Chapter 4
Dynamical Systems with Convex Lyapunov Functions

4.1 Minimization of Convex Functionals

In this section, which is based on [128], we consider a metric space of sequences of continuous mappings acting on a bounded, closed and convex subset of a Banach space, which share a common convex Lyapunov function. We show that for a generic sequence taken from that space the values of the Lyapunov function along all trajectories tend to its infimum.

Assume that $(X, \|\cdot\|)$ is a Banach space with norm $\|\cdot\|$, $K \subset X$ is a bounded, closed and convex subset of X, and $f : K \to R^1$ is a convex and uniformly continuous function. Set

$$\inf(f) = \inf\{f(x) : x \in K\}.$$

Observe that this infimum is finite because K is bounded and f is uniformly continuous. We consider the topological subspace $K \subset X$ with the relative topology. Denote by $\mathcal{A}$ the set of all continuous self-mappings $A : K \to K$ such that

$$f(Ax) \le f(x) \quad \text{for all } x \in K. \tag{4.1}$$

Later in this chapter (see Sect. 4.4), we construct many such mappings.

For the set $\mathcal{A}$ we define a metric $\rho : \mathcal{A} \times \mathcal{A} \to R^1$ by

$$\rho(A, B) = \sup\{\|Ax - Bx\| : x \in K\}, \quad A, B \in \mathcal{A}. \tag{4.2}$$

Clearly, the metric space $\mathcal{A}$ is complete. Denote by $\mathcal{M}$ the set of all sequences $\{A_t\}_{t=1}^{\infty} \subset \mathcal{A}$. Members $\{A_t\}_{t=1}^{\infty}$, $\{B_t\}_{t=1}^{\infty}$ and $\{C_t\}_{t=1}^{\infty}$ of $\mathcal{M}$ will occasionally be denoted by boldface **A**, **B** and **C**, respectively. For the set $\mathcal{M}$ we consider the uniformity determined by the following base:

$$E(N, \varepsilon) = \{(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) \in \mathcal{M} \times \mathcal{M} : \rho(A_t, B_t) \le \varepsilon, t = 1, \dots, N\},$$

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where N is a natural number and $\varepsilon > 0$. Clearly the uniform space $\mathcal{M}$ is metrizable (by a metric $\rho_w : \mathcal{M} \times \mathcal{M} \to R^1$) and complete (see [80]).

From the point of view of the theory of dynamical systems, each element of $\mathcal{M}$ describes a nonstationary dynamical system with a Lyapunov function f. Also, some optimization procedures in Banach spaces can be represented by elements of $\mathcal{M}$ (see the first example in Sect. 4.4 and [97, 98]).

In this section we intend to show that for a generic sequence taken from the space $\mathcal{M}$ the values of the Lyapunov function along all trajectories tend to its infimum.

We now present the two main results of this section. They were obtained in [128]. Theorem 4.1 deals with sequences of operators (the space $\mathcal{M}$), while Theorem 4.2 is concerned with the stationary case (the space $\mathcal{A}$).

Theorem 4.1 *There exists a set $\mathcal{F} \subset \mathcal{M}$, which is a countable intersection of open and everywhere dense sets in $\mathcal{M}$, such that for each $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathcal{F}$ the following assertion holds*:

For each $\varepsilon > 0$, there exist a neighborhood U of $\mathbf{B}$ in $\mathcal{M}$ and a natural number N such that for each $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in U$ and each $x \in K$,

$$f(C_N \cdots C_1 x) \le \inf(f) + \varepsilon.$$

Theorem 4.2 *There exists a set $\mathcal{G} \subset \mathcal{A}$, which is a countable intersection of open and everywhere dense sets in $\mathcal{A}$, such that for each $B \in \mathcal{G}$ the following assertion holds*:

For each $\varepsilon > 0$, there exist a neighborhood U of B in $\mathcal{A}$ and a natural number N such that for each $C \in U$ and each $x \in K$,

$$f\left(C^N x\right) \le \inf(f) + \varepsilon.$$

The following proposition is the key auxiliary result which will be used in the proofs of these two theorems.

Proposition 4.3 *There exists a mapping $A_* \in \mathcal{A}$ with the following property*:

Given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \ge \inf(f) + \varepsilon$, the inequality

$$f(A_* x) \le f(x) - \delta(\varepsilon)$$

is true.

Remark 4.4 If there is $x_{min} \in K$ for which $f(x_{min}) = \inf(f)$, then we can set $A_*(x) = x_{min}$ for all $x \in K$.

Section 4.2 contains the proof of Proposition 4.3. Proofs of Theorems 4.1 and 4.2 are given in Sect. 4.3. Section 4.4 is devoted to two examples.

4.2 Proof of Proposition 4.3

By Remark 4.4, we may assume that

$$\{x \in K : f(x) = \inf(f)\} = \emptyset. \tag{4.3}$$

For each $x \in K$, define an integer $p(x) \geq 1$ by

$$p(x) = \min\{i : i \text{ is a natural number and } f(x) \geq \inf(f) + 2^{-i}\}. \tag{4.4}$$

By (4.3), the function $p(x)$ is well defined for all $x \in K$. Now we will define an open covering $\{V_x : x \in K\}$ of K. For each $x \in K$, there is an open neighborhood V_x of x in K such that:

$$|f(y) - f(x)| \leq 8^{-p(x)-1} \quad \text{for all } y \in V_x \tag{4.5}$$

and

$$\text{if } p(x) > 1 \text{ then } f(y) < \inf(f) + 2^{-p(x)+1} \text{ for all } y \in V_x. \tag{4.6}$$

For each $x \in K$, choose $a_x \in K$ such that

$$f(a_x) \leq \inf(f) + 2^{-p(x)-9}. \tag{4.7}$$

Clearly, $\bigcup\{V_x : x \in K\} = K$ and $\{V_x : x \in K\}$ is an open covering of K.

Lemma 4.5 *Let $x \in K$. Then for all $y \in V_x$,*

$$f(y) \geq \inf(f) + 2^{-p(x)-1} \tag{4.8}$$

and

$$|p(y) - p(x)| \leq 1. \tag{4.9}$$

Proof Let $y \in V_x$. Then (4.8) follows from (4.5) and (4.4). The definition of $p(x)$ (see (4.4)) and (4.8) imply that $p(y) \leq p(x) + 1$. Now we will show that $p(y) \geq p(x) - 1$. It is sufficient to consider the case $p(x) > 1$. Then by the definition of V_x (see (4.6)) and (4.4), $f(y) < \inf(f) + 2^{-p(x)+1}$ and $p(y) \geq p(x)$. This completes the proof of the lemma. □

Since metric spaces are paracompact, there is a continuous locally finite partition of unity $\{\phi_x\}_{x \in K}$ on K subordinated to $\{V_x\}_{x \in K}$ (namely, $\operatorname{supp} \phi_x \subset V_x$ for all $x \in K$ and $\sum_{x \in K} \phi_x(y) = 1$ for all $y \in K$).

For $y \in K$, define

$$A_* y = \sum_{x \in K} \phi_x(y) a_x. \tag{4.10}$$

Clearly, the mapping A_* is well defined, $A_*(K) \subset K$ and A_* is continuous.

Lemma 4.6 *For each* $y \in K$,

$$f(A_* y) \le f(y) - 2^{-p(y)-1}. \tag{4.11}$$

Proof Let $y \in K$. There is an open neighborhood U of y in K and $x_1, \ldots, x_n \in K$ such that

$$\{x \in K : \operatorname{supp} \phi_x \cap U \neq \emptyset\} = \{x_i\}_{i=1}^n. \tag{4.12}$$

We have

$$A_* y = \sum_{i=1}^n \phi_{x_i}(y) a_{x_i}. \tag{4.13}$$

We may assume that there is an integer $m \in \{1, \ldots, n\}$ such that

$$\phi_{x_i}(y) > 0 \quad \text{if and only if} \quad 1 \le i \le m. \tag{4.14}$$

By (4.12) and (4.14), $\sum_{i=1}^m \phi_{x_i}(y) = 1$. When combined with (4.13) and (4.14), this implies that

$$f(A_* y) \le \max\big\{f(a_{x_i}) : i = 1, \ldots, m\big\}. \tag{4.15}$$

Let $i \in \{1, \ldots, m\}$. It follows from (4.14) and Lemma 4.5 that

$$y \in \operatorname{supp} \phi_{x_i} \subset V_{x_i} \quad \text{and} \quad \big|p(y) - p(x_i)\big| \le 1. \tag{4.16}$$

By (4.7) and (4.16),

$$f(a_{x_i}) \le \inf(f) + 2^{-p(x_i)-9} \le \inf(f) + 2^{-p(y)-8}.$$

Thus, by (4.15),

$$f(A_* y) \le \inf(f) + 2^{-p(y)-8}. \tag{4.17}$$

On the other hand, by (4.4), $f(y) \ge \inf(f) + 2^{-p(y)}$. Together with (4.17) this implies (4.11). The lemma is proved. □

Completion of the proof of Proposition 4.3 Clearly, $A_* \in \mathcal{A}$. Let $\varepsilon > 0$ be given. Choose an integer $j \ge 1$ such that $2^{-j} < \varepsilon$.

Let $x \in K$ satisfy $f(x) \ge \inf(f) + \varepsilon$. Then by (4.4), $p(x) \le j$ and by Lemma 4.6,

$$f(A_* x) \le f(x) - 2^{-p(x)-1} \le f(x) - 2^{-j-1}.$$

This completes the proof of the proposition (with $\delta(\varepsilon) = 2^{-j-1}$). □

Remark 4.7 As a matter of fact, if $\varepsilon \in (0, 1)$, then the proof of Proposition 4.3 shows that it holds with $\delta(\varepsilon) = \varepsilon/4$.

4.3 Proofs of Theorems 4.1 and 4.2

Set

$$r_K = \sup\{\|x\| : x \in K\} \quad \text{and} \quad d_0 = \sup\{|f(x)| : x \in K\}. \tag{4.18}$$

Let $A_* \in \mathcal{A}$ be one of the mappings the existence of which is guaranteed by Proposition 4.3. For each $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ and each $\gamma \in (0, 1)$, we define a sequence of mappings $A_t^{\gamma} : K \to K$, $t = 1, 2, \dots$, by

$$A_t^{\gamma} x = (1-\gamma)A_t x + \gamma A_* x, \quad x \in K, t = 1, 2, \dots. \tag{4.19}$$

It is easy to see that for each $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ and each $\gamma \in (0, 1)$,

$$\{A_t^{\gamma}\}_{t=1}^{\infty} \in \mathcal{M} \quad \text{and} \quad \rho(A_t^{\gamma}, A_t) \le 2\gamma r_K, \quad t = 1, 2, \dots. \tag{4.20}$$

We may assume that the function $\delta(\varepsilon)$ of Proposition 4.3 satisfies $\delta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.

Lemma 4.8 *Assume that $\varepsilon, \gamma \in (0, 1)$, $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ and let an integer $N \ge 4$ satisfy*

$$2^{-1} N\gamma\delta(\varepsilon) > 2d_0 + 1. \tag{4.21}$$

Then there exists a number $\Delta > 0$ such that for each sequence $\{B_t\}_{t=1}^{N} \subset \mathcal{A}$ satisfying

$$\rho(B_t, A_t^{\gamma}) \le \Delta, \quad t = 1, \dots, N, \tag{4.22}$$

it follows that, for each $x \in K$,

$$f(B_N \cdots B_1 x) \le \inf(f) + \varepsilon. \tag{4.23}$$

Proof Since the function f is uniformly continuous, there is $\Delta \in (0, 16^{-1}\delta(\varepsilon))$ such that

$$|f(y_1) - f(y_2)| \le 16^{-1}\gamma\delta(\varepsilon) \tag{4.24}$$

for each $y_1, y_2 \in K$ satisfying $\|y_1 - y_2\| \le \Delta$.

Assume that $\{B_t\}_{t=1}^{N} \subset \mathcal{A}$ satisfies (4.22) and that $x \in K$. We now show that (4.23) holds.

Assume the contrary. Then

$$f(x) > \inf(f) + \varepsilon \quad \text{and} \quad f(B_n \cdots B_1 x) > \inf(f) + \varepsilon, \quad n = 1, \dots, N. \tag{4.25}$$

Set

$$x_0 = x, \qquad x_{t+1} = B_{t+1} x_t, \quad t = 0, 1, \dots, N-1. \tag{4.26}$$

For each $t \geq 0$ satisfying $t \leq N-1$, it follows from (4.22), (4.26) and the definition of Δ (see (4.24)) that

$$\left\| B_{t+1}x_t - A^{\gamma}_{t+1}x_t \right\| \leq \Delta \tag{4.27}$$

and

$$\begin{aligned}\left| f(x_{t+1}) - f\left(A^{\gamma}_{t+1}x_t\right)\right| &= \left| f(B_{t+1}x_t) - f\left(A^{\gamma}_{t+1}x_t\right)\right| \\ &\leq 16^{-1}\gamma\delta(\varepsilon).\end{aligned} \tag{4.28}$$

By (4.19), (4.25), (4.26), the definition of $\delta(\varepsilon)$ and the properties of the mapping A_*, we have for each $t = 0, \dots, N-1$,

$$\begin{aligned}f\left(A^{\gamma}_{t+1}x_t\right) &= f\left((1-\gamma)A_{t+1}x_t + \gamma A_* x_t\right) \\ &\leq (1-\gamma)f(A_{t+1}x_t) + \gamma f(A_* x_t) \leq (1-\gamma)f(x_t) + \gamma\left(f(x_t) - \delta(\varepsilon)\right) \\ &= f(x_t) - \gamma\delta(\varepsilon).\end{aligned}$$

Together with (4.28) this implies that for $t = 0, \dots, N-1$,

$$f(x_{t+1}) \leq 16^{-1}\gamma\delta(\varepsilon) + f(x_t) - \gamma\delta(\varepsilon).$$

By induction we can show that for all $t = 1, \dots, N$,

$$f(x_t) \leq f(x_0) - 2^{-1}\gamma\delta(\varepsilon)t.$$

Together with (4.21) and (4.18) this implies that

$$\begin{aligned}f(B_N \cdots B_1 x) = f(x_N) &\leq f(x_0) - 2^{-1}N\gamma\delta(\varepsilon) \\ &\leq d_0 - 2^{-1}N\gamma\delta(\varepsilon) \leq -d_0 - 1 \leq \inf(f) - 1.\end{aligned}$$

This obvious contradiction proves (4.23) and the lemma itself. □

By Lemma 4.8, for each $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}$, each $\gamma \in (0,1)$ and each integer $q \geq 1$, there exist an integer $N(\mathbf{A}, \gamma, q) \geq 4$ and an open neighborhood $U(\mathbf{A}, \gamma, q)$ of $\{A^{\gamma}_t\}_{t=1}^{\infty}$ in $\mathcal{M}$ such that the following property holds:

(a) For each $\{B_t\}_{t=1}^{\infty} \in U(\mathbf{A}, \gamma, q)$ and each $x \in K$,

$$f(B_{N(\mathbf{A},\gamma,q)} \cdots B_1 x) \leq \inf(f) + 4^{-q}.$$

Proof of Theorem 4.1 It follows from (4.20) that the set

$$\left\{\{A^{\gamma}_t\}_{t=1}^{\infty} : \{A_t\}_{t=1}^{\infty} \in \mathcal{M}, \gamma \in (0,1)\right\}$$

is everywhere dense in $\mathcal{M}$. Define

$$\mathcal{F} = \bigcap_{q=1}^{\infty} \bigcup \left\{U(\mathbf{A}, \gamma, q) : \mathbf{A} \in \mathcal{M}, \gamma \in (0,1)\right\}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open and everywhere dense sets in $\mathcal{M}$.

Assume that $\{B_t\}_{t=1}^{\infty} \in \mathcal{F}$ and that $\varepsilon > 0$. Choose an integer $q \geq 1$ such that

$$4^{-q} < \varepsilon. \tag{4.29}$$

There exist $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ and $\gamma \in (0, 1)$ such that

$$\{B_t\}_{t=1}^{\infty} \in U\big(\{A_t\}_{t=1}^{\infty}, \gamma, q\big). \tag{4.30}$$

It follows from (4.29) and property (a) that for each $\{C_t\}_{t=1}^{\infty} \in U(\mathbf{A}, \gamma, q)$ and each $x \in K$,

$$f(C_{N(\mathbf{A},\gamma,q)} \cdots C_1 x) \leq \inf(f) + 4^{-q} < \inf(f) + \varepsilon.$$

This completes the proof of Theorem 4.1. □

Proof of Theorem 4.2 For each $A \in \mathcal{A}$, define

$$\widehat{A}_t = A, \quad t = 1, 2, \ldots. \tag{4.31}$$

Clearly, $\{\widehat{A}_t\}_{t=1}^{\infty} \in \mathcal{M}$ for $A \in \mathcal{A}$, and for each $A \in \mathcal{A}$ and each $\gamma \in (0, 1)$,

$$\widehat{A}_t^{\gamma} x = (1 - \gamma)Ax + \gamma A_* x, \quad x \in K, t = 1, 2, \ldots \tag{4.32}$$

(see (4.19)). By property (a) (which follows from Lemma 4.8), for each $A \in \mathcal{A}$, each $\gamma \in (0, 1)$ and each integer $q \geq 1$, there exist an integer $N(A, \gamma, q) \geq 4$ and an open neighborhood $U(A, \gamma, q)$ of the mapping $(1 - \gamma)A + \gamma A_*$ in $\mathcal{A}$ such that the following property holds:

(b) For each $B \in U(A, \gamma, q)$ and each $x \in K$,

$$f\big(B^{N(A,\gamma,q)} x\big) \leq \inf(f) + 4^{-q}.$$

Clearly, the set

$$\big\{(1 - \gamma)A + \gamma A_* : A \in \mathcal{A}, \gamma \in (0, 1)\big\}$$

is everywhere dense in $\mathcal{A}$. Define

$$\mathcal{G} = \bigcap_{q=1}^{\infty} \bigcup \big\{U(A, \gamma, q) : A \in \mathcal{A}, \gamma \in (0, 1)\big\}.$$

It is clear that $\mathcal{G}$ is a countable intersection of open and everywhere dense sets in $\mathcal{A}$. Assume that $B \in \mathcal{G}$ and $\varepsilon > 0$. Choose an integer $q \geq 1$ such that (4.29) is valid. There exist $A \in \mathcal{A}$ and $\gamma \in (0, 1)$ such that $B \in U(A, \gamma, q)$. It now follows from (4.29) and property (b) that for each $C \in U(A, \gamma, q)$ and each $x \in K$,

$$f\big(C^{N(A,\gamma,q)} x\big) \leq \inf(f) + 4^{-q} < \inf(f) + \varepsilon.$$

Theorem 4.2 is established. □

4.4 Examples

Let $(X, \|\cdot\|)$ be a Banach space. In this section we consider examples of continuous mappings $A : K \to K$ satisfying $f(Ax) \le f(x)$ for all $x \in K$, where K is a bounded, closed and convex subset of X and $f : K \to R^1$ is a convex function.

Example 4.9 Let $f : X \to R^1$ be a convex uniformly continuous function satisfying

$$f(x) \to \infty \quad \text{as } \|x\| \to \infty.$$

Evidently, the function f is bounded from below. For each real number c, let $K_c = \{x \in X : f(x) \le c\}$. Fix a real number c such that $K_c \ne \emptyset$. Clearly, the set K_c is bounded, closed and convex. We assume that the function f is strictly convex on K_c, namely,

$$f\big(\alpha x + (1-\alpha)y\big) < \alpha f(x) + (1-\alpha) f(y)$$

for all $x, y \in K_c$, $x \ne y$, and all $\alpha \in (0, 1)$.

Let $V : K_c \to X$ be any continuous mapping. For each $x \in K_c$, there is a unique solution of the following minimization problem:

$$f(z) \to \min, \quad z \in \big\{x + \alpha V(x) : \alpha \in [0, 1]\big\}.$$

This solution will be denoted by Ax. Since $f(Ax) \le f(x)$ for all $x \in K_c$, we conclude that $A(K_c) \subset K_c$.

We will show that the mapping $A : K_c \to K_c$ is continuous. To this end, consider a sequence $\{x_n\}_{n=1}^{\infty} \subset K_c$ such that $\lim_{n\to\infty} x_n = x_*$. We intend to show that $\lim_{n\to\infty} Ax_n = Ax_*$. For each integer $n \ge 1$, there is $\alpha_n \in [0, 1]$ such that $Ax_n = x_n + \alpha_n Vx_n$. There is also $\alpha_* \in [0, 1]$ such that $Ax_* = x_* + \alpha_* V(x_*)$. We may assume without loss of generality that the limit $\bar{\alpha} = \lim_{n\to\infty} \alpha_n$ exists. By the definition of A,

$$f(Ax_*) \le f\big(x_* + \bar{\alpha} V(x_*)\big).$$

Since the function f is strictly convex, to complete the proof it is sufficient to show that

$$f(Ax_*) = f\big(x_* + \alpha_* V(x_*)\big) = f\big(x_* + \bar{\alpha} V(x_*)\big). \tag{4.33}$$

Assume the contrary. Then

$$\begin{aligned} \lim_{n\to\infty} f\big(x_n + \alpha_* V(x_n)\big) &= f\big(x_* + \alpha_* V(x_*)\big) \\ &< f\big(x_* + \bar{\alpha} V(x_*)\big) = \lim_{n\to\infty} f\big(x_n + \alpha_n V(x_n)\big), \end{aligned}$$

and for all large enough n,

$$f\big(x_n + \alpha_* V(x_n)\big) < f\big(x_n + \alpha_n V(x_n)\big) = f(Ax_n).$$

This contradicts the definition of A. Hence (4.33) is true and the mapping A is indeed continuous.

Example 4.10 Let K be a bounded, closed and convex subset of X and $f : K \to R^1$ be a convex continuous function which is bounded from below. For each $x_0, x_1 \in K$ satisfying $f(x_0) > f(x_1)$, we will construct a continuous mapping $A : K \to K$ such that $f(Ax) \le f(x)$ for all $x \in K$ and $Ax = x_1$ for all x in a neighborhood of x_0.

Indeed, let $x_0, x_1 \in K$ with $f(x_0) > f(x_1)$. There are numbers r_0, ε_0 such that

$$f(x) - \varepsilon_0 > f(x_1) \quad \text{for all } x \in K \text{ satisfying } \|x - x_0\| \le r_0. \tag{4.34}$$

Now we define an open covering $\{V_x : x \in K\}$ of K. Let $x \in K$. If $\|x - x_0\| < r_0$ we set

$$V_x = \{y \in K : \|y - x_0\| < r_0\} \quad \text{and} \quad a_x = x_1.$$

If $\|x - x_0\| \ge r_0$, then there is $r_x \in (0, 4^{-1}r_0)$ and $a_x \in K$ such that

$$f(a_x) \le f(y) \quad \text{for all } y \in \{z \in K : \|z - x\| \le r_x\}. \tag{4.35}$$

In this case we set

$$V_x = \{y \in K : \|y - x\| < r_x\}.$$

Clearly, $\bigcup\{V_x : x \in K\} = K$. There is a continuous locally finite partition of unity $\{\phi_x\}_{x\in K}$ on K subordinated to $\{V_x\}_{x\in K}$ (namely, $\operatorname{supp}\phi_x \subset V_x$ for all $x \in K$). For $y \in K$, define

$$Ay = \sum_{x\in K} \phi_x(y)a_x.$$

Evidently, the mapping A is well defined, $A : K \to X$ and A is continuous. Since $\sum_{x\in K} \phi_x(y) = 1$ for all $y \in K$ and K is convex, we see that $A(K) \subset K$.

We will now show that $f(Ay) \le f(y)$ for all $y \in K$ and that $Ay = x_1$ if $\|y - x_0\| \le 4^{-1}r_0$.

Let $y \in K$. There are $z_1, \dots, z_n \in K$ and a neighborhood U of y in K such that

$$\{z \in K : U \cap \operatorname{supp}\phi_z \ne \emptyset\} = \{z_1, \dots, z_n\}.$$

We have

$$Ay = \sum_{i=1}^n \phi_{z_i}(y)a_{z_i}, \qquad \sum_{i=1}^n \phi_{z_i}(y) = 1, \qquad f(Ay) \le \sum_{i=1}^n \phi_{z_i}(y)f(a_{z_i}). \tag{4.36}$$

We may assume without loss of generality that there is $p \in \{1, \dots, n\}$ such that

$$\phi_{z_i}(y) > 0 \quad \text{if and only if} \quad 1 \le i \le p. \tag{4.37}$$

Let $1 \le i \le p$. Then

$$y \in \operatorname{supp}\phi_{z_i} \subset V_{z_i} \tag{4.38}$$

and by the definition of V_{z_i} and a_{z_i} (see (4.34) and (4.35)), $f(y) \geq f(a_{z_i})$. When combined with (4.36) and (4.37), this implies that $f(Ay) \leq f(y)$.

Assume in addition that $\|y - x_0\| \leq 4^{-1}r_0$. Then it follows from the definition of $\{V_z : z \in K\}$ and (4.38) that $\|z_i - x_0\| < r_0$ and $a_{z_i} = x_1$ for each $i = 1, \dots, p$. By (4.36) and (4.37), $Ay = x_1$. Thus we have indeed constructed a continuous mapping $A : K \to K$ such that $f(Ay) \leq f(y)$ for all $y \in K$, and $Ay = x_1$ for all $y \in K$ satisfying $\|y - x_0\| \leq 4^{-1}r_0$.

4.5 Normal Mappings

Assume that $(X, \|\cdot\|)$ is a Banach space with norm $\|\cdot\|$, $K \subset X$ is a nonempty, bounded, closed and convex subset of X, and $f : K \to R^1$ is a convex and uniformly continuous function. Set

$$\inf(f) = \inf\bigl\{f(x) : x \in K\bigr\}.$$

Observe that this infimum is finite because K is bounded and f is uniformly continuous. We consider the topological subspace $K \subset X$ with the relative topology. Denote by $\mathcal{A}$ the set of all self-mappings $A : K \to K$ such that

$$f(Ax) \leq f(x) \quad \text{for all } x \in K \tag{4.39}$$

and by $\mathcal{A}_c$ the set of all continuous mappings $A \in \mathcal{A}$. In Sect. 4.4 we constructed many mappings which belong to $\mathcal{A}_c$.

We equip the set $\mathcal{A}$ with a metric $\rho : \mathcal{A} \times \mathcal{A} \to R^1$ defined by

$$\rho(A, B) = \sup\bigl\{\|Ax - Bx\| : x \in K\bigr\}, \quad A, B \in \mathcal{A}. \tag{4.40}$$

Clearly, the metric space $\mathcal{A}$ is complete and $\mathcal{A}_c$ is a closed subset of $\mathcal{A}$. In the sequel we will consider the metric space $(\mathcal{A}_c, \rho)$. Denote by $\mathcal{M}$ the set of all sequences $\{A_t\}_{t=1}^{\infty} \subset \mathcal{A}$ and by $\mathcal{M}_c$ the set of all sequences $\{A_t\}_{t=1}^{\infty} \subset \mathcal{A}_c$. Members $\{A_t\}_{t=1}^{\infty}$, $\{B_t\}_{t=1}^{\infty}$ and $\{C_t\}_{t=1}^{\infty}$ of $\mathcal{M}$ will occasionally be denoted by boldface $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, respectively. For the set $\mathcal{M}$ we will consider two uniformities and the topologies induced by them. The first uniformity is determined by the following base:

$$\begin{aligned} E_w(N, \varepsilon) = \Bigl\{\bigl(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\bigr) \in \mathcal{M} \times \mathcal{M} : \\ \rho(A_t, B_t) \leq \varepsilon, t = 1, \dots, N\Bigr\}, \end{aligned} \tag{4.41}$$

where N is a natural number and $\varepsilon > 0$. Clearly the uniform space $\mathcal{M}$ with this uniformity is metrizable (by a metric $\rho_w : \mathcal{M} \times \mathcal{M} \to R^1$) and complete (see [80]). We equip the set $\mathcal{M}$ with the topology induced by this uniformity. This topology will be called weak and denoted by τ_w. Clearly $\mathcal{M}_c$ is a closed subset of $\mathcal{M}$ with the weak topology.

The second uniformity is determined by the following base:

$$E_s(\varepsilon) = \big\{\big(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\big) \in \mathcal{M} \times \mathcal{M} : \rho(A_t, B_t) \le \varepsilon, t \ge 1\big\}, \tag{4.42}$$

where $\varepsilon > 0$. Clearly this uniformity is metrizable (by a metric $\rho_s : \mathcal{M} \times \mathcal{M} \to R^1$) and complete (see [80]). Denote by τ_s the topology induced by this uniformity in $\mathcal{M}$. Since τ_s is clearly stronger than τ_w, it will be called strong. We consider the topological subspace $\mathcal{M}_c \subset \mathcal{M}$ with the relative weak and strong topologies.

In Sects. 4.1–4.3 we showed that for a generic sequence taken from the space $\mathcal{M}_c$, the sequence of values of the Lyapunov function f along any trajectory tends to the infimum of f.

A mapping $A \in \mathcal{A}$ is called normal if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \ge \inf(f) + \varepsilon$, the inequality

$$f(Ax) \le f(x) - \delta(\varepsilon)$$

is true.

A sequence $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ is called normal if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \ge \inf(f) + \varepsilon$ and each integer $t \ge 1$, the inequality

$$f(A_t x) \le f(x) - \delta(\varepsilon)$$

holds.

In this chapter we show that a generic element taken from the spaces $\mathcal{A}$, $\mathcal{A}_c$, $\mathcal{M}$ and $\mathcal{M}_c$ is normal. This is important because it turns out that the sequence of values of the Lyapunov function f along any (unrestricted) trajectory of such an element tends to the infimum of f on K.

For $\alpha \in (0, 1)$, $\mathbf{A} = \{A_t\}_{t=1}^{\infty}$, $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathcal{M}$ define $\alpha\mathbf{A} + (1-\alpha)\mathbf{B} = \{\alpha A_t + (1-\alpha)B_t\}_{t=1}^{\infty} \in \mathcal{M}$.

We can easily prove the following fact.

Proposition 4.11 *Let $\alpha \in (0, 1)$, $\mathbf{A}, \mathbf{B} \in \mathcal{M}$ and let $\mathbf{A}$ be normal. Then $\alpha\mathbf{A} + (1-\alpha)\mathbf{B}$ is also normal.*

In this chapter we will prove the following results obtained in [63].

Theorem 4.12 *Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ be normal and let $\varepsilon > 0$. Then there exists a neighborhood U of $\mathbf{A}$ in $\mathcal{M}$ with the strong topology and a natural number N such that for each $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in U$, each $x \in K$ and each $r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$,*

$$f(C_{r(N)} \cdots C_{r(1)} x) \le \inf(f) + \varepsilon.$$

Theorem 4.13 *Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ be normal and let $\varepsilon > 0$. Then there exists a neighborhood U of $\mathbf{A}$ in $\mathcal{M}$ with the weak topology and a natural number N such that for each $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in U$ and each $x \in K$,*

$$f(C_N \cdots C_1 x) \le \inf(f) + \varepsilon.$$

Theorem 4.14 *There exists a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open and everywhere dense sets in $\mathcal{M}$ with the strong topology and a set $\mathcal{F}_c \subset \mathcal{F} \cap \mathcal{M}_c$ which is a countable intersection of open and everywhere dense sets in $\mathcal{M}_c$ with the strong topology such that each $\mathbf{A} \in \mathcal{F}$ is normal.*

Theorem 4.15 *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open and everywhere dense sets in $\mathcal{A}$ and a set $\mathcal{F}_c \subset \mathcal{F} \cap \mathcal{A}_c$, which is a countable intersection of open and everywhere dense sets in $\mathcal{A}_c$ such that each $\mathbf{A} \in \mathcal{F}$ is normal.*

4.6 Existence of a Normal $A \in \mathcal{A}_c$

If there is $x_{min} \in K$ for which $f(x_{min}) = \inf(f)$, then we can set $A(x) = x_{min}$ for all $x \in K$ and this A is normal. Therefore in order to show the existence of a normal $A \in \mathcal{A}_c$ we may assume that

$$\{x \in K : f(x) = \inf(f)\} = \emptyset. \tag{4.43}$$

The existence of a normal $A \in \mathcal{A}_c$ follows from Michael's selection theorem.

Proposition 4.16 *There exists a normal $A_* \in \mathcal{A}_c$.*

Proof We may assume that (4.43) is true. Define a set-valued map $a : K \to 2^K$ as follows: for each $x \in K$, denote by $a(x)$ the closure (in the norm topology of X) of the set

$$\{y \in K : f(y) < 2^{-1}(f(x) + \inf(f))\}. \tag{4.44}$$

It is clear that for each $x \in K$, the set $a(x)$ is nonempty, closed and convex. We will show that a is lower semicontinuous.

Let $x_0 \in K$, $y_0 \in a(x_0)$ and let $\varepsilon > 0$ be given. In order to prove that a is lower semicontinuous, we need to show that there exists a positive number δ such that for each $x \in K$ satisfying $\|x - x_0\| < \delta$,

$$a(x) \cap \{y \in K : \|y - y_0\| < \varepsilon\} \neq \emptyset.$$

By the definition of $a(x_0)$, there exists a point $y_1 \in K$ such that

$$f(y_1) < 2^{-1}(f(x_0) + \inf(f)) \quad \text{and} \quad \|y_1 - y_0\| < \varepsilon/2.$$

Since the function f is continuous, there is a number $\delta > 0$ such that for each $x \in K$ satisfying $\|x - x_0\| < \delta$,

$$f(y_1) < 2^{-1}(f(x) + \inf(f)).$$

Hence $y_1 \in a(x)$ by definition. Therefore a is indeed lower semicontinuous. By Michael's selection theorem, there exists a continuous mapping $A_* : K \to K$ such that $A_* x \in a(x)$ for all $x \in K$. It follows from the definition of a (see (4.44)) that for each $x \in K$,

$$f(A_* x) \le 2^{-1}\big(f(x) + \inf(f)\big).$$

This implies that A_* is normal. This completes the proof of Proposition 4.16. □

4.7 Auxiliary Results

By Proposition 4.16, there exists a normal mapping $A_* \in \mathcal{A}_c$. For each $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ and each $\gamma \in (0,1)$, we define a sequence of mappings $\mathbf{A}^{\gamma} = \{A_t^{\gamma}\}_{t=1}^{\infty} \in \mathcal{M}$ by

$$A_t^{\gamma} x = (1-\gamma)A_t x + \gamma A_* x, \quad x \in K, t = 1, 2, \ldots. \tag{4.45}$$

Clearly, for each $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}_c$ and each $\gamma \in (0,1)$, $\mathbf{A}^{\gamma} \in \mathcal{M}_c$. By (4.45) and Proposition 4.11, $\mathbf{A}^{\gamma}$ is normal for each $\mathbf{A} \in \mathcal{M}$ and each $\gamma \in (0,1)$. It is obvious that for each $\mathbf{A} \in \mathcal{M}$,

$$\mathbf{A}^{\gamma} \to \mathbf{A} \quad \text{as } \gamma \to 0^{+} \text{ in the strong topology.} \tag{4.46}$$

Lemma 4.17 *Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ be normal and let $\varepsilon > 0$ be given. Then there exist a neighborhood U of $\mathbf{A}$ in $\mathcal{M}$ with the strong topology and a number $\delta > 0$ such that for each $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in U$, each $x \in K$ satisfying*

$$f(x) \ge \inf(f) + \varepsilon \tag{4.47}$$

and each integer $t \ge 1$,

$$f(B_t x) \le f(x) - \delta.$$

Proof Since $\mathbf{A}$ is normal, there is $\delta_0 > 0$ such that for each integer $t \ge 1$ and each $x \in K$ satisfying (4.47),

$$f(A_t x) \le f(x) - \delta_0. \tag{4.48}$$

Since f is uniformly continuous, there is $\delta \in (0, 4^{-1}\delta_0)$ such that

$$\big|f(y) - f(z)\big| \le 4^{-1}\delta_0 \tag{4.49}$$

for each $y, z \in K$ satisfying $\|y - z\| \le 2\delta$. Set

$$U = \big\{\mathbf{B} \in \mathcal{M} : (\mathbf{A}, \mathbf{B}) \in E_s(\delta)\big\}. \tag{4.50}$$

Assume that $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in U$, let $t \ge 1$ be an integer and let $x \in K$ satisfy (4.47). By (4.47) and the definition of δ_0, (4.48) is true. The definitions of δ and U (see (4.49) and (4.50)) imply that

$$\|A_t x - B_t x\| \le \delta \quad \text{and} \quad \big|f(A_t x) - f(B_t x)\big| \le \delta_0/4.$$

When combined with (4.48), this implies that

$$f(B_t x) \leq f(x) + 4^{-1}\delta_0 - \delta_0 \leq f(x) - \delta.$$

This completes the proof of the lemma. □

4.8 Proof of Theorem 4.12

Assume that $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ is normal and let $\varepsilon > 0$ be given. By Lemma 4.17, there exist a neighborhood U of $\mathbf{A}$ in $\mathcal{M}$ with the strong topology and a number $\delta > 0$ such that the following property holds:

(Pi) For each $\{B_t\}_{t=1}^{\infty} \in U$, each integer $t \geq 1$ and each $x \in K$ satisfying (4.47), the inequality

$$f(B_t x) \leq f(x) - \delta \tag{4.51}$$

holds.

Choose a natural number $N \geq 4$ such that

$$\delta N > 2(\varepsilon + 1) + 2\sup\{|f(z)| : z \in K\}. \tag{4.52}$$

Assume that

$$\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in U, \qquad x \in K \quad \text{and} \quad r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}. \tag{4.53}$$

We claim that

$$f(C_{r(N)} \cdots C_{r(1)} x) \leq \inf(f) + \varepsilon. \tag{4.54}$$

Assume the contrary. Then

$$f(x) > \inf(f) + \varepsilon, \qquad f(C_{r(n)} \cdots C_{r(1)} x) > \inf(f) + \varepsilon, \quad n = 1, \ldots, N. \tag{4.55}$$

It follows from (4.55), (4.53) and property (Pi) that

$$f(C_{r(1)} x) \leq f(x) - \delta,$$
$$f(C_{r(n+1)} C_{r(n)} \cdots C_{r(1)} x) \leq f(C_{r(n)} \cdots C_{r(1)} x) - \delta, \quad n = 1, \ldots, N-1.$$

This implies that

$$f(C_{r(n)} \cdots C_{r(1)} x) \leq f(x) - N\delta \leq -2 - \sup\{|f(z)| : z \in K\},$$

a contradiction. Therefore (4.54) is valid and Theorem 4.12 is proved.

4.9 Proof of Theorem 4.13

Assume that $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ is normal and let $\varepsilon > 0$ be given. Since $\mathbf{A}$ is normal, there is $\delta \in (0, 1)$ such that for each integer $t \geq 1$ and each $x \in K$ satisfying

$$f(x) \geq \inf(f) + \varepsilon, \tag{4.56}$$

the following inequality is valid:

$$f(A_t x) \leq f(x) - \delta. \tag{4.57}$$

Choose a natural number $N > 4$ for which

$$N > 4\delta^{-1} + 4\delta^{-1} \sup\{|f(z)| : z \in K\}. \tag{4.58}$$

Since f is uniformly continuous, there is $\Delta \in (0, 4^{-1}\delta)$ such that

$$|f(z) - f(y)| \leq 8^{-1}\delta \tag{4.59}$$

for each $y, z \in K$ satisfying $\|z - y\| \leq 4\Delta$. Set

$$U = \{\mathbf{B} \in \mathcal{M} : (\mathbf{A}, \mathbf{B}) \in E_w(N, \Delta)\}. \tag{4.60}$$

Assume that

$$\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in U \quad \text{and} \quad x \in K. \tag{4.61}$$

We claim that

$$f(C_N \cdots C_1 x) \leq \inf(f) + \varepsilon. \tag{4.62}$$

Assume the contrary. Then

$$f(x) > \inf(f) + \varepsilon, \qquad f(C_n \cdots C_1 x) > \inf(f) + \varepsilon, \quad n = 1, \ldots, N. \tag{4.63}$$

Define $C_0 : K \to K$ by $C_0 x = x$ for all $x \in K$. Let $t \in \{0, \ldots, N - 1\}$. It follows from (4.63) and the definition of δ (see (4.56) and (4.57)) that

$$f(A_{t+1} C_t \cdots C_0 x) \leq f(C_t \cdots C_0 x) - \delta. \tag{4.64}$$

The definition of U (see (4.60)) and (4.61) imply that $\|A_{t+1} C_t \cdots C_0 x - C_{t+1} C_t \cdots C_0 x\| \leq \Delta$. By this inequality and the definition of Δ (see (4.59)),

$$|f(A_{t+1} C_t \cdots C_0 x) - f(C_{t+1} C_t \cdots C_0 x)| \leq 8^{-1}\delta.$$

When combined with (4.64), this implies that

$$f(C_{t+1} C_t \cdots C_0 x) \leq f(C_t \cdots C_0 x) - 2^{-1}\delta.$$

Since this inequality is true for all $t \in \{0, \dots, N-1\}$, we conclude that

$$f(C_N \cdots C_1 x) \le f(x) - 2^{-1} N\delta.$$

Together with (4.58) this implies that

$$\begin{aligned} -\sup\{|f(z)| : z \in K\} &\le \sup\{|f(z)| : z \in K\} - 2^{-1}\delta N \\ &\le -2 - \sup\{|f(z)| : z \in K\}, \end{aligned}$$

a contradiction. Therefore (4.62) does hold and Theorem 4.13 is proved.

4.10 Proof of Theorem 4.14

Let $\mathbf{A} \in \mathcal{M}$, $\gamma \in (0, 1)$ and let $i \ge 1$ be an integer. Consider the sequence $\mathbf{A}^\gamma \in \mathcal{M}$ defined by (4.45). By Proposition 4.11, $\mathbf{A}^\gamma$ is normal. By Lemma 4.17, there exists an open neighborhood $U(\mathbf{A}, \gamma, i)$ of $\mathbf{A}^\gamma$ in $\mathcal{M}$ with the strong topology and a number $\delta(\mathbf{A}, \gamma, i) > 0$ such that the following property holds:

(Pii) For each $\mathbf{B} = \{B_t\}_{t=1}^\infty \in U(\mathbf{A}, \gamma, i)$, each integer $t \ge 1$ and each $x \in K$ satisfying $f(x) \ge \inf(f) + 2^{-i}$,

$$f(B_t x) \le f(x) - \delta(\mathbf{A}, \gamma, i).$$

Define

$$\mathcal{F} = \bigcap_{i=1}^\infty \bigcup \{U(\mathbf{A}, \gamma, i) : \mathbf{A} \in \mathcal{M}, \gamma \in (0, 1)\} \tag{4.65}$$

and

$$\mathcal{F}_c = \left[\bigcap_{i=1}^\infty \bigcup \{U(\mathbf{A}, \gamma, i) : \mathbf{A} \in \mathcal{M}_c, \gamma \in (0, 1)\}\right] \cap \mathcal{M}_c.$$

Clearly, $\mathcal{F}_c \subset \mathcal{F}$, $\mathcal{F}$ is a countable intersection of open and everywhere dense sets in $\mathcal{M}$ with the strong topology, and $\mathcal{F}_c$ is a countable intersection of open and everywhere dense sets in $\mathcal{M}_c$ with the strong topology.

Assume that $\mathbf{B} = \{B_t\}_{t=1}^\infty \in \mathcal{F}$. We will show that $\mathbf{B}$ is normal.

Let $\varepsilon > 0$ be given. Choose an integer $i \ge 1$ such that

$$2^{-i} < \varepsilon/8. \tag{4.66}$$

By (4.65), there exist $\mathbf{A} \in \mathcal{M}$ and $\gamma \in (0, 1)$ such that

$$\mathbf{B} \in U(\mathbf{A}, \gamma, i). \tag{4.67}$$

Let $t \geq 1$ be an integer, $x \in K$, and $f(x) \geq \inf(f) + \varepsilon$. Then by (4.66), (4.67) and property (Pii),

$$f(B_t x) \leq f(x) - \delta(\mathbf{A}, \gamma, i).$$

Thus **B** is indeed normal and Theorem 4.14 is proved.

The proof of Theorem 4.15 is analogous to that of Theorem 4.14.

4.11 Normality and Porosity

In this section, which is based on [133], we continue to consider a complete metric space of sequences of mappings acting on a bounded, closed and convex subset K of a Banach space which share a common convex Lyapunov function f. In previous sections, we introduced the concept of normality and showed that a generic element taken from this space is normal. The sequence of values of the Lyapunov uniformly continuous function f along any (unrestricted) trajectory of such an element tends to the infimum of f on K. In the present section, we first present a convergence result for perturbations of such trajectories. We then show that if f is Lipschitzian, then the complement of the set of normal sequences is σ-porous.

Assume that $(X, \|\cdot\|)$ is a Banach space with norm $\|\cdot\|$, $K \subset X$ is a nonempty, bounded, closed and convex subset of X, and $f : K \to R^1$ is a convex and uniformly continuous function. Observe that the function f is bounded because K is bounded and f is uniformly continuous. Set

$$\inf(f) = \inf\{f(x) : x \in K\} \quad \text{and} \quad \sup(f) = \sup\{f(x) : x \in K\}.$$

We consider the topological subspace $K \subset X$ with the relative topology. Denote by $\mathcal{A}$ the set of all self-mappings $A : K \to K$ such that

$$f(Ax) \leq f(x) \quad \text{for all } x \in K$$

and by $\mathcal{A}_c$ the set of all continuous mappings $A \in \mathcal{A}$.

For the set $\mathcal{A}$ we define a metric $\rho : \mathcal{A} \times \mathcal{A} \to R^1$ by

$$\rho(A, B) = \sup\{\|Ax - Bx\| : x \in K\}, \quad A, B \in \mathcal{A}.$$

It is clear that the metric space $\mathcal{A}$ is complete and $\mathcal{A}_c$ is a closed subset of $\mathcal{A}$. We will study the metric space $(\mathcal{A}_c, \rho)$. Denote by $\mathcal{M}$ the set of all sequences $\{A_t\}_{t=1}^{\infty} \subset \mathcal{A}$ and by $\mathcal{M}_c$ the set of all sequences $\{A_t\}_{t=1}^{\infty} \subset \mathcal{A}_c$. For the set $\mathcal{M}$ we define a metric $\rho_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to R^1$ by

$$\rho_{\mathcal{M}}\big(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\big) = \sup\{\rho(A_t, B_t) : t = 1, 2, \ldots\}, \quad \{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty} \in \mathcal{M}.$$

Clearly, the metric space $\mathcal{M}$ is complete and $\mathcal{M}_c$ is a closed subset of $\mathcal{M}$. We will also study the metric space $(\mathcal{M}_c, \rho_{\mathcal{M}})$.

We recall the following definition of normality.

A mapping $A \in \mathcal{A}$ is called normal if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$, the inequality

$$f(Ax) \leq f(x) - \delta(\varepsilon)$$

is true.

A sequence $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ is called normal if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$ and each integer $t \geq 1$, the inequality

$$f(A_t x) \leq f(x) - \delta(\varepsilon)$$

holds.

We now present two theorems which were obtained in [133]. Their proofs are given in the next two sections.

Theorem 4.18 *Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ be normal and let ε be positive. Then there exist a natural number n_0 and a number $\gamma > 0$ such that for each integer $n \geq n_0$, each mapping $r : \{1, \ldots, n\} \to \{1, 2, \ldots\}$ and each sequence $\{x_i\}_{i=0}^{n} \subset K$ which satisfies*

$$\|x_{i+1} - A_{r(i+1)} x_i\| \leq \gamma, \quad i = 0, \ldots, n-1,$$

the inequality $f(x_i) \leq \inf(f) + \varepsilon$ holds for $i = n_0, \ldots, n$.

Theorem 4.19 *Let $\mathcal{F}$ be the set of all normal sequences in the space $\mathcal{M}$ and let*

$$F = \big\{A \in \mathcal{A} : \{A_t\}_{t=1}^{\infty} \in \mathcal{F} \text{ where } A_t = A, t = 1, 2, \ldots\big\}.$$

Assume that the function f is Lipschitzian. Then the complement of the set $\mathcal{F}$ is a σ-porous subset of $\mathcal{M}$ and the complement of the set $\mathcal{F} \cap \mathcal{M}_c$ is a σ-porous subset of $\mathcal{M}_c$. Moreover, the complement of the set F is a σ-porous subset of $\mathcal{A}$ and the complement of the set $F \cap \mathcal{A}_c$ is a σ-porous subset of $\mathcal{A}_c$.

4.12 Proof of Theorem 4.18

We may assume that $\varepsilon < 1$. Since $\{A_t\}_{t=1}^{\infty}$ is normal, there exists a function $\delta : (0, \infty) \to (0, \infty)$ such that for each $s > 0$, each $x \in K$ satisfying $f(x) \geq \inf(f) + s$ and each integer $t \geq 1$,

$$f(A_t x) \leq f(x) - \delta(s). \tag{4.68}$$

We may assume that $\delta(s) < s$, $s \in (0, \infty)$. Choose a natural number

$$n_0 > 4\big(1 + \sup(f) - \inf(f)\big)\delta\big(8^{-1}\varepsilon\big)^{-1}. \tag{4.69}$$

Since f is uniformly continuous, there exists a number $\gamma > 0$ such that for each $y_1, y_2 \in K$ satisfying $\|y_1 - y_2\| \le \gamma$, the following inequality holds:

$$\big|f(y_1) - f(y_2)\big| \le \delta\big(8^{-1}\varepsilon\big)8^{-1}(n_0+1)^{-1}. \tag{4.70}$$

We claim that the following assertion is true:

(A) Suppose that

$$\{x_i\}_{i=0}^{n_0} \subset K, r : \{1, \dots, n_0\} \to \{1, 2, \dots\},$$
$$\|x_{i+1} - A_{r(i+1)}x_i\| \le \gamma, \quad i = 0, \dots, n_0 - 1. \tag{4.71}$$

Then there exists an integer $n_1 \in \{1, \dots, n_0\}$ such that

$$f(x_{n_1}) \le \inf(f) + \varepsilon/8. \tag{4.72}$$

Assume the contrary. Then

$$f(x_i) > \inf(f) + \varepsilon/8, \quad i = 1, \dots, n_0. \tag{4.73}$$

By (4.73) and the definition of $\delta : (0,\infty) \to (0,\infty)$ (see (4.68)), for each $i = 1, \dots, n_0 - 1$, we have

$$f(A_{r(i+1)}x_i) \le f(x_i) - \delta\big(8^{-1}\varepsilon\big). \tag{4.74}$$

It follows from (4.71) and the definition of γ (see (4.70)) that for $i = 1, \dots, n_0 - 1$,

$$\big|f(x_{i+1}) - f(A_{r(i+1)}x_i)\big| \le \delta\big(8^{-1}\varepsilon\big)8^{-1}(n_0+1)^{-1}.$$

When combined with (4.74), this inequality implies that for $i = 1, \dots, n_0 - 1$,

$$\begin{aligned} f(x_{i+1}) - f(x_i) &\le f(x_{i+1}) - f(A_{r(i+1)}x_i) + f(A_{r(i+1)}x_i) - f(x_i) \\ &\le \delta\big(8^{-1}\varepsilon\big)8^{-1}(n_0+1)^{-1} - \delta\big(8^{-1}\varepsilon\big) \le (-1/2)\delta\big(8^{-1}\varepsilon\big). \end{aligned}$$

This, in turn, implies that

$$\inf(f) - \sup(f) \le f(x_{n_0}) - f(x_1) \le (n_0 - 1)(-1/2)\delta\big(8^{-1}\varepsilon\big),$$

a contradiction (see (4.69)). Thus there exists an integer $n_1 \in \{1, \dots, n_0\}$ such that (4.72) is true. Therefore assertion (A) is valid, as claimed.

Assume now that we are given an integer $n \ge n_0$, a mapping

$$r : \{1, \dots, n\} \to \{1, 2, \dots\} \tag{4.75}$$

and a finite sequence

$$\{x_i\}_{i=0}^{n} \subset K \quad \text{such that} \quad \|x_{i+1} - A_{r(i+1)}x_i\| \le \gamma, \quad i = 0, \dots, n-1. \tag{4.76}$$

It follows from assertion (A) that there exists a finite sequence of natural numbers $\{j_p\}_{p=1}^{q}$ such that

$$\begin{gathered} 1 \le j_1 \le n_0, \qquad 1 \le j_{p+1} - j_p \le n_0 \quad \text{if } 1 \le p \le q-1, n - j_q < n_0, \\ f(x_{j_p}) \le \inf(f) + \varepsilon/8, \quad p = 1, \dots, q. \end{gathered} \tag{4.77}$$

Let $i \in \{n_0, \dots, n\}$. We will show that $f(x_i) \le \inf(f) + \varepsilon/2$. There exists $p \in \{1, \dots, q\}$ such that

$$0 \le i - j_p \le n_0.$$

If $i = j_p$, then by (4.77), $f(x_i) = f(x_{j_p}) \le \inf(f) + \varepsilon/8$. Thus we may assume that $i > j_p$. For all integers $j_p \le s < i$, it follows from (4.76) and the definition of γ (see (4.70)) that

$$f(A_{r(s+1)}x_s) \le f(x_s),$$

$$\bigl|f(x_{s+1}) - f(A_{r(s+1)}x_s)\bigr| \le \delta\bigl(8^{-1}\varepsilon\bigr)8^{-1}(n_0+1)^{-1}$$

and

$$\begin{aligned} f(x_{s+1}) &\le f(A_{r(s+1)}x_s) + \delta\bigl(8^{-1}\varepsilon\bigr)8^{-1}(n_0+1)^{-1} \\ &\le f(x_s) + \delta\bigl(8^{-1}\varepsilon\bigr)8^{-1}(n_0+1)^{-1}. \end{aligned}$$

Thus

$$f(x_{s+1}) - f(x_s) \le \delta\bigl(8^{-1}\varepsilon\bigr)8^{-1}(n_0+1)^{-1}, \quad j_p \le s < i.$$

This implies that

$$\begin{aligned} f(x_i) &\le f(x_{j_p}) + \delta\bigl(8^{-1}\varepsilon\bigr)8^{-1}(n_0+1)^{-1}(n_0+1) \\ &\le \inf(f) + \varepsilon/8 + 8^{-1}\delta\bigl(8^{-1}\varepsilon\bigr) \le \inf(f) + \varepsilon/2. \end{aligned}$$

Therefore $f(x_i) \le \inf(f) + \varepsilon/2$ for all integers $i \in [n_0, n]$ and Theorem 4.18 is proved.

4.13 Proof of Theorem 4.19

Since $f : K \to R^1$ is assumed to be Lipschitzian, there exists a constant $L(f) > 0$ such that

$$\bigl|f(x) - f(y)\bigr| \le L(f)\|x - y\| \quad \text{for all } x, y \in K. \tag{4.78}$$

By Proposition 4.16, there exist a normal continuous mapping $A_* : K \to K$ and a function $\phi : (0, \infty) \to (0, \infty)$ such that for each $\varepsilon > 0$ and each $x \in K$ satisfying $f(x) \ge \inf(f) + \varepsilon$, the inequality $f(A_* x) \le f(x) - \phi(\varepsilon)$ holds.

Let $\varepsilon > 0$ be given. We say that a sequence $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ is (ε)-quasinormal if there exists $\delta > 0$ such that if $x \in K$ satisfies $f(x) \geq \inf(f) + \varepsilon$, then $f(A_t x) \leq f(x) - \delta$ for all integers $t \geq 1$.

Recall that $\mathcal{F}$ is defined to be the set of all normal sequences in $\mathcal{M}$. For each integer $n \geq 1$, denote by $\mathcal{F}_n$ the set of all (n^{-1})-quasinormal sequences in $\mathcal{M}$. Clearly,

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n. \tag{4.79}$$

Set

$$d(K) = \sup\bigl\{\|z\| : z \in K\bigr\}. \tag{4.80}$$

Let $n \geq 1$ be an integer. Choose $\alpha \in (0, 1)$ such that

$$2L(f)\alpha < (1-\alpha)\phi\bigl(n^{-1}\bigr)8^{-1}\bigl(d(K)+1\bigr)^{-1}. \tag{4.81}$$

Assume that $0 < r \leq 1$ and $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$. Set

$$\gamma = (1-\alpha)r8^{-1}\bigl(d(K)+1\bigr)^{-1} \tag{4.82}$$

and define for each integer $t \geq 1$, the mapping $A_{t\gamma} : K \to K$ by

$$A_{t\gamma}x = (1-\gamma)A_t x + \gamma A_* x, \quad x \in K. \tag{4.83}$$

It is clear that $\{A_{t\gamma}\}_{t=1}^{\infty} \in \mathcal{M}$ and

$$\rho_{\mathcal{M}}\bigl(\{A_t\}_{t=1}^{\infty}, \{A_{t\gamma}\}_{t=1}^{\infty}\bigr) \leq 2\gamma \sup\bigl\{\|z\| : z \in K\bigr\} \leq 2\gamma d(K). \tag{4.84}$$

Note that $\{A_{t\gamma}\}_{t=1}^{\infty} \in \mathcal{M}_c$ if $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}_c$ and that $A_{t\gamma} = A_{1\gamma}$, $t = 1, 2, \ldots$, if $A_t = A_1$, $t = 1, 2, \ldots$.

Assume that

$$\{C_t\}_{t=1}^{\infty} \in \mathcal{M} \quad \text{and} \quad \rho_{\mathcal{M}}\bigl(\{A_{t\gamma}\}_{t=1}^{\infty}, \{C_t\}_{t=1}^{\infty}\bigr) \leq \alpha r. \tag{4.85}$$

Then by (4.85), (4.84) and (4.82),

$$\begin{aligned}\rho_{\mathcal{M}}\bigl(\{A_t\}_{t=1}^{\infty}, \{C_t\}_{t=1}^{\infty}\bigr) &\leq \alpha r + 2\gamma d(K) \leq \alpha r + (1-\alpha)r/2 \\ &= r(1+\alpha)/2 < r.\end{aligned} \tag{4.86}$$

Assume now that $x \in K$ satisfies

$$f(x) \geq \inf(f) + n^{-1} \tag{4.87}$$

and that $t \geq 1$ is an integer. By (4.87), the properties of A_* and ϕ, and (4.83),

$$\begin{aligned} f(A_* x) &\leq f(x) - \phi\bigl(n^{-1}\bigr), \\ f(A_{t\gamma}x) &\leq (1-\gamma)f(A_t x) + \gamma f(A_* x) \\ &\leq (1-\gamma)f(x) + \gamma\bigl(f(x) - \phi\bigl(n^{-1}\bigr)\bigr) = f(x) - \gamma\phi\bigl(n^{-1}\bigr).\end{aligned} \tag{4.88}$$

By (4.85), $\|C_t x - A_{t\gamma} x\| \le \alpha r$. Together with (4.78) this inequality yields

$$\big|f(C_t x) - f(A_{t\gamma} x)\big| \le L(f)\alpha r.$$

By the latter inequality, (4.88), (4.82) and (4.81),

$$\begin{aligned} f(C_t x) &\le f(A_{t\gamma} x) + L(f)\alpha r \\ &\le L(f)\alpha r + f(x) - \gamma\phi\big(n^{-1}\big) \\ &\le f(x) - \phi\big(n^{-1}\big)(1-\alpha)r8^{-1}\big(d(K)+1\big)^{-1} + L(f)\alpha r \\ &\le f(x) - L(f)\alpha r. \end{aligned}$$

Thus for each $\{C_t\}_{t=1}^{\infty} \in \mathcal{M}$ satisfying (4.85), inequalities (4.86) hold and $\{C_t\}_{t=1}^{\infty} \in \mathcal{F}_n$. Summing up, we have shown that for each integer $n \ge 1$, $\mathcal{M} \setminus \mathcal{F}_n$ is porous in $\mathcal{M}$, $\mathcal{M}_c \setminus \mathcal{F}_n$ is porous in $\mathcal{M}_c$, the complement of the set

$$\big\{A \in \mathcal{A} : \{A_t\}_{t=1}^{\infty} \in \mathcal{F}_n \text{ with } A_t = A \text{ for all integers } t \ge 1\big\}$$

is porous in $\mathcal{A}$ and the complement of the set

$$\big\{A \in \mathcal{A}_c : \{A_t\}_{t=1}^{\infty} \in \mathcal{F}_n \text{ with } A_t = A \text{ for all integers } t \ge 1\big\}$$

is porous in $\mathcal{A}_c$.

Combining these facts with (4.79), we conclude that $\mathcal{M} \setminus \mathcal{F}$ is σ-porous in $\mathcal{M}$, $\mathcal{M}_c \setminus \mathcal{F}$ is σ-porous in $\mathcal{M}_c$, $\mathcal{A} \setminus F$ is σ-porous in $\mathcal{A}$ and $\mathcal{A}_c \setminus F$ is σ-porous in $\mathcal{A}_c$. This completes the proof of Theorem 4.19.

4.14 Convex Functions Possessing a Sharp Minimum

In this section, which is based on the paper [7], we are given a convex, Lipschitz function f, defined on a bounded, closed and convex subset K of a Banach space X, which possesses a sharp minimum. A minimization algorithm is a self-mapping $A : K \to K$ such that $f(Ax) \le f(x)$ for all $x \in K$. We show that for most of these algorithms A, the sequences $\{A^n x\}_{n=1}^{\infty}$ tend to this sharp minimum (at an exponential rate) for all initial values $x \in K$.

Let $K \subset X$ be a nonempty, bounded, closed and convex subset of a Banach space X. For each $A : K \to X$, set

$$\mathrm{Lip}(A) = \sup\big\{\|Ax - Ay\|/\|x - y\| : x, y \in K \text{ such that } x \ne y\big\}. \tag{4.89}$$

Assume that $f : K \to R^1$ is a convex, Lipschitz function such that $\mathrm{Lip}(f) > 0$. We have

$$\big|f(x) - f(y)\big| \le \mathrm{Lip}(f)\|x - y\| \quad \text{for all } x, y \in K.$$

Assume further that there exists a point $x_* \in K$ and a number $c_0 > 0$ such that

$$\inf(f) := \inf\big\{f(x) : x \in K\big\} = f(x_*)$$

and

$$f(x) \geq f(x_*) + c_0\|x - x_*\| \quad \text{for all } x \in K. \tag{4.90}$$

In other words, we assume that the function f possesses a sharp minimum (cf. [26, 109]).

Denote by $\mathcal{A}$ the set of all self-mappings $A : K \to K$ such that $\mathrm{Lip}(A) < \infty$ and

$$f(Ax) \leq f(x) \quad \text{for all } x \in K. \tag{4.91}$$

We equip the set $\mathcal{A}$ with the uniformity determined by the base

$$\mathcal{E}(\varepsilon) = \big\{(A, B) \in \mathcal{A} \times \mathcal{A} : \|Ax - Bx\| \leq \varepsilon \text{ for all } x \in K \text{ and } \mathrm{Lip}(A - B) \leq \varepsilon\big\},$$

where $\varepsilon > 0$. Clearly, the uniform space $\mathcal{A}$ is metrizable and complete.

Theorem 4.20 *There exists an open and everywhere dense subset $\mathcal{B} \subset \mathcal{A}$ such that for each $B \in \mathcal{B}$, there exist an open neighborhood $\mathcal{U}$ of B in $\mathcal{A}$ and a number $\lambda_0 \in (0, 1)$ such that for each $C \in \mathcal{U}$, each $x \in K$, and each natural number n,*

$$\big\|C^n x - x_*\big\| \leq c_0^{-1}\lambda^n\big(f(x) - f(x_*)\big).$$

Proof Let $\gamma \in (0, 1)$ and $A \in \mathcal{A}$ be given. Set

$$A_\gamma x = (1 - \gamma)Ax + \gamma x_*, \quad x \in K. \tag{4.92}$$

Clearly, for all $x \in K$,

$$f(A_\gamma x) \leq (1 - \gamma)f(Ax) + \gamma f(x_*) \tag{4.93}$$

and

$$A_\gamma \in \mathcal{A}. \tag{4.94}$$

Next, we prove the following lemma.

Lemma 4.21 *Let $A \in \mathcal{A}$, $\gamma \in (0, 1)$ and $B \in \mathcal{A}$. Then for each $x \in K$,*

$$f(Bx) - f(x_*) \leq \big[(1 - \gamma) + \mathrm{Lip}(f)\,\mathrm{Lip}(B - A_\gamma)c_0^{-1}\big]\big(f(x) - f(x_*)\big).$$

Proof Let $x \in K$. By (4.93), the relations $A_\gamma x_* = Bx_* = x_*$ and (4.90),

$$\begin{aligned} f(Bx) - f(x_*) &= f(A_\gamma x) - f(x_*) + f(Bx) - f(A_\gamma x) \\ &\leq (1 - \gamma)\big(f(x) - f(x_*)\big) + \mathrm{Lip}(f)\|Bx - A_\gamma x\| \end{aligned}$$

$$
\begin{aligned}
&\le (1-\gamma)\big(f(x)-f(x_*)\big)+\operatorname{Lip}(f)\operatorname{Lip}(B-A_\gamma)\|x-x_*\| \\
&\le (1-\gamma)\big(f(x)-f(x_*)\big) \\
&\quad +\operatorname{Lip}(f)\operatorname{Lip}(B-A_\gamma)c_0^{-1}\big(f(x)-f(x_*)\big) \\
&\le \big[(1-\gamma)+\operatorname{Lip}(f)\operatorname{Lip}(B-A_\gamma)c_0^{-1}\big]\big(f(x)-f(x_*)\big).
\end{aligned}
$$

The lemma is proved. □

Completion of the proof of Theorem 4.20 Let $A \in \mathcal{A}$ and $\gamma \in (0,1)$ be given. Choose $r(\gamma) > 0$ such that

$$\lambda_\gamma := (1-\gamma)+\operatorname{Lip}(f)r(\gamma)c_0^{-1} < 1. \tag{4.95}$$

Denote by $\mathcal{U}(A,\gamma)$ the open neighborhood of A_γ in $\mathcal{A}$ such that

$$\mathcal{U}(A,\gamma) \subset \big\{B \in \mathcal{A} : (A_\gamma, B) \in \mathcal{E}\big(r(\gamma)\big)\big\}. \tag{4.96}$$

Set

$$\mathcal{B} = \bigcup\big\{\mathcal{U}(A,\gamma) : A \in \mathcal{A}, \gamma \in (0,1)\big\}. \tag{4.97}$$

Clearly, we have for each $A \in \mathcal{A}$,

$$A_\gamma \to A \quad \text{as } \gamma \to 0^+.$$

Therefore $\mathcal{B}$ is an everywhere dense, open subset of $\mathcal{A}$. Let $B \in \mathcal{A}$. There are $A \in \mathcal{A}$ and $\gamma \in (0,1)$ such that

$$B \in \mathcal{U}(A,\gamma). \tag{4.98}$$

Assume that

$$C \in \mathcal{U}(A,\gamma) \quad \text{and} \quad x \in K. \tag{4.99}$$

By Lemma 4.21, (4.99), (4.96) and (4.95),

$$
\begin{aligned}
f(Cx)-f(x_*) &\le \big[(1-\gamma)+\operatorname{Lip}(f)\operatorname{Lip}(C-A_\gamma)c_0^{-1}\big]\big(f(x)-f(x_*)\big) \\
&\le \lambda_\gamma\big(f(x)-f(x_*)\big).
\end{aligned}
$$

This implies that for each $x \in K$ and each natural number n,

$$f\big(C^n x\big)-f(x_*) \le \lambda_\gamma^n\big(f(x)-f(x_*)\big).$$

When combined with (4.90), this last inequality implies, in its turn, that for each $x \in K$ and each integer $n \ge 1$,

$$\big\|C^n x - x_*\big\| \le c_0^{-1}\big(f\big(C^n x\big)-f(x_*)\big) \le c_0^{-1}\lambda_\gamma^n\big(f(x)-f(x_*)\big).$$

This completes the proof of Theorem 4.20. □