CHAPTER 2

Limits and Continuity. One and Several Variables

MATLAB provides commands that allow you to calculate virtually all types of limits. The same functions are used to calculate limits of sequences and limits of functions. The commands for the analysis of one and several variables are similar. In this chapter we will present numerous exercises which illustrate MATLAB's capabilities in this field.

The syntax of the commands concerning limits are presented below:

> **maple ('limit(sequence, n=infinity)') or limit(sequence, n, inf) or limit(sequence, inf) calculates the limit as *n* tends to infinity of the sequence defined by its general term.**
>
> **maple ('limit(function, x=a)') or limit(function, x, a) or limit(function, a) calculates the limit of the function of the variable *x*, indicated by its analytical expression, as the variable *x* tends towards the value *a*.**
>
> **maple ('limit(function, x=a, right)') or limit(function, x, a, 'right') calculates the limit of the function of the variable *x*, indicated by its analytical expression, as *the variable x tends to the value a from the right*.**
>
> **maple ('limit(function, x=a, left)') or limit(function, x, a, 'left') calculates the limit of the function of the variable *x*, indicated by its analytical expression, as *the variable x tends to the value a from the left*.**
>
> **maple ('limit(expr, var=a, complex)') is the complex limit of expr as the variable *var* tends to the value *a*.**
>
> **maple ('limit(expr, {v1=a1, ..., vn=an})') is the n-dimensional limit of *expr* as *v1 tends to a1, v2 tends to a2,..., vn tends to an.***
>
> **maple ('Limit(expr, var=a)') or maple ('Limit(expr, {v1=a1, ..., vn=an})') is the inert limit of the expression *expr* for the specified values of the variable or variables.**

2.1 Limits of Sequences

We present some exercises on the calculation of limits of sequences.

EXERCISE 2-1

Calculate the following limits:

$$\lim_{n\to\infty}\left(\frac{-3+2n}{-7+3n}\right)^4, \lim_{n\to\infty}\frac{1+7n^2+3n^3}{5-8n+4n^3}, \lim_{n\to\infty}\left[\left(\frac{1+n}{2}\right)^4\frac{1+n}{n^5}\right], \lim_{n\to\infty}\sqrt[n]{\frac{1+n}{n^2}}$$

In the first two limits we face the typical uncertainty given by the quotient ∞/∞:

```
>> syms n
>> limit(((2*n-3)/(3*n-7))^4, inf)

ans =

16/81

>> limit((3*n^3+7*n^2+1)/(4*n^3-8*n+5),n,inf)

ans =

3/4
```

The last two limits present an uncertainty of the form $\infty.0$ and ∞^0:

```
>> limit(((n+1)/2)*((n^4+1)/n^5), inf)

ans =

1/2

>> limit(((n+1)/n^2)^(1/n), inf)

ans =

1
```

EXERCISE 2-2

Calculate the following limits:

$$\lim_{n\to\infty}\left(\frac{3+n}{-1+n}\right)^n,\ \lim_{n\to\infty}\left(1-\frac{2}{3+n}\right)^n,\ \lim_{n\to\infty}\sqrt[n]{\frac{1}{n}},\ \lim_{n\to\infty}\frac{-\sqrt[3]{n}+\sqrt[3]{1+n}}{-\sqrt{n}+\sqrt{1+n}},\ \lim_{n\to\infty}\frac{n!}{n^n}$$

The first two examples are indeterminate of the form 1^∞:

```
>> limit(((n+3)/(n-1))^n, inf)

ans =

exp(4)

>> limit((1-2/(n+3))^n, inf)

ans =

exp (- 2)
```

The next two limits are of the form ∞^0 and $(\infty-\infty)/\infty$:

```
>> limit((1/n)^(1/n), inf)

ans =

1

>> maple('limit(((n+1)^(1/3)-n^(1/3))/((n+1)^(1/2)-n^(1/2)),n=infinity)')

ans =

0
```

The last limit is of the form ∞/∞ :

```
>> maple('limit(n!/n^n, n=infinity)')

ans =

0
```

2.2 Limits of Functions. Lateral Limits

To calculate the limits of functions one uses the same MATLAB commands as for limits of sequences. For functions, MATLAB allows you to calculate the limit at a point, and left and right limits (if these limits exist). If a function has a limit at a point then it necessarily has left and right limits at that point, and they coincide. If the left and right limits do not coincide then the function does not have a limit at the given point. Below are several exercises which illustrate how to calculate function limits. Some exercises are accompanied by graphics. The use of graphics is advisable if there are any doubts concerning the results.

EXERCISE 2-3

Calculate the following limits:

$$\lim_{x\to 1}\frac{-1-x}{-1+\sqrt{x}},\ \lim_{x\to 2}\frac{x-\sqrt{2+x}}{-3+\sqrt{1+4x}},\ \lim_{x\to 0}\sqrt[x]{1+x},\ \lim_{x\to 0}\frac{\sin\left[(ax)^2\right]}{x^2},\ \lim_{x\to 0}\frac{e^x-1}{\log(1+x)}.$$

Initially, we have two indeterminates of type 0/0 and one of the form 1^∞:

```
>> syms x
>> limit((x-1)/(x^(1/2)-1),x,1)
2

>> limit((x-(x+2)^(1/2))/((4*x+1)^(1/2)-3),2)

9/8

>> limit((1+x)^(1/x))
exp (1)
```

The last two are indeterminates of the form 0/0:

```
>> syms x a, limit(sin(a*x)^2/x^2,x,0)

a^2

>> numeric(limit((exp(1)^x-1)/log(1+x)))

1
```

EXERCISE 2-4

Calculate the following function limits:

$$\lim_{x\to 1}\frac{|x|}{\sin(x)},\ \lim_{x\to 3}\left|x^2-x-7\right|,\ \lim_{x\to 1}\frac{x-1}{x^n-1},\ \lim_{x\to 0}\sqrt[x]{e}.$$

The first limit is calculated as follows:

```
>> limit(abs(x)/sin(x),x,0)

ans =

NaN

>> limit(abs(x)/sin(x),x,0,'left')

ans =

-1

>> limit(abs(x)/sin(x),x,0,'right')

ans =

1
```

As the lateral boundaries are not equal, the function has no limit at $x = 0$.

If we plot the function (see Figure 2-1), the limits become clear:

```
>> ezplot (abs (x) /sin (x), [- 1, 1])
```

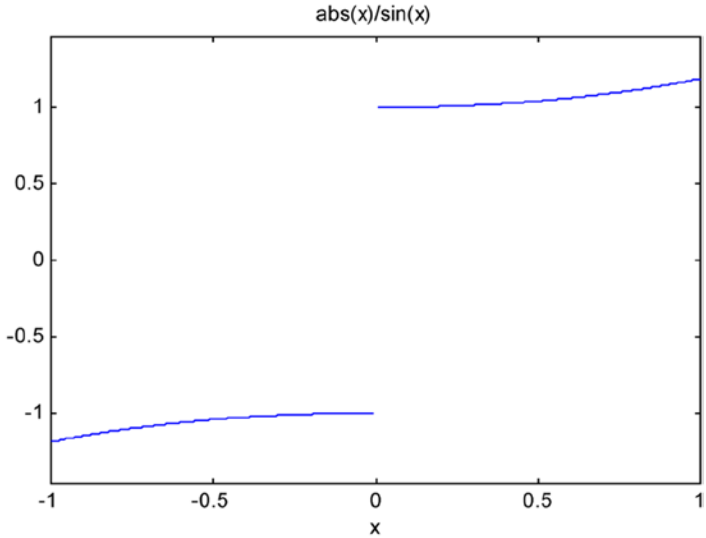

Figure 2-1.

By simple observation, we see that the limit from the right is 1 and the limit from the left is - 1.

For the next two limits we have:

```
>> limit(abs(x^2-x-7),x,3)

ans =

1

>> limit((x-1)/(x^n-1),x,1)

ans =

1/n
```

For the last limit we have the following:

```
>> limit(exp(1)^(1/x),x,0)

ans =

NaN

>> limit(exp(1)^(1/x),x,0,'left')

ans =

0

>> limit(exp(1)^(1/x),x,0,'right')

ans =

INF
```

Then, there is no limit at $x = 0$. If we plot the function (see Figure 2-2), we have:

```
>> ezplot (exp(1)^(1/x), [- 10, 10 - 3, 3])
```

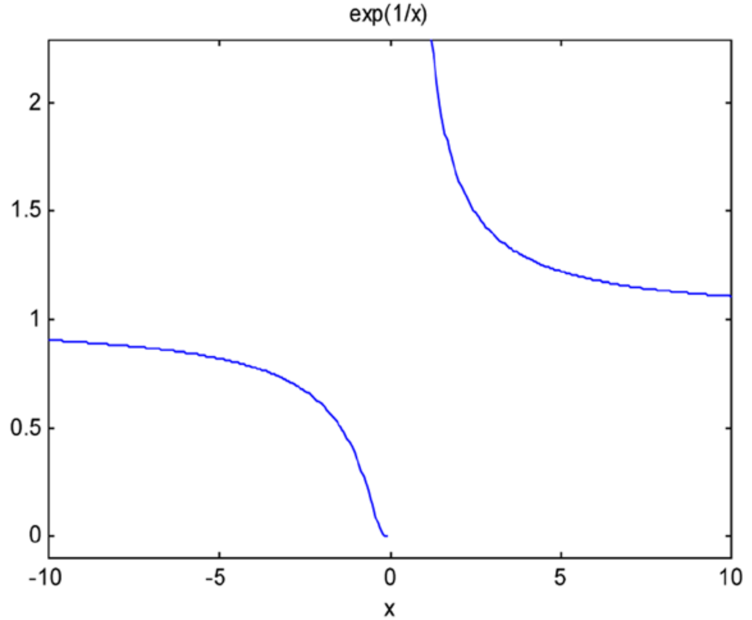

Figure 2-2.

We see that the function becomes (positive) infinite at 0 when approaching from the right, and it tends to 0 when approaching from the left. Thus we conclude that the function has no limit at $x = 0$.

2.3 Sequences of Functions

MATLAB enables you to analyze the convergence of sequences of functions. MATLAB's graphics capabilities can greatly help in understanding the concept of the limit of a sequence of functions and can also aid in the calculation of these limits.

EXERCISE 2-5

Calculate the limit function of the following sequences of functions:

$f_n(x) = x^n/n$, $g_n(x) = x^n/(1 + x^n)$, $h_n(x) = x^n/(n + x^n)$, $k_n(x) = sin^2(\pi x)$ if $1 / (n + 1) \leq x \leq 1/n$ and $k_n(x) = 0$ *if* $(x < 1/(n + 1)$ and $x > 1/n)$. In all cases x varies in [0,1].

First of all, we impose the condition $x \in [0,1]$ using the syntax:

```
>> maple('assume(x, RealRange(0,1))');
```

We then calculate the limits of the function sequences as follows:

```
>> pretty(sym(maple('limit(x^n/n,n=infinity)')))
```

0

```
>> pretty(sym(maple('limit(x^n/(1+x^n),n=infinity)')))
```

0

```
>> pretty(sym(maple('limit(x^n/(n+x^n),n=infinity) ')))
```

0

```
>> pretty(sym(maple('limit(piecewise(x>=1/(n+1) and x<=1/n,
sin(Pi/x)^2,x<1/(n+1) and x<1/n,0),n=infinity) ')))
```

0

Now we support these results with the graphical representations shown in Figures 2-3, 2-4 and 2-5, which in each case illustrates the ***convergence of the sequence of functions towards the constant 0 function***. The syntax for the plots is as follows:

```
>> fplot('[x,x^2/2,x^3/3,x^4/4,x^5/5,x^6/6,x^7/7,x^8/8,x^9/9,x^10/10]',[0,1,-1/2,1])

>> fplot('[x/(1+x),x^2/(1+x^2),x^3/(1+x^3),x^4/(1+x^4),x^5/(1+x^5),x^6/(1+x^6),
         x^7/(1+x^7),x^8/(1+x^8),x^9/(1+x^9),x^10/(1+x^10)]',[0,1,-1/2,1])

>> fplot('[x/(1+x),x^2/(2+x^2),x^3/(3+x^3),x^4/(4+x^4),x^5/(5+x^5),x^6/(6+x^6),
         x^7/(7+x^7),x^8/(8+x^8),x^9/(9+x^9),x^10/(10+x^10)]',[0,1,-1/2,1])
```

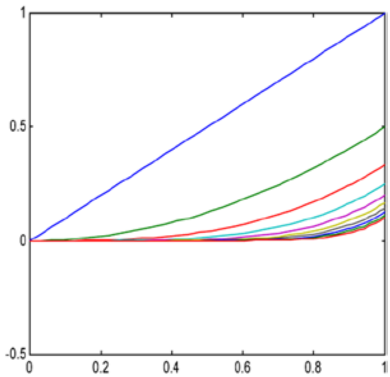

Figure 2-3.

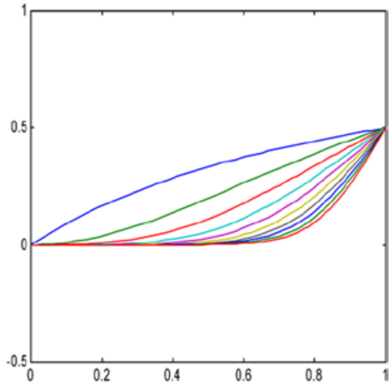

Figure 2-4.

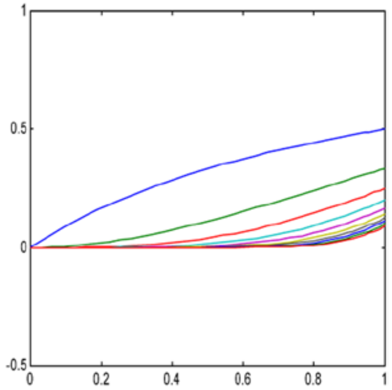

Figure 2-5.

EXERCISE 2-6

Calculate the limit of the following sequences of functions:

$$f_n(x) = (x^2 + nx)/n,\ x \in R$$

$$g_n(x) = 1 \text{ if } -n \le x \le n \text{ and } g_n(x) = 0 \text{ otherwise, } x \in R$$

$$h_n(x) = 2n^2x \text{ if } 0 \le x \le 1/2n,\ h_n(x) = 2n - 2n^2x \text{ if } 1/2n \le x \le n,\ h_n(x) = 0 \text{ if } n \le x \le 1,\ x \in [0,1]$$

The limits of these sequences of functions are calculated as follows:

```
>> pretty(sym(maple('limit((x^2+n*x)/n,n=infinity)')))
```

x

```
>> pretty(sym(maple('limit(piecewise(x>=-n and x<=n,1,0),n=infinity)')))
```

1

```
>> maple('assume(x, RealRange(0,1))');
>> pretty(sym(maple('limit(piecewise(x>=0 and x<=1/(2*n), 2*n^2*x, x>=1/(2*n) and
1/n>=x, 2*n-2*x*n^2, 1/n<=x and x>=1, 0),n=infinity)')))
```

0

Now we support these results with the graphical representations shown in Figures 2-6, 2-7 and 2-8, which illustrate the ***convergence of the first sequence of functions towards the function f(x) = x, the second sequence to the function f(x) = 1 and the third sequence to the function f(x) = 0***. The syntax entered into the *Matlab/Editor debugger* to define the functions via the *M-File* sub-option of the *File* menu in the MATLAB command window is the following:

```
function f=seq1(x,n)
f=(x^2+n*x)/n;

function g=seq2(x,n)
   if x>=-n & x<n
   g=1;
   else g=0;
end

function h=seq3(x,n)
if x>=0 & x<=1/(2*n)
h=2*n^2*x;
elseif x>1/(2*n) & x<=1/n
h=2*n-2*n^2*x;
elseif x>1/n & x<=1
h = 0;
end
```

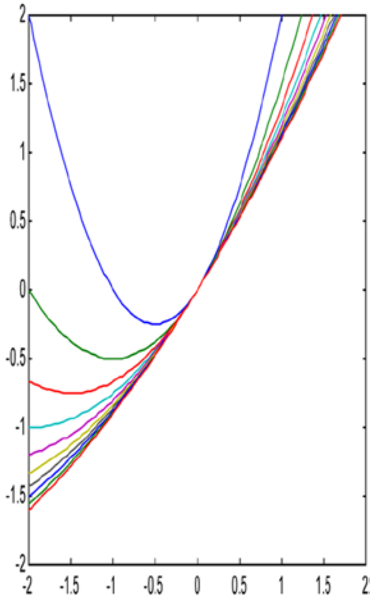

Figure 2-6.

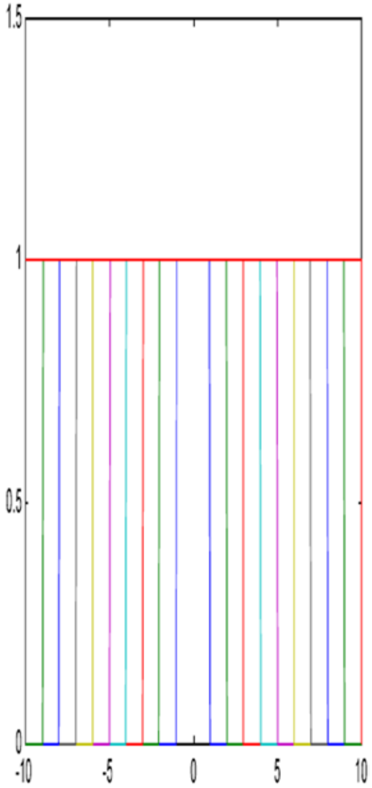

Figure 2-7.

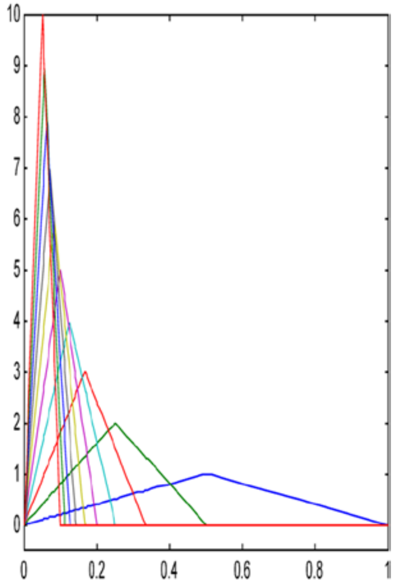

Figure 2-8.

We keep the definitions of the sequences of functions in the M-files named *seq1.m*, *seq2.m* and *seq3.m* respectively. The graphical respresentation of the first 10 functions of each sequence is as follows:

```
>> fplot ('[seq1(x,1), seq1(x,2), seq1(x,3), seq1(x,4), seq1(x,5),])
([seq1(x,6), seq1(x,7), seq1(x,8), seq1(x,9), seq1(x,10)]', [- 2, 2, - 2, 2])

>> fplot ('[seq2(x,1), seq2(x,2), seq2(x,3), seq2(x,4), seq2(x,5),])
([seq2(x,6), seq2(x,7), seq2(x,8), seq2(x,9), seq2(x,10)]'[- 10, 10, 0, 3/2])

>> fplot ('[seq3(x,1), seq3(x,2), seq3(x,3), seq3(x,4), seq3(x,5),])
([seq3(x,6), seq3(x,7), seq3(x,8), seq3(x,9), seq3(x,10)]', [- 0, 1, - 1/2, 10])
```

2.4 Continuity

A function f is continuous at the point $x = a$ if:

$$\lim_{x\to a} f(x) = f(a).$$

Otherwise, it is discontinuous at the point. In other words, in order for a function f to be continuous at a it must be defined at a and the limit of the function at a must be equal to the value of the function at a.

If the limit of $f(x)$ as x tends to a exists but is different to $f(a)$, then f is discontinuous at a, and we say f has an avoidable discontinuity at a. The discontinuity is resolved by redefining $f(a)$ to coincide with the limit.

If the two lateral limits of $f(x)$ at a exist (whether finite or infinite) but are different, then the discontinuity of f at a is said to be of the *first kind*. The difference between the two lateral limits is the *jump*. If the jump is finite then the discontinuity is said to be of the *first kind with finite jump*, otherwise it is of the *first kind with infinite jump*.

If either of the lateral limits do not exist, then the discontinuity is said to be of the *second kind*.

MATLAB provides the following commands relating to continuity:

> **maple ('readlib (discont): discont (function, variable)'): Determines the points of discontinuity of the real valued function given in the specified variable.**
>
> **maple ('readlib (iscont): iscont(function,var=a..b)'): Determines when the given function is continuous on the interval [a, b]. The functional expression can contain piecewise-defined functions.**
>
> **maple ('readlib (iscont): iscont(function,var=a..b, open)'): Determines when the given given function is continuous on the open interval (a, b).**

We illustrate these concepts with several exercises:

EXERCISE 2-7

Study the continuity of the following functions of a real variable:

$$f(x)=\frac{\sin(x)}{x},\ g(x)=\sin\left(\frac{1}{x}\right).$$

```
>> syms x a
>> limit(sin(x)/x,x,a)

ans =

sin(a)/a
```

The function *f* is continuous at any non-zero point *a*, so for any such point we have that $\lim_{x\to a} f(x)=f(a)$.

The problem arises at the point *x = 0*, at which the function *f* is not defined. Therefore, the function is discontinuous at *x = 0*, This discontinuity can be avoided by redefining the function at *x = 0* with a value equal to $\lim_{x\to 0} f(x)$.

```
>> limit(sin(x)/x,x,0)

ans =

1
```

Thus we conclude that the function $f(x)=\sin(x)/x$ presents an avoidable discontinuity at *x* = 0 that is avoided by defining $f(0)=1$. The function is continuous at all non-zero points.

The function *g* is continuous at any non-zero point *a*, so $\lim_{x\to a} g(x)=g(a)$.

```
>> limit (sin (1/x), x, a)

ans =

sin(1/a)
```

The problem arises at the point $x = 0$, where the function g is not defined. Therefore, the function is discontinuous at $x = 0$. To try to avoid the discontinuity, we calculate $\lim_{x \to 0} g(x)$.

```
>> limit(sin(1/x),x,0)

ans =

-1 .. 1

>> limit(sin(1/x),x,0,'left')

ans =

-1 .. 1

>> limit(sin(1/x),x,0,'right')

ans =
-1 .. 1
```

We see that the limit does not exist at $x = 0$ (the limit has to be unique and here the result given is all points in the interval $[-1,1]$), and neither of the lateral limits exist. Thus the function has a discontinuity of the second kind at $x=0$.

MATLAB responded to the calculation of the above limits with the expression *"– 1.. 1"*. This is because the graph of $g(x)$ presents infinitely many oscillations between – 1 and 1. This is illustrated in Figure 2-9:

```
>> fplot ('sin (1/x)', [- 1, 1])
```

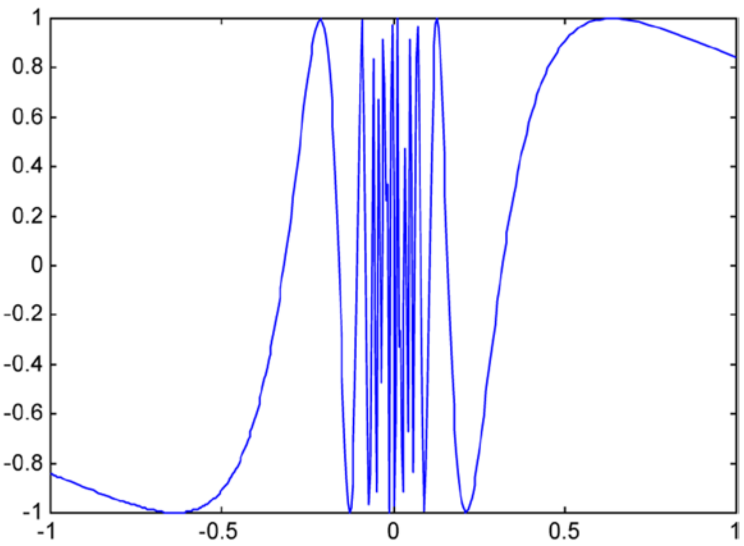

Figure 2-9.

EXERCISE 2-8

Study the continuity of the following function of a real variable:

$$f(x) = \frac{1}{1+\sqrt[x]{e}} \text{ if } x \neq 0 \text{ and } f(x) = 1 \text{ if } x = 0.$$

The only problematic point is $x = 0$. Now, the function does exist at $x = 0$ (it has the value 1). We will try to find the lateral limits as x tends to 0:

```
>> syms x
>> limit(1/(1+exp(1/x)),x,0,'right')

ans =

0

>> limit(1/(1+exp(1/x)),x,0,'left')

ans =

1
```

As the lateral boundaries are different, the limit of the function at 0 does not exist. However, the lateral boundaries are both finite, so the discontinuity at 0 is of the first kind with a finite jump. We illustrate this result in Figure 2-10.

```
>> fplot('1/(1+exp(1/x))',[-5,5])
```

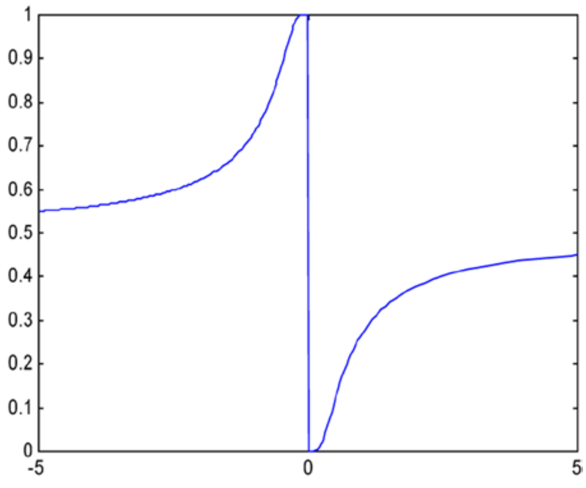

Figure 2-10.

EXERCISE 2-9

Study the continuity of the following function of a real variable:

$$f(x) = \sqrt[x]{e} \text{ if } x \neq 0 \text{ and } f(x) = 1 \text{ if } x = 0.$$

The only problematic point is $x = 0$. The function is defined at $x = 0$ (it has the value 1). We will try to find the lateral limits at 0:

```
>> limit ((exp (x)), x, 0, 'right')

ans =

inf

>> limit((exp(1/x)),x,0, 'left')

ans =

0
```

As the lateral boundaries are different, the limit of the function as *x* tends to 0 does not exist. As the right lateral limit is infinite, the discontinuity of the first kind at $x = 0$ is an infinite jump. We illustrate this result in Figure 2-11 above.

```
>> fplot ('exp (x)', [- 150, 150])
```

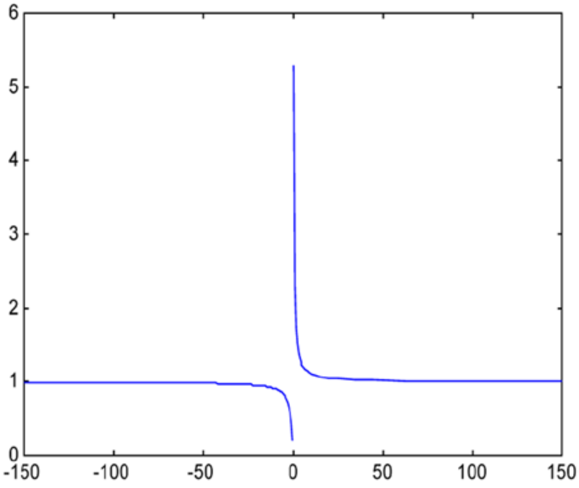

Figure 2-11.

2.5 Several Variables: Limits and Continuity. Characterization Theorems

The sequence $\{(a_{1n}, a_{2n}, \ldots, a_{mn})\}$ of points in m-dimensional space, where n runs through the natural numbers, has as limit the point $(a_1, a_2, \ldots, a_m)$ if, and only if:

$$\lim_{n\to\infty}(a_{1n}) = a_1, \lim_{n\to\infty}(a_{2n}) = a_2, \ldots, \lim_{n\to\infty}(a_{mn}) = a_m.$$

This characterization theorem allows us to calculate limits of sequences of points in m-dimensional space.

There is another theorem, similar to the above, which characterizes the limits of functions between spaces of more than one dimension. This theorem enables us to calculate the limits of multivariable functions.

If $f: R^n \to R^m$ is a function whose m components are $(f_1, \ldots, f_m)$. Then it follows that:

$$\lim_{x_1\to a_1, x_2\to a_2, \ldots, x_n\to a_n}\left(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)\right)$$
$$= (l_1, l_2, \ldots, l_m)$$

if and only if

$$\lim_{x_1\to a_1, x_2\to a_2, \ldots, x_n\to a_n}\left(f_1(x_1, x_2, \ldots, x_n)\right) = l_1,$$
$$\lim_{x_1\to a_1, x_2\to a_2, \ldots, x_n\to a_n}\left(f_2(x_1, x_2, \ldots, x_n)\right) = l_2,$$
$$\ldots,$$
$$\lim_{x_1\to a_1, x_2\to a_2, \ldots, x_n\to a_n}\left(f_m(x_1, x_2, \ldots, x_n)\right) = l_m.$$

EXERCISE 2-10

Calculate the limit of the following three-dimensional sequence:

$$\lim_{n\to\infty}\left[\frac{1+n}{n}, \left(1+\frac{1}{n}\right)^{2n}, \frac{n}{2n-1}\right]$$

```
>> pretty(sym(maple('vector([limit((n+1)/n,n=infinity),limit((1+1/n)^(2*n),
n=infinity),limit(n/(2*n-1),n=infinity)])')))

[1-exp (2) 1/2]
```

EXERCISE 2-11

Calculate the limit as $n \to \infty$ of the following four-dimensional sequence:

$$\lim_{n\to\infty}\left[\sqrt[n]{\frac{n}{1+n^2}},\ \sqrt[n]{\frac{1}{n}},\ \sqrt[n]{5n},\ \frac{1+n^2}{n^2}\right]$$

```
>> pretty(sym(maple('vector([limit((n/(n^2+1))^(1/n),n=infinity),limit((1/n)^(1/n),
n=infinity),limit((5*n)^(1/n) ,n=infinity),limit((n^2+1)/n^2,n=infinity)])')))

[1, 1, 1, 1]
```

EXERCISE 2-12

For the function $f: R \to R^2$ defined below, find $\lim_{x\to 0} f(x)$:

$$f(x) = \left(\frac{\sin(x)}{x},\ \sqrt[x]{1+x}\right).$$

```
>> pretty(sym(maple('vector([limit(sin(x)/x,x=0),limit((1+x)^(1/x),x=0)])')))

[1, exp (1)]
```

EXERCISE 2-13

For the function $f: R^2 \to R^2$ defined below, find $\lim_{(x,y)\to(0,0)} f(x,y)$:

$$f(x,y) = \left[\frac{2(1-\cos(y))}{y^2} + \frac{\sin(x)}{x},\ \sqrt[x]{1+x} - \frac{\tan(y)}{y}\right].$$

```
>> maple ('f:=(x,y) - > sin (x) / x+2 (1-cos (y)) / y ^ 2');

>> maple ('g:=(x,y) - >(1+x) ^ (1/x) - tan (y) / y ');

>> pretty(sym(maple('vector([limit(limit(f(x,y),x=0),y=0),limit(limit(g(x,y),
(((([(x=0), y = 0)])')))

[inf, exp (1) - 1]
```

2.6 Iterated and Directional Limits

Given a function $f: R^n \to R$ an *iterated limit* of f at the point $(a_1, a_2, \ldots, a_n)$ is the value of the limit (if it exists):

$$\lim_{x_1 \to a_1}\left(\lim_{x_2 \to a_2}\left(\ldots \lim_{x_n \to a_n} f(x_1, x_2, \ldots, x_n)\right)\right)$$

or any of the other limits obtained by permuting the order of the component limits.

The *directional limit of f at the point* $(a_1, \ldots, a_n)$ depends on the direction of the curve $h(t) = (h_1(t), h_2(t), \ldots, h_n(t))$, where $h(t_0) = (a_1, a_2, \ldots, a_n)$, and is defined to be the value:

$$\lim_{t \to t_0}(f(h(t)) = \lim_{(x_1, x_2, \ldots, x_n) \to (a_1, a_2, \ldots, a_n)} f(x_1, x_2, \ldots, x_n)$$

A necessary condition for a function of several variables to have a limit at a point is that all the iterated limits have the same value (which will be equal to the value of the limit of the function, if it exists).

It can also happen that the directional limit of a function will vary according to the curve used, so that a different limit exists for different curves, or the limit exists for some curves and not others.

Another necessary condition for a function of several variables to have a limit at a point is that all directional limits, i.e. the limit for all curves, have the same value.

Therefore, to prove that a function has no limit at a point it is enough to show that either an iterated limit does not exist or that two iterated limits have a different value, or we can show that a directional limit does not exist or that two directional limits have different values.

A practical procedure for calculating the limit of a function of several variables is to change from cartesian to polar coordinates.

EXERCISE 2-14

Find $\lim_{(x,y)\to(0,0)} f(x,y)$ for the function $f: R^2 \to R$ defined by:

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

```
>> syms x y
>> limit (limit ((x*y) /(x^2+y^2), x, 0), y, 0)

ans =

0

>> limit (limit ((x*y) /(x^2+y^2), y, 0), x, 0)

ans =

0
```

Thus the two iterated limits are the same. Next we calculate the directional limits corresponding to the family of straight lines $y = mx$:

```
>> syms m
>> limit((m*x^2)/(x^2+(m^2)*(x^2)),x,0)

ans =

m /(1+m^2)
```

The directional limits depend on the parameter m, which will be different for different values of m (corresponding to different straight lines). Thus, we conclude that the function has no limit at (0,0).

EXERCISE 2-15

Find $\lim_{(x,y)\to(0,0)} f(x,y)$ for the function f:$R^2 \to R$ defined by:

$$f(x,y)=\frac{(y^2-x^2)^2}{x^2+y^4}$$

```
>> syms x y
>> limit (limit ((y^2-x^2) ^ 2 /(y^4+x^2), x, 0), y, 0)

ans =

1

>> limit (limit ((y^2-x^2) ^ 2 /(y^4+x^2), y, 0), x, 0)

ans =

0
```

As the two iterated limits are different, we conclude that the function has no limit at the point (0,0).

EXERCISE 2-16

Find $\lim_{(x,y)\to(0,0)} f(x,y)$ for the function $f{:}R^2 \to R$ defined by:

$$f(x,y)=\frac{(y^2-x)^2}{x^2+y^4}$$

```
>> syms x y m, limit (limit ((y^2-x) ^ 2 /(y^4+x^2), y, 0), x, 0)

ans =

1

>> limit (limit ((y^2-x) ^ 2 /(y^4+x^2), x, 0), y, 0)

ans =

1
```

Thus the two iterated limits are the same. Next we calculate the directional limits corresponding to the family of straight lines $y = mx$:

```
>> limit(((m*x)^2-x)^2/((m*x)^4+x^2),x,0)

ans =

1
```

The directional limits corresponding to the family of straight lines $y = mx$ do not depend on m and coincide with the iterated limits. Next we find the directional limits corresponding to the family of parabolas $y \wedge 2 = mx$:

```
>> limit(((m*x)-x)^2/((m*x)^2+x^2),x,0)

ans =

(m-1) ^ 2-/(m^2+1)
```

Thus the directional limits corresponding to this family of parabolas depend on the parameter m, so they are different. This leads us to conclude that the function has no limit at (0,0).

EXERCISE 2-17

Find $\lim_{(x,y)\to(0,0)} f(x,y)$ for the function $f{:}R^2 \to R$ defined by:

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}$$

```
>> syms x y, limit (limit ((x^2*y) /(x^2+y^2), x, 0), y, 0)

ans =

0

>> limit (limit ((x^2*y) /(x^2+y^2), x, 0), y, 0)

ans =

0

>> limit(((x^2)*(m*x))/(x^2+(m*x)^2),x,0)

ans =

0

>> limit (((m*y) ^ 2) * y / ((m*y) ^ 2 + y ^ 2), y, 0)

ans =

0
```

We see that the iterated limits and directional limits corresponding to the given family of lines and parabolas coincide and are all zero. This leads us to suspect that the limit of the function may be zero. To confirm this, we transform to polar coordinates and find the limit:

```
>> syms a r, limit (limit (((r^2) * (cos (a) ^ 2) * (r) * (sin (a))) / ((r^2) *
(cos (a) ^ 2)+(r^2) * (sin (a) ^ 2)), r, 0), a, 0)

ans =

0
```

Therefore we conclude that the limit of the function is zero at the point (0,0).

This is an example where, as a last resort, we had to transform to polar coordinates. In the above examples we used families of lines and parabolas, but other curves can be used. The change to polar coordinates can be crucial in determining limits of functions of several variables. As we have seen, there are sufficient criteria to show that a function has no limit at a point. However, we do not have necessary and sufficient conditions to ensure the existence of the limit.

EXERCISE 2-18

Find $\lim_{(x,y)\to(0,0)} f(x,y)$ for the function f: $R^2 \to R$ defined by:

$$f(x,y)=\frac{(x-1)^2 y^2}{(x-1)^2+y^2}$$

```
>> syms x y m a r
>> limit (limit (y ^ 2 *(x-1) ^ 2 / (y ^ 2 +(x-1) ^ 2), x, 0), y, 0)

ans =

0

>> limit (limit (y ^ 2 *(x-1) ^ 2 / (y ^ 2 +(x-1) ^ 2), y, 0), x, 0)

ans =

0

>> limit((m*x)^2*(x-1)^2/((m*x)^2+(x-1)^2),x,0)

ans =

0

>> limit((m*x)*(x-1)^2/((m*x)+(x-1)^2),x,0)

ans =

0
```

We see that the iterated and directional limits coincide. We calculate the limit by converting to polar coordinates:

```
>> limit (limit ((r ^ 2 * sin (a) ^ 2) * (r * cos (a) - 1) ^ 2 / ((r ^ 2 * sin (a) ^ 2) +
(r * cos (a) - 1) ^ 2), r, 1), a, 0)

ans =

0
```

The limit is zero at the point (1,0). The surface is depicted in Figure 2-12 where we see the function tends to 0 in a neighborhood of (1,0):

```
>> [x, y] = meshgrid(0:0.05:2,-2:0.05:2);
>> z=y.^2.*(x-1).^2./(y.^2+(x-1).^2);
>> mesh(x,y,z), view ([- 23, 30])
```

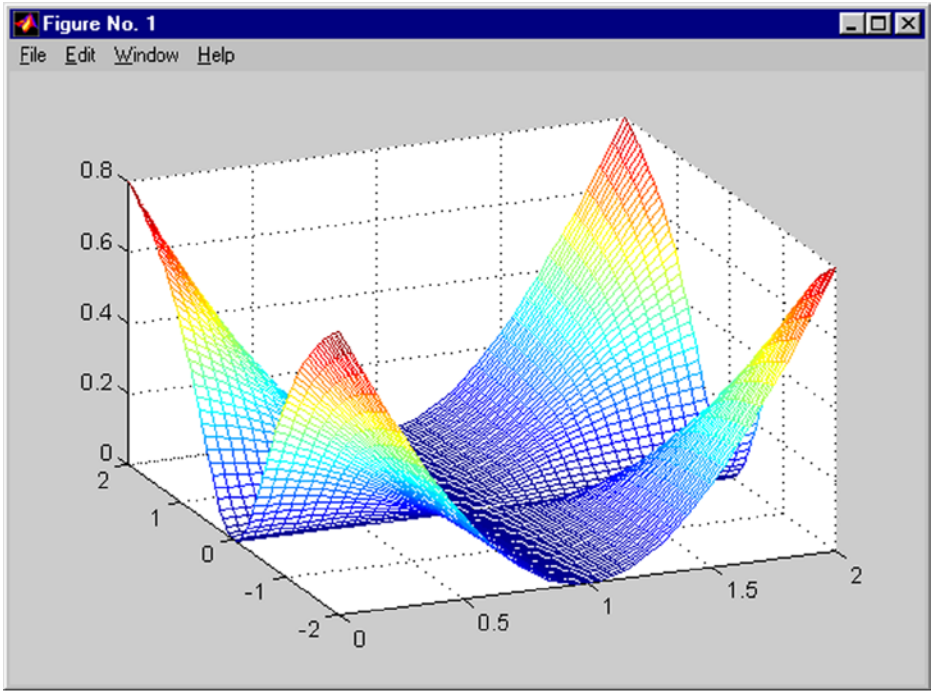

Figure 2-12.

2.7 Continuity in Several Variables

A function $f: R^n \to R^m$ is said to be continuous at the point $(a_1, a_2, \ldots, a_n)$ if:

$$\lim_{x_1 \to a_1, x_2 \to a_2, \ldots, x_n \to a_n} f(a_1, a_2, \ldots, a_n).$$

EXERCISE 2-19

Let the function f: $R^2 \longrightarrow R$ be defined by:

$$f(x,y) = \frac{x^3 + y^3}{x^2 + y^2}.$$

Find $\lim_{(x,y)\to(1,0)} f(x,y)$ and study the continuity of *f*.

The only problematic point is the origin. We will analyze the continuity of the function at the origin by calculating its limit there.

```
>> syms x y m a r
>> limit (limit ((x^3+y^3) /(x^2+y^2), x, 0), y, 0)

ans =

0

>> limit (limit ((x^3+y^3) /(x^2+y^2), y, 0), x, 0)

ans =

0

>> limit((x^3+(m*x)^3)/(x^2+(m*x)^2),x,0)

ans =

0
```

We see that the iterated and the linear directional limits coincide. We try to calculate the limit by converting to polar coordinates:

```
>> maple ('limit (limit (((r * cos (a)) ^ 3 + (r * sin (a)) ^ 3) / ((r * cos (a)) ^ 2 +
(r * sin (a)) ^ 2), r = 0), a = 0)')

ans =

0
```

We see that the limit at (0,0) coincides with f(0,0), so the function is continuous at (0,0). The graph in Figure 2-13 confirms the continuity.

```
>> [x, y] = meshgrid(-1:0.007:1,-1:0.007:1);
>> z=(x.^3+y.^3)./(x.^2+y.^2);
>> mesh (x, y, z)
>> view ([50, - 20])
```

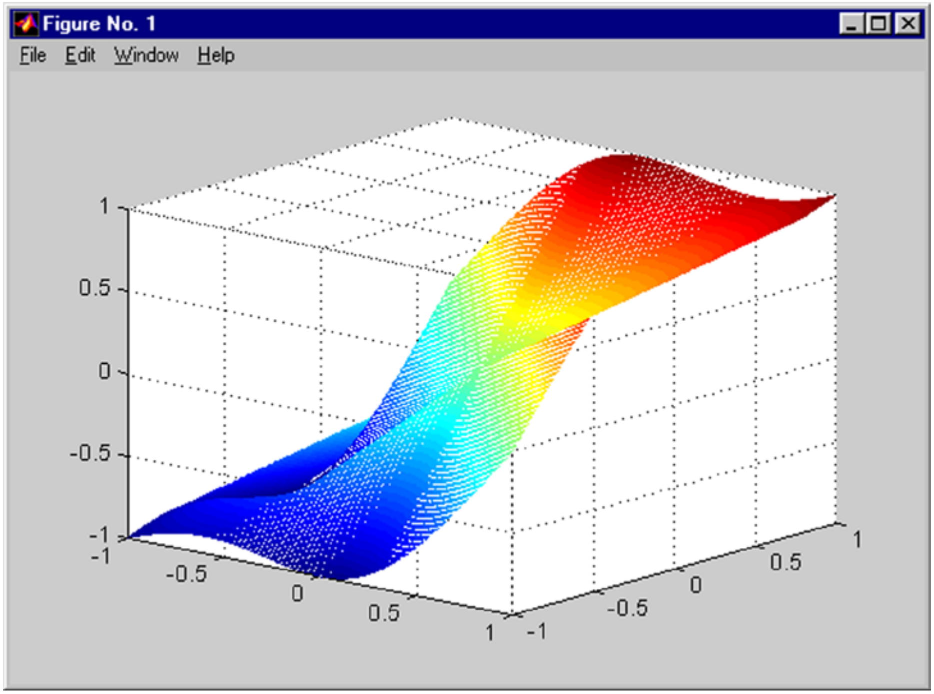

Figure 2-13.

EXERCISE 2-20

Define the function $f: R^2 \to R$ by

$$f(x,y) = x^2 + 2y \text{ if } (x,y) \neq (1,2) \text{ and } f(1,2) = 0$$

and study its continuity at the point (1,2).

```
>> limit (limit ((x^2+2*y), x, 1), y, 2).

ans =

5

>> limit (limit ((x^2+2*y), y, 2), x, 1)

ans =

5
```

We see that if the limit at (1,2) exists, then it should be 5. But the function has the value 0 at the point (1,2). Thus, the function is not continuous at the point (1,2).

EXERCISE 2-21

Consider the function $f: R^2 \to R$ defined by:

$$f(x,y) = \frac{[1-\cos(x)]\sin(y)}{x^3+y^3} \text{ if } (x,y) \neq (0,0) \text{ and } f(0,0) = 0.$$

Study the continuity of *f* at the point (0,0).

```
>> maple ('limit (limit ((1-cos (x)) * sin (y) /(x^3+y^3), x = 0), y = 0)')

ans =

0

>> maple ('limit (limit ((1-cos (x)) * sin (y) /(x^3+y^3), x = 0), y = 0)')

ans =

0

>> maple ('limit ((1-cos (x)) * sin(m*x) / (x ^ 3, +(m*x) ^ 3), x = 0)')

ans =

1/2 * m /(1 + m^3)
```

We see that the limit at (0,0) does not exist, as there are different directional limits for different directions. Thus, the function is not continuous at (0,0). At the rest of the points in the plane, the function is continuous. The plot of the function in a neighborhood of the origin of radius 0.01 shown in Figure 2-14 shows that around (0,0) there are branches of the surface that soar to infinity, which causes the non-existence of the limit at the origin.

```
>> [x,y]=meshgrid(-1/100:0.0009:1/100,-1/100:0.0009:1/100);
>> z=(1-cos(x)).*sin(y)./(x.^3+y.^3);
>> surf (x, y, z)
>> view ([50, - 15])
```

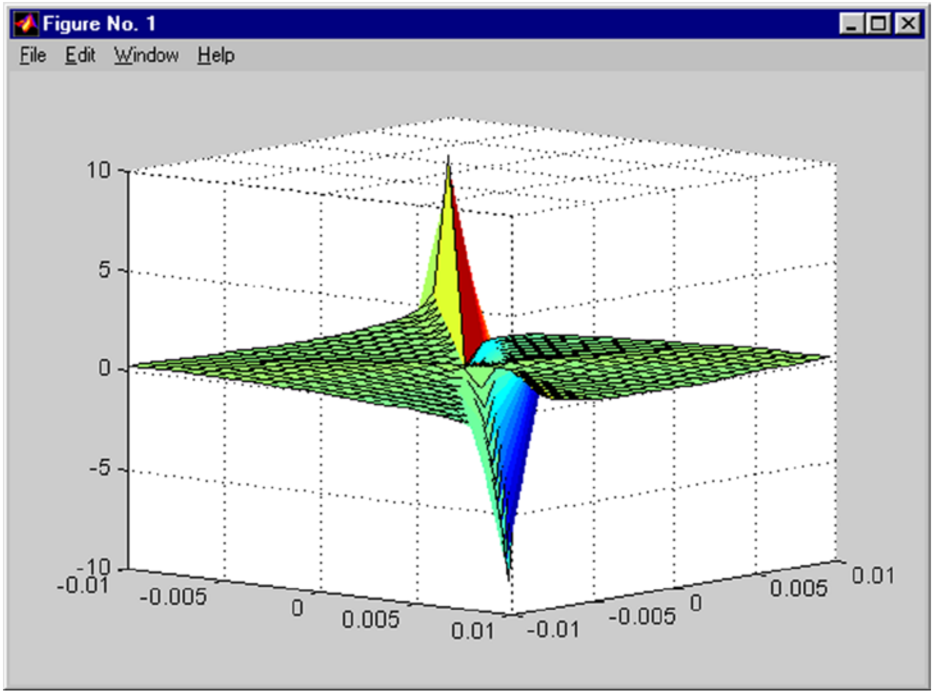
Figure No. 1
File Edit Window Help
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Figure 2-14.