Chapter 3
Mathematics Education in Antiquity

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1 General Presentation

A wide variety of documents relating to mathematics education in the Near East and the Mediterranean basin have survived to the present day. The oldest of these sources are the southern Mesopotamian clay tablets produced in the third millennium before the Common Era. More recent sources were copied in the Byzantine Middle Ages from a long chain of texts which stem back to lost originals. Nonetheless, these late copies provide some evidence of educational activity and pedagogical orientation. As may be seen in the case studies in this chapter, these sources represent a wide chronological distribution but of texts of diverse genres. Some texts, like the tablets made of nearly indestructible clay, survive in great numbers and enable a reconstruction of the mathematical instruction of ancient Mesopotamia. Excavations in Iraq, Iran, and Syria since the late nineteenth century have produced a sufficient number of tablets to permit a detailed reconstruction of the basic mathematics curriculum in the scribal schools of the ancient Near East. Equivalent sources exist for the Greco-Roman world but in much lower numbers, and their state of preservation does not permit many conclusions. The Greco-Roman texts comprise small, disconnected fragments on papyrus, pottery, leather, and even wooden tablets covered with wax, all of which probably served in the teaching of mathematics. By contrast, relatively few texts of the copious Mesopotamian corpus report how the scribes in the ancient Near East conceived of their work, their knowledge, and its transmission, whereas the Greco-Roman texts written on parchment generally resulted from endeavors in copying or translation and only marginally constitute *direct* evidence of scholastic activity. Thus, these later sources shed limited light on the practicalities of transmitting mathematical knowledge from master to disciple in different contexts, and few of these texts detail elementary education. However, these sources do reveal the weighty didactic ideals of the Greco-Roman world which governed the prolific work in philosophy and rhetoric. In some cases, these ideals may be assumed to have been put into practice and corresponded to actual curricula, but this inference is speculative and probably useless. The evidence from Pharaonic

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Egypt represents a nadir of textual preservation and cultural reconstruction: although the series of problems found on ancient papyri probably served pedagogical purposes, the manner of instruction and the institutional setting in which these texts were used remain largely unknown.

The following synthesis is therefore based on sources that are characterized by extreme heterogeneity in their nature as well as in their geographical and chronological distribution. This fundamental fact should always be kept in mind to avoid anachronistic claims. The disparity of sources demands consideration of the varied and unevenly documented diversity of the educational settings and institutions of antiquity. In other words, the ancient sources neither relate to the same environments nor do they refer to the same cultural and institutional codes.[1] The available sources do *not* describe a complete or consistent picture of teaching mathematics in antiquity, but spotlights may be focused on the better documented teaching contexts of Mesopotamia, Egypt, and Greco-Roman world. Even if this disparity limits our actual knowledge of ancient mathematical teaching, these difficulties highlight the fact that both the "positive information" which we can claim to know and *the kind of questions* asked about ancient mathematical education depend strictly on the nature of the surviving sources.

2 Mesopotamia

During excavations conducted in Iraq, Iran, and Syria since the late nineteenth century, archaeologists and illegal diggers have unearthed hundreds of thousands of clay tablets containing texts of all kinds (including administrative records, contracts, letters, literary compositions, medical treatises, astronomical calculations, and mathematical writings). These documents provide evidence about the history of the ancient Near East over a very long period – more than 3,000 years since the beginnings of writing (c. 3300 BCE) until the abandonment of clay for writing at the beginning of the Common Era.

Numerous languages were transcribed in cuneiform writing on clay tablets (Fig. 3.1). Among mathematical texts, Sumerian and Akkadian are used. Sumerian, which was the language of the people of southern Mesopotamia during the third millennium, probably disappeared as a living language before the second millennium but remained the language of scholarship until the end of cuneiform writing. Akkadian is the Semitic language which gradually supplanted Sumerian and had long been the diplomatic language of the ancient Near East and Eastern Mediterranean.

Approximately 2,000 tablets containing mathematical texts are presently known. Most of the mathematical cuneiform texts published during the early twentieth century by Neugebauer, Sachs,

Fig. 3.1 Cuneiform writing (School tablet from Nippur, about 1800 BCE)

[1] This methodological approach is developed in Bernard and Proust (2014); see in particular the introduction.

and Thureau-Dangin were bought from dealers by European and American museums or by private collectors, and their provenances are unknown. However, after the Second World War, archaeologists unearthed new collections of mathematical tablets with clear contexts, notably in the Diyala Valley (northern Mesopotamia) and in Susa (western Iran).

In cases where the provenances of the tablets are well documented, archaeological reports show that the tablets containing high-level mathematics shared the same findspots with elementary school tablets. Thus, the education of young scribes and activities of erudite scholars occurred in the same place, and possibly, the authors of these high-level mathematical texts were involved in teaching. But do these archaeological details indicate that all of the mathematical cuneiform texts were produced for educational purposes? A positive answer is often assumed, more or less tacitly, in recent studies of mathematics in Mesopotamia. However, the situation is probably more complex. Indeed, some of the cuneiform mathematical texts are clearly school exercises. (Some examples are examined below.) Others texts, which contain lists of solved problems, are probably (but not certainly) documents composed and used for advanced mathematical education. However, most of the higher-level mathematical texts do not clearly reveal the exact context in which they were composed or used. It is not always easy to identify the audience of such texts. As far as the cuneiform sources are concerned, more details are clearer here than in the case of Egyptian papyri. The strongest evidence derives from the physical details of the tablets themselves. The very shape, size, and layout of the tablets often reveal the nature of the context in which they were produced.

2.1 The Scribal Schools

Modern historians refer to the places where scribes were educated as "scribal schools." Sometimes, the physical place of the school is well identified. In Nippur, Ur, Mari, and Sippar, for example, traces of teaching activities such as important collections of school exercises or bins used for recycling tablets were found in houses tentatively identified as scribal schools.[2] A particular Sumerian word designates such places as *edubba*, which literally means "house of tablets." Sumerian literature portrays a highly idealized picture of the *edubba*, which appears as prestigious institutions for educating the social elite (Michalowski 1987, p. 63). This image may reflect reality at Nippur, the political and cultural capital of Old Babylonian Mesopotamia, whose schools merited high esteem throughout the ancient Near East. However, the organization of education appears to have varied considerably from one city to another. In some cities, the teaching activities seem to have been limited to the domestic sphere, as shown, for example, in Sippar by Tanret (2002, pp. 153–156). In other cities, priests may have participated in education, as shown in the cases of Ur (Charpin 1986, pp. 420–486) or Tell Haddad, a city of the Diyala Valley (Cavigneaux 1999, p. 257).

Most surviving Old Babylonian mathematical tablets are school tablets. They span a large geographical area (see map Fig. 3.2), but the bulk comes from Nippur. The careful analysis of thousands of tablets of Nippur has allowed historians to reconstruct in great detail the curriculum of mathematical education which took place in the schools of this city and perhaps in other *edubba*.[3]

[2] The presence of school tablets in a house is not always a proof that this house served as a school; in particular, school tablets may have been brought from other places to be reused as construction material. Thus, the archaeological context must be analyzed carefully for each context. See, for example, the case of the "schools" in Ur analyzed by Charpin (1986, pp. 432–434) and Friberg (2000), the case of the houses of "Aire II" of Tell Haddad analyzed by Cavigneaux (1999, pp. 251–252), the case of "House F" in Nippur analyzed by Robson (2001, pp. 39–40), and the case of the house of the "gala-mah" in Sippar-*Amnânum* analyzed by Tanret (2002, p. 5). About bins for recycling tablets, see Tanret (2002, pp. 145–153).

[3] The studies of the curriculum in the *edubba* are mainly based on Nippur sources; from the abundant literature on the subject, see Cavigneaux (1983), Civil (1985), Tinney (1999), Vanstiphout (1996), Veldhuis (1997), Robson (2001), George (2005), Proust (2007), and Delnero (2010).

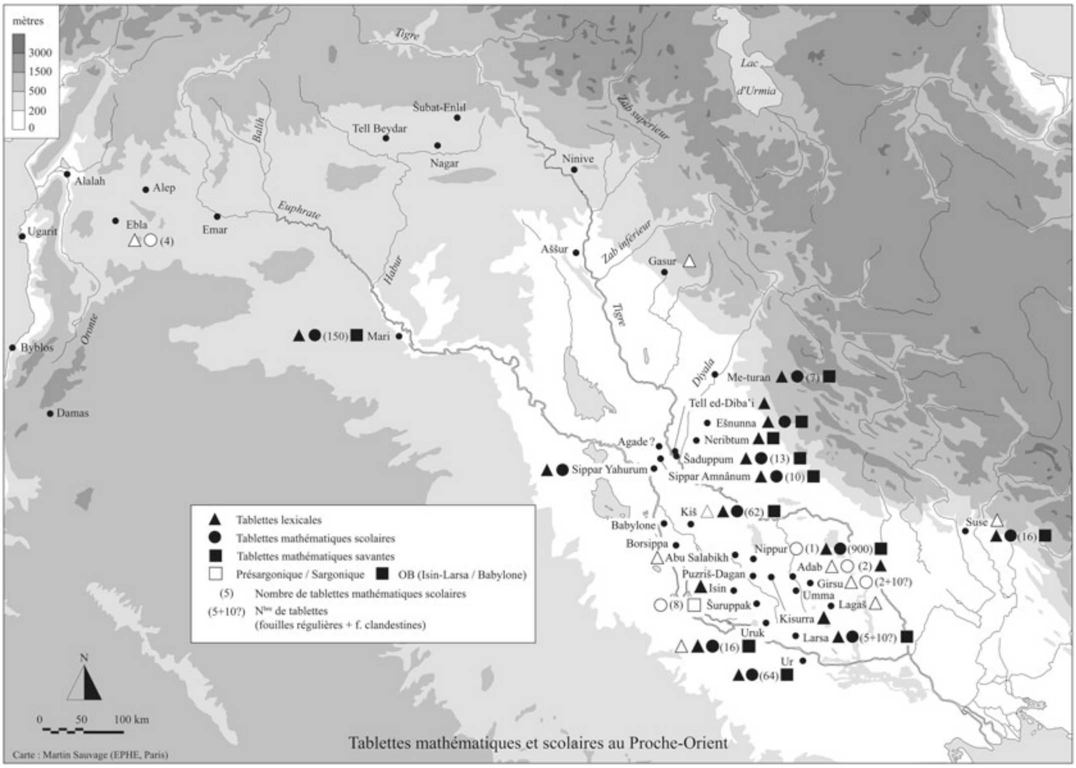


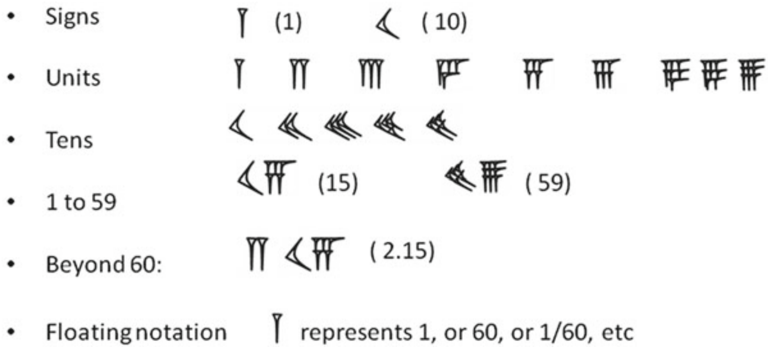
Fig. 3.2 Places where mathematical tablets were found (● = OB elementary mathematical school texts; ∎ = OB advanced mathematical texts) (Map by Martin Sauvage, published in Proust 2007, p. 281)

2.2 *The Elementary Level of Mathematical Education*

The shapes of the tablets provide valuable evidence for the reconstruction of the curriculum. The tablets used most often at Nippur (type II in the typology of Assyriologists) are large rectangular tablets (about 10 × 15 cm), which the young apprentices used in their training to memorize and write a set of standardized texts. These texts included lists of cuneiform signs, Sumerian vocabulary, systems of measurement, and elementary numerical tables. When a long series of lexical lists or mathematical tables had been completely memorized, it was written on large multicolumn tablets known as "type I" or, sometimes, on prisms. These great compositions on prisms may be interpreted as a kind of examination (Veldhuis 1997, p. 31). In addition to these exercises, scribes would sometimes note short excerpts on small single-column rectangular tablets (type III – see Fig. 3.4). The Sumerian name of this type of tablet sometimes appears at the end of the composition, as well as in some literary texts: *imgidda* or "elongated tablets." *Imgidda* tablets were often used to learn multiplication tables (as shown in Fig. 3.4).

Type II tablets (see Fig. 3.3) provided key evidence which allowed historians to identify the exact content of the texts studied by young scribes in the early stage of education and to reconstruct the order in which these texts were learned. Indeed, Veldhuis (1997, pp. 34–36) has shown that the reverse of type II tablets was "used as a repetition of a school text studied at a point earlier in the curriculum" (p. 36). Thus, by comparing the texts written in obverses and reverses of type II tablets, he reconstructed the elementary curriculum (pp. 41–67). Veldhuis focused on lexical texts, but the same

Table 3.1 Sexagesimal place value notation



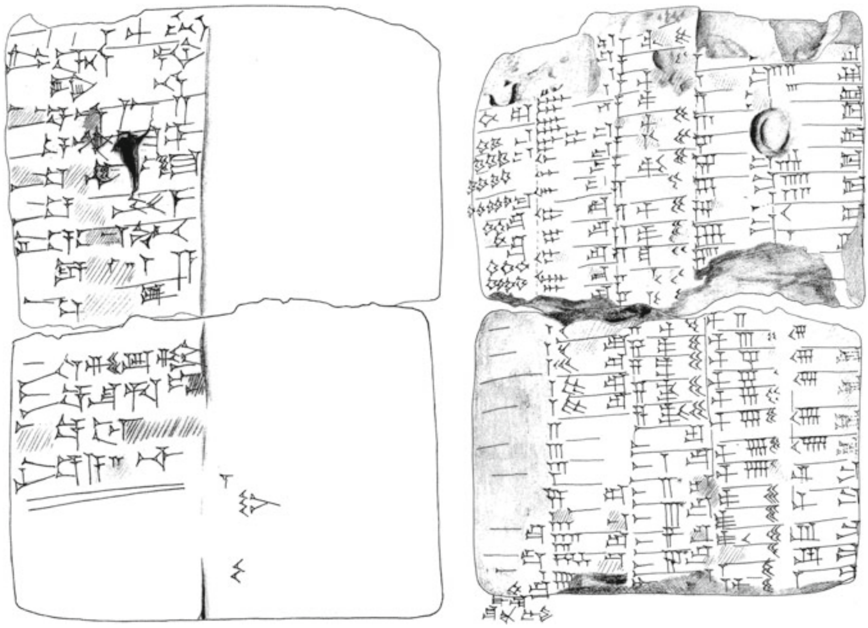


Fig. 3.3 Ni 4840+ +UM 29-13-711, type II tablet. Obverse, lexical list; reverse, measures of capacities (Proust 2007, p. 26)

method can be applied to mathematical texts (Robson 2001, 2002; Proust 2007). A detailed picture of elementary education in Nippur has emerged from these studies. The first level of the mathematical curriculum was devoted to learning the following lists and tables, more or less in the following order: lists enumerating measurements of capacity, weight, surface, and length; tables providing correspondence between the various measures and numbers written in sexagesimal place value notation (see Table 3.1); and numerical tables (tables of reciprocals, multiplication, squares, square roots, and cube roots). All of these elementary lists were probably learned by rote.[4]

Outside of Nippur, the mathematical curriculum cannot be reconstituted in such detail, partly because the number of available tablets is too small for any meaningful statistical consideration. The typology of tablets varies considerably. For example, type II tablets were rarely found outside of Nippur. In the schools of Mari and Ur, mainly small round tablets were used.

[4] About the role of memorization in learning process and transmission, see Veldhuis (1997, pp. 131–132, 148–149) and Delnero (2012).

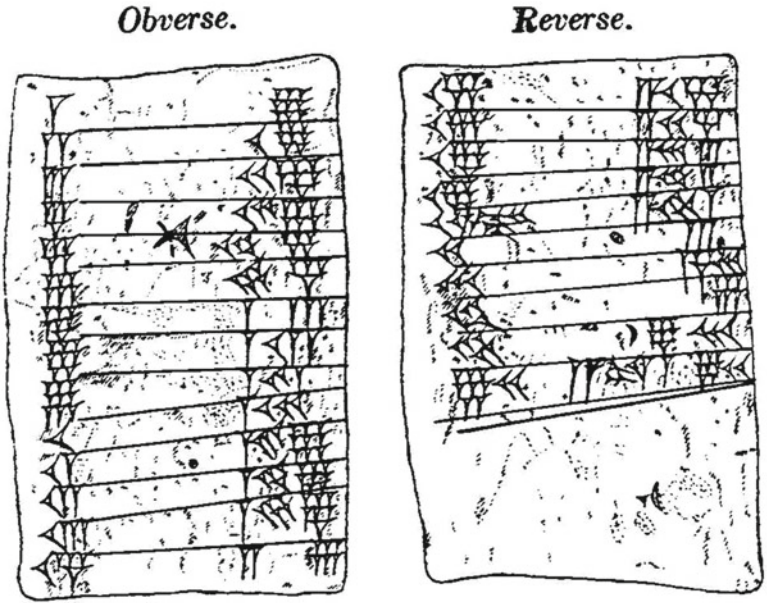


Fig. 3.4 HS 217, type III tablet. Multiplication table (Hilprecht 1906, p. 7)

2.3 The Intermediate Level

After learning the metrological and numerical systems as well as a set of elementary arithmetical results (tables of reciprocals and multiplication tables), the scribes began an intermediate level of education. At this less formalized level, scribes learned the basics of sexagesimal calculation, namely, multiplication and calculation of the reciprocal of large numbers.

This knowledge was then applied to finding areas of squares and other figures. At schools in Nippur, this level of education is documented mainly through exercises noted on square-shaped tablets (see Fig. 3.5).

Table 3.2 summarizes the various aspects of the mathematical curriculum as it could have existed at Nippur and perhaps in other schools.

2.4 The Advanced Level

If the elementary and intermediate levels of mathematical education are well known, at least at Nippur, the context in which advanced mathematical texts from the Old Babylonian period were produced or used is more difficult to reconstruct. Mathematical texts have been previously interpreted as textbooks or as databases compiled for teaching. However, a pragmatic analysis of the texts suggests that the authors had at least some purposes other than teaching.[5] How is it possible to distinguish the tablets used for advanced teaching (written by students or teachers) from those that reflect investigations of pure scholarship? The first type of evidence could be the complexity of the mathematical procedures, but such a criterion can be misleading because what is complex for a modern reader may not have been complex

[5] One example of text not clearly linked with teaching is the famous tablet Plimpton 322 (see Britton et al. 2011); other examples are found among the so-called series texts, which are lists of problem statements written on numbered suites of tablets (see Proust 2012).

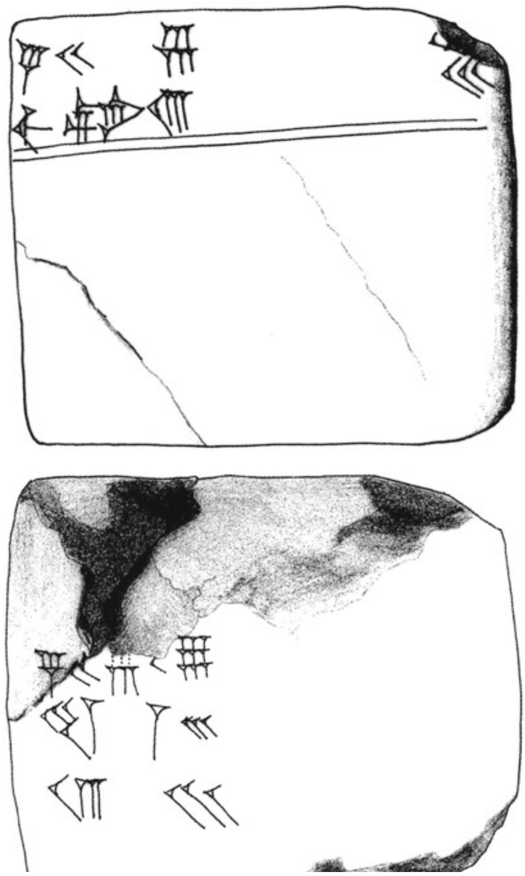


Fig. 3.5 Ni 10241, reciprocal calculation (Proust 2007)

Table 3.2 Mathematical curriculum in OB Nippur

Level	Content	Typology	Examples
Elementary	Metrological lists (capacities, weights, surfaces, lengths)	Types I, II, and III	See Figs. 3.3 and 3.4
	Metrological tables		
	Numerical tables (reciprocals, multiplication, squares)		
	Square and cube roots		
Intermediate	Exercises: multiplications and reciprocals	Square-shaped tablets	See Fig. 3.5
	Surface calculations		

for an ancient scribe and vice versa. Therefore, caution is advised for arguments based on the supposed "level" of a mathematical content. The second type of evidence is linked to material aspects. Very roughly, one can classify the tablets into two types: single-column tablets (type S) and multicolumn tablets (type M).[6] However, the shape of the tablets often conforms to local habits. Since most of the mathematical tablets are from unknown provenance, a general typology cannot be clearly connected with specific pedagogical practices. Thus, only case-by-case examinations are relevant to answer the question of how to distinguish teaching from scholarship. A mathematical text that seems to have been used at the beginning of the advanced level of mathematics presents a useful example.

An example of such a tablet is conserved at Yale University under the inventory number YBC 4663 (see Fig. 3.6). This tablet has an elongated shape and is written in a single column (type S). The tablet

[6] This typology comes from the classification of tablets used in OB Nippur for learning Sumerian literary (Tinney 1999, p. 160).

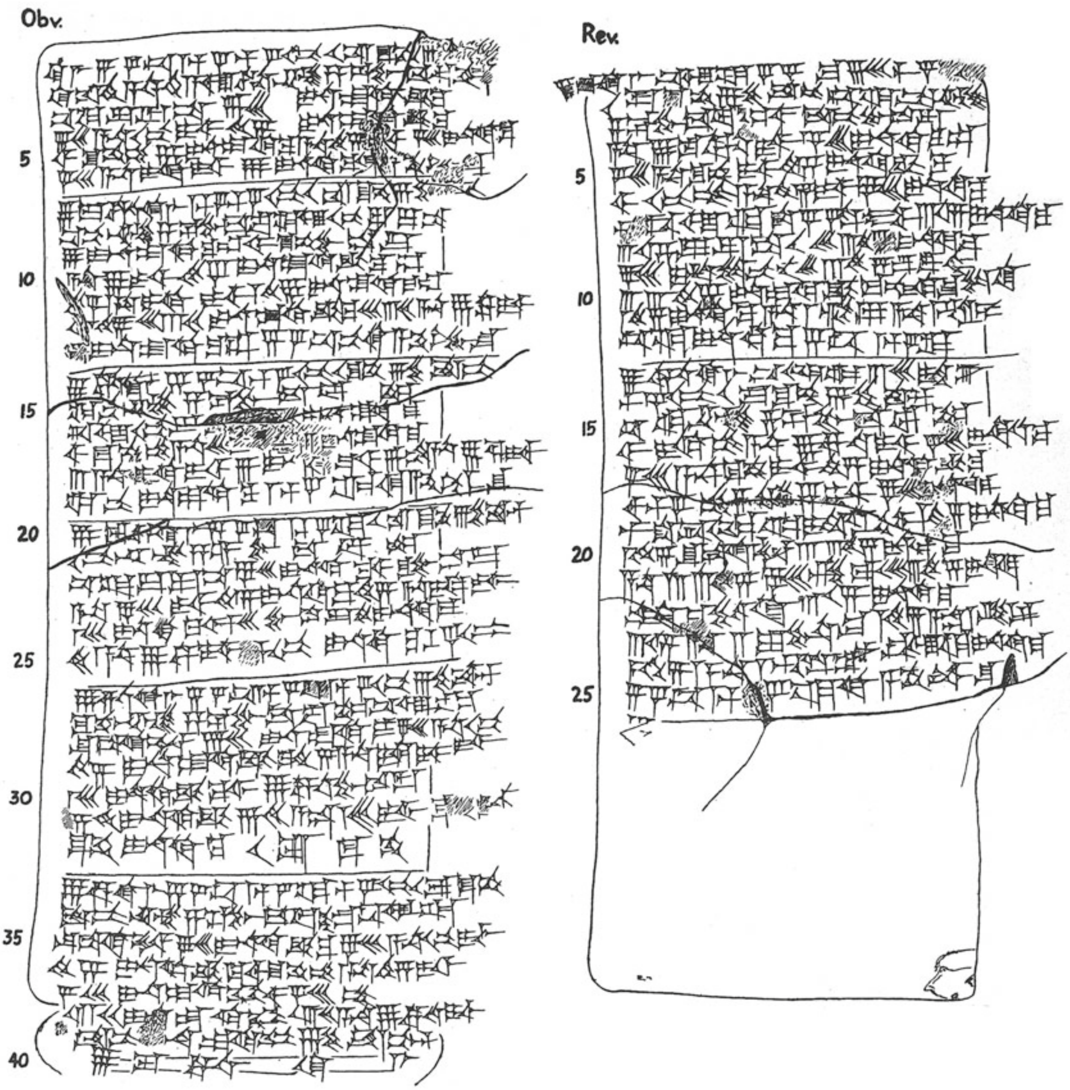


Fig. 3.6 Type S tablet – YBC 4663, Yale University (Courtesy of Benjamin Foster***)

is of unknown origin but probably comes from a city in southern Mesopotamia. It contains a sequence of eight solved problems dealing with digging trenches. The parameters of the problem (data and unknowns) are the dimensions of the trench (length, width, depth), its base, the volume of extracted earth, the number of workers needed for digging, the daily labor assigned to the workers (namely, the volume of earth to be extracted each day by each worker), their daily wage, and the total wages (expressed as a weight of silver). All these parameters are linked by a simple relation that we can represent in modern fashion as follows:

$$\text{Total wage} = \text{daily wage} \times \left(\frac{\text{length} \times \text{width} \times \text{depth}}{\text{daily assigned task}} \right)$$

Examination of the text shows how the procedures implement the computational methods taught at the elementary level of mathematical curriculum. For example, consider the first problem, translated as follows:

Translation of YBC 4663 #1[7]

1. A trench. 5 ninda is the length, 1½ ninda (the width), ½ ninda its depth, 10 (gin) the volume of assignment (for each worker), 6 še (silver) [the wages of a hired man].
2. The area, the volume, the number of workers, and the (total expenses in) silver what? You, in your procedure,
3. the length and the width multiply each other. *This will give you* 7.30.
4. 7.30 *to its depth raise. This will give you* 45.
5. The reciprocal of the assignment detach. *This will give you* 6. *To* 45 *raise*. *This will give you* 4.30.
6. 4.30 *to the wages raise*. *This will give you* 9. *Such is the procedure.*

Note that in the statement of the problem (lines 1–2), the data are expressed in concrete numbers with units of measure, in the same way as in the metrological lists, but in the procedure (lines 3–6), the data appear only as abstract numbers expressed in sexagesimal place value notation (SPVN). Metrological tables had been used to transform measures into abstract numbers for performing calculations. This process is confirmed by the fact that the correspondences between the measures given in the statement and the abstract numbers used in the procedure fit with the correspondences provided by the metrological tables. This observation suggests that the authors of the text used the basic skills taught in the scribal schools. The sequence of problems listed on this tablet provides an opportunity to use all the metrological tables one after another, as well as multiplication tables and calculation techniques taught in the intermediate level (multiplication, inversions, calculating areas and volumes). All knowledge acquired in the early levels of mathematical education is systematically employed. In this way, we can suppose that the text was composed specifically for teaching mathematics. It could have been written by a master or an advanced student. An examination of other tablets similar to YBC 4663, which seem to come from the same city, may show that, in this city, type S tablets were used at the beginning of advanced education.[8]

3 Hellenistic Period

A gap in the preservation of mathematical texts appears after the end of the Old Babylonian period. For subsequent periods, only sporadic examples of metrological and numerical tables are known. Only by the end of the first millennium BCE do coherent sets of mathematical sources reappear. Two small corpuses of mathematics texts dating from the Hellenistic period (c. 300 BC) have been discovered at Uruk and Babylon. It is difficult to know whether these corpuses from late periods reflect a kind of Renaissance after a long eclipse, or if the written mathematical tradition continued through the centuries. The transmission of elements of mathematical tradition over this long period tips the balance toward the second hypothesis. Advanced mathematical texts could have been noted on perishable materials such as leather or papyrus, which would not have resisted time.

The context of the Hellenistic period differs radically from the Old Babylonian world. The mathematical practices were developed by lineages of astrologers and astronomers who were linked to the great temples of Babylon and Uruk. In Hellenistic Mesopotamia, mathematical erudition was closely

[7] Literal translation based on Neugebauer and Sachs (1945, p. 70). The passages written in Sumerian in the cuneiform text are represented by plain font and passages written in Akkadian by italic font. The measurement units used are 1 ninda ≈ 6 m, 1 gin ≈ 1.7 dm^3, and 1 še ≈ 0.04 g.

[8] These tablets are six catalogue texts conserved at Yale University and two related procedure texts (including YBC 4663). See Proust (2012).

associated with the astral sciences.[9] Cuneiform mathematics was no longer taught to children or adolescents acquiring literacy and numeracy, as was the case in the Old Babylonian period, but to young scholars who were probably already literate in Aramaic and perhaps in Greek.

4 Ancient Egypt

Until the establishment of Greek as the solitary administrative language near the end of the second century of the Common Era, the situation regarding Egyptian sources on mathematical instruction inspires less confidence than that of either the cuneiform or Greek sources. First of all, the entire corpus of hieroglyphic and hieratic papyri counts roughly as many texts as the corpus of cuneiform mathematical texts. Among these papyri, only a handful of explicitly mathematical sources have survived.[10] No mathematical texts survive from the earliest periods of Egyptian history, and hieratic texts of Middle Egyptian comprise only three relatively intact papyri. One hieratic mathematical text has also survived on leather. If a wider view of mathematical texts is taken and papyri with calculations are counted, other early mathematical texts include sections of the Reisner Papyrus and a collection of hieratic fragments from Kahun. To these should be added two wooden tablets from Akhmim. After a lapse of more than a millennium, Demotic texts add one complete papyrus and six fragments and three Roman ostraca. In none of these cases have archaeologists established that the papyri survived in a pedagogical setting. The larger, more complete papyri contain collections of solved problems indicative of pedagogical use and fractional tables useful for calculation. Individual fractional tables are preserved among the smaller fragmentary papyri and ostraca, but most contain independent calculations (Table 3.3).

Not only do few mathematical sources survive, but the process of instruction in ancient Egypt is not well known. Because the authors of moral "instructional literature," such as the *Instructions of Ptahhotep* (c. -2880), addressed their readers by familial terms, the earliest mode of Egyptian

Table 3.3 Chronological range of Egyptian mathematical papyri

Name of Egyptian mathematical text	Approximate date
Reisner Papyrus	-1970–-1925
Cairo Cat. 25367/8	-1970–-1925
Kahun Papyri	-1880–-1770
Berlin 6619	-1880–-1700
Moscow E4674	-1770–-1650; original -1990–-1770
BM 10057/10058	-1600; original -1860–-1815
Mathematical Leather Roll	-1650
Cairo JE 89127–30, 89137–43	-300–-200
BM 10794	-331–350
BM 10399	-331–-30
Heidelberg 663	-200–0
Griffith I E.7	-100–100
BM 10520	100–200
Carlsberg 30	100–200
Ostraca Medinet Madi 251	0–200
Ostraca Medinet Madi 720+912	0–200
Theban Ostracon D12	0–200

[9] Rochberg (2004, Chap. 6), Robson (2008, Chap. 8), Clancier (2009, pp. 81–103, 205–211), Steele (2011), Ossendrijver (2012, Chap. 1), and Beaulieu (2006).

[10] For a reliable guide to the bibliography and contents of the specific texts, see Clagett (1999).

instruction has been imagined as a father instructing his children. However, this assumption ignores the possibility that the mode of address merely employs a rhetorical conceit. The first reference to "house of instruction" (ʿ.*t n sbꜣ*) appeared in a Tenth Dynasty (c. -2160–2025) tomb. A composition titled *The Satire of the Trades* (c. -2025–1700) describes a royal school, but because the treatise seeks to esteem learning, the presence of an actual school cannot be assumed. Speculation about Egyptian pedagogy has focused on an element of the temple complexes titled "The House of Life," but whether the curriculum of this place of instruction had a wider applicability outside the temple remains unknown.[11] The *Onomasticon of Amenope* (c. -1187–716) records a list of terms as important to the "scribes of The House of Life" that is similar to the lexicographical lists of Mesopotamia, but Egypt and Mesopotamia seem to have employed different organizational strategies. After he conquered Egypt, Darius endeavored to restore "The House of Life," which may have served as a type of hospital. Again, little information about the Egyptian methods of instruction survives, and mathematics may not have even formed the curriculum of all scribes. Unlike Babylon, no large collections of school texts have survived, although some school texts (from mere onomastica and word lists to literary compositions used as models such as *The Tale of Sinuhe*) have been found. The larger mathematical papyri have been interpreted as pedagogical texts because they presented collections of similar exercises, but whether these writings formed the syllabus of specialists or generalists remains unstated. As shown by ostraca which repeat various phrases for different numbers and genders and others which elucidate the reading of certain hieroglyphs, education in grammar and writing seems to have formed some portion of the activity of the temple complex in the town of Medinet Madi, but these documents were mixed with administrative documents and texts useful for the composition of planetary positions. Unfortunately, some of these ostraca were reused as building materials, and the particular archaeological status of any given text may not be stated with certainty.

The date of composition for the largest of the papyri, the Rhind Mathematical Papyrus, coincides roughly with the first reference to a "place of instruction." Whether or not it was used in such a school, the Rhind Mathematical Papyrus boasts that it contains "the model for enquiring into affairs and for knowing all that exists" but says nothing about the prerequisite knowledge, the intended audience, or the qualifications of a scribe who had mastered the material. Other than the techniques demonstrated, only the basic literacy necessary to read the papyrus may be presumed.

This translation of the title and introduction to the Rhind Mathematical Papyrus (Table 3.4) follows Couchoud's (1993) French translation. Despite the grandiloquent promises of resolving all that is unclear and penetrating every mystery, the 87 surviving applications of Egyptian mathematics concern the doubling of fractions; the division of fractions by 10; the solution of linear polynomials; the unequal distribution of goods; the approximations of area in circles; the geometrical problems with rectangles, triangles, and pyramids; and the problems of exchange and geometrical progressions. The use of masculine, singular pronouns in the second and third person singular conforms to the impression that only males were educated in mathematics, perhaps working singly with the instructor or some other examiner.

The introduction to the most complete Demotic mathematical papyrus no longer survives, but some basic estimation of its pedagogical position may be derived from the fact that it shares a papyrus with a manual of legal formulae. Whether these compositions were textbooks or references remains an open question. Moreover, the juxtaposition of these two texts could be either the accidental result of reuse by a scribe whose training spanned both areas or a deliberate link forged by an instructor who connected land contracts with geometry.

In counterpoint to the pedagogical papyri, the fragmentary papyri may occasionally represent more than mere working notes for the resolution of a commonplace problem. A particular fragment of the Kahun Papyri (Kahun IV.3) could be dismissed as the pedestrian division of commodities, except for

[11] For an accessible discussion of "The House of Life," see Strouhal (1992, pp. 235–243).

Table 3.4 Title and introduction to the Rhind Mathematical Papyrus

1. *Tp ḥsb n ḫȝt m ḫt rḫ ntt nbt snk <.t> … štȝt nbt. ’Iw ist grt*
   1: The model for enquiring into affairs, for knowing all that which is unclear, <and deciphering > every mystery. So, now
2. *sph̲r.n.tw šfdw pn m rnpt 33 íbd 4 ȝḫt <sw ?? nsw> bity ꜥꜣA-wsr-Rʿ di ꜥnḫ m snt r sšw*
   2: this papyrus-roll was copied in Regnal Year 33, month 4 of Flood Season, [day ??] of the King of Upper and Lower Egypt “The Power of Re is Great” (Apophis), may he be given life, in conformity to the writings
3. *n iswt iry m h3wt < n nsw bity Ny-Mꜣ >ꜥt-Rꜥ. In sš ’Iʿh-msw sph̲r snn pn*
   3: of old, made in the time of the King of Upper and Lower Egypt “Belonging to the Justice of Re” (Amenemhat III). It was by the scribe Ahmose that his copy was transcribed

the mathematically playful requirement that the shares of the commodity increase in an arithmetic progression. Although the text may be salvaged from the corpus of “documentary evidence” by this detail, the pedagogical position of the fragment remains unclear. Taken collectively, these texts permit a rough estimation of the range of Egyptian mathematical knowledge and techniques of calculation, but specific details about what constituted basic knowledge, what represented advanced knowledge, how these topics were communicated, and how competency was assessed remain speculative.

5 Greco-Roman World

5.1 The Nature of Greco-Roman Sources on Teaching Mathematics

As evidence from Mesopotamia and Egypt shows, modern understanding of mathematical teaching and learning in antiquity depends on the nature of available sources. The Greco-Roman sources differ significantly from the Mesopotamian and Egyptian sources. For the sake of clarity, the following simplified classification may be proposed. First, texts with highly internally coherent mathematical contents sometimes (but not always) begin with prefaces which announce a pedagogical purpose, eventually oriented by a philosophical position. These texts were transmitted through a long chain of intermediaries and thus became *classical* texts in the long course of Greco-Roman history. This lengthy process incorporates “accompanying texts,” like marginalia or independent commentaries, which can be difficult to distinguish from the “original text.”[12] The writings of Euclid, Archimedes, and Ptolemy exhibit this process of incorporation.

[12] For ancient scholarship and the history of texts and their transmission, see Reynolds, Leighton and Wilson (1968).

Other kinds of texts also relate to mathematics and mathematical education but only in the sense that their content and nature are basically metamathematical: these texts describe philosophical or cultural projects in which mathematics plays an important role. Plato's *Republic*, Vitruvius's *De Architectura*, or Quintilianus's *Institutio Oratoria* shares an esteem of mathematics while not being truly mathematical texts.

Somewhere between these two categories are the literary or philosophical sources which sometimes preserve excerpts of lost mathematical works by inclusion. In this case, the larger, encyclopedic project guides the choice of excerpts. For example, Simplicius preserves lost mathematical texts in his sixth-century commentaries on Aristotle. To this "intermediary" genre also belong texts like the second-century *Introduction to Arithmetic* by Nicomachus or the fifth-century commentary on the first book of Euclid's *Elements* by Proclus, which basically attempt an explicit, philosophical project but introduce mathematical contents to achieve this purpose.

Fourth, direct archaeological evidence of mathematical teaching and learning survives in the form of fragments of papyri, ostraca, and wooden tablets, some of which can be related to a teaching context (although this attribution is often problematic). In the case of ancient mathematical knowledge, such evidence is scanty and consists of disconnected fragments with no clear context. Such sources approach the kind of material that came from Old Babylonian contexts, the richness of which enables a fairly satisfying reconstitution of mathematical curricula.

A fifth, "ghost" category nearly resembles the Egyptian material. This category contains isolated texts, like the problems of the Akhmim papyrus (Baillet 1892) or the problems in the "metrological corpus." This category extends to the sophisticated problems contained in Diophantus's *Arithmetica*, the structure of which strongly evokes pedagogical concerns, but the scholarly context of which remains unknown.

The key fact, then, about these sources is that the three first kinds of sources in the above classification have undergone a long-term process of "classicization" and are by far the best represented, while the other two categories are poorly represented. This does not suggest that nothing is known about mathematical education in praxis, but more precisely that what is "positively" known is necessarily of a different nature than what has been explained above about ancient Mesopotamia and Egypt: the divergence of approaches derives from the fact that the underlying sources differ in kind. This important acknowledgment implies a fundamental bias in our knowledge of mathematical education in this context. Only a specific part of the "ideal" picture may be revealed without access to a much richer range of sources.

For example, the technical texts in the first category, like the extant portions of Euclid, Apollonius, or Archimedes, have survived because they were transmitted, cultivated, commentated, and reused for contexts and purposes beyond the original aims of their Hellenistic authors.[13] Works like the compositions of the Pythagorean school or isolated figures like Hippocrates of Chios are known through the erudition of late scholars like Simplicius of Cilicia, whose work was in turn transmitted and copied in later times. Simplicius continued the tradition of ancient scholars like Vitruvius, Plutarch, Athenaeus, and perhaps even Euclid, who composed works that functioned as encyclopedias of ancient knowledge. This lengthy process of incorporation and citation stands as a general feature of ancient scholarship and extends much beyond the mathematical literature. This process guided the transmission and "classicization" of the ancient heritage of literature, philosophy, and more specialized issues like religious, medical, or mathematical texts. Note that this phenomenon is not proper to Greco-Roman Antiquity : already the Mesopotamian scribes had their own 'classics', as well as 'dead languages' that served as classical references. For classical culture, see Marrou (1965), Hadot (2005). For the Greco-roman heritage constituted as such for mathematics and in modern Europe, cf. Goldstein et al. (1996).

The majority of Greco-Roman sources, then, might be (rightfully) seen as "only" derived products, in contrast to the "direct" evidence of documents in the fourth category. The preserved manuscripts

[13] For the question of our sources of knowledge on ancient mathematics, see Fowler (1999, pp. 199–221), particularly the list in pp. 268–275.

date to the medieval period (eighth or ninth century) for Greek literature and Late Antiquity (fifth century) for Latin technical literature.[14] These manuscripts represent a long transmission of copies, sometimes including transliterations and changes of format (Fowler 1999, pp. 204–221; Chouquet and Favory 2001, Chap. 1). As a result, they contain layers of transformations and annotations made during their history of transmission, often with no solid means to discriminate among variants, distinguish textual traditions, or establish a date for the sources. Unfortunately, *for mathematical sources in particular*, nearly no complementary information survives from other, older sources like scholastic papyri or even detailed accounts about mathematical practice. A larger number of documents of the fourth category would be needed to clarify how students were trained *before* or even *during* their study of more elaborate and erudite works, like Euclid's *Elements*. Despite this lack, textual "transmission" should also be regarded as a *fact* of the utmost importance, for at least three reasons.

First, the process by which these sources were *made* classical[15] can hardly be dissociated from the activities of teaching and learning. This is not to say that any marginal annotation in an ancient manuscript or any commentary automatically relates to a teaching activity; but, in many cases, the activities of commentary and note-taking might have plausibly been related to a scholastic activity. Such operations might have only been practiced at a high level in the curriculum of a literate person (Dorandi 2000). In some cases, then, we may guess at the possible structure or contents of a course for which directly resulting marginal notes were preserved. This relationship should be connected to the fact that in Greco-Roman contexts, advanced, literate education implied the study, oral reading, and excerpting of pieces of the classical corpus that formed "the circle of knowledge" (*enkuklos paideia*).[16]

The second reason is that the activity of reading, excerpting, copying, or commentating on classical sources, be they mathematical in content or not, was valued and formed part of what Ineke Sluiter has called "the didactic tradition" (Sluiter 1999). By this is meant *not* the activity of teaching and its "concrete" tradition, but an idealized set of values that, in a non-negligible number of cases, were spelled out very explicitly. Such is the case in the prologues of redacted commentaries, like Theon's commentary to Ptolemy's *Almagest* (Bernard 2014) or the extremely developed exegetical prologue to Proclus's commentary to Euclid's *Elements* I (Lernoult 2010). Such documents are perhaps less valuable for what they indicate about the scholastic character of the corresponding commentaries than for what they say about the leading ideas and cultural purposes assigned to them. This aspect, in turn, is hardly separable from the existence of the *second* category of sources mentioned above: the Greco-Roman literature, especially in philosophy and rhetoric, contains sophisticated conceptualizations of the general notion of what teaching and learning means and even *should* mean. Famous early examples, as far as mathematics teaching is concerned, include Plato's *Meno*, *Republic*, or *Laws*, Isocrates's *Antidosis*, or for the Roman world Quintilianus's *Institutio Oratoria*. All these works contain partial or extended discussion about what the role and nature of mathematical teaching should be within a general educational framework, the latter giving its full meaning and value to the former.

The third reason directly touches on the bias mentioned above: by their nature, these sources reflect a highly distorted picture of what the *totality* of "ancient mathematical cultures" might have represented, including many presently lost written sources, as well as the totality of non-written cultures. Many (if not most) ancient written sources have been lost by accident. Moreover, these sources could only be used and elaborated upon by persons who belonged to the (highly) restricted elite of "educated people" – *pepaideuomenoi*. Therefore, those milieus with some kind of mathematical

[14] For Greek literature, see Fowler, *op. cit.* For Latin technical literature (corpus agrimensorum, on which more below), see Dilke (1971, 128ff) as well as Chouquer and Favory (2001) (esp. Chap. 1).

[15] Including copying, annotating, and writing memoranda, summaries, and abridgements (*epitomai*)

[16] For more detail, see, for example, Aelius Theon's *Progymnasmata*, which is basically a handbook for teachers of rhetoric.

activity and transmission of knowledge for which a bare trace remains, or which did not successfully highlight their specific skills in the standard terms of the literate culture, are almost totally absent.

The two first aspects should be carefully distinguished from the third, in order *not* to superimpose well-represented idealized descriptions and conceptions of teaching on the actual techniques, which are poorly documented. This precaution also eliminates deeply ingrained confusion between various periods of history or doubtful assimilations, such as the frequent claim that Euclid's fundamental purpose for the *Elements* was pedagogical. The problem with this assertion is that Euclid and his exact purpose cannot be directly known because no document from the Hellenistic period relates to these questions.[17] What is known for certain is that Euclid's purpose for his *Elements* may be interpreted as other than purely didactic[18] and that the first explicit mention of a didactic purpose for the *Elements* only appears some eight centuries later in Proclus's commentary to its first book (Vitrac 1990, pp. 34–40; Bernard 2010b). In the case of commentators like Proclus who were also teachers,[19] this later dimension should probably be interpreted as the direct reflection of their own didactic concerns that they could easily project on the authors for whom they made commentaries (Sluiter 1999).

Another important reason for maintaining the distinction between idealized descriptions and actual techniques of teaching is that, while the actual practice and didactic devices are by nature evanescent, the prefaces, annotations, and such, along with the values they convey, are perennial in that they remain opened to reappropriation in later periods.

Bearing all this in mind, some scarce but interesting indications of the actual practice of teaching mathematics in certain contexts should now be introduced, as a way to convey a sense of the institutional setting for such teaching. This example leads to a summary of the thorny discussion of the (disputed) existence of a scholastic curriculum in Greco-Roman antiquity. Only then may a tentative and differentiated explanation about the meaning of mathematics and mathematics learning for various parts of Greco-Roman society be presented as a conclusion.

5.2 *Three Possible Scenarios for Mathematics Teaching in Late Antiquity and What Can Be Concluded from Them*

In the third book of the so-called Mathematical Collection,[20] the fourth-century polymath Pappus of Alexandria describes an encounter with some students of Pandrosion, a female teacher of geometry and a rival of Pappus. The students he encountered had submitted several challenges to Pappus, who was then encouraged by several of his peers to answer them. The related event is interesting in at least three respects.[21]

Taken first as a straightforward account of the encounter, the anecdote shows that the agonistic and challenging character of ancient Greek culture, noted as early as the classical and first sophistic

[17] On the uncertainty of the date and context of Euclid – uncertainty that already dates from Late Antiquity – see Vitrac (1990, pp. 13–18).

[18] Vitrac (1990, pp. 114–148). Other treatises by Euclid besides the *Elements* might be more legitimately suspected to contain some kind of exercises in demonstration or in the technique of analysis (*Pseudaria* and *Dedomena,* respectively); see Vitrac (1990, pp. 21–23).

[19] Not only were they teachers, but, in the case of late Platonist commentators like Proclus, they considered themselves as the successors (*diadochoi*) of a Platonic tradition that included such famous mathematicians as Euclid or Nicomachus. Thus, according to his biographer Marinus, Proclus believed he was the reincarnation of the latter.

[20] Pappus probably did not author the collection as such, but only the constituent individual treatises which were put together long after Pappus's time.

[21] For more detailed discussions of this event and Pappus's account of it, see Knorr (1989, pp. 63–76), Lloyd 1996, Cuomo (2000, 127ff), and Bernard (2003).

periods, survived well into Late Antique Alexandria and that geometry counted among the possible objects of controversy. Not only the fact that young people challenged Pappus but also his sophisticated answer to the challenge[22] indicates the pervasiveness of the rhetorical model of learning and teaching. This model demanded that the actual exercise of discourse be taught to students to enable them to become immersed, by imitation, in the rules of composition of discourses (Bernard 2003).

Taken now as a literary composition which describes his encounter with students in a slightly idealized manner, the philosophical tale reveals the *values* behind this kind of challenge: for this purpose, Pappus refers to the classical debate between Speusippus (Plato's nephew) and Menaechmus (Eudoxus's student) about the nature of mathematical activity. The first maintained that mathematics was all theory-making, and the other countered that it was all problem-solving. With this classical conflict in the background, Pappus identifies (and praises) the students as followers of Menaechmus (since they proposed a solution to a geometrical problem) and himself as a follower of Speusippus (since Pappus demonstrates his ability to theorize the proposed construction through relevant means). The whole "refutation" of the construction – and its mixture of blame and praise – follows the literary convention for describing such agonistic encounters, with their incumbent heavily charged ethical aspects.

A third aspect of the encounter also demands attention: the entire challenge is based on geometrical figures, handed out in written form to Pappus, which he corrected or completed in his text. This strongly suggests that the actual discussion of these figures, if it ever took place, relied on a physical prop, a figure which, for this discussion, played the same role that the images (*eikones*) played in ancient rhetoric: a pretext for discourse and collective discussion. More than this cannot be said: no traces of any pedagogical device have been retrieved that could help us figure out how geometry was taught or discussed in scholastic assemblies (*sunousiai*).

In his biography of the fifth-century philosopher Proclus of Lycia,[23] Marinus of Neapolis describes (among other stages) both Proclus's training and his teaching methods after he succeeded Syrianus as the head of the Neoplatonist school in Athens. Like Marinus, Proclus in his own time reputedly displayed good enough knowledge of mathematics to have prepared a commentary on the first book of Euclid's *Elements* and knew enough about Ptolemy's *Almagest* to criticize him. As far as Proclus's training is concerned, Marinus makes clear that Proclus's wealthy parents, who were recognized notables, permitted him to travel from master to master, from whom he acquired skills ranging from rhetoric and declamation to mathematics and philosophy. As for mathematics, he is said to have been trained by a certain Hero, named by Marinus as an Alexandrian philosopher (*Vita Procli*, pp. 10–12). From Marinus's description, Hero appears to have probably taught Proclus the neo-Pythagorean mathematics useful to understand Plato's *Timaeus* and theurgic techniques in Hero's own home. The latter aspect is not uncommon: wealthy students traveling from place to place and from one teacher to another often boarded with their teachers and became some kind of spiritual children, called *gnôrimoi* (relatives of the teacher). Moreover, these details show that the kind of mathematics taught probably did not constitute a specialized subject but part of a philosophically oriented teaching, hardly separable from reading Plato.[24]

As an Athenian teacher of philosophy and mathematics, the intellectual activity of Proclus is represented by the two extant commentaries on mathematical authors mentioned above (namely, Euclid and Ptolemy) and also by Marinus's description of his usual pedagogical technique. According to

[22] He answers not only by demonstrating his own capacity to analyze the shortcomings of the construction but also by suggesting that the students could have proceeded otherwise if they had possessed more knowledge of the underlying problems.

[23] Entitled "Proclus or On Happiness" = *Vita Procli*. This "biography" is better termed a hagiography. For the nature of Marinus's discourse, see *Vita Procli* XLI-C (Saffrey and Segonds).

[24] Such an approach to mathematics is already distinctly represented by Theon of Smyrna in the second century A.D. (Delattre 2010). For the noninstitutionalized framework of Late Antique education, see Derda et al. (2007, pp. 177–185) (E. Szabat) See also Watts (2006).

Marinus (*Vita Procli*, pp. 26–27), Proclus met with students for classes where he guided critical discussions on a traditional and varied material. In the evening, he would write down a record of his findings so that the extant commentaries probably represent the redaction of the notes taken from his courses. Again, more than this we cannot say: the archaeological remains of what might have been Proclus's house in Athens (Karivieri 1994) have revealed no special didactic settings. Nonetheless, these details indicate that the study of classics was, as much for mathematics as for other subjects, the core of advanced teaching. The classics discussed in this context, as the extant commentaries make clear, were not "mathematical" in any restrictive sense, but incorporated a much wider circle of knowledge, including Aristotle and Plato's writings.

The picture of a teacher surrounded by soliciting students for whom an extensive knowledge of classical works was prerequisite also seems to underlie the various prefaces written by Theon of Alexandria, a commentator from the second half of the fourth century, who elucidated Ptolemy's *Almagest* (Tihon 1992; Jones 1999; Bernard 2014). Here also, the basic material of the course consists of classical works, not only Ptolemy's treatise but also classical geometrical treatises and other commentaries on Ptolemy, which Theon encourages students to compare with his own (Bernard 2014). In this case, there are good reasons to believe that a significant portion of Theon's audience was comprised of practicing astrologers.

The above examples, however interesting, represent only a small and biased sample of the various didactic settings that might have existed in antiquity. It must be noted that these examples all belong to Late Antiquity, for which we possess a significant number of accounts of teaching, although they are also presented in an idealized way and according to precise literary conventions.[25] Some aspects of these testimonies are nevertheless confirmed by archaeological records, especially discoveries recently made of auditoria at Kom el-Dikka in fifth-century Alexandria (Derda et al. 2007). One should also note that these reports only concern elite teaching and learning. Typically Marinus's account of Proclus's training makes clear that he directly began his schooling *outside home* with a "grammatikos"; any elementary teaching he received must in all probability have been imparted at home, thanks to his wealthy parents, and not in any "primary" school (Kaster 1983, p. 334).

For earlier periods and from other kinds of evidence (like the few surviving papyri), the intrinsically "classical" character of ancient teaching is also confirmed, but tantalizing hints appear about elementary teaching, such as exercises in simple calculations – a venue of mathematical education which is altogether very badly represented in the mathematical works or commentaries (Fowler 1999, pp. 222–262). In Theon's commentaries or in the so-called prolegomena to the *Almagest* (Acerbi et al. 2010), there exists an exposition of calculation techniques, but these are hardly elementary because they relate to numbers expressed in sexagesimal numeration used only for astronomical (and therefore advanced) calculations imported from ancient Mesopotamia. Moreover, precisely because they are explained in such treatises, this style of calculation hardly appears to have been taught at an elementary level. The school exercises retrieved on papyri or *ostraka* are usually very difficult to situate precisely in terms of level and purpose.[26] Some of them, however, must have referred to the professional training of specialized slaves like scribes or calculators.

Finally, archaeological hints evoke very different teaching settings, like Egypt in the Roman period, where temples have existed in which astrological calculations were practiced as they were with astrologers who specialized in astral sciences and mathematics in Mesopotamia during the Hellenistic period (Jones 1994, 1999, p. 157).

[25] This particularity is best explained by the fact that this period is characterized by, among other things, the violent confrontation of various cultural and didactic models, especially between Christian and pagan models, which led each party to highlight and effectively represent these values.

[26] H.I. Marrou, in his short discussion on the teaching of elementary calculation, already warned against the too easy identification of papyri with mathematical content as corresponding to school exercises (Marrou 1964[6], note 10, pp. 398–399). Modern discussions confirm this.

All of these elements, however scarce and limited, indicate that we should certainly not generalize the picture afforded by more literary accounts, such as Pappus, Theon, or the later Neoplatonists. The existence of these accounts stems from the *de facto* selection of sources in proportion to their cultural value and literary sophistication (and therefore their value for the elite society alone) as well as from knowing much more about ancient education in Late Antiquity than about any other period of Greco-Roman antiquity.

In the face of this complexity, two possible approaches offer clarification. First, a widespread scholastic curriculum that would account for the various situations of Greco-Roman antiquity, and especially for the difference between elementary and advanced education, could be reconstructed. While this approach works for other cultural contexts, such as ancient Mesopotamia, this approach will be found to be highly problematic and, to some extent, sterile for Greco-Roman antiquity. Because of these difficulties, a more cautious and "localized" approach, which addresses the various elements of Greco-Roman society potentially concerned with mathematical education, may be adopted.

5.3 Disputes on the Existence and Nature of a Scholastic Curriculum in Greco-Roman Antiquity

For the past century, scholars have debated the existence of an institutionalized educational curriculum in antiquity. In the standard view that once prevailed, this curriculum could be neatly divided into three successive stages, respectively, labeled primary (or elementary), secondary, and higher or tertiary. Each stage had its own kind of teachers and school.[27] If only the contents (and not the institutional background) are considered, the first stage corresponds with the first acquisition of basic literacy (reading and writing skills) and numeracy (calculating skill with simple operations); the second stage with the study of advanced literature, especially poetry, with an emphasis on skillful reading up to the level of literary criticism and eventually including some instruction in higher mathematics; and the last stage with the learning of rhetorical skills or other advanced domains (philosophy, medicine, law).[28] The traditional view gives a straightforward interpretation of these various levels as representative of a progressive curriculum leading from elementary to higher studies, with specific teachers and locales for each level: the "teacher of letters" (*grammatistês* or *ludus litterarii*) for the first level; the "grammarian" (*grammatikos*, *grammaticus*)[29] and perhaps other *professores* (in geometry, arithmetic, astrology?)[30] for the second level; and rhetors, teachers of medicine or law, and philosophers for the last level.[31]

But this standard view, which was already heavily nuanced by its first proponents,[32] has been increasingly challenged since Booth and others have progressively demonstrated that another model

[27] This standard and traditional view is found, among others, in the influential syntheses of Marrou (1965), Bonner (1977), and Clarke (1971).

[28] For a more detailed account of the contents of each level, see Cribiore (2001, Chaps. 6, 7 and 8, pp. 160–244); in those chapters, she focuses on only the basic *contents* of each level. See also the lucid and updated synthesis provided in Szabat (2007), with many references to the debates on these issues.

[29] The term "grammaticus" should not be understood as equivalent to our modern "grammarian," which now designates a distinct discipline. Although the latter was first constructed in antiquity, the competence of the "grammaticus" as a teacher extended much beyond mere "grammatical" analysis of literary and poetic texts: this teaching included a thorough initiation in the reading and analysis of a characteristic corpus of poets and classical writers. See Szebat (2007, pp. 185–187) for a synthetic summary and Kaster (1988) and Cribiore (2001, pp. 185–219) for more detailed explanations.

[30] Some "idealized" accounts allude to the existence of such "separate" professionals, but these accounts are uncertain and ambiguous. There is some, albeit scanty, evidence of such mathematical teaching at the secondary level. See Kaster (1983, p. 335) and Cribiore (2001, pp. 40–42).

[31] For a discussion of the thorny and interesting issue of the "technical" terminology of ancient education, see again Kaster (1983, pp. 329–331) and Szabat (2007), with references to other studies on the same subject.

[32] On the analysis of Marrou's precautions on this issue, see the insightful discussion of Kaster (1983, p. 324).

was applicable in some cases.[33] In this alternative view, elementary teaching should be considered as a basically separate track for people of the lower social level (including slaves), whereas the schools of *grammatikoi* were reserved for a higher elite.

The now accepted view[34] is that no model applies uniformly to all of antiquity or throughout the entire Mediterranean world. Not surprisingly, Booth's model of separate tracks seems better adapted for the big cities of antiquity, whereas the situation in small localities, with few teachers and very specific needs, would have been much more variegated. Thus, the main positive conclusion – perhaps the only indisputable one – of this scholarly debate is that "there were throughout the Empire schools of all shapes and kinds, depending on local needs, expectations, and resources." Kaster rightfully adds that "in a world without centralized direction of education of any sort, that is only what we should expect."[35]

If no single model can be applied to all (known or unknown) teaching situations in antiquity, then it is worthwhile to detail the main reasons why the modern, three-stage curriculum cannot be considered valid. The first reason has already been mentioned: ample reports indicate that, in many cases, the various "stages" of education did not concern the same people, so there may be no stages at all, but only references to different teaching contents for different people.[36] The curriculum of the elite, which is naturally overrepresented in the ancient literature, concerned the same people (the rich and wellborn) and was strongly characterized by a relatively uniform and well-defined idea of literate culture that encompassed a limited set of classical authors and well-identified kinds of exercises practiced on them, from reading to critical analysis.[37] But even then, the order of studies at this stage was not completely fixed and depended on particular teachers and the length of study based on each student's means: a significant rate of attrition existed, and the number of students decreased with the numbers of years of study. This characteristic conforms to the structure and contents of ancient teaching, which sought to deepen the understanding and study of classical texts rather than to attain a definite goal through an accumulative process. The same texts, therefore, were studied again and again but each time in a deeper way until the students, through imitation and impregnation, were able to compose or declaim on their own.[38] Moreover, the number of various elite "professions" that used this secondary curriculum shows that "secondary and tertiary" curricula were not uniform.

The second reason is that there is no clear proof of, and many counterexamples to, a fixed and unambiguous correspondence between these three "levels of teaching" and the competence of particular teachers. Thus, to take up the most discussed issue in the abovementioned studies, the same *names* of teachers, like "grammatistês" (teachers of letters), could actually refer to various contents or levels, from elementary to "secondary" (Kaster 1983, pp. 329–331; Szabat 2007, pp. 181–185). Likewise, *grammatikoi* could prepare students for higher achievements, either rhetorical or philosophical. Thus, the famous philosopher John Philoponus officially was a "grammarian" but was known to be a philosopher and commented on mathematical texts. This also means that the same person could teach at different levels, sometimes at the same time.

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[33] Booth paid attention to the situation in first-century AD Rome. His theses (Booth 1979) are conveniently summarized in Kaster (1983), who expands on his argumentation.

[34] For an efficient summary, see Szabat (2007, pp. 178–181), who draws on previous studies, esp. Kaster (1983).

[35] Kaster (1983, p. 346). The same point is made in Cribiore (2001, Chap. 1) (pp. 15–44) for the sole case of Hellenistic and Roman Egypt and Szabat, *op.cit*. Even the imperial state did not heavily intervene in educational institutions. At best, laws would oblige cities to finance municipal chairs, without intervention in and regulation of their study. The majority of teachers, though, worked privately and directly depended on fees from students and their parents.

[36] Kaster (1983, pp. 337–338) summarizes the "positive" reasons to believe that Booth's model is better adapted in general to antiquity, although it should not be viewed as an alternative model applicable to all ancient situations.

[37] On the uniformity and strong identity of the *grammatikoi*'s teaching, the classic study is Kaster (1988).

[38] This idea of "concentric" studies, in which the same elements and methods are retrieved at each level but with a different depth and difficulty, is central to the argument of Cribiore (2001).

The third reason is that, more often than not, students of various levels learned together, all in the same space, rather than in separate classrooms organized according to the teaching. In general, the locales of teaching, if there were locales at all (some teachers worked in the street), are not easily identified; when they are, they could have belonged to various institutional buildings, from gymnasia to theaters or temples. In fact, what a "school" referred to, in the ancient context, should be understood in personal terms: as a circle of students frequenting a teacher.[39]

Last but not least, the majority of inscriptions or papyri, especially those that can be found in small localities away from the great urban centers of Rome, Alexandria, or Antioch, show that the teachers' presence, competencies, and missions depended heavily on the local context. This is actually the main reason to remain cautious about the "standard," contemporary and (often) prescriptive accounts of ancient education, like in Quintilianus (Kaster 1983).

5.4 Various Kinds of Mathematics Teaching for Various Parts of Society and Professional Circles

Surviving sources preserve mere hints about the particular places and settings in which mathematics was taught in Greco-Roman antiquity. More than this is not really known, as shown by the difficulty of reconstructing a uniform scholastic curriculum in Greco-Roman antiquity. In fact, the various places, periods, and levels in society in which some kind of mathematics was practiced in various ways have left only fragmentary evidence or virtually no trace at all. Despite this dearth of information, some people outside of the elite might have been concerned with mathematical training.

5.4.1 The Elementary or Specialized Teaching of Arithmetical Skills: Slaves and Freedmen, Professional Scribes, and Accountants

As mentioned, little documentary evidence with mathematical content has been found on papyri or *ostraka* that might potentially be interpreted as school exercises. The available evidence[40] has to be checked against the complex and "fuzzy" background of mathematical education. Apart from a few papyri that might be interpreted as stemming from the study of Euclid's *Elements* and therefore as belonging to the "secondary" level, several tables of multiplication and fractions have also been retrieved. But the problem is whether these writing exercises belonged to the elementary level, or whether they should be interpreted, as Cribiore argues,[41] as belonging to a more specialized curriculum for scribes. Written exercises in calculation, especially addition, would by contrast be the exception rather than the rule: elementary operations were probably taught and practiced orally or with an abacus. More advanced exercises might have been executed by writers and calculators already in training, according to the quality of their writing. Because mastering such skills seemed alien to the spirit and contents of "secondary" teaching, this aspect of education might be plausibly interpreted as the production of advanced slaves trained to be *notarii* (professional scribes) or *calculatores* (accountants) who were completing a professional training. Not surprisingly, given the general background and nature of the so-called "elementary" level of teaching as well as uncertainties about its real nature, the mathematical documentation which might be considered "scholastic" is no less

[39] This point is made in Szabat (2007, pp. 180–181) and Cribiore (2001), Chap. 1 (on school accommodations) and Chap. 2 (on teachers).

[40] Fowler (1999)gives the sole extensive discussion on the papyrological evidence concerning mathematics.

[41] Cribiore (2001, pp. 180–183). This short discussion is devoted to the question of the acquisition of numeracy at the elementary level.

fuzzy and uncertain. Part of it, though, might correspond to a specialized curriculum concerning people (including slaves) with a low social status.

In any case, the complete classification and study of those fragments is still an open question, and the achievement of positive results through such a study remains uncertain because of the small number of such documents. Here, the interesting consequence of this general scarcity of sources on the *practical* dimension of the teaching of mathematics, especially at an elementary level, is the confirmation that our documentation is strongly biased. By contrast, sources for the *idealized* curricula are much richer and substantial for Greco-Roman antiquity. Among these sources are highly sophisticated treatises (like Plato's *Republic* or, much later, the commentaries of Proclus), prefaces and introductions (which convey a sense of the didactic tradition), or pedagogically structured treatises (like for Ptolemy's *Almagest*). The variety of these idealized approaches is too vast to be summarized here, and many good studies are available on the subject.

However, one revealing aspect of these works (especially within the prefaces) demands consideration here, namely, the *target audience*. The target audience refers to the milieu that might have been concerned with these ideals and also, to some extent, that were *represented* by them: defining an ideal curriculum expressed the shared cultural values that defined not only the milieu but also its *raison d'être*. One classical example is provided by the case of ancient astrology. For the ancients, astrology implied a demanding and complete *training* and the mastery of a sophisticated cannon and detailed techniques.[42] In general, then, the definition of a culture, including an idealized training system, constituted part of the social identity in antiquity. The various educated circles potentially connected with such a self-definition, and the culture of which might have included some mathematical training, merit a brief review.

5.4.2 Mathematical Training as Part of Philosophical Education

That mathematical subjects might have been considered appropriate for a philosophical curriculum, either an ideal one (as in Plato's *Republic*) or a real one (as with the late Platonists like Iamblichus in the fourth century, or Syrianus and Proclus in the fifth century), is a fact so obvious that it is impossible to review exhaustively the wide range of philosophical positions and schools for which this idea was meaningful.[43] In this long story, Plato and the varied company of Platonists are well represented. Despite the breadth of the topic, the historical importance of three particular ideals merits their acknowledgment and exposition.

The first two of these ideals can probably be regarded as varieties of Platonism. The first ideal is neo-Pythagorean, represented in the second century by Nicomachus and Theon of Smyrna, whose treatises focused on so-called neo-Pythagorean arithmetic and represent a sophisticated philosophical project. Theon particularly relies on a coherent organization of mathematical knowledge, especially the four sciences (arithmetic, geometry, astronomy, and music), which later formed the scholastic *quadrivium* at the edge of the Middle Ages (Hadot 2005, Chaps. III and IV; Vitrac 2005).

The second ideal is that of the Ptolemaic philosophical way of life, centered on the study of mathematics, especially the kind of mathematics related to the movements of the stars and apt to bring the soul of the philosopher closer to this cosmic movement (Taub 1993, Chap. 2 and 5; Sidoli 2004; Feke and Jones 2010; Bernard 2010). While this ideal is basically Platonic in spirit, it does not subordinate the *study* of mathematics to higher studies (like dialectic) but, on the contrary, recommends

[42] For an account of ancient Greek astrology, the standard reference remains Bouché-Leclercq (1899). See also the more recent Barton (1994), especially pp. 134–142 as far as astrological training is concerned.

[43] For an updated extensive study on this question, see Hadot (2005), especially the fourth "étude complémentaire," pp. 431–455, concerning mathematics. Note, however, that Hadot has a tendency to reduce any ancient mathematical teaching to being basically dependent *in all cases* on a philosophical ideal, an idea which is somewhat open to criticism.

mathematics as the highest philosophical study above all others. The *Almagest*, accordingly, is structured as the basis of such a philosophical study and elevation of the mind. This ideal, associated with this impressively well-organized exposition of ancient astronomy in the *Almagest*, proved to be highly influential throughout Late Antiquity, the Middle Ages, and up to the early modern period.

The last philosophical ideal is that of Isocrates, Socrates' other disciple besides Plato. Isocrates proved to be influential on the founders of classical Latin rhetoric, Cicero and Quintilianus. In Isocrates's philosophy, which is not meant as a system but as a particular way to cultivate discourses, mathematical training is essential not for its contents but for the effect it possesses *as a training device*, leading the student to analogous but higher studies. He therefore formed the ideal of mathematics as a preparatory stage in rhetorical education.[44]

5.4.3 Mathematical Training as Part of an Astrological and Astronomical Training

If Ptolemy's ideal can be viewed as one legacy of the literature of philosophy because of his explicit or implicit references to the "grand" philosophical literature (Plato, Aristotle, the Stoics), it might be also be understood in relation to the tradition and culture shared by ancient astrologers. Within the curriculum and ideal training of astrologers, as expressed in Vettius Valens, Firmicus Maternus, Ptolemy, or other authors, calculation of the positions of stars in order to establish an astrological chart is described by these authors[45] as an indispensable first step in the standard training of an astrologer, which expanded to include mastering a body of knowledge that enabled him to interpret the sublunary significance of the astronomical phenomena (Bernard 2010).

Among these ancient representations of astrological training, one recorded in Ptolemy's so-called *Tetrabiblos* is probably the most sophisticated. Here, the astronomical knowledge necessary for the computation is represented not only as an indispensable part of the science concerning the "physical" effects of planetary positions on sublunary events but also as a science desirable in itself (with a reference to the *Almagest*, which presents itself as a self-contained treatise). This simultaneously coherent and "bivalent" system is probably an extreme element within the spectrum of approaches to astrology in the same period. Papyrological evidence has clarified that Greco-Roman astrologers had recourse to calculation techniques other than the cinematic tables advocated by Ptolemy (Jones 1999). Nonetheless, Ptolemy's astronomical text became the subject of commentaries (by Pappus of Theon) by the fourth century, thus consecrating the pedagogical ideal imbedded in the *Almagest* and advancing it into new directions.[46] The importance of Ptolemy's works lies precisely in how he turned a technical treatise of astronomy into classical knowledge, liable to the activities of commentary (Jones 1999, 160ff).

The contents of the commentaries on Ptolemy's *Almagest* and *Handy Tables* are only partly edited, as are the *scholia* to these texts and their commentaries (Tihon 1992). Most of these probably reflect pedagogical activities, the precise nature of which is difficult to reconstitute even if their existence is beyond doubt. The use of geometrical diagrams to explain or detail procedures, the explanation given on "elementary" operations in the sexagesimal system, or the many references to mnemonic schemas most probably reflect teaching activities oriented toward the appropriation of the complex contents of Ptolemy's works (Tihon 1992).

[44] See Isocrates's ideas in *Antid.* pp. 258–269. For Quintilianus, see *Inst. Orat*. I.10, especially pp. 34–49.

[45] Even by those, like Firmicus, who obviously had little command of the mathematical contexts of their art.

[46] These commentaries can indeed be seen, at least in part, as conscious imitations of the *Almagest*; see Bernard(2014) on this point.

5.4.4 Mathematical Training as Part of the Training of Land Surveyors, Engineers, and Architects

Our knowledge of Greco-Roman technology and its specialists, if and when such specialists existed, is complicated by the impressively diverse state of our sources, the differences in the social status of their authors, and the variety of points of view on and classifications of the related subject (Cuomo 2007). Besides highly influential and extant works like Vitruvius's *De Architectura*, some of the sources on these domains, like the pseudo-Heronian and metrological corpus in Greek or the corpus of land surveyors (*corpus agrimensorum* = CA) in Latin, have survived in such a state of confusion that the study, attribution, and characterization of their contents remain a work in progress.[47] Even an attempt to draw sharp distinctions between architecture and the construction of machines for land surveying, for example, leads quickly to thorny problems.[48] Also, the social status of the authors or readership of the related treatises is far from clear. Thus, if there is little doubt that some authors of gromatic literature[49] identified themselves as belonging to a well-defined profession (*mensores*), others, like Frontinus or Vitruvius, had a higher social status with political responsibilities and thus treated a wide range of technical subjects. Even among "professional" Roman land surveyors working during the Roman Empire, there are good reasons to think there was a significant variety of functions and social status with different types of training (Hinrichs 1989, pp. 171–174).

Thus, for the Roman land surveyors working during the imperial period (from which most of our information comes), Hinrichs proposes to distinguish between four categories of mensores, according to their function and social positions: land surveyors (a) who served in the army, (b) who were in the service of the emperor, (c) who were employed (or enslaved) by municipalities, and (d) who were private or independent. The first must have been trained within the army, according to a specific tradition about which we have no details. As for the imperial functionaries, Hinrichs speculates that their typically Greek names and the existence of a Greek "technical" tradition of land surveying indicate that they were slaves or freedmen who received their training in Alexandria; the specific "civil" curriculum of the two last categories is unknown, but there is a good chance that the contents of the CA were actually developed and/or used in this context.

Indeed, while the sources are incredibly varied and complex, there is no doubt that a significant portion related to pedagogical purposes, ideals, or realities.[50] Furthermore, part of the corresponding training included mathematical skills and knowledge of astronomy, geodesy, land surveying, measurement problems, and calculating techniques (Chouquer and Favory 2001, pp. 64–94). Finally, the contents of this corpus were influential on medieval and Renaissance mathematics teaching, and their importance cannot be underestimated.

For the sake of simplicity, some general features of these sources and their underlying teaching background may be sketched broadly. First, these sources contain idealized representations of curricula, like the sophisticated discussions of *De Architectura*, book I and book IX, or Pappus's *Mathematical Collection* VIII, which might be partly derived from the lost introduction to Hero's *Mechanika*. These texts significantly characterize mathematical skills as being basically dependent on

[47] As far as the corpus agrimensorum is concerned, the work of Toneatto (1994-5) has drastically improved our understanding of its history; see Chouquer and Favory (2001, Chap. 1).

[48] Is Hero's *Dioptra*, for example, an exercise in land surveying, as the kinds of problems treated therein strongly suggest, or the skillful description of an instrument, as the preface and many technological details indicate? Is Vitruvius's treatise merely a work on monuments and house-building, or also on machine-building (book 10), the science of sundials and astronomy in general (book 9), and many related subjects, as the contents suggest?

[49] Like Hyginus "gromaticus," the second Hyginus or Siculus Flaccus. For translations of these authors, see Campbell (2000) or the various annotated editions published in French by J.Y. Guillaumin, in particular Guillaumin (2005, 2010).

[50] It has even become commonplace in scholarship on these kinds of sources that they represent didactic efforts and are scholastic "manuals," a qualification difficult to dismiss because of the vague and multifarious meanings of this category. The idea is discussed and nuanced in Chouquet and Favory (2001, p. 38).

the wider culture of the architect and *useful* for that culture. Such discussions about the *utility* of mathematical training within a larger intellectual framework are also characteristic of the introductory material contained in the Greek metrological corpus and in some prefaces in the CA. Vitruvius and Pappus/Hero also insist on the importance of developing creative skills which combine theoretical prerequisites with practical skills.

This last point relates to a second feature of the aforementioned sources: significant parts of these sources are organized as series of problems often arranged in order of growing complexity. This organization bridges from simple, practical motivations to more theoretical or didactic concerns. Problems such as measurement are typically accompanied by algorithms for calculation generally associated with pedagogical activities or purposes, even though the exact connection may not be obvious at the elementary level of teaching.

The third feature has been mentioned already: these corpuses survived in a state of deep confusion and disorder, often explained by the fact that the original texts were significantly changed and reorganized throughout the course of their history, sometimes for pedagogical purposes. Indeed, it is sometimes easier to situate and identify these reorganizations and purposes – for example, late Neoplatonists reworked the metrological corpus[51] – than it is to reconstitute the extent and purpose of the original sources. The same remark potentially holds for the question of illustrations, which are numerous in the CA. Some of these might come from original sources, but many others might have been added later, some for pedagogical purposes.

Finally, scholars often note that these corpuses are often missing significant parts. In the case of Latin *agrimensores*, clear allusions frequently occur to lost Greek technical treatises on related subjects. Hero's *Dioptra* is often mentioned as a possible source, or at least as a treatise similar in kind to this lost material, but in all probability this is only the tip of the iceberg. This is a good example of a domain for which we might well have lost the majority of material available in antiquity.

5.4.5 Mathematical Training as Part of an Unknown Culture of Arithmetical Problem-Solving

Finally, Diophantus's *Arithmetica*, one of the most mysterious mathematical treatises of antiquity, must be addressed. Although there is no certainty, the treatise may have been composed in Late Antiquity, around the third century, yet its author is unknown. The treatise contains 13 books of arithmetical problems (statements and solutions) arranged in progressive order; six books are extant in Greek and four more have been found in Arabic translation. The long preface survives in Greek and is structured around a didactic project: to enable the reader of the treatise to develop the capacity for invention in arithmetical problems (*Arithm.* 2.3–13). The problems and their treatments are indeed arranged in an order that enables the acquisition of a whole range of specific techniques useful for the solution of problems (Bernard and Christianidis 2011, part 4). Moreover, the reference to the rhetorical notion of *invention* (*heurêsis*) points toward the influence of higher culture and rhetorical training on the structure of the treatise. According to Diophantus's own words, his whole treatment is organized as paving a way (*hodos*) to the reader, an example that he might imitate (Christianidis 2007; Bernard and Christianidis 2011, part 5; Bernard 2011).

There is little doubt that the treatise directly relates to a clearly defined didactic strategy akin to the techniques used in ancient rhetorical training. What remains almost entirely absent from the picture is the larger background constituted by the arithmetical problems studied within the "logistic tradition." This expression usually refers to the problem-solving and calculation tradition relative to the kind of arithmetical problems practiced for millennia in ancient Mesopotamia or ancient Egypt or afterwards

[51] Acerbi and Vitrac forthcoming: introduction, A4. A preliminary version of this detailed analysis is available online on hal-SHS http://hal.archives-ouvertes.fr/hal-00473981/fr/ (consulted 5.1.12).

in the Middle Ages. But, as far as Greco-Roman antiquity is concerned, we have only hints and desperately few documents claiming that such a tradition existed[52]: the bulk of it has disappeared. Diophantus's problems might be interpreted as "abstract" problems akin to the scholastic and *ad hoc* rhetorical problems invented for the sake of rhetorical training.

Diophantus's *Arithmetica*, therefore, might be the tip of yet another iceberg, the splendid and isolated outcome of a much larger and widespread tradition of arithmetical teaching through problem-solving, for which there is almost no trace at present.

References

Ancient Sources

Antidosis. 2003. Isocrate: *sur l'échange*. In Isocrates, Discours, tome III, ed. and transl. by G. Mathieu. Paris: Belles Lettres.

Arithm. 1974. *Diophanti Alexandrini Opera Omnia cum Graeciis commentariis*, ed. and Latin Transl. by Paul Tannery, 2 vols. Leipzig: B.G. Teubner 1893–95. Repr. Teubner.

Inst. Orat. 2001. *The Orator's education: Books 1–2,* ed. and transl. by D.A. Russell. Loeb.

Progymn. 1997. Aélius Théon, *Progymnasmata*. Trans. M. Patillon. Paris: Belles Lettres.

Vita Procli. 2002. Marinus: *Proclus ou sur le Bonheur*, text edited, translated and annotated by H.D. Saffrey and A.P. Segonds. Paris: Belles Lettres.

Modern Studies

Acerbi, Fabio, and Bernard Vitrac. Forthcoming. *Héron: Metrica*. Intr., ed., transl. and comm. Fabio Acerbi et Bernard Vitrac. Coll. *Mathematica Graeca Antiqua*. Pisa-Roma: Fabrizio Serra Editore.

Acerbi, Fabio, Nicolas Vinel, and Bernard Vitrac. 2010. Les 'Prolégomènes à l'Almageste'. Une édition à partir des manuscrits les plus anciens. Introduction générale-Parties I-III. *Sciamvs* 11: 53–210.

Baillet, Jules. 1892. *Le papyrus mathématique d'Akhmim*. Paris: E. Leroux.

Barton, Tamsyn. 1994. *Ancient astrology*. London/New York: Routledge.

Beaulieu, Paul-Alain. 2006. De l'Esagil au Mouseion: l'organisation de la recherche scientifique au 4e siècle av. J.C. In *La transition entre l'empire achéménide et les royaumes hellénistiques*, vol. 9, *Persika*, ed. Pierre Briant and Francis Joannès. Paris: De Boccard.

Bernard, Alain. 2003. Sophistic aspects of Pappus's collection. *Archive for History of Exact Sciences* 57(2): 93–150.

Bernard, Alain. 2010a. The historical significance of Ptolemy's Almagest for its early readers. *Revue de synthèse* 131, 6e série: n°4, 495–521.

Bernard, Alain. 2010b. L'arrière-plan rhétorique de la théorie de l'activité mathématique chez Proclus. In *Études sur le commentaire de Proclus au Ier livre des Éléments d'Euclide*, Dir. Alain Lernoult, 67–85. Lille: Presses universitaires du Septentrion.

Bernard, Alain. 2011. Les Arithmétiques de Diophante: Introduction à la lecture d'une oeuvre ancrée dans différentes traditions antiques. In *Circulation, transmission, heritage*, ed. Evelyin Barbin and Pierre Ageron, 557–582. Caen: IREM de Basse-Normandie.

Bernard, Alain. 2014. In what sense did Theon's commentary on the Almagest have a didactic purpose? In ed. Bernard and Proust.

Bernard, Alain, and Christine Proust. 2014. *Studying ancient scientific sources produced in an educational context: Problems and perspectives*. Boston Studies.

Bernard, Alain, and Christine Proust (eds.). Forthcoming. *Scientific sources and teaching contexts throughout history: Problems and perspectives.* Springer.

Bernard, Alain, and Jean Christianidis. 2011. A new analytical framework for the understanding of Diophantus's Arithmetica I–III. *Archive for History of Exact Sciences* 66: 1–69.

[52] For a list of such documents, see Fowler (1999, pp. 269–276). The list is focused on tables rather than problems.

Bonner, Stanley F. 1977. *Education in ancient Rome: From the elder Cato to the younger Pliny*. Berkeley: University of California Press.

Booth, Alain D. 1979. Elementary and secondary education in the Roman Empire. *Florilegium* 1: 1–14.

Bouché-Leclerc, Auguste. 1899. *L'astrologie grecque*. Paris: E. Leroux.

Britton, John P., Christine Proust, and Steve Shnider. 2011. Plimpton 322: A review and a different perspective. *Archive for History of Exact Sciences* 65: 519–566.

Campbell, B. 2000. *The writings of the Roman land surveyors*. Intr., text, transl. and com. by Brian Campbell. London: Society for the Promotion of Roman Studies.

Cavigneaux, Antoine. 1983. Lexikalische listen. *Reallexikon der Assyriologie* 6: 609–641.

Cavigneaux, Antoine. 1999. A scholar's library in Meturan? In *Mesopotamian magic*, ed. T. Abusch and K. van der Toorn. Groningen: Styx Publications.

Charpin, Dominique. 1986. *Le clergé d'Ur au siècle d'Hammurabi*. Genève: Droz.

Chouquet, Gérard, and François Favory. 2001. *L'arpentage romain. Histoire des textes – Droit – Techniques*. Paris: Errance.

Christianidis, Jean. 2007. The way of Diophantus: Some clarifications on Diophantus' method of solution. *Historia Mathematica* 34: 289–305.

Civil, Miguel. 1985. Sur les "livres d'écoliers" à l'époque paléo-babylonienne. In *Miscellanea Babylonica, Mélanges offerts à M. Birot*, ed. J.-M. Durand and J.-R. Kupper. Paris: Editions Recherche sur les Civilisations.

Clagett, Marshall. 1999. *Ancient Egyptian science: A sourcebook*, Ancient Egyptian mathematics, vol. 3. Philadelphia: American Philosophical Society.

Clancier, Philippe. 2009. *Les bibliothèques en Babylonie dans la deuxième moitié du Ier millénaire*, Alter Orient und Altes Testament vol. 363. Münster: Ugarit-Verlag.

Clarke, Martin L. 1971. *Higher education in the ancient world*. London : Routledge and Kegan Paul.

Couchoud, Sylvia. 1993. *Mathématiques égyptiennes*: Editions Le Léopard d'Or.

Cribiore, Raffaella. 2001. *Gymnastics of the mind: Greek education in Hellenistic and Roman Egypt*. Princeton: Princeton University Press.

Cuomo, Serafina. 2000. *Pappus of Alexandria and the mathematics of Late Antiquity*. Cambridge: Cambridge University Press.

Cuomo, Serafina. 2007. *Technology and culture in Greek and Roman antiquity*. Cambridge: Cambridge University Press.

Delattre, Joëlle (trad.). 2010. *Théon de Smyrne: lire Platon*. Trans. Joëlle Delattre. Toulouse: Anacharsis.

Delnero, Paul. 2010. Sumerian extract tablets and scribal education. *Journal of Cuneiform Studies* 62: 53–69.

Delnero, Paul. 2012. Memorization and the transmission of Sumerian literary compositions. *Journal of Near Eastern Studies* 71(2): 189–208.

Derda, Tomasz, Tomasz Markiewicz, and Ewa Wipszycka. 2007. Alexandria: Auditoria of Kom El-Dikka and late antique education. *The Journal of Juristic Papyrology*, Supplement 8. Warsaw: the Raphael Taubenschlag foundation.

Dilke Oswald, A.W. 1971. *The Roman land surveyors*. New York: David and Charles.

Dorandi, Tiziano. 2000. *Le stylet et la tablette, dans le secret des auteurs antiques*. Paris: Belles Lettres.

Feke, Jacqueline, and Alexander Jones. 2010. Ptolemy. In *The Cambridge history of philosophy in Late Antiquity*, ed. Lloyd Gerson, 197–209. Cambridge: Cambridge University Press.

Fowler, David. 1999. *The mathematics of Plato's academy: A new reconstruction*, 2nd ed. New York: Oxford University Press.

Friberg, Jöran. 2000. Mathematics at Ur in the old Babylonian period. *Revue d'Assyriologie* 94: 98–188.

George, Andrew R. 2005. In search of the é.dub.ba.a: The ancient Mesopotamian school in literature and reality. In *An experienced scribe who neglects nothing. Ancient near eastern studies in honor of Jacob Klein*, ed. Yitschak Sefati, Pinhas Artzi, Chaim Cohen, Barry L. Eichler, and Victor A. Hurowitz, 127–137. Bethesda: CDL Press.

Goldstein, Catherine, Jeremy Gray, and Jim Ritter (eds.). 1996. *L'Europe mathématique*. Paris: Maison des Sciences de l'Homme.

Guillaumin, Jean-Yves. 2005. *Les arpenteurs romains1: Hygin le Gromatique, Frontin.* Text establ. and transl. by Jean-Yves Guillaumin. Paris: Belles Lettres.

Guillaumin, Jean-Yves. 2010. *Les arpenteurs romains 2: Hygin, Siculus Flaccus*. Text establ. and transl. by Jean-Yves Guillaumin. Paris: Belles Lettres.

Hadot, Ilsetraut. 2005. *Arts libéraux et philosophie dans la pensée antique*, 2nd ed. Paris: Vrin.

Hilprecht, Hermann V. 1906. *Mathematical, metrological and chronological tablets from the temple library of Nippur*, vol. 20–1. Philadelphia: University of Pennsylvania.

Hinrichs, Focke T. 1989. Histoire des institutions gromatiques*. Die Geschichte der gromatischen Institutionen* (trans: Minary, Daniel). Centre d'histoire ancienne de Besançon. Paris: Geuthner.

Jones, Alexander. 1994. The place of astronomy in Roman Egypt. In *The sciences in Greco-Roman society*, ed. Timothy D. Barnes, 25–54. Edmonton: Academic.

Jones, Alexander. 1999. Uses and users of astronomical commentaries in antiquity, 147–172.

Karivieri, Arja. 1994. The 'House of Proclus' on the Southern slope of the acropolis: A contribution. In *Post-Herulian Athens: Aspects of life and culture in Athens A.D. 267–529,* ed. Paavo Castrén. Helsinki: Suomen Ateenan-instituutin säätiö. 115–139.
Kaster, Robert A. 1983. Notes on "Primary" and "Secondary" schools in Late Antiquity. *Transactions of the American Philological Association* 113: 323–346.
Kaster, Robert A. 1988. *Guardians of language: The grammarian and society in Late Antiquity*. Berkeley: University of California Press.
Knorr, Wilbur R. 1989. *Textual studies in ancient and medieval geometry*. Boston: Birkhaüser.
Lernoult, Alain (ed.). 2010. *Etudes sur le commentaire de Proclus au premier livre des Eléments d'Euclide*. Lille: Presses Universitaires du Septentrion.
Lloyd, Geoffrey E.R. 1996. *Adversaries and authorities. Investigations into ancient Greek and Chinese science*. Cambridge: Cambridge University Press.
Marrou, Henri-Irénée. 1965. *Histoire de l'éducation dans l'Antiquité*, 6th ed. Paris: Seuil.
Michalowski, Piotr. 1987. Charisma and control: On continuity and change in early Mesopotamian bureaucratic systems. In *The organization of power, aspects of bureaucracy in the near east*, ed. McGuire Gibson and Robert D. Biggs, 55–68. Chicago: The Oriental Institute of the University of Chicago.
Neugebauer, Otto, and Abraham J. Sachs. 1945. *Mathematical cuneiform texts*, vol. 29. New Haven: American Oriental Series/American Schools of Oriental Research.
Ossendrijver, Mathieu. 2012. *Babylonian mathematical astronomy: Procedure texts, sources and studies in the history of mathematics and physical sciences*. New York: Springer.
Proust, Christine. 2007. *Tablettes mathématiques de Nippur*, vol. XVIII. Istanbul: IFEA, De Boccard.
Proust, Christine. 2012. Reading colophons from Mesopotamian clay-tablets dealing with mathematics. *NTM Zeitschrift für Geschichte der Wissenschaften, Technik und Medizin* 20: 123–156.
Reynolds, Leighton D., and Nigel G. Wilson. 1968. *Scribes and scholars: A guide to the transmission of Greek and Latin literature*. London: Oxford University Press.
Robson, Eleanor. 2001. The tablet house: A scribal school in old Babylonian Nippur. *Revue d'Assyriologie* 95: 39–66.
Robson, Eleanor. 2002. More than metrology: Mathematics education in an old Babylonian scribal school. In *Under one sky. Astronomy and mathematics in the ancient near east*, ed. John M. Steele and Annette Imhausen. Münster: Ugarit-Verlag.
Robson, Eleanor. 2008. *Mathematics in ancient Iraq: A social history*. Princeton: Princeton University Press.
Rochberg, Francesca. 2004. *The heavenly writing: Divination, horoscopy, and astronomy in Mesopotamian culture*. Cambridge: Cambridge University Press.
Sidoli, Nathan. 2004. *Ptolemy's mathematical approach: Applied mathematics in the second century*. Ph.D. thesis. Available online on the site of the University of Toronto (consulted 19 Jan 2012).
Sluiter, Ineke. 1999. Commentaries and the didactic tradition. In *Commentaries—Kommentare*, Aporemata, Bd. 4, ed. Glenn Most, 173–205. Göttingen: Vandenhoeck und Ruprecht.
Steele, John M. 2011. Astronomy and culture in late Babylonian Uruk. *International Astronomical Union*, 331–342.
Strouhal, Eugen. 1992. *Life of the ancient Egyptians*. Norman: University of Oklahoma Press.
Szabat, Elżbieta. 2007. Teachers in the Eastern Roman Empire, In Derda *et alii*, 177–208.
Tanret, Michel. 2002. *Per aspera ad astra. L'apprentissage du cunéiforme à Sippar-Amnanum pendant la période paléo-babylonienne tardive*, vol. I/2. Gand: Université de Gand.
Taub, Liba C. 1993. *Ptolemy's universe, the natural philosophical and ethical foundations of Ptolemy's astronomy*. Chicago: Open Court.
Tihon, Anne. 1992. Propos sur l'édition de textes astronomiques grecs des IVè et Vè siècles de notre ère. In *Les problèmes posés par l'édition critique des textes anciens et médiévaux*, ed. Jacqueline Hamesse, 113–137. Louvain-la-Neuve: Brepols.
Tinney, Steve. 1999. On the curricular setting of Sumerian literature. *Iraq* 61: 159–172.
Toneatto, Lucio. 1994–1995. *Codices Artis Mensoriæ. I Manoscritti degli Antichi Opuscoli Latini d'Agrimensura (V-XIX SEC.).* Vol. I: Tradizione diretta, Il Medioevo. II: Tradizione diretta, l'età moderna. III: Tradizione indiretta. Spoleto: Centro Italiano di Studi sull'Alto Medioevo.
Vanstiphout, Herman L.J. 1996. On the old Babylonian Eduba curriculum. In *Centres of learning: Learning and location in the pre-modern Europe and the near east*, ed. Jan W. Drijvers and Alasdair A. MacDonald. Leiden/New York/Köln: E.J. Brill.
Veldhuis, Niek. 1997. Elementary education at Nippur, the lists of trees and wooden objects. Ph.D. dissertation thesis. University of Groningen, Groningen.
Vitrac, Bernard. 1990. *Euclide: Les éléments. 1: Introduction. Livres I à IV: Géométrie plane*. French transl. with comm. and notes. Gen. intr. by Maurice Caveing. Paris: PUF.
Vitrac, Bernard. 2005. Les classifications des sciences mathématiques en Grèce ancienne. *Archives de Philosophie* 68: 269–301.
Watts, Edward J. 2006. *City and school in late antique Athens and Alexandria*. Berkeley: Los Angeles.