Chapter 7
Best Approximation

7.1 Well-Posedness and Porosity

Given a nonempty closed subset A of a Banach space $(X, \|\cdot\|)$ and a point $x \in X$, we consider the minimization problem

$$\min\{\|x - y\| : y \in A\}. \tag{P}$$

It is well known that if A is convex and X is reflexive, then problem (P) always has at least one solution. This solution is unique when X is strictly convex.

If A is merely closed but X is uniformly convex, then according to classical results of Stechkin [173] and Edelstein [59], the set of all points in X having a unique nearest point in A is G_δ and dense in X. Since then there has been a lot of activity in this direction. In particular, it is known [84, 88] that the following properties are equivalent for any Banach space X:

(A) X is reflexive and has a Kadec-Klee norm.
(B) For each nonempty closed subset A of X, the set of points in $X \setminus A$ with nearest points in A is dense in $X \setminus A$.
(C) For each nonempty closed subset A of X, the set of points in $X \setminus A$ with nearest points in A is generic (that is, a dense G_δ subset) in $X \setminus A$.

A more recent result of De Blasi, Myjak and Papini [52] establishes well-posedness of problem (P) for a uniformly convex X, closed A and a generic $x \in X$.

In this connection we recall that the minimization problem (P) is said to be well posed if it has a unique solution, say a_0, and every minimizing sequence of (P) converges to a_0.

A more precise formulation of the De Blasi-Myjak-Papini result mentioned above involves the notion of porosity.

Using this terminology and denoting by F the set of all points such that the minimization problem (P) is well posed, we note that De Blasi, Myjak and Papini [52] proved, in fact, that the complement $X \setminus F$ is σ-porous in X.

However, the fundamental restriction in all these results is that they hold only under certain assumptions on the space X. In view of the Lau-Konjagin result

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mentioned above these assumptions cannot be removed. On the other hand, many generic results in nonlinear functional analysis hold in any Banach space. Therefore the following natural question arises: can generic results for best approximation problems be obtained in general Banach spaces? In [138] we answer this question in the affirmative. In this chapter we present the results obtained in [138].

To this end, we change our point of view and consider a new framework. The main feature of this new framework is that the set A in problem (P) may also vary. In our first result (Theorem 7.3) we fix x and consider the space $S(X)$ of all nonempty closed subsets of X equipped with an appropriate complete metric, say h. We then show that the collection of all sets $A \in S(X)$ for which problem (P) is well posed has a σ-porous complement.

In the second result (Theorem 7.4) we consider the space of pairs $S(X) \times X$ with the metric $h(A, B) + \|x - y\|$, where $A, B \in S(X)$ and $x, y \in X$. Once again we show that the family of all pairs $(A, x) \in S(X) \times X$ for which problem (P) is well-posed has a σ-porous complement.

In our third result (Theorem 7.5) we show that for any nonempty, separable and closed subset X_0 of X, there exists a subset $\mathcal{F}$ of $(S(X), h)$ with a σ-porous complement such that any $A \in \mathcal{F}$ has the following property:

There exists a dense G_δ subset F of X_0 such that for any $x \in F$, the minimization problem (P) is well posed.

In order to prove these results we now provide more information on porous sets.

Let (Y, ρ) be a metric space. We denote by $B_\rho(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$.

The following simple observation was made in [180].

Proposition 7.1 *Let E be a subset of the metric space (Y, ρ). Assume that there exist $r_0 > 0$ and $\beta \in (0, 1)$ such that the following property holds*:

(P1) *For each $x \in Y$ and each $r \in (0, r_0]$, there exists $z \in Y \setminus E$ such that $\rho(x, z) \leq r$ and $B_\rho(z, \beta r) \cap E = \emptyset$.*

Then E is porous with respect to ρ.

Proof Let $x \in Y$ and $r \in (0, r_0]$. By property (P1), there exists $z \in Y \setminus E$ such that

$$\rho(x, z) \leq r/2 \quad \text{and} \quad B_\rho(z, \beta r/2) \cap E = \emptyset.$$

Hence $B_\rho(z, \beta r/2) \subset B_\rho(x, r) \setminus E$ and Proposition 7.1 is proved. □

As a matter of fact, property (P1) can be weakened.

Proposition 7.2 *Let E be a subset of the metric space (Y, ρ). Assume that there exist $r_0 > 0$ and $\beta \in (0, 1)$ such that the following property holds*:

(P2) *For each $x \in E$ and each $r \in (0, r_0]$, there exists $z \in Y \setminus E$ such that $\rho(x, z) \leq r$ and $B_\rho(z, \beta r) \cap E = \emptyset$.*

Then E is porous with respect to ρ.

Proof We may assume that $\beta < 1/2$. Let $x \in Y$ and $r \in (0, r_0]$. We will show that there exists $z \in Y \setminus E$ such that

$$\rho(x, z) \le r \quad \text{and} \quad B_\rho(z, \beta r/2) \cap E = \emptyset. \tag{7.1}$$

If $B_\rho(x, r/4) \cap E = \emptyset$, then (7.1) holds with $z = x$. Assume now that $B_\rho(x, r/4) \cap E \neq \emptyset$. Then there exists

$$x_1 \in B_\rho(x, r/4) \cap E. \tag{7.2}$$

By property (P2), there exists $z \in Y \setminus E$ such that

$$\rho(x_1, z) \le r/2 \quad \text{and} \quad B_\rho(z, \beta r/2) \cap E = \emptyset. \tag{7.3}$$

The relations (7.2) and (7.3) imply that

$$\rho(x, z) \le \rho(x, x_1) + \rho(x_1, z) \le 3r/4.$$

Thus there indeed exists $z \in Y \setminus E$ satisfying (7.1). Proposition 7.2 is now seen to follow from Proposition 7.1. □

The following definition was introduced in [180].

Assume that a set Y is equipped with two metrics ρ_1 and ρ_2 such that $\rho_1(x, y) \le \rho_2(x, y)$ for all $x, y \in Y$ and that the metric spaces (Y, ρ_1) and (Y, ρ_2) are complete.

We say that a set $E \subset Y$ is porous with respect to the pair (ρ_1, ρ_2) if there exist $r_0 > 0$ and $\alpha \in (0, 1)$ such that for each $x \in E$ and each $r \in (0, r_0]$, there exists $z \in Y \setminus E$ such that $\rho_2(z, x) \le r$ and $B_{\rho_1}(z, \alpha r) \cap E = \emptyset$.

Proposition 7.2 implies that if E is porous with respect to (ρ_1, ρ_2), then it is porous with respect to both ρ_1 and ρ_2.

A set $E \subset Y$ is called σ-porous with respect to (ρ_1, ρ_2) if it is a countable union of sets which are porous with respect to (ρ_1, ρ_2).

As a matter of fact, it turns out that our results are true not only for Banach spaces, but also for all complete hyperbolic spaces.

Let (X, ρ, M) be a complete hyperbolic space. For each $x \in X$ and each $A \subset X$, set

$$\rho(x, A) = \inf\{\rho(x, y) : y \in A\}.$$

Denote by $S(X)$ the family of all nonempty closed subsets of X. For each $A, B \in S(X)$, define

$$H(A, B) := \max\{\sup\{\rho(x, B) : x \in A\}, \sup\{\rho(y, A) : y \in B\}\} \tag{7.4}$$

and

$$\tilde{H}(A, B) := H(A, B)\big(1 + H(A, B)\big)^{-1}.$$

It is easy to see that $\tilde{H}$ is a metric on $S(X)$ and that the space $(S(X), \tilde{H})$ is complete.

Fix $\theta \in X$. For each natural number n and each $A, B \in S(X)$, we set

$$h_n(A, B) = \sup\{|\rho(x, A) - \rho(x, B)| : x \in X \text{ and } \rho(x, \theta) \le n\} \tag{7.5}$$

and

$$h(A, B) = \sum_{n=1}^{\infty} [2^{-n} h_n(A, B)(1 + h_n(A, B))^{-1}].$$

Once again it is not difficult to see that h is a metric on $S(X)$ and that the metric space $(S(X), h)$ is complete. Clearly,

$$\tilde{H}(A, B) \ge h(A, B) \quad \text{for all } A, B \in S(X).$$

We equip the set $S(X)$ with the pair of metrics $\tilde{H}$ and h.

We now state the following three results which were obtained in [138]. Their proofs are given later in this chapter.

Theorem 7.3 *Let (X, ρ, M) be a complete hyperbolic space and let $\tilde{x} \in X$. Then there exists a set $\Omega \subset S(X)$ such that its complement $S(X) \setminus \Omega$ is σ-porous with respect to the pair $(h, \tilde{H})$ and such that for each $A \in \Omega$, the following property holds*:

(C1) *There exists a unique $\tilde{y} \in A$ such that $\rho(\tilde{x}, \tilde{y}) = \rho(\tilde{x}, A)$. Moreover, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ satisfies $\rho(\tilde{x}, x) \le \rho(\tilde{x}, A) + \delta$, then $\rho(x, \tilde{y}) \le \varepsilon$.*

To state the following result we endow the Cartesian product $S(X) \times X$ with the pair of metrics d_1 and d_2 defined by

$$d_1((A, x), (B, y)) = h(A, B) + \rho(x, y),$$

$$d_2((A, x), (B, y)) = \tilde{H}(A, B) + \rho(x, y), \quad x, y \in X, A, B \in S(X).$$

Theorem 7.4 *Let (X, ρ, M) be a complete hyperbolic space. There exists a set $\Omega \subset S(X) \times X$ such that its complement $[S(X) \times X] \setminus \Omega$ is σ-porous with respect to the pair (d_1, d_2) and such that for each $(A, \tilde{x}) \in \Omega$, the following property holds*:

(C2) *There exists a unique $\tilde{y} \in A$ such that $\rho(\tilde{x}, \tilde{y}) = \rho(\tilde{x}, A)$. Moreover, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $z \in X$ satisfies $\rho(\tilde{x}, z) \le \delta$, $B \in S(X)$ satisfies $h(A, B) \le \delta$, and $y \in B$ satisfies $\rho(y, z) \le \rho(z, B) + \delta$, then $\rho(y, \tilde{y}) \le \varepsilon$.*

In classical generic results the set A was fixed and x varied in a dense G_δ subset of X. In our first two results the set A is also variable. However, in our third result we show that if X_0 is a nonempty, separable and closed subset of X, then for every fixed A in a dense G_δ subset of $S(X)$ with a σ-porous complement, the set of all $x \in X_0$ for which problem (P) is well posed contains a dense G_δ subset of X_0.

Theorem 7.5 *Let (X, ρ, M) be a complete hyperbolic space. Assume that X_0 is a nonempty, separable and closed subset of X. Then there exists a set $\mathcal{F} \subset S(X)$ such that $S(X) \setminus \mathcal{F}$ is σ-porous with respect to the pair $(h, \tilde{H})$ and such that for each $A \in \mathcal{F}$, the following property holds*:

(C3) *There exists a set $F \subset X_0$ which is a countable intersection of open and everywhere dense subsets of X_0 with the relative topology such that for each $\tilde{x} \in F$, there exists a unique $\tilde{y} \in A$ for which $\rho(\tilde{x}, \tilde{y}) = \rho(\tilde{x}, A)$. Moreover, if $\{y_i\}_{i=1}^{\infty} \subset A$ satisfies $\lim_{i\to\infty} \rho(\tilde{x}, y_i) = \rho(\tilde{x}, A)$, then $y_i \to \tilde{y}$ as $i \to \infty$.*

7.2 Auxiliary Results

Let (X, ρ, M) be a complete hyperbolic space and let $S(X)$ be the family of all nonempty closed subsets of X.

Lemma 7.6 *Let $A \in S(X)$, $\tilde{x} \in X$ and let $r, \varepsilon \in (0, 1)$. Then there exists $\bar{x} \in X$ such that $\rho(\bar{x}, A) \le r$ and for the set $\tilde{A} = A \cup \{\bar{x}\}$ the following properties hold*:

$$\rho(\tilde{x}, \bar{x}) = \rho(\tilde{x}, \tilde{A});$$

$$\textit{if } x \in \tilde{A} \textit{ and } \rho(\tilde{x}, x) \le \rho(\tilde{x}, \tilde{A}) + \varepsilon r/4, \textit{ then } \rho(\bar{x}, x) \le \varepsilon.$$

Proof If $\rho(\tilde{x}, A) \le r$, then the lemma holds with $\bar{x} = \tilde{x}$ and $\tilde{A} = A \cup \{\tilde{x}\}$. Therefore we may restrict ourselves to the case where

$$\rho(\tilde{x}, A) > r. \tag{7.6}$$

Choose $x_0 \in A$ such that

$$\rho(\tilde{x}, x_0) \le \rho(\tilde{x}, A) + r/2. \tag{7.7}$$

There exists

$$\bar{x} \in \{\gamma\tilde{x} \oplus (1-\gamma)x_0 : \gamma \in (0, 1)\} \tag{7.8}$$

such that

$$\rho(\bar{x}, x_0) = r \quad \text{and} \quad \rho(\tilde{x}, \bar{x}) = \rho(\tilde{x}, x_0) - r. \tag{7.9}$$

Set $\tilde{A} = A \cup \{\bar{x}\}$. We have by (7.9) and (7.7),

$$\rho(\tilde{x}, \bar{x}) = \rho(\tilde{x}, x_0) - r \le \rho(\tilde{x}, A) + r/2 - r = \rho(\tilde{x}, A) - r/2.$$

Therefore $\rho(\tilde{x}, \bar{x}) = \rho(\tilde{x}, \tilde{A})$, and if $x \in \tilde{A}$ and $\rho(\tilde{x}, x) < \rho(\tilde{x}, \tilde{A}) + r/2$, then $x = \bar{x}$. This completes the proof of Lemma 7.6. □

Before stating our next lemma we choose, for each $\varepsilon \in (0, 1)$ and each natural number n, a number

$$\alpha(\varepsilon, n) \in (0, 16^{-n-2}\varepsilon). \tag{7.10}$$

Lemma 7.7 *Let $A \in S(X)$, $\tilde{x} \in X$ and let $r, \varepsilon \in (0,1)$. Suppose that n is a natural number, let*

$$\alpha = \alpha(\varepsilon, n) \tag{7.11}$$

and assume that

$$\rho(\tilde{x}, \theta) \le n \quad \textit{and} \quad \left\{x \in X : \rho(x, \theta) \le n\right\} \cap A \ne \emptyset. \tag{7.12}$$

Then there exists $\bar{x} \in X$ such that $\rho(\bar{x}, A) \le r$ and such that the set $\tilde{A} = A \cup \{\bar{x}\}$ has the following two properties:

$$\rho(\tilde{x}, \bar{x}) = \rho(\tilde{x}, \tilde{A}); \tag{7.13}$$

if

$$\tilde{y} \in X, \quad \rho(\tilde{y}, \tilde{x}) \le \alpha r, \tag{7.14}$$

$$B \in S(X), \quad h(\tilde{A}, B) \le \alpha r, \tag{7.15}$$

and

$$z \in B, \quad \rho(\tilde{y}, z) \le \rho(\tilde{y}, B) + \varepsilon r/16, \tag{7.16}$$

then

$$\rho(z, \bar{x}) \le \varepsilon. \tag{7.17}$$

Proof By Lemma 7.6, there exists $\bar{x} \in X$ such that

$$\rho(\bar{x}, A) \le r \tag{7.18}$$

and such that for the set $\tilde{A} = A \cup \{\bar{x}\}$, equality (7.13) is true and the following property holds:

$$\text{If } x \in \tilde{A} \text{ and } \rho(\tilde{x}, x) \le \rho(\tilde{x}, \tilde{A}) + \varepsilon r/8, \text{ then } \rho(\bar{x}, x) \le \varepsilon/2. \tag{7.19}$$

Assume that $\tilde{y} \in X$ satisfies (7.14) and $B \in S(X)$ satisfies (7.15). We will show that

$$\rho(\tilde{y}, B) < \rho(\tilde{x}, \tilde{A}) + 4\alpha r 16^n. \tag{7.20}$$

By (7.14),

$$\left|\rho(\tilde{y}, \tilde{A}) - \rho(\tilde{x}, \tilde{A})\right| \le \alpha r.$$

When combined with (7.13), this implies that

$$\left|\rho(\tilde{y}, \tilde{A}) - \rho(\tilde{x}, \bar{x})\right| \le \alpha r. \tag{7.21}$$

Relations (7.13) and (7.12) imply that

$$\rho(\tilde{x}, \bar{x}) \le \rho(\tilde{x}, A) \le 2n \quad \text{and} \quad \rho(\bar{x}, \theta) \le 3n. \tag{7.22}$$

It follows from (7.5) and (7.15) that

$$h_{4n}(\tilde{A}, B)\big(1 + h_{4n}(\tilde{A}, B)\big)^{-1} \le 2^{4n} h(\tilde{A}, B) \le 2^{4n}\alpha r.$$

When combined with (7.10) and (7.11), this inequality implies that

$$h_{4n}(\tilde{A}, B) \le 2^{4n}\alpha r\big(1 - 2^{4n}\alpha r\big)^{-1} < 2^{4n+1}\alpha r. \tag{7.23}$$

Since $\bar{x} \in \tilde{A}$, it now follows from (7.23), (7.22) and (7.5) that $\rho(\bar{x}, B) < 2^{4n+1}\alpha r$ and there exists $\bar{y} \in X$ such that

$$\bar{y} \in B \quad \text{and} \quad \rho(\bar{x}, \bar{y}) < 2\alpha r 16^n. \tag{7.24}$$

By (7.24), (7.14) and (7.13),

$$\begin{aligned}\rho(\tilde{y}, B) &\le \rho(\tilde{y}, \bar{y}) \le \rho(\tilde{y}, \bar{x}) + \rho(\bar{x}, \bar{y}) \\ &< \rho(\tilde{y}, \tilde{x}) + \rho(\tilde{x}, \bar{x}) + 2\alpha r 16^n \\ &\le 2\alpha r 16^n + \alpha r + \rho(\tilde{x}, \tilde{A}).\end{aligned}$$

This certainly implies (7.20), as claimed.

Assume now that $z \in B$ satisfies (7.16). It follows from (7.16), (7.20), (7.11) and (7.10) that

$$\begin{aligned}\rho(\tilde{y}, z) &\le \rho(\tilde{y}, B) + \varepsilon r/16 \le \rho(\tilde{x}, \tilde{A}) + 4\alpha r 16^n + \varepsilon r/16 \\ &\le \rho(\tilde{x}, \tilde{A}) + \varepsilon r/8.\end{aligned} \tag{7.25}$$

Relations (7.25), (7.22) and (7.14) imply that

$$\rho(\tilde{y}, z) \le \rho(\tilde{x}, \tilde{A}) + \varepsilon r/8 \le 2n + r/8. \tag{7.26}$$

By (7.26), (7.14), (7.11) and (7.12),

$$\begin{aligned}\rho(z, \theta) &\le \rho(z, \tilde{y}) + \rho(\tilde{y}, \theta) \le 2n + r/8 + \rho(\tilde{y}, \theta) \\ &\le 2n + r/8 + \rho(\tilde{y}, \tilde{x}) + \rho(\tilde{x}, \theta) \\ &\le 2n + r/8 + \alpha r + n \le 4n.\end{aligned} \tag{7.27}$$

It follows from (7.23), (7.5), (7.16) and (7.27) that

$$\rho(z, \tilde{A}) = \big|\rho(z, \tilde{A}) - \rho(z, B)\big| \le h_{4n}(\tilde{A}, B) < 2\alpha r 16^n.$$

Hence there exists $\tilde{z} \in X$ such that

$$\tilde{z} \in \tilde{A} \quad \text{and} \quad \rho(z, \tilde{z}) < 2\alpha r 16^n. \tag{7.28}$$

By (7.14), (7.28) and (7.16) we have

$$\rho(\tilde{x}, \tilde{z}) \le \rho(\tilde{x}, \tilde{y}) + \rho(\tilde{y}, z) + \rho(z, \tilde{z})$$
$$\le \alpha r + \rho(\tilde{y}, z) + 2\alpha r 16^n$$
$$\le \alpha r + 2\alpha r 16^n + \rho(\tilde{y}, B) + \varepsilon r/16.$$

It follows from this inequality, (7.20), (7.11) and (7.10) that

$$\rho(\tilde{x}, \tilde{z}) \le \alpha r + 2\alpha r 16^n + \varepsilon r/16 + \rho(\tilde{x}, \tilde{A}) + 4\alpha r 16^n$$
$$\le \rho(\tilde{x}, \tilde{A}) + 8\alpha r 16^n + \varepsilon r/16 \le \rho(\tilde{x}, \tilde{A}) + \varepsilon r/8.$$

Thus

$$\rho(\tilde{x}, \tilde{z}) \le \rho(\tilde{x}, \tilde{A}) + \varepsilon r/8.$$

Using this inequality, (7.28) and (7.19), we see that $\rho(\bar{x}, \tilde{z}) \le \varepsilon/2$. Combining this fact with (7.28), (7.11) and (7.10), we conclude that

$$\rho(z, \bar{x}) \le \rho(z, \tilde{z}) + \rho(\tilde{z}, \bar{x}) \le 2\alpha r 16^n + \varepsilon/2 \le \varepsilon.$$

Thus (7.17) holds and Lemma 7.7 is proved. □

7.3 Proofs of Theorems 7.3–7.5

Proof of Theorem 7.3 For each integer $k \ge 1$, denote by Ω_k the set of all $A \in S(X)$ which have the following property:

(P3) There exist $x_A \in X$ and $\delta_A > 0$ such that if $x \in A$ satisfies $\rho(x, \tilde{x}) \le \rho(\tilde{x}, A) + \delta_A$, then $\rho(x, x_A) \le 1/k$.

Clearly, $\Omega_{k+1} \subset \Omega_k$, $k = 1, 2, \dots$. Set

$$\Omega = \bigcap_{k=1}^{\infty} \Omega_k.$$

First we will show that $S(X) \setminus \Omega$ is σ-porous with respect to the pair $(h, \tilde{H})$. To meet this goal it is sufficient to show that $S(X) \setminus \Omega_k$ is σ-porous with respect to $(h, \tilde{H})$ for all sufficiently large integers k.

There exists a natural number k_0 such that $\rho(\theta, \tilde{x}) \le k_0$. Let $k \ge k_0$ be an integer. We will show that the set $S(X) \setminus \Omega_k$ is σ-porous with respect to $(h, \tilde{H})$. For each integer $n \ge k_0$, set

$$E_{nk} = \left\{A \in S(X) \setminus \Omega_k : \left\{z \in X : \rho(z, \theta) \le n\right\} \cap A \ne \emptyset\right\}.$$

By Lemma 7.7, the set E_{nk} is porous with respect to $(h, \tilde{H})$ for all integers $n \ge k_0$. Since $S(X) \setminus \Omega_k = \bigcup_{n=k_0}^{\infty} E_{nk}$, we conclude that $S(X) \setminus \Omega_k$ is σ-porous with respect to $(h, \tilde{H})$. Therefore $S(X) \setminus \Omega$ is also σ-porous with respect to $(h, \tilde{H})$.

Let $A \in \Omega$ be given. We will show that A has property (C1). By the definition of Ω_k and property (P3), for each integer $k \geq 1$, there exist $x_k \in X$ and $\delta_k > 0$ such that the following property holds:

(P4) If $x \in A$ satisfies $\rho(x, \tilde{x}) \leq \rho(\tilde{x}, A) + \delta_k$, then $\rho(x, x_k) \leq 1/k$.

Let $\{z_i\}_{i=1}^{\infty} \subset A$ be such that

$$\lim_{i \to \infty} \rho(\tilde{x}, z_i) = \rho(\tilde{x}, A). \tag{7.29}$$

Fix an integer $k \geq 1$. It follows from property (P4) that for all large enough natural numbers i,

$$\rho(\tilde{x}, z_i) \leq \rho(\tilde{x}, A) + \delta_k$$

and

$$\rho(z_i, x_k) \leq 1/k.$$

Since k is an arbitrary natural number, we conclude that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to some $\tilde{y} \in A$. It is clear that $\rho(\tilde{x}, \tilde{y}) = \rho(\tilde{x}, A)$. If the minimizer $\tilde{y}$ were not unique, we would be able to construct a nonconvergent minimizing sequence $\{z_i\}_{i=1}^{\infty}$. Thus $\tilde{y}$ is the unique solution to problem (P) (with $x = \tilde{x}$) and any sequence $\{z_i\}_{i=1}^{\infty} \subset A$ satisfying (7.29) converges to $\tilde{y}$. This completes the proof of Theorem 7.3. □

Proof of Theorem 7.4 For each integer $k \geq 1$, denote by Ω_k the set of all $(A, \tilde{x}) \in S(X) \times X$ which have the following property:

(P5) There exist $\bar{x} \in X$ and $\bar{\delta} > 0$ such that if $x \in X$ satisfies $\rho(x, \tilde{x}) \leq \bar{\delta}$, $B \in S(X)$ satisfies $h(A, B) \leq \bar{\delta}$, and $y \in B$ satisfies $\rho(y, x) \leq \rho(x, B) + \bar{\delta}$, then $\rho(y, \bar{x}) \leq 1/k$.

Clearly $\Omega_{k+1} \subset \Omega_k$, $k = 1, 2, \ldots$. Set

$$\Omega = \bigcap_{k=1}^{\infty} \Omega_k.$$

First we will show that $[S(X) \times X] \setminus \Omega$ is σ-porous with respect to the pair (d_1, d_2). For each pair of natural numbers n and k, set

$$E_{nk} = \left\{(A, x) \in \left[S(X) \times X\right] \setminus \Omega_k : \rho(x, \theta) \leq n, B_\rho(\theta, n) \cap A \neq \emptyset\right\}.$$

By Lemma 7.7, the set E_{nk} is porous with respect to (d_1, d_2) for all natural numbers n and k. Since

$$\left[S(X) \times X\right] \setminus \Omega = \bigcup_{k=1}^{\infty} \left(\left[S(X) \times X\right] \setminus \Omega_k\right) = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} E_{nk},$$

the set $[S(X) \times X] \setminus \Omega$ is σ-porous with respect to (d_1, d_2), by definition.

Let $(A, \tilde{x}) \in \Omega$. We will show that $(A, \tilde{x})$ has property (C2).

By the definition of Ω_k and property (P5), for each integer $k \geq 1$, there exist $x_k \in X$ and $\delta_k > 0$ with the following property:

(P6) If $x \in X$ satisfies $\rho(x, \tilde{x}) \leq \delta_k$, $B \in S(X)$ satisfies $h(A, B) \leq \delta_k$, and $y \in B$ satisfies $\rho(y, x) \leq \rho(x, B) + \delta_k$, then $\rho(y, x_k) \leq 1/k$.

Let $\{z_i\}_{i=1}^{\infty} \subset A$ be such that

$$\lim_{i\to\infty} \rho(\tilde{x}, z_i) = \rho(\tilde{x}, A). \tag{7.30}$$

Fix an integer $k \geq 1$. It follows from property (P6) that for all large enough natural numbers i,

$$\rho(\tilde{x}, z_i) \leq \rho(\tilde{x}, A) + \delta_k$$

and

$$\rho(z_i, x_k) \leq 1/k.$$

Since k is an arbitrary natural number, we conclude that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to some $\tilde{y} \in A$. Clearly, $\rho(\tilde{x}, \tilde{y}) = \rho(\tilde{x}, A)$. It is not difficult to see that $\tilde{y}$ is the unique solution to the minimization problem (P) with $x = \tilde{x}$.

Let $\varepsilon > 0$ be given. Choose an integer $k > 4/\min\{1, \varepsilon\}$. By property (P6),

$$\rho(\tilde{y}, x_k) \leq 1/k. \tag{7.31}$$

Assume that $z \in X$ satisfies $\rho(z, \tilde{x}) \leq \delta_k$, $B \in S(X)$ satisfies $h(A, B) \leq \delta_k$ and $y \in B$ satisfies $\rho(y, z) \leq \rho(z, B) + \delta_k$. Then it follows from property (P6) that $\rho(y, x_k) \leq 1/k$. When combined with (7.31), this implies that $\rho(y, \tilde{y}) \leq 2/k < \varepsilon$. This completes the proof of Theorem 7.4. □

Proof of Theorem 7.5 Let $\{x_i\}_{i=1}^{\infty} \subset X_0$ be an everywhere dense subset of X_0. For each natural number p, there exists a set $\mathcal{F}_p \subset S(X)$ such that Theorem 7.3 holds with $\tilde{x} = x_p$ and $\Omega = \mathcal{F}_p$. Set $\mathcal{F} = \bigcap_{p=1}^{\infty} \mathcal{F}_p$. Clearly, $S(X) \setminus \mathcal{F}$ is σ-porous with respect to the pair $(h, \tilde{H})$.

Let $A \in \mathcal{F}$ and let $p \geq 1$ be an integer. By Theorem 7.3, which holds with $\tilde{x} = x_p$ and $\Omega = \mathcal{F}_p$, there exists a unique $\bar{x}_p \in A$ such that

$$\rho(x_p, \bar{x}_p) = \rho(x_p, A) \tag{7.32}$$

and the following property holds:

(P7) For each integer $k \geq 1$, there exists $\delta(p, k) > 0$ such that if $x \in A$ satisfies $\rho(x, x_p) \leq \rho(x_p, A) + 4\delta(p, k)$, then $\rho(x, \bar{x}_p) \leq 1/k$.

For each pair of natural numbers p and k, set

$$V(p, k) = \bigl\{z \in X_0 : \rho(z, x_p) < \delta(p, k)\bigr\}.$$

It follows from property (P7) that for each pair of integers $p, k \geq 1$, the following property holds:

(P8) If $x \in A$, $z \in X_0$, $\rho(z, x_p) \leq \delta(p,k)$ and $\rho(z, x) \leq \rho(z, A) + \delta(p,k)$, then $\rho(x, \bar{x}_p) \leq 1/k$.

Set

$$F := \bigcap_{k=1}^{\infty} \Big[\bigcup \big\{ V(p,k) : p = 1, 2, \dots \big\} \Big].$$

Clearly, F is a countable intersection of open and everywhere dense subsets of X_0.

Let $x \in F$ be given. Consider a sequence $\{x_i\}_{i=1}^{\infty} \subset A$ such that

$$\lim_{i \to \infty} \rho(x, x_i) = \rho(x, A). \tag{7.33}$$

Let $\varepsilon > 0$. Choose a natural number $k > 8^{-1}/\min\{1, \varepsilon\}$. There exists an integer $p \geq 1$ such that $x \in V(p,k)$. By the definition of $V(p,k)$, $\rho(x, x_p) < \delta(p,k)$. It follows from this inequality and property (P8) that for all sufficiently large integers i, $\rho(x, x_i) \leq \rho(x, A) + \delta(p,k)$ and $\rho(x_i, \bar{x}_p) \leq 1/k < \varepsilon/2$. Since ε is an arbitrary positive number, we conclude that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to $\tilde{y} \in A$. Clearly, $\tilde{y}$ is the unique minimizer of the minimization problem $z \to \rho(x, z)$, $z \in A$. Note that we have shown that any sequence $\{x_i\}_{i=1}^{\infty} \subset A$ satisfying (7.33) converges to $\tilde{y}$. This completes the proof of Theorem 7.5. □

7.4 Generalized Best Approximation Problems

Given a closed subset A of a Banach space X, a point $x \in X$ and a continuous function $f : X \to R^1$, we consider the problem of finding a solution to the minimization problem $\min\{f(x - y) : y \in A\}$. For a fixed function f, we define an appropriate complete metric space $\mathcal{M}$ of all pairs (A, x) and construct a subset Ω of $\mathcal{M}$, which is a countable intersection of open and everywhere dense sets such that for each pair in Ω, our minimization problem is well posed.

Let $(X, \|\cdot\|)$ be a Banach space and let $f : X \to R^1$ be a continuous function. Assume that

$$\inf\big\{ f(x) : x \in X \big\} \text{ is attained at a unique point } x_* \in X, \tag{7.34}$$

$$\lim_{\|u\| \to \infty} f(u) = \infty, \tag{7.35}$$

$$\text{if } \{x_i\}_{i=1}^{\infty} \subset X \text{ and } \lim_{i \to \infty} f(x_i) = f(x_*), \text{ then } \lim_{i \to \infty} x_i = x_*, \tag{7.36}$$

and that for each integer $n \geq 1$, there exists an increasing function $\phi_n : (0,1) \to (0,1)$ such that

$$f\big(\alpha x + (1-\alpha)x_*\big) \leq \phi_n(\alpha) f(x) + \big(1 - \phi_n(\alpha)\big) f(x_*) \tag{7.37}$$

for all $x \in X$ satisfying $\|x\| \le n$ and all $\alpha \in (0, 1)$. It is clear that (7.37) holds if f is convex.

Given a closed subset A of X and a point $x \in X$, we consider the minimization problem

$$\min\{f(x-y) : y \in A\}. \tag{P}$$

This problem was studied by many mathematicians mostly in the case where $f(x) = \|x\|$. We recall that the minimization problem (P) is said to be well posed if it has a unique solution, say a_0, and every minimizing sequence of (P) converges to a_0. In other words, if $\{y_i\}_{i=1}^{\infty} \subset A$ and $\lim_{i\to\infty} f(x-y_i) = f(x-a_0)$, then $\lim_{i\to\infty} y_i = a_0$.

Note that in the studies of problem (P) [52, 59, 84, 88, 173], the function f is the norm of the space X. There are some additional results in the literature where either f is a Minkowski functional [51, 93] or the function $\|x-y\|$, $y \in A$, is perturbed by some convex function [42].

However, the fundamental restriction in all these results is that they only hold under certain assumptions on either the space X or the set A. In view of the Lau-Konjagin result mentioned above, these assumptions cannot be removed. On the other hand, many generic results in nonlinear functional analysis hold in any Banach space. Therefore a natural question is whether generic existence results for best approximation problems can be obtained for general Banach spaces. Positive answers to this question in the special case where $f = \|\cdot\|$ can be found in Sects. 7.1–7.3. In the next sections, which are based on [143], we answer this question in the affirmative for a general function f satisfying (7.34)–(7.37).

To this end, we change our point of view and consider another framework, the main feature of which is that the set A in problem (P) can also vary. We prove four theorems which were established in [143]. In our first result (Theorem 7.8), we fix x and consider the space $S(X)$ of all nonempty closed subsets of X equipped with an appropriate complete metric, say h. We then show that the collection of all sets $A \in S(X)$ for which problem (P) is well posed contains an everywhere dense G_δ set. In the second result (Theorem 7.9), we consider the space of pairs $S(X) \times X$ with the metric $h(A, B) + \|x - y\|$, $A, B \in S(X)$, $x, y \in X$. Once again, we show that the family of all pairs $(A, x) \in S(X) \times X$ for which problem (P) is well posed contains an everywhere dense G_δ set. In our third result (Theorem 7.10), we show that for any separable closed subset X_0 of X, there exists an everywhere dense G_δ subset $\mathcal{F}$ of $(S(X), h)$ such that any $A \in \mathcal{F}$ has the following property: there exists a G_δ dense subset F of X_0 such that for any $x \in F$, problem (P) is well posed.

In our fourth result (Theorem 7.11), we show that a continuous coercive convex $f : X \to R^1$ which has a unique minimizer and a certain well-posedness property (on the whole space X) has a unique minimizer and the same well-posedness property on a generic closed subset of X.

7.5 Theorems 7.8–7.11

We recall that $(X, \|\cdot\|)$ is a Banach space, $f : X \to R^1$ is a continuous function satisfying (7.34)–(7.36) and that for each integer $n \ge 1$, there exists an increasing function $\phi_n : (0, 1) \to (0, 1)$ such that (7.37) is true.

For each $x \in X$ and each $A \subset X$, set

$$\rho(x, A) = \inf\{\rho(x, y) : y \in A\} \tag{7.38}$$

and

$$\rho_f(x, A) = \inf\{f(x - y) : y \in A\}. \tag{7.39}$$

Denote by $S(X)$ the collection of all nonempty closed subsets of X. For each $A, B \in S(X)$, define

$$H(A, B) := \max\{\sup\{\rho(x, B) : x \in A\}, \sup\{\rho(y, A) : y \in B\}\} \tag{7.40}$$

and

$$\tilde{H}(A, B) := H(A, B)\big(1 + H(A, B)\big)^{-1}.$$

Here we use the convention that $\infty/\infty = 1$.

It is not difficult to see that the metric space $(S(X), \tilde{H})$ is complete.

For each natural number n and each $A, B \in S(X)$, we set

$$h_n(A, B) := \sup\{|\rho(x, A) - \rho(x, B)| : x \in X \text{ and } \|x\| \le n\} \tag{7.41}$$

and

$$h(A, B) := \sum_{n=1}^{\infty}\big[2^{-n} h_n(A, B)\big(1 + h_n(A, B)\big)^{-1}\big].$$

Once again, it is not difficult to see that h is a metric on $S(X)$ and that the metric space $(S(X), h)$ is complete. Clearly, $\tilde{H}(A, B) \ge h(A, B)$ for all $A, B \in S(X)$.

We equip the set $S(X)$ with the pair of metrics $\tilde{H}$ and h. The topologies induced by the metrics $\tilde{H}$ and h on $S(X)$ will be called the strong topology and the weak topology, respectively.

We now state Theorems 7.8–7.11.

Theorem 7.8 *Let $\tilde{x} \in X$. Then there exists a set $\Omega \subset S(X)$, which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$, such that for each $A \in \Omega$, the following property holds:*

(C1) *There exists a unique $\tilde{y} \in A$ such that $f(\tilde{x} - \tilde{y}) = \rho_f(\tilde{x}, A)$. Moreover, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ satisfies $f(\tilde{x} - x) \le \rho_f(\tilde{x}, A) + \delta$, then $\|x - \tilde{y}\| \le \varepsilon$.*

To state our second result we endow the Cartesian product $S(X) \times X$ with the pair of metrics d_1 and d_2 defined by

$$d_1\big((A, x), (B, y)\big) = h(A, B) + \rho(x, y),$$
$$d_2\big((A, x), (B, y)\big) = \tilde{H}(A, B) + \rho(x, y), \quad x, y \in X, A, B \in S(X).$$

We will refer to the topologies induced on $S(X) \times X$ by d_2 and d_1 as the strong and weak topologies, respectively.

Theorem 7.9 *There exists a set $\Omega \subset S(X) \times X$, which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X) \times X$, such that for each $(A, \tilde{x}) \in \Omega$, the following property holds:*

(C2) *There exists a unique $\tilde{y} \in A$ such that $f(\tilde{x} - \tilde{y}) = \rho_f(\tilde{x}, A)$. Moreover, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $z \in X$ satisfies $\|z - \tilde{x}\| \le \delta$, $B \in S(X)$ satisfies $h(A, B) \le \delta$, and $y \in B$ satisfies $f(z - y) \le \rho_f(z, B) + \delta$, then $\|y - \tilde{y}\| \le \varepsilon$.*

In most classical generic results the set A was fixed and x varied in a dense G_δ subset of X. In our first two results the set A is also variable. However, our third result shows that for every fixed A in a dense G_δ subset of $S(X)$, the set of all $x \in X$ for which problem (P) is well posed contains a dense G_δ subset of X.

Theorem 7.10 *Assume that X_0 is a closed separable subset of X. Then there exists a set $\mathcal{F} \subset S(X)$, which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$, such that for each $A \in \mathcal{F}$, the following property holds:*

(C3) *There exists a set $F \subset X_0$, which is a countable intersection of open and everywhere dense subsets of X_0 with the relative topology, such that for each $\tilde{x} \in F$, there exists a unique $\tilde{y} \in A$ for which $f(\tilde{x} - \tilde{y}) = \rho_f(\tilde{x}, A)$. Moreover, if $\{y_i\}_{i=1}^{\infty} \subset A$ satisfies $\lim_{i \to \infty} f(\tilde{x} - y_i) = \rho_f(\tilde{x}, A)$, then $y_i \to \tilde{y}$ as $i \to \infty$.*

Now we will show that Theorem 7.8 implies the following result.

Theorem 7.11 *Assume that $g : X \to R^1$ is a continuous convex function such that $\inf\{g(x) : x \in X\}$ is attained at a unique point $y_* \in X$, $\lim_{\|u\| \to \infty} g(u) = \infty$, and if $\{y_i\}_{i=1}^{\infty} \subset X$ and $\lim_{i \to \infty} g(y_i) = g(y_*)$, then $y_i \to y_*$ as $i \to \infty$. Then there exists a set $\Omega \subset S(X)$, which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$, such that for each $A \in \Omega$, the following property holds:*

(C4) *There is a unique $y_A \in A$ such that $g(y_A) = \inf\{g(y) : y \in A\}$. Moreover, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $y \in A$ satisfies $g(y) \le g(y_A) + \delta$, then $\|y - y_A\| \le \varepsilon$.*

Proof Define $f(x) = g(-x)$, $x \in X$. Clearly, f is convex and satisfies (7.34)–(7.36). Therefore Theorem 7.8 is valid with $\tilde{x} = 0$ and there exists a set $\Omega \subset S(X)$, which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$, such that for each $A \in \Omega$, the following property holds:

There is a unique $\tilde{y} \in A$ such that

$$g(\tilde{y}) = f(-\tilde{y}) = \inf\{f(-y) : y \in A\} = \inf\{g(y) : y \in A\}.$$

Moreover, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ satisfies

$$g(x) = f(-x) \le \rho_f(0, A) + \delta = \inf\{f(-y) : y \in A\} + \delta = \inf\{g(y) : y \in A\} + \delta,$$

then $\|x - \tilde{y}\| \le \varepsilon$. Theorem 7.11 is proved. □

It is easy to see that in the proofs of Theorems 7.8–7.10 we may assume without loss of generality that $\inf\{f(x) : x \in X\} = 0$. It is also not difficult to see that we may assume without loss of generality that $x_* = 0$. Indeed, instead of the function $f(\cdot)$ we can consider $f(\cdot + x_*)$. This new function also satisfies (7.34)–(7.37). Once Theorems 7.8–7.10 are proved for this new function, they will also hold for the original function f because the mapping $(A, x) \to (A, x + x_*)$, $(A, x) \in S(X) \times A$, is an isometry with respect to both metrics d_1 and d_2.

7.6 A Basic Lemma

Lemma 7.12 *Let $A \in S(X)$, $\tilde{x} \in X$, and let $r, \varepsilon \in (0, 1)$. Then there exists $\tilde{A} \in S(X)$, $\bar{x} \in \tilde{A}$, and $\delta > 0$ such that*

$$\tilde{H}(A, \tilde{A}) \le r, \qquad f(\tilde{x} - \bar{x}) = \rho_f(\tilde{x}, \tilde{A}), \tag{7.42}$$

and such that the following property holds:

For each $\tilde{y} \in X$ satisfying $\|\tilde{y} - \tilde{x}\| \le \delta$, each $B \in S(X)$ satisfying $h(B, \tilde{A}) \le \delta$, and each $z \in B$ satisfying

$$f(\tilde{y} - z) \le \rho_f(\tilde{y}, B) + \delta, \tag{7.43}$$

the inequality $\|z - \bar{x}\| \le \varepsilon$ holds.

Proof There are two cases: either $\rho(\tilde{x}, A) \le r$ or $\rho(\tilde{x}, A) > r$. Consider the first case where

$$\rho(\tilde{x}, A) \le r. \tag{7.44}$$

Set

$$\bar{x} = \tilde{x} \quad \text{and} \quad \tilde{A} = A \cup \{\tilde{x}\}. \tag{7.45}$$

Clearly, (7.42) is true. Fix an integer $n > \|\tilde{x}\|$. By (7.36), there is $\xi \in (0, 1)$ such that

$$\text{if } z \in X \text{ and } f(z) \le 4\xi, \text{ then } \|z\| \le \varepsilon/2. \tag{7.46}$$

Using (7.34), we choose a number $\delta \in (0, 1)$ such that

$$\delta < 2^{-n-4} \min\{\varepsilon, \xi\} \tag{7.47}$$

and

$$\text{if } z \in X \text{ and } \|z\| \le 2^{n+4}\delta, \text{ then } f(z) \le \xi. \tag{7.48}$$

Let

$$\tilde{y} \in X, \quad \|\tilde{y} - \tilde{x}\| \le \delta, \qquad B \in S(X), \quad h(B, \tilde{A}) \le \delta \tag{7.49}$$

and let $z \in B$ satisfy (7.43). By (7.49) and (7.41), $h_n(\tilde{A}, B)(1 + h_n(\tilde{A}, B))^{-1} \le 2^n\delta$. This implies that $h_n(\tilde{A}, B)(1 - 2^n\delta) \le 2^n\delta$. When combined with (7.47), this inequality shows that $h_n(\tilde{A}, B) \le 2^{n+1}\delta$. Since $n > \|\tilde{x}\|$, the last inequality, when combined with (7.44) and (7.41), implies that $\rho(\tilde{x}, B) \le 2^{n+1}\delta$. Hence there is $x_0 \in B$ such that $\|\tilde{x} - x_0\| \le 2^{n+2}\delta$. This inequality and (7.49) imply in turn that $\|\tilde{y} - x_0\| \le 2^{n+3}\delta$. The definition of δ (see (7.48)) now shows that $f(\tilde{y} - x_0) \le \xi$. Combining this inequality with (7.43), (7.47) and the inclusion $x_0 \in B$, we see that

$$f(\tilde{y} - z) \le \delta + f(\tilde{y} - x_0) \le \xi + \delta \le 2\xi. \tag{7.50}$$

It now follows from (7.46) that $\|z - \tilde{y}\| \le \varepsilon/2$. Hence (7.47), (7.49) and (7.45) imply that $\|\bar{x} - z\| \le \varepsilon$. This concludes the proof of the lemma in the first case.

Now we turn our attention to the second case where

$$\rho(\tilde{x}, A) > r. \tag{7.51}$$

For each $t \in [0, r]$, set

$$A_t = \{v \in X : \rho(v, A) \le t\} \in S(X) \tag{7.52}$$

and

$$\mu(t) = \rho_f(\tilde{x}, A_t). \tag{7.53}$$

By (7.51) and (7.36),

$$\mu(t) > 0, \quad t \in [0, r]. \tag{7.54}$$

It is clear that $\mu(t)$, $t \in [0, r]$, is a decreasing function. Choose a number

$$t_0 \in (0, r/4) \tag{7.55}$$

such that μ is continuous at t_0. By (7.35), there exists a natural number n which satisfies the following conditions:

$$n > 4\|\tilde{x}\| + 8 \tag{7.56}$$

and

$$\text{if } z \in X,\ f(x) \le \mu(0) + 1,\ \text{ then } \|z\| \le n/4. \tag{7.57}$$

Let $\phi_n : (0, 1) \to (0, 1)$ be an increasing function for which (7.37) is true. Choose a positive number $\gamma \in (0, 1)$ such that

$$\gamma < \mu(t_0)\big(1 - \phi(1 - 2r/n)\big)/8. \tag{7.58}$$

Next, choose a positive number $\delta_0 < 1/4$ such that

$$2^{n+3}\delta_0 < \min\{\varepsilon, \gamma\}, \tag{7.59}$$

$$[t_0 - 4\delta_0, t_0 + 4\delta_0] \subset (0, r/4), \tag{7.60}$$

and

$$\big|\mu(t) - \mu(t_0)\big| \le \gamma, \quad t \in [t_0 - 4\delta_0, t_0 + 4\delta_0]. \tag{7.61}$$

Finally, choose a vector x_0 such that

$$x_0 \in A_{t_0} \quad \text{and} \quad f(\tilde{x} - x_0) \le \mu(t_0) + \gamma. \tag{7.62}$$

It follows from (7.62), (7.52) and (7.55) that

$$\|x_0 - \tilde{x}\| \ge \rho(\tilde{x}, A) - \rho(x_0, A) \ge \rho(\tilde{x}, A) - t_0 \ge \rho(\tilde{x}, A) - r/2, \tag{7.63}$$

and hence by (6.51),

$$\|x_0 - \tilde{x}\| > r/2. \tag{7.64}$$

It follows from (7.62) and (7.57) that

$$\|x_0 - \tilde{x}\| \le n/4. \tag{7.65}$$

There exist $\bar{x} \in \{\alpha x_0 + (1 - \alpha)\tilde{x} : \alpha \in (0, 1)\}$ and $\alpha_0 \in (0, 1)$ such that

$$\|\bar{x} - x_0\| = r/2 \tag{7.66}$$

and

$$\bar{x} = \alpha_0 x_0 + (1 - \alpha_0)\tilde{x}. \tag{7.67}$$

By (7.67) and (7.66), $r/2 = \|\bar{x} - x_0\| = \|\alpha_0 x_0 + (1-\alpha_0)\tilde{x} - x_0\| = (1-\alpha_0)\|\tilde{x} - x_0\|$ and

$$\alpha_0 = 1 - r\big(2\|\tilde{x} - x_0\|\big)^{-1}. \tag{7.68}$$

Relations (7.68) and (7.65) imply that

$$\alpha_0 \le 1 - r/(2n/4) = 1 - 2r/n. \tag{7.69}$$

Set

$$\tilde{A} = A_{t_0} \cup \{\bar{x}\}. \tag{7.70}$$

Now we will estimate $f(\tilde{x} - \bar{x})$. By (7.67), (7.65), (7.37), (7.62) and (7.69),

$$\begin{aligned} f(\tilde{x} - \bar{x}) &= f\big(\tilde{x} - \big(\alpha_0 x_0 + (1 - \alpha_0)\tilde{x}\big)\big) = f\big(\alpha_0(\tilde{x} - x_0)\big) \\ &\le \phi_n(\alpha_0) f(\tilde{x} - x_0) \le \phi_n(\alpha_0)\big(\mu(t_0) + \gamma\big) \\ &\le \phi_n(1 - 2r/n)\big(\mu(t_0) + \gamma\big). \end{aligned}$$

Thus

$$f(\tilde{x} - \bar{x}) \le \phi_n(1 - 2r/n)\big(\mu(t_0) + \gamma\big) \le \mu(t_0)\phi_n(1 - 2r/n) + \gamma. \tag{7.71}$$

By (7.70), (7.53), (7.58) and (7.71), for each $x \in \tilde{A} \setminus \{\bar{x}\} \subset A_{t_0}$,

$$f(\tilde{x} - x) \ge \mu(t_0) > f(\tilde{x} - \bar{x}) \tag{7.72}$$

and therefore

$$f(\tilde{x} - \bar{x}) = \rho_f(\tilde{x}, \tilde{A}). \tag{7.73}$$

There exists $\delta \in (0, \delta_0)$ such that

$$2^{n+4}\delta < \delta_0 \tag{7.74}$$

and

$$\begin{aligned} &\big|f(z) - f(\tilde{x} - \bar{x})\big| \le \gamma/4 \\ &\quad \text{for all } z \in X \text{ satisfying } \big\|z - (\tilde{x} - \bar{x})\big\| \le 2^{n+3}\delta. \end{aligned} \tag{7.75}$$

By (7.70), (7.40), (7.66), (7.62), (7.55) and (7.52),

$$\tilde{H}(\tilde{A}, A) \le H(\tilde{A}, A) \le r. \tag{7.76}$$

Relations (7.76) and (7.73) imply (7.42). Assume now that

$$\tilde{y} \in X, \quad \|\tilde{y} - \tilde{x}\| \le \delta \tag{7.77}$$

and

$$B \in S(X) \quad \text{and} \quad h(\tilde{A}, B) \le \delta. \tag{7.78}$$

First we will show that

$$\rho_f(\tilde{y}, B) \le \mu(t_0)\phi_n(1 - 2r/n) + 2\gamma. \tag{7.79}$$

By (7.78) and the definition of h (see (7.41)), $h_n(\tilde{A}, B)(1 + h_n(\tilde{A}, B))^{-1} \le 2^n\delta$. When combined with (7.74), this inequality implies that

$$h_n(\tilde{A}, B) \le 2^n\delta\big(1 - 2^n\delta\big)^{-1} \le 2^{n+1}\delta. \tag{7.80}$$

It follows from (7.41) and the definition of n (see (7.57), (7.56)) that $\|\tilde{x} - \bar{x}\| \le n/2$ and $\|\bar{x}\| \le n$. When combined with (7.70) and (7.80), this implies that $\rho(\bar{x}, B) \le 2^{n+1}\delta$. Therefore there exists $\bar{y} \in B$ such that $\|\bar{x} - \bar{y}\| \le 2^{n+2}\delta$. Combining this inequality with (7.77), we see that $\|(\bar{y} - \tilde{y}) - (\bar{x} - \tilde{x})\| \le \|\bar{x} - \bar{y}\| + \|\tilde{y} - \tilde{x}\| \le 2^{n+3}\delta$. It follows from this inequality and (7.75) that $f(\tilde{y} - \bar{y}) \le f(\tilde{x} - \bar{x}) + \gamma/4$. By the last inequality and (7.71), $f(\tilde{y} - \bar{y}) \le \mu(t_0)\phi_n(1 - 2r/n) + 2\gamma$. This implies (7.79).

Assume now that $z \in B$ satisfies (7.43). To complete the proof of the lemma it is sufficient to show that $\|\bar{x} - z\| \le \varepsilon$. Assume the contrary. Then

$$\|\bar{x} - z\| > \varepsilon. \tag{7.81}$$

We will show that there exists $\bar{z} \in \tilde{A}$ such that

$$\|z - \bar{z}\| \le 2^{n+2}\delta. \tag{7.82}$$

We have already shown that (7.80) holds. By (7.43), (7.79), (7.58) and (7.74),

$$f(\tilde{y} - z) \le \rho_f(\tilde{y}, B) + \delta \le \phi_n(1 - 2r/n)\mu(t_0) + 2\gamma + \delta \le \mu(0) + 1/2.$$

Hence $\|z - \tilde{y}\| \le n/4$ by (7.57), and by (7.77) and (7.56),

$$\|z\| \le n/4 + \|\tilde{y}\| \le n/4 + \|\tilde{x}\| + \|\tilde{y} - \tilde{x}\| \le n.$$

Thus $\|z\| \le n$. The inclusion $z \in B$ and (7.80) now imply that $\rho(z, \tilde{A}) \le h_n(B, \tilde{A}) \le 2^{n+1}\delta$. Therefore there exists $\bar{z} \in \tilde{A}$ such that (7.82) holds. It follows from (7.82), (7.81), (7.70), (7.74) and (7.59) that

$$\bar{z} \in A_{t_0}. \tag{7.83}$$

By (7.82) and (7.77), $\|z + \tilde{x} - \tilde{y} - \bar{z}\| \le \|\tilde{x} - \tilde{y}\| + \|z - \bar{z}\| \le 2^{n+2}\delta + \delta \le 2^{n+3}\delta$. It follows from this inequality, (7.83), (7.52) and (7.74) that

$$\rho(z + \tilde{x} - \tilde{y}, A) \le \|z + \tilde{x} - \tilde{y} - \bar{z}\| + \rho(\bar{z}, A) \le 2^{n+3}\delta + t_0 \le t_0 + \delta_0.$$

Thus $z + \tilde{x} - \tilde{y} \in A_{t_0+\delta_0}$. By this inclusion, (7.52), (7.53) and (7.61),

$$f(\tilde{y} - z) = f\big(\tilde{x} - (z + \tilde{x} - \tilde{y})\big) \ge \rho_f(\tilde{x}, A_{t_0+\delta_0}) = \mu(t_0 + \delta_0) \ge \mu(t_0) - \gamma.$$

Hence, by (7.43), (7.79), (7.59) and (7.74),

$$\begin{aligned}\mu(t_0) - \gamma &\le f(\tilde{y} - z) \le \rho_f(\tilde{y}, B) + \delta \le \phi_n(1 - 2r/n)\mu(t_0) + 2\gamma + \delta \\ &\le \phi_n(1 - 2r/n)\mu(t_0) + 3\gamma.\end{aligned}$$

Thus $\mu(t_0) - \gamma \le \phi_n(1 - 2r/n)\mu(t_0) + 3\gamma$, which contradicts (7.58). This completes the proof of Lemma 7.12. □

7.7 Proofs of Theorems 7.8–7.11

The cornerstone of our proofs is the property established in Lemma 7.12.

By Lemma 7.12, for each $(A, x) \in S(X) \times X$ and each integer $k \geq 1$, there exist $A(x, k) \in S(X)$, $\bar{x}(A, k) \in A(x, k)$, and $\delta(x, A, k) > 0$ such that

$$\tilde{H}\big(A, A(x, k)\big) \leq 2^{-k}, \qquad f\big(x - \bar{x}(A, k)\big) = \rho_f\big(x, A(x, k)\big), \tag{7.84}$$

and the following property holds:

(P1) For each $y \in X$ satisfying $\|y - x\| \leq 2\delta(x, A, k)$, each $B \in S(X)$ satisfying $h(B, A(x, k)) \leq 2\delta(x, A, k)$ and each $z \in B$ satisfying $f(y - z) \leq \rho_f(y, B) + 2\delta(x, A, k)$, the inequality $\|z - \bar{x}(A, k)\| \leq 2^{-k}$ holds.

For each $(A, x) \in S(X) \times X$ and each integer $k \geq 1$, define

$$V(A, x, k) = \big\{(B, y) \in S(X) \times X : \\ h\big(B, A(x, k)\big) < \delta(x, A, k) \text{ and } \|y - x\| < \delta(x, A, k)\big\} \tag{7.85}$$

and

$$U(A, x, k) = \big\{B \in S(X) : h\big(B, A(x, k)\big) < \delta(x, A, k)\big\}. \tag{7.86}$$

Now set

$$\Omega = \bigcap_{n=1}^{\infty} \bigcup \big\{V(A, x, k) : (A, x) \in S(X) \times X, k \geq n\big\}, \tag{7.87}$$

and for each $x \in X$ let

$$\Omega_x = \bigcap_{n=1}^{\infty} \bigcup \big\{U(A, x, k) : A \in S(X), k \geq n\big\}. \tag{7.88}$$

It is easy to see that $\Omega_x \times \{x\} \subset \Omega$ for all $x \in X$, Ω_x is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$ for all $x \in X$, and Ω is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X) \times X$.

Completion of the proof of Theorem 7.9 Let $(A, \tilde{x}) \in \Omega$. We will show that $(A, \tilde{x})$ has property (C2). By the definition of Ω (see (7.87)), for each integer $n \geq 1$, there exist an integer $k_n \geq n$ and a pair $(A_n, x_n) \in S(X) \times X$ such that

$$(A, \tilde{x}) \in V(A_n, x_n, k_n). \tag{7.89}$$

Let $\{z_i\}_{i=1}^{\infty} \subset A$ be such that

$$\lim_{i \to \infty} f(\tilde{x} - z_i) = \rho_f(\tilde{x}, A). \tag{7.90}$$

Fix an integer $n \geq 1$. It follows from (7.89), (7.85) and property (P1) that for all large enough integers i,

$$f(\tilde{x} - z_i) < \rho_f(\tilde{x}, A) + \delta(x_n, A_n, k_n)$$

and

$$\|z_i - \bar{x}_n(A_n, k_n)\| \leq 2^{-n}.$$

Since $n \geq 1$ is arbitrary, we conclude that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to some $\tilde{y} \in A$. Clearly $f(\tilde{x} - \tilde{y}) = \rho_f(\tilde{x}, A)$. If the minimizer $\tilde{y}$ were not unique we would be able to construct a nonconvergent minimizing sequence $\{z_i\}_{i=1}^{\infty}$. Thus $\tilde{y}$ is the unique solution to problem (P) (with $x = \tilde{x}$).

Let $\varepsilon > 0$ be given. Choose an integer $n > 4/\min\{1, \varepsilon\}$. By property (P1), (7.89) and (7.85),

$$\|\tilde{y} - \bar{x}_n(A_n, k_n)\| \leq 2^{-n}. \tag{7.91}$$

Assume that $z \in X$ satisfies $\|z - \tilde{x}\| \leq \delta(x_n, A_n, k_n)$, $B \in S(X)$ satisfies $h(A, B) \leq \delta(x_n, A_n, k_n)$, and $y \in B$ satisfies $f(z - y) \leq \rho_f(z, B) + \delta(x_n, A_n, k_n)$. Then

$$h\big(B, A_n(x_n, k_n)\big) \leq 2\delta(x_n, A_n, k_n) \quad \text{and} \quad \|z - \bar{x}_n(A_n, k_n)\| \leq 2\delta(x_n, A_n, k_n)$$

by (7.89) and (7.85). Now it follows from property (P1) that

$$\|y - \bar{x}_n(A_n, k_n)\| \leq 2^{-n}.$$

When combined with (7.91), this implies that

$$\|y - \tilde{y}\| \leq 2^{1-n} < \varepsilon.$$

The proof of Theorem 7.9 is complete. □

Theorem 7.8 follows from Theorem 7.9 and the inclusion $\Omega_{\tilde{x}} \times \{\tilde{x}\} \subset \Omega$.

Although a variant of Theorem 7.10 also follows from Theorem 7.9 by a classical result of Kuratowski and Ulam [87], the following direct proof may also be of interest.

Proof of Theorem 7.10 Let the sequence $\{x_i\}_{i=1}^{\infty} \subset X_0$ be everywhere dense in X_0. Set $\mathcal{F} = \bigcap_{p=1}^{\infty} \Omega_{x_p}$. Clearly, $\mathcal{F}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$.

Let $A \in \mathcal{F}$ and let $p, n \geq 1$ be integers. Clearly, $A \in \Omega_{x_p}$ and by (7.88) and (7.86), there exist $A_n \in S(X)$ and an integer $k_n \geq n$ such that

$$h\big(A, A_n(x_p, k_n)\big) < \delta(x_p, A_n, k_n) \quad \text{with } A \in S(X). \tag{7.92}$$

It follows from this inequality and property (P1) that the following property holds:

(P2) For each $y \in X$ satisfying $\|y - x_p\| \le \delta(x_p, A_n, k_n)$ and each $z \in A$ satisfying $f(y - z) \le \rho_f(y, A) + 2\delta(x_p, A_n, k_n)$, the inequality $\|z - \bar{x}_p(A_n, k_n)\| \le 2^{-n}$ holds.

Set $W(p, n) = \{z \in X_0 : \|z - x_p\| < \delta(x_p, A_n, k_n)\}$ and

$$F = \bigcap_{n=1}^{\infty} \bigcup \{W(p, n) : p = 1, 2, \dots\}.$$

It is clear that F is a countable intersection of open and everywhere dense subsets of X_0.

Let $x \in F$ be given. Consider a sequence $\{z_i\}_{i=1}^{\infty} \subset A$ such that

$$\lim_{i \to \infty} f(x - z_i) = \rho_f(x, A). \tag{7.93}$$

Let $\varepsilon > 0$. Choose an integer $n > 8/\min\{1, \varepsilon\}$. There exists an integer $p \ge 1$ such that $x \in W(p, n)$. By the definition of $W(p, n)$, $\|x - x_p\| < \delta(x_p, A_n, k_n)$. It follows from this inequality, (7.93) and property (P2) that for all sufficiently large integers i, $f(x - z_i) \le \rho_f(x, A) + \delta(x_p, A_n, k_n)$ and $\|z_i - \bar{x}_p(A_n, k_n)\| \le 2^{-n} < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to $\tilde{y} \in A$. Clearly, $\tilde{y}$ is the unique minimizer of the minimization problem $z \to f(x - z)$, $z \in A$. Note that we have shown that any sequence $\{z_i\}_{i=1}^{\infty} \subset A$ satisfying (7.93) converges to $\tilde{y}$. This completes the proof of Theorem 7.10. □

7.8 A Porosity Result in Best Approximation Theory

Let D be a nonempty compact subset of a complete hyperbolic space (X, ρ, M) and denote by $S(X)$ the family of all nonempty closed subsets of X. We endow $S(X)$ with a pair of natural complete metrics and show that there exists a set $\Omega \subset S(X)$ such that its complement $S(X) \setminus \Omega$ is σ-porous with respect to this pair of metrics and such that for each $A \in \Omega$ and each $\tilde{x} \in D$, the following property holds: the set $\{y \in A : \rho(\tilde{x}, y) = \rho(\tilde{x}, A)\}$ is nonempty and compact, and each sequence $\{y_i\}_{i=1}^{\infty} \subset A$ which satisfies $\lim_{i \to \infty} \rho(\tilde{x}, y_i) = \rho(\tilde{x}, A)$ has a convergent subsequence. This result was obtained in [147].

Let (X, ρ, M) be a complete hyperbolic space. For each $x \in X$ and each $A \subset X$, set

$$\rho(x, A) = \inf\{\rho(x, y) : y \in A\}.$$

Denote by $S(X)$ the family of all nonempty closed subsets of X. For each $A, B \in S(X)$, define

$$H(A, B) := \max\{\sup\{\rho(x, B) : x \in A\}, \sup\{\rho(y, A) : y \in B\}\} \tag{7.94}$$

and

$$\tilde{H}(A, B) := H(A, B)\big(1 + H(A, B)\big)^{-1}.$$

Here we use the convention that $\infty/\infty = 1$. It is easy to see that $\tilde{H}$ is a metric on $S(X)$ and that the metric space $(S(X), \tilde{H})$ is complete.

Fix $\theta \in X$. For each natural number n and each $A, B \in S(X)$, we set

$$h_n(A, B) = \sup\{|\rho(x, A) - \rho(x, B)| : x \in X \text{ and } \rho(x, \theta) \le n\} \tag{7.95}$$

and

$$h(A, B) = \sum_{n=1}^{\infty} [2^{-n} h_n(A, B)(1 + h_n(A, B))^{-1}].$$

Once again, it is not difficult to see that h is a metric on $S(X)$ and that the metric space $(S(X), h)$ is complete. Clearly,

$$\tilde{H}(A, B) \ge h(A, B) \quad \text{for all } A, B \in S(X).$$

We equip the set $S(X)$ with the pair of metrics $\tilde{H}$ and h and prove the following theorem which is the main result of [147].

Theorem 7.13 *Given a nonempty compact subset D of a complete hyperbolic space (X, ρ, M), there exists a set $\Omega \subset S(X)$ such that its complement $S(X) \setminus \Omega$ is σ-porous with respect to the pair of metrics $(h, \tilde{H})$, and such that for each $A \in \Omega$ and each $\tilde{x} \in D$, the following property holds*:

The set $\{y \in A : \rho(\tilde{x}, y) = \rho(\tilde{x}, A)\}$ is nonempty and compact and each sequence $\{y_i\}_{i=1}^{\infty} \subset A$ which satisfies $\lim_{i\to\infty} \rho(\tilde{x}, y_i) = \rho(\tilde{x}, A)$ has a convergent subsequence.

7.9 Two Lemmata

Let (X, ρ, M) be a complete hyperbolic space and let D be a nonempty compact subset of X. In the proof of Theorem 7.13 we will use the following two lemmata.

Lemma 7.14 *Let q be a natural number, $A \in S(X)$, $\varepsilon \in (0, 1)$, $r \in (0, 1]$, and let $Q = \{\xi_1, \dots, \xi_q\}$ be a finite subset of D. Then there exists a finite set $\{\tilde{\xi}_1, \dots, \tilde{\xi}_q\} \subset X$ such that*

$$\rho(\tilde{\xi}_i, A) \le r, \quad i = 1, \dots, q, \tag{7.96}$$

and such that the set $\tilde{A} := A \cup \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}$ has the following properties:

$$\rho(\xi_i, \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}) = \rho(\xi_i, \tilde{A}), \quad i = 1, \dots, q; \tag{7.97}$$

(P3) *if $i \in \{1, \dots, q\}$, $x \in \tilde{A}$, and $\rho(\xi_i, x) \le \rho(\xi_i, \tilde{A}) + \varepsilon r/4$, then*

$$\rho(x, \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}) \le \varepsilon.$$

Proof Let $i \in \{1, \dots, q\}$. There are two cases: (1) $\rho(\xi_i, A) \le r$; (2) $\rho(\xi_i, A) > r$. In the first case we set

$$\tilde{\xi}_i = \xi_i. \tag{7.98}$$

In the second case, we first choose $x_i \in A$ for which

$$\rho(\xi_i, x_i) \le \rho(\xi_i, A) + r/4, \tag{7.99}$$

and then choose

$$\tilde{\xi}_i \in \{\gamma x_i \oplus (1-\gamma)\xi_i : \gamma \in (0,1)\} \tag{7.100}$$

such that

$$\rho(\tilde{\xi}_i, x_i) = r \quad \text{and} \quad \rho(\tilde{\xi}_i, \xi_i) = \rho(x_i, \xi_i) - r. \tag{7.101}$$

Clearly, (7.96) holds. Consider now the set $\tilde{A} = A \cup \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}$.

Let $i \in \{1, \dots, q\}$. It is not difficult to see that if $\rho(\xi_i, A) \le r$, then the assertion of the lemma is true. Consider the case where $\rho(\xi_i, A) > r$. It follows from (7.99) and (7.101) that

$$\begin{aligned}\rho(\xi_i, \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}) &\le \rho(\xi_i, \tilde{\xi}_i) = \rho(x_i, \xi_i) - r \\ &\le \rho(\xi_i, A) + r/4 - r = \rho(\xi_i, A) - 3r/4.\end{aligned}$$

Therefore

$$\rho(\xi_i, \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}) = \rho(\xi_i, \tilde{A}),$$

and if $x \in \tilde{A}$ and $\rho(\xi_i, x) \le \rho(\xi_i, \tilde{A}) + r/2$, then $x \in \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}$. This completes the proof of Lemma 7.14. □

For each $\varepsilon \in (0,1)$ and each natural number n, choose a number

$$\alpha(\varepsilon, n) \in (0, 16^{-n-2}\varepsilon) \tag{7.102}$$

and a natural number n_0 such that

$$\rho(x, \theta) \le n_0, \quad x \in D. \tag{7.103}$$

Lemma 7.15 *Let $n \ge n_0$ be a natural number, $A \in S(X)$, $\varepsilon \in (0,1)$, $r \in (0,1]$, and*

$$\alpha = \alpha(\varepsilon, n). \tag{7.104}$$

Assume that

$$\{z \in A : \rho(z, \theta) \le n\} \ne \emptyset. \tag{7.105}$$

Then there exist a natural number q and a finite set $\{\tilde{\xi}_1, \dots, \tilde{\xi}_q\} \subset X$ such that

$$\rho(\tilde{\xi}_i, A) \le r, \quad i = 1, \dots, q, \tag{7.106}$$

and if $\tilde{A} := A \cup \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}$, $u \in D$, $B \in S(X)$,

$$h(\tilde{A}, B) \le \alpha r, \tag{7.107}$$

and

$$z \in B, \quad \rho(u, z) \le \rho(u, B) + \varepsilon r/16, \tag{7.108}$$

then

$$\rho\big(z, \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}\big) \le \varepsilon. \tag{7.109}$$

Proof Since D is compact, there are a natural number q and a finite subset $\{\xi_1, \dots, \xi_q\}$ of D such that

$$D \subset \bigcup_{i=1}^{q} \big\{z \in X : \rho(z, \xi_i) < \alpha r\big\}. \tag{7.110}$$

By Lemma 7.14, there exists a finite set $\{\tilde{\xi}_1, \dots \tilde{\xi}_q\} \subset X$ such that (7.106) holds, and the set $\tilde{A} := A \cup \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}$ satisfies (7.97) and has the following property:

(P4) If $i \in \{1, \dots, q\}$, $x \in \tilde{A}$, and $\rho(\xi_i, x) \le \rho(\xi_i, \tilde{A}) + \varepsilon r/8$, then

$$\rho\big(x, \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}\big) \le \varepsilon/2.$$

Assume that $u \in D$, $B \in S(X)$, and that (7.107) holds. By (7.110), there is $j \in \{1, \dots, q\}$ such that

$$\rho(\xi_j, u) < \alpha r. \tag{7.111}$$

We will show that

$$\rho(u, B) < \rho(\xi_j, \tilde{A}) + 4 \cdot 16^n \alpha r. \tag{7.112}$$

Indeed, there exists $p \in \{1, \dots, q\}$ such that

$$\rho(\xi_j, \tilde{\xi}_p) = \rho\big(\xi_j, \{\tilde{\xi}_1, \dots, \tilde{\xi}_q\}\big).$$

By (7.97),

$$\rho(\xi_j, \tilde{\xi}_p) = \rho(\xi_j, \tilde{A}). \tag{7.113}$$

By (7.111),

$$\big|\rho(u, \tilde{A}) - \rho(\xi_j, \tilde{A})\big| \le \alpha r. \tag{7.114}$$

When combined with (7.113), this inequality implies that

$$\big|\rho(u, \tilde{A}) - \rho(\xi_j, \tilde{\xi}_p)\big| \le \alpha r. \tag{7.115}$$

Now (7.113), (7.105) and (7.103) imply that

$$\rho(\xi_j, \tilde{\xi}_p) \le \rho(\xi_j, A) \le 2n \quad \text{and} \quad \rho(\tilde{\xi}_p, \theta) \le 3n. \tag{7.116}$$

It follows from (7.95) and (7.107) that

$$h_{4n}(\tilde{A}, B)\big(1 + h_{4n}(\tilde{A}, B)\big)^{-1} \le 2^{4n} h(\tilde{A}, B) \le 2^{4n}\alpha r,$$

and when combined with (7.104) and (7.102), this inequality yields

$$h_{4n}(\tilde{A}, B) \le 2^{4n}\alpha r\big(1 - 2^{4n}\alpha r\big)^{-1} < 2^{4n+1}\alpha r. \tag{7.117}$$

Since $\tilde{\xi}_p \in \tilde{A}$, it follows from (7.117), (7.116) and (7.97) that $\rho(\tilde{\xi}_p, B) < 2^{4n+1}\alpha r$ and there exists $v \in X$ such that

$$v \in B \quad \text{and} \quad \rho(\tilde{\xi}_p, v) < 2\alpha r 16^n. \tag{7.118}$$

By (7.118), (7.111), (7.113) and (7.118),

$$\begin{aligned}\rho(u, B) &\le \rho(u, v) \le \rho(u, \tilde{\xi}_p) + \rho(\tilde{\xi}_p, v) \le \rho(u, \xi_j) + \rho(\xi_j, \tilde{\xi}_p) + \rho(\tilde{\xi}_p, v)\\ &< \alpha r + \rho(\xi_j, \tilde{A}) + 2 \cdot 16^n \alpha r.\end{aligned}$$

Hence (7.112) is valid.

Now let (7.108) hold. Then by (7.108), (7.112) and (7.102),

$$\begin{aligned}\rho(z, u) &\le \rho(u, B) + \varepsilon r/16 < \rho(\xi_j, \tilde{A}) + 4 \cdot 16^n \alpha r + \varepsilon r/16\\ &< \rho(\xi_j, \tilde{A}) + \varepsilon r/8.\end{aligned} \tag{7.119}$$

Therefore (7.119) and (7.116) imply that

$$\rho(z, u) \le \rho(\xi_j, \tilde{A}) + \varepsilon r/8 \le 2n + r/8.$$

It follows from this inequality, (7.111) and (7.103) that

$$\begin{aligned}\rho(z, \theta) &\le \rho(z, u) + \rho(u, \theta) \le 2n + r/8 + \rho(u, \theta)\\ &\le 2n + r/8 + \rho(u, \xi_j) + \rho(\xi_j, \theta) \le 2n + r/8 + \alpha r + n \le 4n.\end{aligned}$$

Since $z \in B$, it follows from (7.97) and (7.117) that

$$\rho(z, \tilde{A}) = \big|\rho(z, \tilde{A}) - \rho(z, B)\big| \le h_{4n}(\tilde{A}, B) < 2 \cdot 16^n \alpha r.$$

Therefore there exists $\tilde{z} \in \tilde{A}$ such that

$$\rho(z, \tilde{z}) < 2 \cdot 16^n \alpha r. \tag{7.120}$$

By (7.111), (7.120), (7.108), (7.112) and (7.102),

$$\begin{aligned}\rho(\tilde{z}, \xi_j) &\le \rho(\xi_j, u) + \rho(u, z) + \rho(z, \tilde{z}) < \alpha r + \rho(u, z) + 2 \cdot 16^n \alpha r\\ &\le \alpha r + 2 \cdot 16^n \alpha r + \rho(u, B) + \varepsilon r/16\end{aligned}$$

$$< \varepsilon r/16 + \alpha r + 2 \cdot 16^n \alpha r + \rho(\xi_j, \tilde{A}) + 4 \cdot 16^n \alpha r$$
$$\le \rho(\xi_j, \tilde{A}) + 8 \cdot 16^n \alpha r + \varepsilon r/16 \le \rho(\xi_j, \tilde{A}) + \varepsilon r/8$$

and

$$\rho(\tilde{z}, \xi_j) < \rho(\xi_j, \tilde{A}) + \varepsilon r/8. \tag{7.121}$$

Since $\tilde{z} \in \tilde{A}$, it follows from (7.121) and property (P4) that $\rho(\tilde{z}, \{\tilde{\xi}_1, \ldots, \tilde{\xi}_q\}) \le \varepsilon/2$. When combined with (7.120) and (7.102), this inequality implies that

$$\rho\bigl(z, \{\tilde{\xi}_1, \ldots, \tilde{\xi}_q\}\bigr) \le \varepsilon.$$

This completes the proof of Lemma 7.15. □

7.10 Proof of Theorem 7.13

For each integer $k \ge 1$, denote by Ω_k the set of all $A \in S(X)$ which have the following property:

(P5) There exist a nonempty finite set $Q \subset X$ and a number $\delta > 0$ such that if $u \in D$, $x \in A$ and $\rho(u, x) \le \rho(u, A) + \delta$, then $\rho(x, Q) \le 1/k$.

It is clear that $\Omega_{k+1} \subset \Omega_k$, $k = 1, 2, \ldots$. Set $\Omega = \bigcap_{k=1}^{\infty} \Omega_k$.

Let $k \ge n_0$ (see (7.103)) be an integer. We will show that $S(X) \setminus \Omega_k$ is σ-porous with respect to the pair $(h, \tilde{H})$. For any integer $n \ge k$, define

$$E_{nk} = \bigl\{A \in S(X) \setminus \Omega_k : \bigl\{z \in A : \rho(z, \theta) \le n\bigr\} \ne \emptyset\bigr\}.$$

By Lemma 7.15, E_{nk} is porous with respect to the pair $(h, \tilde{H})$ for all integers $n \ge k$. Thus $S(X) \setminus \Omega_k = \bigcup_{n=k}^{\infty} E_{nk}$ is σ-porous with respect to $(h, \tilde{H})$. Hence $S(X) \setminus \Omega = \bigcup_{k=n_0}^{\infty} (S(X) \setminus \Omega_k)$ is also σ-porous with respect to the pair of metrics $(h, \tilde{H})$.

Let $A \in \Omega$. Since $A \in \Omega_k$ for each integer $k \ge 1$, it follows from property (P5) that for any integer $k \ge 1$, there exist a nonempty finite set $Q_k \subset X$ and a number $\delta_k > 0$ such that the following property also holds:

(P6) If $u \in D$, $x \in A$, and $\rho(u, x) \le \rho(x, A) + \delta_k$, then $\rho(x, Q_k) \le 1/k$.

Let $u \in D$. Consider a sequence $\{x_i\}_{i=1}^{\infty} \subset A$ such that $\lim_{i\to\infty} \rho(u, x_i) = \rho(u, D)$. By property (P6), for each integer $k \ge 1$, there exists a subsequence $\{x_i^{(k)}\}_{i=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ such that the following two properties hold:

(i) $\{x_i^{(k+1)}\}_{i=1}^{\infty}$ is a subsequence of $\{x_i^{(k)}\}_{i=1}^{\infty}$ for all integers $k \ge 1$;
(ii) for any integer $k \ge 1$, $\rho(x_j^{(k)}, x_s^{(k)}) \le 2/k$ for all integers $j, s \ge 1$.

These properties imply that there exists a subsequence $\{x_i^*\}_{i=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ which is a Cauchy sequence. Therefore $\{x_i^*\}_{i=1}^{\infty}$ converges to a point $\tilde{x} \in A$ which satisfies $\rho(\tilde{x}, u) = \lim_{i\to\infty} \rho(x_i, u) = \rho(u, D)$. This completes the proof of Theorem 7.13.

7.11 Porous Sets and Generalized Best Approximation Problems

Given a closed subset A of a Banach space X, a point $x \in X$ and a Lipschitzian (on bounded sets) function $f : X \to R^1$, we consider the problem of finding a solution to the minimization problem $\min\{f(x-y) : y \in A\}$. For a fixed function f, we define an appropriate complete metric space $\mathcal{M}$ of all pairs (A, x) and construct a subset Ω of $\mathcal{M}$, with a σ-porous complement $\mathcal{M} \setminus \Omega$, such that for each pair in Ω, our minimization problem is well posed.

Let $(X, \|\cdot\|)$ be a Banach space and let $f : X \to R^1$ be a Lipschitzian (on bounded sets) function. Assume that

$$\inf\big\{f(x) : x \in X\big\} \text{ is attained at a unique point } x_* \in X, \tag{7.122}$$

$$\lim_{\|u\|\to\infty} f(u) = \infty, \tag{7.123}$$

$$\text{if } \{x_i\}_{i=1}^{\infty} \subset X \text{ and } \lim_{i\to\infty} f(x_i) = f(x_*), \text{ then } \lim_{i\to\infty} x_i = x_*, \tag{7.124}$$

$$f\big(\alpha x + (1-\alpha)x_*\big) \le \alpha f(x) + (1-\alpha) f(x_*) \quad \text{for all } x \in X \text{ and all } \alpha \in (0,1), \tag{7.125}$$

and that for each natural number n, there exists $k_n > 0$ such that

$$\big|f(x) - f(y)\big| \le k_n \|x-y\| \quad \text{for each } x, y \in X \text{ satisfying } \|x\|, \|y\| \le n. \tag{7.126}$$

Clearly, (7.125) holds if f is convex.

Given a closed subset A of X and a point $x \in X$, we consider the minimization problem

$$\min\big\{f(x-y) : y \in A\big\}. \tag{P}$$

For each $x \in X$ and each $A \subset X$, set

$$\rho(x, A) = \inf\big\{\|x-y\| : y \in A\big\}$$

and

$$\rho_f(x, A) = \inf\big\{f(x-y) : y \in A\big\}.$$

Denote by $S(X)$ the collection of all nonempty closed subsets of X. For each $A, B \in S(X)$, define

$$H(A, B) := \max\big\{\sup\big\{\rho(x, B) : x \in A\big\}, \sup\big\{\rho(y, A) : y \in B\big\}\big\} \tag{7.127}$$

and

$$\tilde{H}(A, B) := H(A, B)\big(1 + H(A, B)\big)^{-1}.$$

Here we use the convention that $\infty/\infty = 1$.

It is not difficult to see that the metric space $(S(X), \tilde{H})$ is complete.

For each natural number n and each $A, B \in S(X)$, we set

$$h_n(A, B) := \sup\{|\rho(x, A) - \rho(x, B)| : x \in X \text{ and } \|x\| \le n\} \tag{7.128}$$

and

$$h(A, B) := \sum_{n=1}^{\infty} [2^{-n} h_n(A, B)(1 + h_n(A, B))^{-1}].$$

Once again, it is not difficult to see that h is a metric on $S(X)$ and that the metric space $(S(X), h)$ is complete. Clearly, $\tilde{H}(A, B) \ge h(A, B)$ for all $A, B \in S(X)$.

We equip the set $S(X)$ with the pair of metrics $\tilde{H}$ and h. The topologies induced by the metrics $\tilde{H}$ and h on $S(X)$ will be called the strong topology and the weak topology, respectively.

Let $A \in S(X)$ and $\tilde{x} \in X$ be given. We say that the best approximation problem

$$f(\tilde{x} - y) \to \min, \quad y \in A,$$

is strongly well posed if there exists a unique $\bar{x} \in A$ such that

$$f(\tilde{x} - \bar{x}) = \inf\{f(\tilde{x} - y) : y \in A\}$$

and the following property holds:

For each $\varepsilon > 0$, there exists $\delta > 0$ such that if $z \in X$ satisfies $\|z - \tilde{x}\| \le \delta$, $B \in S(X)$ satisfies $h(A, B) \le \delta$, and $y \in B$ satisfies $f(z - y) \le \rho_f(z, B) + \delta$, then $\|y - \bar{x}\| \le \varepsilon$.

We now state four results obtained in [151]. Their proofs will be given in the next sections.

Theorem 7.16 *Let $\tilde{x} \in X$ be given. Then there exists a set $\Omega \subset S(X)$ such that its complement $S(X) \setminus \Omega$ is σ-porous with respect to $(h, \tilde{H})$ and for each $A \in \Omega$, the problem $f(\tilde{x} - y) \to \min$, $y \in A$, is strongly well posed.*

To state our second result, we endow the Cartesian product $S(X) \times X$ with the pair of metrics d_1 and d_2 defined by

$$d_1((A, x), (B, y)) = h(A, B) + \|x - y\|,$$

$$d_2((A, x), (B, y)) = \tilde{H}(A, B) + \|x - y\|, \quad x, y \in X, A, B \in S(X).$$

We will refer to the metrics induced on $S(X) \times X$ by d_2 and d_1 as the strong and weak metrics, respectively.

Theorem 7.17 *There exists a set $\Omega \subset S(X) \times X$ such that its complement $(S(X) \times X) \setminus \Omega$ is σ-porous with respect to (d_1, d_2) and for each $(A, \tilde{x}) \in \Omega$, the minimization problem*

$$f(\tilde{x} - y) \to \min, \quad y \in A,$$

is strongly well posed.

In most classical generic results the set A was fixed and x varied in a dense G_δ subset of X. In our first two results the set A is also variable. However, our third result shows that for every fixed A in a subset of $S(X)$ which has a σ-porous complement, the set of all $x \in X$ for which problem (P) is strongly well posed contains a dense G_δ subset of X.

Theorem 7.18 *Assume that X_0 is a closed separable subset of X. Then there exists a set $\mathcal{F} \subset S(X)$ such that its complement $S(X) \setminus \mathcal{F}$ is σ-porous with respect to $(h, \tilde{H})$ and for each $A \in \mathcal{F}$, the following property holds*:

There exists a set $F \subset X_0$, which is a countable intersection of open and everywhere dense subsets of X_0 with the relative topology, such that for each $\tilde{x} \in F$, the minimization problem

$$f(\tilde{x} - y) \to \min, \quad y \in A,$$

is strongly well posed.

Now we will show that Theorem 7.16 implies the following result.

Theorem 7.19 *Assume that $g : X \to R^1$ is a convex function which is Lipschitzian on bounded subsets of X and that $\inf\{g(x) : x \in X\}$ is attained at a unique point $y_* \in X$, $\lim_{\|u\| \to \infty} g(u) = \infty$, and if $\{y_i\}_{i=1}^{\infty} \subset X$ and $\lim_{i \to \infty} g(y_i) = g(y_*)$, then $y_i \to y_*$ as $i \to \infty$. Then there exists a set $\Omega \subset S(X)$ such that its complement $S(X) \setminus \Omega$ is σ-porous with respect to $(h, \tilde{H})$ and for each $A \in \Omega$, the following property holds*:

There is a unique $y_A \in A$ such that $g(y_A) = \inf\{g(y) : y \in A\}$. Moreover, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $y \in A$ satisfies $g(y) \le g(y_A) + \delta$, then $\|y - y_A\| \le \varepsilon$.

Proof Define $f(x) = g(-x)$, $x \in X$. It is clear that f is convex and satisfies (7.122)–(7.126). Therefore Theorem 7.16 is valid with $\tilde{x} = 0$ and there exists a set $\Omega \subset S(X)$ such that its complement $S(X) \setminus \Omega$ is σ-porous with respect to $(h, \tilde{H})$ and for each $A \in \Omega$, the following property holds:

There is a unique $\tilde{y} \in A$ such that

$$g(\tilde{y}) = f(-\tilde{y}) = \inf\bigl\{f(-y) : y \in A\bigr\} = \inf\bigl\{g(y) : y \in A\bigr\}.$$

Moreover, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $B \in S(X)$ satisfies $h(A, B) \le \delta$ and $x \in B$ satisfies

$$g(x) = f(-x) \le \rho_f(0, B) + \delta = \inf\bigl\{f(-y) : y \in B\bigr\} + \delta = \inf\bigl\{g(y) : y \in B\bigr\} + \delta,$$

then $\|x - \tilde{y}\| \le \varepsilon$. Theorem 7.19 is proved. □

It is easy to see that in the proofs of Theorems 7.16–7.18 we may assume without any loss of generality that $\inf\{f(x) : x \in X\} = 0$. It is also not difficult to see that we may assume without loss of generality that $x_* = 0$. Indeed, instead of the function

$f(\cdot)$ we can consider $f(\cdot + x_*)$. This new function also satisfies (7.122)–(7.126). Once Theorems 7.16–7.18 are proved for this new function, they will also hold for the original function f because the mapping $(A, x) \to (A, x + x_*)$, $(A, x) \in S(X) \times X$, is an isometry with respect to both metrics d_1 and d_2.

7.12 A Basic Lemma

Let m and n be two natural numbers. Choose a number

$$c_m > \sup\{f(u) : u \in X \text{ and } \|u\| \le 2m + 4\} + 2 \tag{7.129}$$

(see (7.126)). By (7.123), there exists a natural number

$$a_m > m + 2$$

such that

$$\text{if } u \in X \text{ and } f(u) \le c_m, \text{ then } \|u\| \le a_m. \tag{7.130}$$

By (7.126), there is $k_m > 1$ such that

$$|f(x) - f(y)| \le k_m \|x - y\| \\ \text{for each } x, y \in X \text{ satisfying } \|x\|, \|y\| \le 4a_m + 4. \tag{7.131}$$

By (7.131), there exists a positive number

$$\alpha(m, n) < 2^{-4a_m - 4} 16^{-1} n^{-1} \tag{7.132}$$

such that

$$\text{if } u \in X \text{ satisfies } f(u) \le 320 a_m \alpha(m, n), \text{ then } \|u\| \le (4n)^{-1}. \tag{7.133}$$

Finally, we choose a positive number

$$\bar{\alpha}(m, n) < \alpha(m, n)\left[(k_m + 1)^{-1} 2^{-4a_m - 16}\right]. \tag{7.134}$$

Lemma 7.20 *Let*

$$\alpha = \alpha(m, n), \qquad \bar{\alpha} = \bar{\alpha}(m, n), \tag{7.135}$$

$A \in S(X)$, $\tilde{x} \in X$, $r \in (0, 1]$, *and assume that*

$$\|\tilde{x}\| \le m \quad \textit{and} \quad \{z \in X : \|z\| \le m\} \cap A \ne \emptyset. \tag{7.136}$$

Then there exists $\bar{x} \in X$ *such that*

$$\rho(\bar{x}, A) \le r/8 \tag{7.137}$$

and for the set $\tilde{A} := A \cup \{\bar{x}\}$, *the following property holds*:

If

$$B \in S(X), \quad h(\tilde{A}, B) \le \bar{\alpha} r, \tag{7.138}$$

$$\tilde{y} \in X, \quad \|\tilde{y} - \bar{x}\| \le \bar{\alpha} r, \tag{7.139}$$

and

$$z \in B, \quad f(\tilde{y} - z) \le \rho_f(\tilde{y}, B) + \alpha r, \tag{7.140}$$

then

$$h(A, B) \le r \tag{7.141}$$

and

$$\|z - \bar{x}\| \le n^{-1}. \tag{7.142}$$

Proof First we choose $\tilde{x} \in X$. There are two cases: (1) $\rho(\tilde{x}, A) \le r/8$; (2) $\rho(\tilde{x}, A) > r/8$. If

$$\rho(\tilde{x}, A) \le r/8, \tag{7.143}$$

then we set

$$\bar{x} = \tilde{x} \quad \text{and} \quad \tilde{A} = A \cup \{\tilde{x}\}. \tag{7.144}$$

Now consider the second case where

$$\rho(\tilde{x}, A) > r/8. \tag{7.145}$$

First, choose $x_0 \in A$ such that

$$f(\tilde{x} - x_0) \le \rho_f(\tilde{x}, A) + \alpha(m, n)r \tag{7.146}$$

and then choose

$$\bar{x} \in \left\{\gamma \tilde{x} + (1 - \gamma)x_0 : \gamma \in (0, 1)\right\} \tag{7.147}$$

such that

$$\|\bar{x} - x_0\| = r/8 \quad \text{and} \quad \|\tilde{x} - \bar{x}\| = \|\tilde{x} - x_0\| - r/8. \tag{7.148}$$

Finally, set

$$\tilde{A} = A \cup \{\bar{x}\}. \tag{7.149}$$

Clearly, there is $\gamma \in (0, 1)$ such that

$$\bar{x} = \gamma \tilde{x} + (1 - \gamma)x_0. \tag{7.150}$$

It is easy to see that in both cases (7.137) holds and

$$\tilde{H}(A, \tilde{A}) \le H(A, \tilde{A}) \le r/8. \tag{7.151}$$

Now assume that $z \in X$ satisfies

$$z \in \tilde{A} \quad \text{and} \quad f(\tilde{x} - z) \le \rho_f(\tilde{x}, \tilde{A}) + 8\alpha(m,n)r. \tag{7.152}$$

We will show that $\|\bar{x} - z\| \le (2n)^{-1}$. First consider case (1). Then by (7.152), (7.144) and (7.149),

$$f(\bar{x} - z) = f(\tilde{x} - z) \le 8\alpha(m,n)r.$$

When combined with (7.133), this inequality implies that

$$\|\bar{x} - z\| \le (4n)^{-1}.$$

Now consider case (2). We first estimate $f(\tilde{x} - \bar{x})$. By (7.150) and (7.125) (with $x_* = 0$ and $f(x_*) = 0$),

$$\begin{aligned} f(\tilde{x} - \bar{x}) &= f\big(\tilde{x} - \gamma\tilde{x} - (1-\gamma)x_0\big) \\ &= f\big((1-\gamma)(\tilde{x} - x_0)\big) \le (1-\gamma) f(\tilde{x} - x_0). \end{aligned} \tag{7.153}$$

By (7.136), there is $z_0 \in X$ such that

$$z_0 \in A \quad \text{and} \quad \|z_0\| \le m. \tag{7.154}$$

Thus (7.146), (7.132), (7.154) and (7.136) imply that

$$\begin{aligned} f(\tilde{x} - x_0) &\le \rho_f(\tilde{x}, \tilde{A}) + 1 \le f(\tilde{x} - z_0) + 1 \\ &\le \sup\big\{f(u) : u \in X,\ \|u\| \le 2m+1\big\} + 1 < c_m. \end{aligned} \tag{7.155}$$

Relations (7.155) and (7.130) imply that

$$\|x_0 - \tilde{x}\| \le a_m. \tag{7.156}$$

It follows from (7.148), (7.150) and (7.156) that

$$\begin{aligned} \|\tilde{x} - x_0\| - r/8 &= \|\tilde{x} - \bar{x}\| = \big\|\tilde{x} - \gamma\tilde{x} - (1-\gamma)x_0\big\| \\ &= (1-\gamma)\|\tilde{x} - x_0\|, \end{aligned}$$

$$\begin{aligned} 1 - \gamma &= \big(\|\tilde{x} - x_0\| - r/8\big)\|\tilde{x} - x_0\|^{-1} = 1 - r\big(8\|\tilde{x} - x_0\|\big)^{-1} \\ &\le 1 - r(8a_m)^{-1} \end{aligned}$$

and that

$$1 - \gamma \le 1 - r(8a_m)^{-1}. \tag{7.157}$$

By (7.153) and (7.157),

$$f(\tilde{x} - \bar{x}) = (1-\gamma) f(\tilde{x} - x_0) \le \big(1 - r(8a_m)^{-1}\big) f(\tilde{x} - x_0). \tag{7.158}$$

Relations (7.152) and (7.158) now imply that

$$f(\tilde{x}-z) \le f(\tilde{x}-\bar{x}) + 8\alpha r \le 8\alpha r + \big(1 - r(8a_m)^{-1}\big) f(\tilde{x}-x_0). \tag{7.159}$$

There are two cases:

$$f(\tilde{x}-x_0) \ge 8 \cdot 18\alpha a_m \tag{7.160}$$

and

$$f(\tilde{x}-x_0) \le 8 \cdot 18\alpha a_m. \tag{7.161}$$

Assume that (7.160) holds. Then it follows from (7.159), (7.146) and (7.160) that

$$\begin{aligned} f(\tilde{x}-z) &\le 8\alpha r + f(\tilde{x}-x_0) - r(8a_m)^{-1} f(\tilde{x}-x_0) \\ &\le 8\alpha r + \rho_f(\tilde{x}, A) + \alpha r - 8^{-1} \cdot 18\alpha r < \rho_f(\tilde{x}, A). \end{aligned}$$

Thus $z \notin A$ and by (7.152) and (7.149),

$$z = \bar{x}. \tag{7.162}$$

Now assume that (7.161) is true. By (7.161) and (7.152),

$$f(\tilde{x}-z) \le f(\tilde{x}-x_0) + 8\alpha r \le 8 \cdot 18\alpha a_m + 8\alpha \le 160\alpha a_m.$$

When combined with (7.133), (7.148) and (7.161), this estimate implies that

$$\|\tilde{x}-z\| \le (4n)^{-1}, \qquad \|\tilde{x}-x_0\| \le (4n)^{-1},$$
$$\|\tilde{x}-\bar{x}\| < \|\tilde{x}-x_0\| < (4n)^{-1},$$

and

$$\|\bar{x}-z\| < (2n)^{-1}.$$

Thus in both cases,

$$\|\bar{x}-z\| < (2n)^{-1}.$$

In other words, we have shown that the following property holds:

(P1) If $z \in X$ satisfies (7.152), then $\|\bar{x}-z\| \le (2n)^{-1}$.

Now assume that (7.138)–(7.140) hold. By (7.136) and (7.139), we have

$$\|\tilde{x}\| \le m \quad \text{and} \quad \|\tilde{y}\| \le m+1. \tag{7.163}$$

Relation (7.136) implies that there is $z_0 \in X$ such that

$$z_0 \in A \quad \text{and} \quad \|z_0\| \le m. \tag{7.164}$$

It follows from (7.128), (7.138), (7.164), (7.134) and (7.128) that

$$h_{4a_m+4}(\tilde{A}, B)\big(1 + h_{4a_m+4}(\tilde{A}, B)\big)^{-1} \le 2^{4a_m+4} h(\tilde{A}, B) \le 2^{4a_m+4}\bar{\alpha} r,$$
$$h_{4a_m+4}(\tilde{A}, B) \le 2^{4a_m+4}\bar{\alpha} r\big(1 - 2^{4a_m+4}\bar{\alpha} r\big) \le 2^{4a_m+5}\bar{\alpha} r \tag{7.165}$$

and

$$\rho(z_0, B) \le \rho(z_0, \tilde{A}) + \big|\rho(z_0, B) - \rho(z_0, \tilde{A})\big|$$
$$\le h_{4a_m+4}(\tilde{A}, B) \le 2^{4a_m+5}\bar{\alpha} r. \tag{7.166}$$

Inequalities (7.166), (7.134) and (7.132) imply that $\rho(z_0, B) < 1$, and that there is $\tilde{z}_0 \in X$ such that

$$\tilde{z}_0 \in B \quad \text{and} \quad \|\tilde{z}_0 - z_0\| < 1. \tag{7.167}$$

Clearly, by (7.164) and (7.167),

$$\|\tilde{z}_0\| < m + 1. \tag{7.168}$$

Let

$$\big\{(L, l)\big\} \in \big\{(\tilde{A}, \tilde{x}), (B, \tilde{y})\big\}. \tag{7.169}$$

By (7.136), (7.163), (7.164), (7.168) and (7.167),

$$\|l\| \le m + 1 \tag{7.170}$$

and there is $\bar{u} \in X$ such that

$$\bar{u} \in L \quad \text{and} \quad \|\bar{u}\| \le m + 1. \tag{7.171}$$

Relations (7.171), (7.170) and (7.129) imply that

$$\rho_f(l, L) \le f(l - \bar{u}) \le \sup\big\{f(u) : u \in X, \|u\| \le 2m + 2\big\} \le c_m - 2. \tag{7.172}$$

Also, relations (7.172), (7.130) and (7.170) imply the following property:

(P2) If $u \in L$ and $f(l - u) \le \rho_f(l, L) + 2$, then $\|l - u\| \le a_m$ and $\|u\| \le \|l\| + a_m \le 2a_m$.

Now assume that $L_i \in S(X)$ and $l_i \in X$, $i = 1, 2$, satisfy

$$\big\{(L_1, l_1), (L_2, l_2)\big\} = \big\{(\tilde{A}, \tilde{x}), (B, \tilde{y})\big\}. \tag{7.173}$$

Let

$$u \in L_1 \text{ be such that } f(l_1 - u) \le \rho_f(l_1, L_1) + 2. \tag{7.174}$$

By (7.174), (7.173) and property (P2),

$$\|u\| \le 2a_m. \tag{7.175}$$

Relations (7.174), (7.173), (7.175), (7.165) and (7.128) imply that

$$\rho(u, L_2) = \left|\rho(u, L_1) - \rho(u, L_2)\right| \le h_{2a_m}(L_1, L_2)$$
$$\le h_{4a_m+4}(\tilde{A}, B) \le 2^{4a_m+5}\bar{\alpha}r.$$

When combined with (7.132) and (7.134), this inequality implies that there is $v \in X$ such that

$$v \in L_2 \quad \text{and} \quad \|u - v\| \in 2^{4a_m+6}\bar{\alpha}r \le 1. \tag{7.176}$$

Inequalities (7.175) and (7.176) imply that

$$\|v\| \le 1 + 2a_m. \tag{7.177}$$

By (7.177), (7.175), (7.173) and (7.163),

$$\|l_1 - u\|, \|l_2 - v\| \le 1 + 2a_m + m + 1 < 3a_m. \tag{7.178}$$

It follows from (7.176), (7.139) and (7.173) that

$$\left\|(l_1 - u) - (l_2 - v)\right\| \le \bar{\alpha}r + 2^{4a_m+6}\bar{\alpha}r. \tag{7.179}$$

By (7.179), (7.178), (7.134) and the definition of k_m (see (7.131)),

$$\left|f(l_1 - u) - f(l_2 - v)\right| \le k_m\left\|(l_1 - u) - (l_2 - v)\right\|$$
$$\le k_m\bar{\alpha}r\left(1 + 2^{4a_m+6}\right) \le r\alpha 2^{-9}. \tag{7.180}$$

Inequalities (7.180) and (7.176) imply that

$$\rho_f(l_2, L_2) \le f(l_2 - v) \le f(l_1 - u) + 2^{-9}\alpha r$$

and

$$\rho_f(l_2, L_2) \le 2^{-9}\alpha r + f(l_1 - u). \tag{7.181}$$

Since (7.181) holds for any u satisfying (7.174), we conclude that

$$\rho_f(l_2, L_2) \le 2^{-9}\alpha r + \rho_f(l_1, L_1).$$

This fact implies, in turn, that

$$\left|\rho_f(l_1, L_1) - \rho_f(l_2, L_2)\right| = \left|\rho_f(\tilde{x}, \tilde{A}) - \rho_f(\tilde{y}, B)\right| \le 2^{-9}\alpha r. \tag{7.182}$$

By property (P2), (7.169) and (7.140),

$$\|\tilde{y} - z\| \le a_m \quad \text{and} \quad \|z\| \le 2a_m. \tag{7.183}$$

It follows from (7.140), (7.183), (7.165) and (7.128) that

$$\rho(z,\tilde{A}) \le \rho(z,B) + \left|\rho(z,B) - \rho(z,\tilde{A})\right|$$
$$= \left|\rho(z,B) - \rho(z,\tilde{A})\right| \le h_{4a_m+4}(\tilde{A},B) \le 2^{4a_m+5}\bar{\alpha}r.$$

Thus there exists $\tilde{z} \in X$ such that

$$\tilde{z} \in \tilde{A} \quad \text{and} \quad \|z - \tilde{z}\| \le 2^{4a_m+6}\bar{\alpha}r. \tag{7.184}$$

By (7.136), (7.183), (7.184), (7.134) and (7.132), we have

$$\|\tilde{x} - \tilde{z}\| \le \|\tilde{x}\| + \|\tilde{z}\| \le m + \|z\| + \|\tilde{z} - z\|$$
$$\le m + 2a_m + 2^{4a_m+6}\bar{\alpha}r \le 3a_m + 1.$$

When combined with (7.134), (7.184), (7.139), (7.140) and (7.182), this inequality implies that

$$f(\tilde{x} - \tilde{z}) \le f(\tilde{y} - z) + \left|f(\tilde{x} - \tilde{z}) - f(\tilde{y} - z)\right|$$
$$\le f(\tilde{y} - z) + k_m\left\|\tilde{x} - \tilde{z} - (\tilde{y} - z)\right\| \le f(\tilde{y} - z)$$
$$\le k_m\|\tilde{x} - \tilde{y}\| + k_m\|\tilde{z} - z\| \le f(\tilde{y} - z) + k_m\bar{\alpha}r + k_m 2^{4a_m+6}\bar{\alpha}r$$
$$\le \rho_f(\tilde{y}, B) + \alpha r + k_m\bar{\alpha}r\left(1 + 2^{4a_m+6}\right)$$
$$\le \alpha r + k_m\bar{\alpha}r\left(1 + 2^{4a_m+6}\right) + \rho_f(\tilde{x},\tilde{A}) + 2^{-9}\alpha r \le \alpha r + \alpha r + \rho_f(\tilde{x},\tilde{A}).$$

Thus we see that

$$f(\tilde{x} - \tilde{z}) \le \rho_f(\tilde{x},\tilde{A}) + 2\alpha r. \tag{7.185}$$

It follows from property (P1), (7.152), (7.185) and (7.184) that

$$\|\tilde{z} - \bar{x}\| \le (2n)^{-1}.$$

When combined with (7.184), (7.134) and (7.132), this inequality implies that

$$\|z - \bar{x}\| \le \|z - \tilde{z}\| + \|\tilde{z} - \bar{x}\| \le 2^{4a_m+6}\bar{\alpha}r + (2n)^{-1} \le n^{-1}.$$

Thus (7.142) is proved. Inequality (7.141) follows from (7.138), (7.151), (7.134) and (7.132). Thus we have shown that (7.138)–(7.140) imply (7.141) and (7.142). Lemma 7.20 is proved. □

7.13 Proofs of Theorems 7.16–7.18

We use the notations and the definitions from the previous section.

For each natural number n, denote by $\mathcal{F}_n$ the set of all $(x, A) \in X \times S(X)$ such that the following property holds:

(P3) There exist $y \in A$ and $\delta > 0$ such that for each $\tilde{x} \in X$ satisfying $\|\tilde{x} - x\| \le \delta$, each $B \in S(X)$ satisfying $h(A, B) \le \delta$, and each $z \in B$ satisfying $f(\tilde{x} - z) \le \rho_f(\tilde{x}, B) + \delta$, the inequality $\|z - y\| \le n^{-1}$ holds.

Set

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n. \tag{7.186}$$

Lemma 7.21 *If*

$$(x, A) \in \mathcal{F}, \tag{7.187}$$

then the problem $f(x - y) \to \min$, $y \in A$, *is strongly well posed.*

Proof Let $(x, A) \in \mathcal{F}$ and let n be a natural number. Since $(x, A) \in \mathcal{F} \subset \mathcal{F}_n$, there exist $x_n \in A$ and $\delta_n > 0$ such that the following property holds:

(P4) For each $\tilde{x} \in X$ satisfying $\|\tilde{x} - x\| \le \delta_n$, each $B \in S(X)$ satisfying $h(A, B) \le \delta_n$, and each $z \in B$ satisfying $f(\tilde{x} - z) \le \rho_f(\tilde{x}, B) + \delta_n$, the inequality $\|z - x_n\| \le n^{-1}$ holds.

Suppose that

$$\{z_i\}_{i=1}^{\infty} \subset A \quad \text{and} \quad \lim_{i \to \infty} f(x - z_i) = \rho_f(x, A). \tag{7.188}$$

Let n be any natural number. By (7.188) and property (P4), for all sufficiently large i we have

$$f(x - z_i) \le \rho_f(x, A) + \delta_n \quad \text{and} \quad \|z_i - x_n\| \le n^{-1}. \tag{7.189}$$

The second inequality of (7.189) implies that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence and there exists

$$\bar{x} = \lim_{i \to \infty} z_i. \tag{7.190}$$

Limits (7.190) and (7.188) imply that

$$f(x - \bar{x}) = \rho_f(x, A).$$

Clearly, $\bar{x}$ is the unique solution of the problem $f(x - z) \to \min$, $z \in A$. Otherwise we would be able to construct a nonconvergent sequence $\{z_i\}_{i=1}^{\infty}$ satisfying (7.188). By (7.190) and (7.189),

$$\|\bar{x} - x_n\| \le n^{-1}, \quad n = 1, 2, \ldots. \tag{7.191}$$

Let $\varepsilon > 0$ be given. Choose a natural number

$$n > 8\varepsilon^{-1}. \tag{7.192}$$

Assume that

$$\tilde{x} \in X, \quad \|\tilde{x} - x\| \le \delta_n, \qquad B \in S(X), \quad h(A, B) \le \delta_n,$$
$$z \in B, \quad \text{and} \quad f(\tilde{x} - z) \le \rho_f(\tilde{x}, B) + \delta_n.$$

By Property (P4), $\|z - x_n\| \le 1/n$. When combined with (7.192) and (7.191), this inequality implies that

$$\|z - \bar{x}\| \le \|z - x_n\| + \|x_n - \bar{x}\| \le (2n)^{-1} < \varepsilon.$$

Thus the problem $f(x - z) \to \min$, $z \in A$, is strongly well posed. Lemma 7.21 is proved. □

Proof of Theorem 7.16 For each integer $n \ge 1$, set

$$\Omega_n := \{A \in S(X) : (\tilde{x}, A) \in \mathcal{F}_n\} \tag{7.193}$$

and let

$$\Omega := \bigcap_{n=1}^{\infty} \Omega_n. \tag{7.194}$$

By Lemma 7.21, (7.193) and (7.194), for each $A \in \Omega$, the problem $f(\tilde{x} - z) \to \min$, $z \in A$, is strongly well posed. In order to prove the theorem, it is sufficient to show that for each natural number n, the set $S(X) \setminus \Omega_n$ is σ-porous with respect to $(h, \tilde{H})$. To this end, let n be any natural number.

Fix a natural number

$$m_0 > \|\tilde{x}\|. \tag{7.195}$$

For each integer $m \ge m_0$, define

$$E_m := \{A \in S(X) : A \cap \{z \in X : \|z\| \le m\} \ne \emptyset\}. \tag{7.196}$$

Since

$$S(X) \setminus \Omega_n = \bigcup_{m=m_0}^{\infty} (E_m \setminus \Omega_n),$$

in order to prove the theorem, it is sufficient to show that for any natural number $m \ge m_0$, the set $E_m \setminus \Omega_n$ is porous with respect to $(h, \tilde{H})$. Let $m \ge m_0$ be a natural number. Define

$$\alpha_* = \bar{\alpha}(m + 1, n)/2 \tag{7.197}$$

(see (7.132) and (7.134)). Let $A \in S(X)$ and $r \in (0, 1]$. There are two cases:

case (1), where

$$A \cap \{z \in X : \|z\| \le m + 1\} = \emptyset \tag{7.198}$$

and case (2), where

$$A \cap \{z \in X : \|z\| \le m+1\} \ne \emptyset. \tag{7.199}$$

Consider the first case.

Let

$$B \in S(X) \text{ be such that } h(A, B) \le 2^{-m-2}. \tag{7.200}$$

We claim that $B \notin E_m$. Assume the contrary. Then there is $u \in X$ such that

$$u \in B \quad \text{and} \quad \|u\| \le m. \tag{7.201}$$

By (7.201) and (7.128),

$$\rho(u, A) \le \rho(u, B) + \left|\rho(u, B) - \rho(u, A)\right| \le h_m(A, B). \tag{7.202}$$

The definition of h_m (see (7.128)) and (7.200) imply that

$$h_m(A, B)\left(1 + h_m(A, B)\right)^{-1} \le h(A, B)2^m \le 2^{-2},$$
$$h_m(A, B) \le h_m(A, B)2^{-2} + 2^{-2}$$

and

$$h_m(A, B) \le 1/3.$$

When combined with (7.202), this implies that there is $v \in A$ such that $\|u - v\| \le 1/2$. Together with (7.201) this inequality implies that $\|v\| \le m + 1/2$, a contradiction (see (7.198)). Therefore $B \notin E_m$, as claimed. Thus we have shown that

$$\{B \in S(X) : h(A, B) \le 2^{-m-2}\} \cap E_m = \emptyset. \tag{7.203}$$

Now consider the second case. Then by Lemma 7.20, (7.195) and (7.199), there exists $\bar{x} \in X$ such that

$$\rho(\bar{x}, A) \le r/8$$

and such that for the set $\tilde{A} = A \cup \{\bar{x}\}$, the following property holds:

(P5) if $B \in S(X)$, $h(\tilde{A}, B) \le \bar{\alpha}(m+1, n)r$, $\tilde{y} \in X$, $\|\tilde{y} - \tilde{x}\| \le \bar{\alpha}(m+1, n)r$, and $z \in B$ satisfies

$$f(\tilde{y} - z) \le \rho_f(\tilde{y}, B) + \bar{\alpha}(m+1, n),$$

then

$$\|z - \bar{x}\| \le n^{-1} \quad \text{and} \quad h(A, B) \le r.$$

Clearly,

$$\tilde{H}(A, \tilde{A}) \le r/8.$$

Property (P5), (7.193) and the definition of $\mathcal{F}_n$ (see (P3)) imply that

$$\{B \in S(X) : h(\tilde{A}, B) \le \bar{\alpha}(m+1, n)r/2\} \subset \Omega_n.$$

Thus in both cases we have

$$\{B \in S(X) : h(\tilde{A}, B) \le \alpha_* r/2\} \cap (E_m \setminus \Omega_n) = \emptyset. \tag{7.204}$$

(Note that in the first case (7.204) is true with $\tilde{A} = A$.)

Therefore we have shown that the set $E_m \setminus \Omega_n$ is porous with respect to $(h, \tilde{H})$. Theorem 7.16 is proved. □

Proof of Theorem 7.17 By Lemma 7.21, in order to prove the theorem, it is sufficient to show that for any natural number n, the set $(X \times S(X)) \setminus \mathcal{F}_n$ is σ-porous in $X \times S(X)$ with respect to $(h, \tilde{H})$. To this end, let n be a natural number. For each natural number m, define

$$E_m = \{(x, A) \in X \times S(X) : \|x\| \le m \text{ and } A \cap \{z \in X : \|z\| \le m\} \ne \emptyset\}. \tag{7.205}$$

Since

$$(X \times S(X)) \setminus \mathcal{F}_n = \bigcup_{m=1}^{\infty} E_m \setminus \mathcal{F}_n,$$

in order to prove the theorem it is sufficient to show that for each natural number m, the set $E_m \setminus \mathcal{F}_n$ is porous in $X \times S(X)$ with respect to $(h, \tilde{H})$.

Let m be a natural number. Define α_* by (7.197). Assume that $(\tilde{x} \times A) \in X \times S(X)$ and $r \in (0, 1]$.

There are three cases:

case (1), where

$$\|\tilde{x}\| > m + 1,$$

case (2), where

$$\|\tilde{x}\| \le m + 1 \quad \text{and} \quad \{z \in A : \|z\| \le m + 1\} = \emptyset, \tag{7.206}$$

and case (3), where

$$\|\tilde{x}\| \le m + 1 \quad \text{and} \quad \{z \in A : \|z\| \le m + 1\} \ne \emptyset. \tag{7.207}$$

In the first case,

$$\{(y, B) \in X \times S(X) : d_1((\tilde{x}, A), (y, B)) \le 2^{-1}\} \cap E_m = \emptyset. \tag{7.208}$$

Next, consider the second case. In the proof of Theorem 7.16 we have shown that

$$\text{if } B \in S(X) \text{ satisfies } h(A, B) \le 2^{-m-2}, \text{ then}$$
$$B \cap \{z \in X : \|z\| \le m\} = \emptyset$$

and

$$\{(y, B) \in X \times S(X) : d_1((y, B), (\tilde{x}, A)) \le 2^{-m-2}\} \cap E_m = \emptyset. \tag{7.209}$$

Finally, consider the third case. Then by Lemma 7.20, there exists $\bar{x} \in X$ such that $\rho(\bar{x}, A) \le r/8$ and such that for the set $\tilde{A} = A \cup \{\bar{x}\}$, property (P5) holds. Clearly,

$$d_2((\tilde{x}, A), (\tilde{x}, \tilde{A})) = \tilde{H}(A, \tilde{A}) \le r/8.$$

Property (P5) implies that

$$\{(\tilde{y}, B) \in X \times S(X) : d_1((\tilde{y}, B), (\tilde{x}, \tilde{A})) \le \bar{\alpha}(m+1, n)r/2\} \subset \mathcal{F}_n.$$

Hence in all three cases we have

$$\{(\tilde{y}, B) \in X \times S(X) : d_1((\tilde{y}, B), (\tilde{x}, \tilde{A})) \le \alpha_* r\} \cap (E_m \setminus \mathcal{F}_n) = \emptyset. \tag{7.210}$$

Note that in the first and second cases, (7.210) is true with $A = \tilde{A}$. Therefore we have shown that the set $E_m \setminus \mathcal{F}_n$ is porous with respect to (d_1, d_2). Theorem 7.17 is proved. □

Proof of Theorem 7.18 Let $\{x_i\}_{i=1}^{\infty}$ be a countable dense subset of X_0. By countable dense subset of X_0. By Theorem 7.16, for each $\mathcal{F}_i \subset S(X)$ such that $S(X) \setminus \mathcal{F}_i$ is σ-porous in $S(X)$ with respect to $(h, \tilde{H})$ and such that for each $A \in S(X)$, the problem $f(x_i - z) \to \min$, $z \in X$, is strongly well posed. Set

$$\mathcal{F} := \bigcap_{i=1}^{\infty} \mathcal{F}_i. \tag{7.211}$$

Clearly, $S(X) \setminus \mathcal{F}$ is a σ-porous subset of $S(X)$ with respect to $(h, \tilde{H})$.

Let $A \in \mathcal{F}$. Assume that n and i are natural numbers. Since the problem $f(x_i - z) \to \min$, $z \in A$, is strongly well posed, there exists a number $\delta_{in} > 0$ and a unique $\bar{x}_i \in A$ such that

$$f(x_i - \bar{x}_i) = \rho_f(x_i, A) \tag{7.212}$$

and the following property holds:

(P6) if $y \in X$ satisfies $\|y - x_i\| \le \delta_{in}$, $B \in S(X)$ satisfies $h(A, B) \le \delta_{in}$, and $z \in B$ satisfies

$$f(y - z) \le \rho_f(y, B) + \delta_{in}, \tag{7.213}$$

then $\|z - \bar{x}_i\| \le (2n)^{-1}$.

Define

$$F = \bigcap_{q=1}^{\infty} \bigcup \{\{z \in X : \|z - x_i\| < \delta_{in}\} : i = 1, 2, \ldots, n = q, q+1, \ldots\} \cap X_0. \tag{7.214}$$

Clearly, F is a countable intersection of open everywhere dense subsets of X_0. Let

$$\tilde{x} \in F. \tag{7.215}$$

For each natural number q, there exist natural numbers $n_q \geq q$ and i_q such that

$$\|\tilde{x} - x_{iq}\| < \delta_{i_q n_q}. \tag{7.216}$$

Assume that

$$\{y_k\}_{k=1}^{\infty} \subset A \quad \text{and} \quad \lim_{k\to\infty} f(\tilde{x} - y_k) = \rho_f(\tilde{x}, A). \tag{7.217}$$

Let q be a natural number. Then for all sufficiently large natural numbers k,

$$f(\tilde{x} - y_k) \leq \rho_f(\tilde{x}, A) + \delta_{i_q n_q},$$

and by property (P6) and (7.216),

$$\|y_k - \bar{x}_{i_q}\| \leq (2n_q)^{-1} \leq (2q)^{-1}. \tag{7.218}$$

This implies that $\{y_k\}_{k=1}^{\infty}$ is a Cauchy sequence and there exists $\bar{x} = \lim_{k\to\infty} y_k$. By (7.217), $f(\tilde{x} - \bar{x}) = \rho_f(\tilde{x}, A)$. Clearly, $\bar{x}$ is the unique minimizer for the problem $f(\tilde{x} - z) \to \min$, $z \in A$. Otherwise, we would be able to construct a nonconvergent sequence $\{y_k\}_{k=1}^{\infty}$. By (7.218),

$$\|\bar{x} - x_{i_q}\| \leq (2q)^{-1}, \quad q = 1, 2, \ldots. \tag{7.219}$$

Let $\varepsilon > 0$ be given. Choose a natural number

$$q > 8\varepsilon^{-1}.$$

Set

$$\delta = \delta_{i_q n_q} - \|\tilde{x} - x_{i_q}\|. \tag{7.220}$$

By (7.216), $\delta > 0$. Assume that

$$y \in X, \quad \|y - \tilde{x}\| \leq \delta, \qquad B \in S(X), \quad h(A, B) \leq \delta, \tag{7.221}$$

and

$$z \in B, \quad f(y - z) \leq \rho_f(y, B) + \delta.$$

By (7.220) and (7.221),

$$\|y - x_{i_q}\| \leq \|y - \tilde{x}\| + \|\tilde{x} - x_{i_q}\| \leq \delta_{i_q n_q}. \tag{7.222}$$

By (7.222), (7.220) and property (P6), $\|z - \bar{x}_{i_q}\| \leq (2q)^{-1}$. When combined with (7.219), this inequality implies that $\|z - \bar{x}\| \leq q^{-1} < \varepsilon$. This completes the proof of Theorem 7.18. □