Chapter 8
A Hidden Markov-Modulated Jump Diffusion Model for European Option Pricing

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Abstract The valuation of a European-style contingent claim is discussed in a hidden Markov regime-switching jump-diffusion market, where the evolution of a hidden economic state process over time is described by a continuous-time, finite-state, hidden Markov chain. A two-stage procedure is used to discuss the option valuation problem. Firstly filtering theory is employed to transform the original market with hidden quantities into a filtered market with complete observations. Then a generalized version of the Esscher transform based on a Doléan-Dade stochastic exponential is employed to select a pricing kernel in the filtered market. A partial-differential-integral equation for the price of a European-style option is presented.

8.1 Introduction

The valuation of contingent claims has long been a very important issue in the theory and practice of finance. The seminal works of Black and Scholes [2] and Merton [27] pioneered the development of option valuation theory and significantly advanced the practice of option valuation in the finance industry. The Black-Scholes-Merton option valuation is deeply immersed in the practice in the finance industry to the extent that it is rather uneasy to find a market practitioner in the City who has never heard of the Black-Scholes-Merton option pricing model. There may be two major reasons why the Black-Scholes-Merton option pricing model is so popular in the finance industry. Firstly, the pricing model is preference-free which means that the price of an option does not depend on the subjective view or risk preference of a market agent. Secondly, there is a closed-form expression for the price of a standard

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European call option which is easy to implement in practice. Despite the popularity of the Black-Scholes-Merton option pricing model, its use has been constantly challenged by both academic researchers and market practitioners. Particularly, the geometric Brownian motion assumption underlying the model cannot explain some important empirical features of asset price dynamics, such as the heavy-tails of the return's distribution, the time-varying volatility, jumps and regime-switchings. Furthermore, the model cannot explain some systematic empirical features of option prices data, such as the implied volatility smile or smirk. There is a large amount of literature which extend the Black-Scholes-Merton model with a view to providing more realistic modeling frameworks for option valuation.

Markovian regime-switching models are one of the major classes of econometric models which can incorporate some stylized facts of asset price dynamics, such as the heavy-tails of the return's distribution, the time-varying volatility and regime-switchings. Though Markovian regime-switching models have a long history in engineering, their general philosophy and principle appeared in some pioneering works in statistics and econometrics. Quandt [31] and Goldfeld and Quandt [18] described nonlinearity in economic data using regime-switching regression models. Tong [37, 38] pioneered the fundamental principle of probability switching in nonlinear time series analysis. Hamilton [20] pioneered and popularized the use of Markov-switching autoregressive time series models in economics and econometrics. Recently much effort has been devoted to the use of Markovian regime-switching models for option valuation. A general belief is that Markovian regime-switching models can incorporate the impact of structural changes in economic conditions on asset prices which is particularly relevant for pricing long-dated options. Some works on option valuation in Markovian regime-switching models include Naik [30], Guo [19], Buffington and Elliott [3], Elliott et al. [9, 13], Siu [33, 34], Siu et al. [35], Elliott and Siu [10, 12], amongst others.

Jump-diffusion models are an important extension of the geometric Brownian motion for modeling asset price dynamics. This class of models captures jumps, or spikes, in returns due to extraordinary market events or news via jump components described by compound Poisson processes. There is a main difference between Markovian regime-switching models and jump-diffusion models. In a Markovian regime-switching model, there are jumps in the model coefficients corresponding to regime switches, but no jumps in the return process. In a jump-diffusion model, there are jumps in the return process, but no jumps in the model coefficients. Merton [28] pioneered the use of a jump-diffusion model for option valuation, where a compound Poisson model with lognormally distributed jump sizes was used to describe the jump component. Kou [25] pioneered option valuation under another jump-diffusion model for option valuation, where the jump amplitudes were exponentially distributed. It seems a general belief that jump-diffusion option valuation models may be suitable for pricing short-lived options by capturing the impact of sudden jumps in the return processes on option prices. Furthermore it is known that jump-diffusion option valuation models can incorporate some empirical features of asset price dynamics, such as jumps, heavy-tails of the return's distribution, and of option prices, such as implied volatility smiles. Bakshi et al. [1] provided a comprehensive empirical study on various option valuation models and found that incorporating both jumps and stochastic volatility is vital for pricing and internal consistency. Pan [32] and Liu et al. [26] provided theoretical and empirical supports on the use of jump-risk premia in explaining systematic empirical behavior of option prices data, respectively.

Both jump-diffusion models and Markovian regime-switching models play an important role in modeling asset price dynamics and option valuation. It may be of interest to combine the two classes of models and establish a class of "second-generation" models, namely, Markovian regime-switching jump-diffusion models. The rationale behind this initiative is to fuse the empirical advantages of the two classes of models so that a generalized option valuation model based on the wider class of "second-generation" models may be suitable for pricing short-lived and long-dated options traded in the finance and insurance industries, respectively. Indeed, this initiative was undertaken by some researchers, for example, Elliott et al. [13] and Siu et al. [35], where Markovian regime-switching jump-diffusion models were used to price financial options and participating life insurance policies, respectively. In both Elliott et al. [13] and Siu et al. [35], the modulating Markov chain governing the evolution of the "true" state of an underlying economy over time was assumed observable. However, in practice, it is difficult, if not impossible, to directly observe the "true" state of the underlying economy. Consequently it is of practical interest to consider a general situation where the modulating Markov chain is hidden or unobservable. In a recent paper, Elliott and Siu [12] considered a hidden Markovian regime-switching pure jump model for option valuation and addressed the corresponding filtering issue.

In this paper, the valuation of a European-style contingent claim in a hidden Markov regime-switching jump-diffusion market is discussed. In such market, the price process of an underlying risky security is described by a generalized jump-diffusion process with stochastic drift and jump intensity being modulated by a continuous-time, finite-state, hidden Markov chain whose states represent different states of a hidden economic environment. A two-stage procedure is used to discuss the option valuation problem. Firstly filtering theory is employed to transform the original market with hidden quantities into a filtered market where the hidden quantities in the original market are replaced by their filtered estimates. Consequently, the filtered market is one with complete observations. Then the option valuation problem is considered in the filtered market which is deemed to be incomplete due to the presence of price jumps in the market. We employ a generalized version of the Esscher transform based on a Doléan-Dade stochastic exponential to select a pricing kernel in the filtered market. A partial-differential-integral equation (PDIE) for the price of a European-style option is presented. This work is different from that in Elliott and Siu [12] in at least two aspects. Firstly, the price process of the risky share we consider here has a diffusion component. Secondly, in Elliott and Siu [12], the selection of a pricing kernel using a generalized version of the Esscher transform was first considered in a market with hidden observations. Filtering theory was then applied to transform the market into one with complete observations. This paper may partly serve as a brief review for some mathematical techniques which are hopefully relevant to pricing European-style options under a hidden regime-switching jump-diffusion model. In Siu [36], an American option pricing problem is considered under the hidden regime-switching jump-diffusion model.

The paper is structured as follows. The next section presents the price dynamics in the hidden Markov regime-switching jump-diffusion market. In Sect. 8.3, we discuss the use of filtering theory to turn the original market into the filtered market and give the filtering equations for the hidden Markov chain. The use of the generalized Esscher transform to select a pricing kernel and the derivation of the PDIE for the price of the European-style option are discussed in Sect. 8.4. The final section gives concluding remarks.

8.2 Hidden Regime-Switching Jump-Diffusion Market

A continuous-time financial market with two primitive investment securities, namely a bond and a share, is considered, where these securities can be traded continuously over time in a finite time horizon $\mathscr{T} := [0,T]$, where $T < \infty$. As usual, uncertainty is described by a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ where $\mathbb{P}$ is a real-world probability measure. The following standard institutional assumptions for the continuous-time financial market are imposed:

1. The market is frictionless, (i.e., there are no transaction costs and taxes in trading the investment securities);
2. Securities are perfectly divisible, (i.e., any fractional units of the securities can be traded);
3. There is a single market interest rate for borrowing and lending;

To describe the evolution of the hidden economic state over time, we consider a continuous-time, finite-state, hidden Markov chain $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathscr{T}\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$. In practice, the "true" state of an underlying economy is not observable. Consequently, it makes practical sense to use a hidden Markov chain to represent different modes of the underlying economic environment. Using the convention in Elliott et al. [8], we identify the state space of the chain $\mathbf{X}$ with a finite set of standard unit vectors $\mathscr{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\}$ in $\Re^N$, where the jth-component of $\mathbf{e}_i$ is the Kronecker product δ_{ij} for each $i, j = 1, 2, \cdots, N$. The space $\mathscr{E}$ is called the canonical state space of the chain $\mathbf{X}$. The statistical laws of the chain $\mathbf{X}$ are described by a family of rate matrices $\{\mathbf{A}(t) | t \in \mathscr{T}\}$, where $\mathbf{A}(t) := [a_{ij}(t)]_{i,j=1,2,\cdots,N}$ and $a_{ij}(t)$ is the instantaneous transition rate of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ at time t. So if $p^i(t) := \mathbb{P}(\mathbf{X}(t) = \mathbf{e}_i)$ and $\mathbf{p}(t) := (p^1(t), p^2(t), \cdots, p^N(t))' \in \Re^N$, then $\mathbf{p}(t)$ satisfies the following Kolmogorov forward equation:

$$\frac{d\mathbf{p}(t)}{dt} = \mathbf{A}(t)\mathbf{p}(t) \ , \quad \mathbf{p}(0) = \mathrm{E}[\mathbf{X}(0)] \ .$$

Here E is an expectation under $\mathbb{P}$.

Let $\mathbb{F}^{\mathbf{X}} := \{\mathscr{F}^{\mathbf{X}}(t) | t \in \mathscr{T}\}$ be the right-continuous, $\mathbb{P}$-complete, natural filtration generated by the chain $\mathbf{X}$. Then Elliott et al. [8] obtained the following semimartingale dynamics for the chain $\mathbf{X}$ under $\mathbb{P}$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}(u)\mathbf{X}(u-)du + \mathbf{M}(t) , \quad t \in \mathscr{T} . \tag{8.1}$$

Here $\mathbf{M} := \{\mathbf{M}(t)|t \in \mathscr{T}\}$ is an $\mathfrak{R}^N$-valued, square-integrable, $(\mathbb{F}^{\mathbf{X}}, P)$-martingale. Since the process $\{\int_0^t \mathbf{A}(u)\mathbf{X}(u-)du | t \in \mathscr{T}\}$ is $\mathbb{F}^{\mathbf{X}}$-predictable, $\mathbf{X}$ is a special semimartingale, and so the above decomposition is unique. This is called the canonical decomposition (see, for example, Elliott [6], Chapter 12 therein).

For each $t \in \mathscr{T}$, let $r(t)$ be the instantaneous interest rate of the bond B at time t, where $r(t) > 0$. Then the price process of the bond $\{B(t)|t \in \mathscr{T}\}$ evolves over time as follows:

$$B(t) = \exp\left(\int_0^t r(u)du\right) , \quad t \in \mathscr{T} ,$$
$$B(0) = 1 .$$

To simplify our discussion, we assume that the interest rate process $\{r(t)|t \in \mathscr{T}\}$ is a deterministic function of time t. In general, one may consider the situation where the interest rate depends on the hidden Markov chain $\mathbf{X}$. However, in this situation, it may be difficult, if not impossible, to use filtering theory, (or in particular the separation principle), to turn the hidden Markovian regime-switching market into one with complete observations. This is one of the main focuses in the paper.

As in Elliott and Siu (2013), we now describe the jump component in the price process of the risky share. Let $Z := \{Z(t)|t \in \mathscr{T}\}$ be a real-valued pure jump process on $(\Omega, \mathscr{F}, \mathbb{P})$ with $Z(0) = 0$, $\mathbb{P}$-a.s. It is clear that

$$Z(t) = \sum_{0<s\leq t} (Z(s) - Z(s-)) .$$

Let $\mathscr{B}(\mathscr{T})$ and $\mathscr{B}(\mathfrak{R}_0)$ be the Borel σ-fields generated by open subsets of $\mathscr{T}$ and $\mathfrak{R}_0 := \mathfrak{R}\backslash\{0\}$, respectively. Suppose $\{\gamma(\cdot,\cdot,\omega)|\omega \in \Omega\}$ is the random measure which selects finite jump times T_k and the corresponding non-zero random jump sizes $\Delta Z(T_k) := Z(T_k) - Z(T_k-)$, $k = 1, 2, \cdots$, of the pure jump process Z. Then

$$\gamma(dt, dz, \omega) = \sum_{k\geq 0} \delta_{(T_k, \Delta Z(T_k))}(dt \times dz) I_{\{T_k<\infty, \Delta Z(T_k)\neq 0\}} ,$$

where $\delta_{(T_k,\Delta Z(T_k))}(dt \times dz)$ is the random Dirac measure, or point mass, at the point $(T_k, \Delta Z(T_k))$ and I_E is the indicator function of the event E. To simplify the notation, we write $\gamma(du, dz)$ for $\gamma(dt, dz, \omega)$ unless otherwise stated.

So, for each $t \in \mathscr{T}$,

$$Z(t) = \int_0^t \int_{\mathfrak{R}_0} z\gamma(du, dz) .$$

To specify the statistical laws of the Poisson random measure $\gamma(dt,dz)$, we consider the hidden Markov regime-switching compensator:

$$\nu_{\mathbf{X}(t-)}(dt,dz) := \sum_{i=1}^{N} \langle \mathbf{X}(t-), \mathbf{e}_i \rangle \lambda_i(t) \eta_i(dz|t) dt \ ,$$

where for each $i = 1,2,\cdots,N$

1. $\{\lambda_i(t)|t \in \mathscr{T}\}$ is the jump intensity process of Z when the economy is in the ith state; we suppose that $\{\lambda_i(t)|t \in \mathscr{T}\}$ is $\mathbb{F}^Z$-predictable, where $\mathbb{F}^Z := \{\mathscr{F}^Z(t)|t \in \mathscr{T}\}$ is the right-continuous, $\mathbb{P}$-complete, natural filtration generated by the pure jump process Z;
2. For each $t \in \mathscr{T}$ and each $i = 1,2,\cdots,N$, $\eta_i(dz|t)$ is the conditional Lévy measure of the random jump size of $\gamma(dt,dz)$ given that there is a jump at time t and that the economy is in the ith state; we assume that $\{\eta_i(dz|t)|t \in \mathscr{T}\}$ is an $\mathbb{F}^Z$-predictable measure-valued process on $(\Omega, \mathscr{F}, \mathbb{P})$.
3. The subscript "$\mathbf{X}(t-)$" is used here to emphasize the dependence of $\nu_{\mathbf{X}(t-)}(dt,dz)$ on $\mathbf{X}(t-)$.

Then the random measure $\tilde{\gamma}(\cdot,\cdot)$ defined by putting:

$$\tilde{\gamma}(dt,dz) := \gamma(dt,dz) - \nu_{\mathbf{X}(t-)}(dt,dz) \ ,$$

is a martingale random measure under $\mathbb{P}$, and hence, it is called the compensated random measure of $\gamma(\cdot,\cdot)$. For discussions on random measures, one may refer to Elliott (1982), Chapter 15 therein.

Let $W := \{W(t)|t \in \mathscr{T}\}$ be a standard Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$ with respect to the $\mathbb{P}$-augmentation of its natural filtration $\mathbb{F}^W := \{\mathscr{F}^W(t)|t \in \mathscr{T}\}$. To simplify our discussion, we assume that W is stochastically independent of $\mathbf{X}$ and Z under $\mathbb{P}$. For each $t \in \mathscr{T}$, let $\alpha^{\mathbf{X}}(t)$ and $\sigma(t)$ be the appreciation rate and the volatility of the risky share at time t, respectively. We suppose that $\alpha^{\mathbf{X}}(t)$ is modulated by the chain $\mathbf{X}$ as:

$$\alpha^{\mathbf{X}}(t) := \langle \alpha(t), \mathbf{X}(t) \rangle \ .$$

Here $\alpha(t) := (\alpha_1(t), \alpha_2(t), \cdots, \alpha_N(t))' \in \Re^N$ such that for each $i = 1,2,\cdots,N$ and each $t \in \mathscr{T}$, $\alpha_i(t) > r(t)$, $\mathbb{P}$-a.s., and $\{\alpha(t)|t \in \mathscr{T}\}$ is an $\mathbb{F}^W$-predictable process; $\alpha_i(t)$ represents the appreciation rate of the risky share at time t when the hidden economy is in the ith-state at that time; the scalar product $\langle \cdot, \cdot \rangle$ selects the component in the vector of the appreciation rates that is in force at a particular time according to the state of the hidden economy at that time; the superscript $\mathbf{X}$ in $\alpha^{\mathbf{X}}$ is used to emphasize the dependence of the appreciation rate $\alpha^{\mathbf{X}}$ on the chain $\mathbf{X}$.

Furthermore, we assume that the volatility process $\{\sigma(t)|t \in \mathscr{T}\}$ is $\mathbb{F}^W$-predictable and that for each $t \in \mathscr{T}$, $\sigma(t) > 0$, $\mathbb{P}$-a.s. In general, one may consider a situation where the volatility depends the hidden Markov chain $\mathbf{X}$. However, there may be two potential concerns about this generalization. Firstly, it complicates the filtering issue and it is difficult, if not impossible, to derive an exact, finite-dimensional filter of the chain $\mathbf{X}$ in this general situation. Secondly, some difficulties may arise in the interpretation of the information structure of the asset price model. Particularly, it was noted in Guo [19] and Gerber and Shiu [17] that in the case of a Markovian regime-switching geometric Brownian motion, the volatility parameter can be completely determined from a given price path of the risky share. More specifically, it can be identified by means of the predictable quadratic variation. Thirdly, it was noted in Merton [29] that appreciation rates of risky securities are a lot harder to estimate than their volatilities. It may not be unreasonable to assume that the volatility does not depend on the hidden Markov chain $\mathbf{X}$. Lastly, if the volatility is assumed to be modulated by the chain $\mathbf{X}$, filtering theory (or in particular the separation principle) may be difficult to apply to turn the hidden Markovian regime-switching market into one with complete observations. One may also refer to Elliott and Siu [11] for related discussions.

We suppose that under the real-world measure $\mathbb{P}$ the price process of the risky share is governed by the following hidden Markovian regime-switching, jump-diffusion model:

$$\frac{dS(t)}{S(t-)} = \left(\propto^{\mathbf{X}}(t) + \sum_{i=1}^{N} (e^z - 1) \langle \mathbf{X}(t), \mathbf{e}_i \rangle \lambda_i(t) \eta_i(dz|t) \right) dt + \sigma(t) dW(t) + \int_{\Re_0} (e^z - 1) \tilde{\gamma}(dt, dz) \ .$$

Write, for each $t \in \mathscr{T}$,

$$Y(t) := \ln(S(t)/S(0)) \ .$$

This is the logarithmic return from the risky share over the time interval $[0,t]$.

Applying Itô's differentiation rule to $Y(t)$ then gives:

$$dY(t) = \left(\propto^{\mathbf{X}}(t) - \frac{1}{2}\sigma^2(t) + \sum_{i=1}^{N} z \langle \mathbf{X}(t), \mathbf{e}_i \rangle \lambda_i(t) \eta_i(dz|t) \right) dt + \sigma(t) dW(t) + \int_{\Re_0} z \tilde{\gamma}(dt, dz) \ .$$

Since the coefficients in the price process, or the return process, of the risky share depends on the hidden Markov chain $\mathbf{X}$, the hidden Markovian regime-switching jump-diffusion market is one with partial observations. In the next section we shall use filtering theory to transform this market into one with complete observations.

We end this section by specifying the information structure of our market model. Let $\mathbb{F}^Y := \{\mathscr{F}^Y(t) | t \in \mathscr{T}\}$ be the $\mathbb{P}$-augmentation of the natural filtration generated by the return process $Y := \{Y(t) | t \in \mathscr{T}\}$. This is the observable filtration in our market model. For each $t \in \mathscr{T}$, let $\mathscr{G}(t) := \mathscr{F}^{\mathbf{X}}(t) \vee \mathscr{F}(t)$. Write $\mathbb{G} := \{\mathscr{G}(t) | t \in \mathscr{T}\}$ representing the full information structure of the model.

8.3 Filtering Theory and Filtered Market

Filtering theory has been widely used by the electrical engineering community to decompose observations from stochastic dynamical systems into signal and noise. Particularly, it has been widely used in signal processing, system and control engineering, radio and telecommunication engineering. In this section we shall first discuss the use of filtering theory to transform the original market with partial observations to a filtered market with complete observations. The general philosophy of this idea is in the spirit of that of the separation principle used in stochastic optimal control theory for partially observed stochastic dynamical systems, see, for example, Fleming and Rishel [15], Kallianpur [22] and Elliott [6]. Then we shall outline the basic idea of a reference probability approach, whose history can be traced back to the work of Zakai [39], to derive a stochastic differential equation for the unnormalized filter of the hidden Markov chain **X** given observations about the return process of the risky asset. This filtering equation is called the Zakai equation in the filtering literature. The derivation of the filtering equation resembles to that in Wu and Elliott [11] and Elliott and Siu [11], so only key steps are presented and the results are stated without giving the proofs. Due to the presence of stochastic integrals in the Zakai equation, its numerical computation may be rather uneasy. From the numerical perspective, it may be more convenient to consider ordinary differential equations than stochastic differential equations. Using the gauge transformation technique in Clark [5], we shall give a (pathwise) linear ordinary differential equation governing the evolution of a "transformed" unnormalized filter of the chain **X** over time. This filter is robust with respect to the observation process in the Skorohod topology and has an advantage from the numerical perspective. Using a version of the Bayes' rule, the normalized filter can be recovered from the (transformed) unnormalized one.

8.3.1 The Separation Principle

The use of the filtering theory to transform the original market to the filtered market involves the use of the innovations approach which is also called the separation principle. This approach has two steps. The first step introduces innovations processes which are adapted to the observable filtration. The second step expresses the price processes with hidden quantities in terms of these innovations processes and filtered estimates of the hidden quantities.

For any integrable, $\mathbb{G}$-adapted process $\{\phi(t)|t \in \mathscr{T}\}$, let $\{\hat{\phi}(t)|t \in \mathscr{T}\}$ be the $\mathbb{F}^Y$-optional projection of $\{\phi(t)|t \in \mathscr{T}\}$ under the measure $\mathbb{P}$. Then, for each $t \in \mathscr{T}$,

$$\hat{\phi}(t) = \mathrm{E}[\phi(t)|\mathscr{F}^Y(t)] \,, \mathscr{P}\text{-a.s.}$$

The optional projection takes into account the measurability in $(t, \omega) \in \mathscr{T} \times \Omega$.

Define, for each $t \in \mathscr{T}$,

$$h(t) := \propto(t) - \frac{1}{2}\sigma^2(t) .$$

Then the return process Y of the risky share is written as:

$$Y(t) = \int_0^t h(u)du + \int_0^t \sigma(u)dW(u) + \int_0^t \int_{\Re_0} z\gamma(du,dz) .$$

Write, for each $t \in \mathscr{T}$,

$$Y_1(t) = \int_0^t h(u)du + \int_0^t \sigma(u)dW(u) , \quad Y_2(t) := \int_0^t \int_{\Re_0} z\gamma(du,dz) ,$$

so that

$$Y(t) = Y_1(t) + Y_2(t) .$$

Consider then the following $\mathbb{F}^Y$-adapted process $\hat{W} := \{\hat{W}(t)|t \in \mathscr{T}\}$:

$$\hat{W}(t) := W(t) + \int_0^t \left(\frac{h(u) - \hat{h}(u)}{\sigma(u)}\right)du , \quad t \in \mathscr{T} .$$

Then following standard filtering theory, (see, for example, [15, 22, 6]), $\hat{W}$ is an $(\mathbb{F}^Y, \mathbb{P})$-standard Brownian motion. The process $\hat{W}$ is called the innovation process for the diffusion part Y_1 of the return process Y of the risky share.

We now define the innovation process for the jump part Y_2 of the return process Y of the risky share. Consider the $\mathbb{G}$-adapted process $Q := \{Q(t)|t \in \mathscr{T}\}$ on $(\Omega, \mathscr{F}, \mathscr{P})$ which is defined by putting:

$$Q(t) := \int_0^t \int_0^\infty z(\gamma(du,dz) - \nu_{\mathbf{X}(u-)}(du,dz)) , \quad t \in \mathscr{T} ,$$

so that Q is a $(\mathbb{G}, \mathbb{P})$-martingale.

Define

$$\hat{\nu}(dt,dz) := \sum_{i=1}^N \langle \hat{\mathbf{X}}(t), \mathbf{e}_i \rangle \lambda_i(t)\eta_i(dz|t)dt ,$$

and

$$\hat{\gamma}(dt,dz) := \gamma(dt,dz) - \hat{\nu}(dt,dz) .$$

The following lemma was due to Elliott [7]. We state the result here without giving the proof.

Lemma 8.1. *Let $\hat{Q} := \{\hat{Q}(t) | t \in \mathscr{T}\}$ be the $\mathbb{F}^Z$-adapted process defined by setting:*

$$\hat{Q}(t) := \int_0^t \int_{\Re_0} z\hat{\gamma}(du, dz) \; .$$

Then $\hat{Q}$ is an $(\mathbb{F}^Y, \mathbb{P})$-martingale.

The process $\hat{Q}$ is then used here as the innovations process of the jump part Y_2 of the return process of the risky share.

The following lemma then gives a representation for the price process of the risky share in terms of the two innovations processes $\hat{W}$ and $\hat{Q}$ under the real-world measure $\mathbb{P}$. Since the result is rather standard, we just state it without giving the proof.

Lemma 8.2. *Under the real-world measure $\mathbb{P}$, the return process of the risky share is:*

$$dY(t) = \left(\hat{h}(t) + \sum_{i=1}^{N} z \left\langle \hat{\mathbf{X}}(t), \mathbf{e}_i \right\rangle \lambda_i(t) \eta_i(dz|t)\right) dt + \sigma(t) d\hat{W}(t)$$
$$+ \int_{\Re_0} z\hat{\gamma}(dt, dz) \; ,$$

and the price process of the risky share is:

$$\frac{dS(t)}{S(t-)} = \left(\propto^{\mathbf{X}}(t) + \sum_{i=1}^{N} (e^z - 1) \left\langle \hat{\mathbf{X}}(t), \mathbf{e}_i \right\rangle \lambda_i(t) \eta_i(dz|t)\right) dt + \sigma(t) d\hat{W}(t)$$
$$+ \int_{\Re_0} (e^z - 1)\hat{\gamma}(dt, dz) \; .$$

It is obvious that the return and price processes of the risky share in Lemma 8.2 only involve observable quantities. Consequently, we adopt these dynamics as the return and price processes of the risky share in the filtered market.

8.3.2 Filtering Equations

The reference probability approach to derive a filtering equation for the chain $\mathbf{X}$ is now discussed. We start with a reference probability measure $\mathbb{P}^\dagger$ on $(\Omega, \mathscr{F})$ under which the return process Y of the risky share becomes simpler and does not depend on the chain $\mathbf{X}$. That is, under $\mathbb{P}^\dagger$,

1. Y_1 is a Brownian motion with $\langle Y_1, Y_1 \rangle (t) = \int_0^t \sigma^2(u) du$, where $\{\langle Y_1, Y_1 \rangle (t) | t \in \mathscr{T}\}$ is the predictable quadratic variation of Y_1;

2. The Poisson random measure $\gamma(dt,dy)$ has an unit intensity for random jump times and the conditional Lèvy measure $\eta(dz|t)$ for random jump sizes so that

$$\tilde{\gamma}^{\dagger}(dt,dz) := \gamma(dt,dz) - \eta(dz|t)dt \ ,$$

is an $(\mathbb{F}^Y, \mathbb{P}^{\dagger})$-martingale random measure;
3. Y_1 and $\gamma(\cdot,\cdot)$ are stochastically independent;
4. The chain $\mathbf{X}$ has the family of rate matrices, $\{\mathbf{A}(t)|t \in \mathscr{T}\}$.

Define

$$\Lambda^1(t) := \exp\left(\int_0^t \sigma^{-2}(u)h(u)dY_1(u) - \frac{1}{2}\int_0^t \sigma^{-2}(u)h^2(u)du\right) ,$$

$$\Lambda^2(t) := \exp\Bigg[-\int_0^t \left(\sum_{i=1}^N \langle \mathbf{X}(u-), \mathbf{e}_i\rangle \int_{\Re_0} (g_i(z|u)-1)\eta(z|u)dz\right)du \\ + \int_0^t \int_{\Re_0} \left(\sum_{i=1}^N \langle \mathbf{X}(u-), \mathbf{e}_i\rangle \log(g_i(z|u))\right)\gamma(du,dz)\Bigg] ,$$

where $g_i(z|t) := \frac{\lambda_i(t)\eta_i(dz|t)}{\eta(dz|t)}$, for each $i = 1,2,\cdots,N$.

Consider the $\mathbb{G}$-adapted process $\Lambda := \{\Lambda(t)|t \in \mathscr{T}\}$ defined by putting:

$$\Lambda(t) := \Lambda^1(t)\cdot\Lambda^2(t) .$$

It is not difficult to check that Λ is a $(\mathbb{G},\mathbb{P})$-martingale, and hence, $\mathrm{E}[\Lambda(T)] = 1$. Consequently the real-world measure $\mathbb{P}$ equivalent to $\mathbb{P}^{\dagger}$ on $\mathscr{G}(T)$ can be reconstructed using the Radon-Nikodym derivative $\Lambda(T)$ as follows:

$$\left.\frac{d\mathbb{P}}{d\mathbb{P}^{\dagger}}\right|_{\mathscr{G}(T)} := \Lambda(T) .$$

Using a version of Girsanov's theorem for jump-diffusion processes, it can be shown that under $\mathbb{P}$,

1. $\Lambda := \{\Lambda(t)|t \in \mathscr{T}\}$ is the unique solution of the following stochastic differential-integral equation:

$$\Lambda(t) = 1 + \int_0^t \Lambda(u)h(u)\sigma^{-2}(u)dY_1(u) \\ + \sum_{i=1}^N \int_0^t \langle \mathbf{X}(u), \mathbf{e}_i\rangle \int_{\Re_0} \Lambda(u-)(g_i(z|u)-1)\tilde{\gamma}^{\dagger}(du,dz) .$$

2. The process

$$W(t) := \int_0^t \sigma^{-1}(u)dY_1(u) - \int_0^t \sigma^{-1}(u)h(u)du \ , \quad t \in \mathscr{T} \ ,$$

is a standard Brownian motion.

3. $\gamma(dt,dz)$ is the Poisson random measure with the compensator:

$$\nu_{\mathbf{X}(t-)}(dt,dz) := \sum_{i=1}^{N} \langle \mathbf{X}(t-), \mathbf{e}_i \rangle \, \lambda_i(t)\eta_i(dz|t)dt \ ,$$

so that

$$\tilde{\gamma}(dt,dz) := \gamma(dt,dz) - \nu_{\mathbf{X}(t-)}(dt,dz) \ ,$$

is a martingale random measure.

Consequently, under $\mathbb{P}$, the return process Y of the risky share is:

$$Y(t) = Y_1(t) + Y_2(t) = \int_0^t h(u)du + \int_0^t \sigma(u)dW(u) + \int_0^t \int_{\Re_0} z\gamma(du,dz) \ .$$

The ultimate goal of filtering is to evaluate $\hat{\mathbf{X}}$ which is an $\mathbb{F}^Y$-optional projection of $\mathbf{X}$ under $\mathbb{P}$. Then, for each $t \in \mathscr{T}$,

$$\hat{\mathbf{X}}(t) = \mathrm{E}[\mathbf{X}(t)|\mathscr{F}^Y(t)] \ , \quad \mathbb{P}\text{-a.s.}$$

Indeed, $\hat{\mathbf{X}}(t)$ is an optimal estimate of $\mathbf{X}(t)$ in the least square sense.

Write, for each $t \in \mathscr{T}$,

$$\mathbf{q}(t) := \mathrm{E}^{\dagger}[\Lambda(t)\mathbf{X}(t)|\mathscr{F}(t)] \ ,$$

where $\mathrm{E}^{\dagger}$ is an expectation under the reference probability measure $\mathbb{P}^{\dagger}$ and $\mathbf{q}(t)$ is called an unnormalized filter of $\mathbf{X}(t)$. Instead of evaluating $\hat{\mathbf{X}}(t)$ directly, a filtering equation governing the evolution of the unnormalized filter $\mathbf{q}(t)$ over time is first derived. Before presenting the filtering equation, we need to define some notation.

For each $t \in \mathscr{T}$, let $Z(t,\cdot) : \Omega \to \Re_0$ be a random variable with a strictly positive conditional Lèvy measure $\eta(dz|t)$ under the reference measure $\mathbb{P}^{\dagger}$. Then, for each $t \in \mathscr{T}$ and each $i = 1, 2, \cdots, N$, the random variable $G_i(t,\cdot) : \Omega \to \Re_+$, where $\Re_+$ is the positive real line, is defined as:

$$G_i(t,\omega) := \frac{\lambda_i(t)\eta_i(Z(t,\omega)|t)}{\eta(Z(t,\omega)|t)} := g_i(Z(t,\omega)|t) \ ,$$

for some measurable function $g_i(\cdot|t)$ on $(\Re_0, \mathscr{B}(\Re_0))$.

Note that $G_i(t,\omega)$ is well-defined since $\eta(dz|t) > 0$. Again, to simplify the notation, we suppress the notation "ω" unless otherwise stated. We now define the following diagonal matrices:

$$\begin{aligned}\mathbf{diag}(\mathbf{G}(t)-\mathbf{1}) &:= \mathbf{diag}(G_1(t)-1, G_2(t)-1, \cdots, G_N(t)-1)\ ,\\ \mathbf{diag}(\lambda(t)-\mathbf{1}) &:= \mathbf{diag}(\lambda_1(t)-1, \lambda_2(t)-1, \cdots, \lambda_N(t)-1)\ .\end{aligned}$$

Here $\mathbf{diag}(\mathbf{y})$ denotes the diagonal matrix with diagonal elements being given by the components in a vector $\mathbf{y}$; $\mathbf{1} := (1,1,\cdots,1)' \in \Re^N$.

For each $i = 1,2,\cdots,N$ and each $t \in \mathscr{T}$, let $h_i(t) := \propto_i(t) - \frac{1}{2}\sigma^2(t)$ and $\mathbf{h}(t) := (h_1(t), h_2(t), \cdots, h_N(t))' \in \Re^N$. Write, for each $t \in \mathscr{T}$,

$$J(t) := \int_0^t \int_{\Re_0} \gamma(du, dz)\ .$$

Then the following theorem is standard and gives the Zakai stochastic differential equation for the unnormalized filter $\mathbf{q}(t)$ (see, for example, Elliott and Siu [11], Theorem 4.1 therein). We state the result without giving the proof.

Theorem 8.1. *For each $t \in \mathscr{T}$, let*

$$\mathbf{B}(t) := \mathbf{diag}(\mathbf{h}(t))\ .$$

Then under $\mathbb{P}^\dagger$, the unnormalized filter $\mathbf{q}(t)$ satisfies the Zakai stochastic differential equation:

$$\begin{aligned}\mathbf{q}(t) &= \mathbf{q}(0) + \int_0^t \mathbf{A}(u)\mathbf{q}(u)du + \int_0^t \mathbf{B}(u)\mathbf{q}(u)\sigma^{-2}(u)dY_1(u)\\ &\quad + \int_0^t \mathbf{diag}(\mathbf{G}(u)-\mathbf{1})\mathbf{q}(u)dJ(u) - \int_0^t \mathbf{diag}(\lambda(u)-\mathbf{1})\mathbf{q}(u)du\ .\end{aligned} \tag{8.2}$$

The filtering equation in Theorem 8.1 involves stochastic integrals. This may be a disadvantage for numerical implementation. Using the gauge transformation technique of Clark [5], the filtering equation can be simplified as a (pathwise) linear ordinary differential equation. The key steps are presented in the sequel.

Define, for each $i = 1,2,\cdots,N$ and each $t \in \mathscr{T}$,

$$\begin{aligned}\gamma_i(t) := \exp\Bigg(&\int_0^t h_i(u)\sigma^{-2}(u)dY_1(u) - \frac{1}{2}\int_0^t h_i^2(u)\sigma^{-4}(u)du\\ &+ \int_0^t (1-\lambda_i(u))du + \int_0^t \log G_i(u)dJ(u)\Bigg)\ .\end{aligned} \tag{8.3}$$

Then the gauge transformation matrix $\Gamma(t)$ is defined as:

$$\Gamma(t) := \mathbf{diag}(\gamma_1(t), \gamma_2(t), \ldots, \gamma_N(t))\ .$$

Write, for each $t \in \mathscr{T}$, $\Gamma^{-1}(t)$ for the inverse of $\Gamma(t)$. The existence of $\Gamma^{-1}(t)$ is guaranteed by the definition of $\Gamma(t)$ and the positivity of $\gamma_i(t)$ for each $i = 1, 2, \cdots, N$.

Take, for each $t \in \mathscr{T}$,

$$\bar{\mathbf{q}}(t) := \Gamma^{-1}(t)\mathbf{q}(t) \ .$$

This is called a transformed unnormalized filter of $\mathbf{X}(t)$.

Then the following theorem gives a (pathwise) linear ordinary differential equation governing the transformed process $\{\bar{\mathbf{q}}(t)|t \in \mathscr{T}\}$. Again we state the result without giving the proof which is rather standard (see, for example, Elliott and Siu [11], Theorem 4.2 therein).

Theorem 8.2. $\bar{\mathbf{q}}$ *satisfies the following first-order linear ordinary differential equation:*

$$\frac{d\bar{\mathbf{q}}(t)}{dt} := \Gamma^{-1}(t)\mathbf{A}(t)\Gamma(t)\bar{\mathbf{q}}(t) \ , \quad \bar{\mathbf{q}}(0) = \mathbf{q}(0) \ .$$

Finally, using a version of the Bayes' rule,

$$\begin{aligned} \hat{\mathbf{X}}(t) &:= \mathrm{E}[\mathbf{X}(t)|\mathscr{F}(t)] \\ &= \frac{\mathrm{E}^{\dagger}[\Lambda(t)\mathbf{X}(t)|\mathscr{F}(t)]}{\mathrm{E}^{\dagger}[\Lambda(t)|\mathscr{F}(t)]} \\ &= \frac{\mathbf{q}(t)}{\langle \mathbf{q}(t), \mathbf{1} \rangle} \\ &:= \frac{\Gamma(t)\bar{\mathbf{q}}(t)}{\langle \Gamma(t)\bar{\mathbf{q}}(t), \mathbf{1} \rangle} \ , \end{aligned}$$

so the normalized filter $\hat{\mathbf{X}}(t)$ can be "recovered" from the (transformed) unnormalized one $\bar{\mathbf{q}}(t)$.

8.4 Generalized Esscher Transform in the Filtered Market

The main theme of this section is to determine a pricing kernel in the filtered market described in Sect. 8.3.1 using a generalized version of the Esscher transform based on a Doléan-Dade stochastic exponential. Firstly, let us recall that in the filtered market, the price process of the risky share under the real-world measure $\mathbb{P}$ is governed by the following stochastic differential equation with jumps:

$$\begin{aligned} \frac{dS(t)}{S(t-)} &= \left(\hat{\propto}(t) + \sum_{i=1}^{N} (e^z - 1) \langle \hat{\mathbf{X}}(t), \mathbf{e}_i \rangle \lambda_i(t)\eta_i(dz|t) \right) dt + \sigma(t) d\hat{W}(t) \\ &\quad + \int_{\Re_0} (e^z - 1)\hat{\gamma}(dt, dz) \ . \end{aligned}$$

The presence of jumps renders the filtered market incomplete. Consequently, there is more than one equivalent martingale measure, or pricing kernel, in the market. Though there are different approaches to select a pricing kernel in an incomplete market, we focus here on the Esscher transformation approach which was pioneered by the seminal work of Gerber and Shiu [16]. Note that the local characteristics of the price process of the risky share in the filtered market are $\mathbb{F}^Y$-predictable processes, so the price process is a semimartingale beyond the class of Lévy processes. Consequently, the original version of the Esscher transform in Esscher [14] and Gerber and Shiu [16] cannot be applied in this situation. Bühlmann et al. [4], Kallsen and Shiryaev [23] and Jacod and Shiryaev [21] considered a generalized version of the Esscher transform for measure changes for general semimartingales and discussed its application for option valuation. This version of the Esscher transform is defined using the concepts of Doléan Dade stochastic exponential and the Laplace cumulant process. It was used in Elliott and Siu [12] to select a pricing kernel in a hidden regime-switching pure jump process. In the sequel, we shall first define the generalized Esscher transform and give the local condition in terms of the local characteristics of the price process of the risky share in the filtered market. Then we present the price dynamics of the risky share under an equivalent (local)-martingale measure selected by the generalized Esscher transform.

Let $\mathscr{L}(Y)$ be the space of processes $\theta := \{\theta(t)|t \in \mathscr{T}\}$ satisfying the following conditions:

1. θ is $\mathbb{F}^Y$-predictable;
2. θ is integrable with respect to the return process Y; that is, the (stochastic) integral process $\{(\theta \cdot Y)(t)|t \in \mathscr{T}\}$, where $(\theta \cdot Y)(t) := \int_0^t \theta(u)dY(u)$ is well-defined.

Consider, for each $\theta \in \mathscr{L}(Y)$, the following exponential process:

$$\mathscr{D}^{\theta}(t) := \exp((\theta \cdot Y)(t)) , \quad t \in \mathscr{T} .$$

Note that $\{(\theta \cdot Y)(t)|t \in \mathscr{T}\}$ is a semimartingale, so $\mathscr{D}^{\theta} := \{\mathscr{D}^{\theta}(t)|t \in \mathscr{T}\}$ is also called an exponential semimartingale.

Define, for each $\theta \in \mathscr{L}(Y)$, the following semimartingale:

$$\begin{aligned}\mathscr{H}^{\theta}(t) := \int_0^t \Big(&\hat{h}(u)\theta(u) + \frac{1}{2}\sigma^2(u)\theta^2(u) \\ &+ \sum_{i=1}^N \int_{\Re_0} (e^z - 1 - \theta(u)z)\left\langle \hat{\mathbf{X}}(u), \mathbf{e}_i \right\rangle \lambda_i(u)\eta_i(dz|u)\Big)du \\ &+ \int_0^t \sigma(u)\theta(u)d\hat{W}(u) + \int_0^t \int_{\Re_0} (e^{\theta(u)z} - 1)\hat{\gamma}(du, dz) .\end{aligned}$$

Using Itô's differentiation rule, it can be shown that

$$\mathscr{D}^{\theta}(t) = 1 + \int_0^t \mathscr{D}^{\theta}(u-)d\mathscr{H}^{\theta}(u) .$$

From Theorem 13.5 in Elliott [6], $\mathscr{D}^\theta$ is the unique solution of the above integral equation. It is the Doléans-Dade exponential of $\mathscr{H}^\theta := \{\mathscr{H}^\theta(t) | t \in \mathscr{T}\}$. Symbolically, it is typified as:

$$\mathscr{D}^\theta(t) = \mathscr{E}(\mathscr{H}^\theta)(t) , \quad t \in \mathscr{T} .$$

Again from Theorem 13.5 in Elliott [6],

$$\mathscr{D}^\theta(t) = \exp\left(\mathscr{H}^\theta(t) - \frac{1}{2} < (\mathscr{H}^\theta)^c, (\mathscr{H}^\theta)^c > (t)\right) \prod_{0<u\leq t} (1+\Delta\mathscr{H}^\theta(u)) e^{-\Delta\mathscr{H}^\theta(u)} ,$$

where

1. $(\mathscr{H}^\theta)^c := \{(\mathscr{H}^\theta)^c(t) | t \in \mathscr{T}\}$ is the continuous part of $\mathscr{H}^\theta$;
2. $\{< (\mathscr{H}^\theta)^c, (\mathscr{H}^\theta)^c > (t) | t \in \mathscr{T}\}$ is the predictable quadratic variation of $(\mathscr{H}^\theta)^c$.

Note that $\mathscr{H}^\theta$ is called the stochastic logarithm of $\mathscr{D}^\theta$ or the exponential transform of $\{(\theta \cdot Y)(t) | t \in \mathscr{T}\}$. Since $\mathscr{H}^\theta$ is a special semimartingale, its predictable part of finite variation, denoted as $\mathscr{K}^\theta := \{\mathscr{K}^\theta(t) | t \in \mathscr{T}\}$, is uniquely determined as:

$$\begin{aligned} \mathscr{K}^\theta(t) = \int_0^t \Bigg(& \hat{h}(u)\theta(u) + \frac{1}{2}\sigma^2(u)\theta^2(u) \\ & + \sum_{i=1}^N \int_{\Re_0} (e^z - 1 - \theta(u)z) \left\langle \hat{\mathbf{X}}(u), \mathbf{e}_i \right\rangle \lambda_i(u) \eta_i(dz|u) \Bigg) du . \end{aligned}$$

It was noted in Kallsen and Shiryaev [23] that the Laplace cumulant process of $\{(\theta \cdot Y)(t) | t \in \mathscr{T}\}$ is the predictable part of finite variation of $\mathscr{H}^\theta$. This is also called the Laplace cumulant process of Y at θ. The Doléan-Dade stochastic exponential $\mathscr{E}(\mathscr{K}^\theta) := \{\mathscr{E}(\mathscr{K}^\theta)(t) | t \in \mathscr{T}\}$ of $\mathscr{K}^\theta$ is the unique solution of the following linear stochastic differential equation:

$$\mathscr{E}(\mathscr{K}^\theta)(t) = 1 + \int_0^t \mathscr{E}(\mathscr{K}^\theta)(u) d\mathscr{K}^\theta(u) .$$

As in Kallsen and Shiryaev [23], the modified Laplace cumulant process of Y at θ is defined by the process $\tilde{\mathscr{K}}^\theta := \{\tilde{\mathscr{K}}^\theta(t) | t \in \mathscr{T}\}$ such that

$$\exp(\tilde{\mathscr{K}}^\theta(t)) = \mathscr{E}(\mathscr{K}^\theta)(t) .$$

By differentiation, $\tilde{\mathscr{K}}^\theta(t) = \mathscr{K}^\theta(t)$, $\mathscr{P}$-a.s., for each $t \in \mathscr{T}$.

Then the density process of the generalized Esscher transform associated with $\theta \in \mathscr{L}(Y)$ is defined as the process $\Lambda^\theta := \{\Lambda^\theta(t) | t \in \mathscr{T}\}$, where

$$\Lambda^\theta(t) := \frac{\exp((\theta \cdot Y)(t))}{\mathscr{E}(\mathscr{K}^\theta)(t)} .$$

Note that the Laplace cumulant process $\mathscr{E}(\mathscr{K}^{\theta})$ is the normalization constant and may be thought of as a generalization of the moment generation function in the "first-generation" of the Esscher transform in Gerber and Shiu [16].

From the definition of $\tilde{\mathscr{K}}^{\theta}$,

$$\begin{aligned}
\Lambda^{\theta}(t) &= \exp\left((\theta \cdot Y)(t) - \tilde{\mathscr{K}}^{\theta}(t)\right) \\
&= \exp\left((\theta \cdot Y)(t) - \mathscr{K}^{\theta}(t)\right) \\
&= \exp\left(-\sum_{i=1}^{N}\int_0^t\int_{\Re_0}(e^z - 1 - \theta(u)z)\left\langle \hat{\mathbf{X}}(u), \mathbf{e}_i\right\rangle \lambda_i(u)\eta_i(dz|u)\right. \\
&\quad \left. + \int_0^t\int_{\Re_0}\theta(u)z\hat{\gamma}(du,dz)\right) \times \exp\left(\int_0^t \theta(u)\sigma(u)d\hat{W}(u)\right. \\
&\quad \left. -\frac{1}{2}\int_0^t \theta^2(u)\sigma^2(u)du\right), \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Using Itô's differentiation rule,

$$\Lambda^{\theta}(t) = 1 + \int_0^t \Lambda^{\theta}(u)\theta(u)\sigma(u)d\hat{W}(u) + \int_0^t\int_{\Re_0}\Lambda^{\theta}(u-)(e^{\theta(u)z} - 1)\hat{\gamma}(du,dz)\,.$$

This is an $(\mathbb{F}^Y, \mathbb{P})$-local martingale. We suppose here that $\theta \in \mathscr{L}(Y)$ is such that Λ^{θ} is an $(\mathbb{F}^Y, \mathbb{P})$-martingale, so $\mathrm{E}[\Lambda^{\theta}(T)] = 1$.

Consequently, for each $\theta \in \mathscr{L}(Y)$, a new probability measure $\mathbb{P}^{\theta} \sim \mathbb{P}$ on $\mathscr{F}^Y(T)$ by putting:

$$\left.\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}}\right|_{\mathscr{F}^Y(T)} := \Lambda^{\theta}(T)\,.$$

To preclude arbitrage opportunities, we must determine $\theta \in \mathscr{L}(Y)$ such that the discounted price process of the risky share $\{\tilde{S}(t)|t \in \mathscr{T}\}$, where $\tilde{S}(t) := \exp(-\int_0^t r(u)du)S(t)$, is an $(\mathbb{F}^Y, \mathbb{P}^{\theta})$-(local)-martingale, (i.e., $\mathbb{P}^{\theta}$ is an equivalent (local)-martingale measure). This is called the (local)-martingale condition. A necessary and sufficient condition for the (local)-martingale condition is given in the following theorem.

Theorem 8.3. *The (local)-martingale condition holds if and only if for each $t \in \mathscr{T}$, there exists an $\mathbb{F}^Y$-progressively measurable process $\{\theta(t)|t \in \mathscr{T}\}$ such that*

$$\hat{\propto}(t) - r(t) + \theta(t)\sigma^2(t) + \sum_{i=1}^{N}\int_{\Re_0}\left\langle \hat{\mathbf{X}}(t), \mathbf{e}_i\right\rangle e^{\theta(t)z}(e^z - 1)\lambda_i(t)\eta_i(dz|t) = 0\,, \quad \mathscr{P}\text{-a.s.} \tag{8.4}$$

Proof. The proof is standard, so only key steps are given. Note that $\{\tilde{S}(t)|t \in \mathscr{T}\}$ is an $(\mathbb{F}^Y, \mathbb{P}^{\theta})$-(local)-martingale if and only if $\{\Lambda^{\theta}(t)\tilde{S}(t)|t \in \mathscr{T}\}$ is an $(\mathbb{F}^Y, \mathbb{P})$-(local)-martingale, for some $\mathbb{F}^Y$-progressively measurable process $\{\theta(t)|t \in \mathscr{T}\}$.

The result then follows by applying Itô's product rule to $\Lambda^{\theta}(t)\tilde{S}(t)$ and equating the finite variation term of $\Lambda^{\theta}(t)\tilde{S}(t)$ to zero. □

From Theorem 8.3, the risk-neutral Esscher parameter $\{\theta(t)|t \in \mathscr{T}\}$ of the generalized Esscher transform Λ^{θ} is obtained by solving Equation (8.4). In the particular case where the jump component is absent, the risk-neutral Esscher parameter is given by:

$$\theta(t) = \frac{r(t) - \hat{\propto}(t)}{\sigma^2(t)} = -\frac{\beta(t)}{\sigma(t)} ,$$

where $\beta(t)$ is the market price of risk of the particular case of the filtered market where jumps are absent and is defined as:

$$\beta(t) := \frac{\hat{\propto}(t) - r(t)}{\sigma(t)} .$$

The following lemma gives the probability laws of the return process Y of the risky share under $\mathbb{P}^{\theta}$. It is a direct consequence of a Girsanov transform for a measure change, so we only state the result.

Lemma 8.3. *The process defined by:*

$$\hat{W}^{\theta}(t) := \hat{W}(t) - \int_0^t \theta(u)\sigma(u)du , \quad t \in \mathscr{T} ,$$

is an $(\mathbb{F}^Y, \mathbb{P}^{\theta})$*-standard Brownian motion. Furthermore, the process defined by:*

$$\hat{\gamma}^{\theta}(dt,dz) := \gamma(dt,dz) - \sum_{i=1}^{N} \langle \hat{\mathbf{X}}(t), \mathbf{e}_i \rangle \lambda_i(t) e^{\theta(t)z} \eta_i(dz|t)dt ,$$

is an $(\mathbb{F}^Y, \mathbb{P}^{\theta})$*-martingale.*

Under $\mathbb{P}^{\theta}$*, the price process of the risky share is given by:*

$$dS(t) = S(t)r(t)dt + S(t)\sigma(t)d\hat{W}^{\theta}(t) + S(t-)\int_{\Re_0} (e^z - 1)\hat{\gamma}^{\theta}(dt,dz) .$$

To simplify our analysis, we suppose here that the probability law of the chain $\mathbf{X}$ remains unchanged after the measure change from $\mathbb{P}$ to $\mathbb{P}^{\theta}$. Consequently, under $\mathbb{P}^{\theta}$, the semimartingale dynamics for the chain $\mathbf{X}$ are:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}(u)\mathbf{X}(u)du + \mathbf{M}(t) , \quad t \in \mathscr{T} .$$

Furthermore, we assume that $\hat{W}^{\theta}$, $\hat{\gamma}^{\theta}$ and $\mathbf{X}$ are stochastically independent under $\mathbb{P}^{\theta}$.

8.5 European-Style Option

In this section, we shall consider the valuation of a standard European-style option in the filtered market and derive a partial differential integral equation (PDIE) for the option price. Consider a European-style option written on the risky share S whose payoff at the maturity time T is $H(S(T)) \in L^2(\Omega, \mathscr{F}^Y(T), \mathbb{P})$, where $L^2(\Omega, \mathscr{F}^Y(T), \mathbb{P})$ is the space of square-integrable random variables on $(\Omega, \mathscr{F}^Y(T), \mathbb{P})$ and $H : \Re \to \Re$ is a measurable function.

Conditional on the observed information $\mathscr{F}^Y(t)$ at the current time t, the price of the European option at time t is given by:

$$V(t) = \mathrm{E}^{\theta}\left[\exp\left(-\int_t^T r(u)du\right)H(S(T))|\mathscr{F}^Y(t)\right] .$$

Here E^{θ} is an expectation under the measure $\mathbb{P}^{\theta}$.

Note that $\{(S(t), \mathbf{q}(t))|t \in \mathscr{T}\}$ is jointly Markovian with respect to the observed filtration $\mathbb{F}^Y$. Consequently if $S(t) = s \in (0, \infty)$ and $\mathbf{q}(t) = \mathbf{q} \in \Re^N$,

$$\begin{aligned} V(t) &= \mathrm{E}^{\theta}\left[\exp\left(-\int_t^T r(u)du\right)H(S(T))|\mathscr{F}^Y(t)\right] \\ &= \mathrm{E}^{\theta}\left[\exp\left(-\int_t^T r(u)du\right)H(S(T))|(S(t), \mathbf{q}(t)) = (s, \mathbf{q})\right] \\ &:= V^{\dagger}(t, s, \mathbf{q}) , \end{aligned}$$

for some function $V^{\dagger} : \mathscr{T} \times (0, \infty) \times \Re^N \to \Re$.

For each $t \in \mathscr{T}$, let $V(t, s, \mathbf{q}) := \exp(-\int_0^t r(u)du)V^{\dagger}(t, s, \mathbf{q})$. Then

$$V(t, s, \mathbf{q}) = \mathrm{E}^{\theta}\left[\exp\left(-\int_0^T r(u)du\right)H(S(T))|\mathscr{F}^Y(t)\right] .$$

This is an $(\mathbb{F}^Y, \mathbb{P}^{\theta})$-martingale.

For each $t \in \mathscr{T}$, let

$$\hat{J}^{\theta}(t) := \int_0^t \int_{\Re_0} \hat{\gamma}^{\theta}(du, dz) .$$

Then using Lemma 8.3 and Theorem 8.1, the unnormalized filter process under the risk-neutral measure $\mathbb{P}^{\theta}$ is given by:

$$\begin{aligned} d\mathbf{q}(t) = \Bigg(&\mathbf{A}(t) + \mathbf{B}(t)\sigma^{-2}(t)(\hat{h}(t) + \sigma^2(t)\theta(t)) \\ &+ \sum_{i=1}^{N} \int_{\Re_0} \mathbf{diag}(\mathbf{G}(t) - \mathbf{1}) \langle \hat{\mathbf{X}}(t), \mathbf{e}_i \rangle e^{\theta(t)z} \lambda_i(t) \eta_i(dz|t) \end{aligned}$$

$$-\mathbf{diag}(\lambda(t)-\mathbf{1})\Big)\mathbf{q}(t)dt+\mathbf{B}(t)\sigma^{-1}(t)\mathbf{q}(t)d\hat{W}^{\theta}(t)$$
$$+\mathbf{diag}(\mathbf{G}(t)-\mathbf{1})\mathbf{q}(t)d\hat{J}^{\theta}(t)\ .$$

To simplify the notation, we write

$$\begin{aligned}\alpha^{\theta}(t) := {} & \mathbf{A}(t)+\mathbf{B}(t)\sigma^{-2}(t)(\hat{h}(t)+\sigma^{2}(t)\theta(t)) \\ & +\sum_{i=1}^{N}\int_{\Re_0}\mathbf{diag}(\mathbf{G}(t)-\mathbf{1})\left\langle \hat{\mathbf{X}}(t),\mathbf{e}_i\right\rangle e^{\theta(t)z}\lambda_i(t)\eta_i(dz|t) \\ & -\mathbf{diag}(\lambda(t)-\mathbf{1})\in\Re^{N}\otimes\Re^{N}\ ,\end{aligned}$$

so

$$d\mathbf{q}(t)=\alpha^{\theta}(t)\mathbf{q}(t)dt+\mathbf{B}(t)\sigma^{-1}(t)\mathbf{q}(t)d\hat{W}^{\theta}(t)+\mathbf{diag}(\mathbf{G}(t)-\mathbf{1})\mathbf{q}(t)d\hat{J}^{\theta}(t)\ .$$

Under $\mathbb{P}^{\theta}$, the price process of the risky asset is:

$$dS(t)=S(t)r(t)dt+S(t)\sigma(t)d\hat{W}^{\theta}(t)+S(t-)\int_{\Re_0}(e^{z}-1)\hat{\gamma}^{\theta}(dt,dz)\ .$$

Suppose $V:\mathscr{T}\times(0,\infty)\times(0,\infty)^{N}\to\Re$ is a function in $\mathscr{C}^{1,2}(\mathscr{T}\times(0,\infty)\times(0,\infty)^{N})$, where $\mathscr{C}^{1,2}(\mathscr{T}\times(0,\infty)\times(0,\infty)^{N})$ is the space of functions $V(t,s,\mathbf{q})$ which are continuously differentiable in $t\in\mathscr{T}$ and twice continuously differentiable in $(s,\mathbf{q})\in(0,\infty)\times(0,\infty)^{N}$.

The following theorem gives the partial differential integral equation (P.D.I.E.) for the price of the European-style option V.

Theorem 8.4. *Let $\mathbf{q}_{-}:=\mathbf{q}(t-)$ and $s_{-}:=S(t-)$, for each $t\in\mathscr{T}$. Write $\mathbf{y}'$ for the transpose of a vector, or matrix, $\mathbf{y}$. Define, for each $t\in\mathscr{T}$,*

$$\beta^{\theta}(t):=\sum_{i=1}^{N}\mathbf{diag}(\mathbf{G}(t)-\mathbf{1})\left\langle \hat{\mathbf{X}}(t),\mathbf{e}_i\right\rangle\lambda_i(t)\int_{\Re_0}e^{\theta(t)z}(e^{z}-1)\eta_i(dz|t)\in\Re^{N}\otimes\Re^{N}\ .$$

Then the option price $V^{\dagger}(t,s,\mathbf{q})$ at time t satisfies:

$$\begin{aligned}&\frac{\partial V^{\dagger}}{\partial t}+\frac{\partial V^{\dagger}}{\partial s}s(\hat{\alpha}(t)+\theta(t)\sigma^{2}(t))+\left\langle\frac{\partial V^{\dagger}}{\partial\mathbf{q}},(\alpha^{\theta}(t)-\beta^{\theta}(t))\mathbf{q}(t)\right\rangle \\ &+\frac{1}{2}\frac{\partial^{2}V^{\dagger}}{\partial s^{2}}\sigma^{2}(t)s^{2}+s\left\langle\frac{\partial^{2}V^{\dagger}}{\partial\mathbf{q}\partial s},\mathbf{B}(t)\mathbf{q}(t)\right\rangle+\frac{1}{2}(\mathbf{B}(t)\mathbf{q}(t))'\frac{\partial^{2}V^{\dagger}}{\partial\mathbf{q}^{2}}(\mathbf{B}(t)\mathbf{q}(t)) \\ &+\sum_{i=1}^{N}\left\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\right\rangle\int_{\Re_0}\Big(V^{\dagger}(t,s_{-}e^{z},\mathbf{q}_{-}\mathbf{diag}(\mathbf{G}(t)))-V^{\dagger}(t,s_{-},\mathbf{q}_{-})\Big) \\ &\times e^{\theta(t)z}\lambda_i(t)\eta_i(dz|t)=r(t)V^{\dagger}\ ,\end{aligned}$$

with the terminal condition $V^{\dagger}(T,S(T),\mathbf{q}(T))=H(S(T))$.

Proof. The proof is standard. For the sake of completeness, we give the proof here. Applying Itô's differentiation rule to $V(t,s,\mathbf{q})$ gives:

$$
\begin{aligned}
V(t,s,\mathbf{q}) = {} & V(0,s_0,\mathbf{q}_0) + \int_0^t \frac{\partial V}{\partial u} du + \int_0^t \frac{\partial V}{\partial s} S(u) \Big(r(u) - \sum_{i=1}^N \langle \hat{\mathbf{X}}(u), \mathbf{e}_i \rangle \lambda_i(u) \\
& \times \int_{\Re_0} e^{\theta(u)z}(e^z - 1)\eta_i(dz|u) \Big) du - \int_0^t \Big\langle \frac{\partial V}{\partial \mathbf{q}}, \mathbf{diag}(\mathbf{G}(u) - \mathbf{1})\mathbf{q}(u) \Big\rangle \\
& \times \sum_{i=1}^N \langle \hat{\mathbf{X}}(u), \mathbf{e}_i \rangle \lambda_i(u) \int_{\Re_0} e^{\theta(u)z}(e^z - 1)\eta_i(dz|u) \Big) du \\
& + \int_0^t \frac{\partial V}{\partial s} S(u)\sigma(u) d\hat{W}^\theta(u) + \int_0^t \Big\langle \frac{\partial V}{\partial \mathbf{q}}, \alpha^\theta(u)\mathbf{q}(u) \Big\rangle du \\
& + \int_0^t \Big\langle \frac{\partial V}{\partial \mathbf{q}}, \mathbf{B}(u)\mathbf{q}(u) \Big\rangle \sigma^{-1}(u) d\hat{W}^\theta(u) + \int_0^t \frac{\partial^2 V}{\partial s^2} \sigma^2(u) S^2(u) du \\
& + \int_0^t S(u) \Big\langle \frac{\partial^2 V}{\partial \mathbf{q} \partial s}, \mathbf{B}(u)\mathbf{q}(u) \Big\rangle du + \frac{1}{2} \int_0^t (\mathbf{B}(u)\mathbf{q}(u))' \frac{\partial^2 V}{\partial \mathbf{q}^2} (\mathbf{B}(u)\mathbf{q}(u)) du \\
& + \sum_{i=1}^N \int_0^t \langle \hat{\mathbf{X}}(u), \mathbf{e}_i \rangle \int_{\Re_0} \Big(V(u, S(u-)e^z, \mathbf{q}(u-)\mathbf{diag}(\mathbf{G}(u))) \\
& - V(u, S(u-), \mathbf{q}(u-)) \Big) e^{\theta(u)z} \lambda_i(u) \eta_i(dz|u) du \\
& + \int_0^t \int_{\Re_0} \Big(V(u, S(u-)e^z, \mathbf{q}(u)) - V(u, S(u-), \mathbf{q}(u-)) \Big) \hat{\gamma}^\theta(du, dz) \,.
\end{aligned}
$$

Rearranging then gives:

$$
\begin{aligned}
V(t,s,\mathbf{q}) = {} & V(0,s_0,\mathbf{q}_0) + \int_0^t \Big[\frac{\partial V}{\partial u} + \frac{\partial V}{\partial s} S(u) \Big(r(u) - \sum_{i=1}^N \langle \hat{\mathbf{X}}(u), \mathbf{e}_i \rangle \lambda_i(u) \\
& \times \int_{\Re_0} e^{\theta(u)z}(e^z - 1)\eta_i(dz|u) \Big) - \Big\langle \frac{\partial V}{\partial \mathbf{q}}, \mathbf{diag}(\mathbf{G}(u) - \mathbf{1})\mathbf{q}(u) \Big\rangle \\
& \times \sum_{i=1}^N \langle \hat{\mathbf{X}}(u), \mathbf{e}_i \rangle \lambda_i(u) \int_{\Re_0} e^{\theta(u)z}(e^z - 1)\eta_i(dz|u) + \Big\langle \frac{\partial V}{\partial \mathbf{q}}, \alpha^\theta(u)\mathbf{q}(u) \Big\rangle \\
& + \frac{1}{2} \frac{\partial^2 V}{\partial s^2} \sigma^2(u) S^2(u) + S(u) \Big\langle \frac{\partial^2 V}{\partial \mathbf{q} \partial s}, \mathbf{B}(u)\mathbf{q}(u) \Big\rangle + \frac{1}{2} (\mathbf{B}(u)\mathbf{q}(u))' \frac{\partial^2 V}{\partial \mathbf{q}^2} (\mathbf{B}(u)\mathbf{q}(u)) \\
& + \sum_{i=1}^N \langle \hat{\mathbf{X}}(u), \mathbf{e}_i \rangle \int_{\Re_0} \Big(V(u, S(u-)e^z, \mathbf{q}(u-)\mathbf{diag}(\mathbf{G}(u))) - V(u, S(u-), \mathbf{q}(u-)) \Big) \\
& \times e^{\theta(u)z} \lambda_i(u) \eta_i(dz|u) \Big] du + \int_0^t \frac{\partial V}{\partial s} S(u)\sigma(u) d\hat{W}^\theta(u) \\
& + \int_0^t \Big\langle \frac{\partial V}{\partial \mathbf{q}}, \mathbf{B}(u)\mathbf{q}(u) \Big\rangle \sigma^{-1}(u) d\hat{W}^\theta(u) \\
& + \int_0^t \int_{\Re_0} \Big(V(u, S(u-)e^z, \mathbf{q}(u)) - V(u, S(u-), \mathbf{q}(u-)) \Big) \hat{\gamma}^\theta(du, dz).
\end{aligned}
$$

Using the martingale condition (8.4) and the definition of $\beta^{\theta}(t)$,

$$
\begin{aligned}
V(t,s,\mathbf{q}) = V(0,s_0,\mathbf{q}_0) &+ \int_0^t \Big[\frac{\partial V}{\partial u} + \frac{\partial V}{\partial s} S(u)(\hat{\propto}(u) + \theta(u)\sigma^2(u)) + \Big\langle \frac{\partial V}{\partial \mathbf{q}}, (\alpha^{\theta}(u) - \beta^{\theta}(u))\mathbf{q}(u) \Big\rangle \\
&+ \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2(u)S^2(u) + S(u)\Big\langle \frac{\partial^2 V}{\partial \mathbf{q}\partial s}, \mathbf{B}(u)\mathbf{q}(u) \Big\rangle + \frac{1}{2}(\mathbf{B}(u)\mathbf{q}(u))' \frac{\partial^2 V}{\partial \mathbf{q}^2}(\mathbf{B}(u)\mathbf{q}(u)) \\
&+ \sum_{i=1}^{N} \langle \hat{\mathbf{X}}(u), \mathbf{e}_i \rangle \int_{\Re_0} \Big(V(u, S(u-)e^z, \mathbf{q}(u-)\mathbf{diag}(\mathbf{G}(u))) - V(u, S(u-), \mathbf{q}(u-)) \Big) \\
&\times e^{\theta(u)z} \lambda_i(u) \eta_i(dz|u) \Big] du + \int_0^t \frac{\partial V}{\partial s} S(u)\sigma(u) d\hat{W}^{\theta}(u) \\
&+ \int_0^t \Big\langle \frac{\partial V}{\partial \mathbf{q}}, \mathbf{B}(u)\mathbf{q}(u) \Big\rangle \sigma^{-1}(u) d\hat{W}^{\theta}(u) \\
&+ \int_0^t \int_{\Re_0} \Big(V(u, S(u-)e^z, \mathbf{q}(u)) - V(u, S(u-), \mathbf{q}(u-)) \Big) \hat{\gamma}^{\theta}(du, dz) .
\end{aligned}
$$

Note that the discounted price process $\{V(t,S(t),\mathbf{q}(t))|t \in \mathscr{T}\}$ is an $(\mathbb{F}^Y, \mathbb{P}^{\theta})$-martingale. It must be a special semimartingale. Consequently, the du-integral terms must sum to zero, and hence,

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial s} S(t)(\hat{\propto}(t) + \theta(t)\sigma^2(t)) + \Big\langle \frac{\partial V}{\partial \mathbf{q}}, (\alpha^{\theta}(t) - \beta^{\theta}(t))\mathbf{q}(t) \Big\rangle \\
&+ \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2(t)S^2(t) + S(t)\Big\langle \frac{\partial^2 V}{\partial \mathbf{q}\partial s}, \mathbf{B}(t)\mathbf{q}(t) \Big\rangle + \frac{1}{2}(\mathbf{B}(t)\mathbf{q}(t))' \frac{\partial^2 V}{\partial \mathbf{q}^2}(\mathbf{B}(t)\mathbf{q}(t)) \\
&+ \sum_{i=1}^{N} \langle \hat{\mathbf{X}}(t), \mathbf{e}_i \rangle \int_{\Re_0} \Big(V(t, S(t-)e^z, \mathbf{q}(t-)\mathbf{diag}(\mathbf{G}(t))) - V(t, S(t-), \mathbf{q}(t-)) \Big) \\
&\times e^{\theta(t)z} \lambda_i(t) \eta_i(dz|t) = 0 .
\end{aligned}
$$

Therefore, the result follows by applying the differentiation rule to

$$
V^{\dagger}(t,s,\mathbf{q}) = \exp\Big(- \int_0^t r(u)du \Big) V(t,s,\mathbf{q})
$$

again. □

The result in Theorem 8.4 may be extended from the class of smooth functions $\mathscr{C}^{1,2}(\mathscr{T} \times (0,\infty) \times (0,\infty)^N)$ to a wider class of functions in which a generalized Itô's differentiation rule holds. The wider class of functions may include differences of two convex functions, (see, for example, [24]).

8.6 Conclusion

We discussed a two-stage approach for pricing a European-style option in a hidden Markovian regime-switching jump-diffusion model. Filtering theory was first used to turn the original market with partial observations to a filtered market with complete observations. Then the option valuation problem was considered in the filtered market where the hidden quantities in the original market were replaced by their filtered estimates. The generalized Esscher transform for semimartingales was used to select a pricing kernel in the incomplete filtered market. By noticing that the price process and the unnormalized filter process of the hidden Markov chain are jointly Markovian with respect to the observed filtration, a partial differential-integral equation governing the price of the European-style option was derived.

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