Chapter 3
Contractive Mappings

In this chapter we consider the class of contractive mappings and show that a typical nonexpansive mapping (in the sense of Baire's categories) is contractive. We also study nonexpansive mappings which are contractive with respect to a given subset of their domain.

3.1 Many Nonexpansive Mappings Are Contractive

Assume that $(X, \|\cdot\|)$ is a Banach space and let K be a bounded, closed and convex subset of X. Denote by $\mathcal{A}$ the set of all operators $A : K \to K$ such that

$$\|Ax - Ay\| \le \|x - y\| \quad \text{for all } x, y \in K. \tag{3.1}$$

In other words, the set $\mathcal{A}$ consists of all the nonexpansive self-mappings of K. Set

$$d(K) = \sup\bigl\{\|x - y\| : x, y \in K\bigr\}. \tag{3.2}$$

We equip the set $\mathcal{A}$ with the metric $h(\cdot,\cdot)$ defined by

$$h(A, B) = \sup\bigl\{\|Ax - Bx\| : x \in K\bigr\}, \quad A, B \in \mathcal{A}.$$

Clearly, the metric space $(\mathcal{A}, h)$ is complete.

We say that a mapping $A \in \mathcal{A}$ is contractive if there exists a decreasing function $\phi^A : [0, d(K)] \to [0, 1]$ such that

$$\phi^A(t) < 1 \quad \text{for all } t \in \bigl(0, d(K)\bigr] \tag{3.3}$$

and

$$\|Ax - Ay\| \le \phi^A\bigl(\|x - y\|\bigr)\|x - y\| \quad \text{for all } x, y \in K. \tag{3.4}$$

The notion of a contractive mapping, as well as its modifications and applications, were studied by many authors. See, for example, [85]. We now quote a convergence result which is valid in all complete metric spaces [114].

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Theorem 3.1 *Assume that $A \in \mathcal{A}$ is contractive. Then there exists $x_A \in K$ such that $A^n x \to x_A$ as $n \to \infty$, uniformly on K.*

In [131] we prove that a generic element in the space of all nonexpansive mappings is contractive. In [137] we show that the set of all noncontractive mappings is not only of the first category, but also σ-porous. Namely, the following result was obtained there.

Theorem 3.2 *There exists a set $\mathcal{F} \subset \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{A}, h)$ and each $A \in \mathcal{F}$ is contractive.*

Proof For each natural number n, denote by $\mathcal{A}_n$ the set of all $A \in \mathcal{A}$ which have the following property:

(P1) There exists $\kappa \in (0, 1)$ such that $\|Ax - Ay\| \le \kappa \|x - y\|$ for all $x, y \in K$ satisfying $\|x - y\| \ge d(K)(2n)^{-1}$.

Let $n \ge 1$ be an integer. We will show that the set $\mathcal{A} \setminus \mathcal{A}_n$ is porous in $(\mathcal{A}, h)$. Set

$$\alpha = 8^{-1} \min\{d(K), 1\}(2n)^{-1}\bigl(d(K) + 1\bigr)^{-1}. \tag{3.5}$$

Fix $\theta \in K$. Let $A \in \mathcal{A}$ and $r \in (0, 1]$. Set

$$\gamma = 2^{-1} r\bigl(d(K) + 1\bigr)^{-1} \tag{3.6}$$

and define

$$A_\gamma x = (1 - \gamma)Ax + \gamma\theta, \quad x \in K. \tag{3.7}$$

Clearly, $A_\gamma \in \mathcal{A}$,

$$h(A_\gamma, A) \le \gamma d(K), \tag{3.8}$$

and for all $x, y \in K$,

$$\|A_\gamma x - A_\gamma y\| \le (1 - \gamma)\|Ax - Ay\| \le (1 - \gamma)\|x - y\|. \tag{3.9}$$

Assume that $B \in \mathcal{A}$ and

$$h(B, A_\gamma) \le \alpha r. \tag{3.10}$$

We will show that $B \in \mathcal{A}_n$.

Let

$$x, y \in K \quad \text{and} \quad \|x - y\| \ge (2n)^{-1} d(K). \tag{3.11}$$

It follows from (3.9) and (3.11) that

$$\|x - y\| - \|A_\gamma x - A_\gamma y\| \ge \gamma\|x - y\| \ge \gamma d(K)(2n)^{-1}. \tag{3.12}$$

By (3.10),

$$\|Bx - By\| \le \|Bx - A_\gamma x\| + \|A_\gamma x - A_\gamma y\| + \|A_\gamma y - By\| \le \|A_\gamma x - A_\gamma y\| + 2\alpha r.$$

When combined with (3.12), (3.6), and (3.5), this implies that

$$\begin{aligned}\|x-y\| - \|Bx - By\| &\ge \|x-y\| - \|A_\gamma x - A_\gamma y\| - 2\alpha r \\ &\ge \gamma d(K)(2n)^{-1} - 2\alpha r \\ &= 2^{-1} r\left[(2n)^{-1} d(K)\big(d(K)+1\big)^{-1} - 4\alpha\right] \\ &\ge 2^{-1} r d(K)(4n)^{-1}\big(d(K)+1\big)^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}\|Bx - By\| &\le \|x-y\| - r d(K)\big(d(K)+1\big)^{-1}(8n)^{-1} \\ &\le \|x-y\|\Big(1 - r(8n)^{-1}\big(d(K)+1\big)^{-1}\Big).\end{aligned}$$

Since this holds for all $x, y \in K$ satisfying (3.11), we conclude that $B \in \mathcal{A}_n$. Thus each $B \in \mathcal{A}$ satisfying (3.10) belongs to $\mathcal{A}_n$. In other words,

$$\big\{B \in \mathcal{A} : h(B, A_\gamma) \le \alpha r\big\} \subset \mathcal{A}_n. \tag{3.13}$$

If $B \in \mathcal{A}$ satisfies (3.10), then by (3.8), (3.5) and (3.6), we have

$$h(A, B) \le h(B, A_\gamma) + h(A_\gamma, A) \le \alpha r + \gamma d(K) \le 8^{-1} r + 2^{-1} r \le r.$$

Thus

$$\big\{B \in \mathcal{A} : h(B, A_\gamma) \le \alpha r\big\} \subset \big\{B \in \mathcal{A} : h(B, A) \le r\big\}.$$

When combined with (3.13), this inclusion implies that $\mathcal{A} \setminus \mathcal{A}_n$ is porous in $(\mathcal{A}, h)$. Set $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{A}_n$. Clearly, $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{A}, h)$. By property (P1), each $A \in \mathcal{F}$ is contractive. □

3.2 Attractive Sets

In this section, we study nonexpansive mappings which are contractive with respect to a given subset of their domain.

Assume that $(X, \|\cdot\|)$ is a Banach space and that K is a closed, bounded and convex subset of X. Once again, denote by $\mathcal{A}$ the set of all mappings $A : K \to K$ such that

$$\|Ax - Ay\| \le \|x - y\| \quad \text{for all } x, y \in K. \tag{3.14}$$

For each $x \in K$ and each subset $E \subset K$, let

$$\rho(x, E) = \inf\big\{\|x - y\| : y \in E\big\}. \tag{3.15}$$

Let F be a nonempty, closed and convex subset of K. Denote by $\mathcal{A}^{(F)}$ the set of all $A \in \mathcal{A}$ such that $Ax = x$ for all $x \in F$. Clearly, $\mathcal{A}^{(F)}$ is a closed subset of $(\mathcal{A}, h)$. In what follows we consider the complete metric space $(\mathcal{A}^{(F)}, h)$.

An operator $A \in \mathcal{A}^{(F)}$ is said to be contractive with respect to F if there exists a decreasing function $\phi^A : [0, d(K)] \to [0, 1]$ such that

$$\phi^A(t) < 1 \quad \text{for all } t \in \big(0, d(K)\big] \tag{3.16}$$

and

$$\rho(Ax, F) \le \phi^A\big(\rho(x, F)\big)\rho(x, F) \quad \text{for all } x \in K. \tag{3.17}$$

We now show that if $\mathcal{A}^{(F)}$ contains a retraction, then the complement of the set of contractive mappings (with respect to F) in $\mathcal{A}^{(F)}$ is σ-porous. This result was also obtained in [137].

Theorem 3.3 *Assume that there exists $Q \in \mathcal{A}^{(F)}$ such that*

$$Q(K) = F. \tag{3.18}$$

Then there exists a set $\mathcal{F} \subset \mathcal{A}^{(F)}$ such that $\mathcal{A}^{(F)} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{A}^{(F)}, h)$ and each $B \in \mathcal{F}$ is contractive with respect to F.

Proof For each natural number n, denote by $\mathcal{A}_n$ the set of all $A \in \mathcal{A}^{(F)}$ which have the following property:

(P2) There exists $\kappa \in (0, 1)$ such that $\rho(Ax, F) \le \kappa\rho(x, F)$ for all $x \in K$ such that $\rho(x, F) \ge \min\{d(K), 1\}/n$. Define

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{A}_n. \tag{3.19}$$

Clearly, each element of $\mathcal{F}$ is contractive with respect to F. We need to show that $\mathcal{A}^{(F)} \setminus \mathcal{A}_n$ is porous in $(\mathcal{A}^{(F)}, h)$ for all integers $n \ge 1$. To this end, let $n \ge 1$ be an integer and set

$$\alpha = \big(d(K) + 1\big)^{-1} \min\big\{d(K), 1\big\}(16n)^{-1}. \tag{3.20}$$

Let $A \in \mathcal{A}^{(F)}$ and $r \in (0, 1]$. Set

$$\gamma = 2^{-1} r\big(d(K) + 1\big)^{-1} \tag{3.21}$$

and define

$$A_\gamma x = (1 - \gamma)Ax + \gamma Qx, \quad x \in K. \tag{3.22}$$

It is obvious that $A_\gamma \in \mathcal{A}^{(F)}$. By (3.22),

$$h(A, A_\gamma) \le \sup\{\|A_\gamma x - Ax\| : x \in K\}$$
$$\le \gamma \sup\{\|Ax - Qx\| : x \in K\} \le \gamma d(K). \tag{3.23}$$

Let $B \in \mathcal{A}^{(F)}$ be such that

$$h(A_\gamma, B) \le \alpha r. \tag{3.24}$$

Then by (3.24), (3.23), (3.21), and (3.20),

$$h(A, B) \le h(A, A_\gamma) + h(A_\gamma, B) \le \gamma d(K) + \alpha r$$
$$< 1/2r + r/2 \le r.$$

Thus (3.24) implies that $h(A, B) \le r$ and

$$\{C \in \mathcal{A}^{(F)} : h(A_\gamma, C) \le \alpha r\}$$
$$\subset \{C \in \mathcal{A}^{(F)} : h(A, C) \le r\}. \tag{3.25}$$

Let $x \in K$ with

$$\rho(x, F) \ge \min\{d(K), 1\}/n. \tag{3.26}$$

For each $\varepsilon > 0$, there exists $z \in F$ such that $\rho(x, F) + \varepsilon \ge \|x - z\|$, and by (3.22) and (3.18),

$$\rho(A_\gamma x, F) = \rho\big((1-\gamma)Ax + \gamma Qx, F\big)$$
$$\le \big((1-\gamma)Ax + Qx\big) - \big((1-\gamma)z + \gamma Qx\big) \le (1-\gamma)\|Ax - z\|$$
$$\le (1-\gamma)\|x - z\| \le (1-\gamma)\rho(x, F) + \varepsilon(1-\gamma).$$

Since ε is an arbitrary positive number, we conclude that

$$\rho(A_\gamma x, F) \le (1-\gamma)\rho(x, F).$$

Since $|\rho(y_1, F) - \rho(y_2, F)| \le \|y_1 - y_2\|$ for all $y_1, y_2 \in K$, it follows from (3.24) that

$$\rho(Bx, F) \le \|A_\gamma x - Bx\| + \rho(A_\gamma x, F) \le \alpha r + \rho(A_\gamma x, F)$$
$$\le \alpha r + (1-\gamma)\rho(x, F),$$

and

$$\rho(Bx, F) \le (1-\gamma)\rho(x, F) + \alpha r.$$

It now follows from this inequality, (3.26), (3.20) and (3.21) that

$$\begin{aligned}\rho(Bx,F) &\le \rho(x,F)\big(1-\gamma+\alpha r\big(\rho(x,F)\big)^{-1}\big)\\ &\le \rho(x,F)\big[1-2^{-1}r\big(d(K)+1\big)^{-1}+\alpha r\big(\min\big\{d(K),1\big\}/n\big)^{-1}\big]\\ &\le \rho(x,F)\big[1-r2^{-1}\big(d(K)+1\big)^{-1}+r\big(16\big(d(K)+1\big)\big)^{-1}\big]\\ &\le \rho(x,F)\big(1-r4^{-1}d(K+1)^{-1}\big).\end{aligned}$$

Thus

$$\rho(Bx,F) \le \rho(x,F)\big(1-r4^{-1}\big(d(K)+1\big)^{-1}\big)$$

for each $x \in K$ satisfying (3.26). This fact implies that $B \in \mathcal{A}_n$. Since this inclusion holds for any B satisfying (3.24), combining it with (3.25) we obtain that

$$\big\{C \in \mathcal{A}^{(F)} : h(A_\gamma, C) \le \alpha r\big\} \subset \big\{C \in \mathcal{A}^{(F)} : h(A,C) \le r\big\} \cap \mathcal{A}_n.$$

This shows that $\mathcal{A}^{(F)} \setminus \mathcal{A}_n$ is indeed porous in $(\mathcal{A}^{(F)}, h)$. □

3.3 Attractive Subsets of Unbounded Spaces

In this section we continue to study nonexpansive mappings which are contractive with respect to a given subset of their domain.

Assume that (X,ρ) is a hyperbolic complete metric space and that K is a closed (not necessarily bounded) and ρ-convex subset of X. Denote by $\mathcal{A}$ the set of all mappings $A : K \to K$ such that

$$\rho(Ax,Ay) \le \rho(x,y) \quad \text{for all } x,y \in K. \tag{3.27}$$

For each $x \in K$ and each subset $E \subset K$, let $\rho(x,E) = \inf\{\rho(x,y) : y \in E\}$. For each $x \in K$ and each $r > 0$, set

$$B(x,r) = \big\{y \in K : \rho(x,y) \le r\big\}. \tag{3.28}$$

Fix $\theta \in K$. For the set $\mathcal{A}$ we consider the uniformity determined by the following base:

$$E(n,\varepsilon) = \big\{(A,B) \in \mathcal{A} \times \mathcal{A} : \rho(Ax,Bx) \le \varepsilon, x \in B(\theta,n)\big\}, \tag{3.29}$$

where $\varepsilon > 0$ and n is a natural number. Clearly the space $\mathcal{A}$ with this uniformity is metrizable and complete. We equip the space $\mathcal{A}$ with the topology induced by this uniformity.

Let F be a nonempty, closed and ρ-convex subset of K. Denote by $\mathcal{A}^{(F)}$ the set of all $A \in \mathcal{A}$ such that $Ax = x$ for all $x \in F$. Clearly, $\mathcal{A}^{(F)}$ is a closed subset of $\mathcal{A}$. We consider the topological subspace $\mathcal{A}^{(F)} \subset \mathcal{A}$ with the relative topology.

An operator $A \in \mathcal{A}^{(F)}$ is said to be contractive with respect to F if for any natural number n there exists a decreasing function $\phi_n^A : [0, \infty) \to [0, 1]$ such that

$$\phi_n^A(t) < 1 \quad \text{for all } t > 0 \tag{3.30}$$

and

$$\rho(Ax, F) \le \phi_n^A\big(\rho(x, F)\big)\rho(x, F) \quad \text{for all } x \in B(\theta, n). \tag{3.31}$$

Clearly, this definition does not depend on our choice of θ.

We begin our discussion of such mappings by proving that the set F attracts all the iterates of A. This result was obtained in [131].

Theorem 3.4 *Let $A \in \mathcal{A}^{(F)}$ be contractive with respect to F. Then there exists $B \in \mathcal{A}^{(F)}$ such that $B(K) = F$ and $A^n x \to Bx$ as $n \to \infty$, uniformly on $B(\theta, m)$ for any natural number m.*

Proof We may assume without loss of generality that $\theta \in F$. Then for each real $r > 0$,

$$C\big(B(\theta, r)\big) \subset B(\theta, r) \quad \text{for all } C \in \mathcal{A}^{(F)}. \tag{3.32}$$

Let r be a natural number. To prove the theorem, it is sufficient to show that there exists $B : B(\theta, r) \to F$ such that

$$A^n x \to Bx \quad \text{as } n \to \infty, \text{ uniformly on } B(\theta, r). \tag{3.33}$$

There exists a decreasing function $\phi_r^A : [0, \infty) \to [0, 1]$ such that

$$\phi_r^A(t) < 1 \quad \text{for all } t > 0 \tag{3.34}$$

and

$$\rho(Ax, F) \le \phi_r^A\big(\rho(x, F)\big)\rho(x, F) \quad \text{for all } x \in B(\theta, r). \tag{3.35}$$

Let $\varepsilon \in (0, 1)$. Choose a natural number $m \ge 4$ such that

$$\phi_r^A(\varepsilon r)^m < 8^{-1}\varepsilon. \tag{3.36}$$

Let $x \in B(\theta, r)$. We will show that

$$\rho\big(A^m x, F\big) < \varepsilon r. \tag{3.37}$$

Assume the contrary. Then for each $i = 0, \ldots, m$, $\rho(A^i x, F) \ge \varepsilon r$, and by (3.35) and (3.32),

$$A^i x \in B(\theta, r), \quad \rho\big(A^{i+1}x, F\big) \le \phi_r^A\big(\rho\big(A^i x, F\big)\big)\rho\big(A^i x, F\big) \\ \le \phi_r^A(\varepsilon r)\rho\big(A^i x, F\big).$$

When combined with (3.36), these inequalities imply that

$$\rho\big(A^m x, F\big) \le \phi_r^A(\varepsilon r)^m \rho(x, F) \le 8^{-1}\varepsilon\rho(x, \theta) \le 8^{-1}\varepsilon r,$$

a contradiction. Therefore (3.27) is valid and for each $x \in B(\theta, r)$, there exists $C_\varepsilon(x) \in F$ such that $\rho(A^m x, C_\varepsilon x) < \varepsilon r$. This implies that for each $x \in B(\theta, r)$,

$$\rho\big(A^i x, C_\varepsilon x\big) < \varepsilon r \quad \text{for all integers } i \ge m. \tag{3.38}$$

Since ε is an arbitrary number in (0, 1), we conclude that for each $x \in B(\theta, r)$, $\{A^i x\}_{i=1}^\infty$ is a Cauchy sequence and there exists $Bx = \lim_{i\to\infty} A^i x$. Clearly,

$$\rho\big(Bx, C_\varepsilon(x)\big) \le \varepsilon r \quad \text{for all } x \in B(\theta, r). \tag{3.39}$$

Since (3.39) is true for any ε in (0, 1), we conclude that $B(B(\theta, r)) \subset F$.

By (3.39) and (3.38), for each $x \in B(\theta, r)$,

$$\rho\big(A^i x, Bx\big) \le 2\varepsilon r \quad \text{for all integers } i \ge m.$$

Finally, since $\varepsilon \in (0, 1)$ is arbitrary, we conclude that (3.33) is valid. This completes the proof of Theorem 3.4. □

Proposition 3.5 *Assume that $A, B \in \mathcal{A}^{(F)}$ and that A is contractive with respect to F. Then AB and BA are also contractive with respect to F.*

Proof We may assume that $\theta \in F$. Then for each real $r > 0$,

$$C\big(B(\theta, r)\big) \subset B(\theta, r) \quad \text{for all } C \in \mathcal{A}^{(F)}. \tag{3.40}$$

Fix $r > 0$. There exists a decreasing function $\phi_r^A : [0, \infty) \to [0, 1]$ such that

$$\phi_r^A(t) < 1 \quad \text{for all } t > 0 \tag{3.41}$$

and

$$\rho(Ax, F) \le \phi_r^A\big(\rho(x, F)\big)\rho(x, F) \quad \text{for all } x \in B(\theta, r). \tag{3.42}$$

By (3.42), for each $x \in B(\theta, r)$,

$$\begin{aligned}\rho(BAx, F) &= \inf\big\{\rho(BAx, y) : y \in F\big\} \le \inf\big\{\rho(Ax, y) : y \in F\big\}\\ &= \rho(Ax, F) \le \phi_r^A\big(\rho(x, F)\big)\rho(x, F).\end{aligned}$$

Therefore BA is contractive with respect to F.

Let now x belong to $B(\theta, r)$. By (3.42) and (3.40), $Bx \in B(\theta, r)$ and

$$\rho(ABx, F) \le \phi_r^A\big(\rho(Bx, F)\big)\rho(Bx, F). \tag{3.43}$$

There are two cases: (1) $\rho(Bx, F) \geq 2^{-1}\rho(x, F)$; (2) $\rho(Bx, F) < 2^{-1}\rho(x, F)$. In the first case, we have by (3.43),

$$\rho(ABx, F) \leq \phi_r^A\big(2^{-1}\rho(x, F)\big)\rho(Bx, F) \leq \phi_r^A\big(2^{-1}\rho(x, F)\big)\rho(x, F),$$

and in the second case, (3.43) implies that

$$\rho(ABx, F) \leq \rho(Bx, F) \leq 2^{-1}\rho(x, F).$$

Thus in both cases we obtain that

$$\begin{aligned}\rho(ABx, F) &\leq \max\big\{\phi_r^A\big(2^{-1}\rho(x, F)\big), 2^{-1}\big\}\rho(x, F)\\ &= \psi\big(\rho(x, F)\big)\rho(x, F),\end{aligned}$$

where $\psi(t) = \max\{\phi_r^A(2^{-1}t), 2^{-1}\}$, $t \in [0, \infty)$. Therefore AB is also contractive with respect to F. Proposition 3.5 is proved. □

We now show that if $\mathcal{A}^{(F)}$ contains a retraction, then almost all the mappings in $\mathcal{A}^{(F)}$ are contractive with respect to F.

Theorem 3.6 *Assume that there exists*

$$Q \in \mathcal{A}^{(F)} \quad \textit{such that} \quad Q(K) = F. \tag{3.44}$$

Then there exists a set $\mathcal{F} \subset \mathcal{A}^{(F)}$ which is a countable intersection of open and everywhere dense sets in $\mathcal{A}^{(F)}$ such that each $B \in \mathcal{F}$ is contractive with respect to F.

Proof We may assume that $\theta \in F$. Then for each real $r > 0$,

$$C\big(B(\theta, r)\big) \subset B(\theta, r) \quad \text{for all } C \in \mathcal{A}^{(F)}. \tag{3.45}$$

For each $A \in \mathcal{A}^{(F)}$ and each $\gamma \in (0, 1)$, define $A_\gamma \in \mathcal{A}^{(F)}$ by

$$A_\gamma x = (1-\gamma)Ax \oplus \gamma Qx, \quad x \in K. \tag{3.46}$$

Clearly, for each $A \in \mathcal{A}^{(F)}$, $A_\gamma \to A$ as $\gamma \to 0^+$ in $\mathcal{A}^{(F)}$. Therefore the set $\{A_\gamma : A \in \mathcal{A}^{(F)}, \gamma \in (0, 1)\}$ is everywhere dense in $\mathcal{A}^{(F)}$.

Let $A \in \mathcal{A}^{(F)}$ and $\gamma \in (0, 1)$. Evidently,

$$\begin{aligned}\rho(A_\gamma x, F) &= \inf_{y\in F}\big\{\rho\big((1-\gamma)Ax \oplus \gamma Qx, y\big)\big\}\\ &\leq \inf_{y\in F}\big\{\rho\big((1-\gamma)Ax \oplus \gamma Qx, (1-\gamma)y \oplus \gamma Qx\big)\big\}\\ &\leq \inf_{y\in F}\big\{(1-\gamma)\rho(Ax, y)\big\} \leq (1-\gamma)\rho(x, F)\end{aligned}$$

for all $x \in K$. Thus

$$\rho(A_\gamma x, F) \le (1-\gamma)\rho(x, F) \quad \text{for all } x \in K. \tag{3.47}$$

For each integer $i \ge 1$, denote by $U(A, \gamma, i)$ an open neighborhood of A_γ in $\mathcal{A}^{(F)}$ for which

$$U(A, \gamma, i) \subset \left\{B \in \mathcal{A}^{(F)} : (B, A_\gamma) \in E\big(2^i, 8^{-i}\gamma\big)\right\} \tag{3.48}$$

(see (3.29)).

We will show that for each $A \in \mathcal{A}^{(F)}$, each $\gamma \in (0,1)$ and each integer $i \ge 1$, the following property holds:

P(2) For each $B \in U(A, \gamma, i)$ and each $x \in B(\theta, 2^i)$ satisfying $\rho(x, F) \ge 4^{-i}$, the inequality $\rho(Bx, F) \le (1 - 2^{-1}\gamma)\rho(x, F)$ is true.

Indeed, let $A \in \mathcal{A}^{(F)}$, $\gamma \in (0,1)$ and let $i \ge 1$ be an integer. Assume that

$$B \in U(A, \gamma, i), \qquad x \in B\big(\theta, 2^i\big) \quad \text{and} \quad \rho(x, F) \ge 4^{-i}. \tag{3.49}$$

Using (3.47), (3.48) and (3.49), we see that

$$\begin{aligned}\rho(Bx, F) &\le \rho(A_\gamma x, F) + 8^{-i}\gamma \le (1-\gamma)\rho(x, F) + 8^{-i}\gamma \\ &\le (1-\gamma)\rho(x, F) + 2^{-1}\gamma\rho(x, F) \le \big(1 - 2^{-1}\gamma\big)\rho(x, F).\end{aligned}$$

Thus property P(2) holds for each $A \in \mathcal{A}^{(F)}$, each $\gamma \in (0,1)$ and each integer $i \ge 1$. Define

$$\mathcal{F} = \bigcap_{q=1}^{\infty} \bigcup \left\{U(A, \gamma, i) : A \in \mathcal{A}^{(F)}, \gamma \in (0,1), i \ge q\right\}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open and everywhere dense sets in $\mathcal{A}^{(F)}$.

Let $B \in \mathcal{F}$. To show that B is contractive with respect to F, it is sufficient to show that for each $r > 0$ and each $\varepsilon \in (0,1)$, there is $\kappa \in (0,1)$ such that

$$\rho(Bx, F) \le \kappa\rho(x, F) \quad \text{for each } x \in B(\theta, r) \text{ satisfying } \rho(x, F) \ge \varepsilon.$$

Let $r > 0$ and $\varepsilon \in (0,1)$. Choose a natural number q such that

$$2^q > 8r \quad \text{and} \quad 2^{-q} < 8^{-1}\varepsilon.$$

There exist $A \in \mathcal{A}^{(F)}$, $\gamma \in (0,1)$ and an integer $i \ge q$ such that $B \in U(A, \gamma, i)$. By property P(2), for each $x \in B(\theta, r) \subset B(\theta, 2^i)$ satisfying $\rho(x, F) \ge \varepsilon > 2^{-i}$, the following inequality holds:

$$\rho(Bx, F) \le \big(1 - 2^{-1}\gamma\big)\rho(x, F).$$

Thus B is contractive with respect to F. This completes the proof of Theorem 3.6. □

3.4 A Contractive Mapping with no Strictly Contractive Powers

Let

$$X = [0, 1] \quad \text{and} \quad \rho(x, y) = |x - y| \quad \text{for each } x, y \in X.$$

In this section, which is based on [155], we construct a contractive mapping $A : [0, 1] \to [0, 1]$ such that none of its powers is a strict contraction.

We begin by setting

$$A(0) = 0. \tag{3.50}$$

Next, we define, for each natural number n, the mapping A on the interval $[(n + 1)^{-1}, n^{-1}]$ by

$$A\big((n+1)^{-1} + t\big) = (n+2)^{-1} + t\big(n^{-1} - (n+1)^{-1}\big)^{-1}\big((n+1)^{-1} - (n+2)^{-1}\big)$$
$$\text{for all } t \in \big[0, n^{-1} - (n+1)^{-1}\big]. \tag{3.51}$$

It is clear that for each natural number n,

$$A\big(n^{-1}\big) = (n+1)^{-1}, \tag{3.52}$$

the restriction of A to the interval $[(n + 1)^{-1}, n^{-1}]$ is affine, and that the mapping $A : [0, 1] \to [0, 1]$ is well defined.

First, we show that A is nonexpansive, that is, $|Ax - Ay| \le |x - y|$ for all $x, y \in [0, 1]$.

Indeed, if $x \in [0, 1]$, then

$$\big|Ax - A(0)\big| \le |x|. \tag{3.53}$$

Assume now that n is a natural number and that

$$x, y \in \big[(n+1)^{-1}, n^{-1}\big]. \tag{3.54}$$

By (3.51) and (3.54),

$$\begin{aligned}
&|Ax - Ay| \\
&\quad = \big|(n+2)^{-1} + \big(x - (n+1)^{-1}\big)\big(n^{-1} - (n+1)^{-1}\big)^{-1}\big((n+1)^{-1} - (n+2)^{-1}\big) \\
&\qquad - \big[(n+2)^{-1} + \big(y - (n+1)^{-1}\big)\big(n^{-1} - (n+1)^{-1}\big)^{-1} \\
&\qquad \times \big((n+1)^{-1} - (n+2)^{-1}\big)\big]\big| \\
&\quad = |x - y|\big(n^{-1} - (n+1)^{-1}\big)^{-1}\big((n+1)^{-1} - (n+2)^{-1}\big) \\
&\quad = |x - y|n(n+1)\big((n+1)(n+2)\big)^{-1} = |x - y|n(n+2)^{-1}.
\end{aligned}$$

Thus for each natural number n and each $x, y \in [(n+1)^{-1}, n^{-1}]$,

$$|Ax - Ay| \le |x - y| n(n+2)^{-1}. \tag{3.55}$$

Together with (3.53) this last inequality implies that

$$|Ax - Ay| \le |x - y| \quad \text{for all } x, y \in [0, 1], \tag{3.56}$$

as claimed.

Next, we show that the power A^m is not a strict contraction for any integer $m \ge 1$. Assume the converse. Then there would exist a natural number m and $c \in (0, 1)$ such that for each $x, y \in [0, 1]$,

$$\left|A^m x - A^m y\right| \le c|x - y|. \tag{3.57}$$

Since

$$(m+i)(m+i+1)i^{-1}(i+1)^{-1} \to 1 \quad \text{as } i \to \infty,$$

there is an integer $p \ge 4$ such that

$$p(p+1) > (p+m)(p+m+1)c. \tag{3.58}$$

By (3.52), (3.50) and (3.58),

$$\begin{aligned} &A^m\left(p^{-1}\right) - A^m\left((p+1)^{-1}\right) \\ &\quad = (p+m)^{-1} - (p+m+1)^{-1} = (p+m)^{-1}(p+m+1)^{-1} \\ &\quad > cp^{-1}(p+1)^{-1} = c\left(p^{-1} - (p+1)^{-1}\right), \end{aligned}$$

which contradicts (3.57).

The contradiction we have reached proves that A^m is not a strict contraction for any integer $m \ge 1$.

Finally, we show that A is contractive. Let $\varepsilon \in (0, 1)$. We claim that there exists $c \in (0, 1)$ such that

$$|Ax - Ay| \le c|x - y| \quad \text{for each } x, y \in [0, 1] \text{ satisfying } |x - y| \ge \varepsilon. \tag{3.59}$$

Indeed, choose a natural number $p \ge 4$ such that

$$p > 18\varepsilon^{-2}, \tag{3.60}$$

and assume that

$$x, y \in [0, 1] \quad \text{and} \quad |x - y| \ge \varepsilon. \tag{3.61}$$

We may assume without loss of generality that

$$y > x. \tag{3.62}$$

There are two cases:

$$x < (4p)^{-1}; \tag{3.63}$$

$$x \geq (4p)^{-1}. \tag{3.64}$$

Assume that (3.63) holds. There exists a natural number n such that

$$(1+n)^{-1} < y \leq n^{-1}. \tag{3.65}$$

By (3.65), (3.62) and (3.61),

$$\varepsilon \leq y \leq 1/n, \qquad (n+2)^{-1} \geq (3n)^{-1} \geq \varepsilon/3. \tag{3.66}$$

By (3.65) and (3.51),

$$\begin{aligned} Ay &= (n+2)^{-1} + \big(y-(n+1)^{-1}\big)\big(n^{-1}-(n+1)^{-1}\big)^{-1}\big((n+1)^{-1}-(n+2)^{-1}\big) \\ &= (n+2)^{-1} + \big(y-(n+1)^{-1}\big)n(n+1)(n+1)^{-1}(n+2)^{-1} \\ &\leq y-(n+1)^{-1}+(n+2)^{-1} \end{aligned}$$

and

$$y - Ay \geq (n+1)^{-1}(n+2)^{-1}.$$

When combined with (3.66), the above inequality implies that

$$Ay - Ax \leq Ay \leq y-(n+1)^{-1}(n+2)^{-1} \leq y-(n+2)^{-2} \leq y-\varepsilon^2/9. \tag{3.67}$$

By (3.63), (3.60) and (3.67),

$$\begin{aligned} \big(1-18^{-1}\varepsilon^2\big)(y-x) &\geq \big(1-18^{-1}\varepsilon^2\big)y-x \geq \big(1-18^{-1}\varepsilon^2\big)y-(4p)^{-1} \\ &\geq y-\varepsilon^2/18-(4p)^{-1} \geq y-\varepsilon^2/18-\varepsilon^2/18 \\ &\geq Ay-Ax. \end{aligned}$$

Thus we have shown that if (3.63) holds, then

$$|Ax-Ay| \leq \big(1-\varepsilon^2/18\big)|x-y|. \tag{3.68}$$

Now assume that (3.64) holds. By (3.64) and (3.62),

$$x, y \in \big[(4p)^{-1}, 1\big].$$

In view of (3.55), the Lipschitz constant of the restriction of A to the interval $[(4p)^{-1}, 1]$ does not exceed $(4p+2)(4p+4)^{-1}$ and therefore we have

$$|Ax-Ay| \leq (4p+2)(4p+4)^{-1}|x-y|.$$

By this inequality and (3.68), we see that, in both cases,

$$|Ax - Ay| \le \max\{(1 - \varepsilon^2/18), (4p+2)(4p+4)^{-1}\}|x - y|.$$

Since this inequality holds for each $x, y \in X$ satisfying (3.61), we conclude that (3.59) is satisfied and therefore A is contractive.

3.5 A Power Convergent Mapping with no Contractive Powers

Let $X = [0, 1]$ and let $\rho(x, y) = |x - y|$ for all $x, y \in X$. In this section, which is based on [155], we construct a mapping $A : [0, 1] \to [0, 1]$ such that

$$|Ax - Ay| \le |x - y| \quad \text{for all } x, y \in [0, 1],$$
$$A^n x \to 0 \quad \text{as } n \to \infty, \text{ uniformly on } [0, 1],$$

and for each integer $m \ge 0$, the power A^m is not contractive.

To this end, let

$$A(0) = 0 \tag{3.69}$$

and for $t \in [2^{-1}, 1]$, set

$$A(t) = t - 1/4. \tag{3.70}$$

Clearly,

$$A(1) = 3/4 \quad \text{and} \quad A(1/2) = 1/4. \tag{3.71}$$

For $t \in [4^{-1}, 2^{-1})$, set

$$A(t) = 4^{-1} - 16^{-1} + (t - 4^{-1})4^{-1}. \tag{3.72}$$

Clearly, A is continuous on $[4^{-1}, 1]$ and

$$A(4^{-1}) = 4^{-1} - 16^{-1}. \tag{3.73}$$

Now let $n \ge 2$ be a natural number. We define the mapping A on the interval $[2^{-2^n}, 2^{-2^{n-1}}]$ as follows. For each $t \in [2^{-2^n+1}, 2^{-2^{n-1}}]$, set

$$A(t) = t - 2^{-2^n}. \tag{3.74}$$

Clearly,

$$A(2^{-2^n+1}) = 2^{-2^n} \quad \text{and} \quad A(2^{-2^{n-1}}) = 2^{-2^{n-1}} - 2^{-2^n}. \tag{3.75}$$

For $t \in [2^{-2^n}, 2^{-2^n+1})$, set

$$\begin{aligned} A(t) &= 2^{-2^n} - 2^{-2^{n+1}} + \left(t - 2^{-2^n}\right)2^{2^n}\left(2^{-2^{n+1}}\right) \\ &= 2^{-2^n} - 2^{-2^{n+1}} + 2^{-2^n}\left(t - 2^{-2^n}\right). \end{aligned} \tag{3.76}$$

It is clear that

$$A\left(2^{-2^n}\right) = 2^{-2^n} - 2^{-2^{n+1}}$$

and

$$\lim_{t \to (2^{-2^n+1})^+} A(t) = 2^{-2^n} - 2^{-2^{n+1}} + 2^{-2^n}\left(2^{-2^n+1} - 2^{-2^n}\right) = 2^{-2^n}. \tag{3.77}$$

It follows from (3.74)–(3.77) that the mapping A is continuous on each one of the intervals $[2^{-2^n}, 2^{-2^{n-1}}]$, $n = 2, 3, \ldots$. It is not difficult to check that A is well defined on $[0, 1]$ and that it is increasing.

By (3.70) and (3.72), for each $x \in [1/4, 1]$ we have $Ax < x$. We will now show that this inequality holds for all $x \in (0, 1]$.

Let $n \geq 2$ be an integer and let $x \in [2^{-2^n}, 2^{-2^{n-1}}]$. It is clear that $Ax < x$ if $x \in [2^{-2^n+1}, 2^{-2^{n-1}}]$. If $x \in [2^{-2^n}, 2^{-2^n+1})$, then by (3.74) and (3.75),

$$Ax < A\left(2^{-2^n+1}\right) \leq 2^{-2^n} \leq x.$$

Thus $Ax < x$ for all $x \in [2^{-2^n}, 2^{-2^{n-1}}]$ and for any integer $n \geq 2$. Therefore we have indeed shown that

$$Ax < x \quad \text{for all } x \in (0, 1], \tag{3.78}$$

as claimed.

Next, we will show that

$$|Ax - Ay| \leq |x - y| \quad \text{for each } x, y \in [0, 1]. \tag{3.79}$$

If $x = 0$ and $y > 0$, then

$$|Ay - Ax| = Ay \leq y = |y - x|. \tag{3.80}$$

Assume that $x, y \in (0, 1]$. Note that the restrictions of the mapping A to the interval $[1/4, 1]$ and to all of the intervals $[2^{-2n}, 2^{-2^{n-1}}]$, where $n \geq 2$ is an integer, are Lipschitz with Lipschitz constant one. This obviously implies that the mapping A is 1-Lipschitz on all of $(0, 1]$. Therefore (3.79) is true.

Let $x \in (0, 1]$. By (3.78), the sequence $\{A^n x\}_{n=1}^{\infty}$ is decreasing and there exists the limit

$$x_* = \lim_{n \to \infty} A^n x.$$

Clearly, $Ax_* = x_*$. If $x_* > 0$, then by (3.78), $Ax_* < x_*$, a contradiction. Thus $x_* = 0$ and $\lim_{n \to \infty} A^n(1) = 0$. Since the mapping A is increasing, this implies that

$$A^n x \to 0 \quad \text{as } n \to \infty, \text{ uniformly on } [0, 1].$$

Finally, we will show that for each integer $m \geq 1$, the power A^m is not contractive.

Indeed, let $m \geq 1$ be an integer. It is sufficient to show that there exist $x, y \in [0, 1]$ such that

$$x \neq y \quad \text{and} \quad \left|A^m - A^m y\right| = |x - y|.$$

To this end, choose a natural number $n \geq m + 4$ such that

$$2^{2^{n-1}} - 3 \geq m + 2. \tag{3.81}$$

Using induction and (3.74), we show that for each integer $i \in \{1, \ldots, 2^{2^{n-1}} - 2\}$,

$$A^i\left(2^{-2^{n-1}}\right) = 2^{-2^{n-1}} - i2^{-2^n} \geq 2^{-2^n+1}$$

and

$$A^i\left(2^{-2^{n-1}}\right) \in \left[2^{-2^n+1}, 2^{-2^n-1}\right].$$

Put

$$x = 2^{-2^{n-1}} \quad \text{and} \quad y = A\left(2^{-2^{n-1}}\right).$$

Then for $i = 1, \ldots, 2^{2^{n-1}} - 3$, we have

$$\left|A^i x - A^i y\right| = |x - y|,$$

and in view of (3.81),

$$\left|A^m x - A^m y\right| = |x - y|.$$

Thus the power A^m is not contractive, as asserted.

3.6 A Mapping with Nonuniformly Convergent Powers

In [155] we proved the following result.

Theorem 3.7 *Let (X, ρ) be a compact metric space, let a mapping $A : X \to X$ satisfy*

$$\rho(Ax, Ay) \leq \rho(x, y) \quad \textit{for each } x, y \in X, \tag{3.82}$$

and let $x_A \in X$ satisfy

$$A^n x \to x_A \quad \textit{as } n \to \infty, \textit{ for each } x \in X.$$

Then $A^n x \to x_A$ as $n \to \infty$, uniformly on X.

Proof Let $\varepsilon > 0$. For each $x \in X$, there is a natural number $n(x)$ such that

$$\rho\big(A^n x, x_A\big) \le \varepsilon/2 \quad \text{for all integers } n \ge n(x). \tag{3.83}$$

Let

$$x, y \in X \quad \text{with} \quad \rho(x, y) < \varepsilon/2. \tag{3.84}$$

By (3.83) and (3.84), for each integer $n \ge n(x)$,

$$\rho\big(A^n y, x_A\big) \le \rho\big(A^n y, A^n x\big) + \rho\big(A^n x, x_A\big) < \varepsilon/2 + \varepsilon/2.$$

Thus the following property holds:

(P) For each $x \in X$, each integer $n \ge n(x)$, and each $y \in X$ satisfying $\rho(x, y) < \varepsilon/2$, we have

$$\rho\big(A^n y, x_A\big) < \varepsilon.$$

Since X is compact, there exist finitely many points $x_1, \dots, x_q \in X$ such that

$$\bigcup_{i=1}^{q} \big\{y \in X : \rho(y, x_i) < \varepsilon/2\big\} = X.$$

Assume that $y \in X$ and that the integer $n \ge \max\{n(x_i) : i = 1, \dots, q\}$. Then there is $j \in \{1, \dots, q\}$ such that $\rho(y, x_j) < \varepsilon/2$. By property (P),

$$\rho\big(A^n y, x_A\big) < \varepsilon.$$

This completes the proof of Theorem 3.7. □

The following example was constructed in [155].

Let X be the set of all sequences $(x_1, x_2, \dots, x_n, \dots)$ such that $\sum_{i=1}^{\infty} |x_i| \le 1$ and set

$$\rho(x, y) = \rho\big((x_i), (y_i)\big) = \sum_{i=1}^{\infty} |x_i - y_i|.$$

In other words, (X, ρ) is the closed unit ball of ℓ_1. Clearly, (X, ρ) is a complete metric space. Define

$$A(x_1, x_2, \dots, x_n, \dots) = (x_2, x_2, \dots, x_n, \dots), \quad x = (x_1, x_2, \dots) \in X.$$

Then the mapping A is nonexpansive, and for each $x \in X$, $A^n x \to 0$ as $n \to \infty$.

However, if n is a natural number and e_n is the n-th unit vector of X, then $\rho(A^n e_{n+1}, 0) = 1$.

3.7 Two Results in Metric Fixed Point Theory

In this section, which is based on [115], we establish two fixed point theorems for certain mappings of contractive type. The first result is concerned with the case where such mappings take a nonempty and closed subset of a complete metric space X into X, and the second with an application of the continuation method to the case where they satisfy the Leray-Schauder boundary condition in Banach spaces.

The following result was obtained in [115].

Theorem 3.8 *Let K be a nonempty and closed subset of a complete metric space (X, ρ). Assume that $T : K \to X$ satisfies*

$$\rho(Tx, Ty) \le \phi\big(\rho(x, y)\big)\rho(x, y) \quad \textit{for each } x, y \in K, \tag{3.85}$$

where $\phi : [0, \infty) \to [0, 1]$ is a monotonically decreasing function such that $\phi(t) < 1$ for all $t > 0$.

Assume that $K_0 \subset K$ is a nonempty and bounded set with the following property:

(P1) *For each natural number n, there exists $x_n \in K_0$ such that $T^i x_n$ is defined for all $i = 1, \dots, n$.*

Then

(A) *the mapping T has a unique fixed point $\bar{x}$ in K;*
(B) *For each $M, \varepsilon > 0$, there exist $\delta > 0$ and a natural number k such that for each integer $n \ge k$ and each sequence $\{x_i\}_{i=0}^n \subset K$ satisfying*

$$\rho(x_0, \bar{x}) \le M \quad \textit{and} \quad \rho(x_{i+1}, Tx_i) \le \delta, \quad i = 0, \dots, n-1,$$

we have

$$\rho(x_i, \bar{x}) \le \varepsilon, \quad i = k, \dots, n. \tag{3.86}$$

Proof of Theorem 3.8(A) The uniqueness of $\bar{x}$ is obvious. To establish its existence, let $x_n \in K_0$ be, for each natural number n, the point provided by property (P1). Fix $\theta_0 \in K$. Since K_0 is bounded, there is $c_0 > 0$ such that

$$\rho(\theta, z) \le c_0 \quad \text{for all } z \in K_0. \tag{3.87}$$

Let $\varepsilon > 0$ be given. We will show that there exists a natural number k such that the following property holds:

(P2) If $n > k$ is an integer and if an integer i satisfies $k \le i < n$, then

$$\rho\big(T^i x_n, T^{i+1} x_n\big) \le \varepsilon. \tag{3.88}$$

Assume the contrary. Then for each natural number k, there exist natural numbers n_k and i_k such that

$$k \le i_k < n_k \quad \text{and} \quad \rho\big(T^{i_k} x_{n_k}, T^{i_k+1} x_{n_k}\big) > \varepsilon. \tag{3.89}$$

Choose a natural number k such that

$$k > \big(\varepsilon\big(1-\phi(\varepsilon)\big)\big)^{-1}\big(2c_0+\rho(\theta, T\theta)\big). \tag{3.90}$$

By (3.89) and (3.85),

$$\rho\big(T^i x_{n_k}, T^{i+1} x_{n_k}\big) > \varepsilon, \quad i = 0, \dots, i_k. \tag{3.91}$$

(Here we use the notation that $T^0 z = z$ for all $z \in K$.) It follows from (3.85), (3.91) and the monotonicity of ϕ that for all $i = 0, \dots, i_k - 1$,

$$\begin{aligned}\rho\big(T^{i+2} x_{n_k}, T^{i+1} x_{n_k}\big) &\le \phi\big(\rho\big(T^{i+1} x_{n_k}, T^i x_{n_k}\big)\big)\rho\big(T^{i+1} x_{n_k}, T^i x_{n_k}\big) \\ &\le \phi(\varepsilon)\rho\big(T^{i+1} x_{n_k}, T^i x_{n_k}\big)\end{aligned}$$

and

$$\begin{aligned}&\rho\big(T^{i+2} x_{n_k}, T^{i+1} x_{n_k}\big) - \rho\big(T^{i+1} x_{n_k}, T^i x_{n_k}\big) \\ &\quad \le \big(\phi(\varepsilon)-1\big)\rho\big(T^{i+1} x_{n_k}, T^i x_{n_k}\big) < -\big(1-\phi(\varepsilon)\big)\varepsilon.\end{aligned} \tag{3.92}$$

Inequalities (3.92) and (3.89) imply that

$$\begin{aligned}-\rho(x_{n_k}, T x_{n_k}) &\le \rho\big(T^{i_k+1} x_{n_k}, T^{i_k} x_{n_k}\big) - \rho(x_{n_k}, T x_{n_k}) \\ &= \sum_{i=0}^{i_k-1}\big[\rho\big(T^{i+2} x_{n_k}, T^{i+1} x_{n_k}\big) - \rho\big(T^{i+1} x_{n_k}, T^i x_{n_k}\big)\big] \\ &\le -\big(1-\phi(\varepsilon)\varepsilon\big)i_k \le -k\big(1-\phi(\varepsilon)\big)\varepsilon\end{aligned}$$

and

$$k\big(1-\phi(\varepsilon)\big)\varepsilon \le \rho(x_{n_k}, T x_{n_k}). \tag{3.93}$$

In view of (3.93), (3.85) and (3.87),

$$\begin{aligned}k\big(1-\phi(\varepsilon)\big)\varepsilon &\le \rho(x_{n_k}, T x_{n_k}) \\ &\le \rho(x_{n_k}, \theta) + \rho(\theta, T\theta) + \rho(T\theta, T x_{n_k}) \le c_0 + \rho(\theta, T\theta) + c_0\end{aligned}$$

and

$$k \le \big(\varepsilon\big(1-\phi(\varepsilon)\big)\big)^{-1}\big(2c_0+\rho(\theta, T\theta)\big).$$

This contradicts (3.90). The contradiction we have reached proves that for each $\varepsilon > 0$, there exists a natural number k such that (P2) holds.

Now let $\delta > 0$ be given. We show that there exists a natural number k such that the following property holds:

(P3) If $n > k$ is an integer and if integers i, j satisfy $k \le i, j < n$, then

$$\rho\big(T^i x_n, T^j x_n\big) \le \delta.$$

To this end, choose a positive number

$$\varepsilon < 4^{-1}\delta\big(1-\phi(\delta)\big). \tag{3.94}$$

We have already shown that there exists a natural number k such that (P2) holds.

Assume that the natural numbers n, i and j satisfy

$$n > k \quad \text{and} \quad k \le i, j < n. \tag{3.95}$$

We claim that $\rho(T^i x_n, T^j x_n) \le \delta$.

Assume the contrary. Then

$$\rho\big(T^i x_n, T^j x_n\big) > \delta. \tag{3.96}$$

By (P2), (3.95), (3.85), (3.96) and the monotonicity of ϕ,

$$\begin{aligned}\rho\big(T^i x_n, T^j x_n\big) &\le \rho\big(T^i x_n, T^{i+1} x_n\big) + \rho\big(T^{i+1} x_n, T^{j+1} x_n\big) + \rho\big(T^{j+1} x_n, T^j x_n\big)\\ &\le \varepsilon + \rho\big(T^{i+1} x_n, T^{j+1} x_n\big) + \varepsilon\\ &\le 2\varepsilon + \phi\big(\rho\big(T^i x_n, T^j x_n\big)\big)\rho\big(T^i x_n, T^j x_n\big)\\ &\le 2\varepsilon + \phi(\delta)\rho\big(T^i x_n, T^j x_n\big).\end{aligned}$$

Together with (3.94) this implies that

$$\rho\big(T^i x_n, T^j x_n\big) \le 2\varepsilon\big(1-\phi(\delta)\big)^{-1} < \delta,$$

a contradiction. Thus we have shown that for each $\delta > 0$, there exists a natural number k such that (P3) holds.

Let $\varepsilon > 0$ be given. We will show that there exists a natural number k such that the following property holds:

(P4) If $n_1, n_2 \ge k$ are integers, then $\rho(T^k x_{n_1}, T^k x_{n_2}) \le \varepsilon$.

Choose a natural number k such that

$$k > \big(\big(1-\phi(\varepsilon)\big)(\varepsilon)\big)^{-1} 4c_0 \tag{3.97}$$

and assume that the integers n_1 and n_2 satisfy

$$n_1, n_2 \ge k. \tag{3.98}$$

We claim that $\rho(T^k x_{n_1}, T^k x_{n_2}) \le \varepsilon$. Assume the contrary. Then

$$\rho\big(T^k x_{n_1}, T^k x_{n_2}\big) > \varepsilon.$$

Together with (3.85) this implies that

$$\rho\big(T^i x_{n_1}, T^i x_{n_2}\big) > \varepsilon, \quad i = 0, \dots, k. \tag{3.99}$$

By (3.85), (3.99) and the monotonicity of ϕ, we have for $i = 0, \dots, k-1$,

$$\rho(T^{i+1}x_{n_1}, T^{i+1}x_{n_2}) \le \phi\big(\rho(T^i x_{n_1}, T^i x_{n_2})\big)\rho(T^i x_{n_1}, T^i x_{n_2}) \le \phi(\varepsilon)\rho(T^i x_{n_1}, T^i x_{n_2})$$

and

$$\rho(T^{i+1}x_{n_1}, T^{i+1}x_{n_2}) - \rho(T^i x_{n_1}, T^i x_{n_2}) \le \big(\phi(\varepsilon)-1\big)\rho(T^i x_{n_1}, T^i x_{n_2}) \le -\big(1-\phi(\varepsilon)\big)\varepsilon.$$

This implies that

$$-\rho(x_{n_1}, x_{n_2}) \le \rho(T^k x_{n_1}, T^k x_{n_2}) - \rho(x_{n_1}, x_{n_2}) = \sum_{i=0}^{k-1}\big[\rho(T^{i+1}x_{n_1}, T^{i+1}x_{n_2}) - \rho(T^i x_{n_1}, T^i x_{n_2})\big] \le -k\big(1-\phi(\varepsilon)\big)\varepsilon.$$

Together with (3.87) this implies that

$$k\big(1-\phi(\varepsilon)\big)\varepsilon \le \rho(x_{n_1}, x_{n_2}) \le \rho(x_{n_1}, \theta) + \rho(\theta, x_{n_2}) \le 2c_0.$$

This contradicts (3.97). Thus we have shown that

$$\rho(T^k x_{n_1}, T^k x_{n_2}) \le \varepsilon.$$

In other words, there exists a natural number k for which (P4) holds.

Let $\varepsilon > 0$ be given. By (P4), there exists a natural number k_1 such that

$$\rho(T^{k_1}x_{n_1}, T^{k_1}x_{n_2}) \le \varepsilon/4 \quad \text{for all integers } n_1, n_2 \ge k_1. \tag{3.100}$$

By (P3), there exists a natural number k_2 such that

$$\rho(T^i x_n, T^j x_n) \le \varepsilon/4 \quad \text{for all natural numbers } n, j, i \text{ satisfying } k_2 \le i, j < n. \tag{3.101}$$

Assume now that the natural numbers n_1, n_2, i and j satisfy

$$n_1, n_2 > k_1 + k_2, \qquad i, j \ge k_1 + k_2, \qquad i < n_1, \qquad j < n_2. \tag{3.102}$$

We claim that

$$\rho(T^i x_{n_1}, T^j x_{n_2}) \le \varepsilon.$$

By (3.100), (3.102) and (3.85),

$$\rho(T^{k_1+k_2}x_{n_1}, T^{k_1+k_2}x_{n_2}) \le \rho(T^{k_1}x_{n_1}, T^{k_1}x_{n_2}) \le \varepsilon/4. \tag{3.103}$$

In view of (3.102) and (3.101),

$$\rho(T^{k_1+k_2}x_{n_1}, T^i x_{n_1}) \le \varepsilon/4 \quad \text{and} \quad \rho(T^{k_1+k_2}x_{n_2}, T^j x_{n_2}) \le \varepsilon/4.$$

Together with (3.103) these inequalities imply that

$$\begin{aligned}
&\rho\big(T^i x_{n_1}, T^j x_{n_2}\big)\\
&\quad\le \rho\big(T^i x_{n_1}, T^{k_1+k_2} x_{n_1}\big) + \rho\big(T^{k_1+k_2} x_{n_1}, T^{k_1+k_2} x_{n_2}\big) + \rho\big(T^{k_1+k_2} x_{n_2}, T^j x_{n_2}\big)\\
&\quad< \varepsilon.
\end{aligned}$$

Thus we have shown that the following property holds:

(P5) For each $\varepsilon > 0$, there exists a natural number $k(\varepsilon)$ such that

$$\rho\big(T^i x_{n_1}, T^j x_{n_2}\big) \le \varepsilon$$

for all natural numbers $n_1, n_2 \ge k(\varepsilon)$, $i \in [k(\varepsilon), n_1)$ and $j \in [k(\varepsilon), n_2)$.

Consider the two sequences $\{T^{n-2} x_n\}_{n=2}^{\infty}$ and $\{T^{n-1} x_n\}_{n=2}^{\infty}$. Property (P5) implies that both of them are Cauchy and that

$$\lim_{n\to\infty} \rho\big(T^{n-1} x_n, T^{n-2} x_n\big) = 0.$$

Therefore there exists $\bar{x} \in K$ such that

$$\lim_{n\to\infty} \rho\big(\bar{x}, T^{n-2} x_n\big) = \lim_{n\to\infty} \rho\big(\bar{x}, T^{n-1} x_n\big) = 0.$$

Since the mapping T is continuous, $T\bar{x} = \bar{x}$ and assertion (A) is proved. □

Proof of Theorem 3.8(B) For each $x \in X$ and $r > 0$, set

$$B(x, r) = \big\{y \in X : \rho(x, y) \le r\big\}. \tag{3.104}$$

Choose $\delta_0 > 0$ such that

$$\delta_0 < M\big(1 - \phi(M/2)\big)/4. \tag{3.105}$$

Assume that

$$y \in K \cap B(\bar{x}, M), \qquad z \in X \quad \text{and} \quad \rho(z, Ty) \le \delta_0. \tag{3.106}$$

By (3.106) and (3.85),

$$\begin{aligned}
\rho(\bar{x}, z) &\le \rho(\bar{x}, Ty) + \rho(Ty, z) \le \rho(T\bar{x}, Ty) + \delta_0\\
&\le \phi\big(\rho(\bar{x}, y)\big)\rho(\bar{x}, y) + \delta_0.
\end{aligned} \tag{3.107}$$

There are two cases:

$$\rho(y, \bar{x}) \le M/2; \tag{3.108}$$

$$\rho(y, \bar{x}) > M/2. \tag{3.109}$$

Assume that (3.108) holds. By (3.107), (3.108) and (3.105),

$$\rho(\bar{x}, z) \le \rho(\bar{x}, y) + \delta_0 \le M/2 + \delta_0 < M. \tag{3.110}$$

If (3.109) holds, then by (3.107), (3.106), (3.109) and the monotonicity of ϕ,

$$\begin{aligned}\rho(\bar{x}, z) &\le \delta_0 + \phi(M/2)\rho(\bar{x}, y) \le \delta_0 + \phi(M/2)M \\ &< (M/4)\big(1 - \phi(M/2)\big) + \phi(M/2)M \le M.\end{aligned}$$

Thus $\rho(\bar{x}, z) \le M$ in both cases.

We have shown that

$$\begin{aligned}&\rho(\bar{x}, z) \le M \quad \text{for each } z \in X \text{ and } y \in K \cap B(\bar{x}, M) \\ &\quad \text{satisfying } \rho(z, Ty) \le \delta_0.\end{aligned} \tag{3.111}$$

Since M is any positive number, we conclude that there is $\delta_1 > 0$ such that

$$\begin{aligned}&\rho(\bar{x}, z) \le \varepsilon \quad \text{for each } z \in X \text{ and } y \in K \cap B(\bar{x}, \varepsilon) \\ &\quad \text{satisfying } \rho(z, Ty) \le \delta_1.\end{aligned} \tag{3.112}$$

Choose a positive number δ such that

$$\delta < \min\big\{\delta_0, \delta_1, \varepsilon\big(1 - \phi(\varepsilon)\big)4^{-1}\big\} \tag{3.113}$$

and a natural number k such that

$$k > 4(M + 1)\big(1 - \phi(\varepsilon)\varepsilon\big)^{-1} + 4. \tag{3.114}$$

Let $n \ge k$ be a natural number and assume that $\{x_i\}_{i=0}^n \subset K$ satisfies

$$\rho(x_0, \bar{x}) \le M \quad \text{and} \quad \rho(x_{i+1}, Tx_i) \le \delta, \quad i = 0, \ldots, n - 1. \tag{3.115}$$

We claim that (3.86) holds. By (3.111), (3.115) and the inequality $\delta < \delta_0$ (see (3.113)),

$$\{x_i\}_{i=0}^k \subset B(\bar{x}, M). \tag{3.116}$$

Assume that (3.86) does not hold. Then there is an integer j such that

$$j \in \{k, n\} \quad \text{and} \quad \rho(x_j, \bar{x}) > \varepsilon. \tag{3.117}$$

By (3.117), (3.115), (3.112) and (3.113),

$$\rho(x_i, \bar{x}) > \varepsilon, \quad i = 0, \ldots, j. \tag{3.118}$$

Let $i \in \{0, \ldots, j - 1\}$. By (3.115), (3.118), the monotonicity of ϕ, (3.113) and (3.85),

$$\begin{aligned}\rho(x_{i+1},\bar{x}) &\le \rho(x_{i+1},Tx_i)+\rho(Tx_i,T\bar{x}) \le \delta+\phi\big(\rho(x_i,\bar{x})\big)\rho(x_i,\bar{x})\\ &\le \delta+\phi(\varepsilon)\rho(x_i,\bar{x})\end{aligned}$$

and

$$\begin{aligned}\rho(x_{i+1},\bar{x})-\rho(x_i,\bar{x}) &\le \delta-\big(1-\phi(\varepsilon)\big)\rho(x_i,\bar{x}) \le \delta-\big(1-\phi(\varepsilon)\big)\varepsilon\\ &\le -\big(1-\phi(\varepsilon)\big)\varepsilon/2.\end{aligned}$$

By (3.115) and (3.117) and the above inequalities,

$$\begin{aligned}-M &\le -\rho(x_0,\bar{x}) \le \rho(x_j,\bar{x})-\rho(x_0,\bar{x})\\ &= \sum_{i=0}^{j-1}\big[\rho(x_{i+1},\bar{x})-\rho(x_i,\bar{x})\big] \le -j\big(1-\phi(\varepsilon)\varepsilon/2\big) \le -k\big(1-\phi(\varepsilon)\big)\varepsilon/2.\end{aligned}$$

This contradicts (3.114). The contradiction we have reached proves (3.86) and assertion (B). □

Let G be a nonempty subset of a Banach space $(Y,\|\cdot\|)$. In [64] J. A. Gatica and W. A. Kirk proved that if $T:\overline{G}\to Y$ is a strict contraction, then T must have a unique fixed point x_1, under the additional assumptions that the origin is in the interior $\mathrm{Int}(G)$ of G and that T satisfies a certain boundary condition known as the Leray-Schauder condition:

$$Tx \neq \lambda x \quad \forall x \in \partial G, \forall \lambda > 1. \tag{L-S}$$

Here G is not necessarily convex or bounded. Their proof was nonconstructive. Later, M. Frigon, A. Granas and Z. E. A. Guennoun [61], and M. Frigon [60] proved that if x_t is the unique fixed point of tT, then, in fact, the mapping $t\to x_t$ is Lipschitz, so it gives a partial way to approximate x_1. Our second result in this section, which was also obtained in [115], extends these theorems to the case where T merely satisfies (3.85).

Theorem 3.9 *Let G be a nonempty subset of a Banach space Y with $0\in\mathrm{Int}(G)$. Suppose that $T:\overline{G}\to X$ is nonexpansive and that it satisfies condition (L-S). Then for each $t\in[0,1)$, the mapping $tT:\overline{G}\to X$ has a unique fixed point $x_t\in\mathrm{Int}(G)$ and the mapping $t\to x_t$ is Lipschitz on $[0,b]$ for any $0<b<1$. If, in addition, T satisfies (3.85), then it has a unique fixed point $x_1\in\overline{G}$ and the mapping $t\to x_t$ is continuous on $[0,1]$. In particular, $x_1=\lim_{t\to 1^-}x_t$.*

Proof In the first part of the proof we assume that T is nonexpansive, i.e., it satisfies (3.85) with ϕ identically equal to one.

Let $S\subset[0,1)$ be the following set:

$$S=\big\{t\in[0,1):tT \text{ has a unique fixed point } x_t\in\mathrm{Int}(G)\big\}.$$

Since tT is a strict contraction for each $t \in [0, 1)$, it has at most one fixed point. In order to prove the first part of this theorem, we have to show that $S = [0, 1)$. Since $0 \in S$ by assumption and since $[0, 1)$ is connected, it is enough to show that S is both open and closed.

1. S is open: Let $t_0 \in S$. From the definition of S it is clear that $t_0 < 1$, so there is a real number q such that $t_0 < q < 1$. Let $x_{t_0} \in \mathrm{Int}(G)$ be the unique fixed point of $t_0 T$.

Since $\mathrm{Int}(G)$ is open, there is $r > 0$ such that the closed ball $B[x_{t_0}, r]$ of radius r and center x_{t_0} is contained in $\mathrm{Int}(G)$. We have, for all $x \in B[x_{t_0}, r]$ and $t \in [0, 1)$,

$$\begin{aligned}\|tTx - x_{t_0}\| &\le \|tTx - tTx_{t_0}\| + |t - t_0|\|Tx_{t_0}\| + \|t_0 Tx_{t_0} - x_{t_0}\| \\ &\le t\|x - x_{t_0}\| + |t - t_0|\|Tx_{t_0}\| \le tr + |t - t_0|\big(\|Tx_{t_0}\| + 1\big). \quad (3.119)\end{aligned}$$

Suppose that $t \in [0, 1)$ satisfies

$$|t - t_0| < \min\left\{\frac{r(1-q)}{1 + \|Tx_{t_0}\|}, q - t_0\right\}. \quad (3.120)$$

Then $t < q$ and

$$|t - t_0| \le \frac{r(1-t)}{1 + \|Tx_{t_0}\|},$$

so $\|tTx - x_{t_0}\| \le r$ by (3.119). Consequently, the closed ball $B[x_{t_0}, r]$ is invariant under tT, and the Banach fixed point theorem ensures that tT has a unique fixed point $x_t \in B[x_{t_0}, r] \subset \mathrm{Int}(G)$. Thus $t \in S$ for all $t \in [0, 1)$ satisfying (3.120).

2. S is closed: Suppose $t_0 \in [0, 1)$ is a limit point of S. We have to prove that $t_0 \in S$, and since $0 \in S$ we can assume that $t_0 > 0$. There is a sequence $(t_n)_n$ in $[0, 1)$ such that $t_0 = \lim_{n\to\infty} t_n$, and since $t_0 < 1$, there is $0 < q < 1$ such that $t_n < q$ for n large enough. Define

$$A_0 := \big\{x_t : t \in S \cap [0, q]\big\}.$$

The set A_0 is not empty since $0 \in A_0$. In addition, if $t \in S \cap [0, q]$, then

$$\|x_t\| = \|tTx_t\| \le q\big(\|Tx_t - T0\| + \|T0\|\big) \le q\phi\big(\|x_t - 0\|\big)\|x_t - 0\| + q\|T0\|.$$

Therefore

$$\|x_t\| \le \frac{q\|T0\|}{1 - \phi(\|x_t\|)q} \le \frac{\|T0\|}{1-q}, \quad (3.121)$$

so A_0 is a bounded set, and since T is Lipschitz, $T(A_0)$ is also bounded, say by M. We will show that $(x_{t_n})_n$ is a Cauchy sequence which converges to the fixed point x_{t_0} of $t_0 T$. Indeed, since x_{t_n} and x_{t_m} are the fixed points of $t_n T$ and $t_m T$, respectively, it follows that

$$\begin{aligned}\|x_{t_n} - x_{t_m}\| &= \|t_n Tx_{t_n} - t_m Tx_{t_m}\| \le |t_n - t_m|\|Tx_{t_n}\| + \|t_m Tx_{t_n} - t_m Tx_{t_m}\| \\ &\le |t_n - t_m|M + t_m\phi\big(\|x_{t_n} - x_{t_m}\|\big)\|x_{t_n} - x_{t_m}\|.\end{aligned}$$

Hence

$$\|x_{t_n} - x_{t_m}\| \le \frac{|t_n - t_m|M}{1 - t_m\phi(\|x_{t_n} - x_{t_m}\|)} \le \frac{|t_n - t_m|M}{1-q}. \tag{3.122}$$

Since $t_n \to t_0$ as $n \to \infty$, we see that $(x_{t_n})_n$ is indeed Cauchy and hence converges to $x_{t_0} \in \overline{G}$. Using again the equality $t_n T x_{t_n} = x_{t_n}$, we obtain

$$\begin{aligned}\|t_0 T x_{t_0} - x_{t_0}\| &\le \|t_0 T x_{t_0} - t_0 T x_{t_n}\| + \|t_0 T x_{t_n} - t_n T x_{t_n}\| + \|t_n T x_{t_n} - x_{t_0}\| \\ &= t_0\|T x_{t_0} - T x_{t_n}\| + |t_0 - t_n|\|T x_{t_n}\| + \|x_{t_n} - x_{t_0}\| \\ &\le \|x_{t_0} - x_{t_n}\| + |t_0 - t_n|M + \|x_{t_n} - x_{t_0}\| \to 0,\end{aligned}$$

so $t_0 T x_{t_0} = x_{t_0}$, i.e., x_{t_0} is indeed a fixed point of $t_0 T$. It remains to show that $x_{t_0} \in \mathrm{Int}(G)$, and this follows from the (L-S) condition: since $T x_{t_0} = \frac{1}{t_0} x_{t_0}$, so (L-S) implies that $x_{t_0} \notin \partial G$ (recall that $0 < t_0 < 1$). Hence S is closed, as claimed.

The fact that the mapping $t \to x_t$ is Lipschitz on the interval $[0, b]$ for any $0 < b < 1$ follows from (3.122).

Suppose now that T satisfies (3.85) with $\phi(t) < 1$ for all positive t. Let $(t_n)_n$ be a sequence in $[0, 1)$ such that $t_n \to t_0 = 1$. The set A_0 (and hence the set $T(A_0)$) remain bounded also when $q = 1$, because if $\|x_t\| \ge 1$, then in (3.121) we get $\|x_t\| \le \frac{\|T0\|}{1-\phi(1)}$, so in any case $\|x_t\| \le \max\left(1, \frac{\|T0\|}{1-\phi(1)}\right)$ (recall that $\phi(t) < 1$). Now, in order to prove that $x_1 := \lim_{t\to 1^{-1}} x_t$ exists, note first that $(x_{t_n})_n$ is Cauchy if $t_n \to 1$, because otherwise there is $\varepsilon > 0$ and a subsequence (call it again t_n) such that $\|x_{t_{2n+1}} - x_{t_{2n+2}}\| \ge \varepsilon$, but from (3.122) we obtain

$$\|x_{t_{2n+1}} - x_{t_{2n+2}}\| \le \frac{|t_{2n+1} - t_{2n+2}|M}{1 - t_{2n+2}\phi(\varepsilon)} \to 0,$$

a contradiction. Now, all these sequences approach the same limit because for any two such sequences

$$(x_{t_n})_n, \qquad (x_{s_n})_n,$$

the interlacing sequence $(t_1, s_1, t_2, s_2, \ldots) \to 1$, so $(x_{t_1}, x_{s_1}, x_{t_2}, x_{s_2}, \ldots)$ is also Cauchy. The fact that x_1 is a fixed point of T is proved as above (here, however, one cannot use (L-S) to conclude that $x_1 \in \mathrm{Int}(G)$, and indeed it may happen that $x_1 \in \partial G$ as the mapping $T : [-1, \infty) \to R$, defined by $Tx = \frac{x-1}{2}$, shows). □

3.8 A Result on Rakotch Contractions

In this section, which is based on [160], we establish fixed point and convergence theorems for certain mappings of contractive type which take a closed subset of a complete metric space X into X.

Let K be a nonempty and closed subset of a complete metric space (X, ρ). For each $x \in X$ and $r > 0$, set

$$B(x, r) = \{y \in X : \rho(x, y) \le r\}.$$

In the following result, which was obtained in [160], we provide a new sufficient condition for the existence and approximation of the unique fixed point of a contractive mapping which maps a nonempty and closed subset of a complete metric space X into X.

Theorem 3.10 *Assume that $T : K \to X$ satisfies*

$$\rho(Tx, Ty) \le \phi\big(\rho(x, y)\big)\rho(x, y) \quad \text{for all } x, y \in K, \tag{3.123}$$

where $\phi : [0, \infty) \to [0, 1]$ is a monotonically decreasing function such that $\phi(t) < 1$ for all $t > 0$.

Assume that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset K$ such that

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0. \tag{3.124}$$

Then there exists a unique $\bar{x} \in K$ such that $T\bar{x} = \bar{x}$.

Proof The uniqueness of $\bar{x}$ is obvious. To establish its existence, let $\varepsilon \in (0, 1)$ be given and choose a positive number γ such that

$$\gamma < \big(1 - \phi(\varepsilon)\big)\varepsilon/8. \tag{3.125}$$

By (3.124), there is a natural number n_0 such that

$$\rho(x_n, Tx_n) < \gamma \quad \text{for all integers } n \ge n_0. \tag{3.126}$$

Assume that the integers $m, n \ge n_0$. We claim that $\rho(x_m, x_n) \le \varepsilon$. Assume the contrary. Then

$$\rho(x_m, x_n) > \varepsilon. \tag{3.127}$$

By (3.125), (3.123), (3.127), the monotonicity of ϕ, and (3.126),

$$\begin{aligned}
\rho(x_m, x_n) &\le \rho(x_m, Tx_m) + \rho(Tx_m, Tx_n) + \rho(Tx_n, x_n) \\
&\le 2\gamma + \phi\big(\rho(x_m, x_n)\big)\rho(x_m, x_n) \le 2\gamma + \phi(\varepsilon)\rho(x_m, x_n) \\
&= \rho(x_m, x_n) - \big(1 - \phi(\varepsilon)\big)\rho(x_m, x_n) + 2\gamma \\
&< \rho(x_m, x_n) - \big(1 - \phi(\varepsilon)\big)\rho(x_m, x_n) + \big(1 - \phi(\varepsilon)\big)\varepsilon/4 \\
&\le \rho(x_m, x_n) - \big(1 - \phi(\varepsilon)\big)\rho(x_m, x_n)(3/4) \\
&= \rho(x_m, x_n)\big[(1/4) + \phi(\varepsilon)(3/4)\big] < \rho(x_m, x_n),
\end{aligned}$$

a contradiction.

The contradiction we have reached proves that $\rho(x_m, x_n) \le \varepsilon$ for all integers $m, n \ge n_0$, as claimed.

Since ε is an arbitrary number in (0, 1), we conclude that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and there exists $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x}$. By (3.123), for all integers $n \geq 1$,

$$\begin{aligned}\rho(T\bar{x}, \bar{x}) &\leq \rho(T\bar{x}, Tx_n) + \rho(Tx_n, x_n) + \rho(x_n, \bar{x}) \\ &\leq 2\rho(x_n, \bar{x}) + \rho(Tx_n, x_n) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This concludes the proof of Theorem 3.10. □

In the following result, which was also obtained in [160], we present another proof of the fixed point theorem established in Theorem 1(A) of [115]. This proof is based on Theorem 3.10.

Theorem 3.11 *Let $T : K \to X$ satisfy*

$$\rho(Tx, Ty) \leq \phi\big(\rho(x, y)\big)\rho(x, y) \quad \textit{for all } x, y \in K,$$

where $\phi : [0, \infty) \to [0, 1]$ is a monotonically decreasing function such that $\phi(t) < 1$ for all $t > 0$.

Assume that $K_0 \subset K$ is a nonempty and bounded set with the following property:

For each natural number n, there exists $y_n \in K_0$ such that $T^i y_n$ is defined for all $i = 1, \ldots, n$.

Then the mapping T has a unique fixed point $\bar{x}$ in K.

Proof By Theorem 3.10, it is sufficient to show that for each $\varepsilon \in (0, 1)$, there is $x \in K$ such that $\rho(x, Tx) < \varepsilon$. Indeed, let $\varepsilon \in (0, 1)$. There is $M > 0$ such that

$$\rho(y_0, y_i) \leq M, \quad i = 1, 2, \ldots. \tag{3.128}$$

By (3.123) and (3.128), for each integer $i \geq 1$,

$$\rho(y_i, Ty_i) \leq \rho(y_i, y_0) + \rho(y_0, Ty_0) + \rho(Ty_0, Ty_i) \leq 2M + \rho(y_0, Ty_0). \tag{3.129}$$

Choose a natural number $q \geq 4$ such that

$$(q-1)\varepsilon\big(1 - \phi(\varepsilon)\big) > 4M + 2\rho(y_0, Ty_0). \tag{3.130}$$

Set $T^0 z = z$, $z \in K$.

We claim that $\rho(T^{q-1}y_q, T^q y_q) < \varepsilon$. Assume the contrary. Then by (3.123),

$$\rho\big(T^i y_q, T^{i+1} y_q\big) \geq \varepsilon, \quad i = 0, \ldots, q-1. \tag{3.131}$$

In view of (3.123), (3.131) and the monotonicity of ϕ, we have for $i = 0, \ldots, q-2$,

$$\begin{aligned}\rho\big(T^{i+1} y_q, T^{i+2} y_q\big) &\leq \phi\big(\rho\big(T^i y_q, T^{i+1} y_q\big)\big)\rho\big(T^i y_q, T^{i+1} y_q\big) \\ &\leq \phi(\varepsilon)\rho\big(T^i y_q, T^{i+1} y_q\big)\end{aligned}$$

and

$$\rho\big(T^i y_q, T^{i+1} y_q\big) - \rho\big(T^{i+1} y_q, T^{i+2} y_q\big) \geq \big(1 - \phi(\varepsilon)\big)\rho\big(T^i y_q, T^{i+1} y_q\big) \\ \geq \big(1 - \phi(\varepsilon)\big)\varepsilon. \tag{3.132}$$

By (3.129) and (3.132),

$$\begin{aligned} 2M + \rho(y_0, Ty_0) &\geq \rho(y_q, Ty_q) - \rho\big(T^{q-1} y_q, T^q y_q\big) \\ &\geq \sum_{i=0}^{q-2} \big[\rho\big(T^i y_q, T^{i+1} y_q\big) - \rho\big(T^{i+1} y_q, T^{i+2} y_q\big)\big] \\ &\geq (q-1)\big(1 - \phi(\varepsilon)\big)\varepsilon \end{aligned}$$

and

$$2M + \rho(y_0, Ty_0) \geq (q-1)\big(1 - \phi(\varepsilon)\big)\varepsilon.$$

This contradicts (3.130). The contradiction we have reached shows that

$$\rho\big(T^{q-1} y_q, T^q y_q\big) < \varepsilon,$$

as claimed. Theorem 3.11 is proved. □

In the following result, also obtained in [160], we establish a convergence result for (unrestricted) infinite products of mappings which satisfy a weak form of condition (3.123).

Theorem 3.12 *Let $\phi : [0, \infty) \to [0, 1]$ be a monotonically decreasing function such that $\phi(t) < 1$ for all $t > 0$.*

Let

$$\bar{x} \in K, \qquad T_i : K \to X, \quad i = 0, 1, \dots, \qquad T_i \bar{x} = \bar{x}, \quad i = 0, 1, \dots, \tag{3.133}$$

and assume that

$$\rho(T_i x, \bar{x}) \leq \phi\big(\rho(x, \bar{x})\big)\rho(x, \bar{x}) \quad \text{for each } x \in K, i = 0, 1, \dots. \tag{3.134}$$

Then for each $M, \varepsilon > 0$, there exist $\delta > 0$ and a natural number k such that for each integer $n \geq k$, each mapping $r : \{0, 1, \dots, n-1\} \to \{0, 1, \dots\}$, and each sequence $\{x_i\}_{i=0}^{n-1} \subset K$ satisfying

$$\rho(x_0, \bar{x}) \leq M \quad \text{and} \quad \rho(x_{i+1}, T_{r(i)} x_i) \leq \delta, \quad i = 0, \dots, n-1,$$

we have

$$\rho(x_i, \bar{x}) \leq \varepsilon, \quad i = k, \dots, n. \tag{3.135}$$

Proof Choose $\delta_0 > 0$ such that

$$\delta_0 < M\big(1 - \phi(M/2)\big)/4. \tag{3.136}$$

Assume that

$$y \in K \cap B(\bar{x}, M), \qquad i \in \{0, 1, \dots\}, \qquad z \in X \quad \text{and} \quad \rho(z, T_i y) \le \delta_0. \tag{3.137}$$

By (3.137) and (3.134),

$$\rho(\bar{x}, z) \le \rho(\bar{x}, T_i y) + \rho(T_i, z) \le \phi\big(\rho(\bar{x}, y)\big)\rho(\bar{x}, y) + \delta_0. \tag{3.138}$$

There are two cases:

$$\rho(y, \bar{x}) \le M/2 \tag{3.139}$$

and

$$\rho(y, \bar{x}) > M/2. \tag{3.140}$$

Assume that (3.139) holds. Then by (3.138), (3.139) and (3.136),

$$\rho(\bar{x}, z) \le \rho(\bar{x}, y) + \delta_0 \le M/2 + \delta_0 < M. \tag{3.141}$$

If (3.140) holds, then by (3.138), (3.137), (3.136) and the monotonicity of ϕ,

$$\begin{aligned} \rho(\bar{x}, z) &\le \delta_0 + \phi(M/2)\rho(\bar{x}, y) \le \delta_0 + \phi(M/2)M \\ &< (M/4)\big(1 - \phi(M/2)\big) + \phi(M/2)M \le M. \end{aligned}$$

Thus $\rho(\bar{x}, z) \le M$ in both cases.

We have shown that

$$\text{if } y \in K \cap B(\bar{x}, M), i \in \{0, 1, \dots\}, z \in X, \rho(z, T_i y) \le \delta_0, \text{ then } \rho(\bar{x}, z) \le M. \tag{3.142}$$

Since M is any positive number, we conclude that there is $\delta_1 > 0$ such that

$$\text{if } y \in K \cap B(\bar{x}, \varepsilon), i \in \{0, 1, \dots\}, z \in X, \rho(z, T_i y) \le \delta_1, \text{ then } \rho(\bar{x}, z) \le \varepsilon. \tag{3.143}$$

Now choose a positive number δ such that

$$\delta < \min\big\{\delta_0, \delta_1, \varepsilon\big(1 - \phi(\varepsilon)\big)4^{-1}\big\} \tag{3.144}$$

and a natural number k such that

$$k > 4(M + 1)\big(\big(1 - \phi(\varepsilon)\big)\varepsilon\big)^{-1} + 4. \tag{3.145}$$

Let $n \ge k$ be a natural number. Assume that $r : \{0, \dots, n-1\} \to \{0, 1, \dots\}$ and that

$$\{x_i\}_{i=0}^{n-1} \subset K$$

satisfies

$$\rho(x_0, \bar{x}) \le M \quad \text{and} \quad \rho(x_{i+1}, T_{r(i)} x_i) \le \delta, \quad i = 0, \dots, n-1. \tag{3.146}$$

We claim that (3.135) holds. By (3.142), (3.146) and the inequality $\delta < \delta_0$,

$$\{x_i\}_{i=0}^{n} \subset B(\bar{x}, M). \tag{3.147}$$

Assume to the contrary that (3.135) does not hold. Then there is an integer j such that

$$j \in \{k, \dots, n\} \quad \text{and} \quad \rho(x_j, \bar{x}) > \varepsilon. \tag{3.148}$$

By (3.148) and (3.134),

$$\rho(x_i, \bar{x}) > \varepsilon, \quad i = 0, \dots, j. \tag{3.149}$$

Let $i \in \{0, \dots, j-1\}$. By (3.146), (3.134) and the monotonicity of ϕ,

$$\begin{aligned}\rho(x_{i+1}, \bar{x}) &\le \rho(x_{i+1}, T_{r(i)} x_i) + \rho(T_{r(i)} x_i, \bar{x}) \le \delta + \phi\big(\rho(x_i, \bar{x})\big)\rho(x_i, \bar{x}) \\ &\le \delta + \phi(\varepsilon)\rho(x_i, \bar{x}).\end{aligned}$$

When combined with (3.144) and (3.49), this implies that

$$\begin{aligned}\rho(x_{i+1}, \bar{x}) - \rho(x_i, \bar{x}) &\le \delta - \big(1 - \phi(\varepsilon)\big)\rho(x_i, \bar{x}) \le \delta - \big(1 - \phi(\varepsilon)\big)\varepsilon \\ &< -\big(1 - \phi(\varepsilon)\big)\varepsilon/2.\end{aligned} \tag{3.150}$$

Finally, by (3.146), (3.150) and (3.148),

$$\begin{aligned}-M &\le -\rho(x_0, \bar{x}) \le \rho(x_j, \bar{x}) - \rho(x_0, \bar{x}) \\ &= \sum_{i=0}^{j-1} \big[\rho(x_{i+1}, \bar{x}) - \rho(x_i, \bar{x})\big] \le -j\big(1 - \phi(\varepsilon)\big)\varepsilon/2 \le -k\big(1 - \phi(\varepsilon)\big)\varepsilon/2.\end{aligned}$$

This contradicts (3.145). The contradiction we have reached proves (3.135) and Theorem 3.12 itself. □

3.9 Asymptotic Contractions

In this section, which is based on [8], we provide sufficient conditions for the iterates of an asymptotic contraction on a complete metric space X to converge to its unique fixed point, uniformly on each bounded subset of X.

Let (X, d) be a complete metric space. The following theorem is the main result of Chen [40]. It improves upon Kirk's original theorem [83]. In this connection, see also [6] and [76].

Theorem 3.13 *Let $T : X \to X$ be such that*

$$d\big(T^n x, T^n y\big) \le \phi_n\big(d(x, y)\big)$$

for all $x, y \in X$ and all natural numbers n, where $\phi_n : [0, \infty) \to [0, \infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0, b]$. Suppose that ϕ is upper semicontinuous and that $\phi(t) < t$ for all $t > 0$. Furthermore, suppose that there exists a positive integer n_* such that ϕ_{n_*} is upper semicontinuous and $\phi_{n_*}(0) = 0$. If there exists $x_0 \in X$ which has a bounded orbit $O(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, then T has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} T^n x = x_*$ for all $x \in X$.

Note that Theorem 3.13 does not provide us with uniform convergence of the iterates of T on bounded subsets of X, although this does hold for many classes of mappings of contractive type (e.g., [23, 114]). This property is important because it yields stability of the convergence of iterates even in the presence of computational errors [35]. In this section we show that this conclusion can be derived in the setting of Theorem 3.13 if for each natural number n, the function ϕ_n is assumed to be bounded on any bounded interval. To this end, we first prove a somewhat more general result (Theorem 3.14) which, when combined with Theorem 3.13, yields our strengthening of Chen's result (Theorem 3.15).

Theorem 3.14 *Let $x_* \in X$ be a fixed point of $T : X \to X$. Assume that*

$$d\big(T^n x, x_*\big) \le \phi_n\big(d(x, x_*)\big) \quad \text{for all } x \in X \text{ and all natural numbers } n, \tag{3.151}$$

where $\phi_n : [0, \infty) \to [0, \infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0, b]$. Suppose that ϕ is upper semicontinuous and $\phi(t) < t$ for all $t > 0$. Then $T^n x \to x_*$ as $n \to \infty$, uniformly on each bounded subset of X.

Theorem 3.15 *Let $T : X \to X$ be such that*

$$d\big(T^n x, T^n y\big) \le \phi_n\big(d(x, y)\big)$$

for all $x, y \in X$ and all natural numbers n, where $\phi_n : [0, \infty) \to [0, \infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0, b]$. Suppose that ϕ is upper semicontinuous and $\phi(t) < t$ for all $t > 0$. Furthermore, suppose that there exists a positive integer n_* such that ϕ_{n_*} is upper semicontinuous and $\phi_{n_*}(0) = 0$. If there exists $x_0 \in X$ which has a bounded orbit $O(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, then T has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} T^n x = x_*$, uniformly on each bounded subset of X.

Proof of Theorem 3.14 We may assume without loss of generality that $\phi(0) = 0$ and $\phi_n(0) = 0$ for all integers $n \geq 1$.

For each $x \in X$ and each $r > 0$, set

$$B(x,r) = \{y \in X : d(x,y) \leq r\}.$$

We first prove three lemmata.

Lemma 3.16 *Let $K > 0$. Then there exists a natural number $\bar{q}$ such that for all integers $s \geq \bar{q}$,*

$$T^s\big(B(x_*, K)\big) \subset B(x_*, K+1).$$

Proof There exists a natural number $\bar{q}$ such that for all integers $s \geq \bar{q}$,

$$\big|\psi_s(t) - \phi(t)\big| < 1 \quad \text{for all } t \in [0, K].$$

Let $s \geq \bar{q}$ be an integer. Then for all $x \in B(x_*, K)$,

$$d\big(T^s x, x_*\big) \leq \phi_s\big(d(x, x_*)\big) < \phi\big(d(x, x_*)\big) + 1 < d(x, x_*) + 1 < K + 1.$$

Lemma 3.16 is proved. □

Lemma 3.17 *Let $0 < \varepsilon_1 < \varepsilon_0$. Then there exists a natural number q such that for each integer $j \geq q$,*

$$T^j\big(B(x_*, \varepsilon_1)\big) \subset B(x_*, \varepsilon_0).$$

Proof There exists an integer $q \geq 1$ such that for each integer $j \geq q$,

$$\big|\phi_j(t) - \phi(t)\big| < (\varepsilon_0 - \varepsilon_1)/2 \quad \text{for all } t \in [0, \varepsilon_0]. \tag{3.152}$$

Assume that

$$j \in \{q, q+1, \ldots\} \quad \text{and} \quad x \in B(x_*, \varepsilon_1).$$

By (3.151) and (3.152),

$$\begin{aligned} d\big(T^j x, x_*\big) &\leq \phi_j\big(d(x, x_*)\big) < \phi\big(d(x, x_*)\big) + (\varepsilon_0 - \varepsilon_1)/2 \\ &\leq \varepsilon_1 + (\varepsilon_0 - \varepsilon_1)/2 = (\varepsilon_0 + \varepsilon_1)/2. \end{aligned}$$

Lemma 3.17 is proved. □

Lemma 3.18 *Let $K, \varepsilon > 0$ be given. Then there exists a natural number q such that for each $x \in B(x_*, K)$,*

$$\min\big\{d\big(T^j x, x_*\big) : j = 1, \ldots, q\big\} \leq \varepsilon.$$

Proof By Lemma 3.16, there is a natural number $\bar{q}$ such that

$$T^n\big(B(x_*, K)\big) \subset B(x_*, K+1) \quad \text{for all natural numbers } n \geq \bar{q}. \tag{3.153}$$

We may assume without loss of generality that $\varepsilon < K/8$. Since the function $t - \phi(t)$, $t \in (0, \infty)$, is lower semicontinuous and positive, there is

$$\delta \in (0, \varepsilon/8) \tag{3.154}$$

such that

$$t - \phi(t) \geq 2\delta \quad \text{for all } t \in [\varepsilon/2, K+1]. \tag{3.155}$$

There is a natural number $s \geq \bar{q}$ such that

$$\big|\phi(t) - \phi_s(t)\big| \leq \delta \quad \text{for all } t \in [0, K+1]. \tag{3.156}$$

By (3.155) and (3.156), we have, for all $t \in [\varepsilon/2, K+1]$,

$$\phi_s(t) \leq \phi(t) + \delta \leq t - 2\delta + \delta = t - \delta. \tag{3.157}$$

In view of (3.156) and (3.154), we have, for all $t \in [0, \varepsilon/2]$,

$$\phi_s(t) \leq \phi(t) + \delta \leq t + \delta \leq \varepsilon/2 + \delta < (3/4)\varepsilon. \tag{3.158}$$

Choose a natural number p such that

$$p > 4 + \delta^{-1}(K+1). \tag{3.159}$$

Let

$$x \in B(x_*, K). \tag{3.160}$$

We will show that

$$\min\big\{d\big(T^j x, x_*\big) : j = 1, 2, \ldots, ps\big\} \leq \varepsilon. \tag{3.161}$$

Assume the contrary. Then

$$d\big(T^j x, x_*\big) > \varepsilon \quad \text{for all } j = s, \ldots, ps. \tag{3.162}$$

By (3.160) and(3.153),

$$T^j x \in B(x_*, K+1), \quad j = s, \ldots, ps. \tag{3.163}$$

Let a natural number i satisfy $i \leq p - 1$. By (3.162) and (3.163),

$$d\big(T^{is} x, x_*\big) > \varepsilon \quad \text{and} \quad d\big(T^{is} x, x_*\big) \leq K+1. \tag{3.164}$$

It follows from (3.151), (3.164) and (3.157) that

$$d\big(T^s\big(T^{is} x\big), x_*\big) \leq \phi_s\big(d\big(T^{is} x, x_*\big)\big) \leq d\big(T^{is} x, x_*\big) - \delta.$$

Thus for each natural number $i \le p-1$,

$$d\bigl(T^{(i+1)s}x, x_*\bigr) \le d\bigl(T^{is}x, x_*\bigr) - \delta.$$

This inequality implies that

$$d\bigl(T^{ps}x, x_*\bigr) \le d\bigl(T^{(p-1)s}x, x_*\bigr) - \delta \le \cdots \le d\bigl(T^{s}x, x_*\bigr) - (p-1)\delta.$$

When combined with (3.163) and (3.159), this implies, in turn, that

$$d\bigl(T^{ps}x, x_*\bigr) \le K + 1 - (p-1)\delta < 0.$$

The contradiction we have reached proves (3.161) and completes the proof of Lemma 3.18. □

Completion of the proof of Theorem 3.14 Let $K, \varepsilon > 0$ be given. Choose $\varepsilon_1 \in (0, \varepsilon)$. By Lemma 3.17, there exists a natural number q_1 such that

$$T^j\bigl(B(x_*, \varepsilon_1)\bigr) \subset B(x_*, \varepsilon) \quad \text{for all integers } j \ge q_1. \tag{3.165}$$

By Lemma 3.18, there exists a natural number q_2 such that

$$\min\bigl\{d\bigl(T^j x, x_*\bigr) : j = 1, \ldots, q_2\bigr\} \le \varepsilon_1 \quad \text{for all } x \in B(x_*, K). \tag{3.166}$$

Assume that

$$x \in B(x_*, K).$$

By (3.166), there is a natural number $j_1 \le q_2$ such that

$$d\bigl(T^{j_1}x, x_*\bigr) \le \varepsilon_1. \tag{3.167}$$

In view of (3.167) and (3.165),

$$T^j\bigl(T^{j_1}x\bigr) \in B(x_*, \varepsilon) \quad \text{for all integers } j \ge q_1. \tag{3.168}$$

Inclusion (3.168) and the inequality $j_1 \le q_2$ now imply that

$$T^i x \in B(x_*, \varepsilon) \quad \text{for all integers } i \ge q_1 + q_2.$$

Theorem 3.14 is proved. □

3.10 Uniform Convergence of Iterates

Let (X, d) be a complete metric space. The following theorem [9] is the main result of this section. In contrast with Theorem 3.14, here we only assume that a subsequence of $\{\phi_n\}_{n=1}^{\infty}$ converges to ϕ.

Theorem 3.19 *Let $x_* \in X$ be a fixed point of $T : X \to X$. Assume that*

$$d\big(T^n x, x_*\big) \le \phi_n\big(d(x, x_*)\big) \tag{3.169}$$

for all $x \in X$ and all natural numbers n, where the functions $\phi_n : [0, \infty) \to [0, \infty)$, $n = 1, 2, \ldots,$ satisfy the following conditions:

(i) *For each $b > 0$, there is a natural number n_b such that*

$$\sup\big\{\phi_n(t) : t \in [0, b] \text{ and all } n \ge n_b\big\} < \infty; \tag{3.170}$$

(ii) *there exist an upper semicontinuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ for all $t > 0$ and a strictly increasing sequence of natural numbers $\{m_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \phi_{m_k} = \phi$, uniformly on any bounded interval $[0, b]$.*

Then $T^n x \to x_*$ as $n \to \infty$, uniformly on any bounded subset of X.

Proof Set $T^0 x = x$ for all $x \in X$. For each $x \in X$ and each $r > 0$, set

$$B(x, r) = \big\{z \in X : d(x, z) \le r\big\}. \tag{3.171}$$

Let $M > 0$ and $\varepsilon \in (0, 1)$ be given. By (i), there are $M_1 > M$ and an integer $n_1 \ge 1$ such that

$$\phi_i(t) \le M_1 \quad \text{for all } t \in [0, M+1] \text{ and all integers } i \ge n_1. \tag{3.172}$$

In view of (3.169) and (3.172), for each $x \in B(x_*, M)$ and each integer $n \ge n_1$,

$$d(T_n x, x_*) \le \phi_n\big(d(x, x_*)\big) \le M_1. \tag{3.173}$$

Since the function $t - \phi(t)$ is lower semicontinuous, there is $\delta > 0$ such that

$$\delta < \varepsilon/8 \tag{3.174}$$

and

$$t - \phi(t) \ge 2\delta, \quad t \in [\varepsilon/8, 4M_1 + 4]. \tag{3.175}$$

By (ii), there is an integer $n_2 \ge 2n_1 + 2$ such that

$$\big|\phi_{n_2}(t) - \phi(t)\big| \le \delta, \quad t \in [0, 4M_1 + 4]. \tag{3.176}$$

Assume that

$$x \in B(x_*, M_1 + 4). \tag{3.177}$$

If $d(x, x_*) \le \varepsilon/8$, then it follows from (3.169), (3.174), (3.176) and (3.177) that

$$d\big(T^{n_2} x, x_*\big) \le \phi_{n_2}\big(d(x, x_*)\big) \le \phi\big(d(x, x_*)\big) + \delta \le d(x, x_*) + \delta < \varepsilon/4.$$

If $d(x, x_*) \geq \varepsilon/8$, then relations (3.169), (3.175), (3.176) and (3.177) imply that

$$d\big(T^{n_2}x, x_*\big) \leq \phi_{n_2}\big(d(x, x_*)\big) \leq \phi\big(d(x, x_*)\big) + \delta \leq d(x, x_*) - 2\delta + \delta = d(x, x_*) - \delta.$$

Thus in both cases we have

$$d\big(T^{n_2}x, x_*\big) \leq \max\big\{d(x, x_*) - \delta, \varepsilon/4\big\}. \tag{3.178}$$

Now choose a natural number $q > 2$ such that

$$q > (8 + 2M_1)\delta^{-1}. \tag{3.179}$$

Assume that

$$x \in B(x_*, M_1 + 4) \quad \text{and} \quad T^{in_2}x \in B(x_*, M_1 + 4), \quad i = 1, \dots, q - 1. \tag{3.180}$$

We claim that

$$\min\big\{d\big(T^{jn_2}x, x_*\big) : j = 1, \dots, q\big\} \leq \varepsilon/4. \tag{3.181}$$

Assume the contrary. Then by (3.178) and (3.180), for each $j = 1, \dots, q$, we have

$$d\big(T^{jn_2}x, x_*\big) \leq d\big(T^{(j-1)n_2}x, x_*\big) - \delta$$

and

$$d\big(T^{qn_2}x, x_*\big) \leq d\big(T^{(q-1)n_2}x, x_*\big) - \delta \leq \cdots \leq d(x, x_*) - q\delta \leq M_1 + 4 - q\delta.$$

This contradicts (3.179). The contradiction we have reached proves (3.181).

Assume that an integer j satisfies $1 \leq j \leq q - 1$ and

$$d\big(T^{jn_2}x, x_*\big) \leq \varepsilon/4.$$

When combined with (3.178) and (3.180), this implies that

$$d\big(T^{(j+1)n_2}x, x_*\big) \leq \max\big\{d\big(T^{jn_2}x, x_*\big) - \delta, \varepsilon/4\big\} \leq \varepsilon/4.$$

It follows from this inequality and (3.181) that

$$d\big(T^{qn_2}x, x_*\big) \leq \varepsilon/4 \tag{3.182}$$

for all points x satisfying (3.177).

Assume now that $x \in B(x_*, M)$ and let an integer s be such that $s \geq n_1 + qn_2$. By (3.173),

$$T^i x \in B(x_*, M_1) \quad \text{for all integers } i \geq n_1$$

and

$$T^{s-qn_2}x \in B(x_*, M_1). \tag{3.183}$$

Since $T^s x = T^{qn_2}(T^{s-qn_2}x)$, it follows from (3.182) and (3.183) that

$$d\big(T^s x, x_*\big) = d\big(T^{qn_2}\big(T^{s-qn_2}x\big), x_*\big) < \varepsilon/4.$$

This completes the proof of Theorem 3.19. □

The following result, which was also obtained in [9], is an extension of Theorem 3.19.

Theorem 3.20 *Let $x_* \in X$ be a fixed point of $T : X \to X$. Assume that $\{m_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers such that*

$$d\big(T^{m_k}x, x_*\big) \le \phi_{m_k}\big(d(x, x_*)\big)$$

for all $x \in X$ and all natural numbers k, where T and the functions $\phi_{m_k} : [0, \infty) \to [0, \infty)$, $k = 1, 2, \ldots,$ satisfy the following conditions:

(i) *For each $M > 0$, there is $M_1 > 0$ such that*

$$T^i\big(B(x_*, M)\big) \subset B(x_*, M_1) \quad \textit{for each integer } i \ge 0;$$

(ii) *there exists an upper semicontinuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ for all $t > 0$ such that $\lim_{k\to\infty} \phi_{m_k} = \phi$, uniformly on any bounded interval $[0, b]$.*

Then $T^n x \to x_*$ as $n \to \infty$, uniformly on any bounded subset of X.

Proof Let i be a natural number such that $i \ne m_k$ for all natural numbers k. For each $t \ge 0$, set

$$\phi_i(t) = \sup\big\{d\big(T^i x, x_*\big) : x \in B(x_*, t)\big\}.$$

Clearly, $\phi_i(t)$ is finite for all $t \ge 0$. It is easy to see that all the assumptions of Theorem 3.19 hold. Therefore Theorem 3.19 implies that $T^n x \to x_*$ as $n \to \infty$, uniformly on all bounded subsets of X. Theorem 3.20 is proved. □

Now we show that Theorem 3.19 has a converse.

Assume now that $T : X \to X$, $x_* \in X$, $T^n x \to x_*$ as $n \to \infty$, uniformly on all bounded subsets of X, and that $T(C)$ is bounded for any bounded $C \subset X$. We claim that T necessarily satisfies all the hypotheses of Theorem 3.19 with an appropriate sequence $\{\phi_n\}_{n=1}^{\infty}$.

Indeed, fix a natural number n and for all $t \ge 0$, set

$$\phi_n(t) = \sup\big\{d\big(T^n x, x_*\big) : x \in B(x_*, t)\big\}.$$

Clearly, $\phi_n(t)$ is finite for all $t \ge 0$ and all natural numbers n, and

$$d\big(T^n x, x_*\big) \le \phi_n\big(d(x, x_*)\big)$$

for all $x \in X$ and all natural numbers n. It is also obvious that $\phi_n \to 0$ as $n \to \infty$, uniformly on any bounded subinterval of $[0, \infty)$, and that for any $b > 0$,

$$\sup\{\phi_n(t) : t \in [0, b], n \geq 1\} < \infty.$$

Thus all the assumptions of Theorem 3.19 hold with $\phi(t) = 0$ identically.

3.11 Well-Posedness of Fixed Point Problems

Let (K, ρ) be a bounded complete metric space. We say that the fixed point problem for a mapping $A : K \to K$ is well posed if there exists a unique $x_A \in K$ such that $Ax_A = x_A$ and the following property holds:

if $\{x_n\}_{n=1}^{\infty} \subset K$ and $\rho(x_n, Ax_n) \to 0$ as $n \to \infty$, then $\rho(x_n, x_A) \to 0$ as $n \to \infty$.

The notion of well-posedness is of central importance in many areas of Mathematics and its applications. In our context this notion was studied in [50], where generic well-posedness of the fixed point problem is established for the space of nonexpansive self-mappings of K.

In this section, which is based on [139], we first show (Theorem 3.21) that the fixed point problem is well posed for any contractive self-mapping of K. Since it is known that in Banach spaces (see Theorem 3.2) almost all nonexpansive mappings are contractive in the sense of Baire's categories, the generic well-posedness of the fixed point problem for the space of nonexpansive self-mappings of K follows immediately in this case. In our second result (Theorem 3.22) we show that the fixed point problem is well posed as soon as the uniformly continuous self-mapping of K has a unique fixed point which is the uniform limit of every sequence of iterates.

Let (K, ρ) be a bounded complete metric space. Define

$$d(K) = \sup\{\rho(x, y) : x, y \in K\}. \tag{3.184}$$

Recall that a mapping $A : K \to K$ is contractive if there exists a decreasing function $\phi : [0, d(K)] \to [0, 1]$ such that

$$\phi(t) < 1, \quad t \in (0, d(K)] \tag{3.185}$$

and

$$\rho(Ax, Ay) \leq \phi(\rho(x, y))\rho(x, y) \quad \text{for all } x, y \in K. \tag{3.186}$$

Theorem 3.21 *Assume that a mapping $A : K \to K$ is contractive. Then the fixed point problem for A is well posed.*

Proof Since the mapping A is contractive, there exists a decreasing function $\phi : [0, d(K)] \to [0, 1]$ such that (3.185) and (3.186) hold. By Theorem 3.1, there exists a unique $x_A \in K$ such that

$$Ax_A = x_A. \tag{3.187}$$

Let $\{x_n\}_{n=1}^{\infty} \subset K$ satisfy

$$\lim_{n\to\infty} \rho(x_n, Ax_n) = 0. \tag{3.188}$$

We claim that $x_n \to x_A$ as $n \to \infty$. Assume the contrary. By extracting a subsequence, if necessary, we may assume without loss of generality that there exists $\varepsilon > 0$ such that

$$\rho(x_n, x_A) \geq \varepsilon \quad \text{for all integers } n \geq 1. \tag{3.189}$$

Then it follows from (3.187), (3.186), (3.189) and the monotonicity of the function ϕ that for all integers $n \geq 1$,

$$\begin{aligned} \rho(x_A, x_n) &\leq \rho(x_A, Ax_n) + \rho(Ax_n, x_n) \leq \rho(Ax_n, x_n) + \phi\bigl(\rho(x_n, x_A)\bigr)\rho(x_n, x_A) \\ &\leq \rho(Ax_n, x_n) + \phi(\varepsilon)\rho(x_A, x_n). \end{aligned} \tag{3.190}$$

Inequalities (3.190) and (3.189) imply that for all integers $n \geq 1$,

$$\varepsilon\bigl(1 - \phi(\varepsilon)\bigr) \leq \bigl(1 - \phi(\varepsilon)\bigr)\rho(x_A, x_n) \leq \rho(Ax_n, x_n),$$

a contradiction (see (3.188)). The contradiction we have reached proves Theorem 3.21. □

Theorem 3.22 *Assume that $A : K \to K$ is a uniformly continuous mapping, $x_A \in K$, $Ax_A = x_A$, and that $A^n x \to x_A$ as $n \to \infty$, uniformly on K. Then the fixed point problem for the mapping A is well posed.*

Proof Let $\varepsilon > 0$ be given. In order to prove this theorem, it is sufficient to show that there exists $\delta > 0$ such that for each $y \in K$ satisfying $\rho(y, Ay) < \delta$, the inequality $\rho(y, x_A) < \varepsilon$ is true.

There exists a natural number $n_0 \geq 3$ such that

$$\rho\bigl(A^n x, x_A\bigr) \leq \varepsilon/8 \quad \text{for any } x \in K \text{ and any integer } n \geq n_0. \tag{3.191}$$

Set

$$\delta_0 = \varepsilon(8n_0)^{-1}. \tag{3.192}$$

Using induction, we define a sequence of positive numbers $\{\delta_i\}_{i=0}^{\infty}$ such that for any integer $i \geq 0$,

$$\delta_{i+1} < \delta_i \tag{3.193}$$

and

$$\text{if } x, y \in K \text{ and } \rho(x, y) \leq \delta_{i+1}, \text{ then } \rho(Ax, Ay) \leq \delta_i. \tag{3.194}$$

We now show that if $y \in K$ satisfies $\rho(y, Ay) < \delta_{n_0}$, then $\rho(y, x_A) < \varepsilon/2$. Indeed, let $y \in K$ satisfy

$$\rho(y, Ay) < \delta_{n_0}. \tag{3.195}$$

It follows from the definition of the sequence $\{\delta_i\}_{i=0}^{\infty}$ (see (3.193), (3.194)) and (3.195) that for any integer $j \in [1, n_0]$,

$$\rho\big(A^j y, A^{j+1} y\big) \le \delta_{n_0 - j}. \tag{3.196}$$

Relations (3.196), (3.193) and (3.192) imply that

$$\rho\big(y, A^{n_0+1} y\big) \le \sum_{j=0}^{n_0} \rho\big(A^j y, A^{j+1} y\big) \le (n_0 + 1)\delta_0 < \varepsilon/4. \tag{3.197}$$

(Here we use the notation $A^0 x = x$ for all $x \in K$.) It follows from (3.197) and the definition of n_0 (see (3.191)) that

$$\rho(y, x_A) \le \rho\big(y, A^{n_0+1} y\big) + \rho\big(A^{n_0+1} y, x_A\big) < \varepsilon/4 + \varepsilon/8 < \varepsilon/2.$$

Thus we have indeed shown that if $y \in K$ satisfies $\rho(y, Ay) < \delta_{n_0}$, then $\rho(y, x_A) < \varepsilon/2$. This completes the proof of Theorem 3.22. □

3.12 A Class of Mappings of Contractive Type

Let (X, ρ) be a complete metric space. In this section, which is based on [158], we present a sufficient condition for the existence and approximation of the unique fixed point of a contractive mapping which maps a nonempty, closed subset of X into X.

Theorem 3.23 *Let K be a nonempty and closed subset of a complete metric space (X, ρ). Assume that $T : K \to X$ satisfies*

$$\rho(Tx, Ty) \le \phi\big(\rho(x, y)\big) \quad \text{for each } x, y \in K, \tag{3.198}$$

where $\phi : [0, \infty) \to [0, \infty)$ is upper semicontinuous and satisfies $\phi(t) < t$ for all $t > 0$.

Assume further that $K_0 \subset K$ is a nonempty and bounded set with the following property:

(P1) *For each natural number n, there exists $x_n \in K_0$ such that $T^n x_n$ is defined.*

Then the following assertions hold.

(A) *There exists a unique $\bar{x} \in K$ such that $T\bar{x} = \bar{x}$.*
(B) *Let $M, \varepsilon > 0$. Then there exist $\delta > 0$ and a natural number k such that for each integer $n \ge k$ and each sequence $\{x_i\}_{i=0}^{n} \subset K$ satisfying*

$$\rho(x_0, \bar{x}) \le M$$

and

$$\rho(x_{i+1}, Tx_i) \le \delta, \quad i = 0, \dots, n-1,$$

the inequality $\rho(x_i, \bar{x}) \le \varepsilon$ *holds for* $i = k, \dots, n$.

Proof (A) The uniqueness of $\bar{x}$ is obvious. To establish its existence, we may and shall assume that $\phi(0) = 0$.

For each natural number n, let x_n be as guaranteed by (P1). Fix $\theta \in K$. Since K_0 is bounded, there is $c_0 > 0$ such that

$$\rho(\theta, z) \le c_0 \quad \text{for all } z \in K_0. \tag{3.199}$$

Let $\varepsilon > 0$ be given. We will show that there exists a natural number k such that the following property holds:

(P2) If n and i are integers such that $k \le i < n$, then

$$\rho\big(T^i x_n, T^{i+1} x_n\big) \le \varepsilon.$$

Assume the contrary. Then for each natural number k, there exist natural numbers n_k and i_k such that

$$k \le i_k < n_k \quad \text{and} \quad \rho\big(T^{i_k} x_{n_k}, T^{i_k+1} x_{n_k}\big) > \varepsilon. \tag{3.200}$$

Since the function $t - \phi(t)$ is positive for all $t > 0$ and lower semicontinuous, there is $\gamma > 0$ such that

$$t - \phi(t) \ge \gamma \quad \text{for all } t \in \big[\varepsilon/2, 2c_0 + \rho(\theta, T\theta) + \varepsilon\big]. \tag{3.201}$$

Choose a natural number k such that

$$k > \gamma^{-1}\big(2c_0 + \rho(\theta, T\theta)\big). \tag{3.202}$$

Then (3.200) holds. By (3.200) and (3.198),

$$\rho\big(T^i x_{n_k}, T^{i+1} x_{n_k}\big) > \varepsilon, \quad i = 0, \dots, i_k. \tag{3.203}$$

(Here we use the convention that $T^0 z = z$ for all $z \in K$.) By (3.198),

$$\rho(x_{n_k}, Tx_{n_k}) \ge \rho\big(T^i x_{n_k}, T^{i+1} x_{n_k}\big) \quad \text{for each integer } i \text{ satisfying } 0 \le i < i_k. \tag{3.204}$$

By (P1), (3.199) and (3.198),

$$\begin{aligned}\rho(x_{n_k}, Tx_{n_k}) &\le \rho(x_{n_k}, \theta) + \rho(\theta, T\theta) + \rho(T\theta, Tx_{n_k}) \\ &\le c_0 + \rho(\theta, T\theta) + c_0.\end{aligned} \tag{3.205}$$

Together with (3.203) and (3.204) this implies that

$$\varepsilon < \rho\big(T^i x_{n_k}, T^{i+1} x_{n_k}\big) \le 2c_0 + \rho(\theta, T\theta) \quad \text{for all } i = 0, \dots, i_k. \tag{3.206}$$

It follows from (3.198), (3.206) and (3.201) that for all $i = 0, \dots, i_k - 1$,

$$\rho\big(T^{i+2} x_{n_k}, T^{i+1} x_{n_k}\big) \le \phi\big(\rho\big(T^{i+1} x_{n_k}, T^i x_{n_k}\big)\big) \le \rho\big(T^{i+1} x_{n_k}, T^i x_{n_k}\big) - \gamma.$$

When combined with (3.205) and (3.200), this implies that

$$\begin{aligned} -\rho(\theta, T\theta) - 2c_0 \le -\rho(x_{n_k}, T x_{n_k}) &\le \rho\big(T^{i_k+1} x_{n_k}, T^{i_k} x_{n_k}\big) - \rho(x_{n_k}, T x_{n_k}) \\ &= \sum_{i=0}^{i_k - 1} \big[\rho\big(T^{i+2} x_{n_k}, T^{i+1} x_{n_k}\big) - \rho\big(T^{i+1} x_{n_k}, T^i x_{n_k}\big)\big] \\ &\le -\gamma i_k \le -k\gamma \end{aligned}$$

and

$$k\gamma \le 2c_0 + \rho(\theta, T\theta).$$

This contradicts (3.202). The contradiction we have reached proves the existence of a natural number k such that property (P2) holds.

Now let $\delta > 0$ be given. We will show that there exists a natural number k such that the following property holds:

(P3) If n, i and j are integers such that $k \le i$, $j < n$, then

$$\rho\big(T^i x_n, T^j x_n\big) \le \delta.$$

Assume to the contrary that there is no natural number k for which (P3) holds.

Then for each natural number k, there exist natural numbers n_k, i_k and j_k such that

$$k \le i_k < j_k < n_k \tag{3.207}$$

and

$$\rho\big(T^{i_k} x_{n_k}, T^{j_k} x_{n_k}\big) > \delta.$$

We may assume without loss of generality that for each natural number k, the following property holds:

If an integer j satisfies $i_k \le j < j_k$, then

$$\rho\big(T^{i_k} x_{n_k}, T^j x_{n_k}\big) \le \delta. \tag{3.208}$$

We have already shown that there exists a natural number k_0 such that (P2) holds with $k = k_0$ and $\varepsilon = \delta$.

Assume now that k is a natural number. It follows from (3.207) and (3.208) that

$$\delta < \rho\big(T^{i_k}x_{n_k}, T^{j_k}x_{n_k}\big) \le \rho\big(T^{j_k}x_{n_k}, T^{j_k-1}x_{n_k}\big) + \rho\big(T^{j_k-1}x_{n_k}, T^{i_k}x_{n_k}\big)$$
$$\le \rho\big(T^{j_k}x_{n_k}, T^{j_k-1}x_{n_k}\big) + \delta. \tag{3.209}$$

By property (P2),

$$\lim_{k\to\infty} \rho\big(T^{j_k}x_{n_k}, T^{j_k-1}x_{n_k}\big) = 0.$$

When combined with (3.209), this implies that

$$\lim_{k\to\infty} \rho\big(T^{i_k}x_{n_k}, T^{j_k}x_{n_k}\big) = \delta. \tag{3.210}$$

By (3.207), for each integer $k \ge 1$,

$$\begin{aligned} \delta &< \rho\big(T^{i_k}x_{n_k}, T^{j_k}x_{n_k}\big) \\ &\le \rho\big(T^{i_k}x_{n_k}, T^{i_k+1}x_{n_k}\big) + \rho\big(T^{i_k+1}x_{n_k}, T^{j_k+1}x_{n_k}\big) + \rho\big(T^{j_k+1}x_{n_k}, T^{j_k}x_{n_k}\big) \\ &\le \rho\big(T^{i_k}x_{n_k}, T^{i_k+1}x_{n_k}\big) + \rho\big(T^{j_k+1}x_{n_k}, T^{j_k}x_{n_k}\big) + \phi\big(\rho\big(T^{i_k}x_{n_k}, T^{j_k}x_{n_k}\big)\big). \end{aligned} \tag{3.211}$$

Since by (P2),

$$\lim_{k\to\infty} \rho\big(T^{i_k}x_{n_k}, T^{i_k+1}x_{n_k}\big) = \lim_{k\to\infty} \rho\big(T^{j_k}x_{n_k}, T^{j_k+1}x_{n_k}\big) = 0,$$

(3.210) and (3.211) imply that $\delta \le \phi(\delta)$, a contradiction.

The contradiction we have reached proves that there exists a natural number k such that (P3) holds.

Let $\varepsilon > 0$ be given. We will show that there exists a natural number k such that the following property holds:

(P4) If the integers $n_1, n_2 > k$, then $\rho(T^k x_{n_1}, T^k x_{n_2}) \le \varepsilon$.

Assume the contrary. Then for each integer $k \ge 1$, there are integers $n_1^{(k)}, n_2^{(k)} > k$ such that

$$\rho\big(T^k x_{n_1^{(k)}}, T^k x_{n_2^{(k)}}\big) > \varepsilon. \tag{3.212}$$

By (P1), (3.198) and (3.199), the sequence

$$\big\{\rho\big(T^k x_{n_1^{(k)}}, T^k x_{n_2^{(k)}}\big)\big\}_{k=1}^{\infty}$$

is bounded. Set

$$\delta = \limsup_{k\to\infty} \rho\big(T^k x_{n_1^{(k)}}, T^k x_{n_2^{(k)}}\big). \tag{3.213}$$

By definition, there exists a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that

$$\delta = \lim_{i\to\infty} \rho\big(T^{k_i} x_{n_1^{(k_i)}}, T^{k_i} x_{n_2^{(k_i)}}\big). \tag{3.214}$$

By (3.212) and (3.213),

$$\delta \geq \varepsilon. \tag{3.215}$$

By (3.198), for each natural number i,

$$\begin{aligned}\rho\big(T^{k_i} x_{n_1^{(k_i)}}, T^{k_i} x_{n_2^{(k_i)}}\big) &\leq \rho\big(T^{k_i+1} x_{n_1^{(k_i)}}, T^{k_i} x_{n_1^{(k_i)}}\big) \\ &\quad + \rho\big(T^{k_i+1} x_{n_1^{(k_i)}}, T^{k_i+1} x_{n_2^{(k_i)}}\big) + \rho\big(T^{k_i+1} x_{n_2^{(k_i)}}, T^{k_i} x_{n_2^{(k_i)}}\big) \\ &\leq \rho\big(T^{k_i+1} x_{n_1^{(k_i)}}, T^{k_i} x_{n_1^{(k_i)}}\big) + \rho\big(T^{k_i+1} x_{n_2^{(k_i)}}, T^{k_i} x_{n_2^{(k_i)}}\big) \\ &\quad + \phi\big(\rho\big(T^{k_i} x_{n_1^{(k_i)}}, T^{k_i} x_{n_2^{(k_i)}}\big)\big).\end{aligned} \tag{3.216}$$

By property (P2),

$$\lim_{i \to \infty} \rho\big(T^{k_i+1} x_{n_j^{(k_i)}}, T^{k_i} x_{n_j^{(k_i)}}\big) = 0, \quad j = 1, 2. \tag{3.217}$$

Now it follows from (3.216), (3.217), (3.204) and (3.215) that $\varepsilon \leq \delta \leq \phi(\delta)$, a contradiction. This contradiction implies that there is indeed a natural number k such that (P4) holds, as claimed.

Let $\varepsilon > 0$ be given. By (P4), there exists a natural number k_1 such that

$$\rho\big(T^{k_1} x_{n_1}, T^{k_1} x_{n_2}\big) \leq \varepsilon/4 \quad \text{for all integers } n_1, n_2 \geq k_1. \tag{3.218}$$

By (P3), there exists a natural number k_2 such that

$$\rho\big(T^i x_n, T^j x_n\big) \leq \varepsilon/4 \quad \text{for all natural numbers } n, i, j \text{ satisfying } k_2 \leq i, j < n. \tag{3.219}$$

Assume that the natural numbers n_1, n_2, i and j satisfy

$$n_1, n_2 > k_1 + k_2, \qquad i, j \geq k_1 + k_2, \qquad i < n_1, \qquad j < n_2. \tag{3.220}$$

We claim that $\rho(T^i x_{n_1}, T^j x_{n_2}) \leq \varepsilon$. By (3.198), (3.218) and (3.220),

$$\rho\big(T^{k_1+k_2} x_{n_1}, T^{k_1+k_2} x_{n_2}\big) \leq \rho\big(T^{k_1} x_{n_1}, T^{k_1} x_{n_2}\big) \leq \varepsilon/4. \tag{3.221}$$

In view of (3.219) and (3.220),

$$\rho\big(T^{k_1+k_2} x_{n_1}, T^i x_{n_1}\big) \leq \varepsilon/4 \quad \text{and} \quad \rho\big(T^{k_1+k_2} x_{n_2}, T^j x_{n_2}\big) \leq \varepsilon/4. \tag{3.222}$$

Inequalities (3.222) and (3.221) imply that

$$\begin{aligned}\rho\big(T^i x_{n_1}, T^j x_{n_2}\big) &\leq \rho\big(T^i x_{n_1}, T^{k_1+k_2} x_{n_1}\big) + \rho\big(T^{k_1+k_2} x_{n_1}, T^{k_1+k_2} x_{n_2}\big) \\ &\quad + \rho\big(T^{k_1+k_2} x_{n_2}, T^j x_{n_2}\big) < \varepsilon.\end{aligned}$$

Thus we have shown that the following property holds:

(P5) For each $\varepsilon > 0$, there exists a natural number $k(\varepsilon)$ such that

$$\rho\bigl(T^i x_{n_1}, T^j x_{n_2}\bigr) \le \varepsilon \quad \text{for all natural numbers } n_1, n_2, i \text{ and } j$$

such that

$$n_1, n_2 > k(\varepsilon), \qquad i \in \bigl[k(\varepsilon), n_1\bigr) \quad \text{and} \quad j \in \bigl[k(\varepsilon), n_2\bigr).$$

Consider now the sequences $\{T^{n-2}x_n\}_{n=3}^{\infty}$ and $\{T^{n-1}x_n\}_{n=3}^{\infty}$. Property (P5) implies that both of them are Cauchy sequences and that

$$\lim_{n\to\infty} \rho\bigl(T^{n-2}x_n, T^{n-1}x_n\bigr) = 0.$$

Hence there exists $\bar{x} \in K$ such that

$$\lim_{n\to\infty} \rho\bigl(\bar{x}, T^{n-2}x_n\bigr) = \lim_{t\to\infty} \rho\bigl(\bar{x}, T^{n-1}x_n\bigr) = 0.$$

Since the mapping T is continuous, it follows that $T\bar{x} = \bar{x}$. Thus part (A) of our theorem is proved.

We now turn to the proof of part (B). Clearly,

$$\inf\bigl\{t - \phi(t) : t \in [M/2, M]\bigr\} > 0.$$

Choose a positive number δ_0 such that

$$\delta_0 < \min\bigl\{M/2, \inf\bigl\{t - \phi(t) : t \in [M/2, M]\bigr\}/4\bigr\}. \tag{3.223}$$

For each $x \in X$ and $r > 0$, set

$$B(x, r) = \bigl\{y \in X : \rho(x, y) \le r\bigr\}.$$

Assume that

$$y \in K \cap B(\bar{x}, M), \qquad z \in X \quad \text{and} \quad \rho(z, Ty) \le \delta_0. \tag{3.224}$$

By (3.224) and (3.198),

$$\rho(\bar{x}, z) \le \rho(\bar{x}, Ty) + \rho(Ty, z) \le \rho(T\bar{x}, Ty) + \delta_0 \le \phi\bigl(\rho(\bar{x}, y)\bigr) + \delta_0. \tag{3.225}$$

There are two cases:

$$\rho(y, \bar{x}) \le M/2; \tag{3.226}$$

$$\rho(y, \bar{x}) > M/2. \tag{3.227}$$

Assume that (3.226) holds. By (3.225), (3.226), (3.198) and (3.223),

$$\rho(\bar{x}, z) \le \rho(\bar{x}, y) + \delta_0 \le M/2 + \delta_0 < M.$$

Assume that (3.227) holds. Then by (3.223), (3.225), (3.224) and (3.227),

$$\rho(\bar{x}, z) \le \delta_0 + \phi\big(\rho(\bar{x}, y)\big) < \big[\rho(\bar{x}, y) - \phi\big(\rho(\bar{x}, y)\big)\big]4^{-1} + \phi\big(\rho(\bar{x}, y)\big)$$
$$< \rho(\bar{x}, y) \le M.$$

Thus $\rho(\bar{x}, z) \le M$ in both cases.

We have shown that

$$\rho(\bar{x}, z) \le M \quad \text{for each } z \in X \text{ such that}$$
$$\text{there exists } y \in K \cap B(\bar{x}, M) \text{ satisfying } \rho(z, Ty) \le \delta_0. \tag{3.228}$$

Since M is an arbitrary positive number, we may conclude that there is $\delta_1 > 0$ so that

$$\rho(\bar{x}, z) \le \varepsilon \quad \text{for each } z \in X \text{ such that}$$
$$\text{there exists } y \in K \cap B(\bar{x}, \varepsilon) \text{ satisfying } \rho(z, Ty) \le \delta_1. \tag{3.229}$$

Choose a positive number δ such that

$$\delta < \min\big\{\delta_0, \delta_1, 4^{-1} \inf\big\{t - \phi(t) : t \in [\varepsilon, M + \varepsilon + 1]\big\}\big\} \tag{3.230}$$

and a natural number k such that

$$k > 2(M + 1)\delta^{-1} + 2. \tag{3.231}$$

Assume that n is a natural number such that $n \ge k$ and that $\{x_i\}_{i=0}^n \subset K$ satisfies

$$\rho(x_0, \bar{x}) \le M, \qquad \rho(x_{i+1}, Tx_i) \le \delta, \quad i = 0, \dots, n-1. \tag{3.232}$$

We claim that

$$\rho(x_i, \bar{x}) \le \varepsilon, \quad i = k, \dots, n. \tag{3.233}$$

By (3.228), (3.230) and (3.232),

$$\{x_i\}_{i=0}^n \subset B(\bar{x}, M). \tag{3.234}$$

Assume that (3.233) does not hold. Then there is an integer j such that

$$j \in \{k, \dots, n\} \quad \text{and} \quad \rho(x_j, \bar{x}) > \varepsilon. \tag{3.235}$$

By (3.229), (3.230) and (3.232),

$$\rho(x_i, \bar{x}) > \varepsilon, \quad i = 0, \dots, j. \tag{3.236}$$

Let $i \in \{0, \dots, j-1\}$. By (3.232), (3.198), (3.234), (3.236) and (3.230),

$$\rho(x_{i+1}, \bar{x}) \le \rho(x_{i+1}, Tx_i) + \rho(Tx_i, T\bar{x}) \le \delta + \phi\big(\rho(x_i, \bar{x})\big)$$
$$< \phi\big(\rho(x_i, \bar{x})\big) + 4^{-1}\big(\rho(x_i, \bar{x}) - \phi\big(\rho(x_i, \bar{x})\big)\big)$$

$$< \phi\big(\rho(x_i, \bar{x})\big) + 2^{-1}\big(\rho(x_i, \bar{x}) - \phi\big(\rho(x_i, \bar{x})\big)\big) - \delta$$
$$\leq \rho(x_i, \bar{x}) - \delta.$$

When combined with (3.232) and (3.235), this implies that

$$-M \leq -\rho(x_0, \bar{x}) \leq \rho(x_j, \bar{x}) - \rho(x_0, \bar{x})$$
$$= \sum_{i=0}^{j-1} \big[\rho(x_{i+1}, \bar{x}) - \rho(x_i, \bar{x})\big] \leq -j\delta \leq -k\delta.$$

Thus

$$k\delta \leq M$$

which contradicts (3.231).

Hence (3.233) is true, as claimed, and part (B) of our theorem is also proved. □

3.13 A Fixed Point Theorem for Matkowski Contractions

Let (X, ρ) be a complete metric space. In this section, which is based on [159], we present a sufficient condition for the existence and approximation of the unique fixed point of a Matkowski contraction [99] which maps a nonempty and closed subset of X into X.

Theorem 3.24 *Let K be a nonempty and closed subset of a complete metric space (X, ρ). Assume that $T : K \to X$ satisfies*

$$\rho(Tx, Ty) \leq \phi\big(\rho(x, y)\big) \quad \text{for each } x, y \in K, \tag{3.237}$$

where $\phi : [0, \infty) \to [0, \infty)$ is increasing and satisfies $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$. Assume that $K_0 \subset K$ is a nonempty and bounded set with the following property:

(P1) *For each natural number n, there exists $x_n \in K_0$ such that $T^n x_n$ is defined.*

Then the following assertions hold.

(A) *There exists a unique $\bar{x} \in K$ such that $T\bar{x} = \bar{x}$.*
(B) *Let $M, \varepsilon > 0$. Then there exists a natural number k such that for each sequence $\{x_i\}_{i=0}^n \subset K$ with $n \geq k$ satisfying*

$$\rho(x_0, \bar{x}) \leq M \quad \text{and} \quad Tx_i = x_{i+1}, \quad i = 0, \ldots, n-1,$$

the inequality $\rho(x_i, \bar{x}) \leq \varepsilon$ holds for all $i = k, \ldots, n$.

Proof For each $x \in X$ and $r > 0$, set

$$B(x,r) = \{y \in X : \rho(x,y) \le r\}. \tag{3.238}$$

(A) Since $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$, and since ϕ is increasing, we have

$$\phi(t) < t \quad \text{for all } t > 0. \tag{3.239}$$

This implies the uniqueness of $\bar{x}$. Clearly, $\phi(0) = 0$.

For each natural number n, let x_n be as guaranteed by property (P1). Fix $\theta \in K$. Since K_0 is bounded, there is $c_0 > 0$ such that

$$\rho(\theta, z) \le c_0 \quad \text{for all } z \in K_0. \tag{3.240}$$

Let $\varepsilon > 0$ be given. We will show that there exists a natural number k such that the following property holds:

(P2) If the integers i and n satisfy $k \le i < n$, then

$$\rho\big(T^i x_n, T^{i+1} x_n\big) \le \varepsilon.$$

By (3.236) and (3.240), for each $z \in K_0$,

$$\begin{aligned}\rho(z, Tz) &\le \rho(z,\theta) + \rho(\theta, T\theta) + \rho(T\theta, Tz)\\ &\le 2\rho(z,\theta) + \rho(\theta, T\theta) \le 2c_0 + \rho(\theta, T\theta).\end{aligned} \tag{3.241}$$

Clearly, there is a natural number k such that

$$\phi^k\big(2c_0 + \rho(\theta, T\theta)\big) < \varepsilon. \tag{3.242}$$

Assume now that the integers i and n satisfy $k \le i < n$.

By (3.236), (3.239), (3.241), the choice of x_n, and (3.242),

$$\begin{aligned}\rho\big(T^i x_n, T^{i+1} x_n\big) &\le \rho\big(T^k x_n, T^{k+1} x_n\big) \le \phi^k\big(\rho(x_n, T x_n)\big)\\ &\le \phi^k\big(2c_0 + \rho(\theta, T\theta)\big) < \varepsilon.\end{aligned}$$

Thus property (P2) holds for this k.

Let $\delta > 0$ be given. We claim that there exists a natural number k such that the following property holds:

(P3) If the integers i, j and n satisfy $k \le i < j < n$, then

$$\rho\big(T^i x_n, T^j x_n\big) \le \delta.$$

Indeed, by (3.239),

$$\phi(\delta) < \delta. \tag{3.243}$$

By (P2) and (3.243), there is a natural number k such that (P2) holds with $\varepsilon = \delta - \phi(\delta)$.

Assume now that the integers i and n satisfy $k \le i < n$. In view of the choice of k and property (P2) with $\varepsilon = \delta - \phi(\delta)$, we have

$$\rho\big(T^i x_n, T^{i+1} x_n\big) \le \delta - \phi(\delta). \tag{3.244}$$

Now let

$$x \in K \cap B\big(T^i x_n, \delta\big). \tag{3.245}$$

It follows from (3.236), (3.244) and (3.245) that

$$\begin{aligned}\rho\big(Tx, T^i x_n\big) &\le \rho\big(Tx, T^{i+1} x_n\big) + \rho\big(T^{i+1} x_n, T^i x_n\big) \le \phi\big(\rho\big(x, T^i x_n\big)\big) + \delta - \phi(\delta)\\ &\le \delta.\end{aligned}$$

Thus

$$T\big(K \cap B\big(T^i x_n, \delta\big)\big) \subset B\big(T^i x_n, \delta\big),$$

and if an integer j satisfies $i < j < n$, then $\rho(T^i x_n, T^j x_n) \le \delta$. Hence property (P3) does hold, as claimed.

Let $\varepsilon > 0$ be given. We will show that there exists a natural number k such that the following property holds:

(P4) If the integers n_1, n_2 and i satisfy $k \le i \le \min\{n_1, n_2\}$, then

$$\rho\big(T^i x_{n_1}, T^i x_{n_2}\big) \le \varepsilon.$$

Indeed, there exists a natural number k such that

$$\phi^i(2c_0) < \varepsilon \quad \text{for all integers } i \ge k. \tag{3.246}$$

Assume now that the natural numbers n_1, n_2 and i satisfy

$$k \le i \le \min\{n_1, n_2\}. \tag{3.247}$$

By (3.236), (3.240) and (3.246),

$$\rho\big(T^i x_{n_1}, T^i x_{n_2}\big) \le \phi^i\big(\rho(x_{n_1}, x_{n_2})\big) \le \phi^i(2c_0) < \varepsilon.$$

Thus property (P4) indeed holds.

Let $\varepsilon > 0$ be given. By (P4), there exists a natural number k_1 such that

$$\begin{aligned}&\rho\big(T^i x_{n_1}, T^i x_{n_2}\big) \le \varepsilon/4 \quad \text{for all integers } n_1, n_2 \ge k_1\\ &\quad \text{and all integers } i \text{ satisfying } k_1 \le i \le \min\{n_1, n_2\}.\end{aligned} \tag{3.248}$$

By property (P3), there exists a natural number k_2 such that

$$\rho\big(T^i x_n, T^j x_n\big) \le \varepsilon/4 \quad \text{for all natural numbers } n, i, j \text{ satisfying } k_2 \le i, j < n. \tag{3.249}$$

Assume that the natural numbers n_1, n_2, i and j satisfy

$$n_1, n_2 > k_1 + k_2, \qquad i, j \geq k_1 + k_2, \qquad i < n_1, \qquad j < n_2. \tag{3.250}$$

We claim that

$$\rho\big(T^i x_{n_1}, T^j x_{n_2}\big) \leq \varepsilon.$$

By (3.238), (3.243), (3.248) and (3.250),

$$\rho\big(T^{k_1+k_2} x_{n_1}, T^{k_1+k_2} x_{n_2}\big) \leq \rho\big(T^{k_1} x_{n_1}, T^{k_1} x_{n_2}\big) \leq \varepsilon/4. \tag{3.251}$$

In view of (3.249) and (3.250),

$$\rho\big(T^{k_1+k_2} x_{n_1}, T^i x_{n_1}\big) \leq \varepsilon/4 \quad \text{and} \quad \rho\big(T^{k_1+k_2} x_{n_2}, T^j x_{n_2}\big) \leq \varepsilon/4.$$

When combined with (3.251), this implies that

$$\begin{aligned} \rho\big(T^i x_{n_1}, T^j x_{n_2}\big) &\leq \rho\big(T^i x_{n_1}, T^{k_1+k_2} x_{n_1}\big) + \rho\big(T^{k_1+k_2} x_{n_1}, T^{k_1+k_2} x_{n_2}\big) \\ &\quad + \rho\big(T^{k_1+k_2} x_{n_2}, T^j x_{n_2}\big) \\ &\leq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 < \varepsilon. \end{aligned}$$

Thus we have shown that the following property holds:

(P5) For each $\varepsilon > 0$, there exists a natural number $k(\varepsilon)$ such that

$$\rho\big(T^i x_{n_1}, T^j x_{n_2}\big) \leq \varepsilon$$

for all natural numbers $n_1, n_2 > k(\varepsilon)$, $i \in [k(\varepsilon), n_1)$ and $j \in [k(\varepsilon), n_2)$.

Consider now the sequences $\{T^{n-2} x_n\}_{n=3}^{\infty}$ and $\{T^{n-1} x_n\}_{n=3}^{\infty}$. Property (P5) implies that these sequences are Cauchy sequences and that

$$\lim_{n\to\infty} \rho\big(T^{n-2} x_n, T^{n-1} x_n\big) = 0.$$

Hence there exists $\bar{x} \in K$ such that

$$\lim_{n\to\infty} \rho\big(\bar{x}, T^{n-2} x_n\big) = \lim_{n\to\infty} \rho\big(\bar{x}, T^{n-1} x_n\big) = 0.$$

Since the mapping T is continuous, $T\bar{x} = \bar{x}$ and part (A) is proved.

(B) Since T is a Matkowski contraction, there is a natural number k such that $\phi^k(M) < \varepsilon$.

Assume that a point $x_0 \in B(\bar{x}, M)$, an integer $n \geq k$, and that $T^i x_0$ is defined for all $i = 0, \dots, n$. Then $T^i x_0 \in K$, $i = 0, \dots, n-1$, and by (3.236),

$$\rho\big(T^k x_0, \bar{x}\big) \leq \phi^k\big(\rho(x_0, \bar{x})\big) \leq \phi^k(M) < \varepsilon.$$

By (3.236) and (3.239), we have for $i = k, \dots, n$,

$$\rho\big(T^i x_0, \bar{x}\big) \leq \rho\big(T^k x_0, \bar{x}\big) \leq \varepsilon.$$

Thus part (B) of our theorem is also proved. □

3.14 Jachymski-Schröder-Stein Contractions

Suppose that (X, d) is a complete metric space, N_0 is a natural number, and $\phi : [0, \infty) \to [0, \infty)$ is a function which is upper semicontinuous from the right and satisfies $\phi(t) < t$ for all $t > 0$. We call a mapping $T : X \to X$ for which

$$\min\{d(T^i x, T^i y) : i \in \{1, \dots, N_0\}\} \le \phi(d(x, y)) \quad \text{for all } x, y \in X \tag{3.252}$$

a Jachymski-Schröder-Stein contraction (with respect to ϕ).

Condition (3.252) was introduced in [78]. Such mappings with $\phi(t) = \gamma t$ for some $\gamma \in (0, 1)$ have recently been of considerable interest [10, 78, 79, 100, 101, 174]. In this section, which is based on [161], we study general Jachymski-Schröder-Stein contractions and prove two fixed point theorems for them (Theorems 3.25 and 3.26 below). In our first result we establish convergence of iterates to a fixed point, and in the second this conclusion is strengthened to obtain uniform convergence on bounded subsets of X. This last type of convergence is useful in the study of inexact orbits [35]. Our theorems contain the (by now classical) results in [23] as well as Theorem 2 in [78]. In contrast with that theorem, in Theorem 3.25 we only assume that ϕ is upper semicontinuous from the right and we do not assume that $\liminf_{t\to\infty}(t - \phi(t)) > 0$. Moreover, our arguments are completely different from those presented in [78], where the Cantor Intersection Theorem was used. We remark in passing that Cantor's theorem was also used in this context in [65] (cf. also [68]).

Theorem 3.25 *Let (X, d) be a complete metric space and let $T : X \to X$ be a Jachymski-Schröder-Stein contraction. Assume there is $x_0 \in X$ such that T is uniformly continuous on the orbit $\{T^i x_0 : i = 1, 2, \dots\}$. Then there exists $\bar{x} = \lim_{i\to\infty} T^i x_0$ in (X, d). Moreover, if T is continuous at $\bar{x}$, then $\bar{x}$ is the unique fixed point of T.*

Proof Set

$$T^0 x = x, \quad x \in X. \tag{3.253}$$

We are going to define a sequence of nonnegative integers $\{k_i\}_{i=0}^{\infty}$ by induction. Set $k_0 = 0$. Assume that $i \ge 0$ is an integer, and that the integer $k_i \ge 0$ has already been defined. Clearly, there exists an integer k_{i+1} such that

$$1 \le k_{i+1} - k_i \le N_0 \tag{3.254}$$

and

$$d(T^{k_{i+1}} x_0, T^{k_{i+1}+1} x_0) = \min\{d(T^{j+k_i} x_0, T^{j+k_i+1} x_0) : j = 1, \dots, N_0\}. \tag{3.255}$$

By (3.252), (3.254) and (3.255), the sequence $\{d(T^{k_j} x_0, T^{k_j+1} x_0)\}_{j=0}^{\infty}$ is decreasing. Set

$$r = \lim_{j\to\infty} d(T^{k_j} x_0, T^{k_j+1} x_0). \tag{3.256}$$

Assume that $r > 0$. Then by (3.252), (3.254) and (3.255), for each integer $j \geq 0$,

$$d\big(T^{k_{j+1}}x_0, T^{k_{j+1}+1}x_0\big) \leq \phi\big(d\big(T^{k_j}x_0, T^{k_j+1}x_0\big)\big).$$

When combined with (3.256), the monotonicity of the sequence

$$\big\{d\big(T^{k_j}x_0, T^{k_j+1}x_0\big)\big\}_{j=0}^{\infty},$$

and the upper semicontinuity from the right of ϕ, this inequality implies that

$$r \leq \limsup_{j\to\infty} \phi\big(d\big(T^{k_j}x_0, T^{k_j+1}x_0\big)\big) \leq \phi(r),$$

a contradiction. Thus $r = 0$ and

$$\lim_{j\to\infty} d\big(T^{k_j}x_0, T^{k_j+1}x_0\big) = 0. \tag{3.257}$$

We claim that, in fact,

$$\lim_{i\to\infty} d\big(T^i x_0, T^{i+1}x_0\big) = 0.$$

Indeed, let $\varepsilon > 0$ be given. Since T is uniformly continuous on the set

$$\Omega := \big\{T^i x_0 : i = 1, 2, \dots\big\}, \tag{3.258}$$

there is

$$\varepsilon_0 \in (0, \varepsilon) \tag{3.259}$$

such that

$$\text{if } x, y \in \Omega, i \in \{1, \dots, N_0\}, d(x, y) \leq \varepsilon_0, \text{ then } d\big(T^i x, T^i y\big) \leq \varepsilon. \tag{3.260}$$

By (3.257), there is a natural number j_0 such that

$$d\big(T^{k_j}x_0, T^{k_j+1}x_0\big) \leq \varepsilon_0 \quad \text{for all integers } j \geq j_0. \tag{3.261}$$

Let p be an integer such that

$$p \geq k_{j_0} + N_0.$$

Then by (3.254) there is an integer $j \geq j_0$ such that

$$k_j < p \leq k_j + N_0. \tag{3.262}$$

By (3.261) and the inequality $j \geq j_0$,

$$d\big(T^{k_j}x_0, T^{k_j+1}x_0\big) \leq \varepsilon_0.$$

Together with (3.262) and (3.261), this implies that

$$d\big(T^p x_0, T^{p+1} x_0\big) \le \varepsilon.$$

Thus this inequality holds for any integer $p \ge k_{j_0} + N_0$ and we conclude that

$$\lim_{p\to\infty} d\big(T^p x_0, T^{p+1} x_0\big) = 0, \tag{3.263}$$

as claimed.

Now we show that $\{T^i x_0\}_{i=1}^{\infty}$ is a Cauchy sequence. Assume the contrary. Then there exists $\varepsilon > 0$ such that for each natural number p, there exist integers $m_p > n_p \ge p$ such that

$$d\big(T^{m_p} x_0, T^{n_p} x_0\big) \ge \varepsilon. \tag{3.264}$$

We may assume without loss of generality that for each natural number p,

$$d\big(T^i x_0, T^{n_p} x_0\big) < \varepsilon \quad \text{for all integers } i \text{ satisfying } n_p < i < m_p. \tag{3.265}$$

By (3.264) and (3.265), for any integer $p \ge 1$,

$$\begin{aligned}\varepsilon &\le d\big(T^{m_p} x_0, T^{n_p} x_0\big) \le d\big(T^{m_p} x_0, T^{m_p-1} x_0\big) + d\big(T^{m_p-1} x_0, T^{n_p} x_0\big) \\ &\le d\big(T^{m_p} x_0, T^{m_p-1} x_0\big) + \varepsilon.\end{aligned}$$

When combined with (3.263), this implies that

$$\lim_{p\to\infty} d\big(T^{m_p} x_0, T^{n_p} x_0\big) = \varepsilon. \tag{3.266}$$

Let $\delta > 0$ be given. By (3.263), there is an integer $p_0 \ge 1$ such that

$$d\big(T^{i+1} x_0, T^i x_0\big) \le \delta (4N_0)^{-1} \quad \text{for all integers } i \ge p_0. \tag{3.267}$$

Let $p \ge p_0$ be an integer. By (3.263), there is $j \in \{1, \dots, N_0\}$ such that

$$d\big(T^{m_p+j} x_0, T^{n_p+j} x_0\big) \le \phi\big(d\big(T^{m_p} x_0, T^{n_p} x_0\big)\big). \tag{3.268}$$

By the inequalities $m_p > n_p \ge p$, (3.267) and (3.268),

$$\begin{aligned} d\big(T^{m_p} x_0, T^{n_p} x_0\big) &\le \sum_{i=0}^{j-1} d\big(T^{m_p+i} x_0, T^{m_p+i+1} x_0\big) + d\big(T^{m_p+j} x_0, T^{n_p+j} x_0\big) \\ &\quad + \sum_{i=0}^{j-1} d\big(T^{n_p+i} x_0, T^{n_p+i+1} x_0\big) \\ &\le 2j\delta (4N_0)^{-1} + \phi\big(d\big(T^{m_p} x_0, T^{n_p} x_0\big)\big) \\ &< \delta + \phi\big(d\big(T^{m_p} x_0, T^{n_p} x_0\big)\big). \end{aligned} \tag{3.269}$$

By (3.266), (3.269), (3.264), and the upper semicontinuity from the right of ϕ,

$$\varepsilon = \lim_{p\to\infty} d\big(T^{m_p}x_0, T^{n_p}x_0\big) \le \delta + \limsup_{p\to\infty} \phi\big(d\big(T^{m_p}x_0, T^{n_p}x_0\big)\big) \le \delta + \phi(\varepsilon).$$

Since δ is an arbitrary positive number, we conclude that $\varepsilon \le \phi(\varepsilon)$. The contradiction we have reached proves that $\{T^i x_0\}_{i=1}^{\infty}$ is indeed a Cauchy sequence. Set

$$\bar{x} = \lim_{i\to\infty} T^i x_0.$$

Clearly, if T is continuous, then $T\bar{x} = \bar{x}$ and $\bar{x}$ is the unique fixed point of T. Theorem 3.25 is proved. □

For each $x \in X$ and $r > 0$, set

$$B(x, r) = \big\{z \in X : \rho(x, z) \le r\big\}.$$

Theorem 3.26 *Let (X, d) be a complete metric space and let $T : X \to X$ be a Jachymski-Schröder-Stein contraction with respect to the function $\phi : [0, \infty) \to [0, \infty)$. Assume that ϕ is upper semicontinuous, T is uniformly continuous on the set $\{T^i x : i = 1, 2, \dots\}$ for each $x \in X$, and that T is continuous on X. Then there exists a unique fixed point $\bar{x}$ of T such that $T^n x \to \bar{x}$ as $n \to \infty$, uniformly on bounded subsets of X.*

Proof By Theorem 3.25, T has a unique fixed point $\bar{x}$ and

$$T^n x \to \bar{x} \quad \text{as } n \to \infty \text{ for all } x \in X. \tag{3.270}$$

Let $r > 0$ be given. We claim that $T^n x \to \bar{x}$ as $n \to \infty$, uniformly on $B(\bar{x}, r)$.

Indeed, let

$$\varepsilon \in (0, r). \tag{3.271}$$

Since T is continuous, there is

$$\varepsilon_0 \in (0, \varepsilon) \tag{3.272}$$

such that

$$\text{if } x \in X, d(x, \bar{x}) \le \varepsilon_0, i \in \{1, \dots, N_0\}, \text{ then } d\big(T^i x, \bar{x}\big) \le \varepsilon. \tag{3.273}$$

Since ϕ is upper semicontinuous, there is

$$\delta \in (0, \varepsilon_0) \tag{3.274}$$

such that

$$\text{if } t \in [\varepsilon_0, r], \text{ then } t - \phi(t) \ge \delta. \tag{3.275}$$

Choose a natural number N_1 such that

$$N_1\delta > 2r. \tag{3.276}$$

Assume that

$$x \in X, \quad d(\bar{x}, x) \le r. \tag{3.277}$$

We will show that

$$d\big(\bar{x}, T^i x\big) \le \varepsilon \quad \text{for all integers } i \ge N_0 + N_0 N_1. \tag{3.278}$$

To this end, set $k_0 = 0$. Define by induction an increasing sequence of integers $\{k_i\}_{i=1}^{\infty}$ such that

$$k_{i+1} - k_i \in [1, N_0], \quad d\big(T^{k_i+1}x, \bar{x}\big) = \min\big\{d\big(T^{j+k_i}x, \bar{x}\big) : j \in \{1, \dots, N_0\}\big\}. \tag{3.279}$$

By (3.252) and (3.279), the sequence $\{d(T^{k_i}x, \bar{x})\}_{i=0}^{\infty}$ is decreasing. We claim that $d(T^{k_{N_1}}x, \bar{x}) \le \varepsilon_0$.

Assume the contrary. Then by (3.277) and (3.252),

$$r \ge d\big(T^{k_j}x, \bar{x}\big) > \varepsilon_0, \quad j = 0, \dots, N_1. \tag{3.280}$$

By (3.279), (3.252), (3.280) and (3.275), we have for $j = 0, \dots, N_1$,

$$d\big(T^{k_j}x, \bar{x}\big) - d\big(T^{k_j+1}x, \bar{x}\big) \ge d\big(T^{k_j}x, \bar{x}\big) - \phi\big(d\big(T^{k_j}x, \bar{x}\big)\big) \ge \delta. \tag{3.281}$$

Together with (3.277), this implies that

$$r \ge d\big(T^{k_0}x, \bar{x}\big) - d\big(T^{k_{N_1+1}}x, \bar{x}\big) \ge \delta(N_1 + 1),$$

which contradicts (3.276). The contradiction we have reached and the monotonicity of the sequence $\{d(T^{k_j}x, \bar{x})\}_{j=0}^{\infty}$ show that there is $p \in \{0, 1, \dots, N_1\}$ such that

$$d\big(T^{k_j}x, \bar{x}\big) \le \varepsilon_0 \quad \text{for all integers } j \ge p. \tag{3.282}$$

Assume that $i \ge N_0 + N_0 N_1$ is an integer. By (3.279), there is an integer $j \ge 0$ such that

$$k_j \le i < k_{j+1}. \tag{3.283}$$

By (3.279), (3.283) and the choice of p,

$$(j+1)N_0 > i,$$

$$j + 1 > i/N_0 \ge N_1 + 1,$$

and

$$j > N_1 \ge p. \tag{3.284}$$

By (3.284) and (3.282), $d(T^{k_j}x, \bar{x}) \le \varepsilon_0$. Together with (3.283), (3.279), (3.272) and (3.273), this inequality implies that

$$d\big(\bar{x}, T^i x\big) \le \varepsilon,$$

as claimed. Theorem 3.26 is proved. □

3.15 Two Results on Jachymski-Schröder-Stein Contractions

Suppose that (X, d) is a complete metric space, N_0 is a natural number, and $\phi : [0, \infty) \to [0, \infty)$ is a function. In this section we continue to study Jachymski-Schröder-Stein contractions (with respect to ϕ) $T : X \to X$ for which

$$\min\big\{d\big(T^i x, T^i y\big) : i \in \{1, \dots, N_0\}\big\} \le \phi\big(d(x, y)\big) \quad \text{for all } x, y \in X. \tag{3.285}$$

In the previous section we studied general Jachymski-Schröder-Stein contractions, where ϕ is upper semicontinuous from the right and satisfies $\phi(t) < 1$ for all positive t. In this section, which is based on [162], we study the case where ϕ is increasing and satisfies

$$\lim_{n\to\infty} \phi(t)^n = 0 \tag{3.286}$$

for all $t > 0$. Here $\phi^n = \phi^{n-1} \circ \phi$ for all integers $n \ge 1$. This condition on ϕ originates in Matkowski's fixed point theorem [99].

More precisely, we establish two fixed point theorems (Theorems 3.27 and 3.28 below). In our first result we prove convergence of iterates to a fixed point, and in the second this conclusion is strengthened to obtain uniform convergence on bounded subsets of X.

Theorem 3.27 *Let (X, d) be a complete metric space and $T : X \to X$ be a Jachymski-Schröder-Stein contraction such that ϕ is increasing and satisfies (3.286). Let $x_0 \in X$. Assume there is $x_0 \in X$ such that T is uniformly continuous on the orbit $\{T^i x_0 : i = 1, 2, \dots\}$. Then there exists $\bar{x} = \lim_{i\to\infty} T^i x_0$. Moreover, if T is continuous at $\bar{x}$, then $\bar{x}$ is the unique fixed point of T.*

Proof Since $\phi^n(t) \to 0$ s $n \to \infty$ for $t > 0$,

$$\phi(\varepsilon) < \varepsilon \quad \text{for any } \varepsilon > 0. \tag{3.287}$$

Set $T^0 x = x$, $x \in X$. Using induction, we now define a sequence of nonnegative integers $\{k_i\}_{i=0}^{\infty}$. Set $k_0 = 0$. Assume that $i \ge 0$ is an integer and that the integer $k_i \ge 0$ has already been defined. Clearly, by (3.286) there exists an integer k_{i+1} such that

$$1 \le k_{i+1} - k_i \le N_0 \tag{3.288}$$

and

$$d\big(T^{k_{i+1}}x_0, T^{k_{i+1}+1}x_0\big) = \min\big\{d\big(T^{j+k_i}x_0, T^{j+k_i+1}x_0\big) : i = 1, \dots, N_0\big\}. \quad (3.289)$$

By (3.285), (3.287), (3.288) and (3.289), the sequence $\{d(T^{k_j}x_0, T^{k_j+1}x_0)\}_{j=0}^{\infty}$ is decreasing and for any integer $i \geq 0$,

$$d\big(T^{k_{i+1}}x_0, T^{k_{i+1}+1}x_0\big) \leq \phi\big(d\big(T^{k_i}x_0, T^{k_i+1}x_0\big)\big). \quad (3.290)$$

Since ϕ is indecreasing, it follows from (3.290) and (3.285) that for any integer $j \geq 1$,

$$d\big(T^{k_j}x_0, T^{k_j+1}x_0\big) \leq \phi^j\big(d(x_0, Tx_0)\big) \to 0 \quad \text{as } j \to \infty.$$

Thus

$$\lim_{j\to\infty} d\big(T^{k_j}x_0, T^{k_j+1}x_0\big) = 0. \quad (3.291)$$

We claim that

$$\lim_{i\to\infty} d\big(T^i x_0, T^{i+1}x_0\big) = 0.$$

Let $\varepsilon > 0$ be given. Since T is uniformly continuous on the set

$$\Omega := \big\{T^i x_0 : i = 1, 2, \dots\big\}, \quad (3.292)$$

there is

$$\varepsilon_0 \in (0, \varepsilon) \quad (3.293)$$

such that

$$\text{if } x, y \in \Omega, i \in \{1, \dots, N_0\}, d(x, y) \leq \varepsilon_0, \text{ then } d\big(T^i x, T^i y\big) \leq \varepsilon. \quad (3.294)$$

By (3.291), there is a natural number j_0 such that

$$d\big(T^{k_j}x_0, T^{k_j+1}x_0\big) \leq \varepsilon_0 \quad \text{for all integers } j \geq j_0. \quad (3.295)$$

Consider an integer

$$p \geq k_{j_0} + N_0. \quad (3.296)$$

Then by (3.288) and (3.296), there is an integer $j \geq j_0$ such that

$$k_j < p \leq k_j + N_0. \quad (3.297)$$

By (3.295) and the inequality $j \geq j_0$, we have

$$d\big(T^{k+j}x_0, T^{k_j+1}x_0\big) \leq \varepsilon_0.$$

Together with (3.294) and (3.297) this implies

$$d\big(T^p x_0, T^{p+1}x_0\big) \leq \varepsilon.$$

Since this inequality holds for any integer $p \geq k_{j_0} + N_0$, we conclude that

$$\lim_{p\to\infty} d\big(T^p x_0, T^{p+1} x_0\big) = 0, \tag{3.298}$$

as claimed.

Next we show that $\{T^i x_0\}_{i=1}^{\infty}$ is a Cauchy sequence. To this end, let $\varepsilon > 0$ be given. By (3.287),

$$\phi(\varepsilon) < \varepsilon. \tag{3.299}$$

By (3.299), there exists $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < \big(\varepsilon - \phi(\varepsilon)\big)4^{-1}. \tag{3.300}$$

By (3.298), there exists a natural number n_0 such that

$$\text{if the integers } i, j \geq n_0, |i - j| \leq 2N_0 + 2, \text{ then } d\big(T^i x_0, T^j x_0\big) \leq \varepsilon_0. \tag{3.301}$$

We show that for each pair of integers $i, j \geq n_0$,

$$d\big(T^i x_0, T^j x_0\big) \leq \varepsilon.$$

Assume the contrary. Then there exist integers $p, q \geq n_0$ such that

$$d\big(T^p x_0, T^q x_0\big) > \varepsilon. \tag{3.302}$$

We may assume without loss of generality that

$$p < q.$$

We also may assume without loss of generality that

$$\text{if an integer } i \text{ satisfies } p \leq i < q, \text{ then } d\big(T^i x_0, T^p x_0\big) \leq \varepsilon. \tag{3.303}$$

By (3.302), (3.301) and (3.300),

$$q - p > 2N_0 + 2$$

and

$$q - N_0 > p + N_0 + 2. \tag{3.304}$$

By (3.303) and (3.304),

$$d\big(T^{q-N_0} x_0, T^p x_0\big) \leq \varepsilon. \tag{3.305}$$

There is $s \in \{1, \ldots, N_0\}$ such that

$$d\big(T^{q-N_0+s} x_0, T^{p+s} x_0\big) = \min\big\{d\big(T^{q-N_0+j} x_0, T^{p+j} x_0\big) : j \in \{1, \ldots, N_0\}\big\}. \tag{3.306}$$

By (3.285), (3.305) and (3.306),

$$d\big(T^{q-N_0+s}x_0, T^{p+s}x_0\big) \le \phi\big(d\big(T^{q-N_0}x_0, T^p x_0\big)\big) \le \phi(\varepsilon). \tag{3.307}$$

Hence,

$$\begin{aligned} d\big(T^q x_0, T^p x_0\big) &\le d\big(T^p x_0, T^{p+s}x_0\big) \\ &\quad + d\big(T^{p+s}x_0, T^{q-N_0+s}x_0\big) + d\big(T^{q-N_0+s}x_0, T^q x_0\big) \\ &\le d\big(T^p x_0, T^{p+s}x_0\big) + \phi(\varepsilon) + d\big(T^{q-N_0+s}x_0, T^q x_0\big). \end{aligned} \tag{3.308}$$

By (3.301) and (3.304) and the choice of s,

$$d\big(T^p x_0, T^{p+s}x_0\big), d\big(T^{q-N_0+s}, T^q x_0\big) \le \varepsilon_0. \tag{3.309}$$

By (3.299), (3.300), (3.308) and (3.309),

$$d\big(T^q x_0, T^p x_0\big) \le 2\varepsilon_0 + \phi(\varepsilon) \le 2^{-1}\varepsilon + 2^{-1}\phi(\varepsilon) < \varepsilon.$$

However, the inequality above contradicts (3.302). The contradiction we have reached proves that

$$d\big(T^i x_0, T^j x_0\big) \le \varepsilon \quad \text{for all integers } i, j \ge n_0.$$

Since ε is an arbitrary positive number, we conclude that $\{T^i x_0\}_{i=1}^{\infty}$ is indeed a Cauchy sequence and there exists $\bar{x} = \lim_{i\to\infty} T^i x_0$.

Clearly, if T is continuous, then $\bar{x}$ is a fixed point of T and it is the unique fixed point of T.

This completes the proof of Theorem 3.27. □

Theorem 3.28 *Let (X, d) be a complete metric space and $T : X \to X$ be a Jachymski-Schröder-Stein contraction such that ϕ is increasing and satisfies* (3.286). *Assume that T is continuous on X and uniformly continuous on the orbit $\{T^i x : i = 1, 2, \dots\}$ for each $x \in X$. Then there exists a unique fixed point $\bar{x}$ of T and $T^n x \to \bar{x}$ as $n \to \infty$, uniformly on all bounded subsets of X.*

Proof By Theorem 3.27, there exists a unique fixed point of T. Let $r > 0$ be given. We claim that $T^n x \to \bar{x}$ as $n \to \infty$, uniformly on the ball $B(\bar{x}, r) = \{y \in X : \rho(\bar{x}, y) \le r\}$.

Indeed, let $\varepsilon \in (0, r)$. Clearly, there exists a number $\varepsilon_0 \in (0, \varepsilon)$ such that

$$\text{if } x \in X, d(x, \bar{x}) \le \varepsilon_0, i \in \{1, \dots, N_0\}, \text{ then } d\big(T^i x, \bar{x}\big) \le \varepsilon. \tag{3.310}$$

By (3.286), there is a natural number n_0 such that

$$\phi^{n_0}(r) < \varepsilon_0. \tag{3.311}$$

Let $x \in X$ satisfy $d(x, \bar{x}) \le r$. Set $k_0 = 0$. We now define by induction an increasing sequence of integers $\{k_i\}_{i=0}^{\infty}$ such that for all integers $i \ge 0$,

$$k_{i+1} - k_i \in [1, N_0],$$
$$d\bigl(T^{k_{i+1}} x, \bar{x}\bigr) = \min\bigl\{d\bigl(T^{k_i + j} x, \bar{x}\bigr) : j \in \{1, \dots, N_0\}\bigr\}. \tag{3.312}$$

By (3.312), (3.285) and (3.287), the sequence $\{d(T^{k_i} x, \bar{x})\}_{i=1}^{\infty}$ is decreasing.

For each integer $i \ge 0$,

$$d\bigl(T^{k_{i+1}} x, \bar{x}\bigr) \le \phi\bigl(d\bigl(T^{k_i} x, \bar{x}\bigr)\bigr). \tag{3.313}$$

By (3.313) and the choice of x, for each integer $m \ge 1$,

$$d\bigl(T^{k_m} x, \bar{x}\bigr) \le \phi^m\bigl(d(x, \bar{x})\bigr) \le \phi^m(r).$$

By (3.287) and (3.311), for each integer $m \ge n_0$,

$$d\bigl(T^{k_m} x, \bar{x}\bigr) \le \phi^m(r) \le \phi^{n_0}(r) < \varepsilon_0. \tag{3.314}$$

Assume now that $i \ge N_0(n_0 + 2)$ is an integer. By (3.312), there is an integer $j \ge 0$ such that

$$k_j \le i < k_{j+1}. \tag{3.315}$$

By (3.312) and (3.315),

$$(j+1)N_0 > i, \qquad j + 1 > i N_0^{-1} \ge n_0 + 2, \qquad j > n_0.$$

Together with (3.314) this implies that

$$d\bigl(T^{k_j} x, \bar{x}\bigr) < \varepsilon_0.$$

When combined with (3.315), (3.312) and (3.310), this implies that

$$d\bigl(T^i x, \bar{x}\bigr) > \varepsilon.$$

Theorem 3.28 is proved. □