porteus (2008) for the proof of the same result under more general conditions. □

Lemma 4 *Let f(x) be a twice-continuously-differentiable function of a single real variable defined on [a,∞). Suppose that* $f''(x) < 0$ *at any* $x \geq a$ *that satisfies* $f'(x) = 0$. *Then:*

(a) *(Petruzzi and Dada1999) If* $f'(a) > 0$ *and f(x) is strictly decreasing in x as x tends to infinity, then there exists a unique* $x^* > a$ *that satisfies* $f'(x) = 0$, *and* x^* *maximizes f(x).*
(b) *If* $f'(a) \leq 0$ *then f(x) is non-increasing for all* $x \geq a$, *and* $x^* = a$ *maximizes f(x).*

Proof of Lemma 4 Omitted. Lemma 4(a) is due to Petruzzi and Dada (1999). See Aydin and Porteus (2008) for a detailed proof. The proof of part (b) is very similar. □

Lemma 5 *In a two-player game, let* $g_i(x_1, x_2)$ *be the payoff function of player* $i = 1,2$ *when the strategies chosen by players 1 and 2 are* x_1 *and* x_2, *respectively. The strategy space for player i is* $X_i := \{x : \underline{x}_i \leq x \leq \overline{x}_i\}$. *Suppose that* g_{i} *is continuous and quasi-concave with respect to* x_{i}, $i = 1,2$. *Let* $x_i^*(x_j)$ *be the best response of player i when player j chooses strategy* x_{j}; *i.e.,* $x_i^*(x_j) = \mathrm{argmax}_{x_i}(g_i(x_1, x_2))$. *Then:*

(a) *There exists at least one pure strategy Nash equilibrium.*
(b) *If* $\frac{dx_1^*(x_2)}{dx_2} \frac{dx_2^*(x_1)}{dx_1} < 1$, *then there exists a unique pure strategy Nash equilibrium.*

Proof of Lemma 5 Omitted. See Cachon and Netessine (2004) for a summary of standard results in game theory. □

Lemma 6 *Suppose (A1) through (A4) hold. Let* $\Pi_R(p, r_{\mathrm{C}}, r_{\mathrm{R}})$ *and* $\Pi_M(p, r_{\mathrm{C}}, r_{\mathrm{R}})$ *be as defined by* (13.5) *and* (13.6) *, respectively. Then:*

(a)
$$\nabla_1 \Pi_R(p, r_{\mathrm{C}}, r_{\mathrm{R}}) = \int_0^{\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v}} z d\xi + \frac{w-v}{p+r_{\mathrm{R}}-v} y^* - \gamma(p + r_{\mathrm{R}} - v) \int_0^{\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v}} z d\xi$$

(b)
$$\nabla_2 \Pi_M(p, r_{\mathrm{C}}, r_{\mathrm{R}}) = (w-c)\alpha\gamma y^* - [\beta + \alpha\gamma(r_{\mathrm{R}} + \beta r_{\mathrm{C}})] \left(\int_0^{\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v}} z d\xi + \frac{w-v}{p+r_{\mathrm{R}}-v} y^* \right),$$

(c)

$$\nabla_3 \Pi_M(p,r_C,r_R) = \left[(w-c)-(r_R+\beta r_C)\frac{w-v}{p+r_R-v}\right]\frac{f}{(p+r_R-v)h} - \int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi - \frac{w-v}{p+r_R-v}y^*$$

(d) *For* $p + r_R > v$*:*

$$\nabla^2_{11}\Pi_R(p,r_C,r_R)\big|_{\nabla_1 \Pi_R=0} = -\gamma\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v}\nabla_1 y^* - (p+r_R-v)\gamma' \int_0^{\frac{p+r_R-w}{p+r_R-v}} z < 0$$

(e) *For* $p + r_R > v$*:*

$$\nabla^2_{22}\Pi_M(p,r_C,r_R)\big|_{\nabla_2 \Pi_M=0} = -\alpha\beta\gamma\left(\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v}y^*\right) + \left[(w-c)y^* - (r_R+\beta r_C)\left(\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v}y^*\right)\right](\alpha^2\theta - \alpha^2\gamma^2)$$

(f) *For* $p + r_R > v$*:*

$$\nabla^2_{33}\Pi_M(p,r_C,r_R)\big|_{\nabla_3 \Pi_M=0} = -\frac{1}{p+r_R-v}\left[\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v}y^*\right] - \left[(w-c)-(r_R+\beta r_C)\frac{w-v}{p+r_R-v}\right]\frac{fh'}{(p+r_R-v)^2 h^3} + \left[-2\frac{w-v}{p+r_R-v} + (r_R+\beta r_C)\frac{w-v}{(p+r_R-v)^2}\right] \frac{f}{(p+r_R-v)h} < 0$$

(g) *For* $p + r_R > v$*:*

$$\nabla^2_{23}\Pi_M(p,r_C,r_R)\big|_{\nabla_2 \Pi_M=0} = -\frac{w-v}{p+r_R-v}\frac{\beta f}{(p+r_R-v)h} < 0$$

(h) *For* $p + r_{\mathrm{R}} > v$*:*

$$\nabla^2_{13}\Pi_R(p,r_{\mathrm{C}},r_{\mathrm{R}})\big|_{\nabla_1\Pi_R=0} = \frac{w-v}{p+r_{\mathrm{R}}-v}\nabla_1 y^* - \gamma\int_0^{\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v}} z d\xi < 0$$

(i) *For* $p + r_{\mathrm{R}} > v$*:*

$$\nabla^2_{13}\Pi_M(p,r_{\mathrm{C}},r_{\mathrm{R}})\big|_{\nabla_3\Pi_M=0} = -\left[(w-c)-(r_{\mathrm{R}}+\beta r_{\mathrm{C}})\frac{w}{p+r_{\mathrm{R}}-v}\right]\frac{fh'}{(p+r_{\mathrm{R}}-v)^2h^3} - \left[(2w-c)-2(r_{\mathrm{R}}+\beta r_{\mathrm{C}})\frac{w-v}{p+r_{\mathrm{R}}-v}\right]\frac{f}{(p+r_{\mathrm{R}}-v)^2h} < 0$$

(j) *For* $p + r_{\mathrm{R}} > v$*:*

$$\nabla^2_{12}\Pi_R(p,r_{\mathrm{C}},r_{\mathrm{R}})\big|_{\nabla_1\Pi_R=0} = \alpha(p+r_{\mathrm{R}}-v)\gamma'\int_0^{\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v}} z d\xi > 0$$

Proof of Lemma 6 *Proof of (a)* The result follows from partial differentiation of $\Pi_R(p,r_{\mathrm{C}},r_{\mathrm{R}})$ [defined by (13.5)] with respect to p and substituting for $\nabla_1 z$ using Lemma 1(a).

Proof of (b) The result follows by partial differentiation of $\Pi_M(p,r_{\mathrm{C}},r_{\mathrm{R}})$ [defined by (13.6)] with respect to r_{C} and substituting for $\nabla_2 z$ and $\nabla_2 y^*$ from Lemma 1 (b) and from Lemma 2(b).

Proof of (c) The result follows by partial differentiation of $\Pi_M(p,r_{\mathrm{C}},r_{\mathrm{R}})$ [defined by (13.6)] with respect to r_{R} and substituting for $\nabla_3 y^*$ from Lemma 2(c).

Proof of (d) The second partial of $\Pi_R(p,r_{\mathrm{C}},r_{\mathrm{R}})$ with respect to p is given, after substituting for $\nabla_1 z$ and $\nabla^2_{11} z$ using Lemma 1(a) and (c), by

$$\nabla^2_{11}\Pi_R(p,r_{\mathrm{C}},r_{\mathrm{R}}) = -2\gamma\int_0^{\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v}} z d\xi + \frac{w-v}{p+r_{\mathrm{R}}-v}\nabla_1 y^* - \frac{w-v}{p+r_{\mathrm{R}}-v}\gamma y^* + (p+r_{\mathrm{R}}-v)\theta\int_0^{\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v}} z d\xi$$

Thus, when $\nabla_1\Pi_R = 0$, using part (a) of the lemma, we have

$$\nabla^2_{11}\Pi_R(p,r_{\mathrm{C}},r_{\mathrm{R}}) = -\gamma\int_0^{\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v}} z d\xi + \frac{w-v}{p+r_{\mathrm{R}}-v}\nabla_1 y^* + (p+r_{\mathrm{R}}-v)\left(\theta-\gamma^2\right)\int_0^{\frac{p+r_{\mathrm{R}}-w}{p+r_{\mathrm{R}}-v}} z d\xi,$$

which is strictly negative, by Lemma 3 and since $\gamma > 0$ [by ($A3$)] and $\gamma' = \gamma^2 - \theta \geq 0$ [by ($A4$)].

Proof of (e) The second partial of $\Pi_M(p, r_C, r_R)$ with respect to r_C is given, after substituting for $\nabla_2 z$, $\nabla_{22}^2 z$, $\nabla_2 y^*$ and $\nabla_{22}^2 y^*$ from Lemma 1(b) and (d), and from Lemma 2(b) and (e), by

$$\nabla_{22}^2 \Pi_M(p, r_C, r_R) = (w-c)\alpha^2\theta y^* - [2\alpha\beta\gamma + (r_R + \beta r_C)\alpha^2\theta] \left(\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v} y^* \right)$$

Thus, when $\nabla_2 \Pi_M = 0$, using part (b) of the lemma, we have

$$\begin{aligned} \nabla_{22}^2 \Pi_M(p, r_C, r_R) = & -\alpha\beta\gamma \left(\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v} y^* \right) \\ & + \left[(w-c)y^* - (r_R + \beta r_C) \left(\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v} y^* \right) \right] \\ & (\alpha^2\theta - \alpha^2\gamma^2), \end{aligned}$$

which is strictly negative since $\gamma > 0$ [by ($A3$)], $\gamma' = \gamma^2 - \theta \geq 0$ [by ($A4$)] and the term in brackets is Π_M which should be positive when $\nabla_2 \Pi_M = 0$.

Proof of (f) The second partial of $\Pi_M(p, r_C, r_R)$ with respect to r_R is given, after substituting for $\nabla_3 y^*$ and $\nabla_{33}^2 y^*$ from Lemma 2(c) and (f), by

$$\begin{aligned} \nabla_{33}^2 \Pi_M(p, r_C, r_R) = & \left[(w-c) - (r_R + \beta r_C) \frac{w-v}{p+r_R-v} \right] \\ & \left[-\frac{f}{(p+r_R-v)^2 h} - \frac{f h'}{(p+r_R-v)^2 h^3} \right] \\ & + \left[-2\frac{w-v}{p+r_R-v} + (r_R + \beta r_C) \frac{w-v}{(p+r_R-v)^2} \right] \frac{f}{(p+r_R-v)h}. \end{aligned}$$

Thus, when $\nabla_3\Pi_M = 0$, using part (c) of the lemma, we have

$$\nabla^2_{33}\Pi_M(p,r_C,r_R) = -\frac{1}{p+r_R-v}\left[\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v}y^*\right]$$
$$-\left[(w-c)-(r_R+\beta r_C)\frac{w-v}{p+r_R-v}\right]\frac{fh'}{(p+r_R-v)^2h^3}$$
$$+\left[-2\frac{w-v}{p+r_R-v}+(r_R+\beta r_C)\frac{w-v}{(p+r_R-v)^2}\right]\frac{f}{(p+r_R-v)h}.$$

In order to show $\nabla^2_{33}\Pi_M(p,r_C,r_R)\big|_{\nabla_3\Pi_M=0} < 0$, first note that, by Lemma 6(c), if $\nabla_3\Pi_M(p,r_C,r_R) = 0$, then we must have $(w-c)-(r_R+\beta r_C)\frac{w-v}{p+r_R-v} > 0$, in which case we will also have $-2\frac{w-v}{p+r_R-v}+(r_R+\beta r_C)\frac{w-v}{(p+r_R-v)^2} < 0$. (This can be verified through some algebra.) After making these observations, the desired result now follows since $h' > 0$ by assumption ($A2$).

Proof of (g) It can be verified that the cross-partial $\nabla_{23}\Pi_M(p,r_C,r_R)$ is given, after substituting for $\nabla_2 z$ from Lemma 1(b) and for $\nabla_2 y^*$, $\nabla_3 y^*$ and $\nabla_{23} y^*$ from Lemma 2(b), (c) and (g), by

$$\nabla^2_{23}\Pi_M(p,r_C,r_R) = -\left[(w-c)-(r_R+\beta r_C)\frac{w-v}{p+r_R-v}\right]\alpha\gamma\frac{f}{(p+r_R-v)h}$$
$$-\alpha\gamma\left(\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v}y^*\right) - \beta\frac{w-v}{p+r_R-v}\frac{f}{(p+r_R-v)h}$$

Thus, when $\nabla_3\Pi_M = 0$, using part (c) of the lemma, we have

$$\nabla^2_{23}\Pi_M(p,r_C,r_R) = -\beta\frac{w-v}{p+r_R-v}\frac{f}{(p+r_R-v)h}$$

Proof of (h) It can be verified that $\nabla^2_{13}\Pi_R(p,r_C,r_R)$ is given, after substituting for $\nabla_1 z$ from Lemma 1(a) and $\nabla_3 y^*$ from Lemma 2(c), by

$$\nabla^2_{13}\Pi_R(p,r_C,r_R) = \frac{w-v}{p+r_R-v}\left(-\gamma y^* + \frac{f}{(p+r_R-v)h}\right) - \gamma\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi$$

Now, from part (a) of Lemma 2, we note that $-\gamma y^* + \frac{f}{h(p+r_R-v)} = \nabla_1 y^*$. The desired conclusion on the sign follows from $\gamma > 0$ [by ($A3$)] and Lemma 3.

Proof of (i) It can be verified that $\nabla_{13}\Pi_M(p, r_C, r_R)$ is given, after substituting for $\nabla_1 z$ from Lemma 1(a) and for $\nabla_1 y^*$, $\nabla_3 y^*$, and $\nabla_{13} y^*$ from Lemma 2(a), (c) and (h), by

$$\begin{aligned}
\nabla^2_{13}\Pi_M(p, r_C, r_R) = & \left[(w-c) - (r_R + \beta r_C)\frac{w-v}{p+r_R-v}\right] \\
& \left[-\gamma\frac{f}{(p+r_R-v)h} - \frac{f}{(p+r_R-v)^2 h} - \frac{fh'}{(p+r_R-v)^2 h^3}\right] \\
& + \gamma\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi - \frac{w-v}{p+r_R-v}\left[-\gamma y^* + \frac{f}{(p+r_R-v)h}\right] \\
& + (r_R + \beta r_C)\frac{w-v}{(p+r_R-v)^2}\frac{f}{(p+r_R-v)h}
\end{aligned}$$

Now, using part (c) of the lemma and the above expression, one can verify through some algebra that the following is true when $\nabla_3 \Pi_M = 0$:

$$\begin{aligned}
\nabla^2_{13}\Pi_M(p, r_C, r_R) = & -\left[(w-c) - (r_R + \beta r_C)\frac{w-v}{p+r_R-v}\right]\frac{fh'}{(p+r_R-v)^2 h^3} \\
& - \left[(2w-c) - 2(r_R + \beta r_C)\frac{w-v}{p+r_R-v}\right]\frac{f}{(p+r_R-v)^2 h}
\end{aligned}$$

In order to show that $\nabla^2_{13}\Pi_M(p, r_C, r_R)\big|_{\nabla_3 \Pi_M = 0} < 0$, note that, by part (c) of the lemma, if $\nabla_3 \Pi_M(p, r_C, r_R) = 0$, then we must have $(w-c) - (r_R + \beta r_C)\frac{w}{p+r_R-v} > 0$, in which case we will also have $(2w-c) - (r_R + \beta r_C)\frac{w}{p+r_R-v} > 0$. The desired result now follows since $h' > 0$ by assumption ($A2$).

Proof of (j) It can be verified that $\nabla^2_{12}\Pi_R(p, r_C, r_R)$ is given, after substituting for $\nabla_2 z$ and $\nabla^2_{12} z$ from Lemma 1(b) and (e) and for $\nabla_2 y^*$ from Lemma 2(b) by

$$\nabla^2_{12}\Pi_R(p, r_C, r_R) = \alpha\gamma\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v}\alpha\gamma y^* - \alpha\theta(p+r_R-v)\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi$$

Now, when $\nabla_1 \Pi_R = 0$, the following relationship can be verified through algebra, using part (a) of the lemma and the above expression:

$$\nabla^2_{12}\Pi_R(p, r_C, r_R) = \alpha(\gamma^2 - \theta)(p + r_R - v)\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi,$$

which is strictly positive since $\gamma' = \gamma^2 - \theta > 0$ by virtue of ($A4$). □

Proof of Proposition 1 Using Lemma 6(a), one can verify that $\nabla_1\Pi_R(w - r_R, r_C, r_R) > 0$. Again using Lemma 6(a), one can also verify that $\nabla_1\Pi_R(p, r_C, r_R) < 0$ as $p \to \infty$. Given these observations, the result now follows from Lemmas 4(a) and 6(d). □

Proof of Proposition 2 *Proof of (a)* Given p and r_R, using Lemma 6(b), one can verify that $\nabla_2\Pi_M(p, r_C, r_R) < 0$ as $r_C \to \infty$. From Lemma 6(e), we know that $\nabla^2_{22}\Pi_M(p, r_C, r_R)|_{\nabla_2\Pi_M=0} < 0$. The result now follows by applying parts (a) and (b) of Lemma 4.

Proof of (b) We can focus on r_R such that $p + r_R \geq w$ (and, hence, $p + r_R \geq v$), since the retailer would stock zero units otherwise, and the manufacturer would make zero profits. Given p and r_C, using Lemma 6(c), one can verify that $\nabla_3\Pi_M(p, r_C, r_R) < 0$ as $r_R \to \infty$. From Lemma 6(f), we know that $\nabla^2_{33}\Pi_M(p, r_C, r_R)|_{\nabla_3\Pi_M=0} < 0$. The result now follows by applying parts (a) and (b) of Lemma 4.

Proof of (c) When $\nabla_2\Pi_M(p, r_C, r_R) = 0$, we can use Lemma 6(b) to write

$$\beta\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \beta\frac{w-v}{p+r_R-v}y^* = \left[(w-c) - (r_R + \beta r_C)\frac{w-v}{p+r_R-v}\right]\alpha\gamma y^* - (r_R + \beta r_C)\alpha\gamma\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi.$$

Note that for the above equality to hold, we need to have $(w-c) - (r_R + r_C)\frac{w-v}{p+r_R-v} > 0$ [since $\gamma > 0$ by assumption ($A3$)]. Similarly, when $\nabla_3\Pi_M(p, r_C, r_R) = 0$, we can use Lemma 6(c) to write

$$\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi + \frac{w-v}{p+r_R-v}y^* = \left[(w-c) - (r_R + \beta r_C)\frac{w-v}{p+r_R-v}\right]\frac{f}{(p+r_R-v)h}.$$

Again, note that for the above equality to hold, we need to have $(w-c) - (r_R + r_C)\frac{w-v}{p+r_R-v} > 0$. By using the last two equalities, we obtain:

$$-(r_R + \beta r_C)\alpha\gamma\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi = \left[(w-c) - (r_R + \beta r_C)\frac{w-v}{p+r_R-v}\right]\left[\frac{\beta f}{(p+r_R-v)h} - \alpha\gamma y^*\right]$$

Now, note that the second term in brackets on the right-hand side of the equality above is $-\nabla_2 y^* + \beta\nabla_3 y^*$ (from parts (b) and (c) of Lemma 2). Also, as noted above, we must have $(w-c)-(r_\mathrm{R}+r_\mathrm{C})\frac{w-v}{p+r_\mathrm{R}-v}>0$. The term on the left-hand side is negative [since $\gamma>0$ by assumption ($A3$)]. The desired result now follows. □

Proof of Proposition 3 Under an equilibrium solution $(\tilde p,\tilde r_\mathrm{C},\tilde r_\mathrm{R})$ with $\tilde r_\mathrm{C}>0$, we need to have $\nabla_2\Pi_M(\tilde p,\tilde r_\mathrm{C},\tilde r_\mathrm{R})=\nabla_1\Pi_R(\tilde p,\tilde r_\mathrm{C},\tilde r_\mathrm{R})=0$ (by Proposition 1 and part (a) of Proposition 2). Since $\nabla_2\Pi_M(\tilde p,\tilde r_\mathrm{C},\tilde r_\mathrm{R})=0$, we know from Lemma 6(b) that

$$\begin{aligned}\beta\frac{w-v}{\tilde p+\tilde r_\mathrm{R}-v}y^*+\beta\int_0^{\frac{\tilde p+\tilde r_\mathrm{R}-w}{\tilde p+\tilde r_\mathrm{R}-v}}zd\xi &= (w-c)\alpha\gamma y^*\\ &\quad -(\tilde r_\mathrm{R}+\beta\tilde r_\mathrm{C})\alpha\gamma\left(\int_0^{\frac{\tilde p+\tilde r_\mathrm{R}-w}{\tilde p+\tilde r_\mathrm{R}}-v}zd\xi+\frac{w-v}{\tilde p+\tilde r_\mathrm{R}-v}y^*\right),\\ &= \alpha\gamma\Pi_M(\tilde p,\tilde r_\mathrm{C},\tilde r_\mathrm{R})\text{ by (6)}\end{aligned}\tag{13.10}$$

Also, since $\nabla_1\Pi_R(\tilde p,\tilde r_\mathrm{C},\tilde r_\mathrm{R})=0$, we know from Lemma 6(a) that

$$\begin{aligned}\frac{w-v}{\tilde p+\tilde r_\mathrm{R}-v}y^*+\int_0^{\frac{\tilde p+\tilde r_\mathrm{R}-w}{\tilde p+\tilde r_\mathrm{R}-v}}zd\xi &= (\tilde p+\tilde r_\mathrm{R}-v)\gamma\int_0^{\frac{\tilde p+\tilde r_\mathrm{R}-w}{\tilde p+\tilde r_\mathrm{R}-v}}zd\xi,\\ &= \gamma\Pi_R(\tilde p,\tilde r_\mathrm{C},\tilde r_\mathrm{R})\text{ by (5)}\end{aligned}\tag{13.11}$$

Now, (13.10) and (13.11) together allow us conclude $\frac{\Pi_M(\tilde p,\tilde r_\mathrm{C},\tilde r_\mathrm{R})}{\Pi_R(\tilde p,\tilde r_\mathrm{C},\tilde r_\mathrm{R})}=\frac{\beta}{\alpha}$. □

Proof of Proposition 4 *Proof of (a)* Throughout the proof, recall that $p^*(r_\mathrm{R})$ will satisfy $\nabla_1\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})=0$ at any given r_R (by Proposition 1). By implicit differentiation of this identity with respect to r_R, we obtain $\frac{dp^*(r_\mathrm{R})}{dr_\mathrm{R}}=-\frac{\nabla^2_{13}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})}{\nabla^2_{11}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})}$. Hence, we will conclude the proof of part (a) if we can show that $\nabla^2_{11}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})\le\nabla^2_{13}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})<0$. From Lemma 6 (d) and (h), we know that $\nabla^2_{11}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})<0$ and $\nabla^2_{13}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})<0$. Again, from Lemma 6(d) and (h), note that:

$$\nabla^2_{11}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})-\nabla^2_{13}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})=-(p+r_\mathrm{R}-v)\gamma'\int_0^{\frac{p+r_\mathrm{R}-w}{p+r_\mathrm{R}-v}}z\le 0,\tag{13.12}$$

where the inequality follows from $\gamma'\ge 0$ [by ($A4$)]. Thus, we are able to conclude that

$$\nabla^2_{11}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})\le\nabla^2_{13}\Pi_R(p^*(r_\mathrm{R}),0,r_\mathrm{R})<0,$$

which concludes the proof of part (a).

Proof of (b) Given p, w and $r_C = 0$, it follows from Proposition 2(b) that either ${r_R}^*(p) = 0$ or $r_R^*(p) > 0$ in which case $r_R^*(p)$ satisfies $\nabla_3 \Pi_M(p, 0, r_R^*(p)) = 0$. If ${r_R}^*(p) = 0$ for all $p > 0$, then part (b) holds trivially. Suppose now there exists a p at which ${r_R}^*(p) > 0$ and satisfies $\nabla_3 \Pi_M(p, 0, r_R^*(p)) = 0$. By implicit differentiation of this identity with respect to p, we obtain $\dfrac{dr_R^*(p)}{dp} = -\dfrac{\nabla_{13}^2 \Pi_M(p, 0, r_R^*(p))}{\nabla_{33}^2 \Pi_M(p, 0, r_R^*(p))}$.
We already know from Lemma 6(f) and (i) that $\nabla_{33}^2 \Pi_M(p, 0, r_R^*(p)) < 0$ and $\nabla_{13}^2 \Pi_M(p, 0, r_R^*(p)) < 0$. Furthermore, again from Lemma 6(f) and (i), one can verify that

$$\begin{aligned}\nabla_{13}^2 \Pi_M(p, r_C, r_R)\big|_{\nabla_3 \Pi_M = 0} &= \nabla_{33}^2 \Pi_M(p, r_C, r_R)\big|_{\nabla_3 \Pi_M = 0} + \frac{w - v}{p + r_R - v} \frac{f}{(p + r_R - v)h} \\ &\quad - \frac{1}{p + r_R - v}\left\{ -\int_0^{\frac{p + r_R - w}{p + r_R - v}} z d\xi - \frac{w - v}{p + r_R - v} y^* \right. \\ &\quad \left. + \left[(w - c) - (r_R + \beta r_C)\frac{w - v}{p + r_R - v}\right] \frac{f}{(p + r_R - v)h} \right\}\end{aligned}$$

From Lemma 6(c), we observe that the term in curly brackets above is in fact $\nabla_3 \Pi_M(p, r_C, r_R)$. Therefore, from the above expression, we obtain:

$$\nabla_{13}^2 \Pi_M(p, r_C, r_R)\big|_{\nabla_3 \Pi_M = 0} = \nabla_{33}^2 \Pi_M(p, r_C, r_R)\big|_{\nabla_3 \Pi_M = 0} + \frac{w - v}{p + r_R - v} \frac{f}{(p + r_R - v)h}$$

Hence, from the last equality, we conclude that $\nabla_{33}^2 \Pi_M(p, 0, r_R^*(p)) < \nabla_{13}^2 \Pi_M(p, 0, r_R^*(p))$, which, along with $\nabla_{33}^2 \Pi_M(p, 0, r_R^*(p)) < 0$ and $\nabla_{13}^2 \Pi_M(p, 0, r_R^*(p)) < 0$, allows us to conclude that $-1 < \frac{dr_R^*(p)}{dp} < 0$. Recall that we assumed p is such that $r_R^*(p) > 0$. For some p', we will have $r_R^*(p') = 0$, and ${r_R}^*(p)$ will remain zero for all $p > p'$, and hence $\frac{dr_R^*(p)}{dp}$ will be zero for all $p > p'$. (If ${r_R}^*(p)$ were to become positive for some $p'' > p'$, this would be a contradiction to the result that $\frac{dr_R^*(p)}{dp} < 0$ when $r_R^*(p) > 0$.)

Proof of (c) The existence of the Nash equilibrium follows from Lemma 5(a), Propositions 1 and 2(b). The uniqueness of the Nash equilibrium follows from Lemma 5(b) and parts (a) and (b) of this proposition. (Note that, in order to apply Lemma 5, we need upper bounds on the decision variables of the retailer and the manufacturer, p and r_R, respectively. We could satisfy this requirement by picking arbitrarily large numbers to bound the feasible choices for p and r_R.) □

Proof of Proposition 5 Throughout the proof, let $p^*(r_R)$ denote the optimal retail price chosen by the retailer at a given r_R when $r_C = 0$. *Proof of (a)* Note that $p_o = p^*(0)$ whereas $\tilde{p} = p^*(\tilde{r}_R)$. Therefore, $\tilde{p} - p_o = \int_0^{\tilde{r}_R} \frac{dp^*(r_R)}{dr_R} dr_R$. By Proposition 4(a), $-1 < \frac{dp^*(r_R)}{dr_R} < 0$. The desired result follows.

Proof of (b) Note that $y_o = y^*(p^*(0), 0, 0)$ whereas $\tilde{y} = y^*(p^*(\tilde{r}_R), 0, \tilde{r}_R)$. Now, $\tilde{y} - y_o = \int_0^{\tilde{r}_R} \frac{dy^*(p^*(r_R), 0, r_R)}{dr_R} dr_R$. Therefore, we will conclude the proof if we can show that $\frac{dy^*(p^*(r_R), 0, r_R)}{dr_R} > 0$. Note that $\frac{dy^*(p^*(r_R), 0, r_R)}{dr_R} = \nabla_3 y^*(p^*(r_R), 0, r_R) + \frac{dp^*(r_R)}{dr_R} \nabla_1 y^*(p^*(r_R), 0, r_R)$. Now, $\nabla_3 y^*(p^*(r_R), 0, r_R) > 0$ from Lemma 2(c), $\frac{dp^*(r_R)}{dr_R} < 0$ from of Proposition 4(a) and $\nabla_1 y^*(p^*(r_R), 0, r_R) < 0$ from Lemma 3. (To see why Lemma 3 can be applied here, recall that $\nabla_1 \Pi_R(p^*(r_R), 0, r_R) = 0$ by Proposition 1 since $p^*(r_R)$ optimizes Π_R.) These observations imply that $\frac{dy^*(p^*(r_R), 0, r_R)}{dr_R} > 0$, which yields the desired result.

Proof of (c) Note that $\Pi_{SC}(\tilde{p}, 0, \tilde{r}_R) = \Pi_{SC}(p^*(\tilde{r}_R), 0, \tilde{r}_R)$ and $\Pi_{SC}(p_o, 0, 0) = \Pi_{SC}(p^*(0), 0, 0)$. Therefore, $\Pi_{SC}(\tilde{p}, 0, \tilde{r}_R) - \Pi_{SC}(p_o, 0, 0) = \int_0^{\tilde{r}_R} \frac{d\Pi_{SC}(p^*(r_R), 0, r_R))}{dr_R} dr_R$. Hence, if we can show that $\Pi_{SC}(p^*(r_R), 0, r_R)$ is increasing in r_R for $r_R \leq w - c$, then the desired result will follow. Hence, we want to show that

$$\frac{d\Pi_{SC}(p^*(r_R), 0, r_R)}{dr_R} = \frac{dp^*(r_R)}{dr_R} \nabla_1 \Pi_{SC}(p^*(r_R), 0, r_R) + \nabla_3 \Pi_{SC}(p^*(r_R), 0, r_R)$$

is positive. The following equalities can be verified using (13.5) and (13.6):

$$\begin{aligned}
\nabla_1 \Pi_{SC}(p, r_C, r_R) &= \nabla_1 \Pi_R(p, r_C, r_R) + \nabla_1 \Pi_M(p, r_C, r_R) \\
&= \nabla_1 \Pi_R(p, r_C, r_R) + \left[w - c - (r_R + \beta r_C) \frac{w - v}{p + r_R - v} \right] \nabla_1 y^* \\
&\quad -(r_R + \beta r_C) \int_0^{\frac{p + r_R - w}{p + r_R - v}} \nabla_1 z d\xi \\
\nabla_3 \Pi_{SC}(p, r_C, r_R) &= \nabla_3 \Pi_R(p, r_C, r_R) + \nabla_3 \Pi_M(p, r_C, r_R) \\
&= \left[w - c - (r_R + \beta r_C) \frac{w - v}{p + r_R - v} \right] \nabla_3 y^*
\end{aligned}$$

Note that $\nabla_1 \Pi_R(p^*(r_R), 0, r_R) = 0$ by definition of $p^*(r_R)$ and Proposition 1. Thus, after substitution and rearranging terms, we get

$$\begin{aligned}\frac{d\Pi_{SC}(p^*(r_R),0,r_R)}{dr_R} &= \left(1+\frac{dp^*(r_R)}{dr_R}\right)\left[w-c-r_R\frac{w-v}{p^*(r_R)+r_R-v}\right]\nabla_3 y^* \\ &\quad + \frac{dp^*(r_R)}{dr_R}\left\{\left[w-c-r_R\frac{w-v}{p^*(r_R)+r_R-v}\right](\nabla_1 y^*-\nabla_3 y^*)\right. \\ &\quad \left. - r_R\int_0^{\frac{p^*(r_R)+r_R-w}{p^*(r_R)+r_R-v}}\nabla_1 z d\xi\right\}\end{aligned}$$

By Lemma 2(c) and Proposition 4(a), the first term above is positive. We show that the second term is also positive, to conclude the proof. Since $\frac{dp^*(r_R)}{dr_R}<0$, all we need to show is

$$\left[w-c-r_R\frac{w-v}{p^*(r_R)+r_R-v}\right](\nabla_1 y^*-\nabla_3 y^*)-r_R\int_0^{\frac{p^*(r_R)+r_R-w}{p^*(r_R)+r_R-v}}\nabla_1 z d\xi<0. \tag{13.13}$$

Now, for $\xi \le \dfrac{p^*(r_R)+r_R-w}{p^*(r_R)+r_R-v}$,

$$\begin{aligned}\nabla_1 y^*-\nabla_3 y^* &= -\gamma y^* \\ &= \nabla_1 z\left(p^*(r_R),0,\frac{p^*(r_R)+r_R-w}{p+r_R-v}\right) \\ &< \nabla_1 z(p^*(r_R),0,\xi)\end{aligned}$$

The first equality follows from Lemma 2(a) and (c), the second from Lemma 1(a), and the inequality holds because $\xi \le \dfrac{p^*(r_R)+r_R-w}{p^*(r_R)+r_R-v}$. Thus, using $r_R \le w-c$, (13.13) holds. □

Proof of Proposition 6 *Proof of (a)* Note that $p^*(r_C)$ will satisfy $\nabla_1\Pi_R(p^*(r_C), r_C,0)=0$ at any given r_C (by Proposition 1). By implicit differentiation of this identity with respect to r_C, we obtain $\frac{dp^*(r_C)}{dr_C}=-\frac{\nabla^2_{12}\Pi_R(p^*(r_C),r_C,0)}{\nabla^2_{11}\Pi_R(p^*(r_C),r_C,0)}$. We know from Lemma 6(d) and (j) that

$$\nabla^2_{11}\Pi_R(p^*(r_C),r_C,0)<0 \text{ and } \nabla^2_{12}\Pi_R(p^*(r_C),r_C,0)>0.$$

Therefore, it follows that $\frac{dp^*(r_C)}{dr_C}>0$. Furthermore, from Lemma 6(d) and (j), we can write:

$$\nabla^2_{12}\Pi_R(p,r_C,r_R)\big|_{\nabla_1\Pi_R=0} = -\alpha\nabla^2_{11}\Pi_R(p,r_C,r_R)\big|_{\nabla_1\Pi_R=0} + \alpha\frac{w-v}{p+r_R-v}\nabla_1 y^* \\ -\alpha\gamma\int_0^{\frac{p+r_R-w}{p+r_R-v}} z d\xi$$

From the equality above, since $\nabla_1 y^* < 0$ when $\nabla_1 \Pi_R = 0$ (from Lemma 3) and $\gamma > 0$ [by ($A3$)], we have $\nabla^2_{12}\Pi_R(p,r_C,r_R)\big|_{\nabla_1\Pi_R=0} < -\alpha\nabla^2_{11}\Pi_R(p,r_C,r_R)\big|_{\nabla_1\Pi_R=0}$. Therefore, we have

$$\nabla_{12}\Pi_R(p^*(r_C),r_C,0) < -\alpha\nabla_{11}\Pi_R(p^*(r_C),r_C,0).$$

This observation yields $\frac{dp^*(r_C)}{dr_C} < \alpha$.

Proof of (b) The existence of the Nash equilibrium follows from Lemma 5(a), Propositions 1 and 2(a). (Note that, in order to apply Lemma 5, we need upper bounds on the decision variables of the retailer and the manufacturer, p and r_C, respectively. We could satisfy this requirement by picking arbitrarily large numbers to bound the feasible choices for p and r_C.) □

Proof of Proposition 7 Throughout the proof, let $p^*(r_C)$ denote the optimal retail price chosen by the retailer at a given r_C when $r_R = 0$. *Proof of (a)* Note that $p_o = p^*(0)$ whereas $\tilde{p} = p^*(\tilde{r}_C)$. Therefore, $\tilde{p} - p_o = \int_0^{\tilde{r}_C} \frac{dp^*(r_C)}{dr_C} dr_C$. By Proposition 6, $0 < \frac{dp^*(r_C)}{dr_C} < \alpha$. The desired result follows.

Proof of (b) Note that $y_o = y^*(p^*(0),0,0)$ whereas $\tilde{y} = y^*(p^*(\tilde{r}_C),\tilde{r}_C,0)$. Now, $\tilde{y} - y_o = \int_0^{\tilde{r}_C} \frac{dy^*(p^*(r_C),r_C,0)}{dr_C} dr_C$. We will conclude the proof if we can show that $\frac{dy^*(p^*(r_C),r_C,0)}{dr_C} > 0$. Note that $\frac{dy^*(p^*(r_C),r_C,0)}{dr_C} = \nabla_2 y^*(p^*(r_C),r_C,0) + \frac{dp^*(r_C)}{dr_C}\nabla_1 y^*(p^*(r_C),r_C,0)$. Since $0 < \frac{dp(r_C)}{dr_C} < \alpha$ by Proposition 6 and $\nabla_1 y^*(p^*(r_C),r_C,0) < 0$ by Lemma 3, we obtain $\frac{dy^*(p^*(r_C),r_C,0)}{dr_C} > \nabla_2 y^*(p^*(r_C),r_C,0) + \alpha\nabla_1 y^*(p^*(r_C),r_C,0)$. Using this last inequality and substituting for $\nabla_1 y^*(p^*(r_C),r_C,0)$ from Lemma 2(a) and for $\nabla_2 y^*(p^*(r_C),r_C,0)$ from Lemma 2(b), we can deduce that $\frac{dy^*(p^*(r_C),r_C,0)}{dr_C} > 0$, which concludes the proof of this part.

Proof of (c) As in the proof of part (c) of Proposition 5, we will show that $\Pi_{SC}(p^*(r_C),r_C,0)$ is increasing in r_C for $r_C \le w - c$ when $\alpha \ge \beta$. The desired result would then follow. Now, the following equalities can be verified by partial differentiation of (13.5) and (13.6):

$$
\begin{aligned}
\frac{d\Pi_{SC}(p^*(r_C), r_C, 0)}{dr_C} &= \frac{dp^*(r_C)}{dr_C} \nabla_1 \Pi_{SC}(p^*(r_C), r_C, 0) + \nabla_2 \Pi_{SC}(p^*(r_C), r_C, 0) \\
&= \frac{dp^*(r_C)}{dr_C} \nabla_1 \Pi_R(p^*(r_C), r_C, 0) \\
&\quad + \left(w - c - \beta r_C \frac{w-v}{p^*(r_C) - v}\right)\left(\nabla_2 y^* + \frac{dp^*(r_C)}{dr_C} \nabla_1 y^*\right) \\
&\quad + \beta r_C \left(\int_0^{\frac{p^*(r_C)-w}{p^*(r_C)-v}} \nabla_2 z d\xi - \frac{dp^*(r_C)}{dr_C} \int_0^{\frac{p^*(r_C)-w}{p^*(r_C)-v}} \nabla_1 z d\xi \right) \\
&\quad + p^*(r_C) \int_0^{\frac{p^*(r_C)-w}{p^*(r_C)-v}} \nabla_2 z d\xi \\
&\quad - \beta \left(\int_0^{\frac{p^*(r_C)-w}{p^*(r_C)-v}} z d\xi + \frac{w-v}{p^*(r_C)-v} y^* \right)
\end{aligned}
$$

Now, the first term is zero, by definition of $p^*(r_C)$. The second term is positive, because, as in the proof of part (b), $\nabla_2 y^*(p^*(r_C), r_C, 0) + \frac{dp^*(r_C)}{dr_C} \nabla_1 y^*(p^*(r_C), r_C, 0) > 0$, and $r_C \leq w - c$. The third term is positive by virtue of Lemma 1(a)–(b) and Proposition 6(a). Using Lemma 6 (a), Lemma 1(a) and (b) and the fact that $\nabla_1 \Pi_R(p^*(r_C), r_C, 0) = 0$ we get that

$$
p^*(r_C) \int_0^{\frac{p^*(r_C)-w-v}{p^*(r_C)-v}} \nabla_2 z d\xi = \alpha \left(\int_0^{\frac{p^*(r_C)-w}{p^*(r_C)-v}} z d\xi + \frac{w-v}{p^*(r_C)-v} y^* \right).
$$

Thus, the sum of the last two terms can be written as

$$
(\alpha - \beta) \left(\int_0^{\frac{p^*(r_C)-w}{p^*(r_C)-v}} z d\xi + \frac{w-v}{p^*(r_C)-v} y^* \right),
$$

which is positive because $\alpha \geq \beta$.

Proof of (d) The proof of this part is almost identical to the analogous result in Proposition 3. Set $\tilde{r}_R = 0$ and the proof follows the same line of argument. □

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