Chapter 6
Finite Element Equations

Finite element equations capture the characteristics of the field equations. Their derivation is based on either the governing differential equation or the global energy balance of the physical problem. The approach involving the governing differential equation is referred to as the *method of weighted residuals* or *Galerkin's method*. The approach utilizing the global energy balance is referred to as the *variational method* or *Rayleigh-Ritz method*.

6.1 Method of Weighted Residuals

The method of weighted residuals involves the approximation of the functional behavior of the dependent variable in the governing differential equation (Finlayson 1972). When substituted into the governing differential equation, the approximate form of the dependent variable leads to an error called the "residual." This residual error is required to vanish in a weighted average sense over the domain. If the weighting functions are chosen to be the same as the element shape (interpolation) functions used in the element approximation functions, the method of weighted residuals is referred to as Galerkin's method.

The governing differential equation for the physical problem in domain D described in Fig. 6.1 can be expressed in the form

$$L(\phi) - f = 0 \tag{6.1}$$

where ϕ is a dependent variable and f is a known forcing function. The ordinary or partial differential operator, L whose order is specified by p, can be linear or nonlinear. The boundary conditions are given by

$$B_j(\phi) = g_j \text{ on } C_1 \tag{6.2}$$

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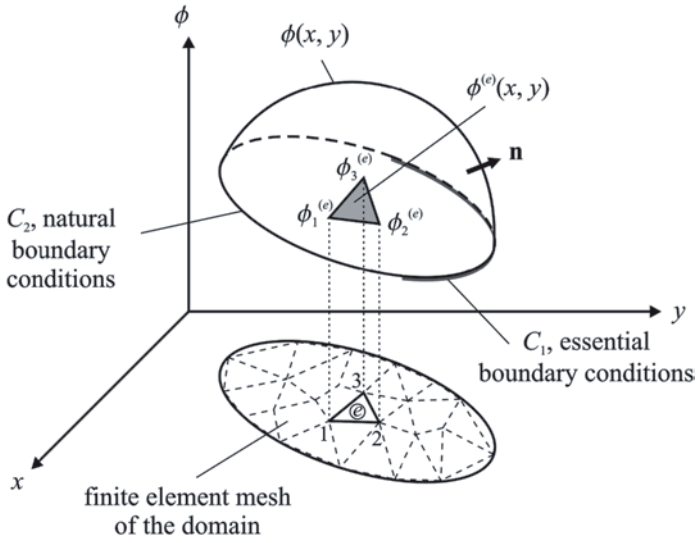

Fig. 6.1 Variation of the dependent (field) variable over a two-dimensional domain under specified boundary conditions

and

$$E_j(\phi) = h_j \text{ on } C_2 \tag{6.3}$$

in which B_j and E_j are operators, with $j = 1, 2, 3, \ldots, p$. The known functions g_j and h_j prescribe the boundary conditions on the dependent vari-able and its derivatives, respectively. The conditions on the dependent variable over C_1 are referred to as *essential* or *forced* boundary conditions, and the ones involving the derivatives of the dependent variable over C_2 are referred to as *natural* boundary conditions.

The method of weighted residuals requires that

$$\int_D \left[L(\tilde{\phi}) - f \right] W_k \, dD = 0, \quad \text{with} \quad k = 1, 2, 3, \ldots, n \tag{6.4}$$

where W_k are the weighting functions approximating the dependent variable as

$$\phi \approx \tilde{\phi} = \sum_{k=1}^{n} \alpha_k W_k \tag{6.5}$$

while satisfying the essential boundary conditions on C_1. The unknown coefficients, α_k, are determined by solving for the resulting system of algebraic equations.

Since the governing differential equation is valid for the entire domain, D, partitioning the domain into subdomains or elements, $D^{(e)}$, and applying Galerkin's method with weighting functions $W_k = N_k^{(e)}$ over the element domain results in

$$\sum_{e=1}^{E} \int_{D^{(e)}} \mathbf{N}^{(e)} \left(L\left(\tilde{\phi}^{(e)}\right) - f \right) dD = 0 \tag{6.6}$$

in which E is the number of elements and the superscript "e" denotes a specific element whose domain is $D^{(e)}$. The approximation to the dependent variable within the element can be expressed as

$$\tilde{\phi}^{(e)} = \sum_{i=1}^{n} N_i^{(e)} \phi_i^{(e)} \tag{6.7}$$

or

$$\tilde{\phi}^{(e)} = \mathbf{N}^{(e)T} \boldsymbol{\varphi}^{(e)} \tag{6.8}$$

where

$$\mathbf{N}^{(e)T} = \left\{ N_1^{(e)} \quad N_2^{(e)} \quad N_3^{(e)} \quad \dots \quad N_n^{(e)} \right\} \tag{6.9}$$

and

$$\boldsymbol{\varphi}^{(e)T} = \left\{ \phi_1^{(e)} \quad \phi_2^{(e)} \quad \phi_3^{(e)} \quad \dots \quad \phi_n^{(e)} \right\} \tag{6.10}$$

with n representing the number of nodes associated with element e. The nodal unknowns and shape functions are denoted by $\phi_i^{(e)}$ and $N_i^{(e)}$, with $i = 1, 2, .., n$, respectively. The shape functions need not satisfy the boundary conditions; however, they satisfy the inter-element continuity conditions necessary for assembly of the element equations. The essential boundary conditions are imposed after assembling the global matrix. The natural boundary conditions are not imposed directly. However, their influence emerges in the derivation of the element equations.

The required order of the element continuity is equal to one less than the highest derivative of the dependent variable appearing in the integrand. This requirement is relaxed by applying integration by parts in the minimization procedure of the residual error in Galerkin's method.

6.1.1 Example: One-Dimensional Differential Equation with Line Elements

The application of Galerkin's method is introduced by considering the ordinary differential equation given by

$$\frac{d^2\phi(x)}{dx^2} + \phi(x) - f(x) = 0 \tag{6.11}$$

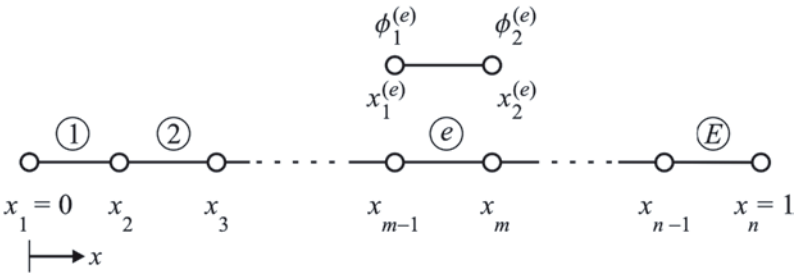

Fig. 6.2 Domain of the one-dimensional differential equation, discretized into E elements

in domain D defined by $0 \le x \le 1$. The known forcing function is given by

$$f(x) = -x \tag{6.12}$$

The boundary conditions, identified as the essential type, are $\phi(0) = 0$ and $\phi(1) = 0$. As shown in Fig. 6.2, the domain can be discretized with E linear line elements, each having two nodes ($n = 2$). There are a total of N nodes, and global coordinates of each node in domain D are specified by x_i, with $i = 1, 2, \ldots, N$. Nodal values of the dependent variable associated with element e are specified at its first and second nodes by $\phi_1^{(e)}$ and $\phi_2^{(e)}$, respectively.

The linear approximation function for the dependent variable in element e can be expressed in the form

$$\tilde{\phi}^{(e)} = N_1^{(e)}\phi_1^{(e)} + N_2^{(e)}\phi_2^{(e)} \tag{6.13}$$

or

$$\tilde{\phi}^{(e)} = \mathbf{N}^{(e)T}\boldsymbol{\varphi}^{(e)} \tag{6.14}$$

where

$$\mathbf{N}^{(e)T} = \left\{ N_1^{(e)} \quad N_2^{(e)} \right\} \text{ and } \boldsymbol{\varphi}^{(e)T} = \left\{ \phi_1^{(e)} \quad \phi_2^{(e)} \right\} \tag{6.15}$$

in which the shape functions are given by

$$N_1^{(e)} = \frac{x_2^{(e)} - x}{x_2^{(e)} - x_1^{(e)}} \text{ and } N_2^{(e)} = \frac{x - x_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \tag{6.16}$$

They are the same as the length coordinates given by Eq. (3.9). Applying Galerkin's method by Eq. (6.6) leads to

$$\sum_{e=1}^{E} \int_{x_1^{(e)}}^{x_2^{(e)}} \mathbf{N}^{(e)} \left(\frac{d^2\tilde{\phi}^{(e)}(x)}{dx^2} + \tilde{\phi}^{(e)}(x) - f(x) \right) dx = 0 \tag{6.17}$$

Integrating the first term in the integral by parts results in

$$\sum_{e=1}^{E}\left[\mathbf{N}^{(e)}\left.\frac{d\tilde{\phi}^{(e)}(x)}{dx}\right|_{x_1^{(e)}}^{x_2^{(e)}}-\int_{x_1^{(e)}}^{x_2^{(e)}}\frac{d\mathbf{N}^{(e)}}{dx}\frac{d\tilde{\phi}^{(e)}}{dx}dx+\int_{x_1^{(e)}}^{x_2^{(e)}}\mathbf{N}^{(e)}\tilde{\phi}^{(e)}(x)dx-\int_{x_1^{(e)}}^{x_2^{(e)}}\mathbf{N}^{(e)}f(x)dx\right]=0 \tag{6.18}$$

Substituting for the element approximation function ($\tilde{\phi}^{(e)}=\mathbf{N}^{(e)T}\boldsymbol{\varphi}^{(e)}$) yields

$$\sum_{e=1}^{E}\mathbf{k}^{(e)}\boldsymbol{\varphi}^{(e)}=\sum_{e=1}^{E}\mathbf{f}^{(e)} \tag{6.19}$$

where

$$\mathbf{k}^{(e)}=-\int_{x_1^{(e)}}^{x_2^{(e)}}\frac{d\mathbf{N}^{(e)}}{dx}\frac{d\mathbf{N}^{(e)T}}{dx}dx+\int_{x_1^{(e)}}^{x_2^{(e)}}\mathbf{N}^{(e)}\mathbf{N}^{(e)T}dx \tag{6.20}$$

and

$$\mathbf{f}^{(e)}=\int_{x_1^{(e)}}^{x_2^{(e)}}\mathbf{N}^{(e)}f(x)dx-\left[\mathbf{N}^{(e)}\frac{d\tilde{\phi}^{(e)}(x)}{dx}\right]_{x_1^{(e)}}^{x_2^{(e)}} \tag{6.21}$$

After substituting for the shape functions and their derivatives, as well as the forcing function, the expressions for the element characteristic matrix, $\mathbf{k}^{(e)}$, and the right-hand-side vector, $\mathbf{f}^{(e)}$, become

$$\mathbf{k}^{(e)}=-\frac{1}{\left(x_2^{(e)}-x_1^{(e)}\right)^2}\int_{x_1^{(e)}}^{x_2^{(e)}}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}dx+\int_{x_1^{(e)}}^{x_2^{(e)}}\begin{bmatrix}N_1^{(e)}N_1^{(e)}&N_1^{(e)}N_2^{(e)}\\N_2^{(e)}N_1^{(e)}&N_2^{(e)}N_2^{(e)}\end{bmatrix}dx \tag{6.22}$$

$$\mathbf{f}^{(e)} = -\int_{x_1^{(e)}}^{x_2^{(e)}} \begin{Bmatrix} N_1^{(e)} \\ N_2^{(e)} \end{Bmatrix} x\,dx - \left[\begin{Bmatrix} N_1^{(e)} \\ N_2^{(e)} \end{Bmatrix} \frac{d\tilde{\phi}^{(e)}(x)}{dx} \right]_{x_1^{(e)}}^{x_2^{(e)}} \tag{6.23}$$

Evaluation of these integrals leads to the final form of the element characteristic matrix, $\mathbf{k}^{(e)}$, and the right-hand-side vector, $\mathbf{f}^{(e)}$

$$\mathbf{k}^{(e)} = -\frac{1}{\left(x_2^{(e)} - x_1^{(e)}\right)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\left(x_2^{(e)} - x_1^{(e)}\right)}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{6.24}$$

and

$$\begin{aligned} \mathbf{f}^{(e)} = & -\frac{1}{6}\left(x_2^{(e)} - x_1^{(e)}\right) \begin{Bmatrix} 2x_1^{(e)} + x_2^{(e)} \\ x_1^{(e)} + 2x_2^{(e)} \end{Bmatrix} \\ & - \left[\frac{d\tilde{\phi}^{(e)}\left(x_2^{(e)}\right)}{dx} \begin{Bmatrix} N_1^{(e)}\left(x_2^{(e)}\right) \\ N_2^{(e)}\left(x_2^{(e)}\right) \end{Bmatrix} - \frac{d\tilde{\phi}^{(e)}\left(x_1^{(e)}\right)}{dx} \begin{Bmatrix} N_1^{(e)}\left(x_1^{(e)}\right) \\ N_2^{(e)}\left(x_1^{(e)}\right) \end{Bmatrix} \right] \end{aligned} \tag{6.25}$$

or

$$\mathbf{f}^{(e)} = -\frac{1}{6}\left(x_2^{(e)} - x_1^{(e)}\right) \begin{Bmatrix} 2x_1^{(e)} + x_2^{(e)} \\ x_1^{(e)} + 2x_2^{(e)} \end{Bmatrix} - \begin{Bmatrix} -\dfrac{d\tilde{\phi}^{(e)}}{dx}\left(x_1^{(e)}\right) \\ \dfrac{d\tilde{\phi}^{(e)}}{dx}\left(x_2^{(e)}\right) \end{Bmatrix} \tag{6.26}$$

The local and global nodes for the domain discretized with three elements, $E = 3$, and four nodes, $N = 4$, are numbered as shown in Table 6.1.

With the appropriate value of the nodal coordinates from Eq. (6.24) and (6.26), the element characteristic matrices and vectors are calculated as

$$\mathbf{k}^{(1)} = \frac{1}{18} \begin{bmatrix} 52 & -55 \\ -55 & 52 \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \end{matrix} \qquad \begin{matrix} \boxed{1} & \boxed{2} \end{matrix} \tag{6.27}$$

$$\mathbf{k}^{(2)} = \frac{1}{18} \begin{bmatrix} 52 & -55 \\ -55 & 52 \end{bmatrix} \begin{matrix} \boxed{2} \\ \boxed{3} \end{matrix} \qquad \begin{matrix} \boxed{2} & \boxed{3} \end{matrix} \tag{6.28}$$

Table 6.1 Element connectivity and nodal coordinates

Element Number (e)	Node 1	Node 2	$x_1^{(e)}$	$x_2^{(e)}$
1	1	2	0	1/3
2	2	3	1/3	2/3
3	3	4	2/3	1

$$\mathbf{k}^{(3)} = \frac{1}{18}\begin{bmatrix} 52 & -55 \\ -55 & 52 \end{bmatrix} \begin{matrix} \boxed{3} \\ \boxed{4} \end{matrix} \qquad \begin{matrix} \boxed{3} & \boxed{4} \end{matrix} \tag{6.29}$$

$$\mathbf{f}^{(1)} = \frac{1}{54}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} + \begin{Bmatrix} -\dfrac{d\tilde{\phi}^{(1)}(0)}{dx} \\ \dfrac{d\tilde{\phi}^{(1)}(1/3)}{dx} \end{Bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \end{matrix} \tag{6.30}$$

$$\mathbf{f}^{(2)} = \frac{1}{54}\begin{Bmatrix} 4 \\ 5 \end{Bmatrix} + \begin{Bmatrix} -\dfrac{d\tilde{\phi}^{(2)}(1/3)}{dx} \\ \dfrac{d\tilde{\phi}^{(2)}(2/3)}{dx} \end{Bmatrix} \begin{matrix} \boxed{2} \\ \boxed{3} \end{matrix} \tag{6.31}$$

$$\mathbf{f}^{(3)} = \frac{1}{54}\begin{Bmatrix} 7 \\ 8 \end{Bmatrix} + \begin{Bmatrix} -\dfrac{d\tilde{\phi}^{(3)}(2/3)}{dx} \\ \dfrac{d\tilde{\phi}^{(3)}(1)}{dx} \end{Bmatrix} \begin{matrix} \boxed{3} \\ \boxed{4} \end{matrix} \tag{6.32}$$

As reflected by the element connectivity in Table 6.1, the boxed numbers indicate the rows and columns of the global matrix, $\mathbf{K}$, and global right-hand-side vector, $\mathbf{F}$, to which the individual coefficients are added. The global coefficient matrix, $\mathbf{K}$, and the global right-hand-side vector, $\mathbf{F}$, are obtained from the "expanded" element coefficient matrices, $\mathbf{k}^{(e)}$, and the element right-hand-side vectors, $\mathbf{f}^{(e)}$, by summation in the form

$$\mathbf{K} = \sum_{e=1}^{E} \mathbf{k}^{(e)} \quad \text{and} \quad \mathbf{F} = \sum_{e=1}^{E} \mathbf{f}^{(e)} \tag{6.33}$$

The "expanded" element matrices are the same size as the global matrix but have rows and columns of zeros corresponding to the nodes not associated with ele-

ment (e). Specifically, the expanded form of the element stiffness and load vector becomes

$$\mathbf{k}^{(1)} = \frac{1}{18}\begin{matrix}\\ \boxed{1}\\ \boxed{2}\\ \boxed{3}\\ \boxed{4}\end{matrix}\begin{matrix}\begin{matrix}\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4}\end{matrix}\\ \begin{bmatrix} 52 & -55 & 0 & 0\\ -55 & 52 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{matrix} \tag{6.34}$$

$$\mathbf{f}^{(1)} = \frac{1}{54}\begin{Bmatrix}1\\2\\0\\0\end{Bmatrix} + \begin{Bmatrix} -\dfrac{d\tilde{\phi}^{(1)}(0)}{dx} \\ \dfrac{d\tilde{\phi}^{(1)}(1/3)}{dx} \\ 0 \\ 0 \end{Bmatrix}\begin{matrix}\boxed{1}\\ \boxed{2}\\ \boxed{3}\\ \boxed{4}\end{matrix} \tag{6.35}$$

$$\mathbf{k}^{(2)} = \frac{1}{18}\begin{matrix}\\ \boxed{1}\\ \boxed{2}\\ \boxed{3}\\ \boxed{4}\end{matrix}\begin{matrix}\begin{matrix}\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4}\end{matrix}\\ \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 52 & -55 & 0\\ 0 & -55 & 52 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{matrix} \tag{6.36}$$

$$\mathbf{f}^{(2)} = \frac{1}{54}\begin{Bmatrix}0\\4\\5\\0\end{Bmatrix} + \begin{Bmatrix} 0 \\ -\dfrac{d\tilde{\phi}^{(2)}(1/3)}{dx} \\ \dfrac{d\tilde{\phi}^{(2)}(2/3)}{dx} \\ 0 \end{Bmatrix}\begin{matrix}\boxed{1}\\ \boxed{2}\\ \boxed{3}\\ \boxed{4}\end{matrix} \tag{6.37}$$

$$\mathbf{k}^{(3)} = \frac{1}{18}\begin{matrix}\\ \boxed{1}\\ \boxed{2}\\ \boxed{3}\\ \boxed{4}\end{matrix}\begin{matrix}\begin{matrix}\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4}\end{matrix}\\ \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 52 & -55\\ 0 & 0 & -55 & 52\end{bmatrix}\end{matrix} \tag{6.38}$$

$$\mathbf{f}^{(3)} = \frac{1}{54}\begin{Bmatrix}0\\0\\7\\8\end{Bmatrix} + \begin{Bmatrix}0 & \boxed{1}\\0 & \boxed{2}\\-\dfrac{d\tilde{\phi}^{(3)}(2/3)}{dx} & \boxed{3}\\\dfrac{d\tilde{\phi}^{(3)}(1)}{dx} & \boxed{4}\end{Bmatrix} \tag{6.39}$$

In accordance with Eq. (6.33) and (6.19), the assembly of the element characteristic matrices and vectors results in the global equilibrium equations

$$\frac{1}{18}\begin{bmatrix}52 & -55 & 0 & 0\\-55 & 52+52 & -55 & 0\\0 & -55 & 52+52 & -55\\0 & 0 & -55 & 52\end{bmatrix}\begin{Bmatrix}\phi_1 = \phi_1^{(1)}\\\phi_2 = \phi_2^{(1)} = \phi_1^{(2)}\\\phi_3 = \phi_2^{(2)} = \phi_1^{(3)}\\\phi_4 = \phi_2^{(3)}\end{Bmatrix} = \frac{1}{54}\begin{Bmatrix}1\\2+4\\5+7\\8\end{Bmatrix}$$
$$+\begin{Bmatrix}-\dfrac{d\tilde{\phi}^{(1)}(0)}{dx}\\\dfrac{d\tilde{\phi}^{(1)}(1/3)}{dx} - \cancelto{0}{\dfrac{d\tilde{\phi}^{(2)}(1/3)}{dx}}\\\dfrac{d\tilde{\phi}^{(2)}(2/3)}{dx} - \cancelto{0}{\dfrac{d\tilde{\phi}^{(3)}(2/3)}{dx}}\\\dfrac{d\tilde{\phi}^{(3)}(1)}{dx}\end{Bmatrix} \tag{6.40}$$

or

$$\frac{1}{18}\begin{bmatrix}52 & -55 & 0 & 0\\-55 & 104 & -55 & 0\\0 & -55 & 104 & -55\\0 & 0 & -55 & 52\end{bmatrix}\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix} = \frac{1}{54}\begin{Bmatrix}1\\6\\12\\8\end{Bmatrix} + \begin{Bmatrix}-\dfrac{d\tilde{\phi}^{(1)}(0)}{dx}\\0\\0\\\dfrac{d\tilde{\phi}^{(3)}(1)}{dx}\end{Bmatrix} \tag{6.41}$$

or

$$\mathbf{K}\Phi = \mathbf{F} \tag{6.42}$$

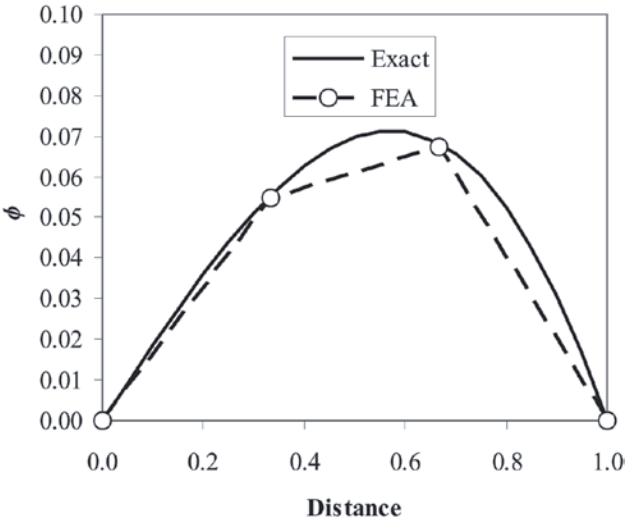

Fig. 6.3 Comparison of the exact and FEA (approximate) solutions to the 1D differential equation

After imposing the essential boundary conditions, $\phi_1 = 0$ and $\phi_4 = 0$, the global system of equations is reduced by deleting the row and column corresponding to ϕ_1 and ϕ_4, leading to

$$\frac{1}{18}\begin{bmatrix} 104 & -55 \\ -55 & 104 \end{bmatrix}\begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \frac{1}{54}\begin{Bmatrix} 6 \\ 12 \end{Bmatrix} \tag{6.43}$$

Its solution yields

$$\begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0.05493 \\ 0.06751 \end{Bmatrix} \tag{6.44}$$

The exact solution to the differential equation given by

$$\phi(x) = \frac{\sin(x)}{\sin(1)} - x \tag{6.45}$$

provides

$$\begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0.0555 \\ 0.0682 \end{Bmatrix} \tag{6.46}$$

The exact and FEM calculations of ϕ along the x-axis are shown in Fig. 6.3.

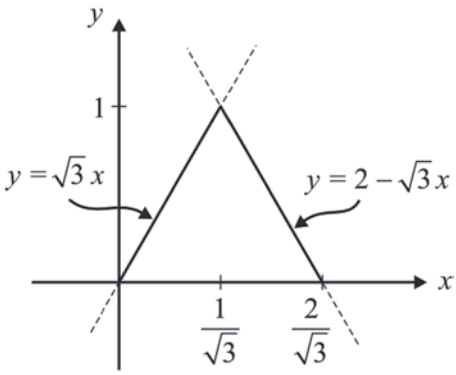

Fig. 6.4 The equilateral triangular domain

6.1.2 Example: Two-Dimensional Differential Equation with Linear Triangular Elements

6.1.2.1 Galerkin's Method

The application of Galerkin's method in solving two-dimensional problems with linear triangular elements is demonstrated by considering the partial differential equation given by

$$\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{\partial^2\phi(x,y)}{\partial y^2}+A=0 \tag{6.47}$$

in domain D, defined by the intersection of $y=0, y=2-\sqrt{3}x$, and $y=\sqrt{3}x$ (as shown in Fig. 6.4), where $A=1$.

The boundary conditions are specified as

$$-n_y\frac{\partial\phi(x,y=0)}{\partial y}=[\phi(x,y=0)-(B=1)] \text{ for } 0\le x\le 2/\sqrt{3} \tag{6.48}$$

$$\phi(x,y=\sqrt{3}x)=0 \quad \text{for} \quad 0\le x\le 1/\sqrt{3} \tag{6.49}$$

$$\phi(x,y=2-\sqrt{3}x)=0 \text{ for } 1/\sqrt{3}\le x\le 2/\sqrt{3} \tag{6.50}$$

When independent of time, these equations provide the temperature field, $\phi(x,y)$, due to heat conduction in a domain having a heat generation of A with one of its boundaries subjected to a convective heat transfer. The thermal conductivity and the film (surface) heat transfer coefficient are equal to unity, and the temperature of the surrounding medium is B.

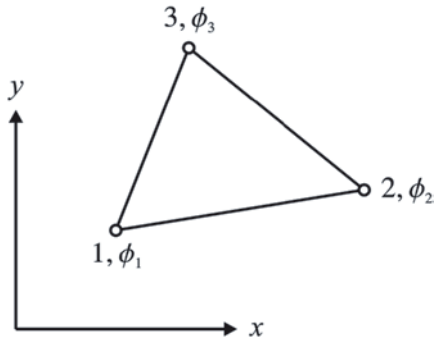

Fig. 6.5 Local node numbering for the linear triangular element

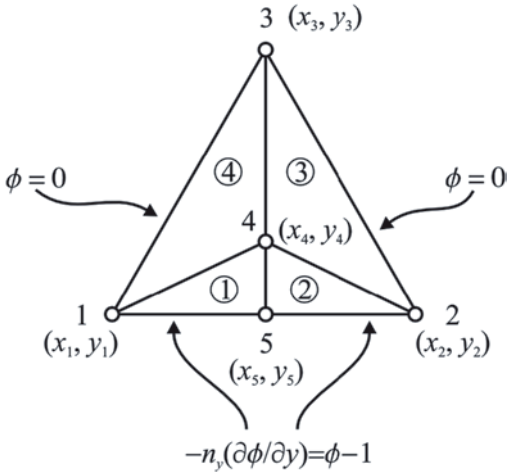

Fig. 6.6 Finite element discretization of the domain

The triangular domain can be discretized into four linear triangular elements, each having three nodes identified as 1, 2, and 3 (local node numbering), as illustrated in Fig. 6.5.

As shown in Fig. 6.6, the global coordinates of each node in domain D are specified by (x_i, y_i), with $i = 1, 2, 3, 4$, and 5. These coordinates are presented in Table 6.2.

The nodal values of the dependent variable associated with the global coordinates are denoted by ϕ_i ($i = 1, 2, 3, 4$, and 5). As shown in Fig. 6.5, the nodal values of the dependent variable associated with element e are specified at its first, second, and third nodes by $\phi_1^{(e)}$, $\phi_2^{(e)}$, and $\phi_3^{(e)}$, respectively.

The linear element approximation function for the dependent field variable in a triangular element "e" is written as

$$\tilde{\phi}^{(e)} = N_1^{(e)}\phi_1^{(e)} + N_2^{(e)}\phi_2^{(e)} + N_3^{(e)}\phi_3^{(e)} \tag{6.51}$$

Table 6.2 Nodal coordinates

Global node number	Nodal coordinates	Nodal unknowns
1	$(0,0)$	ϕ_1
2	$(2/\sqrt{3},0)$	ϕ_2
3	$(1/\sqrt{3},1)$	ϕ_3
4	$(1/\sqrt{3},1/3)$	ϕ_4
5	$(1/\sqrt{3},0)$	ϕ_5

or

$$\tilde{\phi}^{(e)} = \mathbf{N}^{(e)T}\boldsymbol{\varphi}^{(e)} \tag{6.52}$$

As derived in Chap. 3, the element shape functions in Eq. (3.17) are taken as

$$\begin{Bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \end{Bmatrix} = \frac{1}{2\Delta^{(e)}} \begin{bmatrix} (x_2y_3 - x_3y_2) & y_{23} & x_{32} \\ (x_3y_1 - x_1y_3) & y_{31} & x_{13} \\ (x_1y_2 - x_2y_1) & y_{12} & x_{21} \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \tag{6.53}$$

where $x_{mn} = x_m - x_n$, $y_{mn} = y_m - y_n$, and $\Delta^{(e)}$ is the area of the element computed by

$$2\Delta^{(e)} = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{6.54}$$

Applying Eq. (6.6), Galerkin's method, leads to

$$\sum_{e=1}^{E} \int_{D^{(e)}} \mathbf{N}^{(e)} \left(\frac{\partial^2 \tilde{\phi}^{(e)}(x,y)}{\partial x^2} + \frac{\partial^2 \tilde{\phi}^{(e)}(x,y)}{\partial y^2} + A \right) dx\, dy = 0 \tag{6.55}$$

Since the element approximation function is C^0 continuous, the second-order derivatives in the integrand must be reduced by one so that the inter-element continuity is achieved during the assembly of the global matrix. This reduction is achieved by observing that

$$\mathbf{N}^{(e)} \frac{\partial^2 \tilde{\phi}^{(e)}}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\mathbf{N}^{(e)} \frac{\partial \tilde{\phi}^{(e)}}{\partial x}(x,y) \right) - \frac{\partial \mathbf{N}^{(e)}}{\partial x} \frac{\partial \tilde{\phi}^{(e)}}{\partial x}(x,y) \tag{6.56}$$

and

$$\mathbf{N}^{(e)}\frac{\partial^2\tilde{\boldsymbol{\phi}}^{(e)}}{\partial y^2}(x,y)=\frac{\partial}{\partial y}\left(\mathbf{N}^{(e)}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial y}(x,y)\right)-\frac{\partial\mathbf{N}^{(e)}}{\partial y}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial y}(x,y) \tag{6.57}$$

Their substitution into the integrand in Eq. (6.55) and rearrangement of the terms result in

$$\sum_{e=1}^{E}\left\{\int_{D^{(e)}}\left[\frac{\partial}{\partial x}\left(\mathbf{N}^{(e)}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mathbf{N}^{(e)}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial y}\right)\right]dx\,dy\right.$$
$$\left.+\int_{D^{(e)}}\left[-\frac{\partial\mathbf{N}^{(e)}}{\partial x}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial x}-\frac{\partial\mathbf{N}^{(e)}}{\partial y}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial y}+\mathbf{N}^{(e)}A\right]dx\,dy\right\}=0 \tag{6.58}$$

Applying the divergence theorem to the first integral renders the domain integral to the boundary integral, and it yields

$$\sum_{e=1}^{E}\left\{\oint_{C^{(e)}}\left[\left(\mathbf{N}^{(e)}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial x}\right)n_x^{(e)}+\left(\mathbf{N}^{(e)}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial y}\right)n_y^{(e)}\right]ds\right.$$
$$\left.+\int_{D^{(e)}}\left[-\frac{\partial\mathbf{N}^{(e)}}{\partial x}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial x}-\frac{\partial\mathbf{N}^{(e)}}{\partial y}\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial y}+\mathbf{N}^{(e)}A\right]dx\,dy\right\}=0 \tag{6.59}$$

where $n_x^{(e)}$ and $n_y^{(e)}$ are, respectively, the x- and y-components of the outward normal vector along the closed boundary defining the area of the element, $C^{(e)}$.

Substituting for the element approximation function yields

$$\sum_{e=1}^{E}\left\{\oint_{C^{(e)}}\mathbf{N}^{(e)}\left[\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial x}n_x^{(e)}+\frac{\partial\tilde{\boldsymbol{\phi}}^{(e)}}{\partial y}n_y^{(e)}\right]ds\right.$$
$$-\int_{D^{(e)}}\left[\frac{\partial\mathbf{N}^{(e)}}{\partial x}\frac{\partial\mathbf{N}^{(e)T}}{\partial x}+\frac{\partial\mathbf{N}^{(e)}}{\partial y}\frac{\partial\mathbf{N}^{(e)T}}{\partial y}\right]dx\,dy\boldsymbol{\varphi}^{(e)} \tag{6.60}$$
$$\left.+\int_{D^{(e)}}\mathbf{N}^{(e)}A\,dx\,dy\right\}=0$$

This equation can be recast in matrix form as

$$\sum_{e=1}^{E}(-\mathbf{k}^{(e)}\boldsymbol{\varphi}^{(e)}+\mathbf{f}^{(e)}+\mathbf{Q}^{(e)})=0 \tag{6.61}$$

where

$$\mathbf{k}^{(e)}=\int_{D^{(e)}}\left[\frac{\partial\mathbf{N}^{(e)}}{\partial x}\frac{\partial\mathbf{N}^{(e)T}}{\partial x}+\frac{\partial\mathbf{N}^{(e)}}{\partial y}\frac{\partial\mathbf{N}^{(e)T}}{\partial y}\right]dx\,dy \tag{6.62}$$

$$\mathbf{f}^{(e)}=\int_{D^{(e)}}\mathbf{N}^{(e)}A\,dx\,dy \tag{6.63}$$

$$\mathbf{Q}^{(e)}=\oint_{C^{(e)}}\mathbf{N}^{(e)}\left[\frac{\partial\tilde{\phi}^{(e)}}{\partial x}n_x^{(e)}+\frac{\partial\tilde{\phi}^{(e)}}{\partial y}n_y^{(e)}\right]ds \tag{6.64}$$

in which $\mathbf{k}^{(e)}$ is the element characteristic matrix, $\mathbf{f}^{(e)}$ is the element right-hand-side vector, and $\mathbf{Q}^{(e)}$ is often referred to as the inter-element vector that includes the derivative terms along the boundary of the element. The boundary integral around each element is evaluated in a counterclockwise direction, i.e., this boundary integral is the sum of three integrals taken along each side of the element.

Depending on whether the element has an exterior boundary or not, the inter-element vector is divided into two parts

$$\mathbf{Q}^{(e)}=\mathbf{Q}_e^{(e)}+\mathbf{Q}_i^{(e)} \tag{6.65}$$

in which $\mathbf{Q}_e^{(e)}$ represents the contribution of the derivative terms specified along the external boundary of the element $C_e^{(e)}$, and $\mathbf{Q}_i^{(e)}$ represents the contribution from the internal boundaries of the element shared with other adjacent elements. Because each of the boundary integrals is evaluated in a counterclockwise direction, the contributions coming from the vector $\mathbf{Q}_i^{(e)}$ vanish when the global system of equations are assembled, thus no further discussion is necessary. However, in the case of specified derivative boundary conditions, the contribution coming from $\mathbf{Q}_e^{(e)}$ must be included.

In view of the boundary conditions given by Eq. (6.48) and the discretization of the domain, the 1–5 side of element 1 and the 5–2 side of element 2 are subjected to derivative boundary conditions.

With $n_x^{(1)}=n_x^{(2)}=0$ and $n_y^{(1)}=n_y^{(2)}=-1$, the contribution of the derivative boundary conditions appearing in Eq. (6.64) leads to the inter-element vectors as

$$\mathbf{Q}_e^{(1)}=\oint_{C_{1-5}^{(1)}}\mathbf{N}^{(1)}\left[B-\phi_C\right]ds \quad\text{and}\quad \mathbf{Q}_e^{(2)}=\oint_{C_{5-2}^{(2)}}\mathbf{N}^{(2)}\left[B-\phi_C\right]ds \tag{6.66}$$

where ϕ_C is the *unknown* value of the field variable on the external boundary of the element C_e, along which the derivative boundary condition is specified.

Approximating the unknown field variable, ϕ_C, by $\tilde{\phi}^{(e)} = \mathbf{N}^{(e)T}\boldsymbol{\varphi}^{(e)}$ in these equations leads to

$$\mathbf{Q}_e^{(1)} = \oint_{C_{1-5}^{(1)}} \mathbf{N}^{(1)}\left[B - \mathbf{N}^{(1)T}\boldsymbol{\varphi}^{(1)}\right]ds \tag{6.67a}$$

and

$$\mathbf{Q}_e^{(2)} = \oint_{C_{5-2}^{(2)}} \mathbf{N}^{(2)}\left[B - \mathbf{N}^{(2)T}\boldsymbol{\varphi}^{(2)}\right]ds \tag{6.67b}$$

which can be rewritten as

$$\mathbf{Q}_e^{(1)} = \oint_{C_{1-5}^{(1)}} \mathbf{N}^{(1)}Bds - \left\{\oint_{C_{1-5}^{(1)}} \mathbf{N}^{(1)}\mathbf{N}^{(1)T}ds\right\}\boldsymbol{\varphi}^{(1)} \text{ or } \mathbf{Q}_e^{(1)} = \mathbf{g}^{(1)} - \mathbf{h}^{(1)}\boldsymbol{\varphi}^{(1)} \tag{6.68}$$

and

$$\mathbf{Q}_e^{(2)} = \oint_{C_{5-2}^{(2)}} \mathbf{N}^{(2)}Bds - \left\{\oint_{C_{5-2}^{(2)}} \mathbf{N}^{(2)}\mathbf{N}^{(2)T}Bds\right\}\boldsymbol{\varphi}^{(2)} \tag{6.69a}$$

or

$$\mathbf{Q}_e^{(2)} = \mathbf{g}^{(2)} - \mathbf{h}^{(2)}\boldsymbol{\varphi}^{(2)} \tag{6.69b}$$

where

$$\mathbf{h}^{(1)} = \oint_{C_{1-5}^{(1)}} \mathbf{N}^{(1)}\mathbf{N}^{(1)T}ds \quad \text{and} \quad \mathbf{h}^{(2)} = \oint_{C_{2-5}^{(2)}} \mathbf{N}^{(2)}\mathbf{N}^{(2)T}ds \tag{6.70}$$

and

$$\mathbf{g}^{(1)} = \oint_{C_{1-5}^{(1)}} \mathbf{N}^{(1)}Bds \text{ and } \mathbf{g}^{(2)} = \oint_{C_{2-5}^{(2)}} \mathbf{N}^{(2)}Bds \tag{6.71}$$

With this representation of the inter-element vector, the element equilibrium equations given by Eq. (6.61) can be rewritten in their final form as

$$\begin{aligned}
&\left(\mathbf{k}^{(1)} + \mathbf{h}^{(1)}\right)\boldsymbol{\varphi}^{(1)} = \mathbf{f}^{(1)} + \mathbf{g}(1)\\
&\left(\mathbf{k}^{(2)} + \mathbf{h}^{(2)}\right)\boldsymbol{\varphi}^{(2)} = \mathbf{f}^{(2)} + \mathbf{g}^{(2)}\\
&\mathbf{k}^{(3)}\boldsymbol{\varphi}^{(3)} = \mathbf{f}^{(3)}\\
&\mathbf{k}^{(4)}\boldsymbol{\varphi}^{(4)} = \mathbf{f}^{(4)}
\end{aligned} \tag{6.72}$$

With the derivatives of the shape functions obtained as

$$\begin{Bmatrix} \dfrac{\partial N_1^{(e)}}{\partial x} \\ \dfrac{\partial N_2^{(e)}}{\partial x} \\ \dfrac{\partial N_3^{(e)}}{\partial x} \end{Bmatrix} = \frac{1}{2\Delta^{(e)}} \begin{Bmatrix} y_{23} \\ y_{31} \\ y_{12} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dfrac{\partial N_1^{(e)}}{\partial y} \\ \dfrac{\partial N_2^{(e)}}{\partial y} \\ \dfrac{\partial N_3^{(e)}}{\partial y} \end{Bmatrix} = \frac{1}{2\Delta^{(e)}} \begin{Bmatrix} x_{32} \\ x_{13} \\ x_{21} \end{Bmatrix} \tag{6.73}$$

the evaluation of the area integrals in Eq. (6.62) and (6.63) by using Eq. (3.19) leads to the final form of the element coefficient matrix, $\mathbf{k}^{(e)}$, and right-hand-side vector, $\mathbf{f}^{(e)}$

$$\mathbf{k}^{(e)} = \frac{1}{4\Delta^{(e)}} \begin{bmatrix} x_{32}^2 + y_{23}^2 & x_{32}x_{13} + y_{23}y_{31} & x_{32}x_{21} + y_{23}y_{12} \\ x_{32}x_{13} + y_{23}y_{31} & x_{13}^2 + y_{31}^2 & x_{13}x_{21} + y_{31}y_{12} \\ x_{32}x_{21} + y_{23}y_{12} & x_{13}x_{21} + y_{31}y_{12} & x_{21}^2 + y_{12}^2 \end{bmatrix} \tag{6.74}$$

and

$$\mathbf{f}^{(e)} = \frac{A\Delta^{(e)}}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{6.75}$$

Their numerical evaluation results in

$$\mathbf{k}^{(1)} = \frac{\sqrt{3}}{6} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & -3 \\ 0 & -3 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{f}^{(1)} = \frac{1}{18\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{6.76}$$

$$\mathbf{k}^{(2)} = \frac{\sqrt{3}}{6} \begin{bmatrix} 4 & -1 & -3 \\ -1 & 1 & 0 \\ -3 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{f}^{(2)} = \frac{1}{18\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{6.77}$$

$$\mathbf{k}^{(3)} = \frac{\sqrt{3}}{12} \begin{bmatrix} 4 & 2 & -6 \\ 2 & 4 & -6 \\ -6 & -6 & 12 \end{bmatrix} \quad \text{and} \quad \mathbf{f}^{(3)} = \frac{1}{9\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{6.78}$$

$$\mathbf{k}^{(4)} = \frac{\sqrt{3}}{12}\begin{bmatrix} 4 & 2 & -6 \\ 2 & 4 & -6 \\ -6 & -6 & 12 \end{bmatrix} \quad \text{and} \quad \mathbf{f}^{(4)} = \frac{1}{9\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{6.79}$$

in which the area of each element is computed as

$$\Delta^{(1)} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 1/3 \end{vmatrix} = \frac{1}{6\sqrt{3}} \tag{6.80}$$

$$\Delta^{(2)} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 1/3 \end{vmatrix} = \frac{1}{6\sqrt{3}} \tag{6.81}$$

$$\Delta^{(3)} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1 & 1/3 \end{vmatrix} = \frac{1}{3\sqrt{3}} \tag{6.82}$$

$$\Delta^{(4)} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1 & 0 & 1/3 \end{vmatrix} = \frac{1}{3\sqrt{3}} \tag{6.83}$$

Associated with the inter-element vector, the boundary integrals in Eq. (6.70) and (6.71) are rewritten as

$$\mathbf{h}^{(1)} = \oint_{C_{1-5}^{(1)}} \begin{bmatrix} N_1^{(1)}N_1^{(1)} & N_1^{(1)}N_2^{(1)} & N_1^{(1)}N_3^{(1)} \\ N_2^{(1)}N_1^{(1)} & N_2^{(1)}N_2^{(1)} & N_2^{(1)}N_3^{(1)} \\ N_3^{(1)}N_1^{(1)} & N_3^{(1)}N_2^{(1)} & N_3^{(1)}N_3^{(1)} \end{bmatrix} ds \tag{6.84a}$$

$$\mathbf{g}^{(1)} = \oint_{C_{1-5}^{(1)}} \begin{Bmatrix} N_1^{(1)} \\ N_2^{(1)} \\ N_3^{(1)} \end{Bmatrix} ds \tag{6.84b}$$

and

$$\mathbf{h}^{(2)} = \int_{C_{5-2}^{(2)}} \begin{bmatrix} N_1^{(2)}N_1^{(2)} & N_1^{(2)}N_2^{(2)} & N_1^{(2)}N_3^{(2)} \\ N_2^{(2)}N_1^{(2)} & N_2^{(2)}N_2^{(2)} & N_2^{(2)}N_3^{(2)} \\ N_3^{(2)}N_1^{(2)} & N_3^{(2)}N_2^{(2)} & N_3^{(2)}N_3^{(2)} \end{bmatrix} ds \tag{6.85a}$$

$$\mathbf{g}^{(2)} = \int_{C_{5-2}^{(2)}} \begin{Bmatrix} N_1^{(2)} \\ N_2^{(2)} \\ N_3^{(2)} \end{Bmatrix} ds \tag{6.85b}$$

in which $N_3^{(1)}$ and $N_3^{(2)}$ are zero along side 1–5 (with length L_{1-5}) and along side 5–2 (with length L_{2-5}), respectively. The remaining shape functions $N_1^{(1)}$, $N_2^{(1)}$, $N_1^{(2)}$, and $N_2^{(2)}$ reduce to a one-dimensional form as

$$N_1^{(1)} = \frac{L_{1-5} - s}{L_{1-5}} \quad \text{and} \quad N_2^{(1)} = \frac{s}{L_{1-5}} \tag{6.86}$$

$$N_1^{(2)} = \frac{L_{5-2} - s}{L_{5-2}} \quad \text{and} \quad N_2^{(2)} = \frac{s}{L_{5-2}} \tag{6.87}$$

in which s is the local coordinate in the range of $(0 \le s \le L_{1-5})$ along side 1–5 and $(0 \le s \le L_{5-2})$ along side 5–2, $L_{1-5} = 1/\sqrt{3}$, and $L_{5-2} = 1/\sqrt{3}$. With these shape functions, the evaluation of $\mathbf{h}^{(1)}$, $\mathbf{g}^{(1)}$, $\mathbf{h}^{(2)}$, and $\mathbf{g}^{(2)}$ leads to

$$\mathbf{h}^{(1)} = \frac{1}{6\sqrt{3}} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{g}^{(1)} = \frac{1}{2\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{6.88}$$

and

$$\mathbf{h}^{(2)} = \frac{1}{6\sqrt{3}} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{g}^{(2)} = \frac{1}{2\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{6.89}$$

Considering the correspondence between the local and global node numbering presented in Table 6.3, the element characteristic matrices and vectors can be rewritten as

Table 6.3 Element connectivity

Element number (e)	Node 1	Node 2	Node 3
1	1	5	4
2	5	2	4
3	2	3	4
4	3	1	4

Element 1:

$$\begin{matrix} \boxed{1} & \boxed{5} & \boxed{4} \end{matrix}$$

$$\begin{bmatrix} k_{11}^{(1)}+h_{11}^{(1)} & k_{12}^{(1)}+h_{12}^{(1)} & k_{13}^{(1)}+h_{13}^{(1)} \\ k_{21}^{(1)}+h_{21}^{(1)} & k_{22}^{(1)}+h_{22}^{(1)} & k_{23}^{(1)}+h_{23}^{(1)} \\ k_{31}^{(1)}+h_{31}^{(1)} & k_{32}^{(1)}+h_{32}^{(1)} & k_{33}^{(1)}+h_{33}^{(1)} \end{bmatrix} \begin{Bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \\ \phi_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)}+g_1^{(1)} \\ f_2^{(1)}+g_2^{(1)} \\ f_3^{(1)}+g_3^{(1)} \end{Bmatrix} \begin{matrix} \boxed{1} \\ \boxed{5} \\ \boxed{4} \end{matrix} \tag{6.90}$$

Element 2:

$$\begin{matrix} \boxed{5} & \boxed{2} & \boxed{4} \end{matrix}$$

$$\begin{bmatrix} k_{11}^{(2)}+h_{11}^{(2)} & k_{12}^{(2)}+h_{12}^{(2)} & k_{13}^{(2)}+h_{13}^{(2)} \\ k_{21}^{(2)}+h_{21}^{(2)} & k_{22}^{(2)}+h_{22}^{(2)} & k_{23}^{(2)}+h_{23}^{(2)} \\ k_{31}^{(2)}+h_{31}^{(2)} & k_{32}^{(2)}+h_{32}^{(2)} & k_{33}^{(2)}+h_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \\ \phi_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(2)}+g_1^{(2)} \\ f_2^{(2)}+g_2^{(2)} \\ f_3^{(2)}+g_3^{(2)} \end{Bmatrix} \begin{matrix} \boxed{5} \\ \boxed{2} \\ \boxed{4} \end{matrix} \tag{6.91}$$

Element 3:

$$\begin{matrix} \boxed{2} & \boxed{3} & \boxed{4} \end{matrix}$$

$$\begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} & k_{13}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} & k_{23}^{(3)} \\ k_{31}^{(3)} & k_{32}^{(3)} & k_{33}^{(3)} \end{bmatrix} \begin{Bmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \\ \phi_3^{(3)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(3)} \\ f_2^{(3)} \\ f_3^{(3)} \end{Bmatrix} \begin{matrix} \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} \tag{6.92}$$

Element 4:

$$\begin{matrix} \boxed{3} & \boxed{1} & \boxed{4} \end{matrix}$$

$$\begin{bmatrix} k_{11}^{(4)} & k_{12}^{(4)} & k_{13}^{(4)} \\ k_{21}^{(4)} & k_{22}^{(4)} & k_{23}^{(4)} \\ k_{31}^{(4)} & k_{32}^{(4)} & k_{33}^{(4)} \end{bmatrix} \begin{Bmatrix} \phi_1^{(4)} \\ \phi_2^{(4)} \\ \phi_3^{(4)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(4)} \\ f_2^{(4)} \\ f_3^{(4)} \end{Bmatrix} \begin{matrix} \boxed{3} \\ \boxed{1} \\ \boxed{4} \end{matrix} \tag{6.93}$$

In the assembly of the element characteristic matrices and vectors, the boxed numbers indicate the rows and columns of the global matrix, **K**, and global right-hand-side vector, **F**, to which the individual coefficients are added, resulting in

$$\mathbf{K\Phi} = \mathbf{F} \tag{6.94}$$

where

$$\mathbf{K} = \begin{bmatrix} k_{11}^{(1)} + h_{11}^{(1)} + k_{22}^{(4)} & 0 & k_{21}^{(4)} \\ 0 & k_{22}^{(2)} + h_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ k_{12}^{(4)} & k_{21}^{(3)} & k_{22}^{(3)} + k_{11}^{(4)} \\ k_{31}^{(1)} + h_{31}^{(1)} + k_{32}^{(4)} & k_{32}^{(2)} + h_{32}^{(2)} + k_{31}^{(3)} & k_{32}^{(3)} + k_{31}^{(4)} \\ k_{21}^{(1)} + h_{21}^{(1)} & k_{12}^{(2)} + h_{12}^{(2)} & 0 \end{bmatrix} \tag{6.95a}$$

$$\begin{bmatrix} k_{13}^{(1)} + h_{13}^{(1)} + k_{23}^{(4)} & k_{12}^{(1)} + h_{12}^{(1)} \\ k_{23}^{(2)} + h_{23}^{(2)} + k_{13}^{(3)} & k_{21}^{(2)} + h_{21}^{(2)} \\ k_{23}^{(3)} + k_{13}^{(4)} & 0 \\ k_{33}^{(1)} + h_{33}^{(1)} + k_{33}^{(2)} + h_{33}^{(2)} + k_{33}^{(3)} + k_{33}^{(4)} & k_{32}^{(1)} + h_{32}^{(1)} + k_{31}^{(2)} + h_{31}^{(2)} \\ k_{23}^{(1)} + h_{23}^{(1)} + k_{13}^{(2)} + h_{13}^{(2)} & k_{22}^{(1)} + h_{22}^{(1)} + k_{11}^{(2)} + h_{11}^{(2)} \end{bmatrix}$$

$$\mathbf{F} = \begin{Bmatrix} f_1^{(1)} + g_1^{(1)} + f_2^{(4)} \\ f_2^{(2)} + g_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} + f_1^{(4)} \\ f_3^{(1)} + g_3^{(1)} + f_3^{(2)} + g_3^{(2)} + f_3^{(3)} + f_3^{(4)} \\ f_2^{(1)} + g_2^{(1)} + f_1^{(2)} + g_1^{(2)} \end{Bmatrix} \tag{6.95b}$$

$$\Phi = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} \tag{6.95c}$$

After imposing the essential boundary conditions, the global system of equations are reduced by deleting the rows and columns corresponding to ϕ_1, ϕ_2, and ϕ_3, leading to

$$\begin{bmatrix} k_{33}^{(1)} + h_{33}^{(1)} + k_{33}^{(2)} + h_{33}^{(2)} + k_{33}^{(3)} + k_{33}^{(4)} & k_{32}^{(1)} + h_{32}^{(1)} + k_{31}^{(2)} + h_{31}^{(2)} \\ k_{23}^{(1)} + h_{23}^{(1)} + k_{13}^{(2)} - h_{13}^{(2)} & k_{22}^{(1)} + h_{22}^{(1)} + k_{11}^{(2)} + h_{11}^{(2)} \end{bmatrix} \begin{Bmatrix} \phi_4 \\ \phi_5 \end{Bmatrix}$$
$$= \begin{Bmatrix} f_3^{(1)} + g_3^{(1)} + f_3^{(2)} + g_3^{(2)} + f_3^{(3)} + f_3^{(4)} \\ f_2^{(1)} + g_2^{(1)} + f_1^{(2)} + g_1^{(2)} \end{Bmatrix} \tag{6.96}$$

With the explicit values of the coefficients, the nodal unknowns, ϕ_4 and ϕ_5, are determined as

$$\phi_4 = \frac{4}{27} = 0.14815 \tag{6.97a}$$

$$\phi_5 = \frac{1}{3} = 0.33333 \tag{6.97b}$$

The expressions for $\mathbf{h}^{(1)}$ and $\mathbf{h}^{(2)}$ in Eq. (6.70) are derived based on a formulation consistent with the derivation of the element coefficient matrices, $\mathbf{k}^{(e)}$. An alternative to the consistent formulation is the use of lumped diagonal matrices and expressing $\mathbf{h}^{(1)}$ and $\mathbf{h}^{(2)}$ in the form

$$\mathbf{h}^{(1)} = \int_{C_{1-5}^{(1)}} \begin{bmatrix} N_1^{(1)} & 0 & 0 \\ 0 & N_2^{(1)} & 0 \\ 0 & 0 & N_3^{(1)} \end{bmatrix} ds = \frac{1}{6\sqrt{3}} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.98}$$

and

$$\mathbf{h}^{(2)} = \int_{C_{5-2}^{(2)}} \begin{bmatrix} N_1^{(2)} & 0 & 0 \\ 0 & N_2^{(2)} & 0 \\ 0 & 0 & N_3^{(2)} \end{bmatrix} ds = \frac{1}{6\sqrt{3}} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.99}$$

Replacing the components of $\mathbf{h}^{(1)}$ and $\mathbf{h}^{(2)}$ in Eq. (6.96) with the values obtained in Eq. (6.98) and (6.99), the nodal unknowns ϕ_4 and ϕ_5 are determined as

$$\phi_4 = \frac{5}{36} = 0.13889 \tag{6.100a}$$

$$\phi_5 = \frac{11}{36} = 0.30556 \tag{6.100b}$$

Note that the discrepancy in the value of ϕ_4 and ϕ_5 obtained from the two methods is due to the small number of elements in the discretization of the domain.

6.1.2.2 ANSYS Solution

The governing equations for a steady-state heat transfer, described by Eq. (6.47) through (6.50), also can be solved using ANSYS. The solution procedure is outlined as follows:

Model Generation

- Specify the element type (**ET** command) using the following menu path:

Main Menu > Preprocessor > Element Type > Add/Edit/Delete

  - Click on ***Add***.
  - Select ***Solid*** immediately below ***Thermal Mass*** from the left list and ***Quad 4node 55*** from the right list; click on ***OK***.
  - Click on ***Close***.

- Specify material properties (**MP** command) using the following menu path:

Main Menu > Preprocessor > Material Props > Material Models

  - In the *Define Material Model Behavior* dialog box, in the right window, successively left-click on ***Thermal***, ***Conductivity***, and, finally, ***Isotropic***, which brings up another dialog box.
  - Enter ***1*** for ***KXX***, and click on ***OK***.
  - Close the *Define Material Model Behavior* dialog box by using the following menu path:

Material > Exit

- Create nodes (**N** command) using the following menu path:

Main Menu > Preprocessor > Modeling > Create > Nodes > In Active CS

  - A total of 5 nodes are created (Table 6.2).
  - Referring to Table 6.2, enter x- and y-coordinates of node 1, and click on ***Apply***. This action keeps the *Create Nodes in Active Coordinate System* dialog box open. If the ***Node number*** field is left blank, then ANSYS assigns the lowest available node number to the node that is being created.
  - Repeat the same procedure for the nodes 2 through 5.
  - After entering the x- and y-coordinates of node 5, click on ***OK*** (instead of ***Apply***).
  - The nodes should appear in the *Graphics Window*, as shown in Fig. 6.7.

- Create elements (**E** command) using the following menu path:

Main Menu > Preprocessor > Modeling > Create > Elements > Auto Numbered > Thru Nodes

  - *Pick Menu* appears; refer to Fig. 6.8 to create elements by picking ***three*** nodes at a time and clicking on ***Apply*** in between.
  - Observe the elements created after clicking on ***Apply*** in the *Pick Menu*.

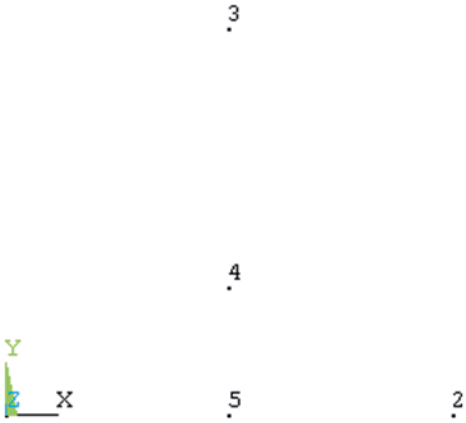

Fig. 6.7 Generation of nodes

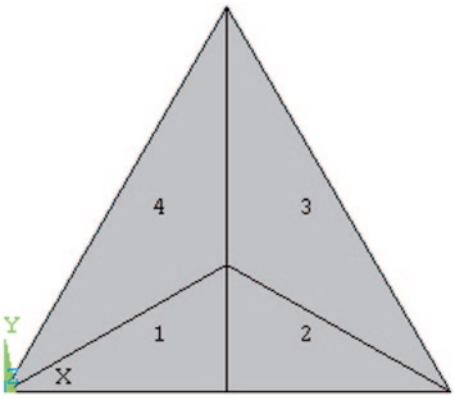

Fig. 6.8 Generation of elements

- Repeat until the last element is created.
- Click on ***OK*** when the last element is created.

- Review elements:
  - Turn on element numbering using the following menu path:

Utility Menu > PlotCtrls > Numbering

- Select ***Element numbers*** from the first pull-down menu; click on ***OK***.
- Plot elements (**`EPLOT`** command) using the following menu path:

Utility Menu > Plot > Elements

- Figure 6.8 shows the outcome of this action as it appears in the *Graphics Window*.
- Turn off element numbering and turn on node numbering using the following menu path:

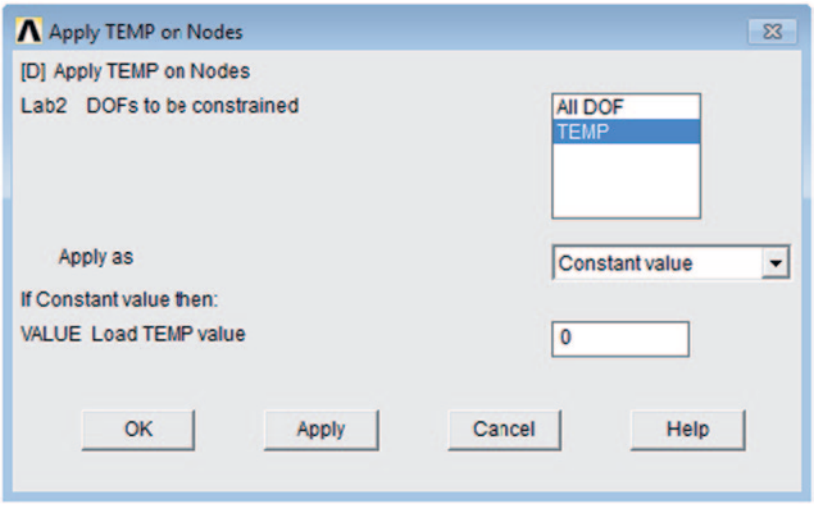

Fig. 6.9 Application of temperature boundary conditions on nodes

Utility Menu > PlotCtrls > Numbering

- Place a *checkmark* by clicking on the empty box next to ***NODE Node numbers***.
- Select ***No numbering*** from the first pull-down menu.
- Click on ***OK***.
- Plot nodes (**`NPLOT`** command) using the following menu path:

Utility Menu > Plot > Nodes

- Figure 6.7 shows the outcome of this action as it appears in the *Graphics Window*.

Solution

- Apply temperature boundary conditions (Dcommand) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Thermal > Temperature > On Nodes

- *Pick Menu* appears; pick nodes 1, 2, and 3 (Fig. 6.7); click on ***OK*** on *Pick Menu.*
- Highlight ***TEMP*** and enter **0** for ***VALUE***; click on ***OK*** (Fig. 6.9).
- Apply convection boundary conditions (**`SF`** command) using the following menu path:

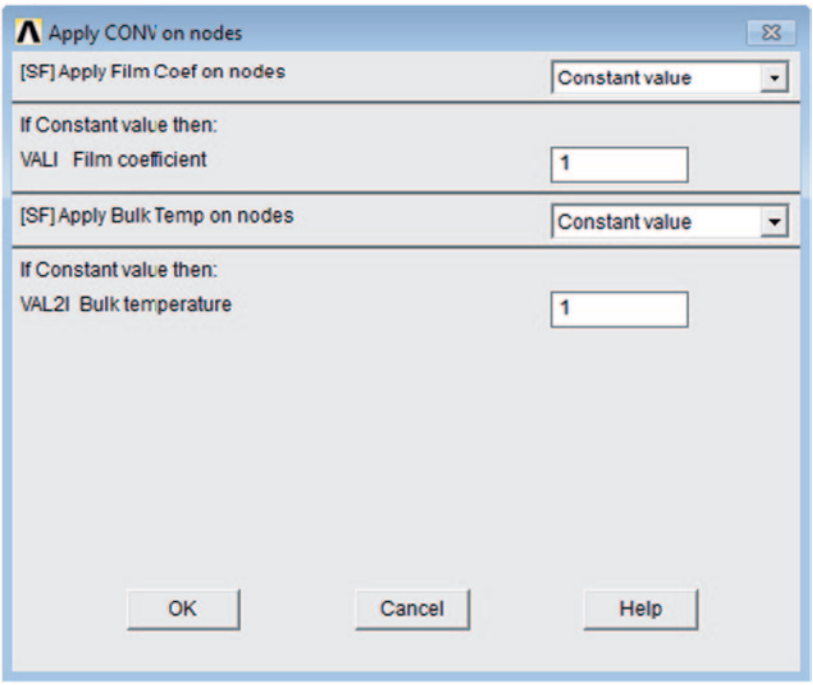

Fig. 6.10 Application of convection boundary conditions on nodes

Main Menu > Solution > Define Loads > Apply > Thermal > Convection > On Nodes

- *Pick Menu* appears; pick nodes 1, 2 and 5 along the boundary (Fig. 6.7); click on ***OK*** on *Pick Menu*.
- Enter ***1*** for both ***VALI Film coefficient*** and ***VAL2I Bulk temperature***; click on ***OK*** (Fig. 6.10).

- Apply body load on elements (**`BFE`** command) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Thermal > Heat Generat > On Elements

- *Pick Menu* appears; click on ***Pick All***.
- Enter ***1*** for ***VAL1*** leave other fields untouched, as shown in Fig. 6.11.
- Click on ***OK***.

- Obtain solution (**`SOLVE`** command) using the following menu path:

Main Menu > Solution > Solve >Current LS

- *Confirmation Window* appears along with *Status Report Window*.
- Review status. If OK, close the *Status Report Window* and click on ***OK*** in *Confirmation Window*.
- Wait until ANSYS responds with ***Solution is done!***

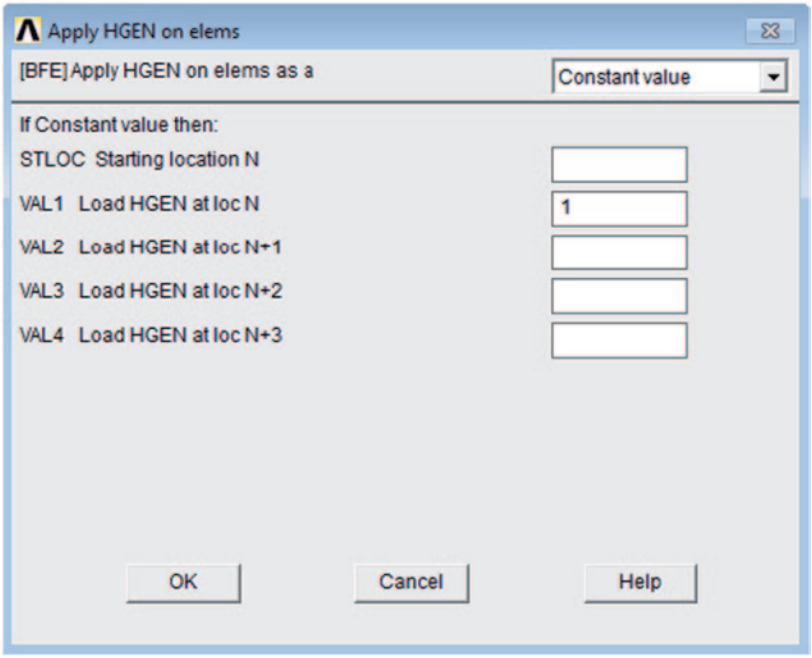

Fig. 6.11 Application of heat generation condition on elements

Postprocessing

- Review temperature values (**PRNSOL** command) using the following menu path:

Main Menu > General Postproc > List Results > Nodal Solution

  - Click on ***DOF Solution*** and ***Nodal Temperature***; click on ***OK***.
  - The list appears in a new window, as shown in Fig. 6.12.

6.1.3 Example: Two-Dimensional Differential Equation with Linear Quadrilateral Elements

6.1.3.1 Galerkin's Method

In solving two-dimensional problems with quadrilateral isoparametric elements, Galerkin's method is demonstrated by considering the partial differential equation given by

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} - A = 0 \tag{6.101}$$

PRNSOL Command

File

```
PRINT TEMP NODAL SOLUTION PER NODE

 ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

 LOAD STEP=     1  SUBSTEP=     1
  TIME=    1.0000      LOAD CASE=   0

   NODE      TEMP
      1    0.0000
      2    0.0000
      3    0.0000
      4   0.13889
      5   0.30556

MAXIMUM ABSOLUTE VALUES
NODE          5
VALUE    0.30556
```

Fig. 6.12 Nodal solution for temperature

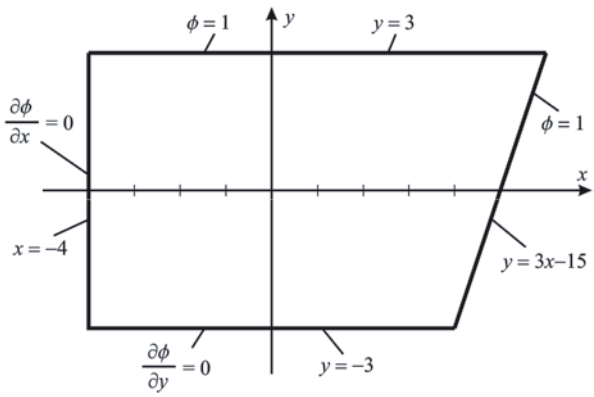

Fig. 6.13 Description of domain, and boundary conditions

in domain D defined by the intersection of $y = -3$, $x = -4$, $y = 3$, and $y = 3x - 15$. The constant, A, is known. As shown in Fig. 6.13, the flux vanishes along the boundary of the domain specified by $y = -3$ and $x = -4$, and along the remaining part of the boundary specified by $y = 3$, and $y = 3x - 15$, the dependent variable, $\phi(x, y)$, has a value of unity. These boundary conditions are expressed as

$$\phi(x, y) = 1 \quad \text{for} \quad 4 \le x \le 6, y = 3x - 15 \tag{6.102}$$

$$\frac{\partial}{\partial x}\phi(x, y = -3) = 0 \quad \text{for} \quad -4 \le x \le 4 \tag{6.103}$$

$$\frac{\partial}{\partial x}\phi(x = -4, y) = 0 \quad \text{for} \quad -3 \le y \le 3 \tag{6.104}$$

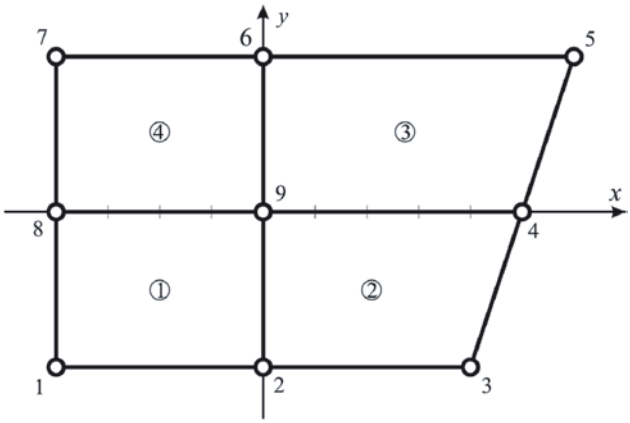

Fig. 6.14 FEM discretization of the domain into four quadrilaterals

$$\phi(x, y = 3) = 1 \quad \text{for} \quad -4 \le x \le 6 \tag{6.105}$$

The domain is discretized with four linear quadrilateral isoparametric elements, each having four nodes identified as 1, 2, 3, and 4, shown in Fig. 6.14. The nodal values of the dependent variable associated with element e are specified at its first, second, third, and fourth nodes by $\phi_1^{(e)}$, $\phi_2^{(e)}$, $\phi_3^{(e)}$, and $\phi_4^{(e)}$, respectively. The discretization of the domain with global node numbering is shown in Fig. 6.14. The global coordinates of the nodal values of the dependent variable denoted by ϕ_i $(i = 1, 2, \ldots, 9)$ are presented in Table 6.4.

The linear element approximation function for the dependent field variable in a quadrilateral isoparametric element "e" is written as

$$\tilde{\phi}^{(e)} = N_1^{(e)}\phi_1^{(e)} + N_2^{(e)}\phi_2^{(e)} + N_3^{(e)}\phi_3^{(e)} + N_4^{(e)}\phi_4^{(e)} \tag{6.106}$$

Table 6.4 Nodal coordinates

Global node number	Nodal coordinates	Nodal variables
1	(−4, −3)	ϕ_1
2	(0, −3)	ϕ_2
3	(4, −3)	ϕ_3
4	(5,0)	ϕ_4
5	(6,3)	ϕ_5
6	(0,3)	ϕ_6
7	(−4,3)	ϕ_7
8	(−4,0)	ϕ_8
9	(0,0)	ϕ_9

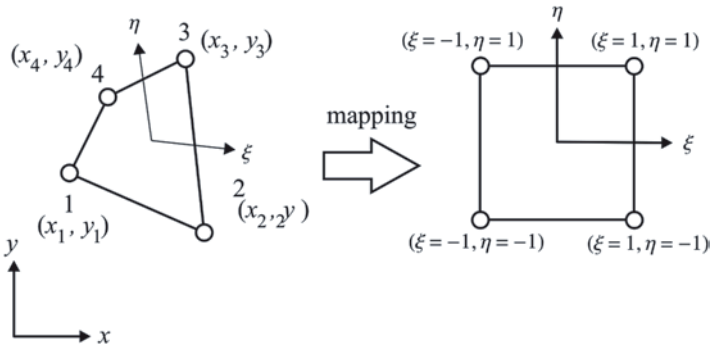

Fig. 6.15 Local node numbering for a linear isoparametric quadrilateral element

or

$$\tilde{\phi}^{(e)} = \mathbf{N}^{(e)T}\boldsymbol{\varphi}^{(e)} \tag{6.107}$$

where

$$\mathbf{N}^{(e)} = \begin{Bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \end{Bmatrix} \quad \text{and} \quad \boldsymbol{\varphi}^{(e)} = \begin{Bmatrix} \phi_1^{(e)} \\ \phi_2^{(e)} \\ \phi_3^{(e)} \\ \phi_4^{(e)} \end{Bmatrix} \tag{6.108}$$

in which the shape functions $N_1^{(e)}$, $N_2^{(e)}$, $N_3^{(e)}$, and $N_4^{(e)}$ are expressed in terms of the centroidal or natural coordinates, (ξ,η), shown in Fig. 6.15. For a linear (straight-sided) quadrilateral illustrated in Fig. 6.15, they are of the form

$$N_i^{(e)} = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \quad \text{with} \quad i = 1,2,3,4 \tag{6.109}$$

where ξ_i and η_i represent the coordinates of the corner nodes in the natural coordinate system, $(\xi_1 = -1, \eta_1 = -1)$, $(\xi_2 = 1, \eta_2 = -1)$, $(\xi_3 = 1, \eta_3 = 1)$, and $(\xi_4 = -1, \eta_4 = 1)$.

Applying Eq. (6.6), Galerkin's method, leads to

$$\sum_{e=1}^{E} \int_{D^{(e)}} \mathbf{N}^{(e)} \left(\frac{\partial^2 \tilde{\phi}^{(e)}(x,y)}{\partial x^2} + \frac{\partial^2 \tilde{\phi}^{(e)}(x,y)}{\partial y^2} - A \right) dx\, dy = 0 \tag{6.110}$$

Since the element approximation function is C^0 continuous, the second-order derivatives in the integrand must be reduced by one so that inter-element continuity is achieved during the assembly of the global matrix. This reduction is achieved by observing that

$$\mathbf{N}^{(e)} \frac{\partial^2 \tilde{\boldsymbol{\phi}}^{(e)}}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\mathbf{N}^{(e)} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial x}(x,y) \right) - \frac{\partial \mathbf{N}^{(e)}}{\partial x} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial x}(x,y) \tag{6.111}$$

and

$$\mathbf{N}^{(e)} \frac{\partial^2 \tilde{\boldsymbol{\phi}}^{(e)}}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\mathbf{N}^{(e)} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial y}(x,y) \right) - \frac{\partial \mathbf{N}^{(e)}}{\partial y} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial y}(x,y) \tag{6.112}$$

Their substitution into the integrand in Eq. (6.110) and rearrangement of the terms result in

$$\begin{aligned} \sum_{e=1}^{E} \Bigg\{ & \int_{D^{(e)}} \left[\frac{\partial}{\partial x}\left(\mathbf{N}^{(e)} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial x} \right) + \frac{\partial}{\partial y}\left(\mathbf{N}^{(e)} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial y} \right) \right] dx\, dy \\ & + \int_{D^{(e)}} \left[-\frac{\partial \mathbf{N}^{(e)}}{\partial x} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial x} - \frac{\partial \mathbf{N}^{(e)}}{\partial y} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial y} + \mathbf{N}^{(e)} A \right] dx\, dy \Bigg\} = 0 \end{aligned} \tag{6.113}$$

Applying the divergence theorem to the first integral renders the domain integral to the boundary integral, and it yields

$$\begin{aligned} \sum_{e=1}^{E} \Bigg\{ & \oint_{C^{(e)}} \left[\left(\mathbf{N}^{(e)} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial x} \right) n_x^{(e)} + \left(\mathbf{N}^{(e)} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial y} \right) n_y^{(e)} \right] ds \\ & + \int_{D^{(e)}} \left[-\frac{\partial \mathbf{N}^{(e)}}{\partial x} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial x} - \frac{\partial \mathbf{N}^{(e)}}{\partial y} \frac{\partial \tilde{\boldsymbol{\phi}}^{(e)}}{\partial y} + \mathbf{N}^{(e)} A \right] dx\, dy \Bigg\} = 0 \end{aligned} \tag{6.114}$$

where $n_x^{(e)}$ and $n_y^{(e)}$ are, respectively, the x- and y-components of the outward normal vector along the closed boundary defining the area of the element $C^{(e)}$.

Substituting for the element approximation function yields

$$\sum_{e=1}^{E}\left\{\oint_{C^{(e)}}\mathbf{N}^{(e)}\left[\frac{\partial\tilde{\phi}^{(e)}}{\partial x}n_x^{(e)}+\frac{\partial\tilde{\phi}^{(e)}}{\partial y}n_y^{(e)}\right]ds\right.$$
$$+\int_{D^{(e)}}\left[-\frac{\partial\mathbf{N}^{(e)}}{\partial x}\frac{\partial\mathbf{N}^{(e)T}}{\partial x}-\frac{\partial\mathbf{N}^{(e)}}{\partial y}\frac{\partial\mathbf{N}^{(e)T}}{\partial y}\right]dx\,dy\boldsymbol{\varphi}^{(e)} \tag{6.115}$$
$$\left.-\int_{D^{(e)}}\mathbf{N}^{(e)}A\,dx\,dy\right\}=0$$

This equation can be recast in matrix form as

$$\sum_{e=1}^{E}\left(\mathbf{k}^{(e)}\boldsymbol{\varphi}^{(e)}-\mathbf{f}^{(e)}+\mathbf{Q}^{(e)}\right)=0 \tag{6.116}$$

where

$$\mathbf{k}^{(e)}=-\int_{D^{(e)}}\left[\frac{\partial\mathbf{N}^{(e)}}{\partial x}\frac{\partial\mathbf{N}^{(e)T}}{\partial x}+\frac{\partial\mathbf{N}^{(e)}}{\partial y}\frac{\partial\mathbf{N}^{(e)T}}{\partial y}\right]dx\,dy \tag{6.117}$$

$$\mathbf{f}^{(e)}=\int_{D^{(e)}}A\mathbf{N}^{(e)}dx\,dy \tag{6.118}$$

$$\mathbf{Q}^{(e)}=\oint_{C^{(e)}}\mathbf{N}^{(e)}\left[\frac{\partial\tilde{\phi}^{(e)}}{\partial x}n_x^{(e)}+\frac{\partial\tilde{\phi}^{(e)}}{\partial y}n_y^{(e)}\right]ds \tag{6.119}$$

in which $\mathbf{k}^{(e)}$ is the element characteristic matrix, $\mathbf{f}^{(e)}$ is the element right-hand-side vector, and $\mathbf{Q}^{(e)}$ is often referred to as the inter-element vector that includes the derivative terms along the boundary of the element. The boundary integral around each element is evaluated in a counterclockwise direction, i.e., this boundary integral is the sum of four integrals taken along each side of the element.

Because the specified derivatives have zero values along the element boundaries, the inter-element vector, $\mathbf{Q}^{(e)}$ vanishes, i.e., $\mathbf{Q}^{(e)}=0$, thus reducing the element equilibrium equations to

$$\sum_{e=1}^{E}\left(\mathbf{k}^{(e)}\boldsymbol{\varphi}^{(e)}-\mathbf{f}^{(e)}\right)=0 \tag{6.120}$$

The integrals contributing to the characteristic element matrix, $\mathbf{k}^{(e)}$, and the right-hand-side vector, $\mathbf{f}^{(e)}$, are evaluated over a square region in the natural coordinate system after an appropriate coordinate transformation given by

$$x = \sum_{i=1}^{4} N_i^{(e)}(\xi,\eta)x_i^{(e)} \quad \text{and} \quad y = \sum_{i=1}^{4} N_i^{(e)}(\xi,\eta)y_i^{(e)} \tag{6.121}$$

Application of the chain rule of differentiation yields

$$\begin{Bmatrix} \dfrac{\partial N_i^{(e)}}{\partial \xi} \\ \dfrac{\partial N_i^{(e)}}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i^{(e)}}{\partial x} \\ \dfrac{\partial N_i^{(e)}}{\partial y} \end{Bmatrix} \quad \text{with} \quad i = 1,2,3,4 \tag{6.122}$$

or

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} N_i^{(e)} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} N_i^{(e)} \tag{6.123}$$

where **J** is called the Jacobian matrix. It can be expressed as

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{6.124}$$

in which

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{1}{4}\left\{-(1-\eta)x_1^{(e)} + (1-\eta)x_2^{(e)} + (1+\eta)x_3^{(e)} - (1+\eta)x_4^{(e)}\right\} \tag{6.125}$$

$$J_{12} = \frac{\partial y}{\partial \xi} = \frac{1}{4}\left\{-(1-\eta)y_1^{(e)} + (1-\eta)y_2^{(e)} + (1+\eta)y_3^{(e)} - (1+\eta)y_4^{(e)}\right\} \tag{6.126}$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{1}{4}\left\{-(1-\xi)x_1^{(e)} - (1+\xi)x_2^{(e)} + (1+\xi)x_3^{(e)} + (1-\xi)x_4^{(e)}\right\} \tag{6.127}$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \frac{1}{4}\left\{-(1-\xi)y_1^{(e)} - (1+\xi)y_2^{(e)} + (1+\xi)y_3^{(e)} + (1-\xi)y_4^{(e)}\right\} \tag{6.128}$$

Also, the Jacobian can be rewritten in the form

$$\mathbf{J}=\begin{bmatrix} \dfrac{\partial N_1^{(e)}}{\partial \xi} & \dfrac{\partial N_2^{(e)}}{\partial \xi} & \dfrac{\partial N_3^{(e)}}{\partial \xi} & \dfrac{\partial N_4^{(e)}}{\partial \xi} \\ \dfrac{\partial N_1^{(e)}}{\partial \eta} & \dfrac{\partial N_2^{(e)}}{\partial \eta} & \dfrac{\partial N_3^{(e)}}{\partial \eta} & \dfrac{\partial N_4^{(e)}}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^{(e)} & y_1^{(e)} \\ x_2^{(e)} & y_2^{(e)} \\ x_3^{(e)} & y_3^{(e)} \\ x_4^{(e)} & y_4^{(e)} \end{bmatrix} \tag{6.129}$$

or

$$\mathbf{J}=\frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} x_1^{(e)} & y_1^{(e)} \\ x_2^{(e)} & y_2^{(e)} \\ x_3^{(e)} & y_3^{(e)} \\ x_4^{(e)} & y_4^{(e)} \end{bmatrix} \tag{6.130}$$

Because the transformation between the natural and global coordinates has a one-to-one correspondence, the inverse of the Jacobian exists, and it can be expressed as

$$\mathbf{J}^{-1}=\frac{1}{|\mathbf{J}|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \tag{6.131}$$

When the element is degenerated into a triangle by increasing an internal angle to 180°, $\mathbf{J}$ is singular at that corner. The inverse of the Jacobian matrix permits the expression for the derivatives in terms of global coordinates

$$\begin{Bmatrix} \dfrac{\partial N_i^{(e)}}{\partial x} \\ \dfrac{\partial N_i^{(e)}}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial N_i^{(e)}}{\partial \xi} \\ \dfrac{\partial N_i^{(e)}}{\partial \eta} \end{Bmatrix} \tag{6.132}$$

Defining the element shape matrix $\mathbf{B}^{(e)}$ as

$$\mathbf{B}^{(e)}=\begin{bmatrix} \dfrac{\partial N_1^{(e)}}{\partial x} & \dfrac{\partial N_2^{(e)}}{\partial x} & \dfrac{\partial N_3^{(e)}}{\partial x} & \dfrac{\partial N_4^{(e)}}{\partial x} \\ \dfrac{\partial N_1^{(e)}}{\partial y} & \dfrac{\partial N_2^{(e)}}{\partial y} & \dfrac{\partial N_3^{(e)}}{\partial y} & \dfrac{\partial N_4^{(e)}}{\partial y} \end{bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \mathbf{N}^{(e)T} \tag{6.133}$$

permits the element matrix $\mathbf{k}^{(e)}$ be written in the form

$$\mathbf{k}^{(e)}=-\int_{D^{(e)}} \mathbf{B}^{(e)T}\mathbf{B}^{(e)}\,dx\,dy=-\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{(e)T}\mathbf{B}^{(e)}\,|\mathbf{J}|\,d\xi\,d\eta \tag{6.134}$$

A similar operation is performed for evaluation of $\mathbf{f}^{(e)}$

$$\mathbf{f}^{(e)} = A \int_{D^{(e)}} \mathbf{N}^{(e)} dx\, dy = A \int_{-1}^{1} \int_{-1}^{1} \mathbf{N}^{(e)} |\mathbf{J}| d\xi\, d\eta \tag{6.135}$$

Due to the difficulty of obtaining an analytical expression for the determinant and inverse of the Jacobian matrix, these integrals are evaluated numerically by the Gaussian integration technique described in detail in Sec. 3.6.

Prior to the calculation of the element characteristic matrices, their Jacobian matrices are obtained for each element using Eq. (6.130) as

$$\mathbf{J}^{(1)} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} -4 & -3 \\ 0 & -3 \\ 0 & 0 \\ -4 & 0 \end{bmatrix} \tag{6.136}$$

$$= \frac{1}{4}\begin{bmatrix} 8 & 0 \\ 0 & 6 \end{bmatrix} \qquad \text{with } |\mathbf{J}^{(1)}| = 3$$

$$\mathbf{J}^{(2)} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} 0 & -3 \\ 4 & -3 \\ 5 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.137}$$

$$= \frac{1}{4}\begin{bmatrix} 9+\eta & 0 \\ 1+\xi & 6 \end{bmatrix} \qquad \text{with } |\mathbf{J}^{(2)}| = \frac{3}{8}(9+\eta)$$

$$\mathbf{J}^{(3)} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 5 & 0 \\ 6 & 3 \\ 0 & 3 \end{bmatrix} \tag{6.138}$$

$$= \frac{1}{4}\begin{bmatrix} 11+\eta & 0 \\ 1+\xi & 6 \end{bmatrix} \qquad \text{with } |\mathbf{J}^{(3)}| = \frac{3}{8}(11+\eta)$$

$$\mathbf{J}^{(4)} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 0 \\ 0 & 3 \\ -4 & 3 \end{bmatrix} \tag{6.139}$$

$$= \frac{1}{4}\begin{bmatrix} 8 & 0 \\ 0 & 6 \end{bmatrix} \qquad \text{with } |\mathbf{J}^{(4)}| = 3$$

The inverse of the Jacobian matrices are obtained as

$$\left[\mathbf{J}^{(1)}\right]^{-1} = 4\begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \tag{6.140}$$

$$\left[\mathbf{J}^{(2)}\right]^{-1} = \frac{4}{6(9+\eta)}\begin{bmatrix} 6 & 0 \\ -(1+\xi) & 9+\eta \end{bmatrix} \tag{6.141}$$

$$\left[\mathbf{J}^{(3)}\right]^{-1} = \frac{4}{6(11+\eta)}\begin{bmatrix} 6 & 0 \\ -(1+\xi) & 11+\eta \end{bmatrix} \tag{6.142}$$

$$\left[\mathbf{J}^{(4)}\right]^{-1} = 4\begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \tag{6.143}$$

The element shape matrices $\mathbf{B}^{(e)}$ are obtained as

$$\mathbf{B}^{(1)} = \begin{bmatrix} -\frac{1}{8}(1-\eta) & \frac{1}{8}(1-\eta) & \frac{1}{8}(1+\eta) & -\frac{1}{8}(1+\eta) \\ -\frac{1}{6}(1-\xi) & -\frac{1}{6}(1+\xi) & \frac{1}{6}(1+\xi) & \frac{1}{6}(1-\xi) \end{bmatrix} \tag{6.144}$$

$$\mathbf{B}^{(2)} = \frac{1}{9+\eta}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -\frac{1}{3}(4-5\xi+\eta) & -\frac{5}{3}(1+\xi) & \frac{4}{3}(1+\xi) & \frac{1}{3}(5-4\xi+\eta) \end{bmatrix} \tag{6.145}$$

$$\mathbf{B}^{(3)} = \frac{1}{11+\eta}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -\frac{1}{3}(5-6\xi+\eta) & -2(1+\xi) & \frac{5}{3}(1+\xi) & \frac{1}{3}(6-5\xi+\eta) \end{bmatrix} \tag{6.146}$$

$$\mathbf{B}^{(4)} = \begin{bmatrix} -\frac{1}{8}(1-\eta) & \frac{1}{8}(1-\eta) & \frac{1}{8}(1+\eta) & -\frac{1}{8}(1+\eta) \\ -\frac{1}{6}(1-\xi) & -\frac{1}{6}(1+\xi) & \frac{1}{6}(1+\xi) & \frac{1}{6}(1-\xi) \end{bmatrix} \tag{6.147}$$

Numerical evaluation of the element characteristic matrices results in

$$\mathbf{k}^{(1)} = \begin{bmatrix} \frac{25}{36} & -\frac{1}{36} & -\frac{25}{72} & -\frac{23}{72} \\ -\frac{1}{36} & \frac{25}{36} & -\frac{23}{72} & -\frac{25}{72} \\ -\frac{25}{72} & -\frac{23}{72} & \frac{25}{36} & -\frac{1}{36} \\ -\frac{23}{72} & -\frac{25}{72} & -\frac{1}{36} & \frac{25}{36} \end{bmatrix} \tag{6.148}$$

$$\mathbf{k}^{(2)} = \begin{bmatrix} 0.688943 & -0.0222762 & -0.282179 & -0.384488 \\ -0.0222762 & 0.85561 & -0.384488 & -0.448846 \\ -0.282179 & -0.384488 & 0.60759 & 0.0590766 \\ -0.384488 & -0.448846 & 0.0590766 & 0.774257 \end{bmatrix} \tag{6.149}$$

$$\mathbf{k}^{(3)} = \begin{bmatrix} 0.753348 & 0.0799856 & -0.316655 & -0.516679 \\ 0.0799856 & 0.920014 & -0.516679 & -0.483321 \\ -0.316655 & -0.516679 & 0.680566 & 0.152768 \\ -0.516679 & -0.483321 & 0.152768 & 0.847232 \end{bmatrix} \tag{6.150}$$

$$\mathbf{k}^{(4)} = \begin{bmatrix} \frac{25}{36} & -\frac{1}{36} & -\frac{25}{72} & -\frac{23}{72} \\ -\frac{1}{36} & \frac{25}{36} & -\frac{23}{72} & -\frac{25}{72} \\ -\frac{25}{72} & -\frac{23}{72} & \frac{25}{36} & -\frac{1}{36} \\ -\frac{23}{72} & -\frac{25}{72} & -\frac{1}{36} & \frac{25}{36} \end{bmatrix} \tag{6.151}$$

Similarly, the right-hand-side vectors are calculated as

$$\mathbf{f}^{(1)} = A\begin{Bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{Bmatrix}, \quad \mathbf{f}^{(2)} = A\begin{Bmatrix} 3.25 \\ 3.25 \\ 3.5 \\ 3.5 \end{Bmatrix}, \quad \mathbf{f}^{(3)} = A\begin{Bmatrix} 4 \\ 4 \\ 4.25 \\ 4.25 \end{Bmatrix}, \quad \mathbf{f}^{(4)} = A\begin{Bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{Bmatrix}, \tag{6.152}$$

The element definitions (or connectivity of elements), as shown in Fig. 6.14, are presented in Table 6.5.

Table 6.5 Element connectivity

Element number (e)	Node 1	Node 2	Node 3	Node 4
1	1	2	9	8
2	2	3	4	9
3	9	4	5	6
4	8	9	6	7

Considering the correspondence between the local and global node numbering as shown in Table 6.5, the element equations can be rewritten as

Element 1:
$$\begin{matrix} \boxed{1} & \boxed{2} & \boxed{9} & \boxed{8} \end{matrix}$$
$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} & k_{34}^{(1)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)} \end{bmatrix} \begin{Bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \\ \phi_3^{(1)} \\ \phi_4^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \\ f_4^{(1)} \end{Bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{9} \\ \boxed{8} \end{matrix} \tag{6.153}$$

Element 2:
$$\begin{matrix} \boxed{2} & \boxed{3} & \boxed{4} & \boxed{9} \end{matrix}$$
$$\begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} \end{bmatrix} \begin{Bmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \\ \phi_3^{(2)} \\ \phi_4^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \\ f_4^{(2)} \end{Bmatrix} \begin{matrix} \boxed{2} \\ \boxed{3} \\ \boxed{4} \\ \boxed{9} \end{matrix} \tag{6.154}$$

Element 3:
$$\begin{matrix} \boxed{9} & \boxed{4} & \boxed{5} & \boxed{6} \end{matrix}$$
$$\begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} & k_{13}^{(3)} & k_{14}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} & k_{23}^{(3)} & k_{24}^{(3)} \\ k_{31}^{(3)} & k_{32}^{(3)} & k_{33}^{(3)} & k_{34}^{(3)} \\ k_{41}^{(3)} & k_{42}^{(3)} & k_{43}^{(3)} & k_{44}^{(3)} \end{bmatrix} \begin{Bmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \\ \phi_3^{(3)} \\ \phi_4^{(3)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(3)} \\ f_2^{(3)} \\ f_3^{(3)} \\ f_4^{(3)} \end{Bmatrix} \begin{matrix} \boxed{9} \\ \boxed{4} \\ \boxed{5} \\ \boxed{6} \end{matrix} \tag{6.155}$$

Element 4:
$$\begin{matrix} \boxed{8} & \boxed{9} & \boxed{6} & \boxed{7} \end{matrix}$$
$$\begin{bmatrix} k_{11}^{(4)} & k_{12}^{(4)} & k_{13}^{(4)} & k_{14}^{(4)} \\ k_{21}^{(4)} & k_{22}^{(4)} & k_{23}^{(4)} & k_{24}^{(4)} \\ k_{31}^{(4)} & k_{32}^{(4)} & k_{33}^{(4)} & k_{34}^{(4)} \\ k_{41}^{(4)} & k_{42}^{(4)} & k_{43}^{(4)} & k_{44}^{(4)} \end{bmatrix} \begin{Bmatrix} \phi_1^{(4)} \\ \phi_2^{(4)} \\ \phi_3^{(4)} \\ \phi_4^{(4)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(4)} \\ f_2^{(4)} \\ f_3^{(4)} \\ f_4^{(4)} \end{Bmatrix} \begin{matrix} \boxed{8} \\ \boxed{9} \\ \boxed{6} \\ \boxed{7} \end{matrix} \tag{6.156}$$

In the assembly of the element characteristic matrices and vectors, the boxed numbers indicate the rows and columns of the global matrix, $\mathbf{K}$, and global right-hand-side vector, $\mathbf{F}$, to which the individual coefficients are added, resulting in

$$\begin{bmatrix}
k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 & 0 & 0 & 0 \\
k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & 0 & 0 & 0 \\
0 & k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & 0 & 0 & 0 \\
0 & k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)}+k_{22}^{(3)} & k_{23}^{(3)} & k_{24}^{(3)} & 0 \\
0 & 0 & 0 & k_{32}^{(3)} & k_{33}^{(3)} & k_{34}^{(3)} & 0 \\
0 & 0 & 0 & k_{42}^{(3)} & k_{43}^{(3)} & k_{44}^{(3)}+k_{33}^{(4)} & k_{34}^{(4)} \\
0 & 0 & 0 & 0 & 0 & k_{43}^{(4)} & k_{44}^{(4)} \\
k_{41}^{(1)} & k_{42}^{(1)} & 0 & 0 & 0 & k_{13}^{(4)} & k_{14}^{(4)} \\
k_{31}^{(1)} & k_{32}^{(1)}+k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)}+k_{12}^{(3)} & k_{13}^{(3)} & k_{14}^{(3)}+k_{23}^{(4)} & k_{24}^{(4)}
\end{bmatrix} \tag{6.157}$$

$$\begin{bmatrix}
k_{14}^{(1)} & k_{13}^{(1)} \\
k_{24}^{(1)} & k_{23}^{(1)}+k_{14}^{(2)} \\
0 & k_{24}^{(2)} \\
0 & k_{34}^{(2)}+k_{21}^{(3)} \\
0 & k_{31}^{(3)} \\
k_{31}^{(4)} & k_{41}^{(3)}+k_{32}^{(4)} \\
k_{41}^{(4)} & k_{42}^{(4)} \\
k_{44}^{(1)}+k_{11}^{(4)} & k_{43}^{(1)}+k_{12}^{(4)} \\
k_{34}^{(1)}+k_{21}^{(4)} & k_{33}^{(1)}+k_{44}^{(2)}+k_{11}^{(3)}+k_{22}^{(4)}
\end{bmatrix}
\begin{Bmatrix}
\phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9
\end{Bmatrix} =
\begin{Bmatrix}
f_1^{(1)} \\
f_2^{(1)}+f_1^{(2)} \\
f_2^{(2)} \\
f_3^{(2)}+f_2^{(3)} \\
f_3^{(3)} \\
f_4^{(3)}+f_3^{(4)} \\
f_4^{(4)} \\
f_4^{(1)}+f_1^{(4)} \\
f_3^{(1)}+f_4^{(2)}+f_1^{(3)}+f_2^{(4)}
\end{Bmatrix}$$

or

$$\mathbf{K\Phi} = \mathbf{F} \tag{6.158}$$

the global stiffness matrix and right-hand-side vector are numerically evaluated as

$$\mathbf{K} = \begin{bmatrix} 0.694444 & -0.0277778 & 0 & 0 & 0 \\ -0.0277778 & 1.38339 & -0.0222762 & -0.282179 & 0 \\ 0 & -0.0222762 & 0.85561 & -0.384488 & 0 \\ 0 & -0.282179 & -0.384488 & 1.5276 & -0.516679 \\ 0 & 0 & 0 & -0.516679 & 0.680566 \\ 0 & 0 & 0 & -0.483321 & 0.152768 \\ 0 & 0 & 0 & 0 & 0 \\ -0.319444 & -0.347222 & 0 & 0 & 0 \\ -0.347222 & -0.703932 & -0.448846 & 0.139062 & -0.316655 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -0.319444 & -0.347222 \\ 0 & 0 & -0.347222 & -0.703932 \\ 0 & 0 & 0 & -0.448846 \\ -0.483321 & 0 & 0 & 0.139062 \\ 0.152768 & 0 & 0 & -0.316655 \\ 1.54168 & -0.0277778 & -0.347222 & -0.836123 \\ -0.0277778 & 0.694444 & -0.319444 & -0.347222 \\ -0.347222 & -0.319444 & 1.38889 & -0.0555556 \\ -0.836123 & -0.347222 & -0.0555556 & 2.91649 \end{bmatrix} \tag{6.159}$$

and

$$\mathbf{F} = \begin{Bmatrix} 3 \\ 6.25 \\ 3.25 \\ 7.5 \\ 4.25 \\ 7.25 \\ 3 \\ 6 \\ 13.5 \end{Bmatrix} \tag{6.160}$$

After imposing the essential boundary conditions, i.e., $\phi_3 = \phi_4 = \phi_5 = \phi_6 = \phi_7 = 1$, the global system of equations is reduced by deleting the rows and columns corresponding to ϕ_3, ϕ_4, ϕ_5, ϕ_6, and ϕ_7, leading to

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{14}^{(1)} & k_{13}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)} & k_{24}^{(1)} & k_{23}^{(1)}+k_{14}^{(2)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{44}^{(1)}+k_{11}^{(4)} & k_{43}^{(1)}+k_{12}^{(4)} \\ k_{31}^{(1)} & k_{32}^{(1)}+k_{41}^{(2)} & k_{34}^{(1)}+k_{21}^{(4)} & k_{33}^{(1)}+k_{44}^{(2)}+k_{11}^{(3)}+k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_8 \\ \phi_9 \end{Bmatrix}$$

$$= \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)}+f_1^{(2)}-k_{12}^{(2)}-k_{13}^{(2)} \\ f_4^{(1)}+f_1^{(4)}-k_{13}^{(4)}-k_{14}^{(4)} \\ f_3^{(1)}+f_4^{(2)}+f_1^{(3)}+f_2^{(4)}-k_{42}^{(2)}-\left(k_{43}^{(2)}+k_{12}^{(3)}\right) \\ -k_{13}^{(3)}-\left(k_{14}^{(3)}+k_{23}^{(4)}\right)-k_{24}^{(4)} \end{Bmatrix} \tag{6.161}$$

which is numerically evaluated as

$$\mathbf{K} = \begin{bmatrix} 0.694444 & -0.0277778 & -0.319444 & -0.347222 \\ -0.0277778 & 1.38339 & -0.347222 & -0.703932 \\ -0.319444 & -0.347222 & 1.38889 & -0.0555556 \\ -0.347222 & -0.703932 & -0.0555556 & 2.91649 \end{bmatrix} \tag{6.162}$$

and

$$\mathbf{F} = \begin{Bmatrix} 3 \\ 6.55446 \\ 6.66667 \\ 15.3098 \end{Bmatrix} \tag{6.163}$$

Finally, the solution of the reduced global system yields

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_8 \\ \phi_9 \end{Bmatrix} = \begin{Bmatrix} 15.8119 \\ 13.5401 \\ 12.2471 \\ 10.6332 \end{Bmatrix} \tag{6.164}$$

6.1.3.2 ANSYS Solution

The governing equations for a steady-state heat transfer, described by Eq. (6.101) through (6.105), also can be solved using ANSYS. The solution procedure is outlined as follows:

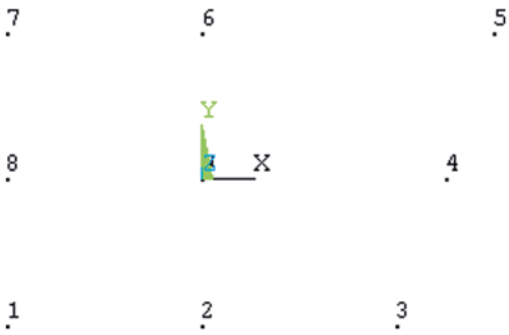

Fig. 6.16 Generation of nodes

Model Generation

- Specify the element type (**ET** command) using the following menu path:

Main Menu > Preprocessor > Element Type > Add/Edit/Delete

  - Click on ***Add***.
  - Select ***Solid*** immediately below ***Thermal Mass*** from the left list and ***Quad 4node 55*** from the right list; click on ***OK***.
  - Click on ***Close***.

- Specify material properties (**MP** command) using the following menu path:

Main Menu > Preprocessor > Material Props > Material Models

  - In the Define Material Model Behavior dialog box, in the right window, successively left-click on Thermal, Conductivity, and, finally, Isotropic, which brings up another dialog box.
  - Enter ***1*** for ***KXX***, and click on ***OK***.
  - Close the Define Material Model Behavior dialog box by using the following menu path:

Material > Exit

- Create nodes (**N** command) using the following menu path:

Main Menu > Preprocessor > Modeling > Create > Nodes > In Active CS

  - A total of 9 nodes will be created (Table 6.4).
  - Referring to Table 6.4, enter x- and y-coordinates of node 1, and Click on Apply. This action will keep the Create Nodes in Active Coordinate System dialog box open. If the Node number field is left blank, then ANSYS will assign the lowest available node number to the node that is being created.
  - Repeat the same procedure for the nodes 2 through 9.
  - After entering the x- and y-coordinates of node 9, click on OK (instead of App***ly***).
  - The nodes should appear in the *Graphics Window*, as shown in Fig. 6.16.

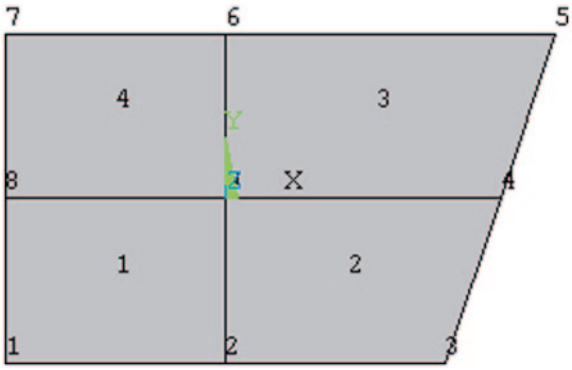

Fig. 6.17 Generation of elements

- Create elements (**E** command) using the following menu path:

Main Menu > Preprocessor > Modeling > Create > Elements > Auto Numbered > Thru Nodes

  - *Pick Menu* appears; refer to Fig. 6.17 to create elements by picking ***four*** nodes at a time and clicking on ***Apply*** in between.
  - Observe the elements created after clicking on ***Apply*** in the *Pick Menu*.
  - Repeat until the last element is created.
  - Click on ***OK*** when the last element is created.

- Review elements:

  - Turn on element numbering using the following menu path:

Utility Menu > PlotCtrls > Numbering

  - Select ***Element numbers*** from the first pull-down menu; click on ***OK***.
  - Plot elements (**EPLOT** command) using the following menu path:

Utility Menu > Plot > lements

  - Figure 6.17 shows the outcome of this action as it appears in the *Graphics Window*.
  - Turn off element numbering and turn on node numbering using the following menu path:

Utility Menu > PlotCtrls > Numbering

  - Place a *checkmark* by clicking on the empty box next to ***NODE Node numbers***.
  - Select ***No numbering*** from the first pull-down menu.
  - Click on ***OK***.
  - Plot nodes (**NPLOT** command) using the following menu path:

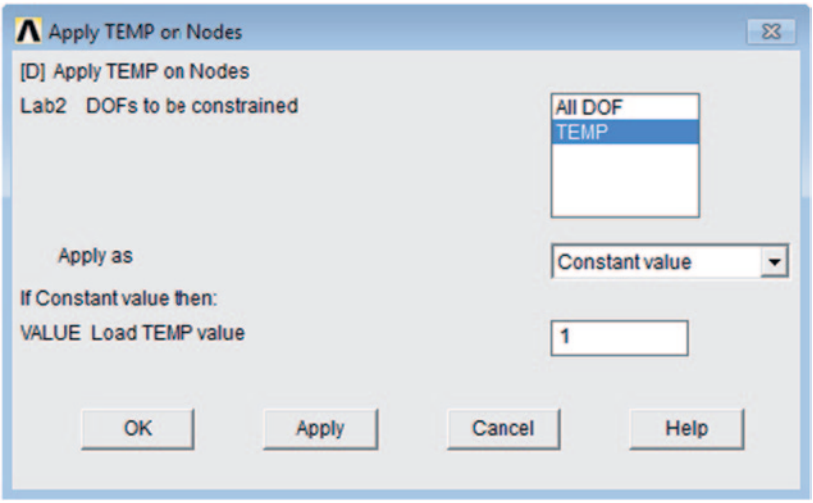

Fig. 6.18 Application of temperature boundary conditions on nodes

Utility Menu > Plot > Nodes

– Figure 6.16 shows the outcome of this action as it appears in the *Graphics Window*.

Solution

- Apply temperature boundary conditions (**D** command) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Thermal > Temperature > On Nodes

– *Pick Menu* appears; pick nodes 3 through 7 along the boundary (Fig. 6.16) and click on ***OK*** on *Pick Menu*.
– Highlight ***TEMP*** and enter ***1*** for ***VALUE***; click on ***OK*** (Fig. 6.18).

- Apply body load on elements (**BFE** command) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Thermal > Heat Generat > On Elements

– *Pick Menu* appears; click on ***Pick All***.
– Enter ***1*** for ***VAL1*** (leave other fields untouched, as shown in Fig. 6.19).
– Click on ***OK***.

- Obtain solution (**SOLVE** command) using the following menu path:

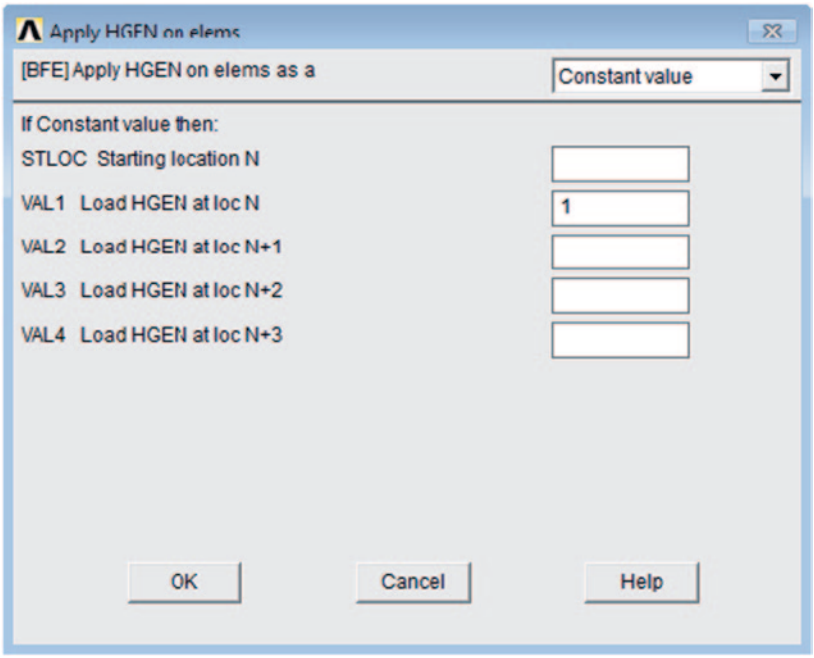

Fig. 6.19 Application of heat generation condition on elements

Main Menu > Solution > Solve > Current LS

- *Confirmation Window* appears along with *Status Report Window*.
- Review status/ If OK, close the *Status Report Window* and click on ***OK*** in the *Confirmation Window*.
- Wait until ANSYS responds with ***Solution is done!***

Postprocessing

- Review temperature values (**`PRNSOL`** command) using the following menu path:

Main Menu > General Postproc > List Results > Nodal Solution

- Click on ***DOF Solution*** and ***Nodal Temperature***; click on ***OK***.
- The list will appear in a new window, as shown in Fig. 6.20.

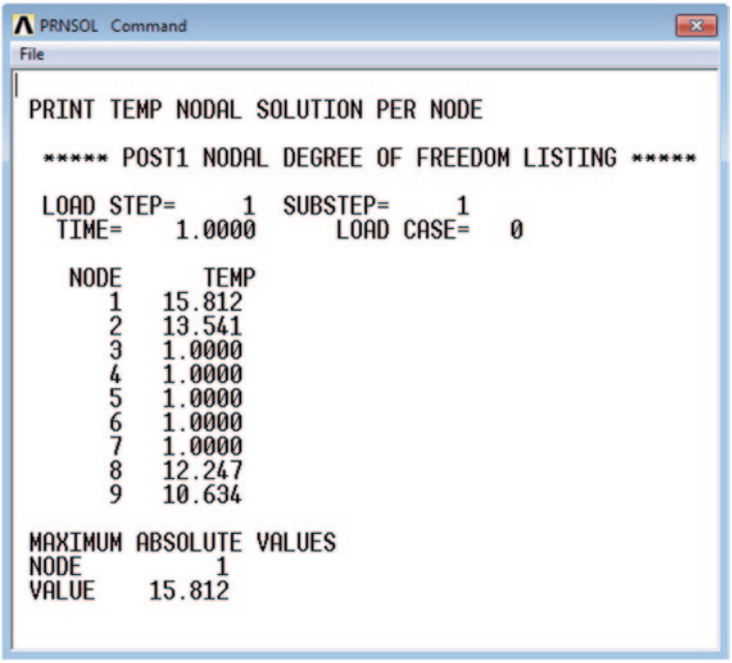

Fig. 6.20 Nodal solution for temperature

6.2 Principle of Minimum Potential Energy

Galerkin's method is not always suitable for all structural problems because of difficulties in mathematically describing the structural geometry and/or the boundary conditions. An alternative to Galerkin's method is the principle of minimum potential energy (Washizu 1982; Dym and Shames 1973).

The energy method involves determination of the stationary values of the global energy. This requires the approximation of the functional behavior of the dependent variable so that the global energy becomes stationary. The stationary value can be a maximum, a minimum or a neutral point. With an understanding of variational calculus, the minimum stationary value leading to stable equilibrium (Fig. 6.21) is obtained by requiring the first variation of the global energy to vanish.

Avoiding the details of variational calculus, the concepts of differential calculus can be used to perform the minimization of the global energy. In solid mechanics, this is known as the principle of minimum potential energy, which states that among all compatible displacement fields satisfying the boundary conditions (kinematically admissible), the correct displacement field satisfying the equilibrium equations is the one that renders the potential energy an absolute minimum. A solution satisfying both equilibrium equations and boundary conditions is, of course, "exact"; however, such solutions are difficult, if not impossible, to construct for complex problems. Therefore, approximate solutions are obtained by assuming kinematically admissible displacement fields with unknown coefficients. The values of

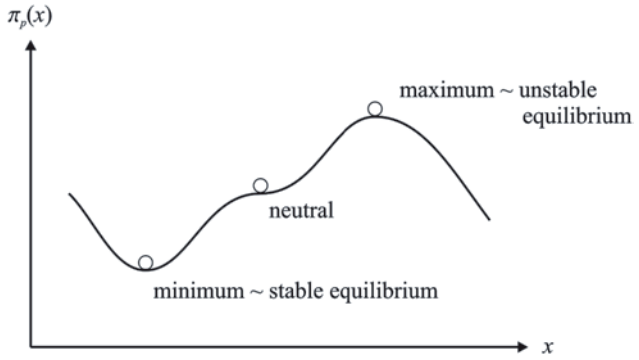

Fig. 6.21 Schematics of stable, neutral, and unstable equilibrium points of the global energy

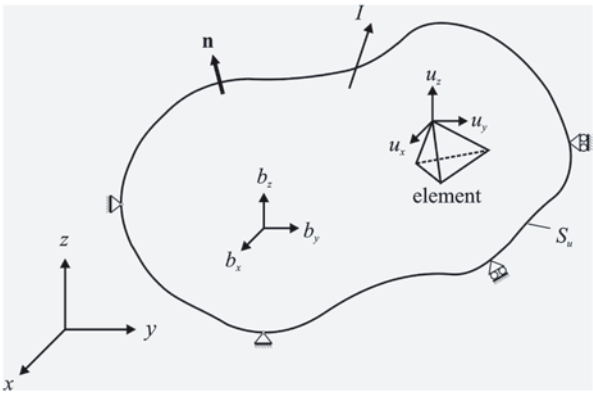

Fig. 6.22 A 3D body with displacement constraints, body and concentrated forces, and surface tractions

these coefficients are determined in such a way that the total potential energy of the system is a minimum.

The principle of virtual work is applicable for any material behavior, whereas the principle of minimum potential energy is applicable only for elastic materials. However, both principles yield the same element equations for elastic materials.

The total potential energy of the structural system shown in Fig. 6.22 is defined as

$$\pi_p = W + \Omega \tag{6.165}$$

in which W is the strain energy and $\boldsymbol{\Omega}$ is the potential energy arising from the presence of body forces, surface tractions, and the initial residual stresses. Strain energy is the capacity of the internal forces (or stresses) to do work through strains in the structure.

For a linear elastic material, the strain energy of the deformed structure is given by

$$W = \frac{1}{2}\int_V \left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^*\right)^T \boldsymbol{\sigma}\, dV \tag{6.166}$$

where $\mathbf{A}$ is the vector of stress components arising from the difference between the total strains, $\boldsymbol{\varepsilon}$, and initial strains, $\boldsymbol{\varepsilon}^*$. It can be expressed as

$$\boldsymbol{\sigma} = \mathbf{D}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^*\right) \tag{6.167}$$

in which

$$\boldsymbol{\sigma}^T = \left\{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{yz} \quad \sigma_{zx}\right\} \tag{6.168}$$

and

$$\boldsymbol{\varepsilon}^T = \left\{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}\right\} \tag{6.169}$$

and the material property matrix

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{(1-2\nu)}{2} \end{bmatrix} \tag{6.170}$$

where σ_{ij} and ε_{ij} represent the stress and strain components, with $i, j = x, y, z$ being the Cartesian coordinates. The elastic modulus and Poisson's ratio are denoted by E and ν, respectively. In the presence of temperature change, the initial strains can be expressed as

$$\boldsymbol{\varepsilon}^{*T} = \left\{\alpha\Delta T \quad \alpha\Delta T \quad \alpha\Delta T \quad 0 \quad 0 \quad 0\right\} \tag{6.171}$$

where α is the coefficient of thermal expansion and ΔT is the temperature change with respect to a reference state.

The potential energy arising from the presence of body forces, $\mathbf{b}$, surface tractions, $\mathbf{T}$, and the initial residual stresses, $\boldsymbol{\sigma}^*$, is given by

$$\Omega = -\int_V \mathbf{u}^T \mathbf{b} dV - \int_{S_\sigma} \mathbf{u}^T \mathbf{T} dS + \int_V \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}^* dV \tag{6.172}$$

with

$$\mathbf{b}^T = \left\{b_x \quad b_y \quad b_z\right\} \tag{6.173}$$

$$\mathbf{T}^T = \begin{Bmatrix} T_x & T_y & T_z \end{Bmatrix} \tag{6.174}$$

$$\mathbf{u}^T = \begin{Bmatrix} u_x & u_y & u_z \end{Bmatrix} \tag{6.175}$$

in which b_x, b_y, and b_z are the components of body force (in units of force per unit volume), and T_x, T_y, and T_z represent the components of the applied traction vector (in units of force per unit area) over the surface defined by S_σ. The entire surface of the body having a volume of V is defined by S, with segments S_u and S_σ subjected to displacement and traction conditions, respectively. The displacement components are given by u_x, u_y, and u_z in the x-, y-, and z-directions, respectively. Also, included in the expression for the total potential is the initial residual stresses denoted by $\boldsymbol{\sigma}^*$. The initial stresses could be measured, but their prediction without full knowledge of the material's history is impossible.

After partitioning the entire domain occupied by volume V into E number of elements with volume V^e, the total potential energy of the system can be rewritten as

$$\pi_p(u_x, u_y, u_z) = \sum_{e=1}^{E} \pi_p^{(e)}(u_x, u_y, u_z) \tag{6.176}$$

in which

$$\begin{aligned} \pi_p^{(e)} = \frac{1}{2}\int_{V^{(e)}} \boldsymbol{\varepsilon}^T \mathbf{D}\boldsymbol{\varepsilon}\, dV - \int_{V^{(e)}} \boldsymbol{\varepsilon}^T \mathbf{D}\boldsymbol{\varepsilon}^*\, dV + \frac{1}{2}\int_{V^{(e)}} \boldsymbol{\varepsilon}^{*T} \mathbf{D}\boldsymbol{\varepsilon}^*\, dV \\ - \int_{V^{(e)}} \mathbf{u}^T \mathbf{b}\, dV - \int_{S_\sigma^{(e)}} \mathbf{u}^T \mathbf{T}\, dS + \int_{V^{(e)}} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}^*\, dV \end{aligned} \tag{6.177}$$

where the superscript e denotes a specific element.

Based on kinematical considerations, the components of the total strain vector, $\boldsymbol{\varepsilon}$, in terms of the displacement components are expressed as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \quad \text{or} \quad \boldsymbol{\varepsilon} = \mathbf{L}\,\mathbf{u} \tag{6.178}$$

in which **L** is the differential operator matrix.

The finite element process seeks a minimum in the potential energy based on the approximate form of the dependent variables (displacement components) within each element. The greater the number of degrees of freedom associated with the element (usually means increasing the number of nodes), the more closely the solution will approximate the true equilibrium position. Within each element, the approximation to the displacement components can be expressed as

$$\begin{aligned} u_x^{(e)} &\approx \tilde{u}_x^{(e)} = \sum_{r=1}^{n} N_r^{(e)} u_{x_r}^{(e)} \\ u_y^{(e)} &\approx \tilde{u}_y^{(e)} = \sum_{r=1}^{n} N_r^{(e)} u_{y_r}^{(e)} \\ u_z^{(e)} &\approx \tilde{u}_z^{(e)} = \sum_{r=1}^{n} N_r^{(e)} u_{z_r}^{(e)} \end{aligned} \tag{6.179}$$

with n representing the number of nodes associated with element e. The nodal unknowns and shape functions are denoted by $u_{x_r}^{(e)}$, $u_{y_r}^{(e)}$, $u_{z_r}^{(e)}$, and $N_r^{(e)}$, respectively. In matrix form, the approximate displacement components can be expressed as

$$\tilde{\mathbf{u}}^{(e)} = \mathbf{N}^{(e)T}\mathbf{U}^{(e)} \tag{6.180}$$

in which

$$\tilde{\mathbf{u}}^{(e)T} = \left\{ \tilde{u}_x^{(e)} \quad \tilde{u}_y^{(e)} \quad \tilde{u}_z^{(e)} \right\} \tag{6.181}$$

$$\mathbf{N}^{(e)T} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots & 0 & 0 & N_n \end{bmatrix}_{3\times 3n} \tag{6.182}$$

$$\mathbf{U}^{(e)T} = \left\{ u_{x_1}^{(e)} \quad u_{y_1}^{(e)} \quad u_{z_1}^{(e)} \quad u_{x_2}^{(e)} \quad u_{y_2}^{(e)} \quad u_{z_2}^{(e)} \quad \cdots \quad u_{x_n}^{(e)} \quad u_{y_n}^{(e)} \quad u_{z_n}^{(e)} \right\} \tag{6.183}$$

With the approximate form of the displacement components, the strain components within each element can be expressed as

$$\boldsymbol{\varepsilon} \approx \mathbf{B}^{(e)}\mathbf{U}^{(e)} \tag{6.184}$$

where

$$\mathbf{B}^{(e)} = \mathbf{L}\mathbf{N}^{(e)T} \tag{6.185}$$

leading to the expression for the total potential in terms of element nodal displacements, $\mathbf{U}^{(e)}$

$$\pi_p^{(e)} = \frac{1}{2}\mathbf{U}^{(e)T}\mathbf{k}^{(e)}\mathbf{U}^{(e)} - \mathbf{U}^{(e)T}\mathbf{p}^{(e)} + \frac{1}{2}\int_{V^{(e)}} \boldsymbol{\varepsilon}^{*T}\mathbf{D}\boldsymbol{\varepsilon}^{*}\,dV \tag{6.186}$$

in which the element stiffness matrix, $\mathbf{k}^{(e)}$, and the element force vector, $\mathbf{p}^{(e)}$, are defined as

$$\mathbf{k}^{(e)} = \int_{V^{(e)}} \mathbf{B}^{(e)T}\mathbf{D}\mathbf{B}^{(e)}\,dV \tag{6.187}$$

and

$$\mathbf{p}^{(e)} = \mathbf{p}_{\mathbf{b}}^{(e)} + \mathbf{p}_{\mathbf{T}}^{(e)} + \mathbf{p}_{\varepsilon^*}^{(e)} - \mathbf{p}_{\sigma^*}^{(e)} \tag{6.188}$$

with $\mathbf{p}_{\mathbf{b}}^{(e)}$, $\mathbf{p}_{\mathbf{T}}^{(e)}$, $\mathbf{p}_{\varepsilon^*}^{(e)}$, and $\mathbf{p}_{\sigma^*}^{(e)}$ representing the element load vectors due to body forces, surface tractions (forces), initial strains, and initial stresses, respectively, defined by

$$\begin{aligned}
\mathbf{p}_{\mathbf{b}}^{(e)} &= \int_{V^{(e)}} \mathbf{N}^{(e)}\mathbf{b}\,dV \\
\mathbf{p}_{\mathbf{T}}^{(e)} &= \int_{S_\sigma^{(e)}} \mathbf{N}^{(e)}\mathbf{T}\,dS \\
\mathbf{p}_{\varepsilon^*}^{(e)} &= \int_{V^{(e)}} \mathbf{B}^{(e)T}\,\mathbf{D}\,\boldsymbol{\varepsilon}^{*}\,dV \\
\mathbf{p}_{\sigma^*}^{(e)} &= \int_{V^{(e)}} \mathbf{B}^{(e)T}\,\boldsymbol{\sigma}^{*}\,dV
\end{aligned} \tag{6.189}$$

Evaluation of these integrals results in the statically equivalent nodal forces in the elements affected by the body force, the surface tractions, and the initial strains and initial stresses. In the presence of external concentrated forces acting on various nodes, the potential energy is modified as

$$\begin{aligned}
\pi_p = \frac{1}{2}\mathbf{U}^T\left\{\sum_{e=1}^{E}\mathbf{k}^{(e)}\right\}\mathbf{U} - \mathbf{U}^T\left\{\sum_{e=1}^{E}\left(\mathbf{p}_{\mathbf{b}}^{(e)} + \mathbf{p}_{\mathbf{T}}^{(e)} + \mathbf{p}_{\varepsilon^*}^{(e)} - \mathbf{p}_{\sigma^*}^{(e)}\right) - \mathbf{P}_{\mathbf{c}}\right\} \\
+ \frac{1}{2}\sum_{e=1}^{E}\int_{V^{(e)}} \boldsymbol{\varepsilon}^{*T}\,\mathbf{D}\,\boldsymbol{\varepsilon}^{*}\,dV
\end{aligned} \tag{6.190}$$

where $\mathbf{P}_{\mathbf{c}}$ is the vector of nodal forces and $\mathbf{U}$ represents the vector of nodal displacements for the entire structure. Note that each component of the element nodal dis-

placement vector, $\mathbf{U}^{(e)}$, appears in the global (system) nodal displacement vector, $\mathbf{U}$. Therefore, the element nodal displacement vector $\mathbf{U}^{(e)}$ can be replaced by $\mathbf{U}$ with the appropriate enlargement of the element matrices and vectors in the expression for the potential energy by adding the required number of zero elements and rearranging. The summation in the expression for the potential energy implies the expansion of the element matrices to the size of the global (system) matrix while collecting the overlapping terms.

Minimization of the total potential energy requires that

$$\left\{\frac{\partial \pi_p}{\partial \mathbf{U}}\right\} = 0 \tag{6.191}$$

leading to the system (global) equilibrium equations in the form

$$\mathbf{KU} = \mathbf{P} \tag{6.192}$$

in which $\mathbf{K}$ and $\mathbf{P}$ are the assembled (global) stiffness matrix and the assembled (global) nodal load vector, respectively, defined by

$$\mathbf{K} = \sum_{e=1}^{E} \mathbf{k}^{(e)} \tag{6.193}$$

and

$$\mathbf{P} = \sum_{e=1}^{E} (\mathbf{p}_{\mathbf{b}}^{(e)} + \mathbf{p}_{\mathbf{T}}^{(e)} + \mathbf{p}_{\varepsilon^*}^{(e)} - \mathbf{p}_{\sigma^*}^{(e)}) - \mathbf{P}_{\mathbf{c}} \tag{6.194}$$

This global equilibrium equation cannot be solved unless boundary constraints are imposed to suppress the rigid-body motion. Otherwise, the global stiffness matrix becomes singular.

After obtaining the solution to the nodal displacements of the system equilibrium equations, the stresses within the element can be determined from

$$\boldsymbol{\sigma} = \mathbf{DB}^{(e)}\mathbf{U}^{(e)} - \mathbf{D}\boldsymbol{\varepsilon}^* + \boldsymbol{\sigma}^* \tag{6.195}$$

The global stiffness matrix and the load vector require the evaluation of the integrals associated with the element stiffness matrix and the element nodal load vector.

6.2.1 Example: One-Dimensional Analysis with Line Elements

The application of this approach is demonstrated by computing the displacements and strains in a rod constructed of three concentric sections of different materials. As shown in Fig. 6.23, the rod has a uniform cross section and is subjected to a

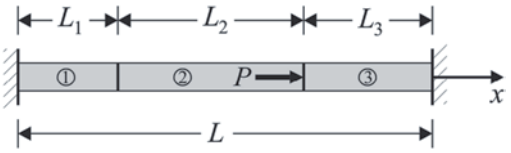

Fig. 6.23 A rod constrained at both ends, subjected to a concentrated force

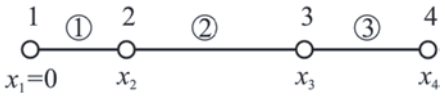

Fig. 6.24 Finite element discretization of the rod with three elements

concentrated horizontal load, P, at the second joint, and the boundary conditions are specified as $u_x(x=0)=0$ and $u_x(x=L)=0$.

The domain is discretized with 3 linear line elements having two nodes, as shown in Fig. 6.24. The global coordinates of each node in domain D are specified by x_i, with $i=1,2,3,4$. The nodal values of the dependent variable associated with element e are specified at its first and second nodes by $u_{x_i}^{(e)}$and $u_{x_j}^{(e)}$, respectively.

For the domain discretized with three elements and four nodes, the local and global nodes are numbered as shown in Table 6.6.

Within each element, the approximation to the displacement component can be expressed as

$$u_x^{(e)} \approx \tilde{u}_x^{(e)} = \sum_{r=1}^{2} N_r^{(e)} u_{x_r}^{(e)} \tag{6.196}$$

The nodal unknowns and shape functions are denoted by $u_{x_r}^{(e)}$ and $N_r^{(e)}$, respectively. In matrix form, the approximate displacement components can be expressed as

$$\tilde{\mathbf{u}}^{(e)} = \mathbf{N}^{(e)T}\mathbf{U}^{(e)} \tag{6.197}$$

with

$$\mathbf{N}^{(e)T} = \left\{ N_1^{(e)} \quad N_2^{(e)} \right\} \quad \text{and} \quad \mathbf{U}^{(e)} = \begin{Bmatrix} u_{x_1}^{(e)} \\ u_{x_2}^{(e)} \end{Bmatrix} \tag{6.198}$$

in which the shape functions are

$$N_1^{(e)} = \frac{x_2^{(e)} - x}{x_2^{(e)} - x_1^{(e)}} \quad \text{and} \quad N_2^{(e)} = \frac{x - x_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \tag{6.199}$$

With the approximate form of the displacement components and $\mathbf{L} = \partial / \partial x$, the shape matrix can be obtained from

Table 6.6 Local and global node numbers

Element number (e)	Node 1	Node 2
1	1	2
2	2	3
3	3	4

$$\mathbf{B}^{(e)} = \frac{\partial}{\partial x}\begin{bmatrix} N_1^{(e)} & N_2^{(e)} \end{bmatrix} \tag{6.200}$$

For a constant cross section, $A^{(e)}$, and elastic modulus, $E^{(e)}$, in each element, the element stiffness matrix is

$$\begin{aligned} \mathbf{k}^{(e)} &= \int_{V^{(e)}} \mathbf{B}^{(e)T}\mathbf{D}\,\mathbf{B}^{(e)}\,dV \\ &= A^{(e)} \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{\partial}{\partial x}\begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \end{bmatrix} E^{(e)} \frac{\partial}{\partial x}\begin{bmatrix} N_1^{(e)} & N_2^{(e)} \end{bmatrix} dx \end{aligned} \tag{6.201}$$

Substituting for the shape functions, the element stiffness matrix becomes

$$\mathbf{k}^{(e)} = -\frac{A^{(e)}E^{(e)}}{\left(x_2^{(e)} - x_1^{(e)}\right)^2} \int_{x_1^{(e)}}^{x_2^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx \tag{6.202}$$

Integration along the element length results in

$$\begin{aligned} \mathbf{k}^{(e)} &= -\frac{A^{(e)}E^{(e)}}{\left(x_2^{(e)} - x_1^{(e)}\right)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = -\frac{A^{(e)}E^{(e)}}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= -\alpha^{(e)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned} \tag{6.203}$$

in which $L^{(e)} = (x_2^{(e)} - x_1^{(e)})$and $\alpha^{(e)} = A^{(e)}E^{(e)} / L^{(e)}$. The element stiffness matrices are computed as

$$\mathbf{k}^{(1)} = \begin{array}{c} \begin{matrix} \boxed{1} & \boxed{2} \end{matrix} \\ \begin{bmatrix} \alpha^{(1)} & -\alpha^{(1)} \\ -\alpha^{(1)} & \alpha^{(1)} \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \end{matrix} \end{array} \tag{6.204}$$

Fig. 6.25 A typical linear line element with two nodes

$$\mathbf{k}^{(2)} = \begin{bmatrix} \boxed{2} & \boxed{3} \\ \alpha^{(2)} & -\alpha^{(2)} \\ -\alpha^{(2)} & \alpha^{(2)} \end{bmatrix} \begin{matrix} \boxed{2} \\ \boxed{3} \end{matrix} \tag{6.205}$$

$$\mathbf{k}^{(3)} = \begin{bmatrix} \boxed{3} & \boxed{4} \\ \alpha^{(3)} & -\alpha^{(3)} \\ -\alpha^{(3)} & \alpha^{(3)} \end{bmatrix} \begin{matrix} \boxed{3} \\ \boxed{4} \end{matrix} \tag{6.206}$$

The element load vector, $\mathbf{p}_{\mathbf{T}}^{(e)}$, due to the unknown nodal forces, T_{x_i} and T_{x_j} at nodes i and j, respectively (Fig. 6.25), can be obtained from

$$\begin{aligned} \mathbf{p}_{\mathbf{T}}^{(e)} &= \int_{S_\sigma^{(e)}} \mathbf{N}^{(e)} \mathbf{T}\, dS = \begin{Bmatrix} N_1^{(e)}\left(x = x_1^{(e)}\right) \\ N_2^{(e)}\left(x = x_1^{(e)}\right) \end{Bmatrix} T_{x_1} \\ &+ \begin{Bmatrix} N_1^{(e)}\left(x = x_2^{(e)}\right) \\ N_2^{(e)}\left(x = x_2^{(e)}\right) \end{Bmatrix} T_{x_2} \end{aligned} \tag{6.207}$$

Evaluating the shape functions results in a load vector of the form

$$\begin{aligned} \mathbf{p}_{\mathbf{T}}^{(e)} &= \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} T_{x_1} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} T_{x_2} \\ &= \begin{Bmatrix} -T_{x_1} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ T_{x_2} \end{Bmatrix} = \begin{Bmatrix} -T_{x_1} \\ T_{x_2} \end{Bmatrix} \end{aligned} \tag{6.208}$$

Associated with each element, the load vectors become

$$\mathbf{p}_{\mathbf{T}}^{(1)} = \begin{Bmatrix} -T_{x_1} \\ T_{x_2} \end{Bmatrix}, \quad \mathbf{p}_{\mathbf{T}}^{(2)} = \begin{Bmatrix} -T_{x_2} \\ T_{x_3} + P \end{Bmatrix}, \quad \mathbf{p}_{\mathbf{T}}^{(3)} = \begin{Bmatrix} -T_{x_3} \\ T_{x_4} \end{Bmatrix} \tag{6.209}$$

The global coefficient matrix, $\mathbf{K}$, and the load vector, $\mathbf{P}_{\mathbf{T}}$, are obtained from the "expanded" element coefficient matrices, $\mathbf{k}^{(e)}$, and the element load vectors, $\mathbf{p}_{\mathbf{T}}^{(e)}$, by summation in the form

$$\mathbf{K} = \sum_{e=1}^{E} \mathbf{k}^{(e)} \quad \text{and} \quad \mathbf{P_T} = \sum_{e=1}^{E} \mathbf{p}_{\mathbf{T}}^{(e)} \tag{6.210}$$

The "expanded" element matrices are the same size as the global matrix but have rows and columns of zeros corresponding to the nodes not associated with element (e). Specifically, the expanded form of the element stiffness and load vector becomes

$$\mathbf{k}^{(1)} = \begin{matrix} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \end{matrix} \begin{bmatrix} \alpha^{(1)} & -\alpha^{(1)} & 0 & 0 \\ -\alpha^{(1)} & \alpha^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix}; \quad \mathbf{p}_{\mathbf{T}}^{(1)} = \begin{matrix} \boxed{1} \end{matrix} \begin{bmatrix} -T_{x_1} \\ T_{x_2} \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} \tag{6.211}$$

$$\mathbf{k}^{(2)} = \begin{matrix} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha^{(2)} & -\alpha^{(2)} & 0 \\ 0 & -\alpha^{(2)} & \alpha^{(2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix}; \quad \mathbf{p}_{\mathbf{T}}^{(2)} = \begin{matrix} \boxed{1} \end{matrix} \begin{bmatrix} 0 \\ -T_{x_2} \\ T_{x_3} + P \\ 0 \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} \tag{6.212}$$

$$\mathbf{k}^{(3)} = \begin{matrix} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^{(3)} & -\alpha^{(3)} \\ 0 & 0 & -\alpha^{(3)} & \alpha^{(3)} \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix}; \quad \mathbf{p}_{\mathbf{T}}^{(3)} = \begin{matrix} \boxed{1} \end{matrix} \begin{bmatrix} 0 \\ 0 \\ -T_{x_3} \\ T_{x_4} \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} \tag{6.213}$$

In accordance with Eq. (6.210) and (6.192), the global equilibrium equations can be written as

$$\begin{bmatrix} \alpha^{(1)} & -\alpha^{(1)} & 0 & 0 \\ -\alpha^{(1)} & \left(\alpha^{(1)} + \alpha^{(2)}\right) & -\alpha^{(2)} & 0 \\ 0 & -\alpha^{(2)} & \left(\alpha^{(2)} + \alpha^{(3)}\right) & -\alpha^{(3)} \\ 0 & 0 & -\alpha^{(3)} & \alpha^{((3))} \end{bmatrix} \begin{Bmatrix} u_{x_1} \\ u_{x_2} \\ u_{x_3} \\ u_{x_4} \end{Bmatrix} = \begin{Bmatrix} -T_{x_1} \\ T_{x_2} - T_{x_2} \\ T_{x_3} + P - T_{x_3} \\ T_{x_4} \end{Bmatrix} \tag{6.214}$$

Enforcing the boundary conditions of $u_{x_1} = u_{x_4} = 0$ leads to

$$\begin{bmatrix} \alpha^{(1)} & -\alpha^{(1)} & 0 & 0 \\ -\alpha^{(1)} & \left(\alpha^{(1)}+\alpha^{(2)}\right) & -\alpha^{(2)} & 0 \\ 0 & -\alpha^{(2)} & \left(\alpha^{(2)}+\alpha^{(3)}\right) & -\alpha^{(3)} \\ 0 & 0 & -\alpha^{(3)} & \alpha^{(3)} \end{bmatrix} \begin{Bmatrix} 0 \\ u_{x_2} \\ u_{x_3} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -T_{x_1} \\ 0 \\ P \\ T_{x_4} \end{Bmatrix} \tag{6.215}$$

This system of equations can be partitioned in the form

$$\begin{bmatrix} -\alpha^{(1)} & \left(\alpha^{(1)}+\alpha^{(2)}\right) & -\alpha^{(2)} & 0 \\ 0 & -\alpha^{(2)} & \left(\alpha^{(2)}+\alpha^{(3)}\right) & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{x2} \\ u_{x3} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \tag{6.216a}$$

or

$$\begin{bmatrix} \left(\alpha^{(1)}+\alpha^{(2)}\right) & -\alpha^{(2)} \\ -\alpha^{(2)} & \left(\alpha^{(2)}+\alpha^{(3)}\right) \end{bmatrix} \begin{Bmatrix} u_{x2} \\ u_{x3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \tag{6.216b}$$

and

$$\begin{bmatrix} \alpha^{(1)} & -\alpha^{(1)} & 0 & 0 \\ 0 & 0 & -\alpha^{(3)} & \alpha^{(3)} \end{bmatrix} \begin{Bmatrix} 0 \\ u_{x_2} \\ u_{x_3} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -T_{x_1} \\ T_{x_4} \end{Bmatrix} \tag{6.217a}$$

or

$$T_{x_1} = \alpha^{(1)} u_{x_2} \text{ and } T_{x_4} = -\alpha^{(3)} u_{x_3} \tag{6.217b}$$

Solution to nodal displacements results in

$$u_{x_2} = \frac{\alpha^{(2)}}{\left(\alpha^{(1)}\alpha^{(2)} + \alpha^{(1)}\alpha^{(3)} + \alpha^{(2)}\alpha^{(3)}\right)} P \tag{6.218}$$

$$u_{x_3} = \frac{\alpha^{(1)} + \alpha^{(2)}}{\left(\alpha^{(1)}\alpha^{(2)} + \alpha^{(1)}\alpha^{(3)} + \alpha^{(2)}\alpha^{(3)}\right)} P \tag{6.219}$$

With these nodal displacements, the reaction forces are computed as

$$T_{x_1} = \frac{\alpha^{(1)}\alpha^{(2)}}{\left(\alpha^{(1)}\alpha^{(2)} + \alpha^{(1)}\alpha^{(3)} + \alpha^{(2)}\alpha^{(3)}\right)} P \tag{6.220}$$

$$T_{x_4} = -\frac{\alpha^{(3)}\left(\alpha^{(1)} + \alpha^{(2)}\right)}{\left(\alpha^{(1)}\alpha^{(2)} + \alpha^{(1)}\alpha^{(3)} + \alpha^{(2)}\alpha^{(3)}\right)} P \tag{6.221}$$

Finally, the strains are computed as

$$\varepsilon_{xx}^{(1)} = \frac{1}{L^{(1)}}\left(u_{x_2} - u_{x_1}\right) = \frac{\alpha^{(2)}}{\left(\alpha^{(1)}\alpha^{(2)} + \alpha^{(1)}\alpha^{(3)} + \alpha^{(2)}\alpha^{(3)}\right)L^{(1)}} P \tag{6.222}$$

$$\varepsilon_{xx}^{(2)} = \frac{1}{L^{(2)}}\left(u_{x_3} - u_{x_2}\right) = \frac{\alpha^{(1)}}{\left(\alpha^{(1)}\alpha^{(2)} + \alpha^{(1)}\alpha^{(3)} + \alpha^{(2)}\alpha^{(3)}\right)L^{(2)}} P \tag{6.223}$$

$$\varepsilon_{xx}^{(3)} = \frac{1}{L^{(3)}}\left(u_{x_4} - u_{x_3}\right) = -\frac{\left(\alpha^{(1)} + \alpha^{(2)}\right)}{\left(\alpha^{(1)}\alpha^{(2)} + \alpha^{(1)}\alpha^{(3)} + \alpha^{(2)}\alpha^{(3)}\right)L^{(3)}} P \tag{6.224}$$

6.2.2 Two-Dimensional Structural Analysis

The three-dimensional analysis of either "thin" or "long" components subjected to in-plane external loading conditions can be reduced to a two-dimensional analysis under certain assumptions referred to as "plane stress" and "plane strain" conditions.

6.2.2.1 Plane Stress Conditions

A state of plane stress exists for thin components subjected only to in-plane external loading, i.e., no lateral loads (Fig. 6.26). Due to a small thickness-to-characteristic length ratio and in-plane external loading only, there is no out-of-plane displacement component, u_z, and the shear strain components associated with the thickness direction, γ_{xz} and γ_{yz}, are very small and assumed to be zero. Therefore, the stress components, σ_{zz}, σ_{xz}, and σ_{yz}, associated with the thickness direction vanish. Under these assumptions, the displacement, $\mathbf{u}$, stress, $\mathbf{\dot{A}}$, strain, $\boldsymbol{\varepsilon}$, and traction, $\mathbf{T}$, vectors, and material property matrix, $\mathbf{D}$, reduce to

$$\begin{aligned}
\mathbf{u}^T &= \left\{u_x \quad u_y\right\} \\
\boldsymbol{\sigma}^T &= \left\{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy}\right\} \\
\boldsymbol{\varepsilon}^T &= \left\{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}\right\} \\
\mathbf{T}^T &= \left\{T_x \quad T_y\right\}
\end{aligned} \tag{6.225}$$

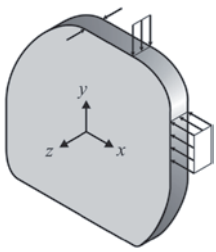

Fig. 6.26 Thin body with in-plane loading; suitable for plane stress idealization

and

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix} \tag{6.226}$$

with

$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) \tag{6.227}$$

The initial strains arising from ΔT, the temperature change with respect to the reference state, can be expressed as

$$\boldsymbol{\varepsilon}^{*T} = \begin{bmatrix} \alpha\Delta T & \alpha\Delta T & 0 \end{bmatrix} \tag{6.228}$$

6.2.2.2 Plane Strain Conditions

A state of plane strain exists for a cylindrical component that is either "long" or fully constrained in the length direction under the action of only uniform lateral external loads (two examples are shown in Fig. 6.27). Because the ends of the cylindrical component are prevented from deforming in the thickness direction, it is assumed that the displacement component u_z vanishes at every cross section of the body. The uniform loading and cross-sectional geometry eliminates any variation in the length direction, leading to $\partial() / \partial z = 0$. Also, planes perpendicular to the thickness direction before deformation remain perpendicular to the thickness direction after deformation. These assumptions result in zero transverse shear strains, $\gamma_{xz} = \gamma_{yz} = 0$. Under these assumptions, the displacement, **u**, stress, **A**, strain, **ε**, and traction, **T**, vectors, and material property matrix, **D,** reduce to

$$\begin{aligned} \mathbf{u}^T &= \begin{Bmatrix} u_x & u_y \end{Bmatrix} \\ \boldsymbol{\sigma}^T &= \begin{Bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{Bmatrix} \\ \boldsymbol{\varepsilon}^T &= \begin{Bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{Bmatrix} \\ \mathbf{T}^T &= \begin{Bmatrix} T_x & T_y \end{Bmatrix} \end{aligned} \tag{6.229}$$

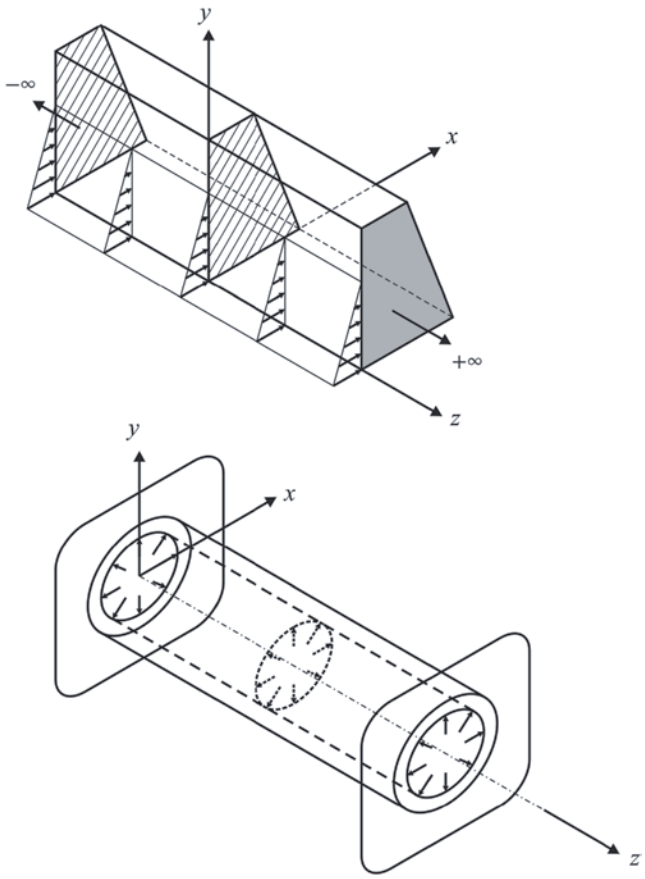

Fig. 6.27 Long bodies with in-plane loading; suitable for plane strain idealization

and

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{(1-2\nu)}{2} \end{bmatrix} \tag{6.230}$$

The initial strain vector due to this temperature change can be expressed as

$$\boldsymbol{\varepsilon}^{*T} = \begin{bmatrix} (1+\nu)\alpha\Delta T & (1+\nu)\alpha\Delta T & 0 \end{bmatrix} \tag{6.231}$$

where ΔT is the temperature change with respect to a reference state.

The material property matrices for both plane stress and strain conditions have the same form, and it is convenient to present it in the form

$$\mathbf{D} = \begin{bmatrix} D_1 & D_1 D_2 & 0 \\ D_1 D_2 & D_1 & 0 \\ 0 & 0 & D_{12} \end{bmatrix} \tag{6.232}$$

where

$$D_{12} = \frac{D_1(1-D_2)}{2} \tag{6.233}$$

with $D_1 = E/(1-\nu)^2$ and $D_2 = \nu$ for plane stress, and $D_1 = E(1-\nu)/\ (1+\nu)(1-2\nu)$ and $D_2 = \nu/(1-\nu)$ for plane strain.

6.2.2.3 Finite Element Equations with Linear Triangular Elements

The displacement components u_x and u_y within a triangular element can be approximated as

$$\begin{aligned} u_x^{(e)} &= \tilde{u}_x^{(e)} = N_1^{(e)} u_{x_1}^{(e)} + N_2^{(e)} u_{x_2}^{(e)} + N_3^{(e)} u_{x_3}^{(e)} \\ u_y^{(e)} &= \tilde{u}_y^{(e)} = N_1^{(e)} u_{y_1}^{(e)} + N_2^{(e)} u_{y_2}^{(e)} + N_3^{(e)} u_{y_3}^{(e)} \end{aligned} \tag{6.234}$$

in which $N_1^{(e)}$, $N_2^{(e)}$, and $N_3^{(e)}$ are the linear shape functions and $(u_{x_1}^{(e)}, u_{y_1}^{(e)})$, $(u_{x_2}^{(e)}, u_{y_2}^{(e)})$, and $(u_{x_3}^{(e)}, u_{y_3}^{(e)})$ are the nodal unknowns (degrees of freedom) associated with first, second, and third nodes, respectively. An example of a triangular element with its nodal degrees of freedom and local nodal numbering is shown in Fig. 6.28. In matrix form, the approximate displacement components become

$$\tilde{\mathbf{u}}^{(e)} = \mathbf{N}^{(e)T} \mathbf{U}^{(e)} \tag{6.235}$$

in which

$$\tilde{\mathbf{u}}^{(e)T} = \begin{Bmatrix} \tilde{u}_x^{(e)} & \tilde{u}_y^{(e)} \end{Bmatrix} \tag{6.236}$$

and

$$\mathbf{N}^{(e)T} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \tag{6.237}$$

and

$$\mathbf{U}^{(e)T} = \begin{Bmatrix} u_{x_1}^{(e)} & u_{y_1}^{(e)} & u_{x_2}^{(e)} & u_{y_2}^{(e)} & u_{x_3}^{(e)} & u_{y_3}^{(e)} \end{Bmatrix} \tag{6.238}$$

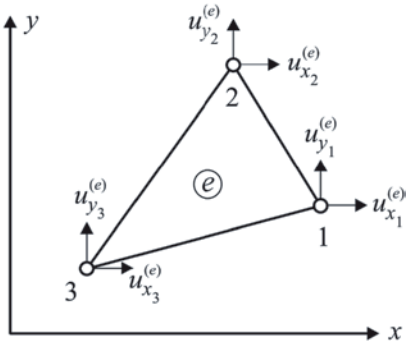

Fig. 6.28 Typical linear triangular element with nodal degrees of freedom

The element shape matrix, $\mathbf{B}^{(e)}$, becomes

$$\mathbf{B}^{(e)} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \tag{6.239}$$

Substituting for the derivatives of the shape functions, this matrix simplifies to

$$\mathbf{B}^{(e)} = \frac{1}{2\Delta^{(e)}} \begin{bmatrix} y_{23}^{(e)} & 0 & y_{31}^{(e)} & 0 & y_{12}^{(e)} & 0 \\ 0 & x_{32}^{(e)} & 0 & x_{13}^{(e)} & 0 & x_{21}^{(e)} \\ x_{32}^{(e)} & y_{23}^{(e)} & x_{13}^{(e)} & y_{31}^{(e)} & x_{21}^{(e)} & y_{12}^{(e)} \end{bmatrix} \tag{6.240}$$

Both the element shape and material property matrices are independent of the spatial coordinates, x and y, thus leading to the evaluation of the element stiffness matrix, $\mathbf{k}^{(e)}$, as

$$\mathbf{k}^{(e)} = \mathbf{B}^{(e)T}\mathbf{D}\mathbf{B}^{(e)}V^{(e)} \tag{6.241}$$

where $V^{(e)} = t\Delta^{(e)}$, with element area $\Delta^{(e)}$ and constant thickness t. The evaluation of the load vectors, $\mathbf{p}_\mathbf{b}^{(e)}$ and $\mathbf{p}_\mathbf{T}^{(e)}$, arising from the body forces and surface tractions (forces), respectively, involve integrals of the form

$$\int dx\, dy, \int x dx\, dy, \int y\, dx\, dy \tag{6.242}$$

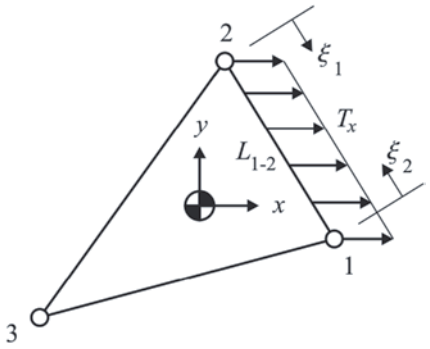

Fig. 6.29 Surface force along side 1–2 of the triangular element

By choosing the centroid of the triangle as the origin of the (x, y) coordinate system, the integrals involving either x or y in the integrand vanish. The load vector arising from the body forces can be obtained from

$$\mathbf{p}_{\mathbf{b}}^{(e)} = \int_{V^{(e)}} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dV = \int_{V^{(e)}} \begin{Bmatrix} N_1 b_x \\ N_1 b_y \\ N_2 b_x \\ N_2 b_y \\ N_3 b_x \\ N_3 b_y \end{Bmatrix} dV \tag{6.243}$$

reducing to

$$\mathbf{p}_{\mathbf{b}}^{(e)T} = \frac{t\Delta^{(e)}}{3} \begin{bmatrix} b_x & b_y & b_x & b_y & b_x & b_y \end{bmatrix} \tag{6.244}$$

in which b_x and b_y are the components of the body force vector.

The evaluation of the element load vector due to the applied traction forces (distributed loads as shown in Fig. 6.29) requires their explicit variation along the edges of the element. For an element of constant thickness subjected to uniform load of T_x in the x-direction along its 1–2 edge, the vector $\mathbf{p}_{\mathbf{T}}^{(e)}$ can be written as

$$\mathbf{p}_{\mathbf{T}}^{(e)} = t \int_{L_{1-2}} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} T_x \\ 0 \end{Bmatrix} dl = t \int_{L_{1-2}} \begin{Bmatrix} N_1 T_x \\ 0 \\ N_2 T_x \\ 0 \\ 0 \\ 0 \end{Bmatrix} dl \tag{6.245}$$

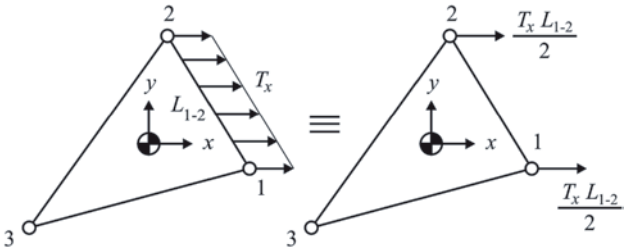

Fig. 6.30 Equivalent nodal forces for the surface force along side 1–2 of the triangular element

in which $N_3 = 0$ along the 1–2 edge and L_{1-2} is the length of the 1–2 edge. Since N_1 and N_2 vary linearly along the 1–2 edge, they can be expressed in terms of the natural coordinates, ξ_1 and ξ_2, as derived in Chap. 3

$$N_1 = \xi_1 = \frac{x_2^{(e)} - x}{x_2^{(e)} - x_1^{(e)}} \quad \text{and} \quad N_2 = \xi_2 = \frac{x - x_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} \tag{6.246}$$

The integrals in the expression for $\mathbf{p}_{\mathbf{T}}^{(e)}$ are evaluated as

$$\begin{aligned} \int_{L_{1-2}} N_1 \, T_x dl &= \int_0^1 \xi_1 \, T_x L_{1-2} d\xi_1 = \frac{T_x \, L_{1-2}}{2} \\ \int_{L_{1-2}} N_2 \, T_x dl &= \int_0^1 \xi_2 \, T_x L_{1-2} d\xi_2 = \frac{T_x \, L_{1-2}}{2} \end{aligned} \tag{6.247}$$

Thus, the load vector, $\mathbf{p}_{\mathbf{T}}^{(e)}$, takes the form

$$\mathbf{p}_{\mathbf{T}}^{(e)T} = t \frac{T_x \, L_{1-2}}{2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{6.248}$$

as illustrated in Fig. 6.30.

Note that this result corresponds to equivalent point forces acting at the first and second nodes. The element load vectors arising from the initial strains and stresses can be written as

$$\begin{aligned} \mathbf{p}_{\varepsilon^*}^{(e)} &= \mathbf{B}^{(e)T} \mathbf{D} \boldsymbol{\varepsilon}^* \, V^{(e)} \\ \mathbf{p}_{\sigma^*}^{(e)} &= \mathbf{B}^{(e)T} \boldsymbol{\sigma}^* \, V^{(e)} \end{aligned} \tag{6.249}$$

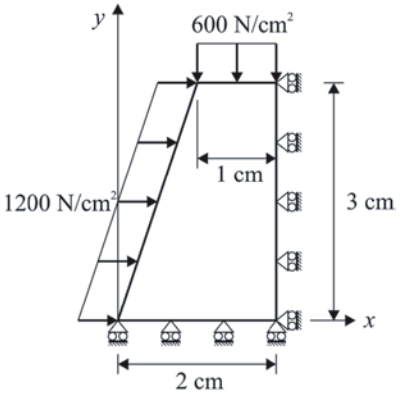

Fig. 6.31 Geometry and loading of the problem

6.2.2.4 Example of a Plane Stress Analysis with Linear Triangular Elements

Derivation of a System of Equations and Its Solution

Using linear triangular elements, determine the nodal displacements and the element stresses in a thin plate subjected to displacement constraints and surface tractions as shown in Fig. 6.31. Also, the plate is exposed to a temperature change of $10\ ^{\circ}\mathrm{C}$ from the reference temperature. The plate thickness is 0.5 cm and the Young's modulus, E, and the Poisson's ratio, ν, are $15\times10^{6}\,\mathrm{N/cm^2}$ and 0.25, respectively. The coefficient of thermal expansion is $6\times10^{-6}/\,^{\circ}\mathrm{C}$.

In order to illustrate the solution method, the plate is discretized into two triangular elements, as shown in Fig. 6.32.

The global coordinates of each node are specified by (x_p, y_p), with $p = 1,2,3,4$, and are presented in Table 6.7.

The global unknown nodal displacement vector is given by

$$\mathbf{U}^T = \left\{ u_{x_1} \quad u_{y_1} \quad u_{x_2} \quad u_{y_2} \quad u_{x_3} \quad u_{y_3} \quad u_{x_4} \quad u_{y_4} \right\} \tag{6.250}$$

Considering the correspondence between the local and global node numbering schemes, the elements are defined (connected) as shown in Table 6.8.

The areas of each element are calculated to be

$$\Delta^{(1)} = 3\ \mathrm{cm}^2 \quad \text{and} \quad \Delta^{(2)} = 3/2\ \mathrm{cm}^2 \tag{6.251}$$

Under plane stress assumptions, the material property matrix becomes

$$\mathbf{D} = 10^6 \begin{bmatrix} 16 & 4 & 0 \\ 4 & 16 & 0 \\ 0 & 0 & 6 \end{bmatrix} \mathrm{N/cm^2} \tag{6.252}$$

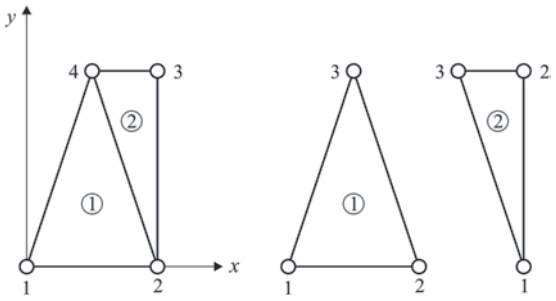

Fig. 6.32 Global and local numbering of nodes and elements

The initial strains arising from the temperature change is written as

$$\boldsymbol{\varepsilon}^{*T} = 10^{-6}\begin{bmatrix}60 & 60 & 0\end{bmatrix} \tag{6.253}$$

The element load vectors arising from the applied tractions are

$$\mathbf{p}_{\mathbf{T}}^{(1)T} = t\frac{T_x L_{1-4}}{2}\begin{bmatrix}1 & 0 & 0 & 0 & 1 & 0\end{bmatrix} \tag{6.254}$$

$$\mathbf{p}_{\mathbf{T}}^{(2)T} = t\frac{T_y L_{3-4}}{2}\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 1\end{bmatrix} \tag{6.255}$$

With the specified values of the thickness and the distributed loads, these element load vectors become

$$\mathbf{p}_{\mathbf{T}}^{(1)T} = 300\sqrt{10}\begin{bmatrix}1 & 0 & 0 & 0 & 1 & 0\end{bmatrix}N \tag{6.256}$$

and

$$\mathbf{p}_{\mathbf{T}}^{(2)T} = -150\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 1\end{bmatrix}N \tag{6.257}$$

For the first element, $e = 1$, the components of the element shape matrix $\mathbf{B}^{(1)}$ are computed as

$$\begin{aligned} y_{23}^{(1)} &= y_2^{(1)} - y_3^{(1)} = y_2 - y_4 = -3\,, & x_{32}^{(1)} &= x_3^{(1)} - x_2^{(1)} = x_4 - x_2 = -1 \\ y_{31}^{(1)} &= y_3^{(1)} - y_1^{(1)} = y_4 - y_1 = 3\,, & x_{13}^{(1)} &= x_1^{(1)} - x_3^{(1)} = x_1 - x_4 = -1 \\ y_{12}^{(1)} &= y_1^{(1)} - y_2^{(1)} = y_1 - y_2 = 0\,, & x_{21}^{(1)} &= x_2^{(1)} - x_1^{(1)} = x_2 - x_1 = 2 \end{aligned} \tag{6.258}$$

leading to

$$\mathbf{B}^{(1)} = \frac{1}{6}\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0\end{bmatrix} \tag{6.259}$$

Table 6.7 Global nodal coordinates

Global node number	Nodal coordinates	Nodal unknowns
1	(0,0)	u_{x_1}, u_{y_1}
2	(2,0)	u_{x_2}, u_{y_2}
3	(2,3)	u_{x_3}, u_{y_3}
4	(1,3)	u_{x_4}, u_{y_4}

Table 6.8 Element connectivity

Element Number (e)	Node 1	Node 2	Node 3
1	1	2	4
2	2	3	4

For the second element, $e = 2$, the components of the element shape matrix $\mathbf{B}^{(2)}$ are computed as

$$\begin{aligned} y_{23}^{(2)} &= y_2^{(2)} - y_3^{(2)} = y_3 - y_4 = 0\,, \quad & x_{32}^{(2)} &= x_3^{(2)} - x_2^{(2)} = x_4 - x_3 = -1 \\ y_{31}^{(2)} &= y_3^{(2)} - y_1^{(2)} = y_4 - y_2 = 3\,, \quad & x_{13}^{(2)} &= x_1^{(2)} - x_3^{(2)} = x_2 - x_4 = 1 \\ y_{12}^{(2)} &= y_1^{(2)} - y_2^{(2)} = y_2 - y_3 = -3\,, \quad & x_{21}^{(2)} &= x_2^{(2)} - x_1^{(2)} = x_3 - x_2 = 0 \end{aligned} \tag{6.260}$$

leading to

$$\mathbf{B}^{(2)} = \frac{1}{3}\begin{bmatrix} 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 3 & 0 & -3 \end{bmatrix} \tag{6.261}$$

The evaluation of the stiffness matrices, $\mathbf{k}^{(1)}$ and $\mathbf{k}^{(2)}$, requires the products of $\mathbf{B}^{(1)T}\mathbf{D}$ and $\mathbf{B}^{(2)T}\mathbf{D}$. Also, these products appear in the evaluation of the element load vectors arising from the temperature change. Therefore,

$$\mathbf{B}^{(1)T}\mathbf{D} = \frac{10^6}{6}\begin{bmatrix} -48 & -12 & -6 \\ -4 & -16 & -18 \\ 48 & 12 & -6 \\ -4 & -16 & 18 \\ 0 & 0 & 12 \\ 8 & 32 & 0 \end{bmatrix} \tag{6.262}$$

$$\mathbf{B}^{(2)T}\mathbf{D} = \frac{10^6}{3}\begin{bmatrix} 0 & 0 & -6 \\ -4 & -16 & 0 \\ 48 & 12 & 6 \\ 4 & 16 & 18 \\ -48 & -12 & 0 \\ 0 & 0 & -18 \end{bmatrix} \tag{6.263}$$

The element stiffness matrices become

$$\mathbf{k}^{(1)} = \frac{10^6}{12}\begin{bmatrix} \boxed{i=1} & & \boxed{j=2} & & \boxed{k=4} & \\ 75 & 15 & -69 & -3 & -6 & -12 \\ 15 & 35 & 3 & -19 & -18 & -16 \\ -69 & 3 & 75 & -15 & -6 & 12 \\ -3 & -19 & -15 & 35 & 18 & -16 \\ -6 & -18 & -6 & 18 & 12 & 0 \\ -12 & -16 & 12 & -16 & 0 & 32 \end{bmatrix} \tag{6.264}$$

and

$$\mathbf{k}^{(2)} = \frac{10^6}{12}\begin{bmatrix} \boxed{i=2} & & \boxed{j=3} & & \boxed{k=4} & \\ 6 & 0 & -6 & -18 & 0 & 18 \\ 0 & 16 & -12 & -16 & 12 & 0 \\ -6 & -12 & 150 & 30 & -144 & -18 \\ -18 & -16 & 30 & 70 & -12 & -54 \\ 0 & 12 & -144 & -12 & 144 & 0 \\ 18 & 0 & -18 & -54 & 0 & 54 \end{bmatrix} \tag{6.265}$$

The boxed numbers above each column pair indicate the nodal order of degrees of freedom in each element stiffness matrix.

The thermal load vectors associated with each element are obtained as

$$\mathbf{p}^{(1)}_{\varepsilon^*} = \begin{Bmatrix} -900 \\ -300 \\ 900 \\ -300 \\ 0 \\ 600 \end{Bmatrix} N \quad \text{and} \quad \mathbf{p}^{(2)}_{\varepsilon^*} = \begin{Bmatrix} 0 \\ -300 \\ 900 \\ 300 \\ -900 \\ 0 \end{Bmatrix} N \tag{6.266}$$

Rewriting the element stiffness matrices and the load vectors, in the expanded order and rearranged form according to the increasing nodal degrees of freedom of the global stiffness matrix, $\mathbf{K}$ yields

Associated with the first element:

$$\mathbf{k}^{(1)} = \frac{10^6}{12}\begin{array}{c} \begin{matrix} \boxed{1} & & \boxed{2} & & \boxed{3} & & \boxed{4} & \end{matrix} \\ \begin{bmatrix} 75 & 15 & -69 & -3 & 0 & 0 & -6 & -12 \\ 15 & 35 & 3 & -19 & 0 & 0 & -18 & -16 \\ -69 & 3 & 75 & -15 & 0 & 0 & -6 & 12 \\ -3 & -19 & -15 & 35 & 0 & 0 & 18 & -16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -18 & -6 & 18 & 0 & 0 & 12 & 0 \\ -12 & -16 & 12 & -16 & 0 & 0 & 0 & 32 \end{bmatrix} \end{array} \tag{6.267}$$

$$\mathbf{p}_{\mathbf{T}}^{(1)} = 300\sqrt{10}\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} N \quad \text{and} \quad \mathbf{p}_{\varepsilon^*}^{(1)} = \begin{Bmatrix} -900 \\ -300 \\ 900 \\ -300 \\ 0 \\ 0 \\ 0 \\ 600 \end{Bmatrix} N \tag{6.268}$$

Associated with the second element:

$$\mathbf{k}^{(2)} = \frac{10^6}{12}\begin{array}{c} \begin{matrix} \boxed{1} & & \boxed{2} & & \boxed{3} & & \boxed{4} & \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & -6 & -18 & 0 & 18 \\ 0 & 0 & 0 & 16 & -12 & -16 & 12 & 0 \\ 0 & 0 & -6 & -12 & 150 & 30 & -144 & -18 \\ 0 & 0 & -18 & -16 & 30 & 70 & -12 & -54 \\ 0 & 0 & 0 & 12 & -144 & -12 & 144 & 0 \\ 0 & 0 & 18 & 0 & -18 & -54 & 0 & 54 \end{bmatrix} \end{array} \tag{6.269}$$

$$\mathbf{p}_{\mathbf{T}}^{(2)} = -150\begin{Bmatrix}0\\0\\0\\0\\0\\1\\0\\1\end{Bmatrix}N \quad \text{and} \quad \mathbf{p}_{\varepsilon^*}^{(2)} = \begin{Bmatrix}0\\0\\0\\-300\\900\\300\\-900\\0\end{Bmatrix}N \tag{6.270}$$

Summation of the element stiffness matrices

$$\mathbf{K} = \sum_{e=1}^{E} \mathbf{k}^{(e)} \tag{6.271}$$

and load vectors

$$\mathbf{P} = \sum_{e=1}^{E=2} \left(\mathbf{p}_{\mathbf{T}}^{(e)} + \mathbf{p}_{\varepsilon^*}^{(e)} \right) \tag{6.272}$$

results in the global stiffness matrix and the global load vector as

$$\mathbf{K} = \frac{10^6}{12}\begin{bmatrix} 75 & 15 & -69 & -3 & 0 & 0 & -6 & -12 \\ 15 & 35 & 3 & -19 & 0 & 0 & -18 & -16 \\ -69 & 3 & (75+6) & -15 & -6 & -18 & -6 & (12+18) \\ -3 & -19 & -15 & (35+6) & -12 & -16 & (18+12) & -16 \\ 0 & 0 & -6 & -12 & 150 & 30 & -144 & -18 \\ 0 & 0 & -18 & -16 & 30 & 64 & -12 & -48 \\ -6 & -18 & -6 & (18+12) & -144 & -12 & (12+144) & 0 \\ -12 & -16 & (12+18) & -16 & -18 & -48 & 0 & (32+48) \end{bmatrix} \tag{6.273}$$

and

$$\mathbf{P} = \begin{Bmatrix} (300\sqrt{10}-900) \\ -300 \\ 900 \\ (-300-300) \\ 900 \\ (-150+300) \\ (300\sqrt{10}-900) \\ -150+600 \end{Bmatrix}N \tag{6.274}$$

The final form of the global system of equations becomes

$$\frac{10^6}{12}\begin{bmatrix} 75 & 15 & -69 & -3 & 0 & 0 & -6 & -12 \\ 15 & 35 & 3 & -19 & 0 & 0 & -18 & -16 \\ -69 & 3 & (75+6) & -15 & -6 & -18 & -6 & (12+18) \\ -3 & -19 & -15 & (35+6) & -12 & -16 & (18+12) & -16 \\ 0 & 0 & -6 & -12 & 150 & 30 & -144 & -18 \\ 0 & 0 & -18 & -16 & 30 & 70 & -12 & -54 \\ -6 & -18 & -6 & (18+12) & -144 & -12 & (12+144) & 0 \\ -12 & -16 & (12+18) & -16 & -18 & -54 & 0 & (32+54) \end{bmatrix} \times \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{Bmatrix} = \begin{Bmatrix} (300\sqrt{10}-900) \\ -300 \\ 900 \\ (-300-300) \\ 900 \\ (-150+300) \\ (300\sqrt{10}-900) \\ -150+600 \end{Bmatrix} \tag{6.275}$$

Applying the prescribed values of the displacement components leads to

$$\frac{10^6}{12}\begin{bmatrix} 75 & 15 & -69 & -3 & 0 & 0 & -6 & -12 \\ 15 & 35 & 3 & -19 & 0 & 0 & -18 & -16 \\ -69 & 3 & (75+6) & -15 & -6 & -18 & -6 & (12+18) \\ -3 & -19 & -15 & (35+6) & -12 & -16 & (18+12) & -16 \\ 0 & 0 & -6 & -12 & 150 & 30 & -144 & -18 \\ 0 & 0 & -18 & -16 & 30 & 70 & -12 & -54 \\ -6 & -18 & -6 & (18+12) & -144 & -12 & (12+144) & 0 \\ -12 & -16 & (12+18) & -16 & -18 & -54 & 0 & (32+54) \end{bmatrix} \times \begin{Bmatrix} u_{x_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{Bmatrix} = \begin{Bmatrix} (300\sqrt{10}-900) \\ -300 \\ 900 \\ (-300-300) \\ 900 \\ (-150+300) \\ (300\sqrt{10}-900) \\ -150+600 \end{Bmatrix} \tag{6.276}$$

Eliminating the rows and columns corresponding to zero displacement components simplifies the global system of equations to

$$\frac{10^6}{12}\begin{bmatrix} 75 & 0 & -6 & -12 \\ 0 & 70 & -12 & -54 \\ -6 & -12 & (12+144) & 0 \\ -12 & -54 & 0 & (32+54) \end{bmatrix}\begin{Bmatrix} u_{x_1} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{Bmatrix} = \begin{Bmatrix} (300\sqrt{10}-900) \\ (-150+300) \\ (300\sqrt{10}-900) \\ (-150+600) \end{Bmatrix} \tag{6.277}$$

The solution to this system of equations results in the values for the unknown displacement components as

$$\begin{Bmatrix} u_{x_1} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{Bmatrix} = \begin{Bmatrix} 0.0000357839 \\ 0.000157003 \\ 0.0000171983 \\ 0.000166367 \end{Bmatrix} \text{cm} \tag{6.278}$$

6.2.2.5 ANSYS Solution

The nodal displacements of the plate subjected to uniform temperature can also be obtained using ANSYS. The solution procedure is outlined as follows:

Model Generation

Specify the element type (**ET** command) using the following menu path:

Main Menu > Preprocessor > Element Type > Add/Edit/Delete

- Click on ***Add***.
- Select ***Solid*** immediately below ***Structural Mass*** from the left list and ***Quad 4node 182*** from the right list; click on ***OK***.
- Click on ***Options***.
- In order to specify the 2-D idealization as plane stress with thickness, in the newly appeared dialog box pull down the menu for ***Element behavior K3*** and select ***Plane strs w/thk***; click on ***OK*** (Fig. 6.33).
- Click on ***Close***.

- Specify real constants (**R** command) using the following menu path:

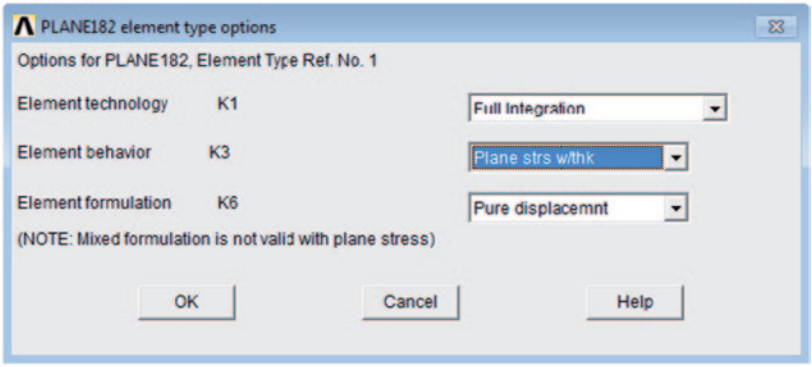

Fig. 6.33 Specification of element options

Main Menu > Preprocessor > Real Constants > Add/Edit/Delete

- Click on ***Add***.
- Click on ***OK***.
- Enter ***5e−3*** for ***Thickness THK***; click on ***OK***.
- Click on ***Close***.

- Specify material properties (**MP** command) using the following menu path:

Main Menu > Preprocessor > Material Props > Material Models

- In the *Define Material Model Behavior* dialog box, in the right window, successively left-click on ***Structural***, ***Linear, Elastic***, and, finally, ***Isotropic***, which will bring another dialog box.
- Enter ***150e9*** for ***EX***, and ***0.25*** for ***PRXY***; click on ***OK***.
- In the *Define Material Model Behavior* dialog box, in the right window, under ***Structural*** find ***Thermal Expansion, Secant Coefficient***, and ***Isotropic***, which will bring another dialog box (Fig. 6.34).
- Enter ***6e−6*** for ***APLX***; click on ***OK***.
- Close the *Define Material Model Behavior* dialog box by using the following menu path:

Material > Exit

- Create nodes (**N** command) using the following menu path:

Main Menu > Preprocessor > Modeling > Create > Nodes > In Active CS

- A total of 4 nodes will be created (Table 6.7).
- Referring to Table 6.7, enter *x*- and *y*-coordinates of node 1 (be sure to convert the coordinates to meters), and Click on ***Apply***. This action will keep the *Create Nodes in Active Coordinate System* dialog box open. If the ***Node number*** field is left blank, then ANSYS will assign the lowest available node number to the node that is being created.

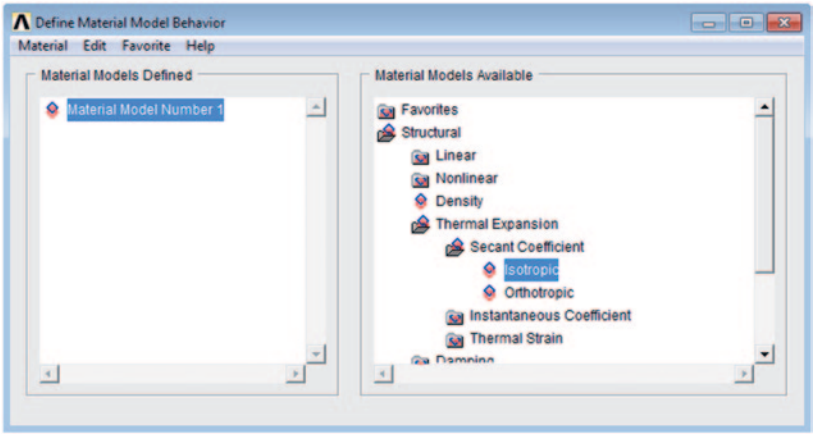

Fig. 6.34 Specification of material behavior

Fig. 6.35 Generation of nodes

- Repeat the same procedure for the nodes 2 through 4.
- After entering the *x*- and *y*-coordinates of node 4, click on ***OK*** (instead of ***Apply***).
- The nodes should appear in the *Graphics Window*, as shown in Fig. 6.35.

- Create elements (**E** command) using the following menu path:

Main Menu > Preprocessor > Modeling > Create > Elements > Auto Numbered > Thru Nodes

- *Pick Menu* appears; refer to Fig. 6.36 to create elements by picking ***three*** nodes at a time and clicking on ***Apply*** in between.
- Observe the elements created after clicking on ***Apply*** in the *Pick Menu.*
- Repeat until the last element is created.
- Click on ***OK*** when the last element is created.

- Review elements:

- Turn on element numbering using the following menu path:

Utility Menu > PlotCtrls > Numbering

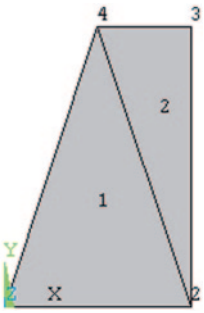

Fig. 6.36 Generation of elements

- Select ***Element numbers*** from the first pull-down menu; click on ***OK***.
- Plot elements (**`EPLOT`** command) using the following menu path:

Utility Menu > Plot > Elements

- Figure 6.36 shows the outcome of this action as it appears in the *Graphics Window*.
- Turn off element numbering and turn on node numbering using the following menu path:

Utility Menu > PlotCtrls > Numbering

- Place a *checkmark* by clicking on the empty box next to ***NODE Node numbers***.
- Select ***No numbering*** from the first pull-down menu.
- Click on ***OK***.
- Plot nodes (**`NPLOT`** command) using the following menu path:

Utility Menu > Plot > Nodes

- Figure 6.35 shows the outcome of this action as it appears in the *Graphics Window*.

Solution

- Apply displacement boundary conditions (**`D`** command) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Structural > Displacement > On Nodes

- *Pick Menu* appears; pick nodes 1 and 2 along the bottom horizontal boundary (Fig. 6.35) and click on ***OK*** on *Pick Menu.*
- Highlight ***UY*** and enter ***0*** for ***VALUE***; click on ***Apply***.
- ***Pick Menu*** reappears; pick nodes 2 and 3 along the right vertical boundary (Fig. 6.35) and click on ***OK*** on *Pick Menu.*
- Highlight ***UX*** and remove the highlight on ***UY***; enter ***0*** for ***VALUE***; click on ***OK***.

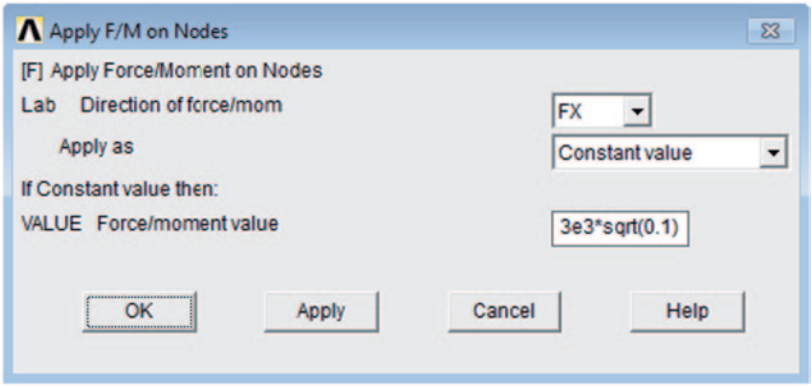

Fig. 6.37 Application of external loads

- Apply force boundary conditions on nodes (**F** command) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Structural > Force/Moment > On Nodes

  - *Pick Menu* appears; pick nodes 1 and 4 along the slanted boundary; click on ***OK***.
  - Enter ***3e3*sqrt(0.1)*** for ***VALUE*** (Fig. 6.37).
  - Click on ***Apply***.
  - *Pick Menu* reappears; pick nodes 4 and 3 along the top horizontal boundary; click on ***OK***.
  - Pull down the menu for ***Direction of force/mom*** and select ***FY***; Enter − ***150*** for ***VALUE***; click on ***OK***.

- Apply thermal load (**TUNIF** command) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Structural > Temperature > Uniform Temp

  - *Uniform Temperature* dialog box appears; Enter ***10*** for ***Uniform temperature***.
  - Click on ***OK***.

- Obtain solution (**SOLVE** command) using the following menu path:

Main Menu > Solution > Solve > Current LS

  - *Confirmation Window* appears along with *Status Report Window*.
  - Review status. If OK, close the *Status Report Window* and click on ***OK*** in *Confirmation Window*.
  - Wait until ANSYS responds with ***Solution is done!***

Postprocessing

- Review deformed shape (**PLDISP** command) using the following menu path:

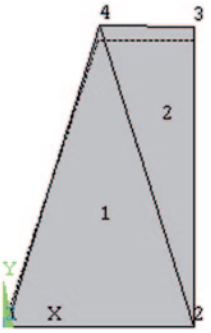

Fig. 6.38 Deformed configuration

Main Menu > General Postproc > Plot Results > Deformed Shape

- In the *Plot Deformed Shape* dialog box, choose the radio-button for ***Def+undef edge***; click on ***OK***.
- The deformed shape will appear in the *Graphics Window*, as shown in Fig. 6.38.

- Review displacement values (**PRNSOL** command) using the following menu path:

Main Menu > General Postproc > List Results > Nodal Solution

- Under ***Nodal Solution***, click on ***DOF Solution*** and ***Displacement vector sum***; click on ***OK***.
- The list will appear in a new window, as shown in Fig. 6.39.

PRNSOL Command

File

```
PRINT DOF  NODAL SOLUTION PER NODE

 ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

 LOAD STEP=     1  SUBSTEP=     1
  TIME=    1.0000      LOAD CASE=   0

 THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

   NODE      UX          UY
      1  0.35784E-06  0.0000
      2   0.0000      0.0000
      3   0.0000     0.15700E-05
      4  0.17198E-06 0.16637E-05

MAXIMUM ABSOLUTE VALUES
NODE         1           4
VALUE   0.35784E-06 0.16637E-05
```

Fig. 6.39 List of nodal displacements

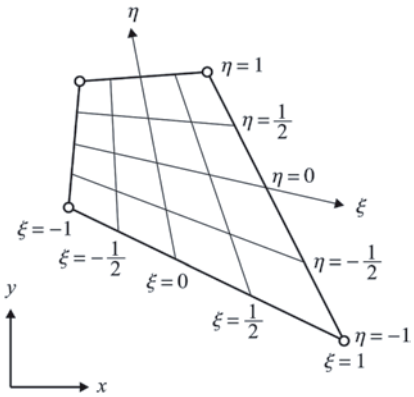

Fig. 6.40 Variation of the natural coordinates in a typical quadrilateral element

Finite Element Equations with Linear Quadrilateral Isoparametric Elements

The displacement components u_x and u_y within a quadrilateral element can be approximated as

$$\begin{aligned} u_x^{(e)} &= \tilde{u}_x^{(e)} = N_1^{(e)} u_{x_1}^{(e)} + N_2^{(e)} u_{x_2}^{(e)} + N_3^{(e)} u_{x_3}^{(e)} + N_4^{(e)} u_{x_4}^{(e)} \\ u_y^{(e)} &= \tilde{u}_y^{(e)} = N_1^{(e)} u_{y_1}^{(e)} + N_2^{(e)} u_{y_2}^{(e)} + N_3^{(e)} u_{y_3}^{(e)} + N_4^{(e)} u_{y_4}^{(e)} \end{aligned} \tag{6.279}$$

in which $N_1^{(e)}$, $N_2^{(e)}$, $N_3^{(e)}$, and $N_4^{(e)}$ are the linear shape functions and $(u_{x_1}^{(e)}, u_{y_1}^{(e)})$, $(u_{x_2}^{(e)}, u_{y_2}^{(e)})$, $(u_{x_3}^{(e)}, u_{y_3}^{(e)})$, and $(u_{x_4}^{(e)}, u_{y_4}^{(e)})$ are the nodal unknowns (degrees of freedom) associated with first, second, third, and fourth nodes, respectively. The shape functions for the linear (straight-sided) quadrilateral shown in Fig. 6.40 are defined in terms of the centroidal or natural coordinates, (ξ, η), as

$$N_p = \frac{1}{4}(1 + \xi\xi_p)(1 + \eta\eta_p) \quad \text{with} \quad p = 1,2,3,4 \tag{6.280}$$

where ξ_p and η_p represent the coordinates of the corner nodes in the natural coordinate system, $(\xi_1 = -1, \eta_1 = -1)$, $(\xi_2 = 1, \eta_2 = -1)$, $(\xi_3 = 1, \eta_3 = 1)$, and $(\xi_4 = -1, \eta_4 = 1)$.

In matrix form, the approximate displacement components become

$$\tilde{\mathbf{u}}^{(e)} = \mathbf{N}^{(e)T} \mathbf{U}^{(e)} \tag{6.281}$$

in which

$$\tilde{\mathbf{u}}^{(e)T} = \left\{ \tilde{u}_x^{(e)} \quad \tilde{u}_y^{(e)} \right\} \tag{6.282}$$

and

$$\mathbf{N}^{(e)T} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{6.283}$$

and

$$\mathbf{U}^{(e)T} = \left\{ u_{x_1}^{(e)} \quad u_{y_1}^{(e)} \quad u_{x_2}^{(e)} \quad u_{y_2}^{(e)} \quad u_{x_3}^{(e)} \quad u_{y_3}^{(e)} \quad u_{x_4}^{(e)} \quad u_{y_4}^{(e)} \right\} \tag{6.284}$$

The element shape matrix $\mathbf{B}^{(e)}$ can be expressed as

$$\mathbf{B}^{(e)} = \mathbf{L}\mathbf{N}^{(e)T} \tag{6.285}$$

in which the differential operator matrix is

$$\mathbf{L} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix} \tag{6.286}$$

The element shape matrix can be rewritten as

$$\mathbf{B}^{(e)} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 & \dfrac{\partial N_4}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} & 0 & \dfrac{\partial N_4}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x} & \dfrac{\partial N_4}{\partial y} & \dfrac{\partial N_4}{\partial x} \end{bmatrix} \tag{6.287}$$

However, the shape functions are defined in terms of the centroidal or natural coordinates, (ξ, η). Therefore, they cannot be differentiated directly with respect to the x- and y-coordinates. In order to overcome this difficulty, the global coordinates are expressed in terms of the shape functions in the form

$$x = \sum_{p=1}^{4} N_p(\xi,\eta) x_p \quad \text{and} \quad y = \sum_{p=1}^{4} N_p(\xi,\eta) y_p \tag{6.288}$$

With this transformation utilizing the same shape functions as those used for the displacement components, the concept of *isoparametric* element emerges, and the element is referred to as an *isoparametric* element.

The derivatives of the shape functions can be obtained as

$$\begin{aligned} \frac{\partial N_p}{\partial x} &= \frac{\partial N_p}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_p}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_p}{\partial y} &= \frac{\partial N_p}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_p}{\partial \eta}\frac{\partial \eta}{\partial y} \end{aligned} \quad \text{with} \quad p = 1,2,3,4 \tag{6.289}$$

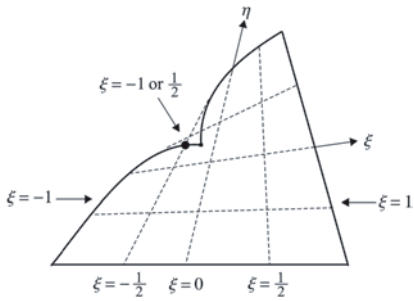

Fig. 6.41 Internal angle exceeding 180°

Application of the chain rule of differentiation yields

$$\begin{aligned}\frac{\partial N_p}{\partial \xi} &= \frac{\partial N_p}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_p}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial N_p}{\partial \eta} &= \frac{\partial N_p}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_p}{\partial y}\frac{\partial y}{\partial \eta}\end{aligned} \quad \text{with} \quad p = 1,2,3,4 \tag{6.290}$$

In matrix form, it can be expressed as

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \tag{6.291}$$

where **J** is called the Jacobian matrix, whose inverse does not exist if there is excessive distortion of the element leading to the intersection of lines of constant ξ and η inside or on the element boundaries, as illustrated in Fig. 6.41. If the quadrilateral element is degenerated into a triangle by increasing an internal angle to 180°, then **J** is singular at that corner. It is possible to obtain the element stiffness because **J** is still unique at the Gaussian integration points. However, the stresses at that corner are indeterminate. A similar situation occurs when two adjacent corner nodes are made coincident to produce a triangular element. Therefore, any internal angle of each corner node should be less than 180°,and there is a loss of accuracy as the internal angle approaches 180°.

In the absence of excessive distortion, the transformation between the natural and global coordinates has a one-to-one correspondence and $\mathbf{J}^{-1}$ inverse exists. It can be expressed as

$$\mathbf{J}^{-1} = \frac{1}{|\mathbf{J}|}\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \tag{6.292}$$

where the determinant of the Jacobian matrix is

$$|\mathbf{J}| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \tag{6.293}$$

in which

$$\begin{aligned}
\frac{\partial x}{\partial \xi} &= \sum_{p=1}^{4}\frac{\partial N_p}{\partial \xi}x_p = \frac{1}{4}\{-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\} \\
\frac{\partial y}{\partial \xi} &= \sum_{p=1}^{4}\frac{\partial N_p}{\partial \xi}y_p = \frac{1}{4}\{-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\} \\
\frac{\partial x}{\partial \eta} &= \sum_{p=1}^{4}\frac{\partial N_p}{\partial \eta}x_p = \frac{1}{4}\{-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\} \\
\frac{\partial y}{\partial \eta} &= \sum_{p=1}^{4}\frac{\partial N_p}{\partial \eta}y_p = \frac{1}{4}\{-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\}
\end{aligned} \tag{6.294}$$

Substituting for the derivatives and rearranging the terms permit the Jacobian to be rewritten in the form

$$\mathbf{J} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{6.295}$$

or

$$\mathbf{J} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{6.296}$$

Its determinant can be expressed in the form

$$|\mathbf{J}| = \frac{1}{8}\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 0 & 1-\eta & -\xi+\eta & -1+\xi \\ -1+\eta & 0 & 1+\xi & -\xi-\eta \\ \xi-\eta & -1-\xi & 0 & 1+\eta \\ 1-\xi & \xi+\eta & -1-\eta & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \tag{6.297}$$

In a concise form, the determinant can be also rewritten as

$$|\mathbf{J}| = \frac{1}{8}[(x_{31}y_{42} - x_{42}y_{31}) + \xi(x_{12}y_{23} - x_{23}y_{12}) + \eta(x_{41}y_{32} - x_{32}y_{41})] \tag{6.298}$$

where

$$x_{ij} = x_i - x_j \quad \text{and} \quad y_{ij} = y_i - y_j \tag{6.299}$$

Determination of the inverse of the Jacobian matrix permits the expression for the derivatives of the natural coordinates in terms of the global coordinates, x and y

$$\begin{Bmatrix} \dfrac{\partial \xi}{\partial x} \\ \dfrac{\partial \xi}{\partial y} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \xi}{\partial \xi} \\ \dfrac{\partial \xi}{\partial \eta} \end{Bmatrix} \tag{6.300a}$$

and

$$\begin{Bmatrix} \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \eta}{\partial y} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \eta}{\partial \xi} \\ \dfrac{\partial \eta}{\partial \eta} \end{Bmatrix} \tag{6.300b}$$

By substituting for the derivatives of the global coordinates in terms of the natural coordinates, these expressions can be rewritten as

$$\begin{aligned} \frac{\partial \xi}{\partial x} &= \frac{1}{|\mathbf{J}|} \sum_{p=1}^{4} \frac{\partial N_p}{\partial \eta} y_p \quad \text{and} \quad \frac{\partial \xi}{\partial y} = -\frac{1}{|\mathbf{J}|} \sum_{p=1}^{4} \frac{\partial N_p}{\partial \eta} x_p \\ \frac{\partial \eta}{\partial x} &= -\frac{1}{|\mathbf{J}|} \sum_{p=1}^{4} \frac{\partial N_p}{\partial \xi} y_p \quad \text{and} \quad \frac{\partial \eta}{\partial y} = \frac{1}{|\mathbf{J}|} \sum_{p=1}^{4} \frac{\partial N_p}{\partial \xi} x_p \end{aligned} \tag{6.301}$$

Finally, the derivatives in the shape matrix becomes

$$\begin{aligned} \frac{\partial N_p}{\partial x} &= \frac{1}{|\mathbf{J}|} \left\{ \frac{\partial N_p}{\partial \xi} \sum_{q=1}^{4} \frac{\partial N_q}{\partial \eta} y_q - \frac{\partial N_p}{\partial \eta} \sum_{q=1}^{4} \frac{\partial N_q}{\partial \xi} y_q \right\} \\ \frac{\partial N_p}{\partial y} &= \frac{1}{|\mathbf{J}|} \left\{ -\frac{\partial N_p}{\partial \xi} \sum_{q=1}^{4} \frac{\partial N_q}{\partial \eta} x_q + \frac{\partial N_p}{\partial \eta} \sum_{q=1}^{4} \frac{\partial N_q}{\partial \xi} x_q \right\} \end{aligned} \quad \text{with} \quad p = 1,2,3,4 \tag{6.302}$$

These explicit expressions for the derivatives appearing in the element shape matrix permit the determination of the element stiffness matrix, $\mathbf{k}^{(e)}$, defined as

$$\mathbf{k}^{(e)} = \int_{V^{(e)}} \mathbf{B}^{(e)T} \mathbf{D} \mathbf{B}^{(e)} \, dV \tag{6.303}$$

in which $V^{(e)} = t\Delta^{(e)}$, with $\Delta^{(e)}$ and t representing the element area and constant element thickness. It can be rewritten in the form

$$\mathbf{k}^{(e)} = t \int_{\Delta^{(e)}} \mathbf{B}^{(e)T} \mathbf{D} \mathbf{B}^{(e)} \, dA \tag{6.304}$$

The material property matrix $\mathbf{D}$ is usually independent of the spatial coordinates, x and y, while the element shape matrix $\mathbf{B}^{(e)}$ requires differentiation of the shape functions with respect to x and y. In order to overcome this difficulty, the integrals are evaluated over a square region in the natural coordinate system, with the transformation of coordinates given by

$$x = \sum_{p=1}^{4} N_p(\xi, \eta) x_p \quad \text{and} \quad y = \sum_{p=1}^{4} N_p(\xi, \eta) y_p \tag{6.305}$$

With this transformation and utilizing the following relation

$$\iint_A dx \, dy = \int_{-1}^{1} \int_{-1}^{1} |\mathbf{J}| \, d\xi \, d\eta \tag{6.306}$$

the element stiffness matrix, $\mathbf{k}^{(e)}$, can be rewritten as

$$\mathbf{k}^{(e)} = t \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^{(e)T} \mathbf{D} \mathbf{B}^{(e)} |\mathbf{J}| \, d\xi \, d\eta \tag{6.307}$$

Due to the difficulty of obtaining analytical expression for the determinant and inverse of the Jacobian matrix, these integrals are evaluated numerically by the Gaussian integration technique. The element stiffness matrix can be evaluated numerically as

$$\mathbf{k}^{(e)} = t \sum_{p=1}^{P} \sum_{q=1}^{Q} w_p w_q \mathbf{B}^{(e)}(\xi_p, \eta_q)^T \mathbf{D} \mathbf{B}^{(e)}(\xi_p, \eta_q) |\mathbf{J}(\xi_p, \eta_q)| \tag{6.308}$$

in which w_p and w_q are the weights and ξ_p and η_q are the integration points of the Gaussian integration technique explained in Sec. 3.6. For this quadrilateral isoparametric element, $P = 2$ and $Q = 2$ are sufficient for accurate integration.

For an element of constant thickness subjected to a uniform load of T_x and T_y in the x- and y-directions, respectively, along its 1–2 edge, the vector $\mathbf{p}_{\mathbf{T}}^{(e)}$, arising from tractions can be written as

$$\mathbf{p}_{\mathbf{T}}^{(e)} = t \int_{L_{1-2}} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix} \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} dl = t \int_{L_{1-2}} \begin{Bmatrix} N_1\ T_x \\ N_1\ T_y \\ N_2\ T_x \\ N_2\ T_y \\ N_3\ T_x \\ N_3 T_y \\ N_4\ T_x \\ N_4\ T_y \end{Bmatrix} dl \tag{6.309}$$

Referring to Fig. 6.40, along the 1–2 edge whose length is L_{1-2}, the coordinate η has a constant value of -1 and ξ varies between -1 and 1, leading to

$$\mathbf{p}_{\mathbf{T}}^{(e)} = t \frac{L_{1-2}}{2} \int_{-1}^{1} \begin{Bmatrix} N_1\ T_x \\ N_1\ T_y \\ N_2\ T_x \\ N_2\ T_y \\ N_3\ T_x \\ N_3 T_y \\ N_4\ T_x \\ N_4\ T_y \end{Bmatrix} d\xi \tag{6.310}$$

Along $\xi = -1$ to 1 and $\eta = -1$,

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-\eta) = \frac{1}{2}(1-\xi) \\ N_2 &= \frac{1}{4}(1+\xi)(1-\eta) = \frac{1}{2}(1+\xi) \\ N_3 &= \frac{1}{4}(1+\xi)(1+\eta) = 0 \\ N_4 &= \frac{1}{4}(1-\xi)(1+\eta) = 0 \end{aligned} \tag{6.311}$$

The integrals in the expression for $\mathbf{p}_{\mathbf{T}}^{(e)}$ are evaluated as

$$t\frac{L_{1-2}}{2}\int_{-1}^{1} N_1\, T_x d\xi = t\frac{L_{1-2}}{4}\int_{-1}^{1}(1-\xi)T_x d\xi = t\frac{L_{1-2}}{2}T_x \tag{6.312}$$

and

$$t\frac{L_{1-2}}{2}\int_{-1}^{1} N_2\, T_y d\xi = t\frac{L_{1-2}}{4}\int_{-1}^{1}(1+\xi)T_y d\xi = t\frac{L_{1-2}}{2}T_y \tag{6.313}$$

Thus, the load vector, $\mathbf{p}_{\mathbf{T}}^{(e)}$, takes the form

$$\mathbf{p}_{\mathbf{T}}^{(e)T} = t\frac{L_{1-2}}{2}\begin{bmatrix} T_x & T_y & T_x & T_y & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.314}$$

Note that this result implies that the applied load is distributed equally at the first and second nodes of the 1–2 edge. This is a result of the linear variation of the shape function along the edges.

As carried out in the derivation of the element stiffness matrix, the load vectors due to body forces, initial strains, and initial stresses can be rewritten as

$$\mathbf{p}_{\mathbf{b}}^{(e)} = t\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^{(e)}\mathbf{b}\left|\mathbf{J}\right| d\xi d\eta \tag{6.315}$$

$$\mathbf{p}_{\varepsilon^*}^{(e)} = t\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^{(e)T}\mathbf{D}\boldsymbol{\varepsilon}^*\left|\mathbf{J}\right| d\xi d\eta \tag{6.316}$$

$$\mathbf{p}_{\sigma^*}^{(e)} = t\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^{(e)T}\boldsymbol{\sigma}^*\left|\mathbf{J}\right| d\xi d\eta \tag{6.317}$$

Application of the Gaussian integration technique leads to the evaluation of these load vectors in the form

$$\mathbf{p}_{\mathbf{b}}^{(e)} = t\sum_{p=1}^{P}\sum_{q=1}^{Q} w_p w_q \mathbf{N}^{(e)}(\xi_p,\eta_q)\mathbf{b}\left|\mathbf{J}(\xi_p,\eta_q)\right| \tag{6.318}$$

$$\mathbf{p}_{\varepsilon^*}^{(e)} = t\sum_{p=1}^{P}\sum_{q=1}^{Q} w_p w_q \mathbf{B}^{(e)}(\xi_p,\eta_q)^T\mathbf{D}\boldsymbol{\varepsilon}^*\left|\mathbf{J}(\xi_p,\eta_q)\right| \tag{6.319}$$

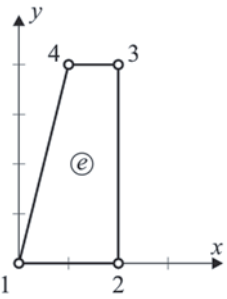

Fig. 6.42 Local numbering scheme of the FEM discretization with a quadrilateral element

$$\mathbf{p}_{\sigma^*}^{(e)} = t\sum_{p=1}^{P}\sum_{q=1}^{Q} w_p w_q \mathbf{B}^{(e)}(\xi_p,\eta_q)^T \boldsymbol{\sigma}^* \left|\mathbf{J}(\xi_p,\eta_q)\right| \tag{6.320}$$

in which w_p and w_q are the weights and ξ_p and η_q are the integration points of the Gaussian integration technique.

6.2.2.6 Example of a Plane Stress Analysis with Linear Quadrilateral Isoparametric Elements

Derivation of a System of Equations and Its Solution

The previous example discussed in Sec. 6.2.2.4 is reconsidered to compute the nodal displacements and the element stresses. In order to illustrate the finite element solution method, the plate is discretized into one quadrilateral isoparametric element, as shown in Fig. 6.42.

The global coordinates of each node are specified by (x_p, y_p), with $p = 1,2,3,4$, and are tabulated in Table 6.9.

The global unknown nodal displacement vector is given by

$$\mathbf{U}^T = \left\{u_{x_1} \quad u_{y_1} \quad u_{x_2} \quad u_{y_2} \quad u_{x_3} \quad u_{y_3} \quad u_{x_4} \quad u_{y_4}\right\} \tag{6.321}$$

Considering the correspondence between the local and global node numbering schemes the elements are defined in Table 6.10.

For this element, $e = 1$, the coefficients of the Jacobian matrix are determined from

$$\frac{\partial x}{\partial \xi} = \frac{1}{4}\left\{-(1-\eta)(x_1 = 0) + (1-\eta)(x_2 = 2) + (1+\eta)(x_3 = 2) - (1+\eta)(x_4 = 1)\right\} = \frac{1}{4}(3-\eta) \tag{6.322a}$$

$$\frac{\partial y}{\partial \xi} = \frac{1}{4}\left\{-(1-\eta)(y_1 = 0) + (1-\eta)(y_2 = 0) + (1+\eta)(y_3 = 3) - (1+\eta)(y_4 = 3)\right\} = 0 \tag{6.322b}$$

Table 6.9 Global nodal coordinates

Global node number	Nodal coordinates	Nodal unknowns
1	$(x_1 = 0, y_1 = 0)$	u_{x_1}, u_{y_1}
2	$(x_2 = 2, y_2 = 0)$	u_{x_2}, u_{y_2}
3	$(x_3 = 2, y_3 = 3)$	u_{x_3}, u_{y_3}
4	$(x_4 = 1, y_4 = 3)$	u_{x_4}, u_{y_4}

Table 6.10 Element connectivity

Element number (e)	Node 1	Node 2	Node 3	Node 4
1	1	2	3	4

$$\frac{\partial x}{\partial \eta} = \frac{1}{4}\{-(1-\xi)(x_1 = 0) - (1+\xi)(x_2 = 2) + (1+\xi)(x_3 = 2) + (1-\xi)(x_4 = 1)\} = \frac{1}{4}(1-\xi) \tag{6.322c}$$

$$\frac{\partial y}{\partial \eta} = \frac{1}{4}\{-(1-\xi)(y_1 = 0) - (1+\xi)(y_2 = 0) + (1+\xi)(y_3 = 3) + (1-\xi)(y_4 = 3) = \frac{6}{4} \tag{6.322d}$$

leading to the Jacobian matrix given by

$$\mathbf{J} = \begin{bmatrix} \frac{1}{4}(3-\eta) & 0 \\ \frac{1}{4}(1-\xi) & \frac{6}{4} \end{bmatrix} \tag{6.323}$$

with its determinant

$$|\mathbf{J}| = \frac{3}{8}(3-\eta) \tag{6.324}$$

The inverse of the Jacobian matrix becomes

$$\mathbf{J}^{-1} = \begin{bmatrix} \frac{4}{(3-\eta)} & 0 \\ -\frac{2(1-\xi)}{3(3-\eta)} & \frac{2}{3} \end{bmatrix} \tag{6.325}$$

The determinant of the Jacobian matrix can be also determined from

$$|\mathbf{J}| = \frac{1}{8}[(x_{31}y_{42} - x_{42}y_{31}) + \xi(x_{12}y_{23} - x_{23}y_{12}) + \eta(x_{41}y_{32} - x_{32}y_{41})] \quad (6.326)$$

in which

$$\begin{aligned} x_{31} &= x_3 - x_1 = 2 & x_{43} &= x_4 - x_3 = -1 & x_{32} &= x_3 - x_2 = 0 \\ y_{31} &= y_3 - y_1 = 3 & y_{43} &= y_4 - y_3 = 0 & y_{32} &= y_3 - y_2 = 3 \\ y_{42} &= y_4 - y_2 = 3 & x_{21} &= x_2 - x_1 = 2 & x_{41} &= x_4 - x_1 = 1 \\ x_{42} &= x_4 - x_2 = -1 & y_{21} &= y_2 - y_1 = 0 & y_{41} &= y_4 - y_1 = 3 \end{aligned} \quad (6.327)$$

Substituting for the following derivatives

$$\begin{aligned} \sum_{p=1}^{4} \frac{\partial N_p}{\partial \xi} x_p &= \frac{\partial x}{\partial \xi} = \frac{1}{4}(3-\eta) \\ \sum_{p=1}^{4} \frac{\partial N_p}{\partial \xi} y_p &= \frac{\partial y}{\partial \xi} = 0 \\ \sum_{p=1}^{4} \frac{\partial N_p}{\partial \eta} x_p &= \frac{\partial x}{\partial \eta} = \frac{1}{4}(1-\xi) \\ \sum_{p=1}^{4} \frac{\partial N_p}{\partial \eta} y_p &= \frac{\partial y}{\partial \eta} = \frac{3}{2} \end{aligned} \quad (6.328)$$

permits the derivatives of the shape functions as

$$\begin{aligned} \frac{\partial N_p}{\partial x} &= \frac{8}{3(3+\eta)}\left\{-\frac{6}{4}\frac{\partial N_p}{\partial \xi}\right\} = -\frac{4}{(3+\eta)}\frac{\partial N_p}{\partial \xi} \\ \frac{\partial N_p}{\partial y} &= \frac{2(1+\xi)}{3(3+\eta)}\frac{\partial N_p}{\partial \xi} - \frac{2}{3}\frac{\partial N_p}{\partial \eta} \end{aligned} \quad \text{with} \quad p = 1,2,3,4 \quad (6.329)$$

Thus, the components of the element shape matrix, $\mathbf{B}^{(1)}$ are computed as

$$\frac{\partial N_1}{\partial x} = -\frac{(1-\eta)}{(3-\eta)}, \quad \frac{\partial N_2}{\partial x} = \frac{(1-\eta)}{(3-\eta)},$$
$$\frac{\partial N_3}{\partial x} = \frac{(1+\eta)}{(3-\eta)}, \quad \frac{\partial N_4}{\partial x} = -\frac{(1+\eta)}{(3-\eta)}$$
$$\frac{\partial N_1}{\partial y} = -\frac{1}{3}\frac{(1-\xi)}{(3-\eta)}, \quad \frac{\partial N_2}{\partial y} = -\frac{(2+\xi-\eta)}{3(3-\eta)}, \tag{6.330}$$
$$\frac{\partial N_3}{\partial y} = \frac{(1+2\xi-\eta)}{3(3-\eta)}, \quad \frac{\partial N_4}{\partial y} = \frac{2(1-\xi)}{3(3-\eta)}$$

$$\mathbf{B}^{(1)} = \begin{bmatrix} -\frac{(1-\eta)}{(3-\eta)} & 0 & -\frac{(1-\eta)}{(3-\eta)} & 0 \\ 0 & -\frac{1}{3}\frac{(1-\xi)}{(3-\eta)} & 0 & -\frac{(2+\xi-\eta)}{3(3-\eta)} \\ -\frac{1}{3}\frac{(1-\xi)}{(3-\eta)} & -\frac{(1-\eta)}{(3-\eta)} & -\frac{(2+\xi-\eta)}{3(3-\eta)} & -\frac{(1-\eta)}{(3-\eta)} \\ \frac{(1+\eta)}{(3-\eta)} & 0 & -\frac{(1+\eta)}{(3-\eta)} & 0 \\ 0 & \frac{(1+2\xi-\eta)}{3(3-\eta)} & 0 & \frac{2(1-\xi)}{3(3-\eta)} \\ \frac{(1+2\xi-\eta)}{3(3-\eta)} & \frac{(1+\eta)}{(3-\eta)} & \frac{2(1-\xi)}{3(3-\eta)} & -\frac{(1+\eta)}{(3-\eta)} \end{bmatrix} \tag{6.331}$$

Under plane stress assumptions, the material property matrix, **D** becomes

$$\mathbf{D} = 10^6 \begin{bmatrix} 16 & 4 & 0 \\ 4 & 16 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{N/cm}^2 \tag{6.332}$$

The element stiffness matrix, $\mathbf{k}^{(1)}$, is computed as

$$\mathbf{k}^{(1)} = 10^6 \begin{bmatrix} 4.8666 & 0.76713 & -4.3666 & 0.23287 \\ 0.76713 & 2.3545 & 0.73287 & -1.0211 \\ -4.3666 & 0.73287 & 5.3666 & -1.7329 \\ 0.23287 & -1.0211 & -1.7329 & 3.6878 \\ -2.7668 & -0.96574 & 2.2668 & -0.034264 \\ -0.96574 & -1.1244 & -0.53426 & -0.20891 \\ 2.2668 & -0.53426 & -3.2668 & 1.5343 \\ -0.034264 & -0.20891 & 1.5343 & -2.4578 \end{bmatrix} \begin{bmatrix} -2.7668 & -0.96574 & 2.2668 & -0.034264 \\ -0.96574 & -1.1244 & -0.53426 & -0.20891 \\ 2.2668 & -0.53426 & -3.2668 & 1.5343 \\ -0.034264 & -0.20891 & 1.5343 & -2.4578 \\ 6.9663 & 0.56853 & -6.4663 & 0.43147 \\ 0.56853 & 3.5845 & 0.93147 & -2.2512 \\ -6.4663 & 0.93147 & 7.4663 & -1.9315 \\ 0.43147 & -2.2512 & -1.9315 & 4.9178 \end{bmatrix} \tag{6.333}$$

The initial strains arising from the temperature change are included in the vector $\boldsymbol{\varepsilon}^*$ as

$$\boldsymbol{\varepsilon}^{*T} = 10^{-6} \begin{bmatrix} 60 & 60 & 0 \end{bmatrix} \tag{6.334}$$

The element load vectors, $\mathbf{p}_{\mathbf{T}}^{(1)}{}_{1-4}$ and $\mathbf{p}_{\mathbf{T}}^{(1)}{}_{3-4}$, arising from the applied tractions are

$$\mathbf{p}_{\mathbf{T}}^{(1)T}{}_{1-4} = t \frac{T_x L_{1-4}}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{6.335}$$

$$\mathbf{p}_{\mathbf{T}}^{(1)T}{}_{3-4} = t \frac{T_y L_{3-4}}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \tag{6.336}$$

With the specified values of the thickness and the distributed loads, these element load vectors become

$$\mathbf{p}_{\mathbf{T}}^{(1)T}{}_{1-4} = 300\sqrt{10} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{N} \tag{6.337}$$

$$\mathbf{p}_{\mathbf{T}}^{(1)T}{}_{3-4} = -150 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \text{N} \tag{6.338}$$

The element load vector from all the applied tractions is

$$\mathbf{p}_{\mathbf{T}}^{(1)}{}_{1-4} + \mathbf{p}_{\mathbf{T}}^{(1)}{}_{3-4} = \begin{Bmatrix} 300\sqrt{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ -150 \\ 300\sqrt{10} \\ -150 \end{Bmatrix} \text{N} \tag{6.339}$$

The thermal load vector of the element, $\mathbf{p}_{\varepsilon^*}^{(1)}$, is obtained as

$$\mathbf{p}_{\varepsilon^*}^{(1)} = \begin{Bmatrix} -900 \\ -300 \\ 900 \\ -600 \\ 900 \\ 300 \\ -900 \\ 600 \end{Bmatrix} \text{N} \tag{6.340}$$

Thus, the total element load vector, **P** is

$$\mathbf{P} = \begin{Bmatrix} (300\sqrt{10} - 900) \\ -300 \\ 900 \\ (-300 - 300) \\ 900 \\ (-150 + 300) \\ (300\sqrt{10} - 900) \\ -150 + 600 \end{Bmatrix} \text{N} \tag{6.341}$$

After applying the boundary conditions, the global stiffness matrix is reduced to

$$\mathbf{K} = 10^6 \begin{bmatrix} 4.8666 & -0.96574 & 2.2668 & -0.034264 \\ -0.96574 & 3.5845 & 0.93147 & -2.2512 \\ 2.2668 & 0.93147 & 7.4663 & -1.9315 \\ -0.034264 & -2.2512 & -1.9315 & 4.9178 \end{bmatrix} \tag{6.342}$$

and the reduced load vector is

$$\mathbf{P} = \begin{Bmatrix} (300\sqrt{10} - 900) \\ 150 \\ (300\sqrt{10} - 900) \\ 450 \end{Bmatrix} \text{N} \tag{6.343}$$

The solution is given by

$$\begin{Bmatrix} u_{x_1} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{Bmatrix} = \begin{Bmatrix} 0.0000307806 \\ 0.000150801 \\ 0.0000222016 \\ 0.000169468 \end{Bmatrix} \text{cm} \tag{6.344}$$

ANSYS Solution

The nodal displacements of the plate subjected to uniform temperature can also be obtained using ANSYS. The solution procedure is outlined as follows:

Model Generation

- Specify the element type (**ET** command) using the following menu path:

Main Menu > Preprocessor > Element Type > Add/Edit/Delete

  - Click on ***Add***.
  - Select ***Solid*** immediately below ***Structural Mass*** from the left list and ***Quad 4node 182*** from the right list; click on ***OK***.
  - Click on ***Options***.
  - In order to specify the 2-D idealization as plane stress with thickness, in the newly appeared dialog box, pull down the menu for ***Element behavior K3*** and select ***Plane strs w/thk***; click on ***OK*** (Fig. 6.43).
  - Click on ***Close***.

- Specify real constants (**R** command) using the following menu path:

Main Menu > Preprocessor > Real Constants > Add/Edit/Delete

  - Click on ***Add***.
  - Click on ***OK***.
  - Enter ***5e−3*** for ***Thickness THK***; click on ***OK***.
  - Click on ***Close***.

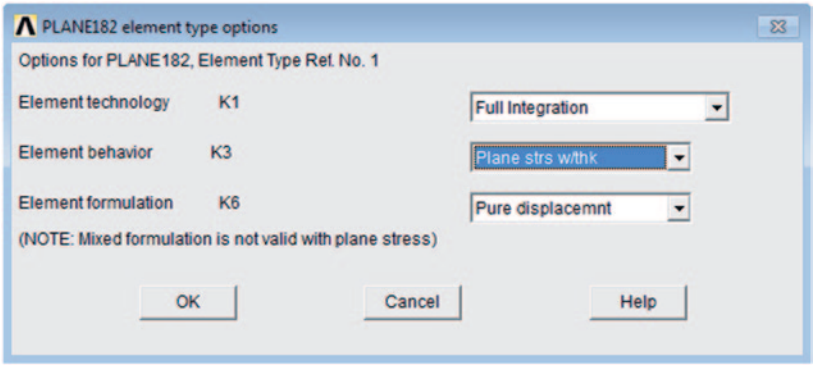

Fig. 6.43 Specification of element options

- Specify material properties (**MP** command) using the following menu path:

Main Menu > Preprocessor > Material Props > Material Models

  - In the *Define Material Model Behavior* dialog box, in the right window, successively left-click on ***Structural***, ***Linear, Elastic***, and, finally, ***Isotropic***, which will bring another dialog box.
  - Enter ***150e9*** for ***EX***, and ***0.25*** for ***PRXY***; click on ***OK***.
  - In the *Define Material Model Behavior* dialog box, in the right window, under ***Structural***, find ***Thermal Expansion, Secant Coefficient***, and ***Isotropic***, which will bring another dialog box (Fig. 6.44).
  - Enter ***6e−6*** for ***APLX***; click on ***OK***.
  - Close the *Define Material Model Behavior* dialog box by using the following menu path:

Material >Exit

- Create nodes (**N** command) using the following menu path:

Main Menu > Preprocessor > Modeling > Create > Nodes > In Active CS

  - A total of four nodes will be created (Table 6.7).
  - Referring to Table 6.7, enter *x*- and *y*-coordinates of node 1 (be sure to convert the coordinates to meters), and Click on ***Apply***. This action will keep the *Create Nodes in Active Coordinate System* dialog box open. If the ***Node number*** field is left blank, then ANSYS will assign the lowest available node number to the node that is being created.
  - Repeat the same procedure for the nodes 2 through 4.
  - After entering the *x*- and *y*-coordinates of node 4, click on ***OK*** (instead of ***Apply***).
  - The nodes should appear in the *Graphics Window*, as shown in Fig. 6.45.

- Create one element (**E** command) using the following menu path:

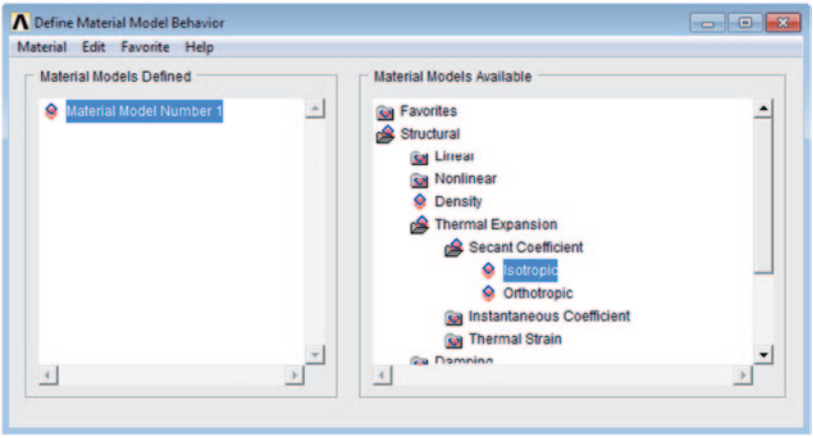

Fig. 6.44 Specification of material behavior

Fig. 6.45 Generation of nodes

Main Menu > Preprocessor > Modeling > Create > Elements > Auto Numbered > Thru Nodes

- *Pick Menu* appears; pick ***four*** nodes in a clockwise (or counterclockwise) order.
- Click on ***OK***.

Solution

- Apply displacement boundary conditions (`D` command) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Structural > Displacement > On Nodes

- *Pick Menu* appears; pick nodes 1 and 2 along the bottom horizontal boundary (Fig. 6.45) and click on ***OK*** on *Pick Menu*.
- Highlight ***UY*** and enter ***0*** for ***VALUE***; click on ***Apply***.
- *Pick Menu* reappears; pick nodes 2 and 3 along the right vertical boundary (Fig. 6.45); click on ***OK*** on *Pick Menu*.

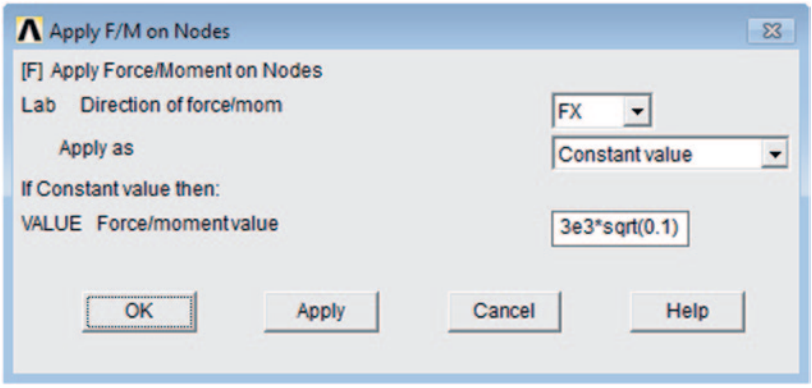

Fig. 6.46 Application of external loads

- Highlight ***UX*** and remove the highlight from ***UY***; Enter ***0*** for ***VALUE***; click on ***OK***.

- Apply force boundary conditions on nodes (**F** command) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Structural > Force/Moment > On Nodes

- *Pick Menu* appears; pick nodes 1 and 4 along the slanted boundary; click on ***OK***.
- Enter ***3e3*sqrt(0.1)*** for ***VALUE*** (Fig. 6.46).
- Click on ***Apply***.
- *Pick Menu* reappears; pick nodes 4 and 3 along the top horizontal boundary; click on ***OK***.
- Pull down the menu for ***Direction of force/mom*** and select ***FY***; Enter − ***150*** for ***VALUE***; click on ***OK.***

- Apply thermal load (**TUNIF** command) using the following menu path:

Main Menu > Solution > Define Loads > Apply > Structural > Temperature > Uniform Temp

- *Uniform Temperature* dialog box appears; Enter ***10*** for ***Uniform temperature***.
- Click on ***OK***.

- Obtain solution (**SOLVE** command) using the following menu path:

Main Menu > Solution > Solve > Current LS

- *Confirmation Window* appears along with *Status Report Window*.
- Review status. If OK, close the Status *Report Window* and click on ***OK*** in *Confirmation Window*.
- Wait until ANSYS responds with ***Solution is done!***

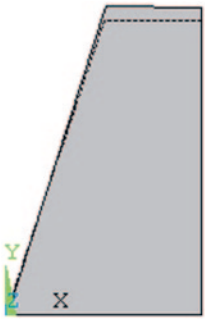

Fig. 6.47 Deformed configuration

Postprocessing

- Review deformed shape (**PLDISP** command) using the following menu path:

Main Menu > General Postproc > Plot Results > Deformed Shape

  - In the *Plot Deformed Shape* dialog box, choose the radio-button for ***Def+undef edge***; click on ***OK***.
  - The deformed shape will appear in the *Graphics Window*, as shown in Fig. 6.47.

- Review displacement values (**PRNSOL** command) using the following menu path:

Main Menu > General Postproc > List Results > Nodal Solution

  - Click on ***DOF Solution*** and ***Displacement vector sum***; click on ***OK***.
  - The list will appear in a new window, as shown in Fig. 6.48.

PRNSOL Command

File

```
PRINT DOF  NODAL SOLUTION PER NODE

 ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

 LOAD STEP=     1  SUBSTEP=     1
  TIME=    1.0000      LOAD CASE=   0

 THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

   NODE      UX          UY
      1  0.30635E-06  0.0000
      2   0.0000      0.0000
      3   0.0000     0.15062E-05
      4  0.22347E-06 0.16956E-05

MAXIMUM ABSOLUTE VALUES
NODE         1           4
VALUE   0.30635E-06 0.16956E-05
```

Fig. 6.48 List of nodal displacements

6.3 Problems

6.1 Construct the finite element equations for the solution of the linear second-order ordinary differential equation given in the form

$$p(x)\frac{d^2u(x)}{dx^2}+\frac{dp(x)}{dx}\frac{du(x)}{dx}+q(x)u(x)=f(x)$$

subject to the conditions given as

$$u(x_0)=A,\ u(x_n)=B$$

by using the Galerkin technique within the realm of finite element method with linear interpolation functions.

6.2 By using a one-dimensional (line) C^1 continuous cubic element, derive the element coefficient matrix for the solution of the differential equation given as

$$\frac{d^2u(x)}{dx^2}=f(x)$$

Assume equally spaced nodal points.

6.3 By using quadratic interpolation functions, derive the element coefficient matrix for the solution of the differential equation given as

$$\frac{d^2u}{dx^2}=e^x$$

subject to the conditions

$$u(0)=1 \text{ and } \frac{du}{dx}(4)=0$$

Also, explicitly assemble both the global coefficient matrix and the right-hand vector for equally spaced nodal points located at $x=0$, 1, 2, 3, and 4.

6.4 Without giving any consideration to the boundary conditions, write down the contribution from the four elements, shown in Fig. 6.49, in the finite element formulation for the Poisson equation $\nabla^2\phi=C$. Denote all entries in the element coefficient matrices symbolically and write your answer in the form $[\mathbf{K}]\{\boldsymbol{\varphi}\}+\{\mathbf{F}\}=\{\mathbf{0}\}$.

6.5 In Problem 6.4, note that the interaction of the internal node 5 with all the adjacent elements is included in forming the equation arising from the field variable

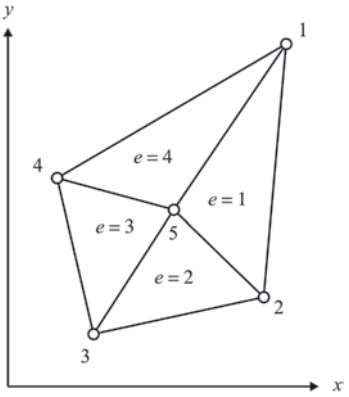

Fig. 6.49 Four linear triangular elements forming a quadrilateral element

ϕ_5 associated with the 5th node. In the absence of external loads, the last row of the vector-matrix expression in the previous problem may be set directly equal to zero. Using the resulting equation, eliminate ϕ_5 from the remaining four rows of the vector-matrix expression to obtain the element coefficient matrix and the contribution to the right-hand-side vector of a *quadrilateral* element made up of four simpler triangular elements.

6.6 Suppose a collection of elements (part of some larger collection) has a total of *n* interior nodes and *m* exterior (or boundary) nodes. The contribution from this collection to the global finite element equations can be written as

$$[\mathbf{K}]^e\{\varphi\}^e + \{\mathbf{f}\}^e$$

The contributions from the exterior nodes, $\phi_i^E \ (i = 1, 2, \ldots, m)$, and the interior nodes, $\phi_i^I \ (i = m+1, \ldots, n+m)$, may be partitioned as

$$\begin{bmatrix} \mathbf{K}^E & \mathbf{K}^* \\ \mathbf{K}^{*T} & \mathbf{K}^I \end{bmatrix} \begin{Bmatrix} \varphi^E \\ \varphi^I \end{Bmatrix} + \begin{Bmatrix} \mathbf{f}^E \\ \mathbf{f}^I \end{Bmatrix}$$

where $[\mathbf{K}^E]$ is an $m \times m$ submatrix, $[\mathbf{K}^I]$is an $n \times n$ submatrix, etc. Consideration of all of the contributions to the interior nodes results in

$$[\mathbf{K}^*]^T\{\varphi^E\} + [\mathbf{K}^I]\{\varphi^I\} + \{\mathbf{f}^I\} = \{\mathbf{0}\}$$

Proceeding from this point, eliminate the quantities φ_i^I from the remaining equations to express the contribution from this collection of elements in the form

$$[\mathbf{K}^R]^e\{\varphi^E\} + \{\mathbf{f}^R\}$$

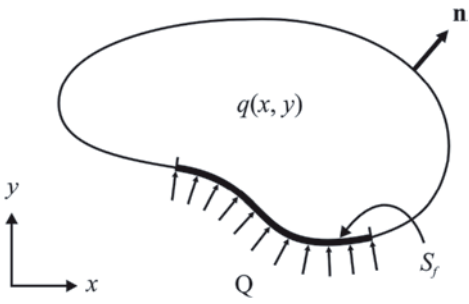

Fig. 6.50 Heat generation within the body and flux boundary condition along S_f

where $[\mathbf{K}^R]$ is an $m \times m$ matrix. This technique is called substructuring.

6.7 For two-dimensional heat transfer in an isotropic body, the governing equation is

$$\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right)+q(x,y)=0$$

where T is temperature, K is thermal conductivity, and $q(x, y)$ is the heat generation rate over the domain. Suppose the heat flux out of some portion, S_f, of the boundary is specified to have a constant value, Q, as shown in Fig. 6.50. Then, the boundary condition over S_f becomes

$$K\left(\frac{\partial T}{\partial \nu}\right)+Q=K\left[\left(\frac{\partial T}{\partial x}\right)n_x+\left(\frac{\partial T}{\partial y}\right)n_y\right]+Q=0$$

where $\mathbf{n} =< n_x, n_y >$ is the unit normal vector to the boundary. Using the *Galerkin technique*, show in a general way how this boundary condition enters the right-hand-side vector.

6.8 Suppose that the heat flux is specified to be Q over the side 4–5 of the domain as shown in Fig. 6.51. Find *explicitly* the contribution of the interpolating function associated with node 4 to the right-hand-side vector in the system of equations derived in Problem 6.7:

a. for the case where element 3 is a linear triangular element.
b. for the case where element 3 is a quadratic triangular element with a mid-side node between nodes 4 and 5.

Hint: Use a local coordinate, s, directed along the side of the triangle from node 4 to node 5. Note that the interpolating function associated with node 4 is linear in s for linear interpolation and quadratic for quadratic interpolation.

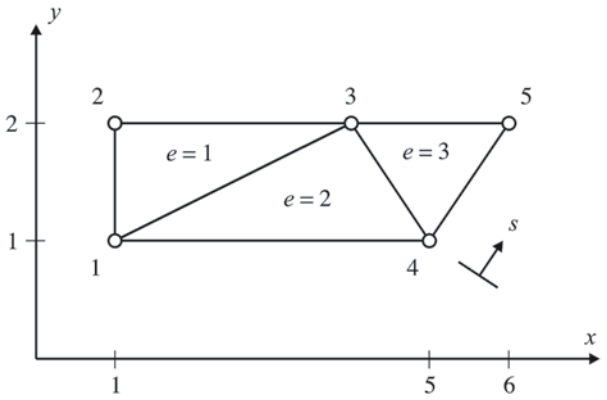

Fig. 6.51 Domain discretized with three triangular elements

6.9 Explicitly evaluate the element coefficient matrix for the problem

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = G$$

using 2×2 Gaussian integration for a 4-noded quadrilateral element whose nodal point locations are given by

Node No.	x	y
1	6.0	3.0
2	−4.0	3.0
3	−5.0	−3.0
4	4.0	−3.0

6.10 Using quadratic interpolation over a 6-noded triangle (shown in Fig. 6.52), derive explicit expressions for the entries K_{11}, K_{44}, and K_{15} in the element coefficient matrix for the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y)$$

6.11 Consider the 3-noded triangular element subjected to traction boundary conditions along the 2–3 side as shown in Fig. 6.53. Assuming plane stress idealization with thickness $t = 0.01$ m, $E = 200$ GPa, and $\nu = 0.25$, construct:

a. the stiffness matrix.
b. the equivalent nodal force vector.

6.12 Assume that the nodal displacement components of the triangular element considered in Problem 6.11 are as follows:

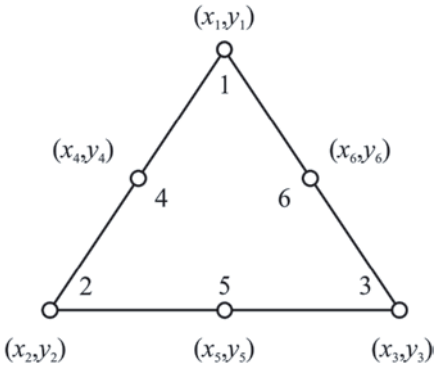

Fig. 6.52 A six-noded triangular element

$$u_1 = 0 \qquad v_1 = 0$$
$$u_2 = 3.30078\times10^{-4}\,\text{m} \qquad v_2 = 0$$
$$u_3 = 1.85937\times10^{-4}\,\text{m} \qquad v_3 = 4.6875\times10^{-6}\,\text{m}$$

Find the stress components (σ_{xx}, σ_{yy} and σ_{xy}).

6.13 Assuming that the triangular element considered in Problem 6.11 is subjected to gravitational acceleration in the negative y-direction with mass density $\rho = 7850\text{kg/m}^3$, find the equivalent nodal force vector.

6.14 Derive the equivalent nodal force vector for a 3-noded triangular element when it is subjected to a uniform temperature change of ΔT. The coefficient of thermal expansion of the material is α.

6.15 The equations governing the *time-dependent* motion of an elastic body are

$$\frac{\partial}{\partial x_j}[\sigma_{ij}] - \rho\frac{\partial^2 u_i}{\partial t^2} = 0$$

where ρ is the mass density of the body. The term $\rho\partial^2 u_i / \partial t^2$ may be interpreted as an "inertia" force, which is a special type of body force.

a. Identifying the inertia force as a body force with $F_i = -\rho\partial^2 u_i / \partial t^2$, derive the contribution from a single element to the global finite element formulation for the case of plane strain.

b. If no tractions are specified over the surface of the body, write down the general form of the global finite element equations. Assuming

$$\{\mathbf{u}\} = \{\tilde{\mathbf{u}}\}\, e^{i\omega t}$$

write down an equation for ω, the natural frequencies of vibration.

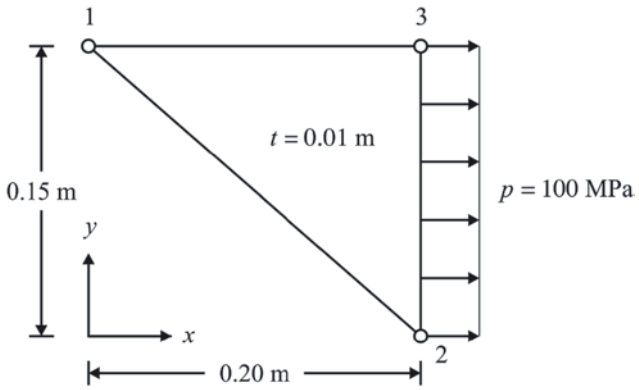

Fig. 6.53 Three-noded triangular element under uniform traction

6.16 A two-dimensional situation that is often of theoretical interest (although less seldom of practical interest) is that of *antiplane strain*, in which $u_1 = u_2 = 0$ and $u_3 = u_3(x_1, x_2)$. Hence, the only non-zero components of strain are ε_{13} and ε_{23} and those of stress are σ_{13} and σ_{23}, which are related by Hooke's law:

$$\sigma_{13} = \frac{E\varepsilon_{13}}{(1+\nu)}, \sigma_{23} = \frac{E\varepsilon_{23}}{(1+\nu)}$$

Find the element coefficient matrix for this problem for the linear triangle (3-noded) using the integration formulas for area coordinates given previously.

6.17 Newton's method is a familiar recursive technique for finding the roots of a transcendental equation. Suppose the roots of n transcendental equations, $\{g_i(a_j)\} = 0$, in n unknowns are to be found. Then, Newton's method can be generalized to

$$\{x_i\}^{(m+1)} = \{x_i\}^{(m)} - \left(\left[\frac{\partial g_i}{\partial x_j} \right]^{-1} \right)^{(m)} \{g_i\}^{(m)}$$

where

$$\left[\frac{\partial g_i}{\partial x_j} \right]^{(m)} = \begin{bmatrix} \frac{\partial g_1}{\partial a_1} & \frac{\partial g_1}{\partial a_2} & \cdots & \frac{\partial g_1}{\partial a_n} \\ \frac{\partial g_2}{\partial a_1} & \frac{\partial g_2}{\partial a_2} & \cdots & \frac{\partial g_2}{\partial a_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial a_1} & \frac{\partial g_n}{\partial a_2} & \cdots & \frac{\partial g_n}{\partial a_n} \end{bmatrix}^{(m)}$$

and $\{g_i\}^{(m)}$ and $[\partial g_i / \partial a_j]$ are evaluated at $\{a_i\}^{(m)}$.
The finite element equations resulting from the nonlinear two-point boundary value problem

$$\frac{d^2u}{dx^2} + g(u, x) = 0$$

have the form

$$[K_{ij}]\{a_i\} + \{f_i(a_j)\} = \{0\}(i = 1, 2, \ldots, n)$$

where $\{a_i\}$ are the nodal values and $\{f_i(a_j)\}$ is some nonlinear function of the nodal values. Apply Newton's method to this problem to obtain a recursive formula for the nodal values. What is the major drawback of this approach?