Alexander Karp · Gert Schubring
Editors

Handbook on the History of Mathematics Education

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Introduction

This Handbook strives to present the history of teaching and learning mathematics over the various epochs and civilizations, cultures, and countries. This comprehensive approach became possible only now because of the recent development of research. The aim of this Handbook is, on the one hand, to reflect the current state of the history of mathematics education and to make more accessible the results of existing research, and on the other hand, thereby to facilitate the further development of this field, drawing attention to that which has not yet been studied.

Our conviction is that concern about the future of mathematics education is impossible without an understanding of what is going on in the present, which in turn is impossible without an understanding (and consequently the study) of the past. We would like to instill this conviction in our readers as well, and not limit ourselves merely to providing them with a reference book that contains needed information (although this aim is important for us, too).

The manner in which mathematics education developed is important for today's mathematics educators, but it is likewise important for researchers of the history of education, of which mathematics education is a part. Our view is that, even more broadly, for researchers of cultural history and even of social history as a whole, an appreciation of the historical development of mathematics education will be useful also.

It should be emphasized that at present our knowledge (and hence our understanding) is limited. In certain cases, the spread of knowledge is hampered by linguistic barriers – important and substantive studies remain unread even by those who would be receptive to them and find them interesting. In other cases, there have simply been no studies – it would be no mistake to say that educational documents lying in archives practically in any country have not been sufficiently researched.

We should say at once that, in discussing the history of mathematics teaching and learning, we are mainly concentrating on that which in recent centuries has been called pre-university education. This terminology is naturally not applicable, say, to Antiquity, or even to later periods, but nonetheless it may be said that education which in one way or another corresponds to "higher level education" is usually discussed in this Handbook only in order to gain a better general picture of education at preceding levels (for example, when the discussion concerns mathematics teacher education). Being even more specific, we can say that typically Handbook chapters focus mainly on secondary education (or its equivalents) rather than on primary.

Mathematics education is a complex phenomenon, and therefore an attempt has been made in this Handbook to investigate it from different angles. Consequently, the Handbook is divided into parts. Part I is devoted to the history of mathematics education as a scientific field, with a discussion of its scholarly literature and methodologies. Parts II, III and IV are organized along chronological and geographical lines, containing analysis of mathematics education during different periods and in different regions. Part V is devoted to the study of various mathematical subjects and teaching practices.

Finally, Part VI examines processes that are common to different countries – the emergence of international cooperation, the introduction of technology, and the spread of teacher preparation.

Different chapters in the Handbook are written from different viewpoints and reflect different existing approaches to historical research in mathematics education. Nor did the editors strive to achieve complete uniformity in more technical aspects of the text – in particular, in the spelling of non-English words, as different systems of transliteration have developed for historical reasons, and not infrequently even the same name in different cases is pronounced in English in different ways. Attempts to achieve a uniform approach might have rendered the text incomprehensible.

It should be noted that certain chapters consist of subchapters that were written separately from one another. In such cases, the authors of these subchapters are indicated in the text. All of the authors of such subchapters are also listed at the beginning of the chapter and in the table of contents in alphabetical order. Other chapters were worked on jointly by their authors, in such cases the authors of chapters' parts are not indicated.

In conclusion, it should be reiterated that this book does not and cannot claim to be absolutely complete. Readers (as well as editors) may justifiably express regret that for one or another reason not enough has been said about one or another country or one or another phenomenon. If the recognition of such insufficiencies leads to new research, however, the editors will consider this an important achievement of the present volume.

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Part I
History and Methodology of the Field

Chapter 1
On Historiography of Teaching and Learning Mathematics

Gert Schubring

1 The Evolution of the Field

Research on the history of teaching and learning mathematics constitutes a field, on one hand, with a considerable tradition and, on the other hand, with only recent rapid developments. In fact, during the nineteenth century, numerous pertinent studies had already been published, the best known of which, so far, are from Germany. One focus of these publications was the history of mathematics teaching at particular schools. Perhaps the earliest of such studies dates from 1843, which assessed the evolution of the mathematics curriculum from the Enlightenment-minded reforms to the Prussian Gymnasium reforms, at the Gymnasium in Arnsberg, a town in Prussian Westphalia (Fisch 1843). Another focus of these studies was on teaching methods; in 1888, Jänicke published the continuously valuable study on methods of teaching arithmetic (Jänicke 1888).

The first monograph devoted to this area was, perhaps, the 1887 book by Siegmund Günther on the history of mathematics teaching in Germany during the Middle Ages (Günther 1887). This book was followed by a doctoral dissertation, also published as a book, on mathematics teaching in the German state of Saxony in the eighteenth and nineteenth centuries (Starke 1897).

Studies in the nineteenth century did not remain restricted to Germany; another study in book format was by Florian Cajori describing the history of mathematics education in the USA up to the end of the nineteenth century, including a detailed report on the situation at the end of that period based on a large questionnaire survey (Cajori 1890). A book by Christensen on the history of mathematics in Denmark and Norway in the eighteenth century also dealt with the history of mathematics education in these countries (Christensen 1895). Analogously, a book on the history of mathematics in Finland until about 1800 also considered mathematics teaching (Dahlin 1897).

At the beginning of the twentieth century, Grosse's book on arithmetic textbooks since the sixteenth century inaugurated the subarea of research on schoolbooks (Grosse 1901). The following years, until World War I, were marked by a rather dynamic development. To a considerable degree, the new dynamic arose from the initiatives of Felix Klein to reform mathematics teaching and, particularly from his agenda for an international reform movement, which he promoted as president of IMUK (*Internationale Mathematische Unterrichts-Kommission*), founded in 1908.[1] Klein himself wrote an account of the last 100 years of teaching mathematics in Germany (Klein 1904). Moreover,

[1] Its French name and acronym then was CIEM, *Commission Internationale de l'Enseignement Mathématique*. It became ICMI, in English, only during the 1950s.

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in the series of German reports for IMUK on the state of mathematics teaching in Germany, several pertinent monographs were published: Schimmack's study on the evolution of the reform movement (Schimmack 1911), Lorey's study on the training of mathematics teachers (Lorey 1911), and his later study on the mathematics taught at nineteenth-century universities (Lorey 1916). Moreover, the numerous reports on the actual state of mathematics teaching in the various German states today constitute excellent sources for research on this period, particularly Lietzmann's well-documented books on the teaching of geometry and of arithmetic (Lietzmann 1912a, b).

Similarly, the reports for IMUK from the other participating countries,[2] although in general not historical, provide valuable sources today for studies on this period, with one remarkable exception: the series of British reports were so highly specialized and written without any look beyond their perimeter that they presented no comprehensible picture of the state of mathematics teaching there. As a result, Klein sent an "observer" to the British Isles who succeeded in fact to analyze and illuminate the multifaceted system so effectively that his book remains valuable to this day (Wolff 1913). Another revealing book on England, dedicated to the school history of various disciplines, includes 150 pages on arithmetic, geometry, and astronomy teaching (Watson 1909).

Importantly, the first doctoral dissertations in mathematics education defended in the USA under the supervision of David Eugene Smith were devoted to the history of mathematics education (Jackson 1906; Walker Stamper 1909).

The period before World War I witnessed first attempts at generalizing studies. Pahl (1913) wrote a book on the history of teaching mathematics and the sciences in Germany. Even more ambitious was Timerding's approach to survey the history of mathematics teaching, from Egypt and the Greeks to the early twentieth century, with special emphasis on the teaching of mathematics in Germany during the nineteenth century (Timerding 1914).

The interwar period was not as rich in publications, but several in-depth studies were conducted. Most notable was Laumann's study on geometry teaching in the sixteenth century (Laumann 1918/19) and in particular a two-volume study on teaching mathematics in Germany from Charlemagne to the beginning of the nineteenth century (Grundel 1928, 1929). In addition was the book on Scottish history (Wilson 1935) and on teaching arithmetic in England (Yeldham 1936).

After World War II, studies intensified and gradually covered more countries. The first book of this era appeared in 1945, on geometry teaching in Finland (Nykänen 1945). In 1947 and 1948, A. P. Jushkevich published a series of papers on mathematics teaching in Russia, from the seventeenth to the nineteenth centuries (Jushkevich 1947–1948). A book by Prudnikov on Russian mathematics educators in the eighteenth and nineteenth centuries followed in 1956 (Prudnikov 1956).

An important impact was made when the NCTM, the US mathematics teachers association, published two volumes in 1970, thus giving institutional promotion to this field of study: the first volume was a reader with selections from major documents spanning the period 1831–1959 in the USA (Bidwell and Clason 1970), and the other was the NCTM Yearbook for 1970, edited by Jones and Coxford, with research studies on primary and secondary education in the USA and Canada (Jones and Coxford 1970).

From the 1980s, one notes a fairly continuous flow of publications, on ever more countries, now representing various trends. On one hand, there are more specialized studies for a given country, such as Howson's book on mathematics education in England (Howson 1982). On the other hand, there are new attempts to investigate international history, such as Schubring's (1984) study of the history from Antiquity and of various civilizations according to theoretical categories until Modern Times and Miorim's (1998) book chronicling the development from Antiquity up until Modern Times (focused specifically on Brazil). The third and latest trends are methodologically reflected approaches to go beyond the surface of administrative facts and decisions, with the objective of unraveling the reality

[2] See the list of all reports elaborated for IMUK: *L'Enseignement Mathématique*, vol. 21, 1920, 319–342.

of teaching in school practice. These approaches rely on extensive archival research and on interdisciplinary methodology. The first such study by Schubring (1983[1], 1991[2]) analyzed the reality of the emerging profession of mathematics teachers in Prussia. A following study was done by Siegbert Schmidt who analyzed the reality of teacher training for primary schools in a specific region of Prussia (Schmidt 1991). The approach then was applied to the Netherlands where mathematics turned during the nineteenth century from an unwanted intruder in classical secondary schools to a major discipline in a new type of school (Smid 1997).

More recently, there has been substantial growth in interest in the national histories of mathematics teaching and learning as well. Among other publications are the histories of mathematics education in the USA (Stanic and Kilpatrick 2003) and Russia (Karp and Vogeli 2010, 2011). Similarly, while the traditional focus of research had been secondary schools (and their functional equivalents), due mainly to the existence of specialized teachers for this subject, studies on primary schooling became more numerous.

Thus far, all these activities were mainly individual initiatives. However, this changed decisively with ICME 10, held in Copenhagen in 2004, when the field became internationally institutionalized the first time, as with the Topic Study Group 29 on the History of Teaching and Learning Mathematics. In preparing this symposium, the first international bibliography of relevant publications became elaborated, thanks to the cooperation of researchers from many countries. The bibliography is accessible online (Bibliography: 2004). Not only were the contributions of TSG 29 published as a special issue of the journal *Paedagogica Historica* (Schubring 2006), but another important sequel to this internationalization occurred: the foundation of the first journal dedicated to this field of research, the *International Journal for the History of Mathematics Education*, published since 2006 and now in its eighth volume.

While a number of activities are ongoing at national levels – for example, the Dutch group W*erkgroep* G*eschiedenis van het* R*eken*-W*iskunde*O*nderwijs* organizes yearly meetings, the Italian *Associazione Subalpina Mathesis* is very active in searching documents and making them available online, and the *Japan Society for Historical Studies in Mathematics Education*, founded in 2000, organizing yearly meetings and publishing the *Journal for Historical Studies in Mathematics Education* – these activities have been complemented by international meetings since several years. The first *International Conference on On-Going Research in the History of Mathematics Education* was organized in 2009 at Garðabær, near Reykjavik (Iceland); the *Second International Conference on the History of Mathematics Education* followed in 2011, in Lisbon (Portugal); at present, the third conference is being prepared for September 2013 in Uppsala (Sweden). Proceedings of the first two conferences were published (Bjarnadottir et al. 2009, 2012). In addition to these international meetings, there are more regional meetings as well. For example, the *I Congresso Ibero-Americano de História da Educação Matemática* was held in 2011 in Portugal, with participants from Spain, Portugal, and various Latin-American countries; at present, its second meeting is being prepared for Mexico.

2 Connections

Research on the history of mathematics teaching and learning is not self-contained nor is it autonomous; rather, it constitutes a profoundly interdisciplinary activity. It has major intersections at the very least with history, history of education, sociology, and history of mathematics.

History proves to be most relevant with regard to methodology of research; one has to rely on its methods for all issues of searching for sources and interpreting and evaluating them, for determining the historical contexts of given situations, for developing methods to elaborate on biographies, and in particular for applying the more recent method of prosopography, that is, collective biographies of relevant groups of persons – which proved fruitful when studying professional profiles of mathematics teachers (see Shapin and Thackray 1974).

The history of education is of the utmost importance for assessing the general development of the educational system(s) in which mathematics education occurs. Clearly, there is an interrelation between the two areas since results from the history of mathematics education can enrich and concretize the history of education or even indicate lacunae of research in the history of education. In general, the history of education is not specific for the contents of teaching, but a subarea has developed since the 1980s called the history of school disciplines and is particularly prominent in France. One typical example is Chervel's (1988) paper, which launched this conception. Its basic notion is "school culture"; school disciplines are studied as expressions of school culture and exclusively in relation with society. Typically, disciplines studied within this conception are humanities, mother language, history and geography, and religion. The focus is on socializing effects, on enculturation. School disciplines are regarded as creations of society; relations to their scientific disciplines are not approached. According to this conception, school disciplines are characterized by autonomies, in particular by autonomy regarding other school disciplines (Chervel 1988, p. 73). However, the history of mathematics education shows that the status and the concrete forms of teaching mathematics decisively depend upon its position within the complex of all subjects to be taught within a given school type.

Usually, the history of education is confined to history within a given civilization, country, or nation, as are in general studies on the history of mathematics education. However, since the quality of research results for a given nation is enhanced when reflecting on the generalizable patterns of such specific cases, comparative education provides essential tools for widening the perspectives to international comparisons. Moreover, such comparative issues from mathematics education might enrich research within comparative education, revealing critical open questions there.

Given the essentially social character of the evolution of school systems and in particular of the value attributed to specific school disciplines, it is evident that sociology provides numerous approaches and conceptions for researching their history. This applies especially to mathematics, which has always functioned at the crossroads between general education and professional training, thus relating its teaching history even to the labor market and thus widening the sociological dimensions implied.

The history of mathematics and the history of mathematics education have experienced considerably extensive intersections as well. For the early periods until the Middle Ages, one can even say that both histories deal with nearly the same documents and personalities yet are studied with different approaches. A recent revealing discussion on the relation between mathematics and its teaching is pertinent for this intersection: it has been argued that this relation can no longer be understood as traditionally being between production and reproduction but that teaching systematically constitutes one of the basic factors for new developments within mathematics (Schubring 2001).

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Chapter 2
The History of Mathematics Education: Developing a Research Methodology

Alexander Karp

The aim of this chapter is to discuss answers to two basic questions: what does the history of mathematics education investigate, and how does it carry out its investigation? It is not enough to say, following and paraphrasing Leopold von Ranke, that the goal is to determine how mathematics education really happened: first, one must determine what is meant by the history of mathematics education (in other words, one must determine what pertains to the history of mathematics education) and, most importantly, what it means to determine something and how exactly this can be done.

Schoenfeld (2007) formulated the following three questions, which may be asked of any study:

- Why should one believe what the author says? (the issue of trustworthiness)
- What situations or contexts does the research really apply to? (the issue of generality; or scope)
- Why should one care? (the issue of importance) (p. 81)

These questions must be answered for historical studies, too. Possible answers to these questions (along with possible criticism of them) will be addressed in the discussion below.

The history of mathematics education is a branch of research that is still only taking shape, and consequently its methodology, too, is still only in its formative stage. Studies directly focused on the methodology of the history of mathematics education are comparatively few, and below they will be discussed in sections that pertain to their subject matter. At the same time, this comparatively new field inevitably inherits the techniques and methods developed both in mathematics education and in history (including the history of science).

In history, methodological discussions have been going on for centuries if not millennia, and at first glance they may seem quite distant from anything that one might argue about in the history of mathematics education, a field in which everything appears more modest and concrete. On the other hand, research methodologies in mathematics education have developed to a very considerable extent under the influence of psychological research and very frequently have been connected with quantitative methods, whose use in historical studies is often problematic, if only because much information has simply not survived. The use of qualitative research methodologies, which have become more popular in recent decades, in mathematics education likewise needs to be made more precise when research begins to address periods that lie hundreds of years in the past. Nonetheless, one cannot speak about the methodology of the history of mathematics education without touching on the two fields just mentioned, if only because it is from these fields that future researchers in the history of mathematics education usually emerge.

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Let us say at once that we see the history of mathematics education as a part of social history, which becomes meaningful only when it includes social analysis and examines what happened in mathematics education in connection with the processes that were taking place in society around it. The development of mathematics teaching is an element of the social subsystem of schools, which, in turn, is a part of the broader subsystem of education, which interacts with other social subsystems such as the labor market (Schubring 2006). Mathematics education has appeared and developed not in isolation, but in response to various social needs.

These needs cannot be understood in a primitive manner, for example, as needs of a purely economic nature. During certain periods, it was in fact the need to prepare a sufficient number of engineers, technicians, and scientists capable of developing new technological applications and organizing their production that was the main stimulus for the development of mathematics education. During other periods, by contrast, the connection with economic development was less simple – it is easy to cite instances when attention to mathematics education was predicated on the need to prepare users of already existing technologies or simply administrators. The development of mathematics education has been influenced by political views as well as philosophical and religious ones. Identifying and elucidating connections with the general course of the development of society is a crucial problem for the historian of mathematics education.

At the same time, historical analysis is often based on the analysis of texts, and since the texts that must be analyzed in this field are most often of a mathematical-pedagogical nature, it is impossible to analyze them without a knowledge and understanding of the relevant mathematical-pedagogical issues (Karp 2011). This two-sided nature of the field under discussion is precisely what makes it interesting, we would argue, although also difficult for research.

1 Historical Research: Who Cares?

The history of mathematics education is sometimes conceived of as a kind of preamble to current research. Demonstrating in two or three paragraphs that the problems that are of interest to them have occupied people for a long time and in this way showing the importance of their own research, some authors seem to say: “well, that was all history, and now to business” – and then proceed to that which really interests them, which is to say their own lucubrations. The thoughtful reader, however, might pause to wonder why a given problem was recognized when it was recognized rather than earlier or later and what happened with previous attempts to solve this problem and even why no one had previously thought of proposing what is being proposed now by this or that author. In general, the relationship between history and “business” – that is, contemporary problems – turns out to be very complicated.

It would be naïve, of course, to rely on the old maxim “*historia est magistra vitae”* and expect that answers to questions arising today can always be taken directly from history (contrary to those who say, as it were, “let’s take a textbook from the last century, or even better, from the century before last, and everything will work out” – see, e.g., Kostenko n.d.). Such a search for direct guidance in the past is akin to the widespread faith in direct borrowings from abroad, a faith that virtually everyone appears to denounce, justly pointing to cultural and social differences, but which now here, now there, leads educators to use textbooks from Singapore or proudly announce that in one or another American school, everything is done as it is, say, in China.

History, however, knows numerous instances in which foreign textbooks were used to teach students – and quite successfully – for example, British and French textbooks in the United States. Why, then, did this work in some cases and in others seem laughable and naïve? Attempts to formulate general and permanent rules concerning what can and what cannot be done in mathematics education come to naught: everything depends on the specific nature of the case at hand, on the specific process of which the phenomenon under investigation constitutes a part.

Schubring (1987) justly wrote about the deficiencies of the widespread methodology in which one or another textbook is considered in isolation from everything else or is only nominally compared with certain other textbooks. Textbooks are written and used in the context of a specific system – the textbook's author consciously or unconsciously expects that teachers and students will behave in a certain manner, consciously or unconsciously relies on certain habits that influence how educational texts are read; on certain traditions and tastes that influence how assignments are used; on certain life plans on the part of the students and a certain educational background on the part of the teachers; on certain roles that will be played by parents, the school administration, and the higher educational authorities; on a certain system of testing both teachers and students; and so on and so forth. All of these habits and traditions accumulate and change (as they undoubtedly do change) gradually. Attempts to introduce something extraneous and thus to implement instantaneous transformations are, of course, different from using materials that are close to one another in spirit and style or even materials that are new for a country but used in a context in which not textbooks alone but the whole educational system is brought over from elsewhere, sometimes with teachers included. Steps that superficially seem to resemble each other turn out to be completely different when their historical context is analyzed.

And here we come to the main point, which is that one can achieve an understanding of what is happening only on the basis of studying and understanding the processes of which it is a part, which is precisely the concern of history. The answer to Schoenfeld's third question – "Why should one care?" – can be briefly formulated for historical research as a whole in the following way: "In order to understand what is going on." And this answer pertains by no means only to the use of foreign textbooks, which was mentioned above. The complexity of historical research, however, consists in the fact that the same event forms part of different processes and may itself be considered from different angles.

In our view, it is not only mathematics educators who would do well to care about the history of mathematics education. Contemporary historians try to understand how people lived in the past. Attempts to reconstruct the psychology of the people of another time (Huizinga 1996) or to study their everyday life (Braudel 1974) have long ago become classics of historical research. Education in general and education in mathematics in particular are a part of this daily life, and it was not for nothing that already Weber (2003) drew connections between the development of mathematics (and thus also mathematics education) and the general processes of "disenchantment" taking place during a certain period. If we try to imagine a group of seventeenth-century French schoolchildren sitting on the floor with their teacher and discussing how to fill in a table of expenditures incurred on a journey from one city to another and how to construct a pie chart based on this table, we will immediately sense the impossibility of such a thing ever having taken place – it will seem as impossible to us, perhaps, as picturing the same group of students using iPads, although by contrast with iPads lessons in arithmetic already existed in the seventeenth century, of course, and people did have to travel from city to city. The goals of teaching, the place of mathematics education in general education, anything that may be meant by the words "style of mathematics education," reveal a great deal not only about mathematics education in and of itself but also about the period under investigation as a whole.

2 The Subject of Study

Schubring (1988) once noted that

> [t]raditional historiography used to focus on administrative policy and its operationalisation by decrees, time tables and the weekly portion of mathematics instruction and by the syllabus but did not bother much about real school life and about epistemological dimensions of school knowledge. (p. 1)

Indeed, today when we speak about curricula, we distinguish between professed, intended, and enacted or implemented curricula (Stein et al. 2007); we discuss students' beliefs, with the understanding that this is one of the key components of the learning process (Leder et al. 2002); we conduct detailed analyses of various models of teacher behavior and compositions of lesson plans; and so on. In historical studies, such approaches are used, at least at present, to a far more modest degree. Justly noting that the administrative side of education is much easier to study, which is one of the reasons for its dominance in the history of mathematics education, Schubring (1988) pointed to yet another aspect of the problem: the lack of worked-out theoretical categories for analysis, in other words, the absence of clear questions that one must try to answer. It may be argued that the limitations of historical studies in mathematics education also stem from the limitations of research in mathematics education in general: many of today's problems are understood too statically and one-sidedly.

We will limit ourselves to one example. As has already been said, the role of beliefs and attitudes in mathematics education is today universally recognized. Quite often one hears it is said that, for example, the characteristic trait of Western education and specifically British and American education is the striving to make learning pleasurable (Leung 2001; Leung et al. 2006). It is natural to ask, even if this is the case, whether it has always been the case. Could "making learning pleasurable" have possibly been the goal of the sadistic teachers whose likenesses have been preserved for us in classic British literature? The reality is far more complex: in Britain itself, several different traditions in education have existed and continued to exist in parallel – traditions that have developed in different ways and that have at different times been more or less influential for different reasons. The tendency toward simplification prevalent in studies devoted to the present day – the extraction of some single aspect or feature from its whole framework and context – influences historical studies also: a great deal of research begins to seem like useless digging around in details of a kind to which no one gives any thought even when analyzing contemporary or recent times, for which the collection of the relevant data is far easier than it is for long-gone epochs.

By contrast, by transferring into the past today's conception of mathematics education as a complex phenomenon, one that is by no means reducible to a list of topics found in textbooks or even to a description of lesson formats recommended by someone or others, we see the multifacetedness of our objects of study in the past also. Effectively any topic of contemporary research (for instance, any topic mentioned in the name of a topic study group at international congresses on mathematics education) may be studied from a historical point of view. The history of the development and formation of any school mathematics subject or of teacher education or, on the other hand, the history of the development and change of beliefs and attitudes toward mathematics and its teaching, and many other topics can and should be subjects of study. Moreover, since the processes through which education developed were by no means necessarily identical in different countries, specific national characteristics in teaching (beginning from the time when such specific national characteristics first emerged) are of considerable interest, as are interactions between different countries and transmission between different cultures, about which not a little has already been said above.

Following mainly Schubring (1988), let us list several types of crosscutting studies that might be relevant for different times and different countries:

- *The state of mathematics within general education for all and within professional education.* The role of mathematics education in different societies has varied quite widely. Today, on the wall of virtually every mathematics classroom in Russia, one can read the famous words of Lomonosov (1711–1765): "Mathematics must be studied if only because it puts the mind in order." However, in Lomonosov's time, mathematics in Russia was studied not at all in order to put the mind in order, but first and foremost for practical needs. On the other hand, the Western European tradition of liberal arts allocated a place for mathematics in a system constructed – at least in principle – for the school, not for life. The development and interaction of different traditions is quite relevant for an understanding of different processes taking place in mathematics education.

- *Divisions, interactions, and influences between different stages in education, including secondary and elementary and secondary and higher education.* Today's system, in which a university student, before enrolling in the university, has gone through secondary and elementary schools along with others who did not go on to a university, is the product of a comparatively recent past. Differences between secondary and higher education have not always been as clear as they are today (and even today, they are often easier to trace organizationally, than in terms of curricula). Elementary schools, on the other hand, often made no provisions at all for the continuation of education beyond them. A certain degree of interaction, balance, and even mutual influence nonetheless existed among different institutions.
- *The organization of the instructional process, teaching practices, the role of the teacher, and the function of textbooks.* It would not be wrong to say that historical studies in these topics in many countries are only beginning. The identification of the specific character of the development of mathematics as an educational subject against the background of the development of education as a whole needs to continue.
- *The professional role of the mathematics teacher, teacher education, and its influence on the instructional process.* The teacher of mathematics in a school, a familiar figure today, also appeared historically not that long ago, and the appearance of this figure and the changes in the professional life of the teacher, including the appearance of a special mathematics teacher education, are all important topics for research.
- *The relation between scientific knowledge and school knowledge.* The content of mathematics education, including the mathematical form in which it is presented, despite widespread views to the contrary, has by no means been absolutely stable. How it changed and what roles have been played by conceptions of mathematics as a science and by conceptions of its role as a science are all topics that require further study.
- *Local, national, and international in mathematics education*. The development of mathematics education, while reflecting the local context in which it takes place, is at the same time subject to powerful influences from international processes. Such influences are evident and regularly discussed today, but they existed in the past as well, as has already been pointed out. Meanwhile, the manner in which various national and regional systems of mathematics education interacted with one another in the past has not yet been studied in the majority of cases (Schubring 2009).

The topics listed above, which pertain to the place of mathematics education, interactions among its various parts, the contents of education, the nature of instruction, and those who implement this instruction, are important for characterizing the state and development of mathematics education. They do not exhaust all possible research topics, of course, and each of them furthermore contains numerous subtopics while being in itself connected to broader topics, such as *mathematics education and religion; the cultural determination of school knowledge; the contribution of mathematics education, and specifically textbooks, to the social history of ideas; mathematics education and political movements;* and so on.

3 The Theoretical and the Descriptive

Studies in topics listed above and others may be carried out in different ways. Going back to Schubring's (1988) proposition concerning the lack of worked-out theoretical categories, we would argue that theorization in general is sometimes seen as being opposed to dispassionate research and historical generalization and even the use of general concepts – whether Marxist socioeconomic formations or the *mentalité* of the Annales school – as a distortion of reality. It is, indeed, not difficult to cite instances in which historical theory did not grow out of facts, but preceded them, and the expert scientist was invited only in order to find facts that could confirm one or another set of propositions from, say, the famous Soviet *Short Course in the History of the Communist Party (Bolsheviks)* or some other collection of wise remarks by a political leader (the history of mathematics education, naturally, does not contain as many examples of this kind as does political history, but they also exist and will be addressed later). In opposition to this, there has been a tendency to avoid theory and to present "just the facts."

Without question, the collection and publication of existing evidence, that is, of surviving documents, is undoubtedly useful, and one can only welcome, say, such collections of documents on administrative history as D'Enfert et al. (2003). At the same time, as was justly remarked by Andrey Kiselev, Russian author of mathematics textbooks who was himself involved in politics, "not everything by which the common man lives is based only on laws, not everything is prescribed in them" (cited in Karp 2002, p. 15). By limiting ourselves to the decrees of the central government or even if we include in our studies the directives of local governments, we automatically paint an incomplete picture.

In most cases, however, it is impossible to present all of the facts: sometimes a great quantity of materials has survived (official memoranda from a large school district alone can fill up many volumes), and sometimes, conversely, the most important information has not reached us. Conscientious researchers take into account the greatest possible number of the materials that are known to them and seem relevant to them, but for purposes of publication, they almost always make a selection from them (this is the case in all, not just historical, research). Writing about the methodology of all research in mathematics education, Schoenfeld notes: "These acts of selection/rejection are consequential for the subsequent representation and analysis of those data" (p. 71).

Below, we will have an opportunity to discuss which incomplete presentations of data seem possible in historical research and which do not seem possible. Here, we will merely state once again that, in any case, it is still impossible to make do without general theoretical positions. They are expressed both in the selection of materials and in their presentation. "Theoretical" studies, which represent new conceptions and new approaches toward describing and understanding the changes that have taken place, are accordingly no worse (although not necessarily any better) than "descriptive" ones, whose authors do not explicitly state what it was that led them to give some particular description and what conclusions they reached on the basis of this description or have not yet found an adequate form in which to express their understanding.

At this point, however, it must be said that the very notion of process and evolution is understood in historical literature in various different ways. Bayly (2011) writes:

> Late twentieth century postmodernist scholars rejected the whole idea of the historical evolution of forms, stressing instead the fragment, the unique experience, and arguing that far from an objective, evolving process, history was constituted by the discourses of the present. World history therefore became, following Foucault, the history of discourses of global powers. (p. 13)

As part of the ongoing process of change in the interests of historians (a process whose existence no one appears to deny, although one might also speak about it in terms of changing discourses), the notion of causation itself is rejected (Wong 2011) and not just some primitive conception of this notion, but the very existence of cause-effect connections between historical events. Indeed, the very notion of a fact becomes open to doubt, because historians allegedly see not facts themselves, but their reflections in the perceptions of various people, who inevitably perceive reality in different ways.[1]

Russian researcher Andrey Zaliznyak (2010) objects quite sharply to such views, writing that the "paradigm of postmodernism, initially enthusiastically perceived as a sign of new freedom, in reality now leads to many destructive consequences," causing people "gradually to forget how to draw rigid distinctions between the true and the false, the accurate and the inaccurate."

[1] In one respected American pedagogical journal, I have had occasion recently to read a teacher's disquisition about how she tried to open the eyes of 6-year-old children to the fact that one should not say that the three little pigs from the well-known story are good and the wolf is bad. In reality, they just have different perspectives. (In support of this proposition, it was pointed out how pleasant it can be to eat some good ham.)

What is undoubtedly true and what is false is more difficult to establish in history than it is in mathematics. Schoenfeld (2007) quotes Henry Pollak: "there are no theorems in mathematics education" (p. 92) – although even in mathematics the level of a proof's strictness has not always been the same and consequently could be criticized (Grabiner 1974). It is noteworthy that an orientation toward examining the "unique experience" is sometimes used to deny connections and dependencies and indeed all attempts at generalization in the spirit of mathematical reasoning, in which, as is known, one counterexample suffices to disprove a proposition.

Mathematics education by its very nature has to do with specific individuals and individual lives. The specific characteristics of individuals can sometimes contradict any observable tendency, which undoubtedly implies that no such tendency can be represented as an iron law that is always in effect. At the same time, neither can the existence of such a tendency and the possibility of a theoretical generalization be denied on the basis of one or even several counterexamples. Analyzing the biographies of outstanding Russian mathematicians who have received Fields Medals (Karp 2011), one can observe that very many among them were winners of high-level mathematics Olympiads and attended famous mathematical schools. This is not true for all of them, however. It would therefore be as wrong to claim that a Russian who did not attend such a school cannot become a major mathematician as it would be to claim that whether or not a person attended such a school has no importance.

In those cases in which insufficient information has survived, it will be more difficult to identify a tendency than in cases for which we have all the data. Indeed, even when we have all the data, establishing such important characteristics of education as the existence of restrictions for various categories of citizens, for example, often meets with objections: what kind of restrictions on education can there be, it is asked, when so many people have obtained it? Such facts are even more difficult to demonstrate when the discussion concerns the past. For example, even the relatively obvious fact that individuals of non-noble origins in Russia in the 1830s were restricted in obtaining an education in general and a mathematics education in particular can be argued with by citing examples of people who did obtain it. The lives of such individuals are undoubtedly worthy of study, and references to this or that set of laws are insufficient, if only because laws, generally speaking, are not necessarily obeyed,[2] but the fact that someone managed to overcome the restrictions does not mean that they did not exist. In general, identifying tendencies and generalizing them is quite a demanding task – which does not mean, however, that it ought not to be undertaken.

Generalization and theorization, however, must always take into account the complex nature of the object being studied – mathematics education – which is, on the one hand, truly international, but on the other hand, not even national, but regional and local. American researchers of contemporary mathematics education usually understand that observations made at a suburban school cannot be generalized for all cases. In historical studies, it is not easy to answer the question posed above – "What situations or contexts does the research really apply to?" – if only because first one must determine what factors constitute the situations and what defines the contexts, and this requires a deep immersion into the history and culture of the period in question; the difference between suburban and urban districts is today known to practically everyone, but certain differences between regions, obvious to contemporaries, may get lost, particularly if they are not economic in nature, but based on some other parameter that many people today are not familiar with.[3]

[2] In reality, analysis of the biographies of persons who did obtain an education shows that the authorities monitored adherence to the established order quite closely: when, for example, a graduate from St. Petersburg's Third Gymnasium was discovered to have been originally admitted to the school from among the low-level social strata without the requisite forms and permissions, the director was quite severely reprimanded (No author 1835).

[3] The Russian writer Alexander Herzen wrote: "The power of the governor in general grows in direct proportion to the distance from St. Petersburg, but it grows geometrically in gubernias where there is no gentry" (Herzen 1956, p. 237). The researcher of education in Novgorod and Vyatka during the 1830s must take these considerations into account.

Robson (2002) justly noted, when discussing the history of Babylonian mathematics (which is, as a matter of fact, practically inseparable for that time from the history of mathematics education), that the popular metaphor of Sherlock Holmes traveling to the ancient world and understanding what is going on around him – by using his wits and powers of observation – is completely wrong. One cannot get by on acumen alone; one has to understand the conditions in which one or another text was written, and only by relying on such an understanding can one formulate one's hypotheses. (Conan Doyle's Holmes, incidentally, kept an abundance of manuals and never denied the importance of specialized knowledge.)

Undoubtedly, one can imagine studies in which tables of contents from various textbooks are ingenuously analyzed and, for example, chapters and sections that were missing at a certain stage but appeared at later stages are identified. Moreover, such a study, unencumbered by considerations about the era in which these textbooks were used, can possess a certain value, but only, unfortunately, as a preliminary step. Subsequently, the researcher will need to try to understand why these changes took place, and at this point he or she will have no choice but to go beyond the perusal of tables of contents.

4 Sources: Their Identification, Analysis, and Interpretation

Contemporary readers who for one reason or another have never before given any thought to the question "What is the historical method?" will most likely look on the Internet and learn that the historical method consists first and foremost in working with primary sources, analyzing them, criticizing them, and interpreting them. This is true of studies in the history of mathematics education as well. It is another matter that both in history in general and in the history of mathematics education, the difficulty consists precisely in determining which sources may be used and how to use them.

Without even attempting to enter into questions of historiography here, let us recall that attitudes toward evidence from the past have varied from almost total trust to almost total distrust, and the change in these attitudes has not by any means always been steady and in the same direction: quite recently, studies that took everything in the sources they used at face value have appeared alongside of hypercritical works (probably the most vivid examples of the latter, actually lying beyond the bounds of scientific literature, are the works of the mathematician Anatoly Fomenko and his collaborators, which prove that Classical Antiquity never existed, that Pope Gregory VII and Jesus Christ were really the same person, etc., and that at one time people simply fabricated a great deal of counterfeit evidence about life in the past). The specific character of the history of mathematics education as a scholarly discipline is such that, due to a certain incompleteness in its development, already mentioned above, the question of what could be considered a source has also often been understood in a narrow way, so that the issue of critiquing and comparing sources often did not arise at all. Therefore, we must begin by discussing possible primary sources.

Again, textbooks and the Internet report that primary sources are usually divided into "relics" and "narratives." In our field, relics consist of what are also called "tools of mathematics education" (Kidwell et al. 2008). This includes manipulatives, blackboards, computers and calculators, models, and even textbooks. The appearance of such tools and their technological development are significant for the development of mathematics education (one might compare it to the influence of the development of musical instruments on the development of music, which was studied by Weber 1958). The technical parameters of the instruments used in schools themselves contain quite a bit of information; still, in discussing blackboards or models (textbooks will have to be discussed separately – their role is more complicated), we cannot limit ourselves to such parameters. Naturally, the fact that blackboards replaced slates in classrooms and the ways in which they came to be improved demonstrates that the nature of work in mathematics classes changed over time, but we will be able to get serious insights into the changes that occurred only if we make use of people's testimony – only if we make use of narratives.

This category of sources encompasses a great deal. Effectively any text that concerns mathematics education and, even more broadly, that concerns the life of a person involved in the development of mathematics education can under certain circumstances serve as a source in the history of mathematics education. Naturally, this includes official documents pertaining to mathematics education, for example, testing documents, complaints, or internal reviews of textbooks; but useful information can also be derived from strictly personal diaries and memoirs, newspapers and journals in which advertisements or announcements of various materials were published, novels that depict classes and teachers, poems and songs composed by students about their education, and much else besides (some examples are given in Karp 2008).

The notion that sources in the history of mathematics education can be listed once and for all is erroneous. Marc Bloch once wrote: "even those texts and archeological finds which seem the clearest and the most accommodating will speak only when they are properly questioned" (Bloch 2004, p. 53). The authors of various diaries would probably be surprised to learn that they have provided evidence about mathematics education, since they wrote about something completely different. Nonetheless, sometimes indirect evidence turns out to be substantive and indispensable. Therefore, a text may be useful within the framework of one approach and useless within the framework of another. Consequently, texts that had not been considered sources and had not been used in research previously may turn out to be useful in the future.

Researchers studying the work of the already mentioned Russian author of mathematics textbooks Andrey Kiselev may, generally speaking, limit themselves to analyzing how his famous geometry textbook is constructed, how it differs from the previously most popular Russian textbooks (e.g., the textbook by Davidov), and how it is connected with textbooks from other countries, above all French textbooks. Further research might include an analysis of the changes that took place from one edition to the next (in fact, memoir accounts have survived about Kiselev attending teachers' meetings and writing down the suggestions and comments of working teachers) and above all the changes introduced into the Soviet editions of the textbook, edited by N.A. Glagolev and published in enormous numbers. Finally, another further step might be connected with interpreting what happened with Kiselev's textbooks in the context of the changes taking place in mathematics education as a whole, which reflected the radical changes taking place in the country during the late 1920s and early 1930s.

For the first of these studies, researchers could limit themselves to Kiselev's textbook and several others, although, of course, it would be useful to examine reviews or transcripts of teachers' meetings, as well as surviving normative documents. The second study would necessarily have to include such materials. The third study would necessarily have to rely on a very broad range of materials, among which, for example, it would be very sensible to include prerevolutionary newspaper publications portraying Kiselev as a conservative political figure, consequently very far from the revolutionary ideology of 1917–1918, but fitting in much more among those who gained ascendancy under Stalin.

Analyzing and interpreting relevant documents means being able to read a cultural code. The cultural historian Yuri Lotman (1992) wrote about this notion, analyzing the comments of a French traveler who met Russians from different circles, accurately reported what happened and what was said, but drew absolutely mistaken conclusions, precisely due to a lack of understanding of the attendant circumstances. The French traveler saw and noted that various individuals chuckled at the czar but drew the conclusion that they were freethinkers, while they were on the contrary the czar's closest collaborators and precisely as such permitted themselves to smirk and to grumble. Historians of mathematics education (like all cultural historians) must connect their observations with a general culturological and social analysis of the era; a simple reproduction of the judgments expressed by various mathematics educators, various mission statements, or even official statistical data that does not take into account the circumstances of the time in which they were made can be misleading.[4]

This is all the more true because the meanings of terms used in the past, even if they continue to be employed, usually change. Today's mathematics educators who wish to promote problem solving do not necessarily mean by this term the same thing that was meant by their colleagues 30 years ago (and even today "problem solving" often means different things to different people), while differences between what is meant by "equality in education" today and what was meant by this phrase during the French Revolution are altogether glaring. The genuine meanings of words emerge in the course of analysis that combines the special and the general, sociohistorical.

Schubring (1987) suggested that researchers choose a basic "unit" for studies "where at least some of the relevant dimensions can already be seen in interaction." As one such unit, he proposed looking at the life of the individual mathematics educator, in which working with teachers, writing textbooks, and simply the ordinary life of a person in a given era are all intertwined. As another technique, he proposed studying different editions of the same textbook: by connecting the changes taking place in life and pedagogy as a whole with the changes introduced into different editions of a textbook, he argued, one can arrive at a better understanding of what has taken place.

The historian of mathematics education in general must as far as possible strive to achieve a kind of competition among different sides; one must always try to find sources representing different sides and presenting mathematics education in different ways. For example, complaints about lowered grades or investigations of conflicts between teachers make it possible to portray existing practices, prevailing values, and the outlines of the contradictions between them. Indeed, the comparison of sources constitutes the main technique for critiquing and analyzing them.

A source in the history of mathematics education can be as unreliable as any other historical source. The teacher of mathematics who cleans up mistakes on students' tests in order not to be scolded for poor instruction by a visiting inspector is, of course, far less ambitious than the author of the Pseudo-Isidorean Decretals. Nonetheless, in both cases historians have to deal with falsifications. Evidence about the teaching of mathematics is no more reliable than evidence about any other human activity – here, too, one can find deliberate lies, and here, too, one can find honest mistakes, confusions between what is desirable and what is real, and so on. Critical analysis is indispensable in such cases, too.

The trouble, however, consists in the fact that sources rarely come to us fully complete – for various reasons, a great deal turns out to be inaccessible for researchers. It is unlikely that one could give a universal solution for overcoming this incompleteness: everything depends on the question being examined. For example, when we examine memoirs about school years and the teaching of mathematics, we evidently have to do with authors who, of course, cannot be considered representative of all students of a given time – far from everyone who writes memoirs. It would therefore be incorrect automatically to consider the assessments expressed in surviving memoirs (concerning, say, the importance of various sections of the school curriculum or different school subjects), even if all surviving memoirs express the same views, to have been shared by all the students of the time. On the other hand, if all memoirs describe, for example, certain lesson formats or certain typical homework assignments, this can be regarded as substantial evidence of the fact that the memoirists' schools did indeed employ such lesson formats and homework assignments.

[4] The same Herzen relates how the German traveler Humboldt, who was accompanied by a Cossack, inquired of the latter about the temperature of the water in a spring that they had come to. The Cossack "put up a stony front and replied: 'whatever duty demands, Your Grace – and we are glad to do what we can,' since to himself he thought: 'No, sir, you won't put anything over on me'" (Herzen 1956, p. 126). And the question in this case concerned merely a measurement of temperature, something far more simple than any measurement of education.

In studying the history of mathematics education, just as in any other kind of historical research, one might be confronted with the question of whether it is admissible to use evidence drawn from sources that are not entirely reliable. Discussing this issue, the Russian historian Yakov Lurie (2011) wrote that only by putting together all the facts that pertain to a given source (such as its overall structure and its textual history), along with observations concerning the specific items in it, and thus developing a general preliminary picture of the source as a whole can one acquire a footing for assessing the veracity of particular facts. The historian noted that, instead of doing this, researchers quite often leap from observations contained in the source directly to assessing their likelihood (and in addition sometimes interpreting their likelihood as proof of the fact that what is described in the source actually took place). Lurie analyzes transcripts of interrogations of heretics in Western Europe and Russia, but methodologically similar situations arise in studies of far more peaceful circumstances.

For example, in a diary from 1892 that has survived in manuscript form, St. Petersburg resident Alexandra M. describes how she and her girlfriends called on Yakov Gurevich, the director of a gymnasium and the editor of an influential journal, in order to persuade him to give them copies of a mathematics examination paper before the examination was given, acquaintance with which would have naturally been useful to her brother. According to her, Gurevich heard the girls out, remaining well disposed (in particular, he led them away from some doormen who could have informed on them (M., Alexandra 1892)). Not a little surviving evidence points to the fact that indeed the pedagogues of that time did not take the secrecy of exams too seriously (e.g., Brushtein (1985) tells a story pretty similar to the one just mentioned). At the same time, an analysis of the text as a whole – which abounds in such phrases as "I coldly and somewhat derisively asked" or "we were as if drunk," as well as self-congratulatory observations by the author of the diary about her and her girlfriends' cleverness – suggests that in this particular instance the story recounted by the author nonetheless could be pure fantasy.

By casting doubt on the veracity of the story told in this diary about Gurevich, we by no means must reject the diary as a whole as a source: considered in the context of other sources, it confirms that in and of itself procuring copies of an examination paper in advance of the exam was a rather quotidian affair. An interpretation of a source that is grounded in a criticism of the source is not reducible to a paraphrase of that information in the source which criticism recognizes to be reliable but possess a far more complex character.

The analysis of a source's general characteristics, including its history, sometimes reveals significant facts in and of itself. Let us mention one more Russian example. An inspector's report from 1946, which has survived, tells of third-grade students who made a mistake in copying a text from the blackboard, omitting a word, and thereby making the copied phrase "politically harmful." Probably even more useful than the details of the inspector's report for understanding the system of education that existed at that time is the fact that this report ended up in the Leningrad City Committee of the Communist Party, which found time, while governing a city with a population of several million, for investigating the causes of the politically harmful behavior of third graders (Karp 2010).

The incompleteness of sources mentioned above often prevents researchers from carrying out quantitative analyses, which are in general so popular in mathematics education. At the same time, in certain (and even in many) cases, such analyses are in fact possible. For example, for a relatively large number of nineteenth-century secondary schools, teachers' journals have survived, and consequently, it is quite possible to determine quantitatively how grades in mathematics were given in various places, at various times, and sometimes even to follow the individual lives of good students and bad ones. Surviving examination statistics sometimes make it possible to discover typical mistakes made by students or to carry out other kinds of studies along the lines of those which are conducted using the results of today's examinations.

Quantitative methods in historical research, of course, are quite feasible and useful. Yet one ought not to contrast them with other kinds of methods and assume that quantitative methods are more reliable. Without qualitative analysis and comparisons with other facts, computations are unlikely to be meaningful (e.g., going back to the example of the analysis of school grades, it is important not only to examine their distribution but also to understand what value grades possessed in general and what value grades given for specific assignments possessed in particular).

In general, to repeat, the methods of the history of mathematics education are first and foremost historical methods. It is clear, however, that much greater mathematical preparedness is demanded of the researcher in the history of mathematics education than of the general historian: the texts studied are often mathematical texts.

Grattan-Guinness (2009) remarked that the history of mathematics is "too mathematical for historians and too historical for mathematicians." This is at least to a certain extent true of the history of mathematics education as well, although for some reason it is sometimes assumed that school-level mathematics is something that everyone knows. A superficial knowledge of it leads researchers to confine their analysis of mathematical texts – textbooks or examination materials – to superficial approaches, such as subdividing problems only into obvious categories, for example, problems that require proofs or word problems. A deeper analysis often makes it possible to see how problems become more or less difficult over time, the changes in the principles underlying their selection, and the influence of various sources or social forces. This last point may seem like an exaggeration: such a title as "The Social History of Quadratic Equations" sounds like a parody, since we are accustomed to think that quadratic equations are in no way connected to social life and that the solving of quadratic equations, by contrast, may serve as an example of a lack of social activism. This, however, is not the case: the authors of textbooks or exam problems write in response to certain demands, defining both the form and content of their assignments in a corresponding manner (Karp 2011). The historian of mathematics education must be able to see such connections and to command both historical and mathematical-methodological methods of analysis.

Summing up, and attempting, as on an exam, to give a one-sentence response to Schoenfeld's question: "Why should one believe what the author [of a historical study] says?" – we can say that the answer often comes down to the support (direct or indirect) found in different sources for the conclusions drawn by the author and to their general conformity with the existing understanding of what took place at a particular time. However, people by no means believe only that for which support may be found.

5 Myths in the History of Mathematics Education

The mythologies that arise on the basis of history are studied by many researchers. For example, in a recently published book, Margaret MacMillan (2010) lists different instances of the "abuse of history," in which invented or exaggerated historical facts have been used for political purposes, including in this last category all possible group interests as well. The history of mathematics education may seem too small a field to possess its own mythology: it is a discipline, one might say, that is not big enough for such britches. This, however, is not the case. Mathematics education lies at the intersection of a large number of politically important topics, an intersection large enough for people to seek to falsify its history.

Furthermore, a mythology is not always invented deliberately in someone's interests: sometimes it arises simply from a striving for simplification. The noble baron in Mark Twain's *A Connecticut Yankee in King Arthur's Court* (1889) cannot find the product of 9 and 6 on an exam, but obtains the position he desires because all of his ancestors were also noble barons. The difference between the positions of a baron and a weaver in the Middle Ages was undoubtedly enormous, but it would be wrong to project contemporary notions of discriminatory procedures into the past: barons did not take exams in mathematics (let alone together with weavers). Mark Twain deliberately modernized the past – no less in the given instance than when describing the use of a telegraph in King Arthur's court – but sometimes the past is seriously conceived of as being not very different from the present day.

The striving for simplification likewise reinforces the most widespread myths, which may be called "the myth of the blessed past" and "the myth of inevitable progress." These two myths are in a certain sense opposed to one another. The Russian mathematics educator Yuri Kolyagin (2001) opens his book with remarks about the rapid growth and improvement of Russian education during the years of Soviet rule (particularly during the Stalinist period) and its subsequent decline due in large part to the machinations of the CIA (on p. 7, Kolyagin reproduces excerpts from the so-called Dulles Declaration, a long-discredited forgery). The book also informs us, however, that even prior to Soviet rule, Russian schools were excellent and all was well with the world – but eventually troublemakers came along and ruined everything. This book, of course, represents a kind of extreme case, but the conviction that in the past everything was perfect and that people then started changing everything for some reason can be found in milder formulations in different countries. Above, we have already mentioned the naive belief that when old textbooks were being used, everyone knew and understood everything.

Examples demonstrating successes achieved in the past are indeed sometimes not difficult to find, as are situations in which something appears to us to be better than what it subsequently turned into. The creators and propagators of the myth, however, usually take an absolutist stance toward such examples, ignoring the existence of other aspects of what took place. The Hungarian educators Halmos and Varga (1978) once described the state of affairs in a school undergoing reforms as follows: "It goes without saying that the last four years of this general school could offer less to every pupil than what the first four years of the earlier eight-grade secondary school could give to a highly selected population of the same age" (p. 225). This sentence underscores the complexity of the processes taking place – some things improved, some things got worse. A given change for the worse may be obvious, but the process is far more complex than the creators of myths like to recognize. Even leaving aside Dulles, and without going into other details, let us say that a good prerevolutionary mathematics education in Russia was accessible to very few individuals, while mass-scale Soviet education developed against the background of the difficult predicament in which other school subjects found themselves, which in itself makes it impossible to characterize those years as a happy time even for school education.

If the myth about the good old days is often fueled by reactionary political views (while at the same time serving such views), then the myth of progress often arises from a transfer into education of what is observed in technology. Computers over the past 20 years have become thousands of times more effective; it is natural (although mistaken) to think about improvements, even if not such rapid ones, in other fields. Unfortunately, there are no grounds whatsoever for this view, just as there are no grounds to think that the teaching of mathematics over the course of centuries has invariably grown better.

Moreover, myths about improvement and deterioration usually presuppose the existence of some common, universal scale, using which one can determine, for all times, what is good and what is bad – which is obviously far from reality. But the challenge for the history of mathematics education is sometimes seen as consisting precisely in finding "good practices" and sometimes even in extolling them. This becomes especially clear when attempts are made to use the history of mathematics education for nationalist motives, by creating and sustaining a *myth about the special role of national mathematics education.*

This myth comes in different versions: one might see the system of education in one's country as the cradle of international education, claiming that it was specifically in a given country and a given place that the most important ideas were born; or one might see the system of education in one's country as a bastion of international education, contending that even if mathematics education developed later in one's country than in some other places, it nevertheless attained greater heights; or one might not even make this claim and criticize the system of education in one's country, simultaneously assuming, however, that it is the most important system of education all the same, since the country as a whole is very important.

Such mathematics education nationalism becomes more acute during periods of conflicts and wars (Karp 2007) but is by no means limited to such periods, manifesting itself during quite peaceful times as well. In reality, the processes of development are far more complex; even when it became possible to speak about national education (which happened relatively recently – certainly not before the formation of nation states), processes and ideas from different countries often echoed each other, and different ideas were first realized in different countries. This does not mean that one can never or that one should not strive to determine when exactly they first appeared, but it rules out the use of such information to fuel national pride. Of particular interest in this connection are instances when ardent patriots struggling for their country's national independence preferred foreign curricula or textbooks in mathematics, assuming that they would be of greater benefit to their homeland and not subscribing to the view that their own was invariably better (Zuccheri and Zudini 2007).

Each country has its own history of the development of mathematics and mathematics education, and this history has different pages. Already in the nineteenth century, Dmitry Tolstoy (1885), a Russian political figure and historian of education, indignantly quoted a learned German of the eighteenth century who had argued that not all nations were capable of genuinely scientific undertakings and therefore that Russians, who were incapable of reaching the true summits of learning, should devote themselves to lower concerns, namely, mathematics. In the nineteenth century, and even more so in the twentieth, mathematics ceased to be regarded as an insufficiently scientific enterprise; however, the belief that not all nations were capable of reaching the true summits of learning (now by learning mathematics) was one that people sometimes continued to espouse and to express.

The assertion that Russians or Americans are incapable of learning mathematics, made in the past, could only make us laugh today. Meanwhile, we know that in France and Germany there were both major mathematicians and a quite developed system of mathematics education at a time when nothing of the kind existed in Russia, while by the time that Russia could boast of the names of Lobachevsky or Chebyshev and of substantive courses in gymnasia and universities, the United States could only look forward to anything comparable. One might think that this alone should make highly suspect any contemporary claims about representatives of countries which today occupy less prominent positions in the world of mathematics than Russia and the United States and their alleged inability to learn mathematics. However, one often encounters as a counterweight to such claims what may be called the *myth of universal simultaneous development*. The fact that certain techniques, ideas, and organizational structures were borrowed from abroad by all countries begins to be considered invariably offensive, and people begin to claim that every place had its own indigenous system, which not only deserves to be taken into account and respected but in general requires no additions or improvements whatsoever.

The myths listed above, along with others not mentioned, are most often found in popular literature, but they exert an influence on scholarly literature as well, or more precisely, on the people who study it. Determining the truth turns out to be less important than not contradicting existing viewpoints – which calls for a completely different methodology. Historians feel obligated to seek supporting evidence for these viewpoints and are afraid of the conclusions that can be reached on the basis of the documents they study. As MacMillan (2010) writes, these are indeed "dangerous games."

6 Conclusion

The aim of this chapter has not been to list all possible research methodologies in the history of mathematics education. Oral history alone – and consequently the practice of interviewing, a powerful research tool in the history of mathematics education, in which even the relatively recent past has not been sufficiently documented – has been the subject of numerous books and articles. Considerable literature has also been devoted to other methodologies of historical research and qualitative research methodologies in mathematics education, including methodologies used in the studies discussed in this handbook.

The goal of this chapter has been to describe the twofold nature of the field: historical in terms of methodologies and mathematical-pedagogical in terms of the objects of study. Overcoming a simplistic understanding of both these objects and these methodologies, which reduces research to reprinting tables of contents from textbooks and the like, may be the most important methodological objective of all.

As has already been repeatedly said, the history of mathematics education is still in its formative stages as a scientific discipline. Over time, it will likely become enriched by vivid examples and model studies (although not so little has already been done), while its methodology will expand and acquire new resources. The principles and spirit of conscientious research based on all available information and aiming to reconstruct a realistic picture of what has taken place will, one would like to hope, remain unchanged.

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Part II
Mathematics Education in Different Epochs and in Different Regions: Antiquity and the Middle Ages

Chapter 3
Mathematics Education in Antiquity

Alain Bernard, Christine Proust, and Micah Ross

1 General Presentation

A wide variety of documents relating to mathematics education in the Near East and the Mediterranean basin have survived to the present day. The oldest of these sources are the southern Mesopotamian clay tablets produced in the third millennium before the Common Era. More recent sources were copied in the Byzantine Middle Ages from a long chain of texts which stem back to lost originals. Nonetheless, these late copies provide some evidence of educational activity and pedagogical orientation. As may be seen in the case studies in this chapter, these sources represent a wide chronological distribution but of texts of diverse genres. Some texts, like the tablets made of nearly indestructible clay, survive in great numbers and enable a reconstruction of the mathematical instruction of ancient Mesopotamia. Excavations in Iraq, Iran, and Syria since the late nineteenth century have produced a sufficient number of tablets to permit a detailed reconstruction of the basic mathematics curriculum in the scribal schools of the ancient Near East. Equivalent sources exist for the Greco-Roman world but in much lower numbers, and their state of preservation does not permit many conclusions. The Greco-Roman texts comprise small, disconnected fragments on papyrus, pottery, leather, and even wooden tablets covered with wax, all of which probably served in the teaching of mathematics. By contrast, relatively few texts of the copious Mesopotamian corpus report how the scribes in the ancient Near East conceived of their work, their knowledge, and its transmission, whereas the Greco-Roman texts written on parchment generally resulted from endeavors in copying or translation and only marginally constitute *direct* evidence of scholastic activity. Thus, these later sources shed limited light on the practicalities of transmitting mathematical knowledge from master to disciple in different contexts, and few of these texts detail elementary education. However, these sources do reveal the weighty didactic ideals of the Greco-Roman world which governed the prolific work in philosophy and rhetoric. In some cases, these ideals may be assumed to have been put into practice and corresponded to actual curricula, but this inference is speculative and probably useless. The evidence from Pharaonic

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Egypt represents a nadir of textual preservation and cultural reconstruction: although the series of problems found on ancient papyri probably served pedagogical purposes, the manner of instruction and the institutional setting in which these texts were used remain largely unknown.

The following synthesis is therefore based on sources that are characterized by extreme heterogeneity in their nature as well as in their geographical and chronological distribution. This fundamental fact should always be kept in mind to avoid anachronistic claims. The disparity of sources demands consideration of the varied and unevenly documented diversity of the educational settings and institutions of antiquity. In other words, the ancient sources neither relate to the same environments nor do they refer to the same cultural and institutional codes.[1] The available sources do *not* describe a complete or consistent picture of teaching mathematics in antiquity, but spotlights may be focused on the better documented teaching contexts of Mesopotamia, Egypt, and Greco-Roman world. Even if this disparity limits our actual knowledge of ancient mathematical teaching, these difficulties highlight the fact that both the "positive information" which we can claim to know and *the kind of questions* asked about ancient mathematical education depend strictly on the nature of the surviving sources.

2 Mesopotamia

During excavations conducted in Iraq, Iran, and Syria since the late nineteenth century, archaeologists and illegal diggers have unearthed hundreds of thousands of clay tablets containing texts of all kinds (including administrative records, contracts, letters, literary compositions, medical treatises, astronomical calculations, and mathematical writings). These documents provide evidence about the history of the ancient Near East over a very long period – more than 3,000 years since the beginnings of writing (c. 3300 BCE) until the abandonment of clay for writing at the beginning of the Common Era.

Numerous languages were transcribed in cuneiform writing on clay tablets (Fig. 3.1). Among mathematical texts, Sumerian and Akkadian are used. Sumerian, which was the language of the people of southern Mesopotamia during the third millennium, probably disappeared as a living language before the second millennium but remained the language of scholarship until the end of cuneiform writing. Akkadian is the Semitic language which gradually supplanted Sumerian and had long been the diplomatic language of the ancient Near East and Eastern Mediterranean.

Approximately 2,000 tablets containing mathematical texts are presently known. Most of the mathematical cuneiform texts published during the early twentieth century by Neugebauer, Sachs,

Fig. 3.1 Cuneiform writing (School tablet from Nippur, about 1800 BCE)

[1] This methodological approach is developed in Bernard and Proust (2014); see in particular the introduction.

and Thureau-Dangin were bought from dealers by European and American museums or by private collectors, and their provenances are unknown. However, after the Second World War, archaeologists unearthed new collections of mathematical tablets with clear contexts, notably in the Diyala Valley (northern Mesopotamia) and in Susa (western Iran).

In cases where the provenances of the tablets are well documented, archaeological reports show that the tablets containing high-level mathematics shared the same findspots with elementary school tablets. Thus, the education of young scribes and activities of erudite scholars occurred in the same place, and possibly, the authors of these high-level mathematical texts were involved in teaching. But do these archaeological details indicate that all of the mathematical cuneiform texts were produced for educational purposes? A positive answer is often assumed, more or less tacitly, in recent studies of mathematics in Mesopotamia. However, the situation is probably more complex. Indeed, some of the cuneiform mathematical texts are clearly school exercises. (Some examples are examined below.) Others texts, which contain lists of solved problems, are probably (but not certainly) documents composed and used for advanced mathematical education. However, most of the higher-level mathematical texts do not clearly reveal the exact context in which they were composed or used. It is not always easy to identify the audience of such texts. As far as the cuneiform sources are concerned, more details are clearer here than in the case of Egyptian papyri. The strongest evidence derives from the physical details of the tablets themselves. The very shape, size, and layout of the tablets often reveal the nature of the context in which they were produced.

2.1 The Scribal Schools

Modern historians refer to the places where scribes were educated as "scribal schools." Sometimes, the physical place of the school is well identified. In Nippur, Ur, Mari, and Sippar, for example, traces of teaching activities such as important collections of school exercises or bins used for recycling tablets were found in houses tentatively identified as scribal schools.[2] A particular Sumerian word designates such places as *edubba*, which literally means "house of tablets." Sumerian literature portrays a highly idealized picture of the *edubba*, which appears as prestigious institutions for educating the social elite (Michalowski 1987, p. 63). This image may reflect reality at Nippur, the political and cultural capital of Old Babylonian Mesopotamia, whose schools merited high esteem throughout the ancient Near East. However, the organization of education appears to have varied considerably from one city to another. In some cities, the teaching activities seem to have been limited to the domestic sphere, as shown, for example, in Sippar by Tanret (2002, pp. 153–156). In other cities, priests may have participated in education, as shown in the cases of Ur (Charpin 1986, pp. 420–486) or Tell Haddad, a city of the Diyala Valley (Cavigneaux 1999, p. 257).

Most surviving Old Babylonian mathematical tablets are school tablets. They span a large geographical area (see map Fig. 3.2), but the bulk comes from Nippur. The careful analysis of thousands of tablets of Nippur has allowed historians to reconstruct in great detail the curriculum of mathematical education which took place in the schools of this city and perhaps in other *edubba*.[3]

[2] The presence of school tablets in a house is not always a proof that this house served as a school; in particular, school tablets may have been brought from other places to be reused as construction material. Thus, the archaeological context must be analyzed carefully for each context. See, for example, the case of the "schools" in Ur analyzed by Charpin (1986, pp. 432–434) and Friberg (2000), the case of the houses of "Aire II" of Tell Haddad analyzed by Cavigneaux (1999, pp. 251–252), the case of "House F" in Nippur analyzed by Robson (2001, pp. 39–40), and the case of the house of the "gala-mah" in Sippar-*Amnânum* analyzed by Tanret (2002, p. 5). About bins for recycling tablets, see Tanret (2002, pp. 145–153).

[3] The studies of the curriculum in the *edubba* are mainly based on Nippur sources; from the abundant literature on the subject, see Cavigneaux (1983), Civil (1985), Tinney (1999), Vanstiphout (1996), Veldhuis (1997), Robson (2001), George (2005), Proust (2007), and Delnero (2010).

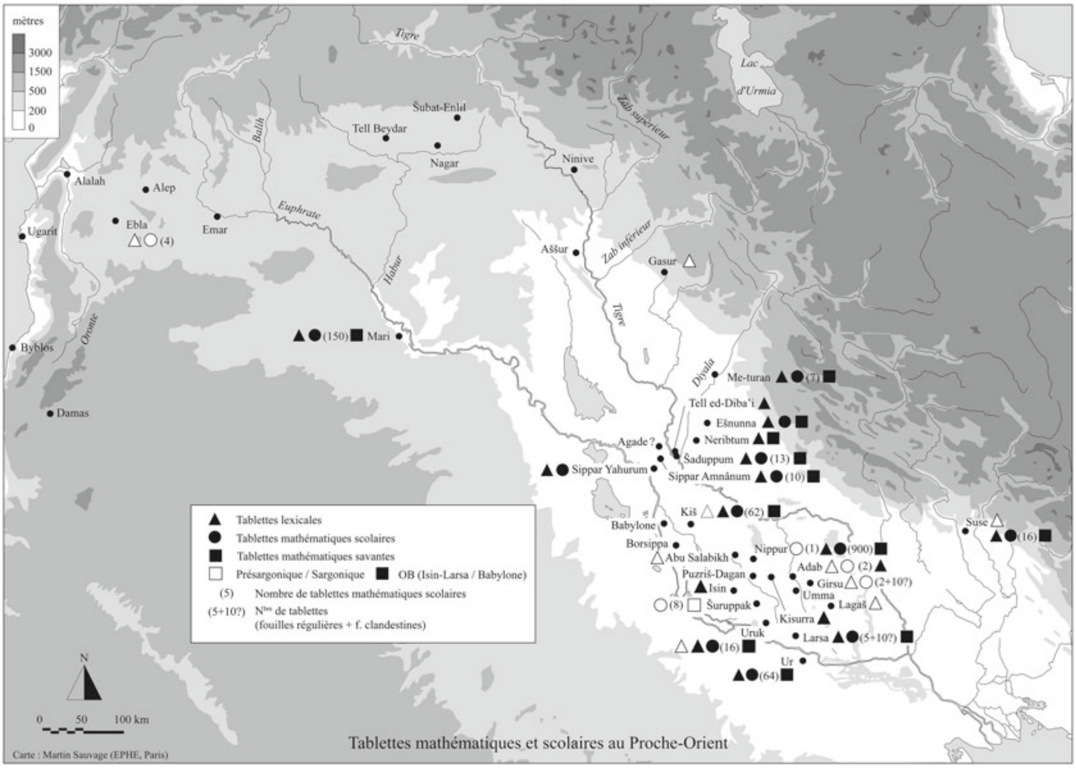


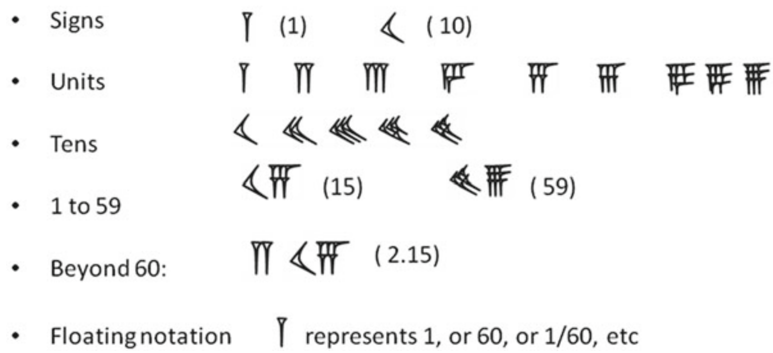
Fig. 3.2 Places where mathematical tablets were found (● = OB elementary mathematical school texts; ▮ = OB advanced mathematical texts) (Map by Martin Sauvage, published in Proust 2007, p. 281)

2.2 *The Elementary Level of Mathematical Education*

The shapes of the tablets provide valuable evidence for the reconstruction of the curriculum. The tablets used most often at Nippur (type II in the typology of Assyriologists) are large rectangular tablets (about 10 × 15 cm), which the young apprentices used in their training to memorize and write a set of standardized texts. These texts included lists of cuneiform signs, Sumerian vocabulary, systems of measurement, and elementary numerical tables. When a long series of lexical lists or mathematical tables had been completely memorized, it was written on large multicolumn tablets known as "type I" or, sometimes, on prisms. These great compositions on prisms may be interpreted as a kind of examination (Veldhuis 1997, p. 31). In addition to these exercises, scribes would sometimes note short excerpts on small single-column rectangular tablets (type III – see Fig. 3.4). The Sumerian name of this type of tablet sometimes appears at the end of the composition, as well as in some literary texts: *imgidda* or "elongated tablets." *Imgidda* tablets were often used to learn multiplication tables (as shown in Fig. 3.4).

Type II tablets (see Fig. 3.3) provided key evidence which allowed historians to identify the exact content of the texts studied by young scribes in the early stage of education and to reconstruct the order in which these texts were learned. Indeed, Veldhuis (1997, pp. 34–36) has shown that the reverse of type II tablets was "used as a repetition of a school text studied at a point earlier in the curriculum" (p. 36). Thus, by comparing the texts written in obverses and reverses of type II tablets, he reconstructed the elementary curriculum (pp. 41–67). Veldhuis focused on lexical texts, but the same

Table 3.1 Sexagesimal place value notation



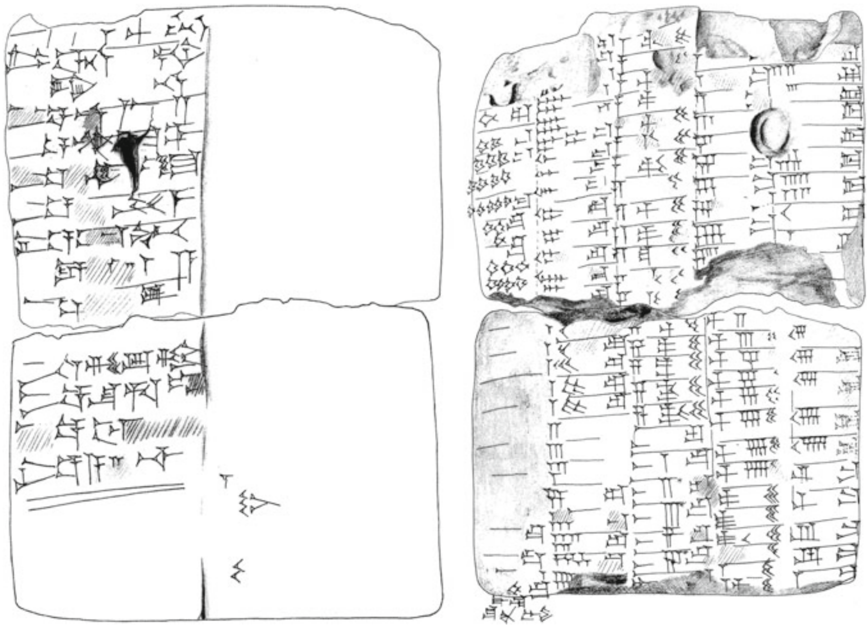


Fig. 3.3 Ni 4840+ +UM 29-13-711, type II tablet. Obverse, lexical list; reverse, measures of capacities (Proust 2007, p. 26)

method can be applied to mathematical texts (Robson 2001, 2002; Proust 2007). A detailed picture of elementary education in Nippur has emerged from these studies. The first level of the mathematical curriculum was devoted to learning the following lists and tables, more or less in the following order: lists enumerating measurements of capacity, weight, surface, and length; tables providing correspondence between the various measures and numbers written in sexagesimal place value notation (see Table 3.1); and numerical tables (tables of reciprocals, multiplication, squares, square roots, and cube roots). All of these elementary lists were probably learned by rote.[4]

Outside of Nippur, the mathematical curriculum cannot be reconstituted in such detail, partly because the number of available tablets is too small for any meaningful statistical consideration. The typology of tablets varies considerably. For example, type II tablets were rarely found outside of Nippur. In the schools of Mari and Ur, mainly small round tablets were used.

[4] About the role of memorization in learning process and transmission, see Veldhuis (1997, pp. 131–132, 148–149) and Delnero (2012).

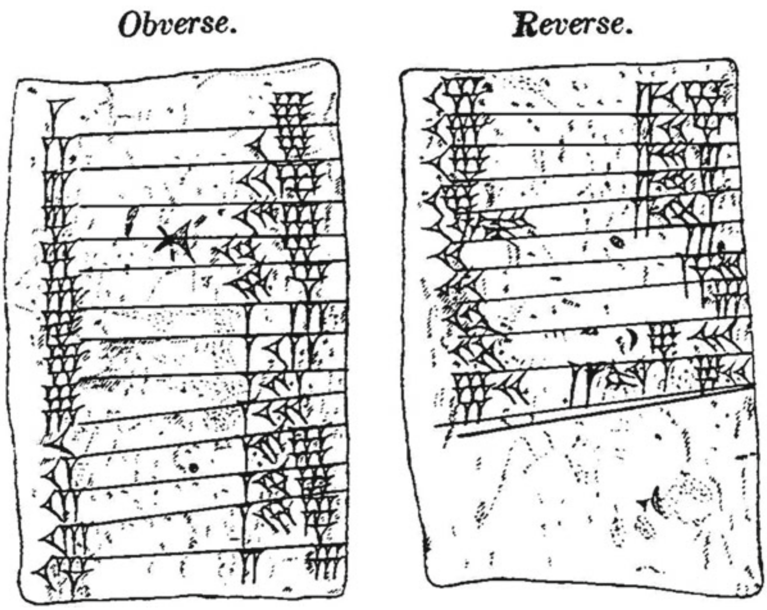


Fig. 3.4 HS 217, type III tablet. Multiplication table (Hilprecht 1906, p. 7)

2.3 The Intermediate Level

After learning the metrological and numerical systems as well as a set of elementary arithmetical results (tables of reciprocals and multiplication tables), the scribes began an intermediate level of education. At this less formalized level, scribes learned the basics of sexagesimal calculation, namely, multiplication and calculation of the reciprocal of large numbers.

This knowledge was then applied to finding areas of squares and other figures. At schools in Nippur, this level of education is documented mainly through exercises noted on square-shaped tablets (see Fig. 3.5).

Table 3.2 summarizes the various aspects of the mathematical curriculum as it could have existed at Nippur and perhaps in other schools.

2.4 The Advanced Level

If the elementary and intermediate levels of mathematical education are well known, at least at Nippur, the context in which advanced mathematical texts from the Old Babylonian period were produced or used is more difficult to reconstruct. Mathematical texts have been previously interpreted as textbooks or as databases compiled for teaching. However, a pragmatic analysis of the texts suggests that the authors had at least some purposes other than teaching.[5] How is it possible to distinguish the tablets used for advanced teaching (written by students or teachers) from those that reflect investigations of pure scholarship? The first type of evidence could be the complexity of the mathematical procedures, but such a criterion can be misleading because what is complex for a modern reader may not have been complex

[5] One example of text not clearly linked with teaching is the famous tablet Plimpton 322 (see Britton et al. 2011); other examples are found among the so-called series texts, which are lists of problem statements written on numbered suites of tablets (see Proust 2012).

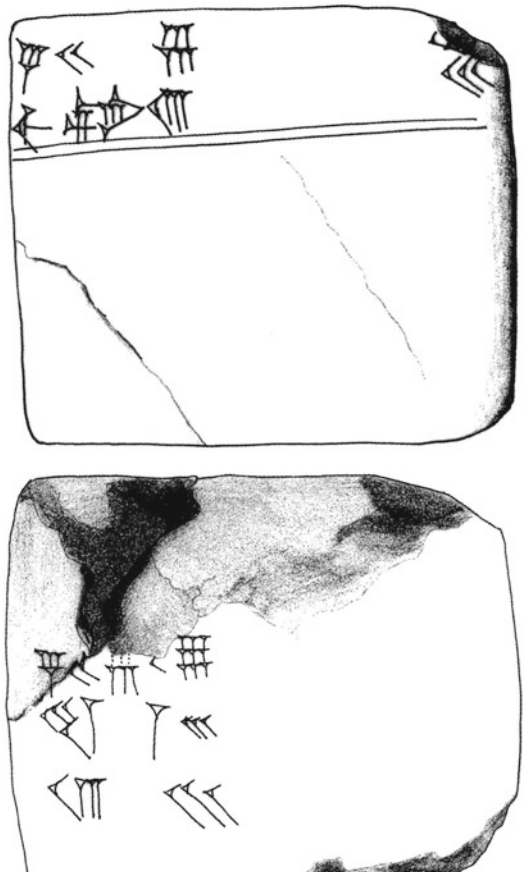


Fig. 3.5 Ni 10241, reciprocal calculation (Proust 2007)

Table 3.2 Mathematical curriculum in OB Nippur

Level	Content	Typology	Examples
Elementary	Metrological lists (capacities, weights, surfaces, lengths)	Types I, II, and III	See Figs. 3.3 and 3.4
	Metrological tables		
	Numerical tables (reciprocals, multiplication, squares)		
	Square and cube roots		
Intermediate	Exercises: multiplications and reciprocals	Square-shaped tablets	See Fig. 3.5
	Surface calculations		

for an ancient scribe and vice versa. Therefore, caution is advised for arguments based on the supposed "level" of a mathematical content. The second type of evidence is linked to material aspects. Very roughly, one can classify the tablets into two types: single-column tablets (type S) and multicolumn tablets (type M).[6] However, the shape of the tablets often conforms to local habits. Since most of the mathematical tablets are from unknown provenance, a general typology cannot be clearly connected with specific pedagogical practices. Thus, only case-by-case examinations are relevant to answer the question of how to distinguish teaching from scholarship. A mathematical text that seems to have been used at the beginning of the advanced level of mathematics presents a useful example.

An example of such a tablet is conserved at Yale University under the inventory number YBC 4663 (see Fig. 3.6). This tablet has an elongated shape and is written in a single column (type S). The tablet

[6] This typology comes from the classification of tablets used in OB Nippur for learning Sumerian literary (Tinney 1999, p. 160).

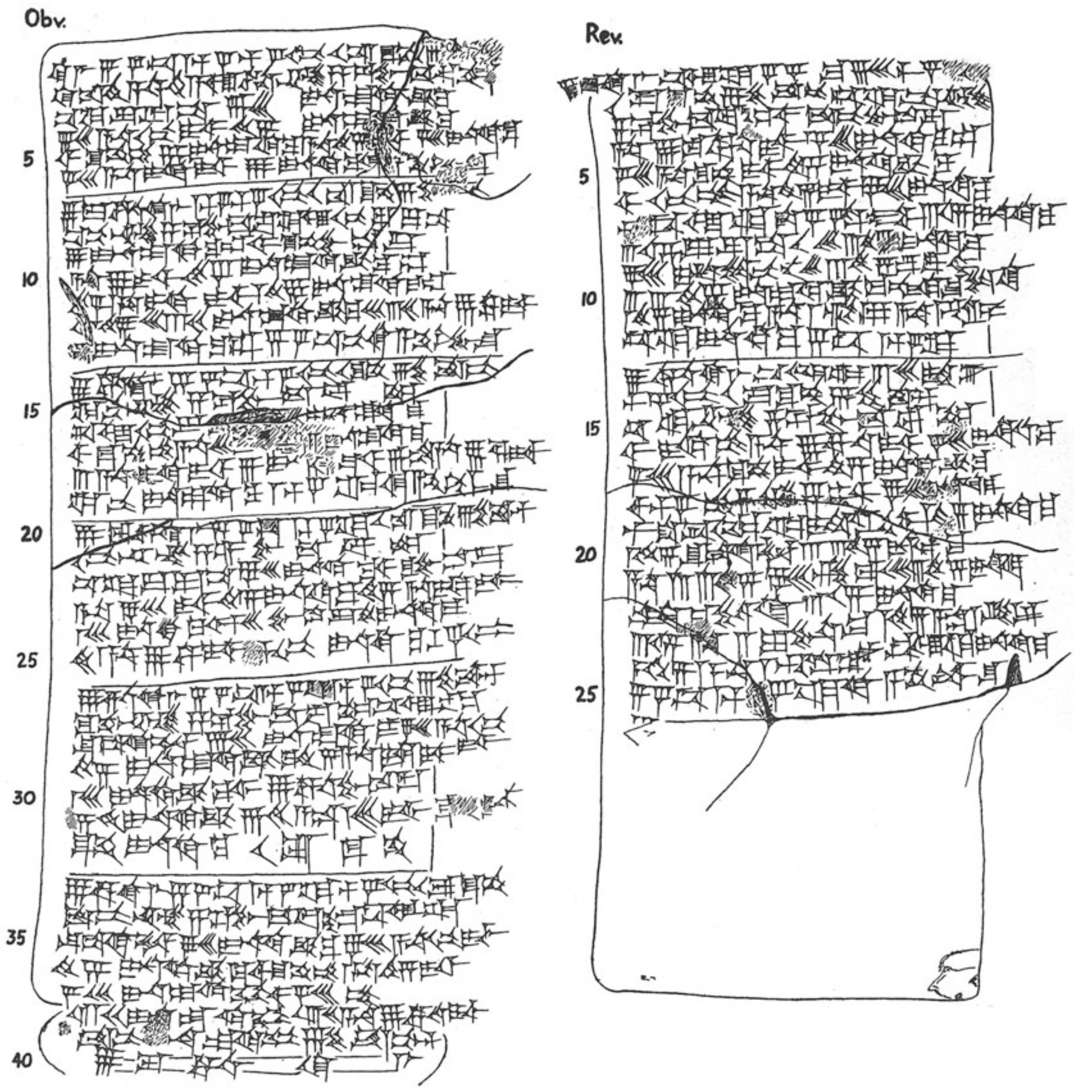


Fig. 3.6 Type S tablet – YBC 4663, Yale University (Courtesy of Benjamin Foster[***])

is of unknown origin but probably comes from a city in southern Mesopotamia. It contains a sequence of eight solved problems dealing with digging trenches. The parameters of the problem (data and unknowns) are the dimensions of the trench (length, width, depth), its base, the volume of extracted earth, the number of workers needed for digging, the daily labor assigned to the workers (namely, the volume of earth to be extracted each day by each worker), their daily wage, and the total wages (expressed as a weight of silver). All these parameters are linked by a simple relation that we can represent in modern fashion as follows:

$$\text{Total wage} = \text{daily wage} \times \left(\frac{\text{length} \times \text{width} \times \text{depth}}{\text{daily assigned task}} \right)$$

Examination of the text shows how the procedures implement the computational methods taught at the elementary level of mathematical curriculum. For example, consider the first problem, translated as follows:

Translation of YBC 4663 #1[7]

1. A trench. 5 ninda is the length, 1½ ninda (the width), ½ ninda its depth, 10 (gin) the volume of assignment (for each worker), 6 še (silver) [the wages of a hired man].
2. The area, the volume, the number of workers, and the (total expenses in) silver what? You, in your procedure,
3. the length and the width multiply each other. *This will give you* 7.30.
4. 7.30 *to its depth raise. This will give you* 45.
5. The reciprocal of the assignment detach. *This will give you* 6. *To* 45 *raise*. *This will give you* 4.30.
6. 4.30 *to the wages raise*. *This will give you* 9. *Such is the procedure.*

Note that in the statement of the problem (lines 1–2), the data are expressed in concrete numbers with units of measure, in the same way as in the metrological lists, but in the procedure (lines 3–6), the data appear only as abstract numbers expressed in sexagesimal place value notation (SPVN). Metrological tables had been used to transform measures into abstract numbers for performing calculations. This process is confirmed by the fact that the correspondences between the measures given in the statement and the abstract numbers used in the procedure fit with the correspondences provided by the metrological tables. This observation suggests that the authors of the text used the basic skills taught in the scribal schools. The sequence of problems listed on this tablet provides an opportunity to use all the metrological tables one after another, as well as multiplication tables and calculation techniques taught in the intermediate level (multiplication, inversions, calculating areas and volumes). All knowledge acquired in the early levels of mathematical education is systematically employed. In this way, we can suppose that the text was composed specifically for teaching mathematics. It could have been written by a master or an advanced student. An examination of other tablets similar to YBC 4663, which seem to come from the same city, may show that, in this city, type S tablets were used at the beginning of advanced education.[8]

3 Hellenistic Period

A gap in the preservation of mathematical texts appears after the end of the Old Babylonian period. For subsequent periods, only sporadic examples of metrological and numerical tables are known. Only by the end of the first millennium BCE do coherent sets of mathematical sources reappear. Two small corpuses of mathematics texts dating from the Hellenistic period (c. 300 BC) have been discovered at Uruk and Babylon. It is difficult to know whether these corpuses from late periods reflect a kind of Renaissance after a long eclipse, or if the written mathematical tradition continued through the centuries. The transmission of elements of mathematical tradition over this long period tips the balance toward the second hypothesis. Advanced mathematical texts could have been noted on perishable materials such as leather or papyrus, which would not have resisted time.

The context of the Hellenistic period differs radically from the Old Babylonian world. The mathematical practices were developed by lineages of astrologers and astronomers who were linked to the great temples of Babylon and Uruk. In Hellenistic Mesopotamia, mathematical erudition was closely

[7] Literal translation based on Neugebauer and Sachs (1945, p. 70). The passages written in Sumerian in the cuneiform text are represented by plain font and passages written in Akkadian by italic font. The measurement units used are 1 ninda ≈ 6 m, 1 gin ≈ 1.7 dm^3, and 1 še ≈ 0.04 g.

[8] These tablets are six catalogue texts conserved at Yale University and two related procedure texts (including YBC 4663). See Proust (2012).

associated with the astral sciences.[9] Cuneiform mathematics was no longer taught to children or adolescents acquiring literacy and numeracy, as was the case in the Old Babylonian period, but to young scholars who were probably already literate in Aramaic and perhaps in Greek.

4 Ancient Egypt

Until the establishment of Greek as the solitary administrative language near the end of the second century of the Common Era, the situation regarding Egyptian sources on mathematical instruction inspires less confidence than that of either the cuneiform or Greek sources. First of all, the entire corpus of hieroglyphic and hieratic papyri counts roughly as many texts as the corpus of cuneiform mathematical texts. Among these papyri, only a handful of explicitly mathematical sources have survived.[10] No mathematical texts survive from the earliest periods of Egyptian history, and hieratic texts of Middle Egyptian comprise only three relatively intact papyri. One hieratic mathematical text has also survived on leather. If a wider view of mathematical texts is taken and papyri with calculations are counted, other early mathematical texts include sections of the Reisner Papyrus and a collection of hieratic fragments from Kahun. To these should be added two wooden tablets from Akhmim. After a lapse of more than a millennium, Demotic texts add one complete papyrus and six fragments and three Roman ostraca. In none of these cases have archaeologists established that the papyri survived in a pedagogical setting. The larger, more complete papyri contain collections of solved problems indicative of pedagogical use and fractional tables useful for calculation. Individual fractional tables are preserved among the smaller fragmentary papyri and ostraca, but most contain independent calculations (Table 3.3).

Not only do few mathematical sources survive, but the process of instruction in ancient Egypt is not well known. Because the authors of moral “instructional literature,” such as the *Instructions of Ptahhotep* (c. -2880), addressed their readers by familial terms, the earliest mode of Egyptian

Table 3.3 Chronological range of Egyptian mathematical papyri

Name of Egyptian mathematical text	Approximate date
Reisner Papyrus	-1970–-1925
Cairo Cat. 25367/8	-1970–-1925
Kahun Papyri	-1880–-1770
Berlin 6619	-1880–-1700
Moscow E4674	-1770–-1650; original -1990–-1770
BM 10057/10058	-1600; original -1860–-1815
Mathematical Leather Roll	-1650
Cairo JE 89127–30, 89137–43	-300–-200
BM 10794	-331–350
BM 10399	-331–-30
Heidelberg 663	-200–0
Griffith I E.7	-100–100
BM 10520	100–200
Carlsberg 30	100–200
Ostraca Medinet Madi 251	0–200
Ostraca Medinet Madi 720+912	0–200
Theban Ostracon D12	0–200

[9] Rochberg (2004, Chap. 6), Robson (2008, Chap. 8), Clancier (2009, pp. 81–103, 205–211), Steele (2011), Ossendrijver (2012, Chap. 1), and Beaulieu (2006).

[10] For a reliable guide to the bibliography and contents of the specific texts, see Clagett (1999).

instruction has been imagined as a father instructing his children. However, this assumption ignores the possibility that the mode of address merely employs a rhetorical conceit. The first reference to "house of instruction" (ꜥ*.t n sb*ꜣ) appeared in a Tenth Dynasty (c. -2160–2025) tomb. A composition titled *The Satire of the Trades* (c. -2025–1700) describes a royal school, but because the treatise seeks to esteem learning, the presence of an actual school cannot be assumed. Speculation about Egyptian pedagogy has focused on an element of the temple complexes titled "The House of Life," but whether the curriculum of this place of instruction had a wider applicability outside the temple remains unknown.[11] The *Onomasticon of Amenope* (c. -1187–716) records a list of terms as important to the "scribes of The House of Life" that is similar to the lexicographical lists of Mesopotamia, but Egypt and Mesopotamia seem to have employed different organizational strategies. After he conquered Egypt, Darius endeavored to restore "The House of Life," which may have served as a type of hospital. Again, little information about the Egyptian methods of instruction survives, and mathematics may not have even formed the curriculum of all scribes. Unlike Babylon, no large collections of school texts have survived, although some school texts (from mere onomastica and word lists to literary compositions used as models such as *The Tale of Sinuhe*) have been found. The larger mathematical papyri have been interpreted as pedagogical texts because they presented collections of similar exercises, but whether these writings formed the syllabus of specialists or generalists remains unstated. As shown by ostraca which repeat various phrases for different numbers and genders and others which elucidate the reading of certain hieroglyphs, education in grammar and writing seems to have formed some portion of the activity of the temple complex in the town of Medinet Madi, but these documents were mixed with administrative documents and texts useful for the composition of planetary positions. Unfortunately, some of these ostraca were reused as building materials, and the particular archaeological status of any given text may not be stated with certainty.

The date of composition for the largest of the papyri, the Rhind Mathematical Papyrus, coincides roughly with the first reference to a "place of instruction." Whether or not it was used in such a school, the Rhind Mathematical Papyrus boasts that it contains "the model for enquiring into affairs and for knowing all that exists" but says nothing about the prerequisite knowledge, the intended audience, or the qualifications of a scribe who had mastered the material. Other than the techniques demonstrated, only the basic literacy necessary to read the papyrus may be presumed.

This translation of the title and introduction to the Rhind Mathematical Papyrus (Table 3.4) follows Couchoud's (1993) French translation. Despite the grandiloquent promises of resolving all that is unclear and penetrating every mystery, the 87 surviving applications of Egyptian mathematics concern the doubling of fractions; the division of fractions by 10; the solution of linear polynomials; the unequal distribution of goods; the approximations of area in circles; the geometrical problems with rectangles, triangles, and pyramids; and the problems of exchange and geometrical progressions. The use of masculine, singular pronouns in the second and third person singular conforms to the impression that only males were educated in mathematics, perhaps working singly with the instructor or some other examiner.

The introduction to the most complete Demotic mathematical papyrus no longer survives, but some basic estimation of its pedagogical position may be derived from the fact that it shares a papyrus with a manual of legal formulae. Whether these compositions were textbooks or references remains an open question. Moreover, the juxtaposition of these two texts could be either the accidental result of reuse by a scribe whose training spanned both areas or a deliberate link forged by an instructor who connected land contracts with geometry.

In counterpoint to the pedagogical papyri, the fragmentary papyri may occasionally represent more than mere working notes for the resolution of a commonplace problem. A particular fragment of the Kahun Papyri (Kahun IV.3) could be dismissed as the pedestrian division of commodities, except for

[11] For an accessible discussion of "The House of Life," see Strouhal (1992, pp. 235–243).

Table 3.4 Title and introduction to the Rhind Mathematical Papyrus

1. *Tp ḥsb n ḫꜣt m ḫt rḫ ntt nbt snk <.t> … štꜣt nbt. ’Iw ist grt*
   1: The model for enquiring into affairs, for knowing all that which is unclear, <and deciphering > every mystery. So, now
2. *spẖr.n.tw šfdw pn m rnpt 33 íbd 4 ꜣḫt <sw ?? nsw> bity ʿꜣA-wsr-Rʿ di ʿnḫ m snt r sšw*
   2: this papyrus-roll was copied in Regnal Year 33, month 4 of Flood Season, [day ??] of the King of Upper and Lower Egypt “The Power of Re is Great” (Apophis), may he be given life, in conformity to the writings
3. *n iswt iry m h3wt < n nsw bity Ny-Mꜣ >ʿt-Rʿ. In sš ’Iʿh-msw spẖr snn pn*
   3: of old, made in the time of the King of Upper and Lower Egypt “Belonging to the Justice of Re” (Amenemhat III). It was by the scribe Ahmose that his copy was transcribed

the mathematically playful requirement that the shares of the commodity increase in an arithmetic progression. Although the text may be salvaged from the corpus of “documentary evidence” by this detail, the pedagogical position of the fragment remains unclear. Taken collectively, these texts permit a rough estimation of the range of Egyptian mathematical knowledge and techniques of calculation, but specific details about what constituted basic knowledge, what represented advanced knowledge, how these topics were communicated, and how competency was assessed remain speculative.

5 Greco-Roman World

5.1 The Nature of Greco-Roman Sources on Teaching Mathematics

As evidence from Mesopotamia and Egypt shows, modern understanding of mathematical teaching and learning in antiquity depends on the nature of available sources. The Greco-Roman sources differ significantly from the Mesopotamian and Egyptian sources. For the sake of clarity, the following simplified classification may be proposed. First, texts with highly internally coherent mathematical contents sometimes (but not always) begin with prefaces which announce a pedagogical purpose, eventually oriented by a philosophical position. These texts were transmitted through a long chain of intermediaries and thus became *classical* texts in the long course of Greco-Roman history. This lengthy process incorporates “accompanying texts,” like marginalia or independent commentaries, which can be difficult to distinguish from the “original text.”[12] The writings of Euclid, Archimedes, and Ptolemy exhibit this process of incorporation.

[12] For ancient scholarship and the history of texts and their transmission, see Reynolds, Leighton and Wilson (1968).

Other kinds of texts also relate to mathematics and mathematical education but only in the sense that their content and nature are basically metamathematical: these texts describe philosophical or cultural projects in which mathematics plays an important role. Plato's *Republic*, Vitruvius's *De Architectura*, or Quintilianus's *Institutio Oratoria* shares an esteem of mathematics while not being truly mathematical texts.

Somewhere between these two categories are the literary or philosophical sources which sometimes preserve excerpts of lost mathematical works by inclusion. In this case, the larger, encyclopedic project guides the choice of excerpts. For example, Simplicius preserves lost mathematical texts in his sixth-century commentaries on Aristotle. To this "intermediary" genre also belong texts like the second-century *Introduction to Arithmetic* by Nicomachus or the fifth-century commentary on the first book of Euclid's *Elements* by Proclus, which basically attempt an explicit, philosophical project but introduce mathematical contents to achieve this purpose.

Fourth, direct archaeological evidence of mathematical teaching and learning survives in the form of fragments of papyri, ostraca, and wooden tablets, some of which can be related to a teaching context (although this attribution is often problematic). In the case of ancient mathematical knowledge, such evidence is scanty and consists of disconnected fragments with no clear context. Such sources approach the kind of material that came from Old Babylonian contexts, the richness of which enables a fairly satisfying reconstitution of mathematical curricula.

A fifth, "ghost" category nearly resembles the Egyptian material. This category contains isolated texts, like the problems of the Akhmim papyrus (Baillet 1892) or the problems in the "metrological corpus." This category extends to the sophisticated problems contained in Diophantus's *Arithmetica*, the structure of which strongly evokes pedagogical concerns, but the scholarly context of which remains unknown.

The key fact, then, about these sources is that the three first kinds of sources in the above classification have undergone a long-term process of "classicization" and are by far the best represented, while the other two categories are poorly represented. This does not suggest that nothing is known about mathematical education in praxis, but more precisely that what is "positively" known is necessarily of a different nature than what has been explained above about ancient Mesopotamia and Egypt: the divergence of approaches derives from the fact that the underlying sources differ in kind. This important acknowledgment implies a fundamental bias in our knowledge of mathematical education in this context. Only a specific part of the "ideal" picture may be revealed without access to a much richer range of sources.

For example, the technical texts in the first category, like the extant portions of Euclid, Apollonius, or Archimedes, have survived because they were transmitted, cultivated, commentated, and reused for contexts and purposes beyond the original aims of their Hellenistic authors.[13] Works like the compositions of the Pythagorean school or isolated figures like Hippocrates of Chios are known through the erudition of late scholars like Simplicius of Cilicia, whose work was in turn transmitted and copied in later times. Simplicius continued the tradition of ancient scholars like Vitruvius, Plutarch, Athenaeus, and perhaps even Euclid, who composed works that functioned as encyclopedias of ancient knowledge. This lengthy process of incorporation and citation stands as a general feature of ancient scholarship and extends much beyond the mathematical literature. This process guided the transmission and "classicization" of the ancient heritage of literature, philosophy, and more specialized issues like religious, medical, or mathematical texts. Note that this phenomenon is not proper to Greco-Roman Antiquity : already the Mesopotamian scribes had their own 'classics', as well as 'dead languages' that served as classical references. For classical culture, see Marrou (1965), Hadot (2005). For the Greco-roman heritage constituted as such for mathematics and in modern Europe, cf. Goldstein et al. (1996).

The majority of Greco-Roman sources, then, might be (rightfully) seen as "only" derived products, in contrast to the "direct" evidence of documents in the fourth category. The preserved manuscripts

[13] For the question of our sources of knowledge on ancient mathematics, see Fowler (1999, pp. 199–221), particularly the list in pp. 268–275.

date to the medieval period (eighth or ninth century) for Greek literature and Late Antiquity (fifth century) for Latin technical literature.[14] These manuscripts represent a long transmission of copies, sometimes including transliterations and changes of format (Fowler 1999, pp. 204–221; Chouquet and Favory 2001, Chap. 1). As a result, they contain layers of transformations and annotations made during their history of transmission, often with no solid means to discriminate among variants, distinguish textual traditions, or establish a date for the sources. Unfortunately, *for mathematical sources in particular*, nearly no complementary information survives from other, older sources like scholastic papyri or even detailed accounts about mathematical practice. A larger number of documents of the fourth category would be needed to clarify how students were trained *before* or even *during* their study of more elaborate and erudite works, like Euclid's *Elements*. Despite this lack, textual "transmission" should also be regarded as a *fact* of the utmost importance, for at least three reasons.

First, the process by which these sources were *made* classical[15] can hardly be dissociated from the activities of teaching and learning. This is not to say that any marginal annotation in an ancient manuscript or any commentary automatically relates to a teaching activity; but, in many cases, the activities of commentary and note-taking might have plausibly been related to a scholastic activity. Such operations might have only been practiced at a high level in the curriculum of a literate person (Dorandi 2000). In some cases, then, we may guess at the possible structure or contents of a course for which directly resulting marginal notes were preserved. This relationship should be connected to the fact that in Greco-Roman contexts, advanced, literate education implied the study, oral reading, and excerpting of pieces of the classical corpus that formed "the circle of knowledge" (*enkuklos paideia*).[16]

The second reason is that the activity of reading, excerpting, copying, or commentating on classical sources, be they mathematical in content or not, was valued and formed part of what Ineke Sluiter has called "the didactic tradition" (Sluiter 1999). By this is meant *not* the activity of teaching and its "concrete" tradition, but an idealized set of values that, in a non-negligible number of cases, were spelled out very explicitly. Such is the case in the prologues of redacted commentaries, like Theon's commentary to Ptolemy's *Almagest* (Bernard 2014) or the extremely developed exegetical prologue to Proclus's commentary to Euclid's *Elements* I (Lernoult 2010). Such documents are perhaps less valuable for what they indicate about the scholastic character of the corresponding commentaries than for what they say about the leading ideas and cultural purposes assigned to them. This aspect, in turn, is hardly separable from the existence of the *second* category of sources mentioned above: the Greco-Roman literature, especially in philosophy and rhetoric, contains sophisticated conceptualizations of the general notion of what teaching and learning means and even *should* mean. Famous early examples, as far as mathematics teaching is concerned, include Plato's *Meno*, *Republic*, or *Laws*, Isocrates's *Antidosis*, or for the Roman world Quintilianus's *Institutio Oratoria*. All these works contain partial or extended discussion about what the role and nature of mathematical teaching should be within a general educational framework, the latter giving its full meaning and value to the former.

The third reason directly touches on the bias mentioned above: by their nature, these sources reflect a highly distorted picture of what the *totality* of "ancient mathematical cultures" might have represented, including many presently lost written sources, as well as the totality of non-written cultures. Many (if not most) ancient written sources have been lost by accident. Moreover, these sources could only be used and elaborated upon by persons who belonged to the (highly) restricted elite of "educated people" – *pepaideuomenoi*. Therefore, those milieus with some kind of mathematical

[14] For Greek literature, see Fowler, *op. cit.* For Latin technical literature (corpus agrimensorum, on which more below), see Dilke (1971, 128ff) as well as Chouquer and Favory (2001) (esp. Chap. 1).

[15] Including copying, annotating, and writing memoranda, summaries, and abridgements (*epitomai*)

[16] For more detail, see, for example, Aelius Theon's *Progymnasmata*, which is basically a handbook for teachers of rhetoric.

activity and transmission of knowledge for which a bare trace remains, or which did not successfully highlight their specific skills in the standard terms of the literate culture, are almost totally absent.

The two first aspects should be carefully distinguished from the third, in order *not* to superimpose well-represented idealized descriptions and conceptions of teaching on the actual techniques, which are poorly documented. This precaution also eliminates deeply ingrained confusion between various periods of history or doubtful assimilations, such as the frequent claim that Euclid's fundamental purpose for the *Elements* was pedagogical. The problem with this assertion is that Euclid and his exact purpose cannot be directly known because no document from the Hellenistic period relates to these questions.[17] What is known for certain is that Euclid's purpose for his *Elements* may be interpreted as other than purely didactic[18] and that the first explicit mention of a didactic purpose for the *Elements* only appears some eight centuries later in Proclus's commentary to its first book (Vitrac 1990, pp. 34–40; Bernard 2010b). In the case of commentators like Proclus who were also teachers,[19] this later dimension should probably be interpreted as the direct reflection of their own didactic concerns that they could easily project on the authors for whom they made commentaries (Sluiter 1999).

Another important reason for maintaining the distinction between idealized descriptions and actual techniques of teaching is that, while the actual practice and didactic devices are by nature evanescent, the prefaces, annotations, and such, along with the values they convey, are perennial in that they remain opened to reappropriation in later periods.

Bearing all this in mind, some scarce but interesting indications of the actual practice of teaching mathematics in certain contexts should now be introduced, as a way to convey a sense of the institutional setting for such teaching. This example leads to a summary of the thorny discussion of the (disputed) existence of a scholastic curriculum in Greco-Roman antiquity. Only then may a tentative and differentiated explanation about the meaning of mathematics and mathematics learning for various parts of Greco-Roman society be presented as a conclusion.

5.2 *Three Possible Scenarios for Mathematics Teaching in Late Antiquity and What Can Be Concluded from Them*

In the third book of the so-called Mathematical Collection,[20] the fourth-century polymath Pappus of Alexandria describes an encounter with some students of Pandrosion, a female teacher of geometry and a rival of Pappus. The students he encountered had submitted several challenges to Pappus, who was then encouraged by several of his peers to answer them. The related event is interesting in at least three respects.[21]

Taken first as a straightforward account of the encounter, the anecdote shows that the agonistic and challenging character of ancient Greek culture, noted as early as the classical and first sophistic

[17] On the uncertainty of the date and context of Euclid – uncertainty that already dates from Late Antiquity – see Vitrac (1990, pp. 13–18).

[18] Vitrac (1990, pp. 114–148). Other treatises by Euclid besides the *Elements* might be more legitimately suspected to contain some kind of exercises in demonstration or in the technique of analysis (*Pseudaria* and *Dedomena,* respectively); see Vitrac (1990, pp. 21–23).

[19] Not only were they teachers, but, in the case of late Platonist commentators like Proclus, they considered themselves as the successors (*diadochoi*) of a Platonic tradition that included such famous mathematicians as Euclid or Nicomachus. Thus, according to his biographer Marinus, Proclus believed he was the reincarnation of the latter.

[20] Pappus probably did not author the collection as such, but only the constituent individual treatises which were put together long after Pappus's time.

[21] For more detailed discussions of this event and Pappus's account of it, see Knorr (1989, pp. 63–76), Lloyd 1996, Cuomo (2000, 127ff), and Bernard (2003).

periods, survived well into Late Antique Alexandria and that geometry counted among the possible objects of controversy. Not only the fact that young people challenged Pappus but also his sophisticated answer to the challenge[22] indicates the pervasiveness of the rhetorical model of learning and teaching. This model demanded that the actual exercise of discourse be taught to students to enable them to become immersed, by imitation, in the rules of composition of discourses (Bernard 2003).

Taken now as a literary composition which describes his encounter with students in a slightly idealized manner, the philosophical tale reveals the *values* behind this kind of challenge: for this purpose, Pappus refers to the classical debate between Speusippus (Plato's nephew) and Menaechmus (Eudoxus's student) about the nature of mathematical activity. The first maintained that mathematics was all theory-making, and the other countered that it was all problem-solving. With this classical conflict in the background, Pappus identifies (and praises) the students as followers of Menaechmus (since they proposed a solution to a geometrical problem) and himself as a follower of Speusippus (since Pappus demonstrates his ability to theorize the proposed construction through relevant means). The whole "refutation" of the construction – and its mixture of blame and praise – follows the literary convention for describing such agonistic encounters, with their incumbent heavily charged ethical aspects.

A third aspect of the encounter also demands attention: the entire challenge is based on geometrical figures, handed out in written form to Pappus, which he corrected or completed in his text. This strongly suggests that the actual discussion of these figures, if it ever took place, relied on a physical prop, a figure which, for this discussion, played the same role that the images (*eikones*) played in ancient rhetoric: a pretext for discourse and collective discussion. More than this cannot be said: no traces of any pedagogical device have been retrieved that could help us figure out how geometry was taught or discussed in scholastic assemblies (*sunousiai*).

In his biography of the fifth-century philosopher Proclus of Lycia,[23] Marinus of Neapolis describes (among other stages) both Proclus's training and his teaching methods after he succeeded Syrianus as the head of the Neoplatonist school in Athens. Like Marinus, Proclus in his own time reputedly displayed good enough knowledge of mathematics to have prepared a commentary on the first book of Euclid's *Elements* and knew enough about Ptolemy's *Almagest* to criticize him. As far as Proclus's training is concerned, Marinus makes clear that Proclus's wealthy parents, who were recognized notables, permitted him to travel from master to master, from whom he acquired skills ranging from rhetoric and declamation to mathematics and philosophy. As for mathematics, he is said to have been trained by a certain Hero, named by Marinus as an Alexandrian philosopher (*Vita Procli*, pp. 10–12). From Marinus's description, Hero appears to have probably taught Proclus the neo-Pythagorean mathematics useful to understand Plato's *Timaeus* and theurgic techniques in Hero's own home. The latter aspect is not uncommon: wealthy students traveling from place to place and from one teacher to another often boarded with their teachers and became some kind of spiritual children, called *gnôrimoi* (relatives of the teacher). Moreover, these details show that the kind of mathematics taught probably did not constitute a specialized subject but part of a philosophically oriented teaching, hardly separable from reading Plato.[24]

As an Athenian teacher of philosophy and mathematics, the intellectual activity of Proclus is represented by the two extant commentaries on mathematical authors mentioned above (namely, Euclid and Ptolemy) and also by Marinus's description of his usual pedagogical technique. According to

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[22] He answers not only by demonstrating his own capacity to analyze the shortcomings of the construction but also by suggesting that the students could have proceeded otherwise if they had possessed more knowledge of the underlying problems.

[23] Entitled "Proclus or On Happiness" = *Vita Procli*. This "biography" is better termed a hagiography. For the nature of Marinus's discourse, see *Vita Procli* XLI-C (Saffrey and Segonds).

[24] Such an approach to mathematics is already distinctly represented by Theon of Smyrna in the second century A.D. (Delattre 2010). For the noninstitutionalized framework of Late Antique education, see Derda et al. (2007, pp. 177–185) (E. Szabat) See also Watts (2006).

Marinus (*Vita Procli*, pp. 26–27), Proclus met with students for classes where he guided critical discussions on a traditional and varied material. In the evening, he would write down a record of his findings so that the extant commentaries probably represent the redaction of the notes taken from his courses. Again, more than this we cannot say: the archaeological remains of what might have been Proclus's house in Athens (Karivieri 1994) have revealed no special didactic settings. Nonetheless, these details indicate that the study of classics was, as much for mathematics as for other subjects, the core of advanced teaching. The classics discussed in this context, as the extant commentaries make clear, were not "mathematical" in any restrictive sense, but incorporated a much wider circle of knowledge, including Aristotle and Plato's writings.

The picture of a teacher surrounded by soliciting students for whom an extensive knowledge of classical works was prerequisite also seems to underlie the various prefaces written by Theon of Alexandria, a commentator from the second half of the fourth century, who elucidated Ptolemy's *Almagest* (Tihon 1992; Jones 1999; Bernard 2014). Here also, the basic material of the course consists of classical works, not only Ptolemy's treatise but also classical geometrical treatises and other commentaries on Ptolemy, which Theon encourages students to compare with his own (Bernard 2014). In this case, there are good reasons to believe that a significant portion of Theon's audience was comprised of practicing astrologers.

The above examples, however interesting, represent only a small and biased sample of the various didactic settings that might have existed in antiquity. It must be noted that these examples all belong to Late Antiquity, for which we possess a significant number of accounts of teaching, although they are also presented in an idealized way and according to precise literary conventions.[25] Some aspects of these testimonies are nevertheless confirmed by archaeological records, especially discoveries recently made of auditoria at Kom el-Dikka in fifth-century Alexandria (Derda et al. 2007). One should also note that these reports only concern elite teaching and learning. Typically Marinus's account of Proclus's training makes clear that he directly began his schooling *outside home* with a "grammatikos"; any elementary teaching he received must in all probability have been imparted at home, thanks to his wealthy parents, and not in any "primary" school (Kaster 1983, p. 334).

For earlier periods and from other kinds of evidence (like the few surviving papyri), the intrinsically "classical" character of ancient teaching is also confirmed, but tantalizing hints appear about elementary teaching, such as exercises in simple calculations – a venue of mathematical education which is altogether very badly represented in the mathematical works or commentaries (Fowler 1999, pp. 222–262). In Theon's commentaries or in the so-called prolegomena to the *Almagest* (Acerbi et al. 2010), there exists an exposition of calculation techniques, but these are hardly elementary because they relate to numbers expressed in sexagesimal numeration used only for astronomical (and therefore advanced) calculations imported from ancient Mesopotamia. Moreover, precisely because they are explained in such treatises, this style of calculation hardly appears to have been taught at an elementary level. The school exercises retrieved on papyri or *ostraka* are usually very difficult to situate precisely in terms of level and purpose.[26] Some of them, however, must have referred to the professional training of specialized slaves like scribes or calculators.

Finally, archaeological hints evoke very different teaching settings, like Egypt in the Roman period, where temples have existed in which astrological calculations were practiced as they were with astrologers who specialized in astral sciences and mathematics in Mesopotamia during the Hellenistic period (Jones 1994, 1999, p. 157).

[25] This particularity is best explained by the fact that this period is characterized by, among other things, the violent confrontation of various cultural and didactic models, especially between Christian and pagan models, which led each party to highlight and effectively represent these values.

[26] H.I. Marrou, in his short discussion on the teaching of elementary calculation, already warned against the too easy identification of papyri with mathematical content as corresponding to school exercises (Marrou 1964[6], note 10, pp. 398–399). Modern discussions confirm this.

All of these elements, however scarce and limited, indicate that we should certainly not generalize the picture afforded by more literary accounts, such as Pappus, Theon, or the later Neoplatonists. The existence of these accounts stems from the *de facto* selection of sources in proportion to their cultural value and literary sophistication (and therefore their value for the elite society alone) as well as from knowing much more about ancient education in Late Antiquity than about any other period of Greco-Roman antiquity.

In the face of this complexity, two possible approaches offer clarification. First, a widespread scholastic curriculum that would account for the various situations of Greco-Roman antiquity, and especially for the difference between elementary and advanced education, could be reconstructed. While this approach works for other cultural contexts, such as ancient Mesopotamia, this approach will be found to be highly problematic and, to some extent, sterile for Greco-Roman antiquity. Because of these difficulties, a more cautious and "localized" approach, which addresses the various elements of Greco-Roman society potentially concerned with mathematical education, may be adopted.

5.3 Disputes on the Existence and Nature of a Scholastic Curriculum in Greco-Roman Antiquity

For the past century, scholars have debated the existence of an institutionalized educational curriculum in antiquity. In the standard view that once prevailed, this curriculum could be neatly divided into three successive stages, respectively, labeled primary (or elementary), secondary, and higher or tertiary. Each stage had its own kind of teachers and school.[27] If only the contents (and not the institutional background) are considered, the first stage corresponds with the first acquisition of basic literacy (reading and writing skills) and numeracy (calculating skill with simple operations); the second stage with the study of advanced literature, especially poetry, with an emphasis on skillful reading up to the level of literary criticism and eventually including some instruction in higher mathematics; and the last stage with the learning of rhetorical skills or other advanced domains (philosophy, medicine, law).[28] The traditional view gives a straightforward interpretation of these various levels as representative of a progressive curriculum leading from elementary to higher studies, with specific teachers and locales for each level: the "teacher of letters" (*grammatistês* or *ludus litterarii*) for the first level; the "grammarian" (*grammatikos*, *grammaticus*)[29] and perhaps other *professores* (in geometry, arithmetic, astrology?)[30] for the second level; and rhetors, teachers of medicine or law, and philosophers for the last level.[31]

But this standard view, which was already heavily nuanced by its first proponents,[32] has been increasingly challenged since Booth and others have progressively demonstrated that another model

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[27] This standard and traditional view is found, among others, in the influential syntheses of Marrou (1965), Bonner (1977), and Clarke (1971).

[28] For a more detailed account of the contents of each level, see Cribiore (2001, Chaps. 6, 7 and 8, pp. 160–244); in those chapters, she focuses on only the basic *contents* of each level. See also the lucid and updated synthesis provided in Szabat (2007), with many references to the debates on these issues.

[29] The term "grammaticus" should not be understood as equivalent to our modern "grammarian," which now designates a distinct discipline. Although the latter was first constructed in antiquity, the competence of the "grammaticus" as a teacher extended much beyond mere "grammatical" analysis of literary and poetic texts: this teaching included a thorough initiation in the reading and analysis of a characteristic corpus of poets and classical writers. See Szebat (2007, pp. 185–187) for a synthetic summary and Kaster (1988) and Cribiore (2001, pp. 185–219) for more detailed explanations.

[30] Some "idealized" accounts allude to the existence of such "separate" professionals, but these accounts are uncertain and ambiguous. There is some, albeit scanty, evidence of such mathematical teaching at the secondary level. See Kaster (1983, p. 335) and Cribiore (2001, pp. 40–42).

[31] For a discussion of the thorny and interesting issue of the "technical" terminology of ancient education, see again Kaster (1983, pp. 329–331) and Szabat (2007), with references to other studies on the same subject.

[32] On the analysis of Marrou's precautions on this issue, see the insightful discussion of Kaster (1983, p. 324).

was applicable in some cases.[33] In this alternative view, elementary teaching should be considered as a basically separate track for people of the lower social level (including slaves), whereas the schools of *grammatikoi* were reserved for a higher elite.

The now accepted view[34] is that no model applies uniformly to all of antiquity or throughout the entire Mediterranean world. Not surprisingly, Booth's model of separate tracks seems better adapted for the big cities of antiquity, whereas the situation in small localities, with few teachers and very specific needs, would have been much more variegated. Thus, the main positive conclusion – perhaps the only indisputable one – of this scholarly debate is that "there were throughout the Empire schools of all shapes and kinds, depending on local needs, expectations, and resources." Kaster rightfully adds that "in a world without centralized direction of education of any sort, that is only what we should expect."[35]

If no single model can be applied to all (known or unknown) teaching situations in antiquity, then it is worthwhile to detail the main reasons why the modern, three-stage curriculum cannot be considered valid. The first reason has already been mentioned: ample reports indicate that, in many cases, the various "stages" of education did not concern the same people, so there may be no stages at all, but only references to different teaching contents for different people.[36] The curriculum of the elite, which is naturally overrepresented in the ancient literature, concerned the same people (the rich and wellborn) and was strongly characterized by a relatively uniform and well-defined idea of literate culture that encompassed a limited set of classical authors and well-identified kinds of exercises practiced on them, from reading to critical analysis.[37] But even then, the order of studies at this stage was not completely fixed and depended on particular teachers and the length of study based on each student's means: a significant rate of attrition existed, and the number of students decreased with the numbers of years of study. This characteristic conforms to the structure and contents of ancient teaching, which sought to deepen the understanding and study of classical texts rather than to attain a definite goal through an accumulative process. The same texts, therefore, were studied again and again but each time in a deeper way until the students, through imitation and impregnation, were able to compose or declaim on their own.[38] Moreover, the number of various elite "professions" that used this secondary curriculum shows that "secondary and tertiary" curricula were not uniform.

The second reason is that there is no clear proof of, and many counterexamples to, a fixed and unambiguous correspondence between these three "levels of teaching" and the competence of particular teachers. Thus, to take up the most discussed issue in the abovementioned studies, the same *names* of teachers, like "grammatistês" (teachers of letters), could actually refer to various contents or levels, from elementary to "secondary" (Kaster 1983, pp. 329–331; Szabat 2007, pp. 181–185). Likewise, *grammatikoi* could prepare students for higher achievements, either rhetorical or philosophical. Thus, the famous philosopher John Philoponus officially was a "grammarian" but was known to be a philosopher and commented on mathematical texts. This also means that the same person could teach at different levels, sometimes at the same time.

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[33] Booth paid attention to the situation in first-century AD Rome. His theses (Booth 1979) are conveniently summarized in Kaster (1983), who expands on his argumentation.

[34] For an efficient summary, see Szabat (2007, pp. 178–181), who draws on previous studies, esp. Kaster (1983).

[35] Kaster (1983, p. 346). The same point is made in Cribiore (2001, Chap. 1) (pp. 15–44) for the sole case of Hellenistic and Roman Egypt and Szabat, *op.cit*. Even the imperial state did not heavily intervene in educational institutions. At best, laws would oblige cities to finance municipal chairs, without intervention in and regulation of their study. The majority of teachers, though, worked privately and directly depended on fees from students and their parents.

[36] Kaster (1983, pp. 337–338) summarizes the "positive" reasons to believe that Booth's model is better adapted in general to antiquity, although it should not be viewed as an alternative model applicable to all ancient situations.

[37] On the uniformity and strong identity of the *grammatikoi*'s teaching, the classic study is Kaster (1988).

[38] This idea of "concentric" studies, in which the same elements and methods are retrieved at each level but with a different depth and difficulty, is central to the argument of Cribiore (2001).

The third reason is that, more often than not, students of various levels learned together, all in the same space, rather than in separate classrooms organized according to the teaching. In general, the locales of teaching, if there were locales at all (some teachers worked in the street), are not easily identified; when they are, they could have belonged to various institutional buildings, from gymnasia to theaters or temples. In fact, what a “school” referred to, in the ancient context, should be understood in personal terms: as a circle of students frequenting a teacher.[39]

Last but not least, the majority of inscriptions or papyri, especially those that can be found in small localities away from the great urban centers of Rome, Alexandria, or Antioch, show that the teachers’ presence, competencies, and missions depended heavily on the local context. This is actually the main reason to remain cautious about the “standard,” contemporary and (often) prescriptive accounts of ancient education, like in Quintilianus (Kaster 1983).

5.4 *Various Kinds of Mathematics Teaching for Various Parts of Society and Professional Circles*

Surviving sources preserve mere hints about the particular places and settings in which mathematics was taught in Greco-Roman antiquity. More than this is not really known, as shown by the difficulty of reconstructing a uniform scholastic curriculum in Greco-Roman antiquity. In fact, the various places, periods, and levels in society in which some kind of mathematics was practiced in various ways have left only fragmentary evidence or virtually no trace at all. Despite this dearth of information, some people outside of the elite might have been concerned with mathematical training.

5.4.1 The Elementary or Specialized Teaching of Arithmetical Skills: Slaves and Freedmen, Professional Scribes, and Accountants

As mentioned, little documentary evidence with mathematical content has been found on papyri or *ostraka* that might potentially be interpreted as school exercises. The available evidence[40] has to be checked against the complex and “fuzzy” background of mathematical education. Apart from a few papyri that might be interpreted as stemming from the study of Euclid’s *Elements* and therefore as belonging to the “secondary” level, several tables of multiplication and fractions have also been retrieved. But the problem is whether these writing exercises belonged to the elementary level, or whether they should be interpreted, as Cribiore argues,[41] as belonging to a more specialized curriculum for scribes. Written exercises in calculation, especially addition, would by contrast be the exception rather than the rule: elementary operations were probably taught and practiced orally or with an abacus. More advanced exercises might have been executed by writers and calculators already in training, according to the quality of their writing. Because mastering such skills seemed alien to the spirit and contents of “secondary” teaching, this aspect of education might be plausibly interpreted as the production of advanced slaves trained to be *notarii* (professional scribes) or *calculatores* (accountants) who were completing a professional training. Not surprisingly, given the general background and nature of the so-called “elementary” level of teaching as well as uncertainties about its real nature, the mathematical documentation which might be considered “scholastic” is no less

[39] This point is made in Szabat (2007, pp. 180–181) and Cribiore (2001), Chap. 1 (on school accommodations) and Chap. 2 (on teachers).

[40] Fowler (1999)gives the sole extensive discussion on the papyrological evidence concerning mathematics.

[41] Cribiore (2001, pp. 180–183). This short discussion is devoted to the question of the acquisition of numeracy at the elementary level.

fuzzy and uncertain. Part of it, though, might correspond to a specialized curriculum concerning people (including slaves) with a low social status.

In any case, the complete classification and study of those fragments is still an open question, and the achievement of positive results through such a study remains uncertain because of the small number of such documents. Here, the interesting consequence of this general scarcity of sources on the *practical* dimension of the teaching of mathematics, especially at an elementary level, is the confirmation that our documentation is strongly biased. By contrast, sources for the *idealized* curricula are much richer and substantial for Greco-Roman antiquity. Among these sources are highly sophisticated treatises (like Plato's *Republic* or, much later, the commentaries of Proclus), prefaces and introductions (which convey a sense of the didactic tradition), or pedagogically structured treatises (like for Ptolemy's *Almagest*). The variety of these idealized approaches is too vast to be summarized here, and many good studies are available on the subject.

However, one revealing aspect of these works (especially within the prefaces) demands consideration here, namely, the *target audience*. The target audience refers to the milieu that might have been concerned with these ideals and also, to some extent, that were *represented* by them: defining an ideal curriculum expressed the shared cultural values that defined not only the milieu but also its *raison d'être*. One classical example is provided by the case of ancient astrology. For the ancients, astrology implied a demanding and complete *training* and the mastery of a sophisticated cannon and detailed techniques.[42] In general, then, the definition of a culture, including an idealized training system, constituted part of the social identity in antiquity. The various educated circles potentially connected with such a self-definition, and the culture of which might have included some mathematical training, merit a brief review.

5.4.2 Mathematical Training as Part of Philosophical Education

That mathematical subjects might have been considered appropriate for a philosophical curriculum, either an ideal one (as in Plato's *Republic*) or a real one (as with the late Platonists like Iamblichus in the fourth century, or Syrianus and Proclus in the fifth century), is a fact so obvious that it is impossible to review exhaustively the wide range of philosophical positions and schools for which this idea was meaningful.[43] In this long story, Plato and the varied company of Platonists are well represented. Despite the breadth of the topic, the historical importance of three particular ideals merits their acknowledgment and exposition.

The first two of these ideals can probably be regarded as varieties of Platonism. The first ideal is neo-Pythagorean, represented in the second century by Nicomachus and Theon of Smyrna, whose treatises focused on so-called neo-Pythagorean arithmetic and represent a sophisticated philosophical project. Theon particularly relies on a coherent organization of mathematical knowledge, especially the four sciences (arithmetic, geometry, astronomy, and music), which later formed the scholastic *quadrivium* at the edge of the Middle Ages (Hadot 2005, Chaps. III and IV; Vitrac 2005).

The second ideal is that of the Ptolemaic philosophical way of life, centered on the study of mathematics, especially the kind of mathematics related to the movements of the stars and apt to bring the soul of the philosopher closer to this cosmic movement (Taub 1993, Chap. 2 and 5; Sidoli 2004; Feke and Jones 2010; Bernard 2010). While this ideal is basically Platonic in spirit, it does not subordinate the *study* of mathematics to higher studies (like dialectic) but, on the contrary, recommends

[42] For an account of ancient Greek astrology, the standard reference remains Bouché-Leclercq (1899). See also the more recent Barton (1994), especially pp. 134–142 as far as astrological training is concerned.

[43] For an updated extensive study on this question, see Hadot (2005), especially the fourth "étude complémentaire," pp. 431–455, concerning mathematics. Note, however, that Hadot has a tendency to reduce any ancient mathematical teaching to being basically dependent *in all cases* on a philosophical ideal, an idea which is somewhat open to criticism.

mathematics as the highest philosophical study above all others. The *Almagest*, accordingly, is structured as the basis of such a philosophical study and elevation of the mind. This ideal, associated with this impressively well-organized exposition of ancient astronomy in the *Almagest*, proved to be highly influential throughout Late Antiquity, the Middle Ages, and up to the early modern period.

The last philosophical ideal is that of Isocrates, Socrates' other disciple besides Plato. Isocrates proved to be influential on the founders of classical Latin rhetoric, Cicero and Quintilianus. In Isocrates's philosophy, which is not meant as a system but as a particular way to cultivate discourses, mathematical training is essential not for its contents but for the effect it possesses *as a training device*, leading the student to analogous but higher studies. He therefore formed the ideal of mathematics as a preparatory stage in rhetorical education.[44]

5.4.3 Mathematical Training as Part of an Astrological and Astronomical Training

If Ptolemy's ideal can be viewed as one legacy of the literature of philosophy because of his explicit or implicit references to the "grand" philosophical literature (Plato, Aristotle, the Stoics), it might be also be understood in relation to the tradition and culture shared by ancient astrologers. Within the curriculum and ideal training of astrologers, as expressed in Vettius Valens, Firmicus Maternus, Ptolemy, or other authors, calculation of the positions of stars in order to establish an astrological chart is described by these authors[45] as an indispensable first step in the standard training of an astrologer, which expanded to include mastering a body of knowledge that enabled him to interpret the sublunary significance of the astronomical phenomena (Bernard 2010).

Among these ancient representations of astrological training, one recorded in Ptolemy's so-called *Tetrabiblos* is probably the most sophisticated. Here, the astronomical knowledge necessary for the computation is represented not only as an indispensable part of the science concerning the "physical" effects of planetary positions on sublunary events but also as a science desirable in itself (with a reference to the *Almagest*, which presents itself as a self-contained treatise). This simultaneously coherent and "bivalent" system is probably an extreme element within the spectrum of approaches to astrology in the same period. Papyrological evidence has clarified that Greco-Roman astrologers had recourse to calculation techniques other than the cinematic tables advocated by Ptolemy (Jones 1999). Nonetheless, Ptolemy's astronomical text became the subject of commentaries (by Pappus of Theon) by the fourth century, thus consecrating the pedagogical ideal imbedded in the *Almagest* and advancing it into new directions.[46] The importance of Ptolemy's works lies precisely in how he turned a technical treatise of astronomy into classical knowledge, liable to the activities of commentary (Jones 1999, 160ff).

The contents of the commentaries on Ptolemy's *Almagest* and *Handy Tables* are only partly edited, as are the *scholia* to these texts and their commentaries (Tihon 1992). Most of these probably reflect pedagogical activities, the precise nature of which is difficult to reconstitute even if their existence is beyond doubt. The use of geometrical diagrams to explain or detail procedures, the explanation given on "elementary" operations in the sexagesimal system, or the many references to mnemonic schemas most probably reflect teaching activities oriented toward the appropriation of the complex contents of Ptolemy's works (Tihon 1992).

[44] See Isocrates's ideas in *Antid.* pp. 258–269. For Quintilianus, see *Inst. Orat*. I.10, especially pp. 34–49.

[45] Even by those, like Firmicus, who obviously had little command of the mathematical contexts of their art.

[46] These commentaries can indeed be seen, at least in part, as conscious imitations of the *Almagest*; see Bernard(2014) on this point.

5.4.4 Mathematical Training as Part of the Training of Land Surveyors, Engineers, and Architects

Our knowledge of Greco-Roman technology and its specialists, if and when such specialists existed, is complicated by the impressively diverse state of our sources, the differences in the social status of their authors, and the variety of points of view on and classifications of the related subject (Cuomo 2007). Besides highly influential and extant works like Vitruvius's *De Architectura*, some of the sources on these domains, like the pseudo-Heronian and metrological corpus in Greek or the corpus of land surveyors (*corpus agrimensorum* = CA) in Latin, have survived in such a state of confusion that the study, attribution, and characterization of their contents remain a work in progress.[47] Even an attempt to draw sharp distinctions between architecture and the construction of machines for land surveying, for example, leads quickly to thorny problems.[48] Also, the social status of the authors or readership of the related treatises is far from clear. Thus, if there is little doubt that some authors of gromatic literature[49] identified themselves as belonging to a well-defined profession (*mensores*), others, like Frontinus or Vitruvius, had a higher social status with political responsibilities and thus treated a wide range of technical subjects. Even among "professional" Roman land surveyors working during the Roman Empire, there are good reasons to think there was a significant variety of functions and social status with different types of training (Hinrichs 1989, pp. 171–174).

Thus, for the Roman land surveyors working during the imperial period (from which most of our information comes), Hinrichs proposes to distinguish between four categories of mensores, according to their function and social positions: land surveyors (a) who served in the army, (b) who were in the service of the emperor, (c) who were employed (or enslaved) by municipalities, and (d) who were private or independent. The first must have been trained within the army, according to a specific tradition about which we have no details. As for the imperial functionaries, Hinrichs speculates that their typically Greek names and the existence of a Greek "technical" tradition of land surveying indicate that they were slaves or freedmen who received their training in Alexandria; the specific "civil" curriculum of the two last categories is unknown, but there is a good chance that the contents of the CA were actually developed and/or used in this context.

Indeed, while the sources are incredibly varied and complex, there is no doubt that a significant portion related to pedagogical purposes, ideals, or realities.[50] Furthermore, part of the corresponding training included mathematical skills and knowledge of astronomy, geodesy, land surveying, measurement problems, and calculating techniques (Chouquer and Favory 2001, pp. 64–94). Finally, the contents of this corpus were influential on medieval and Renaissance mathematics teaching, and their importance cannot be underestimated.

For the sake of simplicity, some general features of these sources and their underlying teaching background may be sketched broadly. First, these sources contain idealized representations of curricula, like the sophisticated discussions of *De Architectura*, book I and book IX, or Pappus's *Mathematical Collection* VIII, which might be partly derived from the lost introduction to Hero's *Mechanika*. These texts significantly characterize mathematical skills as being basically dependent on

[47] As far as the corpus agrimensorum is concerned, the work of Toneatto (1994-5) has drastically improved our understanding of its history; see Chouquer and Favory (2001, Chap. 1).

[48] Is Hero's *Dioptra*, for example, an exercise in land surveying, as the kinds of problems treated therein strongly suggest, or the skillful description of an instrument, as the preface and many technological details indicate? Is Vitruvius's treatise merely a work on monuments and house-building, or also on machine-building (book 10), the science of sundials and astronomy in general (book 9), and many related subjects, as the contents suggest?

[49] Like Hyginus "gromaticus," the second Hyginus or Siculus Flaccus. For translations of these authors, see Campbell (2000) or the various annotated editions published in French by J.Y. Guillaumin, in particular Guillaumin (2005, 2010).

[50] It has even become commonplace in scholarship on these kinds of sources that they represent didactic efforts and are scholastic "manuals," a qualification difficult to dismiss because of the vague and multifarious meanings of this category. The idea is discussed and nuanced in Chouquet and Favory (2001, p. 38).

the wider culture of the architect and *useful* for that culture. Such discussions about the *utility* of mathematical training within a larger intellectual framework are also characteristic of the introductory material contained in the Greek metrological corpus and in some prefaces in the CA. Vitruvius and Pappus/Hero also insist on the importance of developing creative skills which combine theoretical prerequisites with practical skills.

This last point relates to a second feature of the aforementioned sources: significant parts of these sources are organized as series of problems often arranged in order of growing complexity. This organization bridges from simple, practical motivations to more theoretical or didactic concerns. Problems such as measurement are typically accompanied by algorithms for calculation generally associated with pedagogical activities or purposes, even though the exact connection may not be obvious at the elementary level of teaching.

The third feature has been mentioned already: these corpuses survived in a state of deep confusion and disorder, often explained by the fact that the original texts were significantly changed and reorganized throughout the course of their history, sometimes for pedagogical purposes. Indeed, it is sometimes easier to situate and identify these reorganizations and purposes – for example, late Neoplatonists reworked the metrological corpus[51] – than it is to reconstitute the extent and purpose of the original sources. The same remark potentially holds for the question of illustrations, which are numerous in the CA. Some of these might come from original sources, but many others might have been added later, some for pedagogical purposes.

Finally, scholars often note that these corpuses are often missing significant parts. In the case of Latin *agrimensores*, clear allusions frequently occur to lost Greek technical treatises on related subjects. Hero's *Dioptra* is often mentioned as a possible source, or at least as a treatise similar in kind to this lost material, but in all probability this is only the tip of the iceberg. This is a good example of a domain for which we might well have lost the majority of material available in antiquity.

5.4.5 Mathematical Training as Part of an Unknown Culture of Arithmetical Problem-Solving

Finally, Diophantus's *Arithmetica*, one of the most mysterious mathematical treatises of antiquity, must be addressed. Although there is no certainty, the treatise may have been composed in Late Antiquity, around the third century, yet its author is unknown. The treatise contains 13 books of arithmetical problems (statements and solutions) arranged in progressive order; six books are extant in Greek and four more have been found in Arabic translation. The long preface survives in Greek and is structured around a didactic project: to enable the reader of the treatise to develop the capacity for invention in arithmetical problems (*Arithm.* 2.3–13). The problems and their treatments are indeed arranged in an order that enables the acquisition of a whole range of specific techniques useful for the solution of problems (Bernard and Christianidis 2011, part 4). Moreover, the reference to the rhetorical notion of *invention* (*heurêsis*) points toward the influence of higher culture and rhetorical training on the structure of the treatise. According to Diophantus's own words, his whole treatment is organized as paving a way (*hodos*) to the reader, an example that he might imitate (Christianidis 2007; Bernard and Christianidis 2011, part 5; Bernard 2011).

There is little doubt that the treatise directly relates to a clearly defined didactic strategy akin to the techniques used in ancient rhetorical training. What remains almost entirely absent from the picture is the larger background constituted by the arithmetical problems studied within the "logistic tradition." This expression usually refers to the problem-solving and calculation tradition relative to the kind of arithmetical problems practiced for millennia in ancient Mesopotamia or ancient Egypt or afterwards

[51] Acerbi and Vitrac forthcoming: introduction, A4. A preliminary version of this detailed analysis is available online on hal-SHS http://hal.archives-ouvertes.fr/hal-00473981/fr/ (consulted 5.1.12).

in the Middle Ages. But, as far as Greco-Roman antiquity is concerned, we have only hints and desperately few documents claiming that such a tradition existed[52]: the bulk of it has disappeared. Diophantus's problems might be interpreted as "abstract" problems akin to the scholastic and *ad hoc* rhetorical problems invented for the sake of rhetorical training.

Diophantus's *Arithmetica*, therefore, might be the tip of yet another iceberg, the splendid and isolated outcome of a much larger and widespread tradition of arithmetical teaching through problem-solving, for which there is almost no trace at present.

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Chapter 4
Mathematics Education in Oriental Antiquity and Middle Ages

Agathe Keller and Alexei Volkov

This chapter is devoted to the history of mathematics education in Asia during the ancient and medieval periods. As no systematic account of the history of all Asian countries can be given here, this chapter will focus only, on the one hand, on China (more precisely, the political formations that existed in what is today Chinese territory) and certain states that came under its direct cultural influence (what are today Korea, Japan, Vietnam) and, on the other hand, on India.

It should be said at once that even the chronological boundaries of the period under investigation turn out to be difficult to establish precisely and uniformly for all the territories studied: in discussing the Indian subcontinent, it turns out to be convenient in this chapter to examine the so-called premodern period, from the thirteenth to the eighteenth centuries, as a single entity, while for China and Japan the premodern period will be defined in a different way, beginning in the seventeenth century, and discussed in another chapter.

This chapter consists of two sections, written, respectively, by Alexei Volkov and Agathe Keller.

1 Mathematics Education in East- and Southeast Asia

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1.1 Introduction: Recent Publications on Mathematics Education in China

Certain periods of the history of Chinese mathematics education were discussed in publications in Western languages. Excerpts from Chinese historical documents related to mathematics instruction and examinations of the Tang 唐 dynasty (618–907) were translated and commented upon by Robert des Rotours (1932), while an unpublished dissertation of Joseph Wong (1979) contains a detailed description of the educational institutions in China and Japan in the first millennium AD including

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their mathematical curricula.[1] Publications of Siu Man Keung 蕭文強 and the author of this section (Siu 1993, 1995, 1999, 2004, 2009; Siu and Volkov 1999; Volkov 2012a) focus on various aspects of Chinese mathematics instruction of the first millennium AD, while a number of documents pertaining to the mathematics instruction of the Song dynasty have been translated and discussed by Manfred Friedsam (2003). Thomas Lee Hung-Chi's 李弘祺 comprehensive monograph on the history of education in China (Lee 2000) provides a number of useful references, especially in the sections devoted to "technical education"; see also an earlier monograph by the same author (1985) devoted to the education in China during the Song 宋 (960–1279) dynasty.

In Asian languages, the first most substantial contribution to the topic was made by Li Yan 李儼 (1892–1963); in his seminal paper (Li 1933 [1977]) he quoted a large number of excerpts from extant Chinese historical sources relevant to the mathematics education of the Tang, Song, Yuan 元 (1279–1368), and Ming 明 (1368–1644) dynasties. Unfortunately, this impressive collection of quotations was not accompanied by an equally thorough discussion of educational activities and examination procedures. Authors of numerous works on the history of Chinese science and mathematics education that appeared in Mainland China after the publication of Li's paper approached the topic in different ways, often placing mathematics education in China in a larger historical and sociocultural context. For example, Sun (1996) offers a description of mathematics education in China starting from the earliest period of its written history and ending by the introduction of Western mathematics in the seventeenth and eighteenth centuries, while the monograph of Tong et al. (2007) provides a description of mathematics education in traditional China combined with information about the contents of the Chinese mathematical treatises used as textbooks in official educational institutions during the first millennium AD. Some authors focus on more specific questions; for example, Guo Shirong (1991) discusses the role of state-run educational institutions in the advancement of mathematical knowledge.

1.2 Mathematics Education in China Prior to the Mid-First Millennium AD

1.2.1 Mathematics Education in China from Antiquity to the End of the Han Empire

A reference to the earliest system of mathematics education in China is found in the treatise *Zhou li* 周禮 (Rituals of the Zhou [Dynasty]); it mentions *jiu shu* 九數, "nine [kinds of operations with (?)] numbers,"[2] as one of the "six arts" (*liu yi* 六藝), the curriculum used to instruct the sons of aristocratic families and introduced by the first de facto ruler of the Zhou 周 dynasty (1046–256 BC), the Duke of Zhou (Zhou gong 周公). Since the *Zhou li* is a relatively late compilation produced, according to modern Western scholarship, in the fourth or third century BC (Boltz 1993), the mention of mathematics education at the court of the Zhou dynasty is most likely not a contemporaneous description but a later reconstruction based on unidentified sources. In his preface to the mathematical treatise *jiu zhang*

[1] In this chapter, the *pinyin* transliteration system adopted in P.R. of China and European sinology is used for the ancient and medieval Chinese personal names, terms, and titles of treatises. However, the reader should be aware that the reading of Chinese characters went through considerable changes during the period from the late first millennium BC to ca. AD 1500, which render the *pinyin* transliteration a pure convention used solely to facilitate referring to names and titles and irrelevant to their actual historical pronunciation. The names of Chinese authors from Hong Kong and Taiwan are transliterated according to the transliteration systems adopted by the authors themselves. In all cases, I used *traditional* Chinese characters for personal names and titles of publications and provided their simplified versions currently used in the P.R. of China only when they were originally used by the authors.

[2] The term *jiu shu* could be understood as "nine numbers," yet its comparison with the other "arts" such as, for example, *wu she* 五射 (five [styles of] archery) and *liu shu* 六書 (six [styles of] writing), suggests that the word *shu* 數 here most likely meant "operations with numbers" or "numerical procedures"; see below on alternative interpretations.

suan shu 九章筭 (or 算)術 (Computational Procedures of Nine Categories),[3] its commentator, Liu Hui 劉徽 (fl. AD 263), quoted this mention of *jiu shu* found in the *Zhou li* and interpreted the term *jiu shu* as referring to a prototype of the *Jiu zhang suan shu*. In doing so, Liu followed the interpretation of this term offered by Zheng Zhong 鄭衆 (?–83) and later quoted by Zheng Xuan 鄭玄 (127–200) in his commentary on the *Zhou li*. Some authors argued against this interpretation and suggested that the "nine numbers" may have referred to the multiplication table (Liu 1944). It cannot be ruled out either that the term "nine numbers" referred to the numbers from one to nine and, more broadly, to all numbers and operations with them (this interpretation was suggested in Ou et al. 1994, p. 235).

Another oft-quoted yet extremely brief mention of mathematical instruction in ancient China is found in the chapter "Nei ze" 內則 (Models [to follow at] Home) of the *Li ji* 禮記 (Book of Rites) compiled in the first century AD on the basis of earlier materials (Riegel 1993); it reads as follows: "When [children] are five years old,[4] [one] teaches them [operations with?] numbers (*shu* 數) and names of cardinal points."[5] Even though the excerpt allows multiple interpretations of the term *shu*, it makes rather clear that basic mathematical skills were taught to children at a fairly young age.

It appears plausible to suggest that the *Jiu zhang suan shu*, considered until recently the earliest extant Chinese mathematical treatise, was compiled and used for educational purposes. The dates of the compilation of the treatise and of its successive revisions, produced prior to the version commented upon by Liu Hui, remain unknown, yet modern historians agree that the book may have been based, at least partly, on prototypes compiled prior to the late third century BC (see, e.g., Cullen 1993; Chemla and Guo 2004). The hypothesis of the original use of this treatise for educational purposes fits well into its legendary history, again found in Liu Hui's preface; however, Liu's claim is not the main reason to advance it.[6] Instead, an analysis of the very contents of several chapters suggests that the numerical parameters of the problems of the treatise were selected to provide an instructor with a set of didactical tools (Volkov 2012b, 2013).

A mathematical treatise titled *Suan shu shu* 筭數書 (*Scripture on Computations* [*To Be Performed*] *with Counting Rods*)[7] and written on bamboo strips was found in China in 1983 in a tomb sealed no later than 186 BC,[8] and two other ancient mathematical texts were unearthed even more recently. The first of them, titled *Shu* 數 (*Numerical* [*Procedures*] or [*Book on*] *Numbers*), was compiled during the Qin 秦 dynasty, 221–206 BC (Zou 2011), while the second text, titled *Suan shu* 算術 ([*Computational*] *Procedures* [*To Be Performed with*] *Counting Rods*), is dated to the Former Han 前漢 dynasty (206 BC–AD 6).[9] The studies of the *Suan shu shu* as well as available information about the two other

[3] The term *jiu zhang* 九章 that appears in the titles of a number of Chinese mathematical treatises, including that of the extant editions of the *Jiu zhang suan shu*, is usually rendered in the Western historiography as "nine chapters," resulting in such translations as *The Nine Chapters on the Mathematical Art*. For a discussion of the meaning of this term, see Volkov 2010, p. 281, n. 1.

[4] The text literally reads "six years," yet traditional Chinese age count starts from the moment of conception.

[5] *Li ji zhu shu* 禮記注疏 (*Book of Rites, with Commentaries and Explanations*), *Siku quanshu* 四庫全書 edition, *juan* 28, p. 27b.

[6] Cullen (2007, p. 34) strongly doubted the degree of Liu Hui's awareness of the actual early textual history of the treatise.

[7] In this chapter I use the character *suan* 筭 instead of *suan* 算 in the titles of a treatise or the names of institutions if in at least one source the former character was used instead of the latter. The original meanings of these two characters were not the same: according to the etymological dictionary *Shuo wen jie zi* 說文解字 (Explaining [Written] Signs and Analyzing [Compound] Characters) by Xu Shen 許慎 (AD 55?–149?), the character *suan* 筭 meant "counting rods," while *suan* 算 meant "operations performed with the counting rods," that is, "computations."

[8] The treatise was published for the first time in 2000 (SSS 2000) and later reproduced photographically (SSS 2001, 2006); it was translated into Japanese (Jochi 2001; Ohkawa et al. 2006), modern Chinese (Horng et al. 2006), and English (Cullen 2004; Dauben 2008). For references to publications devoted to the *Suan shu shu*, see Ohkawa et al. 2006, pp. 168–169; Dauben 2008, pp. 172–177; and Zou 2008, pp. 95–98.

[9] A short excerpt from the *Suan shu* 算術 was published in 2008 (SS 2008) and studied by Karine Chemla and Ma Biao (2011).

hitherto unpublished texts show that these treatises share a number of common features with the *Jiu zhang suan shu* and are arguably related to traditions of mathematics instruction antedating the late second century BC (Volkov 2012b).

A curious artifact that may bear witness to the procedures of mathematics instruction in ancient China was found in an Eastern Han dynasty (AD 25–220) tomb near the present-day city of Shenzhen in Guangdong province. It is a clay brick upon which a part of a multiplication table was scratched before the brick was put into the oven; the inscription contains two (vertical) lines: the left line reading "9 • 9 = 81, 8 • 9 = 72, 7 • 9 = 63, 6 • 9 = 54, 5 • 9 = 45" and the right line reading "3 • 9 = 27, 2 • 9 = 18, 4 • 9 =36."[10] The person who made this inscription apparently started with the 9 • 9 and reached the edge of the brick when writing the product "45"; after that, he or she started writing to the right from the first line, scratched "4 • 9 =36," and reached the edge again and then started writing from a new position above and wrote two remaining products: 3 • 9 and 2 • 9 (Ou et al. 1994, p. 232). This poor management of writing space, the handwriting style, and an unorthodox graph of numeral "7" in one position suggest that the inscription was made by a child of the artisan who made the brick. The child used the still soft surface of clay to scratch a part of the multiplication table he or she was supposed to learn by heart, and his or her father later put the brick in the oven without noticing the inscription.

1.2.2 Institutionalized Mathematics Education in China in the Early First Millennium AD

A state-run School of Computations (*Suan xue* 筭學) was established during the Northern Zhou 北周 dynasty (557–581) in the capital city of Chang'an 長安 (modern Xi'an 西安); there are reasons to believe that a prototype of this school existed even earlier, during the Northern Wei 北魏 dynasty (386–534).[11]

It seems plausible to conjecture that there was a connection between the institutionalized mathematics instruction that existed under the Northern Zhou dynasty and a set of mathematical treatises produced by Zhen Luan 甄鸞 (fl. ca. 570).[12] Zhen sometimes was mentioned as the author of some of these treatises or their commentator. According to the *Old History of the Tang* [*Dynasty*] (*Jiu Tang shu* 舊唐書) completed in 945 (JTS hereafter), the mathematical treatises compiled or commented upon by Zhen Luan included the *Zhou bi* 周髀 (Gnomon of the Zhou [Dynasty]) in one volume (*juan* 卷),[13] *Jiu zhang suan jing* 九章筭經 (Computational Treatise on Nine Categories [of Mathematical Methods]) in nine volumes,[14] two versions of the *Wu cao suan jing* 五曹筭經 (Computational Treatise of Five Departments) in three and five volumes,[15] *Sunzi suan jing* 孫子筭經 (Computational Treatise of Master Sun) in three volumes (in JTS Zhen Luan is mentioned as its compiler *and* commentator), *Zhang Qiujian suan jing* 張丘建筭經 (Computational Treatise of Zhang Qiujian) in one volume (in JTS Zhen Luan is mentioned as its compiler), *Xiahou Yang suan jing* 夏侯陽筭經 (Computational Treatise of Xiahou Yang) in three volumes (in JTS Zhen Luan is mentioned as commentator of this

[10] The original Chinese text reads "nine nine eighty one eight nine seventy two…"; the symbols of multiplication and equality are added for the convenience of the modern reader.

[11] Sun 2000, p. 138; see also Lee 2000, p. 515, n. 230. Even though Lee did not find any evidence supporting this statement of Sun, he agrees that mathematics was systematically taught in state-run institutions prior to the Northern Wei.

[12] For biographical data of Zhen Luan, see Volkov (1994).

[13] JTS 1975, p. 2036. This treatise in one volume, commented by Zhen Luan, was apparently different from the treatise bearing the same title but subdivided into two volumes and listed on the same page as compiled by Li Chunfeng李淳風 (602–670).

[14] JTS 1975, p. 2039. Zhen Luan is mentioned as the compiler of the treatise.

[15] JTS 1975, p. 2039. Zhen Luan is mentioned as the compiler of both texts. Some authors believe that the catalog contains a scribal error and Zhen Luan was not the compiler of the three-volume treatise; see JTS 1975, p. 2083, n. 6. One cannot rule out the possibility that this entry resulted from an error made by a copyist when writing the title of another treatise presumably authored by Zhen Luan, the *Wu jing suan shu* 五經筭術.

and two following treatises), *Shu shu ji yi* 數術記遺 (Records of the Procedures of Numbering Left Behind for Posterity) in one volume, and *San deng shu* 三等數 (Numbers of Three Ranks) in one volume. Some sources also credit to Zhen Luan's authorship the *Hai dao suan jing* 海島筭經 (Computational Treatise [Beginning with a Problem] about a Sea Island)[16] and *Wu jing suan shu* 五經筭術 (Computational Procedures in the *Five Classical Books*) (Li 1937, pp. 33–35).

According to the abovementioned records, Zhen Luan edited, compiled, or commented upon eight to ten out of the twelve mathematical treatises bearing the titles identical to those of the textbooks used in China in the School of Computations some time later, during the Tang dynasty (618–907) (see below). The fact that the treatises compiled or commented upon by Zhen Luan are listed in the *Jiu Tang shu* may suggest that they were still extant when this text was compiled, that is, in the mid-tenth century AD; therefore, one cannot rule out the possibility that they were used for instruction in the Tang dynasty School of Computations simultaneously with the versions edited by Li Chunfeng 李淳風 (602–670) and his team in 656 (see section 1.3.2 below) or used elsewhere by students and private tutors.

1.3 Mathematics Education in China During the Sui and Tang Dynasties and Its Transmission to Korea and Japan

1.3.1 Mathematics Education in China During the Sui Dynasty

A state-run School of Computations was established during the Sui 隋 dynasty (581–618) that unified China after a long period of disunity. It is unknown whether there was any relationship between this institution and the abovementioned school functioning during the Northern Zhou dynasty, yet such a conjecture sounds highly probable. It is known that the instruction was conducted by one or two "erudites" (*boshi* 博士)[17] and two "teaching assistants" (*zhujiao* 助教) and that the number of students totaled 80.[18] No information about the curriculum and examinations is available, yet the closeness of the titles of the treatises edited, compiled, or commented upon by Zhen Luan and those used for instruction during the Tang dynasty (see below) suggests that at least some of these treatises were used in the Sui dynasty school.

1.3.2 Mathematics Education in China During the Tang Dynasty

The earliest mention of the school is dated 628; in this year, instructors were hired and students admitted.[19] The age of the students entering the School of Computations ranged from 13 to 18.[20] No information is available about the teaching materials used at that stage, yet one can conjecture that textbooks

[16] In JTS 1975 this treatise is mentioned as compiled by Liu Hui.

[17] Different sources provide different figures; see Li 1933 [1977], p. 255. The title *boshi* literally means "serviceman (*shi*) of broad [knowledge] (*bo*)," hence "erudite," as Hucker (1988, p. 389, no. 4746) suggests.

[18] Li 1933 [1977]. This number looks problematic, especially when compared with the number of students enrolled in this program during the Tang dynasty. One cannot rule out the possibility that the number was miswritten by a copyist (who, e.g., wrote 八十, "80", instead of 十八, "18") or that 80 was no more than a projected number of students, while the actual number was smaller.

[19] Li 1933 [1977], p. 261. Des Rotours (1932) also states that mathematics examinations must have been conducted during the period 627–649 (p. 28) and conjectures that they must have started around 629–632 (p. 129, n. 1). Hucker (1988, p. 461, no. 5856), who claimed that "Tang did not duplicate the Sui school until 657," apparently did not take into consideration the works of Li Yan and des Rotours.

[20] Mentioned as "14 to 19" in Chinese sources, see des Rotours 1932, p. 136.

inherited from the School of Computations of the Sui dynasty may have been used. Apparently, some time later the school was closed down, since extant sources mention its "reestablishment" in 656 under the supervision of the governmental agency named "Directorate [of Education of] Sons of State" (*Guo zi jian* 國子監).[21] Also in this year, a new set of mathematical treatises to be used as textbooks in the School of Computations that were edited by a team of functionaries directed by the astronomer Li Chunfeng was presented to the throne for the formal approval of the Emperor.[22] Since the earlier versions of the treatises are no longer extant, it is unknown what modifications were made by the team of editors; however, there are reasons to believe they corrected the treatises as well as their commentaries by earlier authors, unified the mathematical terms used in the treatises and commentaries, and in some cases provided their own commentaries. The extent to which the set of treatises edited and commented by Zhen Luan in the sixth century was used by seventh-century editors is unknown; as descriptions of Tang dynasty textbooks in dynastic histories as well as the extant editions of them reprinted during the Song dynasty suggest, the commentaries of Zhen Luan were simply removed from the books, except in a few cases when they were criticized by Li Chunfeng and his team (this is the case of Zhen Luan's commentary on the *Zhou bi suan jing*).

The set of textbooks to be studied in the school included 12 treatises. These manuscript books apparently must have been copied very frequently because they were used by several generations of students and instructors. The particulars of this process remain unknown, and it is possible that the texts edited in 656 may have been modified when copied. It is not known whether the printed editions of the eleventh and thirteenth centuries were based on the original edition of 656 or on some unidentified later copies that circulated among students and instructors; yet it is unlikely that the editions of the early second millennium took into account any versions of the treatises antedating 656.[23] Some parts of the treatises (especially containing diagrams) may have been removed and published separately (Volkov 2007). The reading of certain mathematical terms found in the treatises most probably changed with time, and there existed special guides explaining how to read and understand them; two such guides for the *Zhou bi suan jing* and *Jiu zhang suan shu* authored by one Li Ji 李籍 are still extant.[24]

The textbooks and the duration of their study as specified in the *Xin Tang shu* 新唐書 (The New History of the Tang [dynasty], completed in 1060) are listed in Table 4.1. Several descriptions exist of the instruction in the School of Computations, however, it remains unknown whether the described educational procedures were set up as late as 656 or existed since the late 620s. These descriptions specify the number of students, the lists of textbooks, the periods of time allotted to the study of each book, and other details.[25] The lists of treatises and the duration of their study specified in extant sources are virtually identical.[26] According to the *Tang liu dian* 唐六典 (Six Codes of the Tang [dynasty]) and to the *Jiu Tang shu*, the students of the school were subdivided into two groups each

[21] Modern authors use various renderings of the name of this institution: Hucker (1988, p. 299) suggests "Directorate of Education" while Lee (2000, *passim*) prefers "Directorate of National Youth."

[22] Martzloff (1997) on p. 123 erroneously claims that editorial work was done during the period 618–627; on p. 125 he contradicts himself when saying, equally erroneously, that this work was carried out from 644 to 648.

[23] With perhaps only one exception: a copy of the treatise *Shu shu ji yi* was found by Bao Huanzhi 鮑澣之 (fl. ca. 1200) in a Daoist monastery; its thirteenth-century printed edition, unlike other extant treatises, does not contain an opening part with the names of the Tang dynasty editors and therefore may have been based on either an incomplete copy of the Tang edition or even on a version based on a pre-Tang edition (Volkov 1994).

[24] The lifetime of Li Ji is unknown; Guo Shuchun argues that Li must have been active after 712 and no later than the early ninth century; see Guo 1989, pp. 198–199.

[25] The descriptions are found in the *Tang liu dian* 唐六典 (Six Codes of the Tang [Dynasty]), completed in 738, in the *Jiu Tang shu*, and in the *Xin Tang shu*; see TLD 1983, Chap. 21, p. 10b; JTS 1975, vol. 6, p. 1892; XTS 1956, Chap. 44, p. 2a.

[26] The *Jiu Tang shu* provides only the titles of the textbooks but not the duration of their study. For a translation of the description found in the *Xin Tang shu*, see des Rotours 1932, pp. 139–142, 154–155; see also Siu 1995, p. 226, Siu and Volkov 1999.

Table 4.1 Mathematical curriculum of the Tang School of Computations

#	Title	Duration of study	Program[a]
1	*Sunzi* 孫子 ([*Treatise of*] *Master Sun*)	1 year for two treatises together	Regular
2	*Wu cao* 五曹 (*Five Departments*)		Regular
3	*Jiu zhang* 九章 (*Nine Categories*)	3 years for twotreatises together	Regular
4	*Hai dao* 海島 (*Sea Island*)		Regular
5	*Zhang Qiujian* 張丘建 ([*Treatise of*] *Zhang Qiujian*)	1 year	Regular
6	*Xiahou Yang* 夏侯陽 ([*Treatise of*] *Xiahou Yang*)	1 year	Regular
7	*Zhou bi* 周髀 (*Gnomon of the Zhou* [*Dynasty*])	1 year for two treatises together	Regular
8	*Wu jing suan* 五經筭 (*Computations in the Five Classical Books*)		Regular
9	*Zhui shu* 綴術 (*Procedures of Mending* [=*Interpolation?*])[b]	4 years	Advanced
10	*Qi gu*[c] 緝古 (*Continuation* [*of Traditions*] *of Ancient* [*Authors*])	3 years	Advanced
11	*Ji yi* 記遺 (*Records Left Behind for Posterity*)	Not specified	Compulsory
12	*San deng shu* 三等數 (*Numbers of Three Ranks*)	Not specified	Compulsory

The order of the books in this tableis that adopted in the *Xin Tang shu*; it is possible that the latter source listed the treatises in the order in which they were actually studied (Volkov 2012a).

[a]The terms "regular" and "advanced" are not found in the original descriptions; they are added for the convenience of the reader. For the explanation of these terms, see below.

[b]This treatise was lost by the early second millennium AD and the meaning of its title remains unclear; see Martzloff 1997, p. 45; Yan 2000, pp. 125–132.

[c]In modern Mandarin dialect, the character 緝 can be read *qi* and *ji*, according to its meanings: "to tie up; to extend [a cord]; to continue" in the former case and "to arrest, to seize" in the latter. I interpret the meaning of the title following E. Berezkina (1975, p. 351) as "Continuation [of the Ancient Traditions]" and therefore transliterate the title as *Qi gu* [*suan jing*]. Des Rotours probably had in mind a similar interpretation because he systematically transliterated the character 緝 as "*ts'i*" (in French transliteration system), which corresponds to "*qi*" in *pinyin* transliteration; see des Rotours 1932, pp. 140, 154–155. The alternative transliteration of the title, i.e., *Ji gu* [*suan jing*], can be found in a number of recent publications; see, for example, Martzloff 1997, p. 140. Martzloff, however, adopts the translation "continuation" of the character 緝. At any rate, as explained earlier, the modern reading of Chinese characters in the Mandarin dialect is not identical with their historical reading.

comprising 15 people[27] and instructed by two "erudites" (JTS 1975, vol. 6, p. 1892) and one "teaching assistant" (Li 1933 [1977], p. 256). The students of the first group studied treatises [1–8], and those of the second one studied treatises [9–10]. In Table 4.1 and below, these treatises are referred to as "regular program" and "advanced program," respectively.[28] The study in each program usually lasted 7 years but in exceptional cases could be extended to 9 years. Treatises [11–12] were studied simultaneously with the other treatises in both programs; the time necessary for their study was not specified.

The 12 textbooks of the curriculum and the extant mathematical treatises with which they are conventionally identified are shown in Table 4.2. Not much is known about the procedures of instruction in the School of Computations; the only element mentioned in the extant sources is "oral explanations" provided by the instructors. There were two kinds of examinations: (1) the quizzes conducted every 10 days and (2) the examination conducted at the end of each year. A quiz included three questions: two on the memorization of a 2,000-word excerpt and one on its "general meaning" (*da yi* 大義). On the yearly examination, students were asked ten questions on the "general meaning" and they were expected to answer them orally. There is no information about any kind of graduation examinations at the end of the entire course (see XTS 1956, Chap. 44, p. 2a; for translation, see des Rotours (1932), pp. 141–142; for a discussion of the procedure, see Siu and Volkov 1999).

[27] Hucker (1988, p. 461, no. 5856) is apparently wrong when claiming that in 657 the "prescribed student enrollment was set at only 10." He probably was misled by the quota of students, 10, adopted in the early ninth century; see below.

[28] No specific names of the programs are provided in the original documents; the adjectives "regular," "advanced," and "compulsory" are added on the basis of the contents of the programs (Siu and Volkov 1999 Volkov 2012a).

Table 4.2 Conventional identification of the Tang dynasty textbooks with the extant mathematical treatises (Volkov 2012a)

#	Treatises as listed in the *Jiu Tang shu* and *Xin Tang shu*	The extant treatises with which the Tang dynasty treatises are conventionally identified	Author	Date of compilation of the extant treatise
1	*Sun zi* 孫子 ([*Treatise of*] *Master Sun*)	*Sun zi suan jing* 孫子筭經 (*Computational Treatise of Master Sun*)	Unknown	Ca. AD 400 (?)
2	*Wu cao* 五曹 (*Five Departments*)	*Wu cao suan jing* 五曹筭經 (*Computational Treatise of Five Departments*)	Unknown[a]	Not earlier than 386 AD
3	*Jiu zhang* 九章 (*Nine Categories*)	*Jiu zhang suan shu* 九章筭術 (*Computational Procedures of Nine Categories*)	Unknown	Prior to the mid-first century AD
4	*Hai dao* 海島 (*Sea Island*)	*Hai dao suan jing* 海島筭經 (*Computational Treatise* [*Beginning with a Problem about a*] *Sea Island*)	Liu Hui	ca. AD 263
5	*Zhang Qiujian* 張丘建 ([*Treatise of*] *Zhang Qiujian*)	*Zhang Qiujian suan jing* 張丘建筭經 (*Computational Treatise of Zhang Qiujian*)	Zhang Qiujian 張丘建 (dates unknown)	Mid-fifth century AD
6	*Xiahou Yang* 夏侯陽 ([*Treatise of*] *Xiahou Yang*)	*Xiahou Yang suan jing* 夏侯陽筭經 (*Computational Treatise of Xiahou Yang*)	Han Yan 韓延 (dates unknown)	763–779
7	*Zhou bi* 周髀 (*Gnomon of the Zhou* [*Dynasty*])	*Zhou bi suan jing* 周髀筭經 (*Computational Treatise on the Gnomon of the Zhou* [*Dynasty*])	Unknown	Early first century AD (?)
8	*Wu jing suan* 五經筭 (*Computations in the Five Classical Books*)	*Wu jing suan shu* 五經筭術 (*Computational Procedures in the Five Classical Books*)	Zhen Luan	ca. AD 570
9	*Zhui shu* 綴術 (*Procedures of Mending* [=*interpolation?*])	*Lost*	Zu Chongzhi 祖沖之 (429–500)[b]	Second half of the fifth century AD
10	*Qi gu* 緝古 (*Continuation* [*of Tradition*] *of Ancient* [*Authors*])	*Qi gu suan jing* 緝古筭經 (*Computational Treatise on the Continuation* [*of Tradition*] *of Ancient* [*authors*])	Wang Xiaotong 王孝通 (b. ?–d. after AD 626)	
11	*Ji yi* 記遺 (*Records Left Behind for Posterity*)	*Shu shu ji yi* 數術記遺 (*Records of the Procedures of Numbering Left Behind for Posterity*)	Xu Yue 徐岳 (b. before 185–d. after 227)	ca. AD 220
12	*San deng shu* 三等數 (*Numbers of Three Ranks*)	*Lost*	Dong Quan 董泉 (dates unknown)	Prior to AD 570

[a]In some sources, the treatise is credited to the authorship of Zhen Luan.
[b]In some sources, this treatise is credited to the authorship of Zu Chongzhi's son, Zu Gengzhi 祖暅之 (b. before ca. 480–d. after 525).

Students who successfully graduated from the school were allowed to take the examination for the doctoral degree *ming suan* 明筭 ("[Person] Understanding Computations")[29] together with the candidates coming from the provinces.[30] The examination included two parts. The tasks for the first part consisted of writing essays to answer ten questions related to one of the two programs, "regular" or

[29] See Lee 2000, p. 138, for a slightly different rendering.

[30] See des Rotours 1932, p. 128, n. 1 for a description of the origin of the candidates.

"advanced."[31] The second part of the examinations in both programs consisted of a test on the memorization of the treatises *San deng shu* and *Shu shu ji yi* held in the form of an "examination by quotation" *tie du* 帖讀 (literally, "strip reading") or *tie jing* 帖經 ("strip [reading] of classics") (on the procedure of the "examination by quotation," see des Rotours 1932, pp. 30–31, 141, n. 2; Siu and Volkov 1999, p. 91, n. 41; see also Lee 2000, p. 142). The *Xin Tang shu* provides the following description of the examination procedure of the first part: all the candidates had to answer ten questions on "general meaning;" in their answers they were supposed (1) to "elucidate the numerical values (?)" of the given problems (*ming shu* 明數) and (2) to "design [new computational] procedures" that would solve these problems (*zao shu* 造術). Moreover, the candidates were asked to "elucidate in detail the internal structure of the [computational] procedures [they designed]" (*xiang ming shu li* 詳明術理; see des Rotours 1932, pp. 154–155; Siu and Volkov 1999, p. 92; Volkov 2012a). In the "regular program" examinations, three questions were related to the *Jiu zhang* and one question to each of the seven remaining treatises; in the "advanced program," seven questions were related to the *Zhui shu* and three to the *Qi gu*. In both programs, the candidates passed the examination if they successfully answered at least six out of ten questions.[32] As for the memorization test included in the examination for both programs, it contained ten questions related to the treatises *Ji yi* and *San deng shu*. The candidates passed if they successfully answered nine questions. The extant descriptions do not specify whether the examination works of the candidates were written in the same format as those on other subjects, such as, for instance, Confucian classics, or whether they had some specific format relevant to the mathematical contents of the treatises. However, it appears plausible to conjecture that during the first part of examinations, the "questions" given to the candidates were mathematical problems *similar* (but not *identical*) to those contained in the treatises of the chosen "program," that is, problems belonging to the categories for which the candidates knew the solutions, yet with *modified* numerical parameters. The change of the parameters may have implied modifications, sometimes considerable, of the known algorithms needed to solve the problems.[33] The candidates thus were asked to design algorithms which were not mere replicas of the algorithms found in the textbooks but rather their generic versions designed according to the modified parameters.[34]

The school was closed in 657 or 658 and reopened in 662 (Li 1933 [1977], p. 260; Wong 1979, p. 28; Hucker 1988, p. 461, no. 5856). It stopped functioning by the mid-eighth century again when the rebellion of An Lushan 安祿山 (703?–757?) interrupted the work of all departments in 755. Attempts were made to revive the university around 766, but in 783 the capital was once again occupied by rebels, and the School of Computations was closed from 780 to 804 (Wong 1979, pp. 58–59). In 807, the school was reopened yet the total number of students decreased considerably; only ten students were enrolled in the branch of the school of the Western capital (i.e., Chang'an, modern Xi'an) and only two students in the branch of the Eastern capital (i.e., Luoyang 洛陽) (Li 1933 [1977], pp. 260–264).

However, the schools of the capitals were not the only institutions where mathematical disciplines were taught. There is evidence that a variety of subjects including mathematics and astronomy were taught in Buddhist monasteries which often possessed large collections of books (Wong 1979, pp. 78–89, 190, n. 2). Mathematical texts found in the Dunhuang 敦煌 monastery seem to support this claim.[35]

[31] One cannot help but notice that the curriculum of the school corresponded to the doctoral exams, formally open for even those candidates who did not graduate from the school, such as "provincial candidates." It remains unknown whether instruction in the provinces was conducted on the basis of the same textbooks or whether the degree examination was designed in this way to give advantage to the graduates of the school.

[32] Some restrictions applied in the advanced program; see des Rotours 1932, p. 155, n. 2.

[33] This hypothesis was advanced by Siu and Volkov (1999) and amply illustrated in Siu (1999, 2004, pp. 174–177).

[34] A piece of evidence supporting this hypothesis was found in a Vietnamese mathematical treatise; see Volkov (2012a).

[35] A brief introduction to nine mathematical manuscripts from Dunhuang reproduced in Guo (1993, pp. 407–420) containing references to works on them published in Chinese is found in Wang 1993. Several mathematical manuscripts from Dunhuang were earlier published by Li Yan (1955, pp. 22–39); for their description and analysis, see Libbrecht 1982.

1.3.3 Transmission to Korea

In 682 the government of the Korean kingdom of Unified Silla 統一新羅 (668–935) established an educational institution for mathematics instruction that imitated the Chinese prototype.[36] The age of the students enrolled in this School of Computations varied from 14 to 29 (!) years, and the duration of studies was limited to 9 years.[37] The curriculum of the Korean mathematical school differed from the Chinese one; it included only four treatises[38]: *Gu jang* 九章 (Chinese *Jiu zhang*), *Cheol sul* 綴術 (Chinese *Zhui shu*), *Ryuk jang* 六章 (Chinese *Liu zhang*, *Six Categories* [*of Mathematical Methods*]), and *Sam gae* 三開 (Chinese *San kai*, *Triple Root Extraction*) (Kim and Kim 1978; Li et al. 1999, pp 73–74; Jun 2006, p. 477). The first two treatises were most likely identical with the Chinese *Jiu zhang* and *Zhui shu* used in Chinese School of Computations (see Table 4.2 above), while the origin and contents of the *Ryuk jang* and *Sam gae* remain unknown. It was believed that they were compiled by Korean authors on the basis of unknown Chinese prototypes (Li et al. 1999, p. 74); however, recently it was argued (Feng and Li 2000) that they were authored by the Chinese mathematician Gao Yun 高允 (390–487). One may suggest that Korean authorities did not have access to the entire set of treatises used as textbooks in the contemporaneous Chinese School of Computations and designed the curriculum on the basis of the mathematical texts they had at their disposal. Extant Korean sources do not describe the procedures of instruction and examinations adopted in this institution. However, since the Chinese system of mathematics education was transmitted to Japan via Korea in the first millennium AD, one may conjecture that the instruction and examination procedures adopted in Korean mathematical schools resembled those in China and Japan.

1.3.4 Transmission to Japan

The first mention of an attempt to establish in Japan a systematic state-sponsored education in mathematics (actually, mathematical astronomy) took place in 602 when the Buddhist monk Gwalleuk (Kwallŭk, Jap. Kanroku) 觀勒 from the Korean kingdom of Baekje (Paekche) 百濟 (18 BC–AD 660) instructed several students at the Japanese court using Chinese books on the calendar and divination (NHS 1985, vol. 2, *juan* 22, p. 140).[39] The next attempt took place during the reign of the Emperor Tenchi (or Tenji) 天智 (626–672, r. 661–672):[40] two erudites *hakase* 博士 (Chinese *boshi*) instructed 20 students, but nothing is known about the curriculum and of the instructors' background (Li 1933 [1977], p. 265).[41] In 702, during the reign of the Emperor Mommu 文武 (683–707, r. 697–707), the

[36] Wong (1979, p. 95) argues that the School did not have instructors of mathematics until 717.

[37] *Samguk sagi* 三國史記 (Historical Records of Three [Korean] Kingdoms), *juan* 38, as quoted in Feng and Li (2000, p. 89). The original document presents the ages in traditional "Chinese" style, that is, actual age plus one year. It is not impossible that the Korean record contains a scribal error, and the limit age, 30 (i.e., 三十), was a miswritten 20 (二十) – that is, the actual limit age of the students was 19 (in Western style), which in this case would be close enough to the limit age in the contemporaneous Chinese school (18).

[38] For Korean names and terms, the Revised Romanization transliteration (adopted in South Korea in 2000) is used; in cases when a name may be known to the Western reader in the McCune-Reischauer transliteration, it will be provided in parentheses.

[39] The fact that mathematical subjects were taught by a Buddhist monk appears particularly interesting in the context of the connections between Buddhist networks and the transmission of mathematical knowledge, as mentioned above.

[40] For Japanese terms, the Hepburn romanization system is used.

[41] At that time a Korean scholar from the Kingdom of Paekche named Gwisil Jipsa (Kwisil Chipsa) 鬼室集斯 (Japanese reading Kishitsu Shushi) served as the head of the newly established National University; see Wong 1979, p. 86. On the role played in educational activities in Japan by instructors and students from the kingdom of Paekche defeated in 660–663 by the allied armies of Silla 新羅 (57 BC–935AD) and the Chinese Tang Empire, see Wong 1979, pp. 88–90.

mathematical school was reestablished and courses on astronomy and the calendar were offered; the number of students totaled 30 (Li 1933 [1977], p. 265).[42] The *Explanations of the Codes* [*of the eras Taihō and Yōrō*] (*Ryō no gige* 令義解) compiled in the early ninth century preserved the curriculum of the school originally recorded in the *Code of the Era Taihō* (*Taihō ritsuryō* 大寶律令, 710) and the *Code of the Era Yōrō* (*Yōrō ritsuryō* 養老律令, 758). The number of treatises used for mathematical instruction totaled nine, including *Sonshi* 孫子 (Chinese *Sunzi*), *Go zō* 五曹 (Chinese *Wu cao*), *Kyū shō* 九章 (Chinese *Jiu zhang*), *Kai tō* 海島 (Chinese *Hai dao*), *Roku shō* 六章 (Chinese *Liu zhang*), *Toji jutsu* 綴術 (Chinese *Zhui shu*), *San kai jū sa* 三開重差 (Chinese *San kai chong cha*, lit. *Triple Root Extraction and* [*Method of*] *Double Difference* (?)),[43] *Shū hi* 周髀 (Chinese *Zhou bi*), and *Kyū shi* 九司 (Chinese *Jiu si*, lit. *Nine Governors* (?)) (RGG 1985, vol. 2, p. 455, quoted in Li 1933 [1977], p. 265; see also Feng and Li 2000, p. 89). Five out of the nine titles were identical with those of the textbooks from the "regular program" of the mathematical school of the Tang dynasty, and one title (*Toji jutsu* 綴術) was identical with that of the treatise *Zhui shu* from the Chinese "advanced program." Nothing is known about the origin and contents of the *Kyū shi*,[44] while the *Roku shō* and *San kai jū sa* may have been identical with the *Liu zhang* and *San kai* of the Chinese scholar Gao Yun adopted for mathematics instruction in Korea (see above). The Japanese text goes on to describe the system of examinations set for the students who were supposed to "discuss and understand/make clear the structure of the procedures [to be used to solve the problems];"[45] the wording of this requirement is textually close to the description of the examinations in the School of Computations of the Tang dynasty (see above). The examination in the "regular program" included three questions related to the *Kyū shō* and one question related to each of the treatises *Sonshi*, *Go zō*, *Kai tō*, *San kai jū sa*, *Shū hi*, and *Kyū shi*. Those who successfully answered all the nine questions obtained an "A" mark, while those who answered only six questions (including at least one question on the *Kyū shō*) were given a "B."[46] The examination in the "advanced program" included six questions related to the *Toji jutsu* and three related to the *Roku shō*; those who successfully answered all nine questions obtained an "A" mark, while those who answered only six questions (including at least one question on the *Roku shō*) were given a "B."[47]

It is unknown when the instruction was interrupted; a catalog of the Imperial Library of the late ninth century lists eight of the nine abovementioned treatises,[48] and a source of the early second millennium mentions the mathematics instruction of the late tenth century as using all seven treatises of the "regular program" (Li 1933 [1977], p. 266).

[42] Wong 1979, p. 95, provides an interesting comment on the number of students (30): "The class of mathematics […] was equal in size to its counterpart in China. However, in terms of the percentage in the whole student body, it was much larger."

[43] *San kai* and *Chong cha* were interpreted by Li Yan as titles of two different treatises; the total number of textbooks in this case should have been equal to 10. Feng and Li 2000 argue that the four characters "*san kai chong cha*" referred to one treatise, and the total number of the textbooks therefore was nine; the same hypothesis was advanced much earlier by Fujiwara (1940). This interpretation appears plausible since it fits into the description of the examination procedure found in the same source (see below).

[44] The title of this treatise (lit. "Nine Governors" or "Nine Powers") may suggest that its contents were somehow related to the Indian astronomical/astrological system *Navagraha* (Nine Celestial Bodies, i.e., five naked-eye planets, the Sun, the Moon, and two "imaginary" planets, Rahu and Ketu).

[45] *Bian ming shu li* 辯明術理 (in Mandarin transliteration), see RGG 1985, vol. 2, p. 456.

[46] The Japanese text uses the cyclic characters *kō* 甲 (Chinese *jia*) and *otsu* 乙 (Chinese *yi*) meaning "first" and "second," respectively. The score of those who answered seven or eight questions is not specified.

[47] RGG 1985, vol. 2, pp. 456–457. This description suggests that the *Roku shō* was considered a relatively difficult treatise.

[48] Li 1933 [1977], p. 267; the only treatise left unmentioned is the *Shū hi*.

1.4 Early Second Millennium AD

1.4.1 Mathematics Education in the Song Dynasty China (960–1279)

In China, the reopening of the School of Computations following the establishment of the Song dynasty (960–1279) was related to two events: in 1084, available experts were examined and hired as "erudites" *boshi* and "school tutors" *xueyu* 學諭 (Li 1933 [1977], pp. 274–275), and in the same year a set of textbooks based on the treatises used for instruction during the Tang dynasty was edited and printed. However, the school was closed in 1086; this event may have been related to the death of the Emperor Shenzong 神宗 (1048–1085, r. 1067–1085) followed by a radical change in the politics of the reforms to which he adhered. The school was reopened again in 1104. The sons of officials as well as of commoners were admitted. The school had three colleges (*she* 舍), external (*wai* 外), internal (*nei* 內), and superior (*shang* 上), with projected quotas of 150, 80, and 30 students, respectively.[49] Graduation from the superior college granted an honorary title and, theoretically, led to an appointment in a governmental office. The number of students in the school was comparable with that in other "technical" schools; for example, in the School of Medicine (*Yi xue* 醫學) at that time, the quotas for the three colleges were 200, 60, and 40 students, respectively (Yang 2003, vol. 2, p. 120). The faculty of the School of Computations included four erudites (two of them were supposed to teach mathematics and two, astronomy and the calendar) and teaching personnel of a lower level (such as one "school tutor" *xueyu* and a number of "study hall tutors" *zhaiyu* 齋諭), whose functions are not specified in the extant documents, as well as a number of nonacademic staff members (one provost *xuelu* 學錄, one accountant *siji* 司計, two registrars *zhixue* 直學, etc.) (Li 1933 [1977], p. 273).

The school functioning was interrupted in 1106 and then resumed later in the same year. In 1110 the school was closed again and the students were transferred to the Office of the Great Astrologer (*Tai shi ju* 太史局); the school was restored in 1113 and closed one more time in 1120 (Lee 1985, p. 102). After the transfer of the capital to Lin'an 臨安 (modern Hangzhou 杭州) in 1127, no attempts to revive the school are known, except the reprinting in 1200–1213 of a set of mathematical textbooks mainly due to the efforts of Bao Huanzhi 鮑澣之 (see below). However, the plan to restart teaching mathematics was not completely abandoned: as the extant documents suggest, mathematics instruction continued in the Astronomical Bureau and combined with instruction in mathematical astronomy and calendrical computations.

It remains unknown how many textbooks were printed in 1084 and whether *all* the textbooks used in the school in the late eleventh and early twelfth century were those printed in 1084; bibliographical sources of the late thirteenth–early fourteenth century mention only the *Sunzi*, *Wu cao*, *Qi gu*, *Hai dao*, and *Xiahou Yang* as printed in 1084 (Li 1933 [1977], pp. 279–280; 1955, p. 90). The *Zhui shu*, *San deng shu*, and *Shu shu ji yi* were lost by 1084 and therefore could not be printed at that time; it is not certain whether the *Wu jing suan shu* was printed during the Song dynasty and, if so, whether it was printed in 1084 or later, in the early 1200s.[50] In turn, the *Qi gu* (later known as *Qi gu suan shu* and *Qi gu suan jing*) was printed in 1084, but it was not mentioned as used for instruction in the early twelfth century.

It is conventionally assumed that the carved wooden blocks used for printing the mathematical treatises in 1084 were lost in 1127 during the sack of Kaifeng 開封, the capital city of the Northern Song dynasty, by the Jurchens (Li 1933 [1977], p. 280; 1955, p. 90). Some time later, the governmental officer Bao Huanzhi attempted to collect and reprint the mathematical treatises used in the School of

[49] Some sources mention 210 as the total number of students, which is not consistent with the quotas of the three colleges listed above; see Yang 2003, vol. 2, p. 128.

[50] As Qian Baocong 錢寶琮 suggested in his introduction to the treatise (Guo 1993, vol. 1, p. 309), the treatise was printed during the Song dynasty; however, Qian does not specify whether the book was printed in 1084 or later. All its printed copies were lost, according to Qian, during the Qing 清 dynasty (1644–1911).

Mathematics during the Tang dynasty. He reprinted the extant mathematical treatises of the edition of 1084 that he managed to find as well as a manuscript copy of the *Shu shu ji yi* found in a Daoist monastery (Volkov 1994). The exact number of the textbooks printed at that time is not known, yet one can argue that it was either nine or ten;[51] the time of printing is known for only three treatises (*Jiu zhang suan jing*, preface 1200; *Shu shu ji yi*, preface 1212; *Zhou bi suan jing*, preface 1213) (Chemla and Guo 2004, p. 72).

The curriculum of the School of Computations is only briefly described in the extant materials. According to the edict of the establishment of the school dated 1104, the two major mathematical manuals to be studied were the *Jiu zhang* and *Zhou bi*; the students were also supposed to study "in addition" the treatises *Hai dao*, *Sunzi*, *Wu cao*, *Zhang Qiujian*, and *Xiahou Yang*. Even though the titles of these treatises are identical with the titles of the treatises used for mathematics instruction during the Tang dynasty, it remains unknown whether the first printed editions of 1084 (supposedly used in the school starting from 1104) were identical with the (manuscript) editions of the textbooks of 656 or were based on somewhat different versions of them. In addition to mathematical treatises, students were supposed to study calendrical computations, the so-called "three [astrological] schemes" or "three cosmic boards" *san shi* 三式, that is, three major methods of divination, as well as other unidentified astrological texts (Lee 1985, p. 96; Friedsam 2003, p. 52).[52]

To enter the school and become a student of the external college, a candidate had to pass an "admission examination" which consisted of three tasks "on the meaning" of the *Jiu zhang* and two tasks on "computational problems" (Li 1933 [1977], p. 273). The "internal examinations," that is, examinations for the promotion of students from external to internal college and from internal to superior College, were conducted in three "stages" (*chang* 場): the examination of the first stage included three tasks on the "meaning" of *Jiu zhang* and *Zhou bi*, as well as two tasks on "computational problems," the examination of the second "stage" included one task on calendrical calculations, and the examination of the third "stage" included one task on the "three [astrological] schemes" (Li 1933 [1977], p. 274; Lee 1985, p. 96; Friedsam 2003, p. 52). One can notice that the five mathematical treatises supposed to be studied "in addition" were actually not subjects of the examinations at any stage, unless the extant descriptions are incomplete.

Not much is known about the procedures of the mathematical instruction in the School of Computations; the only source of information on the topic is the chapter "Xi suan gang mu" 習筭綱目 (Master List [of Topics] for Practicing Computations) opening the mathematical treatise *Suan fa tong bian ben mo* 算法通變本末 (Alfa and Omega of Continuity and Deviation of Counting Methods, 1274) by Yang Hui 楊輝 (fl. 1261–1275) (Guo 1993, vol. 1, pp. 1048–1050). No biographical data on Yang Hui are available, and most probably he was never employed by the Astronomical Bureau or other state agencies related to mathematics instruction. However, he published a number of treatises on the subject in which he demonstrated a thorough knowledge of the works of past mathematicians (Lam 1977).

The chapter "Xi suan gang mu" contains a detailed description of the topics expected to be known by those who studied mathematics (Lam 1977, pp. 11–14; Friedsam 2003, pp. 58–63). The list includes four arithmetical operations, root extraction, numerical solution of quadratic equations of certain types, operations with common fractions, and all methods of the traditional "nine categories

[51] According to Cheng Dawei 程大位 (1533–1606), ten mathematical texts were printed in 1084 and reprinted in the early thirteenth century; they included *Huangdi jiu zhang* 黃帝九章, *Zhou bi suan jing*, *Wu jing suan fa* 五經筭法, *Hai dao suan fa* 海島筭法, *Sunzi suan fa* 孫子筭法, *Zhang Qiujian suan fa* 張丘建筭法, *Wu cao suan fa* 五曹筭法, *Qi gu suan fa* 緝古筭法, *Xiahou Yang suan fa* 夏侯陽筭法, and *Suan shu qia yi* 筭術恰遺 (Li 1933 [1977], p. 280). Some of the titles listed by Cheng slightly differ from those found in the curricula of the School of Computation of the Tang and Song dynasties.

[52] The "three [astrological] schemes" or "three cosmic boards" were the divinatory systems *Tai yi* 太乙, *Qimen dunjia* 奇門遁甲, and *Liu ren* 六壬 (Ho 2003, pp. 36–40, 83–84, 113–119).

of mathematics" (i.e., the methods found in the treatise *Jiu zhang* [*suan shu*]). For each method or group of methods, Yang Hui indicated the number of days necessary to master it; the total number of days for all listed topics was 260 (Zhou 1990, p. 397). It can be argued that Yang Hui did not describe the program of studies adopted in the state School of Computations in the early twelfth century since the textbooks he referred to contain two mathematical treatises (*Ying yong suan fa* 應用算法 and *Zhi nan suan fa* 指南算法) not mentioned in the descriptions of the instruction at the school (Guo 1993, p. 1048); these two treatises were published, according to Cheng Dawei, in the late eleventh or twelfth century (Cheng 1990, p. 991).

1.4.2 Mathematics Education in Non-Chinese States Located Within Boundaries of Present-Day China

Only very limited information is available on scientific education, in particular, astronomical education (which included mathematical subjects) that existed in the areas controlled by the so-called non-Chinese dynasties contemporaneous with the Chinese Song dynasty, the Liao 遼 (the Khitans, 907–1125) and the Jin 金 (the Jurchens, 1115–1234). The rulers of the Liao established an observatory where astronomical and astrological instruction was conducted by Khitan and Han (ethnic Chinese) experts (Hucker 1988, pp. 456–457, nos. 5780, 5783; Cheng 1993, p. 72). In turn, in 1151 the Jin established educational institutions (Directorate of the Sons of State) imitating the Chinese ones (Lee 2000, p. 521; Yang 2003, vol. 2, p. 689). It is also known that 76 Jurchen and Han students aged 14 to 29[53] from the families of both officials and commoners studied in the astronomical observatory (*si tian tai* 司天臺) in the Jin capital (Hucker 1988, p. 457, no. 5783). The studied disciplines included calendrical calculations and various types of astrological divination (Cheng 1993, pp. 75–77; Yang 2003, vol. 2, p. 775). Given the interest of the Jurchens rulers in mathematical astronomy, one can conjecture that some teaching materials left in the School of Mathematics and the Astronomical Bureau of the Northern Song dynasty capital Kaifeng after its conquest by the Jin army in 1127 may have been used for instruction at the Jin court.

1.4.3 Transmission to Vietnam

After Vietnam became independent from China in 939, the Vietnamese rulers established a system of national education including the Directorate [of Education of] Sons of State (*Quốc tử giám*) and an examination system which to a large extent was based on the Chinese blueprint. There are mentions of mathematics examinations conducted in 1077, 1261, 1363, 1403 or 1404, 1477, and 1507 (CM 1969, pp. 697, 984, 1292, 1458, 2253, 2456), yet the contents and procedure of these examinations remain unknown. As the date of the first examinations (1077) suggests, at least in the eleventh century, the Vietnamese educational institutions used for instruction and examinations a set of textbooks that *differed* from the collection of treatises reprinted in the Song dynasty capital in 1084. These textbooks might have been manuscript copies of the mathematical treatises used for instruction in the Tang dynasty that made their way to Vietnam in the late first millennium AD when the country was one of the Chinese provinces.

1.4.4 Korean Mathematics Education of the Early First Millennium

A Directorate [of Education of] Sons of State *Gukjagam* (*Kukchagam*) 國子監 (Chinese *Guo zi jian*) was established in the Korean Goryeo (Koryŏ) 高麗 Kingdom (918–1392) in 992; soon after, a School

[53] To compare with the abovementioned record of the age of students in Korea.

of Computations was open and two erudites *baksa* (*paksa*) (Chinese *boshi*) began to teach there. The textbooks included *Gu jang*, *Cheol gyeong* 綴經 (Chinese *Zhui jing*, *Treatise on* [*Procedures of*] *mending* [=*Interpolation?*]), *Sa ga* 謝家 (Chinese *Xie jia*, The school of [Master] Xie),[54] and *Sam gae* (Li et al. 1999, p. 74; Jun 2006, p. 478).

When a new Yi 李 dynasty (1392–1896) was established and the country was renamed to Joseon (Chosŏn) 朝鮮, officials of the Korean Court were dispatched to China to purchase books for educational purposes. During his reign (1418–1450), the fourth King of Joseon, Sejong 世宗 (1397–1450), reformed and reinforced the state system of mathematics education. Five books were used for instruction; four of them were the Chinese treatises *Yang hwi san beop* 楊輝算法 (Chinese *Yang Hui suan fa*, Computational Methods of Yang Hui, 1275), *San hak gye mong* 算學啟蒙 (Chinese *Suan xue qi meng*, Primer for Schools of Computations, 1299) by Zhu Shijie 朱世傑, *Sang myeong san beop* 詳明算法 (Chinese *Xiang ming suan fa*, Computational Methods, Elucidated in Detail, 1373) by He Pingzi 何平子 (dates unknown), and *O jo san gyeong* 五曹算經 (Chinese *Wu cao suan jing*); one additional treatise was the anonymous *Ji san* 地算 (Chinese *Di suan*, Computations of Terrains) of unknown origin (Li et al. 1999, p. 75; Jun 2006, pp. 478–479). After an educational reform in the fifteenth century, only three former books were used for state mathematical instruction and examinations (Jun 2006, p. 479).

1.5 *The Decline of the State Mathematics Education in China in the Fourteenth–Fifteenth Centuries*

Not much is known about mathematics education in China during the (Mongolian) Yuan dynasty. In conventional Chinese historiography, the Mongol rule has often been pictured as hostile to Chinese culture and learning and the same opinion was shared by some historians of science who claimed that the decline of mathematics was related to the attitude of the Mongolian government (see, e.g., Lam 1977, p. 291). Other historians, however, have a different vision; for example, Lee (2000, p. 523) suggests that "the Yuan government took a rather open and serious attitude toward technical education. […] Students of the Directorate of National Youth [= *Guo zi jian*] were […] formally required to learn mathematics."

Extant documents confirm that mathematical subjects related to astronomy and calendar were taught in the Astronomical Bureau (*si tian jian* 司天監 or *si tian tai* 司天臺, established in 1260, see Hucker 1988, pp. 456–457, nos. 5780, 5783) and probably in the Astrological Commission (*Tai shi yuan* 太史院, established in 1278, see Hucker 1988, p. 482, no. 6220; Sun 1996, p. 517; Lee 2000, p. 521; Sivin 2009, p. 147). Even though the extant *syllabi* of these institutions do not contain any references to the mathematical treatises used in the state Schools of Mathematics of the Tang and Song dynasties (Sun 1996, p. 517; Yang 2003, vol. 2, pp. 819–820), the high level of mathematical expertise manifested in the calendar *Shou shi li* 授時曆 (*Granting the Seasons Calendar* promulgated in 1281) designed by Guo Shoujing 郭守敬 (1231–1316) and his collaborators (Sivin 2009) suggests that a number of professional astronomers obtained comprehensive training in mathematical subjects; one can also suggest that in certain cases they studied mathematics with private teachers (Friedsam 2003, p. 56). The large-scale economic and military activities conducted throughout the Mongol Empire certainly required an advanced mathematical expertise of governmental officers, which would be impossible to acquire without systematic training in such mathematical subjects as those related to taxation, remote surveying, and construction works amply featured in the Chinese textbooks of the first millennium AD. The plausible conjecture thus would be that this kind of mathematical training

[54] It has been suggested that this was the Chinese treatise *Xie Chawei suan fa* 謝察微算法 (Computational Methods of Xie Chawei) compiled in ca. 1050; see Li et al. 1999, p. 74 and Jun 2006, p. 478.

was conducted in the respective governmental offices, while the transmission of advanced mathematical topics, such as polynomial algebra, took place in private schools of which that of Zhu Shijie朱世傑 (fl. 1299–1303) is the best-known example. Zhu authored a treatise with a conspicuous title *Suan xue qi meng*算學啟蒙 (Primer for Schools of Computations), which contained a systematic presentation of contemporaneous mathematics starting with such elementary topics as the multiplication table and ending with polynomial algebra (Friedsam 2003, pp. 65–66); this treatise was later used as one of the main mathematical textbooks in Korea.

Li Yan provided a number of quotations discerning the attempts of Ming dynasty rulers to revive the state mathematics education; these attempts included imperial edicts issued in 1392, 1429, and 1450 stipulating that students of the state-run educational institutions had to study mathematics to take examinations in this subject (Li 1933 [1977], pp. 284–285). However, no information exists on any systematic activities such as curriculum design and reproduction of mathematical textbooks conducted at that time.

1.6 Conclusion

This brief presentation of the history of mathematics education focused on the state mathematics education which was historically incorporated into the Chinese educational system of the early first millennium AD. It remains unknown how precisely the mathematical treatises compiled prior to this time were used, yet one can argue that these books, usually designed as collections of quasi-applied problems, were originally compiled mainly for educational purposes. It is important to stress that the implementation of the mathematical instruction as a part of the state-run educational system resulted in its transmission to countries located within the boundaries of the Chinese cultural *oecumene*.

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2 Mathematics Education in India

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2.1 Introduction

Very little is known of the context in which much of ancient India's scholarly knowledge burgeoned. Part of this ignorance springs precisely from the fact that very little is known about elementary, higher, or specialized education in ancient and medieval India. For ancient and medieval mathematics in the Indian subcontinent, most of the studied textual sources are in Sanskrit, a Brahmanical language which became the scholarly language of an educated cosmopolitan elite. Sources in vernacular languages could provide information on the contents and means of mathematical knowledge transmission in wider circles, but concerning mathematics and astronomy, little-studied and very late texts provide only meager testimonies. Also, archeology has until now given us little information on how mathematics was taught. In the following section, to sketch an uncertain image of mathematical education in India during the period in question, we have structured this section along the three historical periods relevant to the history of mathematics in India: Vedic period (ca. 2500 BCE–500 BCE), Classical and Medieval period (500 BCE–twelfth century), and Pre-Modern period (thirteenth–eighteenth century). However, in fact, the continuous use of ancient texts throughout this period mingled with our ignorance serves to combine it all into one continuous mode of transmission.

2.2 Vedic India (ca. 2500 BCE–500 BCE)

The oldest texts that have come down to us from the Indian subcontinent, the Vedas (ca. 2500–1700 BCE), gave rise to a set of scholarly commentaries that define the knowledge that should be imparted to a Brahman and how this knowledge should be transmitted. According to such texts, the high caste male (and in rare cases, female) should live through four life stages, one being the state of *brahmacārin* or student. The pupil (*śiṣya*) and his teacher or master (*ācārya, guru*) practiced restraint and yogic exercises to develop an inner energy (*tapas*) that was believed to be central to good learning. Given that important religious texts were oral texts, knowledge was seen as being acquired through heard (*śruti*) and remembered (*smṛti*) texts. An important part of the education of a Brahman consisted of learning how to recite specific parts of the Vedas. This implies not only knowing these texts by heart but also knowing how to chant them according to strict metrical rules. In some cases, the recitation involved knowing how to chant Vedic verses following many systematic combinations of their syllables, first in order, then inverting one verse/syllable after another, then reciting it backwards, and so on, so that the recitation itself could be seen as an application of a systematical "mathematical" combination.

To become a student, one had to find a teacher who would accept to perform an *upanayana* ceremony, by which the teacher became symbolically pregnant with his student, who was usually between 8 and 12 years old. Women were not normally made to study, but there are known exceptions. Education then lasted at least 12 years. The teaching season opened with an *upakarman* ceremony on a full-moon day in July or August and lasted for 5 to 6 months. To end the period of apprenticeship, a ceremony was held in which the student offered a present to his teacher. Teaching was probably not performed by an individual preceptor. Texts describe the benefit for a student of having several teachers. Further, small "assemblies" (*śakha, charaṇa, pariṣad,* etc.) were founded around the transmission of a certain number of texts and specific interpretations, housing students and teachers together. Students had the duty to tend the houses, fires, and cattle of these "assemblies" and teachers. Obviously such "assemblies" could gather several members of the same extended family and, reciprocally, such education could be undertaken within a family unit. It was possible for a student to stay for his entire life Scharfe Harmut (2002), Mookerji (1947).

The verb *adhī-* is usually used to indicate how a student should learn the Vedic text (it is referred to sometimes as "one's own lesson" *svādhyāya*), meaning both "learning by heart" and also "seeking," Malamoud Charles (1989). Vedic auxiliaries (*vedāṅga*) proclaimed that they served the purpose of seeking the meaning of the Vedic text. They consisted of five topics: phonetics, metrology, etymology, ritual, and astral science (*jyotiṣa*). Astral science as propounded by the *Jyotiṣavedāṅga* ("Vedic auxiliary on Asterisms") (ca 1200 BCE), while not properly a mathematicized astronomy from the perspective of planet movements, still contained procedures involving elementary arithmetical operations, such as the Rule of Three, Sarma (1985). The *Śulbasūtra*s ("Rules of the Chord") (eighth to fifth century BCE) were sub-parts of larger ritual texts, each belonging to different schools and ascribed to different authors (Baudhāyana, Apāstamba, Kātyāyana, Mānava). Secondary literature on the history of mathematics groups all *Śulbasūtra*s together. They indeed share many rules and topics in common. It is uncertain how such texts should be understood in relation to mathematical education. They describe the construction of Vedic ritual altars and delimit ritual grounds. They provide algorithms to construct with strings and poles, with oriented geometrical figures (square, rectangles, right triangles, etc.) having a given size. Procedures describe how to transform one figure into another with the same area or a given part of this area (a rectangle into a square, an isosceles triangle into a square, etc.). Rules for constructing altars of given shapes (some very complex, such as a hawk with open wings) with a fixed number of bricks are given as well. Each *Śulbasūtra* often gives several separate procedures for a same aim. It also contains rules with a general scope, such as a procedure for the Pythagorean Theorem. However, texts give little information on how to transmit these rules. Each separate text discusses several ritual schools, thus exposing contradictory data on the length of this altar or the size of that brick. The knowledge imparted by the *Śulbasūtras* was probably intended to be mastered by the *adhvaryu* priest, one of the well-versed priests in charge of Vedic sacrifices. According to a hypothesis offered by Chattopadhyaya (1986), the paving knowledge of the *Śulbasūtras* would have been inherited from the artisans who had constructed the remarkable prehistoric cities of the Indus civilizations. This supposes, most provocatively, a transmission of knowledge from non-Brahmins to Brahmins. The knowledge imparted in the *śulbasūtra*s is still brought to life today by a Brahmin cast of Kerala, the Nambudiris, who regularly perform Vedic sacrifices Somayajipad (1983). If priests know the theory of how altars and sacrificial areas should be delimited and constructed, the actual setting can be constructed by specialized artisans of a lower caste but by specialized Brahmins as well. Thus contemporary anthropological evidence could support Chattopadhyaya's hypothesis of a collaboration between artisans and priests, although that would involve the improbable supposition that over thousands of years, roles and channels of transmissions remained unchanged or that, in this respect, random, the situation happens to be the same at the beginning of the twenty-first century as it was 3,000 years ago. Thus little is known about the Vedic education of low-caste. Given the occupational definitions of castes, it is often imagined that the education was taken charge of either by guilds or by a family.

Separate independent schools for astral science probably developed during the end of the Vedic period. Religious sects contesting Vedic values proliferated, such as Buddhism and Jainism. Each would eventually develop a non-Sanskrit scholarly literature. No mathematical or astronomical Buddhist text has been transmitted from that time, although Buddhist texts refer to astronomy and even evoke counting devices of the kind of an abacus. The four branches of Jaina canonical texts include principles of mathematics (*gaṇitānuyoga*), arithmetic (*samkhyāna*), and astral science (*jyotiṣa*). Such texts are known to us in later forms as compilations made during the Classical and Medieval ages. They form a separate part of the history of mathematics in India, although in constant dialog with Hindu lore. We are uncertain, however, how these texts were integrated into the monk's curriculum of the monks. They may have been at some specific moment in time part of the curricula, but this does not mean they were constantly studied.

Thus from the Vedic period is first transmitted a vision of how education is carried out: transmission is oral and seems to be done – with only few exceptions – within each individual's own caste and profession. Scholarly mathematics seems to have been used essentially for ritual, cosmological, and religious purposes: to determine the moment of sacrifice, explain the universe, and help build proper sacrificial altars and grounds.

Also, scholarly knowledge developed into a standard given form: as treatises described as being said orally whose aphorism (*sūtra*s) should be as simple and concise as possible. They were sometimes studied with their written commentaries, often by another author. This form would have an everlasting imprint on the texts transmitted to us in the Classical and Medieval ages of India.

2.3 Classical and Medieval India (500 BCE–Twelfth Century CE)

After the burgeoning of the late Vedic period, 1,000 years of silence followed in the transmission of Sanskrit texts in astral science and mathematics. Then, two kinds of Sanskrit mathematical texts come to light. The first type larger in quantity consists in mathematical chapters of astronomical treatises. The second type is the self-proclaimed "mathematics for worldly affairs" (*loka vyavahāra*), which were often related to Jain lore.

Both kinds of texts are in the form of versified rules, more or less aphoristic, that transmit definitions and procedures. It is likely that these rules did not aim to describe precisely an algorithm to be carried out or a defined object, but rather to coin the important and remarkable elements worth memorizing: the rest were completed either by wit or with the help of a commentary. Thus, secondary literature has often considered these treatises as student manuals. The commentary is consequently seen as something akin to "teacher's notes." These commentaries in prose can stage dialogs, where it is tempting to read a representation of an actual act of teaching. Indeed, the words to coin how these texts are transmitted and conveyed allude to a broad educational context. They thus refer to themselves as showing (*pradṛś-*), indicating (*upadiś-*), and explaining (*pratipād-*); all these verbs can also mean "to teach."

In the fifth century CE, a mathematical astronomy appeared as an already completed body of knowledge through two treatises that were self-proclaimed compilations: the "five astronomical treatises" (*Pañcasiddhānta)* of Varāhamihira (476 CE) and the *Āryabha`īya* of Āryabha`a (499). The latter devoted a chapter to mathematics, giving a definition of the place value notation, evoking derivations of sines, and providing different elements of arithmetics, algebra, and indeterminate analysis. These topics were subsequently constantly reexplored. The *Āryabhaṭīya* gave rise to schools, commentaries, and criticisms – a steady tradition of Sanskrit scholarly astral science including mathematics.

Most of the texts devoted to "worldly mathematics" (also called "board mathematics" – *pā`īgaṇita* – probably a reference to the slab on which computations and drawings could be carried out) Datta Bibhutibhussan and Singh (1935), have survived and reached us by chance. This was the case of the *Bakhshālī Manuscript* (bearing the name of the town where it was excavated), Hayashi Takao (1995), of uncertain date (eighth to twelfth century); it was found by a peasant digging a hole in his field near Peshwar (now in present-day Pakistan) in 1881. Sometimes, we have only a single unique manuscript belonging to one library or collection, such as Ṭhakkura Pheru's *Gaṇitasārakaumudī*, "Moonlight of the Essence of Mathematics" (ca. 1310), or the Pa`an manuscript (later than the fourteenth century), both Jain texts known in a unique recension. The most famous of all such texts were those compiled by Śrīdhāra (ca. ninth century): the *Triśatika* ("Three Hundred [Rules]") and the *Pā`īgaṇita* ("Board Mathematics"). Many of these texts, often self-proclaimed compilations, testify to vernacular lore of all kinds and have strong ties with the Jain tradition, such as the very popular Sanskrit *Gaṇitasārasaṃgraha* ("Collection of the Essence of Mathematics") by Mahāvīra (ca. fl. 850) Rangacarya (1912).

Both mathematical traditions refer to each other as separate but dependent forms. By the twelfth century, a synthesis of both traditions was attempted in certain circles. This is probably the case with Bhāskarācarya's (b. 1114) mathematical texts, the *Līlāvatī* ("With Fun" or maybe the name of the girl to whom the examples in this text were addressed), which was devoted to arithmetics, and the *Bījagaṇita* ("Seed Mathematics"), devoted to algebra; both were integrated as chapters in his astronomical treatise, the *Siddāntaśiromāṇi* ("Crest-Jewel (Among) Astronomical Treatises").

2.3.1 Teaching Elementary Mathematics

Very little testimony exists on what would have been the elements of mathematics taught to any child to whom an education was given.

Medieval nonmathematical texts offer only fleeting information on the mathematical education of non-Brahmins. Thus Kauṭilya's *Arthaśāstra* ("Manual of Statecraft," ca. 100 BCE–100 CE), a sort of Indian Machiavellic law manual for Kings, gives information on regular education. We learn from this text that after the ceremony of tonsure, a child was taught writing and arithmetic (*saṃkhyāna*).

The future king was encouraged to learn accounting, so as not to be easily swindled. Further, the *Arthaśāstra* included a detailed list of measuring units and their shifting values; evidently, it was important for a king to master conversions.

The Pali Buddhist canon (through the *Gaṇakamoggallānasutta* of the *Majjhimanikāya*) compiled in the first centuries before CE described a Brahmin calculator (*gaṇaka*) who took in live-in pupils (*antevāsin*). He started by teaching them how to count to a hundred.

As we will see, scholarly mathematical texts, whether devoted to "worldly mathematics" or to astral science, provide a classification of operations and topics. It is noteworthy that but for one exception, no text describes how one should carry out addition or subtraction with the decimal place value notation. More generally, the algorithms put forth in scholarly mathematical texts seem to take for granted that additions and subtractions on higher numbers, multiplications, divisions, squares, and cubes of digits were known. We do not know if children were made to learn multiplication tables, but this could have been the case; as later (modern) manuscripts of vernacular tables of multiplication, squares, and cubes are known.

According to Hayashi 2001, *saṃkhyāna* could have also included "a sort of statistical estimate of the quantity of nuts, crops, etc."

2.3.2 Board Mathematics: A Prerequisite for Mathematical Astronomy?

Texts of "worldly mathematics" probably testify to what would have been a general mathematical culture – not an elementary one, but not one of high-brow Sanskrit mathematical knowledge either. Much like the mathematical chapters of astronomical texts, these texts present a structure of elementary and/or fundamental operations (*parikarman*) and of "practices" (*vyavahāra*), which express an implicit theory of how such mathematics is organized. They also provide a structured list in which knowledge can be coined and memorized. Bhāskarācarya gives what has in the secondary literature become a sort of canonical subdivision of topics. Arithmetic (*rāśigaṇita*) was subdivided into eight operations (*parikarman)* and eight practices (*vyavahāra)* that included topics we would probably classify in geometry and trigonometry*:*

The operations were	The practices were
1. Addition (*saṇkalita*)	1. Mixtures (*miśraka*)
2. Subtraction (*vyavakalita*)	2. Series (*śreḍhī*)
3. Multiplication (*pratyutpanna*)	3. Figures (*kṣetra*)
4. Division (*bhāgahāra*)	4. Excavations (*khata*)
5. Square (*varga*)	5. Stacks (*citi*)
6. Square root (*vargamūla*)	6. Sawings (*krakacika*)
7. Cube (*ghana*)	7. Grains (*rāśi*)
8. Cube root (*ghanamūla*)	8. Shadow (*chāyā*)

Each operation was defined for integers (actually referred to as regular numbers, *saṇkhyā*), fractions (*bhinna*), and zero (*śunya*). Such an organization was articulated with algebra (*bījagaṇita*). For Bhāskarācarya, this topic was structured by six rules (*vidha*) which correspond to the six first operations of arithmetics. These rules were applied to additives (*dhana*) and subtractives (*ṛna*), zero, undeterminates (e.g., "unknowns," *avyakta*), and surds (*karaṇī*). They were also defined as providing suboperations for rules concerning different topics such as indeterminate linear *(ku``aka)* and quadratic (*vargaprakṛti)* problems and equations with one or more unknowns (*samakaraṇa*). Many variations of the number and contents of both operations and practices for arithmetic (usually rules of proportions including the Rule of Three (*trairāśika*) are included as operations), and types of equations for algebra, are known. However, such an ordered structure that is found in all mathematical texts handed down to us could have been of the progressive learning of mathematics. As we will see in the section on higher education in mathematics, algebra may have been viewed as a more advanced topic in mathematics. Variations then of structures could account for the fact that despite belonging to a common Sanskrit mathematical culture, each school and each family had its own particularities.

The question of the nature of practices (*vyavahāra*) and how they relate to vocational training, for instance, is still very much open to research. Hayashi (2001) provides elements showing that accountants and calculators, as scribes, seemed to have been regularly needed for all sorts of administrative activities. One can thus imagine that these professionals needed to have solid mathematical preparation. However, these worldly mathematics did not provide any rules for accounting. Indeed, each "practice" was a scholarly topic. Bhāskara (628 CE) thus evokes a series of scholars who before him had worked, composed, and compiled treatises, as Sanskrit scholarly knowledge self-proclaims itself:

> Or else, the scope of mathematics is vast, there are eight practices Mixtures, Series, Figures, Excavations, Piles of brick, Sawings, Mounds of grain, and Shadows. (…) For each, rules and books were made and compiled by master Maskari, Pūraṇa, Mudgala, and others. (Keller 2007, p. 30)

In all cases, whether explicitly or implicitly, these topics seemed to have been at least partially mastered by one who was to know astronomy.

Thus at the outset of the mathematical chapter (*gaṇitādhyāya*) of his astronomical treatise *Brahmaspu`asiddhānta*, Brahmagupta (629 CE) claims:

> BSS.12.1 Whoever knows separately the twenty operations (*parikarman*) beginning with addition, and the eight practices (*vyavahāra*) ending with shadows, he is a calculator (*gaṇaka*).[55] (Plofker 2007, p. 421)

Prthudakasvamin, a nineteenth century commentator, specifies: "He has the rank (*adhikārin*) to learn the Sphere." In other words, to learn mathematical astronomy – and specifically the movement of planets, which is the topic called *gola* (the sphere) – one first had to master the different operations of "board mathematics."

Rules and problems given in mathematical texts can contribute to an idea of what was required of a good mathematician – that is, the skills that one learning mathematics should acquire.

2.3.3 Riddle Culture

The Sanskrit elite culture no doubt valued playfulness, which could have also been part of the mathematical pedagogy. Texts of "board mathematics" often contain versified mathematical problems that can be seen as mathematical riddles. Their authorship is often doubtful: similar type problems are known to have traveled from text to text. Further, all mathematical commentaries of astronomical treatises contain list of versified examples, some of which echo those of "board mathematics" texts. They often used the vocative and encouraged rapidity and wit. For example, here is a problem of computations of the purity of gold melting given in the *Līlavātī*:

> L.102. Example: Parcels of gold weighing severally ten, four, two and four *māṣas*, and of fineness thirteen, twelve, eleven and ten respectively, being melted together, tell me quickly, merchant who art conversant with the computation of gold, what is the fineness of the mass? If the twenty *māṣas* of gold be reduced to sixteen by refining, tell me instantly the touch of the purified mass. Or, if its purity when refined be of sixteen, prithee, what is the number to which the twenty *māṣas* are reduced?[56]

Statement	Touch	13	12	11	10
	Weight	10	4	2	4

> Answer: After melting, fineness 12. Weight 20. After refining, the weight being sixteen *māṣas*; the touch is 15. The touch being sixteen, the weight is 15.[57] (Colebrooke 1817, pp. 46–47)

[55] Note that Brahmagupta counts more operations than *Bhāskarācarya*: he adds the Rules of Three, Five, Nine, and Eleven; the Inverse Rule of Three; Five rules to reduce fractions; barter and exchange; and rules to sell living beings.

[56] If xi is the fineness (or "touch") of the ith piece of gold, and yi its purity, x and y their respective value for the melted mass of gold, then xy = ∑xiyi. Therefore, x = ∑/y, and y = ∑/x.

[57] 13*10 + 12*4 + 11*2 + 10*4 = 240 = xy. Therefore in the first case, x = 240/20 = 12. In the second case, y = 240/16 = 15. In the third case, x = 240/15 = 16.

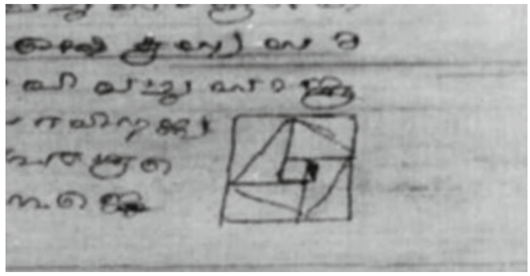

Fig. 4.1 Mss Burnell 517 British Library

Such gold-melting problems can be found in all known "board mathematics" texts, including those mentioned above, the *Bakhshālī Manuscript*, the *Gaṇitasārasaṃgraha* and the *Pā`īgaṇita,* the *Gaṇitasārakaumudī,* and the *Paṭan manuscript*.

The playfulness of the problem can be seen in the ever-ceasing variations of weight or fineness of the gold. Indeed, the problem seemed to deal with an algorithm of adding and dividing rather than with the apprentice of a jeweler or of a superintendent of coins (as described by the rules for alligation in the *Arthaśāstra,* a text on government rule). Such problems, as we can see, addressed a reader or a listener and evoke his or her qualities, the most common one being training in being quick when the rule exemplified in the problem is known.

The problem is set down in a sort of numerical table on a working surface: first, products are taken within the pairs forming each column; then, columns are summed before dividing, respectively, either by the intended fineness or by weight. Such graphic dispositions may have been carried out on a working surface, such as a dust board or slate with chalk. They could have also served the purpose of representing mental computations to be carried out.

Indeed, if the problems evoked the witty quick one who did not need to write computations, the commentaries sometimes hit on the dull-minded one who was too slow and for which an extended explanation was needed. Bhāskara thus described a diagram, probably to illustrate the Pythagorean Theorem, that he will draw for the dull minded.

> When one has sketched an equi-quadrilateral field and divided [it] in eight, one should form four rectangles whose breadth and length are three and four and whose diagonals are five. There, in the same way, stands in the middle a field whose sides are the diagonals of the [four] rectangles which were the selected quadrilaterals [and] which is an equi-quadrilateral field. And the square of the diagonal of a rectangular field there, is the area in the interior equi-quadrilateral field. (…) And a field is sketched in order to convince a dull-minded one (*duḥvigdha*). (Fig. 4.1). (Keller 2006 vol.1, p. 15)

However, mathematical texts did describe tools for drawing diagrams (pairs of compasses, ropes, and chalk) or solid objects made of clay to explain computations involving solids. They often evoked the explanation of an expert to actually "show" (teach, explain, prove) the rule.

2.3.4 Knowing How to Apply the General Rule

Riddles, solved problems, and commentaries were all directed towards a hermeneutic act, the proper interpretation of a rule. Bhāskara, the seventh-century commentator who used drawing for the dull rules, saw the rule provided by the treatise he commented upon as a seed (*bīja*) to be grown. In other words, the rule was a general statement that could encompass many different rules. This no doubt was part of a more general scholarly conception of how a rule should be coined. Looking at this endeavor from the point of view of acquired skills, it is clear that one had to know how to apply such general rules to many different cases. Thus consider, for instance, the rule for summing (*saṇkalita*), such as the one given in the *Brahmaspu`hasiddhānta:*

> BSS. 12.2 Of two quantities the denominators and the numerators when multiplied by the opposite denominators have the same divisor. In addition, the numerators are added together; in subtraction the difference of the numerators is to be computed. (Adapted from Plofker 2007, p. 421)

We immediately read this rule as concerning fractions. Pṛthudhakasvamin, the nineteenth-century commentator, specified that integer numbers can be seen as having a denominator equal to one. Such a rule can thus also be applied to integers. He further extended the rule to series, echoing other rules "for addition" given in worldly mathematical texts. Indeed, through a solved example, he provides an interpretation of the rule as a procedure to sum arithmetic progressions.

2.3.5 Higher Education in Mathematics and Mathematical Astronomy

In the shift from the Vedic period to the Classical and Middle Ages, mutations in the organization of state and local royalties showed the development of learned settlements in association with religious complexes (*ma`ha*s), giving rise to what may have been the oldest universities of the ancient world. Such universities were known to have cultivated the study of vernacular languages not restricting themselves to scholarly Sanskrit. Chinese travelers and Buddhist monks have left us testimonies of great Buddhist centers of learning, such as Nalanda in the Eastern Gangetic plain, but they do not mention the teaching of mathematics or astronomy in this context. Thus Fa-Hien (fl. 399 CE) described Pāṭaliputra also a center of learning on the eastern gangetic plain and its hundreds of monks. The fifth-century astronomer Āryabhaṭa mentioned that he learned astronomy in Kusumpura and presented this as a title of glory. His seventh-century commentator Bhāskara glossed this name with Pāṭaliputra, while others here recognized the name of Kurukṣetra, another center of higher learning.

Astral science, and within it mathematical astronomy, was understood as a scholarly topic with divine origin and a long stream of forefathers, from gods (often Brahma) to seers (men with supernatural powers) to great scholars and gurus.

Varāhamihira in his *Bṛhatsaṃhitā* recorded the names of a large number of astronomers, among whom can be mentioned Garga who will often be recalled as well as a certain number of foreigners such as the recognizable Romaka (Roman?).

Sūryadeva Yajvan, a twelfth-century commentator of Āryabhaṭa, described this process of transmission. The field of astral science was first seen or intuited (*dṛś-*) by Brahma. He then founded the discipline or composed a treatise which he taught to a great scholar, who in turn synthesized it and wrote a new treatise, and taught it to his followers. Compiling, reducing/synthesizing, and transmitting then were the activities/steps which a student used to become a teacher.

We have seen that a minimum of mathematical knowledge was apparently required to study mathematical astronomy. On the other hand, mathematics was often brought up as the ultimate step in mastering astronomy, as it enabled one to ground astronomical rules. For instance, after having evoked what was necessary for preparing a calculator, or a mathematician, on the outset of his algebraic chapter, Brahmagupta states:

> BSS.18.2 « [One becomes] a master (*ācārya*) among experts (*vidyā*) of the doctrine (*tantra*) by knowing the pulverizer, zero, negatives, positives, unknowns, elimination of the middle [term], [reduction to one] colour/unknown, *bhāvita* (multiplication) and the square nature. (Adapted from Plofker 2007, p. 428)

Indeed, for many late medieval authors, algebra was a tool to ground arithmetics. With the Rule of Three and the Pythagorean Theorem, algebra served as the basis of another skill, that of understanding and then producing explanations for rules.

2.3.6 Criticizing

Some of the wry irony of treatises and commentaries support the idea that debates were likely very virulent in astral science and mathematics in India during the classical and medieval times.

Thus, the ultimate scholar, no longer a student, was most probably one who could engage in debates with his teacher, but not criticize him. In Bhāskara's 7th century commentary of the *Āryabha`īya*, Shukla (1976) the author was compared, as he started his treatise, to a warrior on the battlefield raising his sword. The same commentator later comments on a compound (*samavṛttiparidhi*, the circumference of an evenly circular [figure]) which can be understood as either referring to a disk or a circle. He notes specifically about the first meaning, nonsensical within the context:

> This very analysis [of the compound] has been taught by master Prabhākara. Because he is a *guru*, we are not blaming him.

This suggests that such an explanation of the compound is wrong. The early *Arthaśāstra* states thus:

> *5.5* A science imparts discipline to one, whose intellect has (the qualities of) desire to learn, listening, learning, retention, thorough understanding, reflection, rejection and intentness on truth, not to any other person.

It thus defines the ideal scholar by the qualities that a good student needs, including an eventual rejection of bad rules. It also adds:

> 5.6 But training and discipline in the sciences (are acquired) by (accepting) the authoritativeness of the teachers in the respective sciences.

This tension of criticizing and respecting forefathers will be carried over into the premodern period.

2.4 Premodern India (Thirteenth–Seventeenth Century)

By the beginning of the early modern period, information on what the secondary literature calls astronomical "schools" became available. Most Sanskrit mathematical manuscripts handed to us by tradition were copied at the end of this period and afterwards. Through their history and those of family libraries, family lineages running through centuries in astral science (including mathematical astronomy) were revealed. Thus, the twelfth century astronomer and mathematician Bhāskarācarya belonged to a family of astronomers. His sons and nephews were also well-known court astrologers and composed astronomical texts.

The most famous school was initiated by Mādhava (c. 1340–1425) near the town today known as Kochi. Nī lakaṇ`ha (1445–1545) was from a Nambudiri family in Trikkantiyur on the coast of South Kerala. He thus traveled to learn his mathematics from Parameśvara's (fl.ca.1430) son Dāmodara (fl. ca.1460) at Ālattūr (Aśvatthagrāma), Kerala. The "Kerala school" produced astronomers who were also specialists in other fields of knowledge. This can be seen as "premodern" features of scholarly Sanskrit India, where traditional boundaries were breaking down, and scholars writing in vernacular languages sought new structures of knowledge. Nī lakaṇṭha, known also as a philosopher, had many pupils as Śaṇkara (fl. 1550) who produced an important commentary on both the *Līlāvatī* and Nī lakaṇṭha's works: although substantially renewing mathematical tools and theories, the authors positioned themselves in a continuous tradition.

The main turn in premodern mathematics seemed to be the emphasis on providing proofs (*upapatti*) of the rules coined by canonical authors. The "Kerala school" was indeed known for its endeavor at correcting and grounding Āryabhaṭa's parameters, leading to its development of the Taylor series. Commentators of Bhāskarācarya wrote books with algebraical "proofs" of his arithmetical rules. More advanced students no doubt had to learn these explanations first before creating their own grounding of mathematical algorithms.

In the premodern period, we know of rival schools and families. Thus, Divākara (fl.ca.1530), a Gujarati astronomer of the sixteenth century, went to study near Benares with Gaṇeśa (b. 1507), who was famous for his commentary of the *Līlāvatī* (among other texts) which included many proofs. Divākara's descendants became important astrologers in this city, although they came under attack by a rival family of astrologers, notably for their acceptance of Islamic astronomical theories.

Indeed, the premodern period also saw the arrival and installation in most of North India of Muslims from central Asia, Afghanistan, and Persia. They arrived with an Arabic and Persian culture grounded in mathematics and astronomy. However, more even than the mathematical education of Hindu, Buddhist, and Jains, the mathematical education of Indian Muslims is, to my knowledge, mainly uncharted territory. For Muslims of the Indian subcontinent, as well as likely elsewhere in Muslim countries, elementary mathematical education was developed in Madrassas. Under the reign of Sultan Fīrūz Shāh Tughluq (1305–1388) who commissioned many scientific texts and translations, it is known that a number of Madrassas were opened to encourage literacy and numeracy. Arabic and Persian mathematics flourished in the Indian subcontinent: many manuscripts of such texts can be found in Indian libraries, notably those of the Asiatic Society of Mumbai and Calcutta. Certainly, centers of learning in the Indian subcontinent were devoted to studying these texts. Further, during this period, astrolabes and table texts influenced by Arabic and Persian literature were translated into Sanskrit, and in turn many Sanskrit texts were translated into Persian. Court patronages were available for both Hindu and Muslim astrologers and mathematicians: a close study of the courts of such Moghol rulers would probably yield much information on how mathematics and astral sciences were taught. Note that Jain monks were known to have been active players, enabling the cross-fertilization of both mathematical and astronomical traditions. Jain mathematicians then from the late Vedic period to the premodern seemed to have been crucial actors of mathematical activity in the Indian subcontinent. Investigating how mathematics could have been taught to them is yet another venue for further research.

2.5 Conclusion

All texts are intended to impart information: one can read them as instructional and thus, often too quickly, try to reconstruct from them a classroom picture. Our knowledge is overall uncertain, but one might nonetheless imagine an assembly where pupils, teachers, and scholars at different stages of learning worked together, and this representation should extend from the Vedic period to the late modern period. This is a sort of dream, a fantasy such as the mirage of Jai Singh's eighteenth-century court in which we would like to imagine Jesuit priests, Hindu pandits, and Arabic and Persian scholars translating Euclid together. But as the stories goes,[58] while this vision is but a mirage, it is also an incentive for further research.

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Chapter 5
Teaching the Mathematical Sciences in Islamic Societies Eighth–Seventeenth Centuries

Sonja Brentjes

1 Introduction

The mathematical sciences in Islamic societies comprised four theoretical disciplines – number theory, geometry, astronomy, and music (theory of proportions) – and various subdisciplines or branches such as algebra, magic squares, optics, and architecture, although these components varied from author to author, period to period, and place to place. Number theory became soon imbedded in the larger field of *ḥisāb* (arithmetic as the science of calculation) as classifications of the sciences composed in the tenth century and the first chapter on number theory in the *Rasāʾil Ikhwān al-Ṣafāʾ* suggest (Brentjes 1984).[1] Astronomy diversified into *ʿilm al-hayʾa* (usually translated as mathematical cosmography which studied planetary movements), *ʿilm al-mīqāt* (the knowledge of how to determine prayer times, the direction towards Mecca for prayers and other important religious obligations, and the beginning of the new month, in particular that of Ramaḍān), and other domains like *ʿilm al-falak* (the science of the universe, possibly focusing on the compilation of astronomical handbooks and the tables characteristic of this genre), whose precise meaning and content are not well understood for the postclassical period. Professional interests in these fields of mathematical knowledge stretched equally across several areas, in particular law with its branch of inheritance law, surveying, astrology, and music. Students who took classes in some mathematical themes later held positions as judges, witnesses, experts for inheritance law, muezzins, *muwaqqit*s (a new specialization emerging in the late thirteenth century in Mamluk Syria and Egypt), surveyors, *muhandisūn* (constructors of irrigation canals; mechanics responsible for constructing and maintaining fountains, water wheels, and similar machinery; architects), secretaries and tax collectors, astrologers, musicians, and teachers. The institutions they served were mosques, madrasas (schools of higher education), *maktab*s (elementary schools), courts, diwans (bodies of civil and military administration), and, at times, the bazaar.

During the first centuries of active mathematical life in Islamic societies (late eighth to late twelfth centuries), most of the mathematical texts we know of, thanks to historical chronicles, catalogues, or encyclopedias, were translations of Greek, Middle Persian, and Sanskrit writings on geometry, number theory, theoretical music, or astronomy; editions of and commentaries on these translations; as well as

[1] See also below Sect. 5.

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newly composed treatises reflecting the eagerness to learn and the curiosity to discover. Many of these texts represented an advanced level of mathematical knowledge. Those among them that were undoubtedly the result of teaching or of private studies of major textbooks like Euclid's *Elements* focus on specific topics like exercising the ancient methods of *taḥlīl* and *tarkīb* (analysis and synthesis), finding alternative proofs for known theorems, or exercising constructions. They treat mostly elementary mathematical problems suitable for someone who wanted to enter the world of the philosophical sciences to which the mathematical sciences belonged in the view of this period. Such learning activities took often place at courts or in the houses of courtiers.[2] Other elementary texts composed before 1200 reflect the needs of legal scholars for some training in arithmetic and their desire to get acquainted with Indian digits and rules for multiplying, dividing, adding, subtracting, halving, doubling, and extracting square and cubic roots with them that were introduced in the late eighth or early ninth century from Sanskrit sources.[3] Although they are not necessarily presented as classroom texts, they were meant to serve as teaching tools for private studies or in the circles of religious scholars held at mosques. The latter are the only examples of larger public classes known from this period and were focused on religious knowledge. It is unknown how much mathematics was taught in them, and it is unlikely that classes dedicated specifically to one of the mathematical sciences were held at mosques in this period. Mathematical education beyond the basics that were taught during the years of childhood seems to have been delivered by house teachers and itinerant tutors or was acquired as an autodidact.

A few texts preserved from this period reflect the existence of discussion circles in private houses, where people of different backgrounds came together to ponder particular mathematical problems. Mature adults obviously enjoyed spending some hours with friends to learn how to solve such problems or used the occasion to challenge the knowledge of a stranger.

A second major period of mathematical education began at the latest in the twelfth century when the madrasa spread throughout Iran, Iraq, eastern Anatolia, Syria and Palestine, and Egypt.[4] At first apparently meant primarily for promoting the adherents of a particular Sunni law school, namely that named after al-Shafiʿī (767–820), it soon became available for the followers of other law schools too, specifically those of Abū Ḥanīfa (699–767) and Aḥmad b.Ḥanbal (780–855).[5] Several mathematical manuscripts copied during the twelfth and thirteenth centuries indicate that their scribes produced them in such surroundings. One of the three extant copies of Theodosius' *Spherics*, for instance, was copied in 1158 in the Niẓāmiyya madrasa at Mosul (Theodosius 2010, p. 5).[6] On September 9, 1215, a great-great-great-grandson of the Seljuq vizier Niẓām al-Mulk (1018–1092) copied at the first and most famous and influential madrasa founded by his ancestor in Baghdad a collection of mathematical texts by Aḥmad b. Muḥammad b. ʿAbd al-Jalīl al-Sijzī (d. after 998) and Ibn al-Haytham (d. 1048). At the same madrasa, Naṣīr al-Dīn al-Ṭūsī's (1201–1274) introductory textbook on *ʿilm al-hayʾa* was copied in 1283 (Ragep 1993, vol. 1, p. 77).

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[2] See Sect. 2.

[3] It is not clear which Sanskrit sources with arithmetical chapters were translated, and none of the Arabic translations are extant. But the Latin texts derived from Muḥammad b. Mūsā al-Khwārizmī's work/s on Indian arithmetic, as well as the clear labeling of this kind of notation and the procedures related to them as Indian in Arabic sources since the ninth century, leave no doubt that mathematical knowledge from India and, in all likelihood, texts written in Sanskrit about this knowledge provided the basis for the decimal positional system and its fundamental rules of calculation, as described in extant texts of the tenth century by authors like Kūshyār b. Labbān from Gilan or al-Uqlīdisī from Damascus.

[4] In the Maghrib and al-Andalus, this process took place later, perhaps during the thirteenth century.

[5] In the thirteenth century, adherents of the legal doctrines as taught on the basis of Anas b. Mālik's (fl. between 612–712) works and that of his early followers and interpreters also founded madrasas in Egypt and Syria.

[6] These are the only three copies of the ninth-century translation into Arabic that are extant. Other copies are extant from Naṣīr al-Dīn al-Ṭūsī's edition made in the thirteenth century.

Copying mathematical manuscripts in the precincts of a madrasa did not remain the only connection between the new teaching institution and the mathematical sciences. Arithmetic, algebra, geometry, surveying, ʻ*ilm al-hay*ʾ*a*, ʻ*ilm al-mīqāt,* and the construction of instruments soon became topics of classes taught by madrasa teachers inside and outside of this institution. These classes did not pursue a primarily disciplinary approach to teaching but were text centered. Major forms of learning followed those used in the dominant religious fields of madrasa teaching. As a result, most of the texts extant from this second period are elementary school texts meant to introduce the newcomer into the very basics of calculating, solving quadratic equations, constructing plane and solid figures, or solving basic problems of astronomy and astrology.

The sources available for discussing the forms and contents of mathematical teaching are mathematical treatises; chapters in encyclopedias; works on classifying the sciences; pedagogical tracts; instruments and their descriptions; biographical literature including autobiographies, historical chronicles, notebooks, commentaries, and glosses; colophons; and ownership entries. Since they are not preserved for all periods and regions, it is impossible to write a comprehensive history of mathematical teaching and learning for all regions in the same manner. Examples of particular cities, years, people, textbooks, or methods can, however, bring out important aspects of the theme. Even so, the extant sources do not provide all the information one would like to have for going beyond the mere description of the texts that were taught, the teachers and students who took part in a particular class, and the patrons who supported such undertaking materially, financially, or by regulating cases of conflict. In most cases, this is impossible. One reason is that very little research has been done so far on teaching the mathematical sciences in Islamic societies beyond the study of texts. Furthermore, the structure of the reports found in the sources was set up for other purposes than studying the forms and contents of mathematical classes. Hence, bits and pieces of information available for different regions need to be joined together to paint a richer picture. As a rule, such mergers will be limited to the same time periods. Other methodological risks are posed by the ambivalence and multivalence of the terminology used for writing about madrasas, texts, professors, students, patrons, and their respective activities and values. It is not always clear whether learning a mathematical text by heart was more highly regarded than practicing its methods and analyzing its principles in order to acquire the skills needed to solve the posed as well as other mathematical problems independently from one's teacher. Nor is it well understood what the rhetoric of utility meant in the context of madrasa teaching or writing commentaries on mathematical texts. The following sections will offer examples and undertake efforts to describe and clarify the available information in different social contexts and cultural forms.

2 Teaching Mathematics at Courts

Teaching mathematical sciences at courts began at the latest under the sixth ʿAbbasid Caliph al-Muʿtasim billāh (r. 833–834). The caliph's grandson Aḥmad and later Caliph al-Muʿtasim billāh (r. 862–866) was sent to the foremost philosopher of the time who was closely connected with the court, Yaʿqūb b. Isḥāq al-Kindī (d. ca. 866), for instruction in the newly translated Greek philosophical works which covered at least natural philosophy and spherical geometry, according to a manuscript which survives in Istanbul (Matvievskaya and Rozenfel'd 1983, vol. 2, p. 67).[7] Aḥmad was not the only male member of the ʿAbbasid dynasty who studied mathematical sciences with a prominent scholar. Other men like Jaʿfar b. al-Muktafī (10th c) reached the level of becoming writers of mathematical texts themselves. Nor was mathematical education limited to sons of caliphs. ʿAbbasid viziers

[7] The revised and enlarged English translation of this work, which unfortunately contains many typing as well as other errors, is Rosenfeld and Ihsanoğlu 2003. Other additions can be found in Rosenfeld 2004, 2006.

also considered it appropriate to provide their sons with such an education, as illustrated by the example of Abū Muḥammad al- Ḥasan, the son of Caliph al-Muʿtaḍid billāh's (r. 892–902) vizier ʿUbaydallāh b. Sulaymān b. Wahb (d. 901) (Matvievskaya and Rozenfel'd, vol. 2, p. 125).

With the decentralization of the ʿAbbasid caliphate and the loss of its hold over the vast territory between India and the Atlantic in the tenth century, many new, bigger, and smaller domains ruled by local scions, warlords, foreign mercenaries, or members of the urban lower strata were carved from the large body politic. Many new courts of rulers and viziers were set up competing with each other and the caliphs in Baghdad. While the ninth century was primarily shaped by scholarly activities in Baghdad (the ʿAbbasid capital), the tenth century saw a multitude of cities prospering in Iran, Central Asia, northern India, Syria, Egypt, and parts of northern Africa which attracted scholars in search of patrons and financial support. Many well-known writers of mathematical texts like Abū l-Rayḥān al-Bīrūnī (963–1048), Aḥmad b. Muḥammad b. ʿAbd al-Jalīl al-Sijzī, Abū Sahl Wayjan al-Kūhī (ca. 940–1000), Muḥammad al-Khujandī (ca. 940–1000), Abū Jaʿfar al-Khāzin (d. between 961 and 971), Abū l-Wafā'al-Buzjānī (940–998), ʿAbd al-Raḥmān al-Ṣūfī (903–986), or the famous physician, philosopher, and vizier Ibn Sīnā (d. 1037) produced their works at courts linked to one or more patrons. Several of them moved in fast sequence from one court to the next, while others were abducted as spoils of war.

Most of the mathematical production known to us today directly through extant manuscripts and instruments or indirectly through quotes, bibliographies, and other references resulted from this symbiosis between scholars and courts. Teaching consisted of the study of major texts like those of Euclid (*Elements*, *Data*, *Phaenomena*), Theodosius (*Spherics*, *Rising and Setting*, *On Habitations*), Autolykos (*On the Moving Sphere*, *Risings and Settings*), Ptolemy (*Almagest*), Banū Mūsā (*Book on the Knowledge of the Measurement of Plane and Spherical Figures*), or Thābit b. Qurra (*Book about the Premises*) either with a private teacher or alone. The purpose of learning was first and foremost to acquire knowledge of a text. This understanding of how mathematics ought to be taught is expressed clearly in the relief and pride that the Buyid ruler ʿAḍud al-Dawla (r. 948–983) felt when he had finished reading with his teacher the last page of Euclid's *Elements*. This motivated him to spend 20,000 dirham as alms.[8]

Princes, viziers, and officers at courts of several later Islamic dynasties also received a mathematical education, although its content is not always described with sufficient precision to judge whether it went beyond the very basics. Reports that indicate a more advanced level of training are extant for the Ayyubid dynasty (1171–1260), the Ilkhanid dynasty (1256–1336), the Timurid dynasty (ca. 1370–1507), the Khiljī Sultans of Malwa (1436–1531), and the Mughal dynasty (1526–1827). Manuscripts suggesting a more elementary level of mathematical education are extant for the Ak Koyunlu (1278–1508), the Safavid dynasty (1501–1722), and the Ottoman dynasty (ca. 1300–1922). It is, however, most likely that more than the named dynasties included some readings of mathematical literature in the education of their sons. These manuscripts contain texts or chapters on arithmetic, some parts of number theory, in particular the rules for calculating perfect and amicable numbers, algebra, surveying, sometimes some optics, and the basics of astronomy.

3 The Mathematical Sciences in Autobiographies: Ibn Sīnā (d. 1037) and al-Qalaṣādī (1412–1486)

Although less often written than biographies (in particular short entries in biographical dictionaries), autobiographies are not as rare in Arabic literature as one might assume. Most of them were written by scholars whose focus had been on the religious disciplines. The two autobiographies chosen here were

[8] J.L. Berggren, *Patronage of the Mathematical Sciences in the Buyid Courts*, unpublished manuscript. I thank Lennart Berggren for allowing me to use his text.

written by a philosopher and a *mudarris*, a teacher of a madrasa, the former having lived in the classical period, the latter in the postclassical phase. Their reports highlight some of the differences between these two periods as far as mathematical education and its place in the lives of scholars is concerned. The literary output of each of the two scholars included a good number of mathematical works, although mathematical disciplines were not of primary concern in their education. Ibn Sīnā, the most important philosopher of the classical period and the most influential author of philosophical and medical books during the postclassical period, included chapters on the four fundamental mathematical sciences in almost all of his encyclopedias or asked one of his most cherished students, Abū ʿUbayd al-Juzjānī (11th c), to write them in his place. Ibn Sīnā also wrote a few independent treatises on topics of the mathematical sciences – for instance, about an astronomical instrument with which he is credited, about the five simple machines of Antiquity (lever, screw, winch, block, wedge) or about the lifting of weights. Al-Qalaṣādī became well known as an author of elementary mathematical texts which were often used in madrasas for training students of the legal sciences in those basic parts of calculating, solving quadratic equations, and determining areas and volumes which they might need when asked to determine the inheritance shares of the relatives of a deceased person and the bequests he or she had stipulated. Altogether, at least thirteen mathematical texts by al-Qalaṣādī are extant today.

As seems to be typical for the classical period, Ibn Sīnā studied mathematical topics in his childhood and adolescence with a teacher whom his father – an administrator of the Samanid dynasty, first in a village, then in one of the major Central Asian towns (Bukhara) – hired for his extraordinarily talented oldest son. At first, Ibn Sīnā learned geometry, Indian arithmetic, and philosophy during the talks that his father had with his younger brother who adhered to the Shīʿī creed of the Fatimid caliphs in Egypt. "My father was among those who responded to the proselytizer of the Egyptians and was considered one of the Ismāʿīlīs. He had heard from them the account about the soul and the intellect in the way in which they tell it and present it, and so had my brother. They would sometimes discuss this matter [i.e. the account about the soul and the intellect] among themselves, while I would listen to them and comprehend what they were saying, but my soul would not accept it; and they began to summon me to it with constant talk on their tongues about philosophy, geometry, and Indian arithmetic" (Gutas 1988, pp. 23–24). Then, one day, Abū ʿAbdallāh al-Nātilī arrived in town, presenting himself as a philosopher. Ibn Sīnā's father decided to hire him as a house teacher for his son. However, al-Nātili's knowledge of the philosophical sciences was not very advanced, if Ibn Sīnā's autobiography can be trusted in this respect. Gutas (1988, p. 157) who retranslated the preserved Arabic text of the autobiography and pointed to its many problematic parts observed the close affinity between Ibn Sīnā's educational progress through texts and disciplines and the hierarchy of these sciences, as portrayed in contemporary treatises on the classification of the sciences. He suggested that the autobiography may not describe Ibn Sīnā's factual course of education, but an idealized version pretending that Ibn Sīnā followed the proper path of a philosopher from his childhood years. Two mathematical books are explicitly mentioned in the philosopher's autobiography, namely, Euclid's *Elements* and Ptolemy's *Almagest*. His teacher's understanding of geometrical matters was, however, mediocre. Ibn Sīnā described him as pretending to be above the content of the two works, while in fact the boy had to work out things for himself on the basis of his manuscript copies alone.

> As for the *Elements* of Euclid, I read the first five or six propositions with him, and thereafter undertook on my own to solve the entire remainder of the book. Next I moved on to the *Almagest* [of Ptolemy] and when I had finished its introductory sections and reached the geometrical figures, an-Nātilī said to me, 'Take over reading and solving them [i.e. the geometrical figures] by yourself, and then submit them to me so that I can show you what is right and what is wrong.' But the man could not deal with the book, so I made the analysis myself; and many were the figures with which he was unfamiliar until I presented them to him and made him understand them! (Gutas 1988, pp. 26–27)

When describing his childhood upbringing, Ibn Sīnā reported on the talks between his father and his brother on philosophy (theory of the soul) and Indian arithmetic to which he was invited. Indian arithmetic was an important subject matter that the father wished his son to master because he felt that he needed more training in this art than what he could acquire through listening to him and his brother. Thus he sent the boy to a greengrocer in Bukhara who used the Indian numerals and the decimal position system in his business. Ismāʿīlīs were strongly represented among the merchant communities. Ibn Sīnā's father may thus have seen Indian arithmetic as a part of an Ismāʿīlī education.

ʿAlī b. Muḥammad al-Qalaṣādī was born in Basṭa, today Baza, in the Naṣrīd emirate of Granada. His autobiography took the form of a travel account, a *riḥla*, a report about his intellectual upbringing and the scholars who trained him (Marín 2004). As such, it belongs to a genre that often is named dictionary (*muʿjam*), program (*barnāmaj*), or catalogue (*fihrist*), where the names of teachers and books studied with them are listed. However, the different name *riḥla* is fully deserved. Al-Qalaṣādī, perhaps bored by the dry listing of names and titles characteristic for this sub-genre of Arabic autobiographical literature, allowed his readers to see the landscapes through which he wandered and the cities of al-Andalus, North Africa, Egypt, and the Arabian peninsula in which he learned and which he visited as part of a pilgrimage. In this sense, his text also belongs properly to the genre of travel accounts that had known many Andalusian contributors. Al-Qalaṣādī left al-Andalus in 1436, arrived in Mecca in 1447, and returned to Almería, where his travel had started, in 1451. Of these 15 years, he spent 8 in Tlemcen, 4 in Tunis, and less than 1 in Cairo. Three years were needed for the journeys between the various cities. According to Marín (2004, p. 296), al-Qalaṣādī's account is the last representative of this genre composed in al-Andalus. Six years after the scholar's death, the king of Castile conquered the Naṣrīd kingdom, the last Muslim state on the Iberian Peninsula.

Al-Qalaṣādī began his education in his hometown. He studied the Qur'an; the sayings of the Prophet; Arabic grammar and literature; *fiqh* (law), in particular the rules that governed inheritances and bequests (*ʿilm al-farāʾiḍ*); and arithmetic. The boy's interest in arithmetic corresponded well with the interest of numerous Andalusian scholars between the tenth and fifteenth centuries. Al-Qalaṣādī studied and commented upon two works by one of these previous scholars, Abū Muḥammad Ibn al-Yāsamīn's (d. 1204) poems on algebra and on the extraction of roots (Matvievskaya and Rozenfel'd 1983, vol. 2, p. 512). Among al-Qalaṣādī's teachers in Baza were four religious scholars who were interested in arithmetic, algebra, geometry, and astronomy – Abū ʿAbdallāh Muḥammad al-Qusṭurlī (d. 1440/1), Abū Aḥmad Jaʿfar b. Abī Yaḥyā, Abū l-Ḥasan ʿAlī b. Mūsā al-Lakhmī (d. 1440/1), and Abū ʿAbdallāh Muḥammad al-Bayyānī (d. 1472) (Marín 2004, pp. 301–303). Mathematical activities, however, were much more intensive in the Maghrib in al-Qalaṣādī's lifetime than in al-Andalus. Thus, traveling across the sea to further his education was a sensitive decision. Al-Qalaṣādī's biography highlights the long time that it took to become a mature scholar of the "traditional," "rational," and mathematical sciences (independent of how many of those disciplines an individual scholar studied). He was already 24 years old when he decided to travel for the acquisition of additional knowledge and spent about 13 years studying in Tlemcen, Tunis, and Cairo.

Al-Qalaṣādī began his mathematical education parallel to his introduction to the Qur'an. His second teacher, Abū ʿAbdallāh Muḥammad al-Qusṭurlī, read with him Ibn Bannāʾ's (1256–1321) work *al-Maqālāt al-arbaʿa fī ʿilm al-ḥisāb* (The Four Chapters on Arithmetic).[9] Ibn Abī Yaḥyā introduced Ibn al-Bannāʾ's *Talkhīṣ aʿmāl al-ḥisāb* (Epitome of the Operations of Arithmetic) to al-Qalaṣādī and read once more *al-Maqālāt al-arbaʿa* with him. Reading one and the same text with different teachers was not an uncommon feature in postclassical Islamic education. This approach to learning was motivated by at least two prominent factors that spilled over in the teaching of mathematical knowledge. The older of the two was the custom in the "traditional" disciplines, in particular the study of the sayings of the Prophet, to evaluate the soundness of knowledge by the length of the transmitter chain, that is, the length of time between the last recipient of such a transmission and the first transmitter of a particular saying, determined by the number of intermediary members of the chain and their age at the moment when they transmitted their knowledge to the next person in the chain, and by the moral

[9] This text has different titles in different manuscripts (Djebbar and Aballagh 2001, pp. 105–107).

quality of each transmitter as a virtuous and trustworthy Muslim. This custom resulted in an attitude which maintained that it was not the piece of knowledge that mattered most, but the people who transmitted it. Hence, studying the same piece of knowledge with more than one teacher could improve the standing of the recipient in the chain of transmission and thus increase his or her future reputation as a transmitter. The second factor was the trend that emerged within the madrasa system by which it became customary to read less and less an entire book and more and more specific chapters only.[10] This method of teaching and learning reinforced the preference for studying specific works by authors who were alive and of relevance in a certain field of knowledge, while the student listened to or read their works, preferably together with them. The goal of education was not to learn an entire discipline, but to spend time with well-known living scholars reading their works, those of their teachers, and their teachers' teachers.

When al-Qalaṣādī arrived in 1437 in Tlemcen, the city was the capital of a small but independent kingdom coveted by the mightier dynasties of the Marinids (1258–1465) in Fes and the Ḥafsids (1229–1574) in Tunis. Its flourishing economy and culture attracted numerous Andalusian scholars who fled the advance of Castilian armies. Al-Qalaṣādī looked for teachers of the mathematical sciences and found six; he again read Ibn al-Bannāʾ's *Talkhīṣ* with them in addition to further works by the Maghribī scholar (Marín 2004, pp. 306–309).[11] These studies expanded his mathematical knowledge substantially and enabled him to write his first treatise on arithmetic, *al-Tabṣira fi l-ghubār* (Introduction to the Dust [Letters]).[12] One of his teachers, Abū l-ʿAbbās Aḥmad b. Muḥammad al-Maghrāwī, known as Ibn Zāghū (d. 1440/1), taught at the madrasa al-Yaʿqūbiyya according to seasons, as al-Qalaṣādī reports in his *Riḥla*: in winter he taught *tafsīr* (exegesis), *ḥadīth* (the sayings of the Prophet), and *fiqh* (law), while summer classes were devoted to *uṣūl al-fiqh* (the fundamentals of law), *farāʾiḍ* (inheritance calculations), rhetoric, arithmetic, and geometry (Marín 2004, pp. 308–309).

Al-Qalaṣādī continued his education in Tunis and Cairo, spending now some time with a text on the astrolabe. According to Marín (2004, p. 309), al-Qalaṣādī abstained, however, from describing in his travel account any further study of specific texts on the mathematical sciences. Other sources, though, add to our knowledge about his mathematical activities in these two cities. In Tunis, he studied several chapters of a work by the Maghribī scholar Abū Bakr Muḥammad b. ʿAbdallāh al-Ḥassār (12th c). Aballagh believes that this was al-Ḥassār's great work *al-Kāmil fī ṣināʿat al-ʿadad* (The Complete, On the Art of the Number).[13] In Cairo, al-Qalaṣādī taught his own works and apparently gave at least one student a teaching certificate (*ijāza*) (Al-Suyuti 1927, p. 131). Five of his own treatises are exclusively preserved in the National Library of Tunis. This might suggest that he composed them during his stay in the capital of the Ḥafsid kingdom. They discuss conceptual issues such as the meaning of terms like fraction, numerator, compound, and simple, questions of number theory, and the basics of arithmetic (Matvievskaya and Rozenfel'd 1983, vol. 2, pp. 510–512). The other mathematical works of al-Qalaṣādī exist in more than one copy, mostly between 3 and 8. One of his works, *Kashf al-asrār ʿan ʿilm ḥurūf al-ghubār* (Uncovering of the Secrets, about the Science of the Dust [Letters]), is extant even in 37 exemplars. This figure suggests that this work was much higher appreciated by the scholars of Tunis and their students than al-Qalaṣādī's other works.

[10] It has not been studied when, where, and in which fields of knowledge this trend started. According to the sources I am familiar with, it was well on its way in the fourteenth century in Mamluk Egypt and Syria.

[11] Among the additional works of Ibn al-Bannāʾ that al-Qalaṣādī studied in Tlemcen were his *Rafʿ al-ḥijāb* and his work on algebra (Aballagh 1988; Djebbar 1990). I thank Mohamed Aballagh for his generous help in verifying the details of al-Qalaṣādī's education and later teaching positions.

[12] This is the usual title given to this text. The manuscript Tunis, Bibliothèque nationale, 8275, however, gives the following title: *al-Tabṣira al-wāḍiḥa fī masāʾil al-aʿdād al-lāʾiha*. I thank the anonymous francophone reviewer for this information. Each Arabic letter carries a numerical value and served as a numeral.

[13] I thank Mohamed Aballagh for this information taken from Aḥmad Bābā al-Ṭinbuqtī's (1564–1627); al-Ṭinbuktī, Aḥmad Bābā. 1398/1989 biographical dictionary.

4 Aḥmad al-Sijzī: How Does One Become a Productive Mathematician?

In the ʿAbbasid period, mathematical knowledge was literary knowledge for the broader educated public. Those who wished to become experts, however, focused on methods. Some began their education in the workshop of an instrument maker, as was the case of ʻAlī b. ʻĪsā (first half 9th c), or learned from their fathers and uncles. Ibn al-Nadīm (d. 998) reports, for instance, in his *Kitāb al-fihrist* (Book of the Catalogue) about the eminent scholar of the mathematical sciences and chief of protocol at the Buyid court in Baghdad, Abū l-Wafāʾ al-Būzjānī (d. 940): "He studied what there was [to be known] of numbers and arithmetic under his paternal uncle, who was known as Abū ʻAmr al-Mughāzilī, and his maternal uncle, known as Abū ʻAbdallāh Muḥammad ibn ʻAnbasah" (*The Fihrist of al-Nadīm* 1970, vol. 2, p. 667). Others who had already advanced in their training met with famous experts for private discussion sessions. Mathematical questions could also be sent as a letter to an expert who took this occasion to write a little treatise on the topic (Berggren 2003, p. 181). One of the scholars of the tenth century who participated in such sessions, dedicated some of his works to local rulers, and maintained an active exchange of letters on mathematical problems with a few of his contemporaries was Aḥmad b. Muḥammad b. ʻAbd al-Jalīl al-Sijzī. He was also the author of a didactic text in which he outlined his ideas about how a student could acquire the needed skills not merely for solving known problems, but for finding new results. This text, named *Kitāb fī tashīl al-subul li'stikhrāj al-masāʾil al-handasiyya* (Book on Making Easy the Ways of Deriving Geometrical Problems), introduces students to different methods for solving known problems as well as finding new knowledge. According to Hogendijk (al-Sijzī 1996, p. ix), this text can be dated to approximately the year 980, thus constituting a work of the mature Sijzī and reflecting "his own experience with problem-solving in geometry." Sijzī wished to enumerate in it "the rules which will make it easy for the researcher who knows and masters them to derive whatever geometrical constructions he wants" (al-Sijzī 1996, p. 1). After the statement of the book's goal, al-Sijzī criticized some unnamed scholars in a manner that might allude to Ibn Sīnā's theory of induction. "Some people think that there is no way of learning the rules for deriving (new propositions) even with much research, practice, study, and lessons in the elements of geometry, unless a man has an innate natural talent which enables him to discover figures, because study and practice are insufficient" (al-Sijzī 1996, p. 1). Al-Sizjī vehemently disagreed with such a position and insisted on the necessity to study much and work hard to become a good geometer (al-Sijzī 1996, pp. 1–2).

After this rebuke, al-Sizjī took the standard position of tenth-century scholars, according to which the beginner needs to learn first the theorems of Euclid's *Elements*.

> It is necessary for someone who wants to learn this art, to thoroughly master the theorems which Euclid presented in his *Elements*. For between mastering the thing and the thing itself there is a very deep gap. It is necessary that he has a very thorough idea of their general and their special properties, so if he needs to look for their properties he is well-prepared to find them. If he has to do any research, then it is necessary for him to study and visualize in his imagination the preliminaries and the theorems that are of that genus, or that have (something) in common with it. (al-Sijzī 1996, pp. 2–3)

Then the author discussed the meaning of preliminaries, special properties, and other concepts and named the methods which a student needs to learn in order to become a successful mathematician.

> First, cleverness and intelligence, bearing in mind the conditions which the proper order (of the problem) makes necessary.
>
> The second is the profound mastery of the (relevant) theorems and preliminaries.
>
> The third is: following of the methods of them (these theorems and preliminaries) in a profound and successful way, so that you rely not only on the theorems and preliminaries and constructions and arrangements which we mentioned. But you must combine with that (your own) cleverness and guesswork and tricks. The pivotal factor in this art is the application of tricks, and not only (your own) intelligence, but also the thought of the experienced (mathematicians), the skilled, those who use tricks.
>
> The fourth is: information about the common features (of figures), their differences, and their special properties. In this particular approach, the special features, the resemblance and the opposition are (considered by themselves) without enumeration of the theorems and preliminaries.

The fifth is the use of transformation.
The sixth is the use of analysis.
The seventh is the use of tricks, such as Heron used. (al-Sijzī 1996, pp. 4–6)

After having declared what he considers the most important things a beginner needs to learn and the methods for his success as a mathematician, al-Sizjī turned to specific examples to help the reader of his instruction manual truly understand its essence.

"Since we have presented and mentioned these things in a loose manner, let us now bring for each of them examples, so that the researcher learns their true natures. For one can speak about this art in two different ways: first, abstractly, in a deceiving and illusory manner; and secondly in a profound way, with clear explanations and the presentation of examples, so that it is perceived and understood completely" (al-Sijzī 1996, p. 6).

Al-Sijzī's treatise is a rare medieval text on how to find new mathematical results. It shows which methods the scholar (and with him, many of his contemporaries) preferred when doing mathematics. At the same time, he makes unmistakably clear which teaching methods he favored and on which epistemic basis he stood.

5 Classifying the Mathematical Sciences

In the eighth and ninth centuries, scholars of the new Abbasid caliphate also learned about the relationship between different fields of knowledge such as Aristotle's classification of philosophy, the Neoplatonic quadrivium, or Galen's ideas about the preeminence of mathematics and philosophy for a student of medicine. Ordering human knowledge proved an important topic for many centuries to come. Questions about what was divine and what was human knowledge, which human knowledge ranked highest, and which fields belonged to one cluster were debated from the earliest times of the translation movement in the eighth and early ninth centuries. Presenting schemes of knowledge served as propaedeutic literature for the student of philosophy and in later centuries for the student of the "rational" sciences, in particular *uṣūl al-dīn* (fundamentals of religion). These schemes provide insights into their authors' perception of the main clusters of human knowledge including the mathematical sciences. Such classifications often included comments on which mathematical topics, authors, or textbooks had to be studied and were seen as authoritative. Except for variations in the order of the sciences, most authors considered number theory, geometry, astronomy (or the science of the stars), and music (i.e., theory of proportions) as the basic disciplines of mathematics. Some authors integrated into this scheme other disciplines like optics or topics that today we consider part of mechanics as equal components.

Four groups of authors participated in these discussions: philosophers, physicians, administrators, and religious scholars. Their texts were studied not only during their lifetimes in their own circles, but they also inspired generations of later teachers and students of madrasas and cognate teaching institutes. Among the most important representatives of these four groups, we find the philosophers Abū Naṣr al-Fārābī (ca. 870–950) and Ibn Sīnā, the physician Abū Sahl al-Masīḥī (d. 999/1000), the administrator Muḥammad b. Aḥmad al-Khwārazmī (10th c), the eminent religious scholar Fakhr al-Dīn al-Rāzī (1149–1209), and the physician and madrasa teacher Muḥammad b. Ibrāhīm b. al-Akfānī (1283–1348). Except perhaps for al-Khwārazmī's text, all writings on the disciplinary structure of the system of knowledge were undeniably of great relevance to teaching and provided information about important authors and textbooks.

Abū Naṣr al-Fārābī's classification has various features that made it a highly regarded concise survey on the main components of knowledge, recommended to those who wished to become scholars. Concerning the mathematical disciplines, al-Fārābī's insistence on the relevance of practical knowledge and hence the division of each of the seven disciplines which he regarded as mathematical into a theoretical and a practical part was of great importance. The seven mathematical disciplines which al-Fārābī discussed in his book *Iḥṣāʾ al-ʿulūm* (Enumeration of the Sciences) are number theory, geometry, optics, the science of the stars, music, the science of the weights (*ʿilm al-athqāl*), and the science of the artifices (*ʿilm al-ḥiyal*) (Zonta 1992, XIV, p. 49).[14] Practical arithmetic, for instance, studies numbers as they enumerate concrete objects and are used in civil and commercial affairs. It focuses on teaching how to add, subtract, multiply, divide, and extract roots and similar operations (Zonta 1992, pp. 73–74). Theoretical arithmetic, by contrast, treats numbers as abstracted from concrete objects and studies their properties and divisions. The examples al-Fārābī offers here to his readers are primarily taken from Euclid's *Elements* (Zonta 1992, p. 74). In a second paragraph, al-Fārābī names abundant, deficient, amicable, and perfect numbers, thus turning to the other fundamental text on number theory (*Introduction in Arithmetic*) by an ancient author, Nicomachus of Gerasa (2nd c), which in later centuries dominated the teaching of this field at schools.

Ibn Sīnā drew inspiration from al-Fārābī's treatise for his own classification of the sciences called *Aqsām al-ʿulūm al-ʿaqliyya* (The Parts of the Intellectual Sciences), however, without always following the lead of his predecessor (Heath 1992, p. 42–43). Ibn Sīnā's text was studied at Ayyubid and Mamluk madrasas in Cairo and Damascus in the thirteenth and fourteenth centuries. A different view on the place of the mathematical sciences and hence on mathematical education had the physician and teacher of Ibn Sīnā, Abū Sahl al-Masīḥī (d. 999/1000), in his *Kitāb fī aṣnāf al-ʿulūm al-ḥikmiyya* (Book on the Categories of the Philosophical Sciences). He presented five of them – namely, geometry, arithmetic, the science of the stars, music, and optics – as theoretical particular sciences and as propaedeutics for physics and metaphysics, which he saw as universal sciences. Abū Sahl listed authors and their books to be studied for each of these theoretical disciplines: for geometry, works by Euclid, Archimedes, and Apollonios; for arithmetic, Nicomachus of Gerasa, Books VII–IX of the *Elements,* and books on Indian arithmetic and on algebra; for astronomy, Ptolemy's *Almagest* and Abū l-ʿAbbās al-Faḍl al-Nayrīzī's (d. 922) commentary on it, the *Middle Books,* and the astronomical handbooks by Ḥabash al-ḥāsib (9th c), Muḥammad b. Jābir al-Battānī (ca. 850–929), and al-Nayrīzī; and for music, Ptolemy's *Harmonics* and al-Kindī's elaboration and al-Fārābī's exhaustive treatment of the subject. Beyond the five theoretical disciplines, Abū Sahl recognized four particular mathematical sciences that were used in a profession (*miḥna*): ʿ*ilm al-ḥiyal*, medicine, agriculture, and alchemy (Gutas 1988, p. 149). The works listed by Abū Sahl indicate that the main content of mathematical education considered appropriate in the second half of the tenth century derived from ancient Greek works translated during the late eighth and ninth centuries, commentaries and revisions of such works produced by authors during the ninth and early tenth centuries, some new works by such authors, and a few books based on translations from Sanskrit. They also suggest that the level of education reached beyond the elementary and aimed to train scholars. The four disciplines that were exercised in a professional capacity illustrate a much wider understanding of the term "mathematical" in the tenth century than it is understood today.

Fakhr al-Dīn al-Rāzī (1149–1210) had strong interests in astrology, philosophy, and other sciences. He addressed primarily courtly circles. To one of the now Turkic rulers of Khurasan, the Khwārazmshāh ʿAlāʾ al-Dīn Tekesh (r. 1172–1200), he dedicated his encyclopedic survey of 40–60 disciplines called *Jawāmiʿ al-ʿulūm* (Collection of the Sciences) in Arabic (40) and *Ḥadāʾiq al-anwār fī ḥaqāʾiq al-asrār* (The Gardens of Lights, on the Truths of Secrets) in Persian (60). The mathematical sciences in these two encyclopedias comprised the four principal disciplines and a number of branch sciences like various calculation systems (Indian arithmetic, calculating with fingers, sexagesimal system, etc.), algebra, magic squares, surveying, ʿ*ilm al-athqāl*, optics, burning mirrors, and other branches. Al-Rāzī was thus one of the main religious scholars who contributed to a stable dispersion of mathematical and philosophical knowledge in Arabic through his many students who flocked to him and then opened their own teaching circles, such as those in northern Iraq or Syria.

[14] Be aware, however, of the misleading modernization of terms like ʿ*ilm al-nujūm*, ʿ*ilm al-athqāl,* and ʿ*ilm al-ḥiyal* by Zonta replaced here by my own translations.

In the lifetime of al-Rāzī and his students, a major reorganization of the system of disciplinary knowledge seems to have taken place in Islamic societies from Central Asia and India that, with only a short delay, also spread to the Maghrib. The former triangle of religious sciences, ancient sciences, and philological sciences was rearranged as the traditional sciences, the rational sciences, and the mathematical sciences (Brentjes 2008a). These changes seem to have been closely connected with the new teaching possibilities available outside the courtly sphere and the private household which arose in the new teaching institutes. Madrasa teachers, for instance, appeared now as authors of classifications of knowledge. One of them was Muḥammad b. Ibrāhīm b. al-Akfānī (1283–1348) from Sinjar, who at the same time was a successful physician in Cairo – a typical combination in the Mamluk period. His *Irshād al-qāṣid ilā asnā al-maqāṣid* (Guidance of the One Who Aspires to the Most Shining Goals) is a veritable guidebook through the literature for a beginner, a student who had reached some middle level in his training, and a mature student. It presents for each of them a list of books to read for studying any of the disciplines that the author included in his system of knowledge. Altogether, Ibn al-Akfānī presents 600 book titles for 60 sciences.

The classification of the mathematical sciences standard among the philosophers and physicians of the classical period as principal and branch disciplines, while macerated, still lends order to the presentation. The mathematical sciences which come after the natural sciences begin with geometry (Witkam 1989, pp. 45–53), which in turn is followed by *ʿilm al-hayʾa* (mathematical cosmography). This is the main theoretical discipline of astronomy that slowly emerged since the ninth and tenth centuries and focused on the discussion of models for planetary movements, including the calculation of distances between the planets and their sizes as well as basic information about mathematical geography. The other two disciplines are number theory and music (Witkam 1989, p. 54). Thus, Ibn al-Akfānī adopted a new order of the principal sciences, arranging those that worked with geometrical concepts and methods in the first group and those that worked with numbers in the second. The number of branches of each principal science has grown considerably. Geometry is now linked to ten branches (architecture, optics, burning mirrors, centers of gravity, surveying, lifting of water, pulling of weights, sundials, instruments of war, automata) (Witkam 1989, pp. 54–55). Mathematical cosmography is linked to five branches (astronomical handbooks and ephemerides, timekeeping, the quality of observations, projections from the sphere on the plane, gnomons) (Witkam 1989, pp. 57–59). Number theory possesses six branches (fundamental arithmetical operations not using Indian arithmetic; the arithmetic of the dust board, that is, positional decimal system; algebra; the rule of the double false position; calculation of legations and testaments; calculating with dinar and dirham); music has no branches (Witkam 1989, pp. 59–63).

6 Mathematical Education in Biographical Literature

Arabic biographical literature, particularly that written from the twelfth century onwards, is a major source of information on mathematical education in Syria and Egypt under the Ayyubids (1171–1260), the Mamluks (1260–1516), and the Ottomans (late thirteenth century–1922). Most works of this kind were dictionaries devoted to the philosophical or medical sciences, such as those by Ibn al-Qifṭī (1173–1248), an Ayyubid vizier in Aleppo born in Egypt, or Ibn Abī Uṣaybiʿa (1194–1270), a Damascene physician from a family of high-ranking court physicians; or else they were dictionaries dedicated to one of the traditional sciences like a specific law school (Shafiʿī, Ḥanafī, Mālikī, Ḥanbalī, etc.) or *ḥadīth*, for instance, those by Tāj al-Dīn al-Subkī (d. 1370) or Shams al-Dīn al-Dhahabī (1274–1348). Other types of biographical literature with information about the mathematical sciences are a kind of medieval curriculum vitae of individual scholars (*muʿjam*), that is, lists of teachers and books that the compiler of the list had studied under their supervision, or descriptions of two or more generations of teachers (*fihrist* or *barnāmaj*) of the author. Persian biographical literature preferred poets and painters over scholars and preachers. Hence, information on mathematical education is rarely found in Persian biographical dictionaries. Historical chronicles serve here as the main outlet for such reports, while in the Arabic regions they complement the depictions given in biographical dictionaries by information about the scholars' relationship to princes, notables, or warlords.

Basic mathematical skills like counting, calculating, and the elementary properties of geometrical figures like the circle, the triangle, the square, or the cube were taught in childhood by a house teacher if the family was well off to afford one. In other cases, parents sent their children to elementary schools (*maktab*, *dār al-ta*ʻ*līm*). Catholic and Protestant travelers to the Arabic provinces in the Ottoman Empire reported in the sixteenth century that not only did little boys receive such basic education but little girls too (Belon 1553). Dold-Samplonius documented remainders of such a mathematical class in an elementary traditional school in Morocco at the end of the twentieth century. Called *qubba* after its dome, it is a structure which consists of a small domed square room with the tomb of a saint, a walled courtyard with two more tombs and a large brick oven, the living quarters of the saint, and a door to the outside world (Dold-Samplonius 2008). In 1992, when Dold-Samplonius visited this place, the living quarters of the saint were used as a classroom. Several tables with Qur'anic verses stood upright, while a single piece with arithmetical problems hung on the wall. The problems included counting, addition, subtraction, and multiplication in integers. Dold-Samplonius concluded that these problems addressed different age groups and that the students found it difficult to write on one and the same tablet. They made mistakes, wiped out the tablet several times, and wrote one exercise over the other (Dold-Samplonius 2008, p. 384). Whoever taught arithmetic in this small sanctuary was obviously not a skilled pedagogue. Slates like the ones found by Dold-Samplonius in Morocco have also been found in other regions in North and sub-Saharan Africa like Mali from the nineteenth and twentieth centuries. Although the ones shown on the website of the collector Bouwman contain only the Basmalah or Qur'anic verses, their similarity to those found by Dold-Samplonius suggests that slates with arithmetical exercises also existed in those regions and times (Bouwman Oriental Books n.d.). Djebbar and Moyon confirm this for the twentieth century in Mali (Djebbar and Moyon 2011).

Students who wished to acquire more substantial mathematical knowledge usually studied with a madrasa teacher, one or more of their relatives or a specialized house teacher. Ibrāhīm b. ʿUmar al-Sūbīnī studied law and two branches of arithmetic (calculating with dust letters and fractions) in Hama with the city's madrasa teachers. In Cairo, he took classes in algebra, surveying, instruments for timekeeping, and similar topics with the head and law professor of the Sufi lodge al-Jānibakiyya Dāwadāriyya and *muwaqqit* at the Azhar Mosque, Ibn al-Majdī (1366–1447) (al-Sakhāwī n.d., vol. 1, pp. 100–101, 300–302; Charette 2007, pp. 561–562). Ibrāhīm b. ʿAlī, known as al-Zamzamī, studied the following in Mecca with his brother al-Badr Ḥusayn al-Zamzamī: inheritance rules, arithmetic, algebra, mathematical cosmography, geometry, timekeeping, the derivation of an ephemeris from an astronomical handbook, and chronology (al-Sakhāwī n.d., vol. 1, p. 86). Central to the studying were teachers and individual texts, not entire disciplines, as explained in Sect. 3. While the students often learned different texts with different teachers, one and the same text was occasionally read with two different teachers. Important texts were learned by heart, such as *al-Alfiyya* (The Thousand [Verses]) on inheritance rules and *al-Muqni*ʿ (The Evident) on algebra by Ibn al-Hāʾim (1352–1412), a famous teacher in Mamluk Cairo and Jerusalem, and apparently at times also Euclid's *Elements* (al-Sakhāwī n.d., vol. 1, p. 227; vol. 3, p. 115). Abū l-Faraj Ibn al-ʿIbrī (1225 or 6–1286), the Jacobite Patriarch and author of several books in Arabic and Syriac – among them a universal history – reported on 'Izz al-Dīn al-Ḍarīr, one of "the most excellent (men) of this time in the sciences of the ancients," who had memorized the first six books of the *Elements,* including all the diagrams with their letters, and could explain each single proposition (Ibn al-'Ibrī 1997, p. 275).

Other books were read aloud in class by one of the elder students and written down by the newcomers. The teacher checked at the end whether the students had produced correct copies of the treatise, which was mostly the teacher's own commentary on the text of a predecessor or an introduction to arithmetic, algebra, geometry, or methods for determining prayer times or the *qibla*, the direction of prayer, by a leading teacher mostly from the region where the class took place. In Cairo, Ibrāhīm b. Muḥammad al-Dimashqī read with al-Warrāq the latter's commentary on Ibn al-Hāʾim's work *al-Ḥāwī* (The Encompassing), a paraphrase of Ibn al-Bannāʾ's *al-Talkhīṣ* (al-Sakhāwī n.d., vol. 1, p. 128). Had a student passed successfully through the years of learning texts by heart and reading them in class, he could progress to the level of tutor and help younger students with their homework. The final stage was reached when the student became a disciple of a teacher. The methods of study did not change mucḥ The disciple read specific texts with his chosen teacher, deputized for him in one of the smaller madrasas or cognate teaching institutes or served to him in other capacities. He also annotated studied texts with short explanations or wrote his own commentaries. The end of the complete educational cycle consisted of (often public) exams that gave the successful candidate the right to work as a judge and write *fatwa*s.

None of the methods described so far were specific for teaching any of the mathematical disciplines (Makdisi 1981). Rather, they applied to all three clusters of sciences that were taught by madrasa teachers, that is, the traditional, the rational, and the mathematical sciences. This also applies to the manner in which achievements in the mathematical sciences were appreciated or criticized. Authors of biographical dictionaries, historical chronicles, topographies or histories of madrasas, and cognate institutes use the same rhetoric for all three clusters of sciences. The father of the famous Ḥanbalī judge and conservative champion against logic, philosophy, astrology, and innovation, Taqī al-Dīn Ibn Taymiyya (1263–1328), Shihāb al-Dīn Ibn Taymiyya (1230–1283), is said to have had "a long hand in inheritance mathematics, arithmetic and *ʿilm al-hayʾa*" (al-Nuʿaymī 1990, vol. 1, p. 75). A standard entry on a scholar who had studied one or more mathematical texts with usually more than one teacher is that of ʿAbd al-Raḥīm b. Ibrāhīm […] al-Qāhirī […]: "our neighbor […] and he also took from al-Shirwānī *al-uṣūl* and logic and from al-Kāfiyājī *hayʾa*, geometry and other (things) and *farāʾiḍ* and arithmetic together with algebra from al-Sayyid ʿAlī Tilmīdh Ibn Majdī (i.e. Ibn Majdī's student) and metric from al-Abādī or someone else […]" (al-Sakhāwī n.d., vol. 4, pp. 164–165).

The available sources speak rarely of specific methods used in classes or personal sessions for teaching mathematics. One such rare occasion is the brief description of classes taught in Granada and Tlemcen in the first half of the fifteenth century. Abū ʿAbdallāh Muḥammad al-Majārī (d. 1457/8) was an Andalusian who studied in Granada, among other things, arithmetic, geometry, algebra, and inheritance rules. There, he read several works by Ibn al-Bannāʾ, including the *Talkhīṣ* (Epitome), in the manner of "tutoring and practicing" (al-Majārī 1982, p. 128). He also studied the *Tadhkirat al-albāb* (Memoir of the Minds), a book compiled by his teacher Abū ʿAbdallāh Muḥammad b. Shaykh al-Kabayī. In it the author combined number theory, arithmetic, and three ways to teach inheritance rules.

Combining several different fields in one work had become a widespread manner of textbook writing by the twelfth century in Syria and Egypt. There, writers of biographical dictionaries praised this style as being very useful. However, al-Majārī may not have been certain about its appropriateness since he described it as *gharīb al-nawʿ* (of a strange kind) (al-Majārī 1982, p. 128). After years of study in Granada, al-Majārī, like al-Qalaṣādī, traveled to Tlemcen, Tunis, Cairo, and other cities to study law, the fundamentals of religion, and mathematical matters with a variety of teachers (al-Majārī 1982, p. 130). In Tlemcen, he met Abū ʿUthmān Muḥammad b. Muḥammad al-ʿUqbānī with whom he studied, among other texts, a commentary on Ibn al-Bannāʾ's *Talkhīṣ*. His teacher told him that he had studied with ʿAbdallāh b. Sulaymān al-Saṭī, whose competence in geometry was very limited. One day in class, the teacher asked al-Majārī's teacher and his friend to demonstrate that the solution for some problems was correct and to name the necessary theorems from Euclid's *Elements*. "In some cases my friend Abū ʿAbdallāh al-Sharīf (from Tlemcen) was faster than me, in others I preceded him. Several questions we answered in the same time" (al-Majārī 1982, p. 130). Al-Majārī enjoyed these little stories since they made him feel that his teacher "had immersed himself deeply in geometry" (al-Majārī 1982, p. 130). He was also proud to report that his teacher's "transmitter chain" in the mathematical sciences went back to Ibn al-Bannāʾ himself (al-Majārī 1982, p. 131). Despite this praise for al-ʿUqbānī, al-Majārī read Ibn al-Bannāʾ's *Talkhīṣ* again, plus Ibn al-Yāsamīn's poem on algebra, with another teacher, Abū ʿAbdallāh Muḥammad, known as al-Tharghī, with whom he also studied Euclid's *Elements* from Book I to the middle of Book X in the form of visualization (*taṣawwur*). His teacher presumably drew the Euclidean diagrams on a slate. Ibn al-Bannāʾ's and Ibn al Yāsamīn's texts al-Majārī learned from al-Tharghī through "visualization and practice" (al-Majārī 1982, p. 137).

A second of these rare glimpses into the teaching practices comes from seventeenth-century Ottoman Damascus. There, the teacher also aspired to show his students the mathematical objects and activities they were meant to master. Muḥammad Amīn al-Muḥibbī (d. 1699), the author of the biographical dictionary *Khulāṣat al-athar fī aʿyān al-qarn al-ḥādī ʿashr* (The Essence of the Trace, On the Notables of the Eleventh Century), wrote about his teachers and the kind of teaching he received in geometry: "Maḥmūd al-Baṣīr al-Ṣāliḥī al-Dimashqī […], our excellent shaykh […]. He read in Damascus with the most important of the teachers, among them our shaykh, the arch-scholar Ibrāhīm al-Fattāl, and by him he was trained and (educated) in (various) disciplines. He read with him Arabic, rhetoric, and logic. He took the mathematical (sciences) from Shaykh Rajab b. Ḥusayn and metaphysics from Munlā Sharīf al-Kurdī. […] And I took from him logic, geometry, and *kalām* ('rational theology'). While I took geometry, he used tricks for determining their figures (or: theorems) well by giving examples from wax to me. His professor, the mentioned Shaykh Rajab, had imitated them for him. He made them in a very precise manner. While I read geometry with him I was astonished by his visual representation (*taṣwīr*) as he had taken them from his professor. It was said: if the figure appears which he fabricated, then it corresponds with the figure, which is in the book" (al-Muḥibbī n.d. vol. 4, pp. 330–331).

Other forms of teaching need to be gleaned from manuscripts and other objects like slates produced in class or at home. It is well known, and many manuscripts serve as evidence for this evaluation, that the preferred form of working with a written text was to add explanations of terms, methods, or rules at the margins or to add a further example or a quote from another text. Students often used texts written by their predecessors, but some also copied the textbook they studied for themselves and filled the margins only with what they considered relevant. MS Berlin, orient. Quarto 728 is a copy from 1239 h/1823/4 of two texts of the important Safavid religious scholar, courtier, and cleric Bahāʾ al-Dīn al-ʿAmilī (1547–1622): one on astronomy, the *Tashrīḥ al-aflāk* (Anatomy of the Heavens), and the other on arithmetic, algebra, and surveying, the *Khulāṣat al-ḥisāb* (The Essence of Arithmetic). Its predecessor had been copied almost a century earlier (1137 h/1724/5). Its original owner was an Iranian scholar by the name of Muḥammad b. Ḥaqq al-Rūzbihānī. In 1292 h/1875, the manuscript was bought by the scribe Muṣṭafā b. ʿAbd al-Raḥmān al-Sawādī well known for his connection with Mecca. He was a madrasa teacher who apparently taught in Ankara (MS Berlin, or. Quarto 728, cover page).[15] A note in the text indicates that al-ʿAmilī's arithmetic was also taught in a remote fortress called Chavalan (MS Berlin, or. Quarto 728, f 21a).[16] This manuscript includes different kinds of remarks by 13 students. They show that the students did not merely study the works of their teachers and the teachers of their teachers and so forth, but that they also read the explanations compiled by earlier student generations.

The commentaries explain concepts like *takrīr* (repetition) or *kasr aṣamm* (surd fraction, i.e., a fraction whose denominator is a prime number) from earlier mathematical texts and exercises for addition, multiplication, division, and extraction of roots (MS Berlin, or. Quarto 728, ff 12b-13a, 14, 19a-b, 21a, 23a, 25a-b et al.).[17] The earlier mathematical texts quoted by the students are the chapter on number theory of Ibn Sīnāʾ *Kitāb al-shifāʾ* (Book of the Cure); *al-Risāla al-shamsiyya fi l-ḥisāb* (The Epistle for Shams al-Dīn on Arithmetic) by Niẓām al-Dīn al-Ḥasan b. Muḥammad Nīsābūrī (1270–1330), both Iranian authors; and the *Sharḥ al-Lumʿa* (Commentary on The Brilliant) by Sibṭ

[15] The name of the city is barely legible.

[16] Chavalan or Jabalan was, according to Dekhoda's *Lughat-nāma*, a fortress in Yemen. I thank Nasrollah Pourjavady for this information.

[17] This understanding of a surd fraction was already present in the first chapter on number theory in the *Rasāʾil Ikhwān al-Ṣafāʾ* in the late tenth or early eleventh century (Brentjes 1984, pp. 181–274).

al-Māridānī (1423–1501) on a work by Ibn al-Hāʾim, both Mamluk scholars who worked in Cairo and Jerusalem, respectively. The students' reading nicely highlights the historical traditions that shaped mathematical education in the Ottoman Empire. While most of these remarks were marginalia, some of them were too long to fit into the space left by the margin, even if they had been written in all three margins above, beside, and below the main text. Hence, scribes interrupted the main text and included the lengthy commentaries of ʿAbdallāh b. Haydar and ʿAbd al-Samad b. ʿAlī in the center of the page (MS Berlin, or. Quarto 728, ff 13b-14a, 19a-b, 21a-22b, 24a-b).

Other material witnesses to teaching and studying activities are flyleaves of different formats, usually smaller than a manuscript page, glued to a page, or simply left between two of them. These flyleaves were covered with similar comments, examples, and quotes as the marginalia. One result was the emergence of a new genre of text, the collection of *fawāʾid* (advantages). These were quotes from other texts on matters of relevance to the students. In a sense, they were the predecessor of the CliffsNotes. MS We 1713 of the State Library in Berlin (ff 70b,17-71a,1) contains such "CliffsNotes" from the works of Ibn al-Bannāʾ.[18]

Textual forms like the *fawāʾid* or the *naẓm* tell stories of learning. At times they can even reflect political or ideological preferences and objectives of a dynasty newly come to power. Didactic poems like those of Ibn al-Yāsamīn are a case in point. They allow the rapid memorization of knowledge and beliefs that Burhān al-Dīn al-Zarnūjī (12th/13th cc) had loathed and condemned as an unsuitable technique of learning. During the twelve century, however, such didactic poems began to appear in increasing numbers across all fields of knowledge. Fierro suggests that this trend reflects, at least in the Maghrib and al-Andalus, the desire of the new dynasty of the Almohads (ca. 1121–1269) to train their conquered subjects in the kind of knowledge and beliefs they themselves privileged.[19]

Three didactic poems ascribed to Ibn al-Yāsamīn are extant today, two of which were already mentioned with respect to al-Qalaṣādī: one on roots, another on algebra, and the third on the double false position the ascription of which to Ibn al-Yāsamīn is contested.[20] The poem on roots consists of 55 verses. It begins with a line about God as the Creator followed by a tribute to the author's teacher of mathematics, Muḥammad b. Qāsim b. Shalwāsḥ Then Ibn al-Yāsamīn explains his reasons for writing the poem. Forty-one lines deal with different operations of irrational square roots and the definition of a square root. At the end, Ibn al-Yāsamīn concludes with a repetition of his goal when writing the poem, that is, to provide the beginner with an aide-mémoire and the more advanced with a guide and reference book, along with some prayers (Abdeljouad 2004, p. 3). The poem on algebra with 53 lines is of equal length and structure. It introduces algebraic terminology and six standard forms of a quadratic equation and their solution. Ibn al-Yāsamīn dedicated it to his teacher of mathematics (Abdeljouad 2004, p. 4). The third poem with eight lines is much shorter and deals with one method only, that of the double false position (Abdeljouad 2004, p. 3). An example of Ibn al-Yāsamīn's writing style praised by his commentators is this definition of algebra's basic elements *māl* (plural: *amwāl*; literal meaning: wealth, riches; in algebra it means, like the poem specifies: x^2), root (means: x), and absolute number (signifies the third member of a quadratic equation) from the poem on algebra:

Algebra rests upon three	*amwāl*, numbers, then roots
māl is every square number	and its root one of the factors
absolute number relates	neither to the *amwāl*, nor to the roots
	Be it understood!

(Abdeljouad 2004, p. 4)[21]

[18] The work of Ibn al-Bannāʾ extracted by the students is the *Mukhtaṣar fī l-misāḥa* (Souissi 1984, pp. 491–520).

[19] Maribel Fierro, Madrid. Oral communication. I thank Maribel for sharing her insights into Almohad cultural policies with me.

[20] Lamrabet and Djebbar, for instance, reject this attribution to Ibn al-Yāsamīn (Lamrabet 1994, pp. 66–67; Djebbar 2005, pp. 97–132).

[21] Each line of the poem consists of two verses, which are separated physically on paper by an empty space.

Mathematics was not a field of learning in the madrasa that was independent from legal education. Most students studied certain mathematical texts because they needed some mathematical knowledge when they wanted to be appointed as a judge or work in minor legal positions. Those who wished to make a career in the administration as a secretary or a tax calculator needed a more substantial mathematical training. In the classical period, these young men seem to have learned their skills as a craft within their families. In the postclassical period, biographical dictionaries also included entries on secretaries and reported on their education at madrasas. Furthermore, the study of mathematical texts could open up other professional possibilities such as becoming an astrologer. While the more lucrative positions were those at the courts, there were also itinerant and army astrologers, although the latter were rarely mentioned in the sources. In Egypt, Syria, Anatolia, the Balkans, northern Africa, and al-Andalus, further professional options for students with solid mathematical and astronomical training were those of the *muwaqqit*, *mīqātī,* and muezzin (Brentjes 2008b; King 1996). These people were often attached to a madrasa or mosque and received a small stipend for determining prayer times, the beginning of the new month, and the *qibla* (King 2004, pp. 646–647). Some of them supervised the sundials and other clocks, repairing them if necessary (Brentjes 2008b, p. 134). During the Mamluk dynasty, numerous *muwaqqit*s held much better paid positions as teachers at madrasas or Sufi lodges (Brentjes 2008b, pp. 130–135, 146–150).

7 Mathematics in Reflections on Education

The literature on education focused primarily on issues like behavior, age, and time necessary for a successful acquisition of knowledge and on how to organize this period of life. An often copied and well-appreciated little text is that written by the jurist Burhān al-Dīn al-Zarnūjī in 1203 somewhere in northeastern Iran or Central Asia. Its titles *Ta*ʿ*līm al-muta*ʿ*allim*; *Ṭarīq al-ta*ʿ*allum* (Instruction of the Student; The Method of Learning) reflect the program of the author who wished to help students pass their school years fruitfully and successfully. Hence, he does not stop with the description of generalities and abstract recommendations. He tells stories about earlier teachers and students, mainly of his own juridical school, that is, that of Abū Ḥanīfa, offers moral guidelines, explains techniques of learning, and discusses other practical issues such as diet, accommodation, and the preferable company. The primary type of knowledge that should be sought according to the author is religious – the Qur'an, *ḥadīth*, *fiqh*, and related religious disciplines – as well as philological. The mathematical sciences are frowned upon except when they can contribute to improving the reliability of religious activities. Already in Chap. I, *About the Nature and the Merit of Knowledge*, al-Zarnūjī ascertains that his readers know what he thinks is right or wrong, permitted or forbidden. "Astronomy is forbidden with one exception: it is permitted to study that which is necessary for determining the direction towards Mecca and the hours of prayer" (az-Zarnūjī 1991, p. 40). The only non-religious discipline al-Zarnūjī declares without reservation as permitted for studying is medicine because it studies "accidental or secondary causes" (az-Zarnūjī 1991, p. 40).

Beyond the explicit discussion of subject matters that can or should be studied, mathematical knowledge surfaces in an indirect manner when al-Zarnūjī speaks, for instance, about texts that teach the legalities of inheritance and bequests. Other themes of importance for understanding the conditions under which mathematical knowledge was taught and transmitted concern the recommended choices of what kind of knowledge to learn, with whom to study, and how to learn. Al-Zarnūjī is certain that a student should strive first and foremost to acquire "old knowledge." "Remain with old things, avoid new things" (az-Zarnūjī 1991, p. 47). This saying also applies to the choice of teachers. Young teachers are to be avoided, old and pious men to be preferred. On the other hand, patience is a quality that a good student needs to cultivate. He should finish his readings within one domain of knowledge before starting on another one. He also should stay in one place until he acquires there all that can be learned (az-Zarnūjī 1991, p. 49). The most important method for learning that al-Zarnūjī proposes and that indeed dominated the study at madrasas and cognate institutes, including mathematical fields, was memorizing what one had studied. Each day the student was supposed to increase the capacity of his memory by one word in order to guarantee a smooth learning. Memorizing too much at once was counterproductive and the main cause for failure. Once in the appropriate mood and custom, a student should regularly repeat the lectures of the previous days: those of the day before five times, those of the day before that day four times, etc. (az-Zarnūjī 1991, pp. 79–80). Equally, a student was not supposed to write anything down on paper which he had not fully understood. Violating this principle is bad for the student's character, destroys his intelligence, and is a loss of time. Thus, understanding what the teacher offers is the second most important method of learning. Only "when the lecture is limited, but the repetition and the reflection are extensive, will the student arrive at a solid comprehension and understanding" (az-Zarnūjī 1991, pp. 71–72). The third kind of learning methods includes debating, discussing, and questioning. All three should be done with impartiality and without ire because the goal is to establish the truth̩ A discussion which aims to overpower and crush one's opponent, by contrast, is illicit (az-Zarnūjī 1991, p. 74).

A comparison of these recommendations and ideals with descriptions of biographies of scholars who studied mathematical texts indicates that some of these methods also entered into teaching mathematics, while others are only rarely mentioned. In the classical period, for instance, two discursive forms dominated research and teaching texts. One operated without any kind of introduction; the other presented a self-confident author proud of his mathematical achievements in the text. In the postclassical period, the former type continued to be followed by some authors, while the second type was replaced more often than not by the new rhetoric of self-denial. In this rhetoric, the author presented himself as a poor man or a slave in the service of God, a scholar who happily followed his predecessors' ways. Fame was now ascribed mostly to the text itself, not to its author. Memorizing, researching, answering questions, and practicing were words that former students of the mathematical sciences used when describing their class experience. Researching meant in this context searching for a solution of a problem or indeed proposing something new if the student had reached an advanced level answering questions; and practicing were words that former students of the mathematical sciences used when describing their class experience.

8 Which Mathematical Discipline Is Legitimate for a Muslim to Learn, Teach, and Earn a Living With?

If a student had reached the last phase of his education and was a disciple, he had five main options for earning his living if he did not come from an affluent family or if he had not committed himself to poverty. Two of these options were part of the educational system. Disciples often deputized for their teachers at smaller madrasas or Sufi lodges and received a part of the salary that the donation documents had stipulated for the office holder (Chamberlain 1994, pp. 92, 94–95). This could be a very lucrative position (Chamberlain 1994, p. 98–99). The second educational opportunity to earn money consisted of teaching either children in a *maktab* or the poor (al-Sakhāwī n.d., vol. 1, pp. 83, 101–102, 137, 229, 289, 362; vol. 2, p. 7; vol. 3, p. 135). Other possibilities for students or young scholars to earn a living existed in the legal system, commerce, and financial services (al-Sakhāwī n.d., vol. 1, pp. 5, 102, 111, 137, 229, 362; vol. 2, pp. 20, 136, 164–165; vol. 3, pp. 124, 151). Acting as a witness, for instance, was such an income-providing opportunity in the legal system. In the fifteenth century, it was apparently the most often chosen option among scholars who had not yet been appointed to a professorship or a *qāḍī*-ship. If a student had spent some time reading medical texts, he could try to earn money as a physician. This was not always a fortunate choice since not everybody who had read a few medical texts became a successful doctor and found a well-paying clientele (al-Sakhāwī n.d., vol. 3, pp. 115–116). It was also possible to combine a variety of these methods and thus increase one's income (al-Sakhāwī n.d., vol. 1, pp. 137, 229). In addition to these more standard modes of income earning, the biographical dictionaries also report cases which their authors found exceptional, as al-Sakhāwī makes clear in the case of Shihāb al-Dīn Aḥmad b. ʿIbād (d. 1480), who had studied inheritance rules and arithmetic with Ibn al-Majdī and Nāṣir al-Dīn al-Bārinbārī (al-Sakhāwī n.d., vol. 1, p. 320). He began his work life as a shepherd. After he had moved to Cairo, he took care of the city's ventilators before he began studying at the Azhar (al-Sakhāwī n.d., vol. 1, p. 329).

Once a student had become a scholar, he could be appointed by the ruler or his representatives to various posts, usually in the religious or administrative domain, like a *qāḍī*, a madrasa teacher, a librarian, a treasurer, or the administrative head of a *waqf* (religious donation). Since the thirteenth century, the positions at madrasas, cognate institutes like the *dār al-Qurʾān* or *al-ḥadīth* (House (for studying) the Qur'an or the sayings of the Prophet), or mosques were considered very prestigious and lucrative. This led to serious fights over their possession. Many scholars did their utmost to acquire as many teaching posts as possible which they then outsourced often to their advanced students or their sons and other male relatives. They tried hard to assure that their sons, sons-in-law, or preferred students inherited their most prestigious chairs. They manipulated appointments and bargained with governors and judges in order to take away a coveted post from its incumbent. This attitude towards education as a ground for social empowerment and personal gains was already widespread among Mamluk scholars at the end of the thirteenth century (Chamberlain 1994, pp. 91–107). In the thirteenth century, the scholars, according to Chamberlain, did not consider this behavior unsavory, including negotiating offices with the enemies (the Mongols). In the seventeenth-century Ottoman Empire, it was a loathsome fact that professorial positions could be bought and corruption reigned supreme. It was apparently a remarkable exception when a scholar was appointed to a chair without paying bribes (Chamberlain 1994, p. 98; Muḥibbī, pp. 4, 207).

The spread of the madrasa and cognate teaching institutes broadened and regularized the possibilities of a paid teaching position and acted thus to ease earlier rejections of payment for teaching since the sharing of religious knowledge was considered by many to be an obligation and thus was believed to be free. Even after the madrasa system with its paid positions had been widely accepted by different legal schools, scholars, and rulers, some scholars continued to hold on to the belief that all disciplines linked to religious knowledge should be taught for free. Among the mathematical fields, this applied in particular to the skills and themes taught in the legal field of inheritance and bequests. Other mathematical subjects like algebra or geometry, however, could be taught for a fee. This position was taken, for instance, by the Mamluk scholar Zayn al-Dīn b. Ṭalḥa (d. 1430) in his early professional life. The author of the famous history of madrasas and cognate institutes in Ayyubid and Mamluk Egypt and Syria, ʿAbd al-Qādir al-Nuʿaymī (d. 1520), described him as follows: "He studied inheritance mathematics and arithmetic, was excellent in both, […] wrote beautiful books on that, read on it many (lectures), […] but did not take a remuneration for his teachings in inheritance mathematics and arithmetic, while he accepted payment for algebra. […] At the end of his life, he wrote legal opinions on problems in inheritance mathematics and arithmetic and took fees as other followers of this discipline" (al-Nuʿaymī 1990, vol. 1, pp. 88–89).

9 Best Sellers of Education in the Mathematical Sciences

The bestseller in teaching geometry was Naṣīr al-Dīn al-Ṭūsī's edition of Euclid's *Elements,* made in 1248. Its author was believed to have had access to two texts representing the two main traditions of the Greek text in Arabic since the ninth century.[22] The first text Ṭūsī believed to have come from the

[22] All that al-Ṭūsī could access in the middle of the thirteenth century were manuscripts ascribed to either of the two main textual traditions of al-Ḥajjāj b. Yūsuf b. Maṭar (tradition 1) and Isḥāq b. Ḥunayn, edited by Thābit b. Qurra (tradition 2). As we know today, the texts found in all extant manuscripts contain mostly variants of the first tradition, although a good number of these manuscripts are ascribed to the second tradition. It took us more than 100 years of

first translator of the *Elements* into Arabic, al-Ḥajjāj b. Yūsuf b. Maṭar (fl. 780–830). The second text was, in Ṭūsī's opinion, the fruit of the work of Thābit b. Qurra, himself a prolific translator of Greek mathematical works into Arabic and one of the leading scholars of the ninth century. Thābit improved the translation of the *Elements* made in the 870s by Isḥāq b. Ḥunayn (d. 911) by providing additions and information from other Greek manuscripts and possibly corrections of mathematical passages which the translator might have misunderstood. Marking his takeovers from the first with red and those from the second with black, Ṭūsī wished to provide students with a clear understanding of the differences between the traditions with respect to the number and placement of propositions and their status as a general proof or a proof according to particular cases. He added as a rule further alternative proofs either taken from other editions or provided by himself as well as comments mostly of a pedagogical nature. The more than 200 extant copies of his work from the Maghrib to India and Central Asia and the various Persian translations made of it point to the success of his project, although his evaluation of the two traditions is apparently false.[23]

To be added to the *Elements* and studied after them, the texts collected in the *Middle Books* and named above exist in sufficiently many copies to indicate their function in teaching geometry and spherical astronomy. This general statement, however, does not mean that these texts were taught always and everywhere. Dated copies exist from Ilkhanid and Safavid Iran, for example, while for Mamluk Egypt and Syria, no such interest can be proven (Brentjes 2008, pp. 325, 332–333; Idem 2010, pp. 325–402). There, other texts seem to have dominated the classroom, among them Qażī-zāde al-Rūmī's (1360–1437) two commentaries on Shams al-Dīn al-Samarqandī's (second half of the 13th c.) introduction into the first books of Euclid's *Elements* called *Ashkāl al- taʾsīs* (Theorems of the Foundation) and Maḥmūd al-Chaghmīnī's (fl. around 1223) introduction on astronomy called *al-Mulakhkhaṣ fī l-hayʾa* (Short Presentation on Mathematical Cosmography). *Al-Mulakhkhaṣ* itself was the basic text used for teaching the structure of the universe and planetary movements. Qażī-zāde al-Rūmī's second commentary exists, for instance, in more than 300 exemplars, while the first commentary comes "only" a few digits over 200 (*Osmanlı Astronomi Literatürü Tarihi* 1997, vol. 1, pp. 9–21; *Osmanlı Matematik Literatürü Tarihi* 1997, vol. 1, pp. 7–18). However, these numbers are lower boundaries since the catalogue does not list most of Iran's, India's, and Central Asia's manuscript collections. Copies that give place names indicate that the text was read in the Ottoman capital, centers of provinces, small towns, and even occasionally a village. Mathematical education had spread and was accessible if desired in remote corners of the Ottoman Empire. Similar observations apply to other mathematical texts in the Safavid Empire.

In arithmetic and algebra, several texts competed for preeminence. As indicated above, in the fifteenth century, Ibn al-Bannāʾ's treatises dominated teaching in al-Andalus and the Maghrib, making an impact also in Mamluk Egypt, Syria, and beyond. Another Maghribī author studied in the same regions and possibly also further to the east was Ibn al-Yāsamīn, whose poems were commented upon by several students and teachers of the mathematical sciences, among them al-Qalaṣādī, Ibn al-Hāʾim, and Sibṭ al-Māridānī. The most studied Mamluk author in arithmetic and algebra was Ibn al-Hāʾim, who wrote some 18 treatises on arithmetic and algebra. In the seventeenth and eighteenth centuries, the most successful textbook in the Ottoman Empire, India, and Central Asia was Bahāʾ al-Dīn al-ʿAmilī's *Khulāṣat al-ḥisāb*. Of this text, far more than 100 copies exist in libraries in Istanbul and Anatolia alone. Interestingly enough, this text seems to have been more successful in the Ottoman Empire and India than in Iran itself, while on the other hand texts by Mamluk and Ottoman authors seem have been much less used by Ottoman teachers and students than the text from Safavid Iran. These local variations and differences between contemporary Islamic societies deserve serious attention because they

research to discover this mixture of the two traditions. Al-Ṭūsī, having no concept of critical source analysis, could not find out whether an ascribed text did indeed belong to the tradition to which it was attributed.

[23] The two textual versions described by al-Ṭūsī are in the extant manuscripts for Books III-IX, variants of tradition 1 only, although they possess the features ascribed by al-Ṭūsī to tradition 2. This suggests that al-Ṭūsī worked, not surprisingly, with similarly mixed texts ascribed to tradition 2 as those that are extant today.

suggest the existence of local cultures of mathematics in teaching and perhaps also in research, in contrast with the usually assumed unified scientific culture of all Islamic societies, as suggested by the dominant rhetoric of "science in Islam" or "Arabo-Islamic mathematics" and similar phrases.

In addition to these best sellers of mostly elementary character, a few more advanced texts by earlier authors from the Ilkhanid and Timurid dynasties of the postclassical period were read parallel or in sequence to these basic works.[24] This is an important observation because it shows that a number of students could aspire to higher levels of learning and could indeed achieve them. The overwhelming presence of simple elementary teaching texts conceals the existence of such a higher level in the educational practice well into the eighteenth century.

10 Conclusions

The history of teaching mathematics in Islamic societies is marked by one major watershed – the creation of the madrasa and the integration of the nonreligious sciences into the realm of formal education. It is not yet clear when this integration began and how it spread through the different areas of the Islamic world. Neither is it well understood which of the legal schools of Sunni Islam opened its doors first to teaching other topics than law, including mathematical themes. In this survey, I outlined major lines of development derived from primary evidence found in biographical dictionaries, historical chronicles, and mathematical manuscripts. These sources suggest that Syria, northern Iraq, Anatolia, and Iran were of primary importance for these processes during the later twelfth and thirteenth centuries. Since the fourteenth century, Egypt had become an important region for teaching mathematics at madrasas or by madrasa professors. Apparently, the Maghrib followed suit shortly afterwards. Somewhat later, mathematics began to be taught in Sub-Saharan Africa (Djebbar and Moyon 2011). These temporal differences in the spread of mathematical education from capitals to provincial towns and at times even to villages indicate what biographies and mathematical manuscripts confirm for the content of teaching – that there were local cultures of mathematical education, not one big imperial or international system of teaching mathematics, as is often believed. These local cultures were interlinked through traveling students, teachers, patrons, and texts. Some mathematical texts and their authors became canonical authorities in many madrasas in northern Africa, Egypt, Syria, Anatolia, Iran, Central Asia, and India. Authors from Iran and their texts often took the first rank in this canonical literature and spread new debates to the west and the east, as well as elementary summaries of the knowledge expected in astronomy, arithmetic, algebra, or geometry.

The increase of elementary texts as well as commentaries, super-commentaries, and glosses and the decreased importance of texts by ancient and classical authors in the educational cultures of postclassical societies are features that came out of the integration of the mathematical disciplines into the madrasa. These phenomena observable in the mathematical literature of the period reflect their function as introductory and survey texts for a student population that aimed primarily to acquire legal and religious knowledge and studied mathematical texts as an auxiliary means for their later activities as judges and other legal officials or as members of the fiscal administration. Thus, the emergence of a huge quantity of basic, elementary texts on mathematical topics should no longer be interpreted as signs of decline, but recognized as an expression of a profoundly altered social and cultural function of mathematical education.

[24] These more advanced works are in particular the following: ʿImād al-Dīn ʿAbdallāh b. Muḥāmmad al-Khuddāmī (1245–1325), *Fawāʾid al-Bahāʾiyyya fī l-qawāʾid al-ḥisābiyya* (The Advantages of Arithmetical Rules); Niẓām al-Dīn Nīsābūrī, *al-Risāla al-shamsiyya fī l-ḥisāb*; Jamshīd al-Kāshī, *Miftāḥ al-ḥisāb* (Key to Arithmetic); and its *Talkhīṣ* (Epitome) Examples can be found in MS Berlin 1733. Information on the study of Ibn al-Bannāʾ can be found, according to Aballagh, in al-Tinbuktī, Aḥmad Bābā. 1398/1989.

Mathematical education in postclassical societies, however, was not always limited to the elementary rules of arithmetic, algebra, or geometry. As my investigation of manuscripts annotated by students has shown, even as late as the eighteenth century, students studied introductory texts concomitantly with more advanced and at times even high-level texts on arithmetic or astronomy. In much earlier centuries, the interest in higher-level studies was promoted by the emergence of a new specialized domain in astronomy – ʻ*ilm al-mīqāt*, the science of timekeeping. King's and Charette's investigations of the texts and instruments produced by the *muwaqqit*s in Syria and Egypt, the representatives of this special field, during the fourteenth and fifteenth centuries leave no doubt that both teaching and research flourished in this period, region, and discipline. My own studies of this group of experts brought to the fore that their expertise was part and parcel of a larger domain of knowledge that they taught as professors at madrasas and Sufi lodges. Hence, mathematical education, both on the elementary and advanced level, became well incorporated in the sociocultural practices of madrasa professors, their deputies, and their students, that is, of the group of the ʻ*ulamā*', usually understood as religious scholars. This embodiment opened up opportunities and set boundaries. Mathematical topics were increasingly taught like law and other religious disciplines as the many mentioned elementary texts demonstrate. But even these boundaries, with their consequences of subjecting mathematics to the sociocultural standards of law and religion, did not signify the end of all forms of teaching more suitable for mathematics than rote learning; some of the learning and teaching literature from the Maghrib quoted in my survey highlights this. Studying the past of mathematical education in postclassical Islamic societies thus helps to modify the often still too simplistic history of mathematics in Islamic societies as a history of uniform, long-term, and large-scale processes valid for the entire Islamic world.

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Chapter 6
Mathematics Education in the European Middle Ages

Jens Høyrup

1 Periodization and Institutional Types

By lucky but adequate accident, "the Middle Ages" have become plural in English. The millennium that separates the definitive demise of the Western Roman Empire from the discovery of the New World and the Reformation can hardly be understood as one homogeneous period under any point of view – and certainly not if we look at mathematics education.

The earliest phase, lasting until c. 750 CE, is known as "the Dark Ages" – both because surviving sources for this period are rare and because this rarity reflects a very low intensity of literate culture. During the ensuing "Central Middle Ages" (c. 750–1050, the epoch of Charlemagne and the Ottonian emperors), attempts at statal centralization led not only to creation of the cathedral school system but also to corresponding developments of monastic learning. The High (c. 1050–1300) and Late (c. 1300–1500) Middle Ages are characterized by the rise of city culture, which led to the emergence of the university system as well as to the appearance of institutionalized lay education – connected to but not identical with the Renaissance current.

In principle we should also distinguish between partially or fully separate types of education – that of the Latin school and university tradition and those of various kinds of practitioners. Among the latter, however, only the education of Late Medieval merchant youth is well documented in sources – for that of craftsmen we have very little evidence.

2 The Dark Ages

From the point of view of mathematics education, the Dark Ages are even "darker" than other aspects of literate culture. During the centuries after the final collapse of the Western Roman Empire, some members of the social elite of the new barbarian states in Italy, Visigothic Spain and Gaul were still taught Latin letters; notarial and legal services were still needed in royal and (what remained of)

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municipal administration in the same areas; and monks who entered monasteries in not too late age would learn to read at least the psalter book – in some places and periods considerably more and not only sacred literature.[1]

However, if ever the seven liberal arts had been a serious curriculum, already in Augustine's youth (later fourth century) no more than grammar and rhetoric remained – the *quadrivium* (arithmetic, geometry, mathematical astronomy + geography and mathematical musical theory)[2] had neither teachers nor students. Augustine himself had profounder mathematical interests (and had read Euclid on his own), but his *De doctrina christiana* (ed. trans. Robertson 1958) – which set a higher aim than ecclesiastical teaching was able to attain for many centuries – mentions only the need to understand certain numbers in the scriptures (II. 25, III. 51). He points out that mathematical truths are of divine origin – but that is not seen as a reason to pursue them, Augustine only warns against being too interested in them (II. 56–57).

Isidore, the learned Visigothic bishop of Seville (c. 560–636), certainly praised mathematics (more precisely "the science of number") in his monumental *Etymologies*, which was to become one of the most quoted authorities of the Middle Ages, but his own knowledge of the quadrivial disciplines does not go beyond a few ill-digested definitions and a few concepts borrowed from late Latin encyclopediae; no wonder that for more than a century, there is no trace of anybody being *taught* according to his modest programme.

In this as in other domains, however, the Dark Ages are made darker by the absence of sources. Administration, taxation and the household accounting of monasteries were not possible without some calculation and land measurement. We have no traces of how this non-quadrivial mathematics was taught, but we may safely presume that the necessary skills were trained in apprenticeship and "on the job", as they had already been in Antiquity. Merchants – a class that had been reduced but had not disappeared – must also have known how to calculate; even here we have no direct information, but we shall return to a possible indirect trace.

We might have expected some teaching of *computus* (Easter and other sacred calendar reckoning) within monasteries; since the early fourth century, it had been considered a problem that various regions celebrated Easter at different times, and tables as well as (discordant) calculation methods had been developed that should allow the prediction of the right day.[3] It appears, however, that the matter did not enter any monastic teaching programme except in Ireland from the seventh century onwards. From here in the earlier eighth century, it entered Anglo-Saxon Britain – or at least the monastery where the Venerable Bede was teaching and writing.

3 Carolingian to Ottonian Times

During the second half of the eighth century, Charlemagne first took over the Frankish kingdom and next subdued much of Western and Central Europe. Probably as part of an effort to create a stable power structure (but apparently also because of sincere personal concern), he initiated an ecclesiastical,

[1] A detailed investigation (which also makes clear the absence of every kind of mathematical studies) is Riché 1976.

[2] This group of disciplines and the collective name used about it since Boethius are the closest we can get to a *unified concept* of mathematics in the Medieval Latin school tradition at least until the thirteenth century; only Aristotelian philosophy brought in the notion of "more physical" mathematical disciplines (which beyond astronomy included optics and the science of weights). However, when asking about mathematics education – a modern concept – we shall need to include also mathematical activities falling outside the quadrivial framework, such as practical computation. Unfortunately, the sources are mostly mute on this account.

[3] The ways the problem was confronted from the beginning until the early eighth century is accounted for in Jones 1943, pp. 1–114.

liturgical and educational reform after having attached to his court the best scholars he could find in Italy and Britain. A "general admonition" was issued in 789, which called for the creation of schools in all monasteries and bishoprics where (select) boys could be taught hymns, notes, singing, computus and grammar (in this order, which may reflect Charlemagne's concerns).[4] As we see, mathematics only appears in the programme through its service for computus (for which Charlemagne had a passion); we are still left in the dark when asking how the underlying calculational skills were to be taught. However, another circular letter to the clergy exhorted those who had the ability (presumably the scholars at his court) to teach "the liberal arts" to others, and we do indeed have small treatises (mostly primers) written by Alcuin of York, second-generation scholarly descendant of Bede and the central figure of the "palace school" (a "school" about whose character we know nothing precise). They deal with elementary grammar, rhetoric, dialectic and computus; in the treatise concerned with the latter subject, it is again clear that the reader is supposed to know simple calculational operations (including divisions).[5]

One more possible trace of mathematical instruction in the environment exists: a collection of mixed "recreational problems", *Propositions to sharpen the minds of youngsters* ascribed in some manuscripts to Alcuin.[6] Once again, their solution presupposes familiarity with elementary calculation. Some of the problems refer to the monastic environment, many others to the world of trade. The collection may well have been put together by Alcuin or a contemporary, but most of the problems are likely to have circulated since late Antiquity and thus to reflect the teaching of basic arithmetic of young monks and merchant youth.

In the long run, the obligation of bishops to take care of teaching developed into the cathedral school system – but at first the breakdown of Charlemagne's realm after his death did not allow schools to flourish. In some monasteries, however, the attempt to fill out the full gamut of liberal arts gave rise to a hunt for manuscripts. Already in Charlemagne's time, Martianus Capella's *Marriage of Philology and Mercury* turned up (Stahl 1971, pp. 61–64) – a late ancient encyclopedic presentation of the liberal arts (not teaching much mathematical substance, yet more than other encyclopedic works that started to circulate at the same time). In the early ninth century, Boethius's *Arithmetica* and *De musica* (free translations from c. 500 of Greek originals written by Nicomachos around 100) were rediscovered, and the surviving agrimensor writings (Latin treatises on practical surveying) were collected and combined with surviving fragments of a translation of Euclid's *Elements* (or, plausibly, of a digest omitting most proofs) and reclassified as quadrivial geometry (to which surveying geometry had never been reckoned since the ancient invention of the "liberal arts") (Ullman 1964).

Until the early twelfth century, Boethius's *Arithmetic* and the agrimensorial tradition (with the same fragments of Euclid) were the fundament of all teaching of arithmetic and geometry (to the extension such teaching existed). The former offered a philosophical discussion of number, the concepts of odd and even, prime and composite number, figurate numbers and their properties, an extensive classification of ratios[7] and various means (arithmetical, geometric, harmonic and seven more) (no practical computation). The "sub-Euclidean" geometry derived from the agrimensors and from the Euclidean fragments contained Euclidean definitions, postulates, the proposition statements from

[4] The relevant passage of the *Admonition* appears in Riché 1979, 352ff. Brown (1994) is a fine presentation of the whole reform effort.

[5] This picture is confirmed by the *Manual for My Son* written by the mid-ninth-century noblewoman Dhuoda. When speaking about sacred numerology, in Augustinian style, she tacitly assumes the son to understand basic computation (and shows that she does so herself) (Riché 1975, pp. 326–334).

[6] Edition and German translation in Folkerts and Gericke (1993).

[7] A ratio was not understood as the number resulting from a division but as a *relation* between two numbers; it might be *multiplex* (of type $m:1$), *superparticular* (of type $m+1:n$), *multiplex superparticular* ($mn+1:n$), superpartient ($n+p:n$), *multiplex superpartient* (of type $mn+p:n$) or inverses of any of these. Depending on the numbers involved, ratios had specific names – $5:2$ (and $10:4$, etc.), for instance, were "duplex sesquialter".

books I to III and the proofs for the first three propositions – and from the agrimensor side, mainly rules for area calculation (not always correct). Its most mature expression is the eleventh century so-called *"Boethius" Geometry II* (ed. Folkerts 1970).

When this compilation was made, however, the tradition had already developed, primarily in the cathedral schools of Lotharingia.[8] Most important was the introduction of a new type of abacus (plausibly a transformation of a type that was already around), using counters marked by Hindu-Arabic numerals. It may have been designed by Gerbert of Aurillac after his stay in Catalonia (*not* in Muslim Iberia) in the late 960s and was at least taught by him while he was the head of the cathedral school in Rheims (972–982, 984–996).[9] It seems not to have spread outside the monastic and school environment, and it is likely to have served more in teaching than for practical calculation. The topic was pigeonholed under geometry, not arithmetic – perhaps because of its use of a plane surface, perhaps because it would serve the area calculations of sub-Euclidean geometry, perhaps because ancient geometry was known from Martianus Capella to have made its drawings on a sand board similarly designated *abacus*. In any case, a categorization under quadrivial arithmetic would have been no more adequate.

Linked to this abacus by using at times its board but also to the teaching of arithmetic was a newly invented game board *rithmomachia*.[10] The players had to know the Boethian theory of ratios as well as the whole gamut of figurate numbers, and there is little doubt that the game contributed to keeping alive the interest in Boethian arithmetic until the sixteenth century (and vice versa).

Computus was still taught, but astronomy was now more than computus. This *may* have depended on the incipient interest in astrology – the first compilations using Arabic material are from the late tenth century (van de Vyver 1936; Burnett 1987, pp. 141ff). However, Gerbert's teaching of the topic as described by his former student Richer (ed. Bubnov 1899, 379ff) points to Martianus Capella and shows no hint of astrological preoccupations – it deals with the horizon, tropics, ecliptic and other circles of the heavenly sphere. On the other hand, the first treatises on the astrolabe turn up around the same time; one has been ascribed to Gerbert. This instrument came from an area where astrology was a central motivation for work on astronomy; whether its complicated use was taught in any organized way at the moment is dubious.

Even music had changed. In Charlemagne's time, as we saw, it was no mathematical topic at all; with the discovery of Boethius's *De musica*, it once more became a mathematical discipline (albeit hesitatingly), and singing was reclassified as *cantus*. But even theoretical music split in the early eleventh century. Guido di Arezzo, known as the inventor of (the earliest form of) the modern musical notation, used musical theory in the teaching of singing and developed it for that purpose (*musica practica*, in a later term) (Wason 2002); predecessors in the tenth century had started this process. Gerbert, teaching in Rheims, taught *musica theorica*, Boethian theory, which was to remain the music of quadrivial teaching throughout the Middle Ages (whereas a number of outstanding university scholars, some – like Jean de Murs – known as mathematicians, developed theory far beyond Boethius in the thirteenth and fourteenth centuries).

[8] See Bergmann 1985 (to be used with some care).

[9] Since the slave trade route to Muslim Spain passed through Lotharingia, Gerbert's stay in Catalonia was not the only numerate cultural contact at hand. Thompson (1929) lists a number of further contact on the courtly and literate levels. Since the names given to the nine figures seem to be of mixed Magyar-Arabic-Latin-German origin (Köppen 1892, p. 45), the slave traders could be the most likely inspiration.

[10] Arno Borst claims in his fundamental study (1986) that the game can be traced back to c. 1030 and no further. However, Walther von Speyer's *Libellus scolasticus* (ed. trans. Vossen 1962, p. 41, pp. 52ff) clearly speaks of a very similar game played around 970 (without indicating the name, which may indeed be later) on the abacus board and using its counters. Borst dismisses this testimony, asserting that Walther does not understand what he is speaking about, and Vossen because he does not know that the abacus board belonged with geometry.

The didactical use of the game was discussed by Gillian R. Evans (1976). A recent discussion of the game and its survival is Moyer (2001). A short presentation of the way the game was played is in Beaujouan (1972, pp. 644–650).

When Richer speaks of Gerbert's geometry teaching, he only mentions the abacus. Letters written to Gerbert by a former student and correspondences between ex-students (or students' students) who themselves had become school heads show that any further teaching of agrimensorial geometry, if existing, had been in vain.[11] As Paul Tannery (1922, p. 79) says about these correspondences, they belong not to the history of science but to that of ignorance. One correspondent does not understand why the determination of a triangular area from triangular numbers does not coincide with that following from base and height and asks Gerbert (now Pope Sylvester II) for an explanation; two others discuss the meaning of the notion of "exterior angles" of a triangle (which they have found in Boethius) without coming to the correct result; one of the latter also supposes that the Archimedean formula for the circular area[12] had been found by cutting and reassembling a parchment circle. Other writings of theirs show them to have been both well educated and intelligent; their failure thus reveals the absence of any adequate teaching (and the unavailability of relevant manuscripts); simultaneously, the letter exchanges testify to a vivid interest in the topic at least among schoolmasters and former school heads who had risen to the rank of bishops.

Gerbert's fame allows us to conclude that nobody else at the time reached his level. What he did *could* be done in his time; but we should not believe that others did as much at the moment.[13]

4 The "Twelfth-Century Renaissance"

The translation of medical writings from the Arabic began in the later eleventh century, but the golden age of philosophical and scientific translation arrived with the twelfth. However, the same factors as caused this new beginning at first produced a culmination of the autochthonous Latin scholarly tradition. One factor was the growth of towns, of artisanal industry and of urban wealth; another (largely dependent on the first one) was the growth of schools, in absolute number as well as number of students at each. Already because of the latter increase, a single scholast could no longer take care of the whole school. In consequence, masters became free scholars, teaching with permission of the local see but living from the fees of the students. Most famous of these at the time – and one of the most famous philosophers of the Middle Ages and not only because of his love affair with Héloïse – was Abelard, whose reputation contributed to make Paris and Ile de France the school region par excellence (Haskins 1927, pp. 377–379).

However, Paris – one of the most important cities of Europe – had been a school city before Abelard had any influence, and the one where we get information about mathematics *education*. Wealthy burghers wanted their sons to be educated, and the only institutions where education was disbursed were those of the church. In particular, the Saint Victor monastery in Paris had an external school, whose head Hugue wrote a study guide to the arts (liberal as well as mechanical) and to sacred scriptures, the *Didascalicon*. He also produced a *Practica geometriae* (ed. Baron 1956), which we must presume was connected to his teaching. Since the former work was famous enough to be plagiarized around 1500 and the second influenced the terminology of all Latin practical geometry already a few decades after it was written (probably, like the *Didascalicon*, in the 1120s), we must presume the work to have been at the forefront of what was possible at the time in the most advanced region of Western Europe, and not typical – but at least it was sufficiently close to the level of other scholars to be understood. The success shows that subsequent generations soon reached Hugue's level – presumably also in teaching.

[11] The existence of a *Geometria Gerberti* decides nothing, since it may well be a compilation from the later eleventh century.

[12] Known by him from one of Boethius's treatises on Aristotelian logic, even though it is amply used by the agrimensors.

[13] Uta Lindgren (1976, pp. 48–59) discusses some of them in detail and comes substantially to the same result.

When presenting sacred history in the study guide (trans. Taylor 1961, pp. 135–137), Hugue asks (as his students would perhaps do) whether knowledge of this topic is really necessary. He argues that seeming trifles are useful and gives as an example his own boyhood experience: not at all dealing with sacred history but with experiments on numbers, on area measurement, on the sound of strings, and his observations of the stars – that is, the full quadrivium, which he must somehow have been taught as a boy, if we are to believe his words.

The presentation of the quadrivium in the *Didascalicon* is *metatheoretical*, in part arithmological (presenting the meaning of numbers in a perspective derived from ancient Platonizing writers, including Augustine), in part metamathematical (presenting the distinction between continuous and discrete quantity, the view of mathematical objects as abstractions), in part concerned with the division into subdisciplines – and in part simply etymological, explaining the names of the four disciplines. As befits a study guide, it does not enter into the mathematical subject matter, but here the geometry treatise shows what Hugue might teach.

The title presupposes a distinction between theoretical and practical geometry; however, what is said about geometry in the *Didascalicon* corresponds exactly to the practical branch, which we may thus assume to be what was taught (at what age and to whom remains an open question). The subject matter is to a large extent derived from the best of the sub-Euclidean writings, but everything is thought through – Hugue thus begins with explanations of concepts which are *not* repetitions of the familiar Euclidean fragments. Hugue's own contributions are also conspicuous, in particular concerning the section on "cosmimetry", measurement of the (spherical) world.[14]

Abelard and Hugue may stand for the culmination of autochthonous Latin knowledge, but the thirst for more among the brighter scholars of the epoch is already symbolized by the name Héloïse gave to the son she had with Abelard some time around 1120: Astralabius.[15] The astrolabe, though already known in the eleventh century and prescribed by Hugue for the measurement of angles, was first of all the central tool for that "medico-astrological naturalism" which (as made clear by the selection of works to be translated and of their diffusion) was a main motive for the translations from the Arabic and the Greek – the other motive being the desire to get hold of those famous works which were known by name and fame from Martianus Capella and other Latin authors but not in body.

From the perspective of mathematics education, the first important acquisitions were Euclid's *Elements* and the Hindu-Arabic numerals. The first translation of the *Elements* (from the Arabic, known as "Adelard I") was presumably made by Adelard of Bath (probably assisted by somebody who knew Arabic better). Adelard's general orientation was toward naturalism, astrology and magic, and his own mathematical upbringing as reflected in his juvenile *De eodem et diverso* and *Regule abaci* had not gone beyond the traditional quadrivium; he may have worked on the *Elements* because this work was known by Arabic astronomers to be the fundament for the mathematics of the *Almagest* (two other twelfth-century translators of the *Elements*, Gerard of Cremona and an anonymous translator from the Greek, also translated the latter work). In the wake of Adelard I, a family of derived versions emerged (collectively known as "Version II" (ed. Busard and Folkerts 1992)), seemingly produced by an informal network of Adelard's former students (Burnett 1996, pp. 229–234).

Version II is clearly marked by didactical concerns. Instead of giving full proofs, it often just gives hints of how a proof should be made; at this point it is clear that the matter presented in the work had become the primary aim, while further utility for astronomy (and, still further, for astrology) had retreated into the background.

[14] An analysis of this part of the treatise and its inspiration from ancient philosophical sources is in Tannery (1922, pp. 208–210). Tannery rejects Hugue's authorship as probably a thirteenth-century reconstruction (pp. 319–321), but better editions of the texts on which his arguments are based turn the conclusion upside-down – cf. also Baron (1955).

[15] Abelard, *Historia calamitatum* (ed. Muckle 1950).

Hindu-Arabic numerals, however, were introduced and studied (at first outside every formal framework) as a tool for astronomical calculation and for understanding astronomical tables; initially, some writers experimented with alternatives, such as use of the Roman numerals I through IX within a place value system, or of Latin letters as numerals (as known also from the Greek) (Burnett 2010, articles III and X). However, well before the end of the twelfth century, the Hindu-Arabic numerals had forced out these possibilities.

We know nothing about the way these innovations made their way into the schools during the twelfth century; the situation is no different if we think of the *Almagest*. But somehow they must have reached a fair number of students. Indeed, toward the end of the century, the conservative theologian (and head of the school of the St Geneviève monastery in Paris) Étienne de Tournais (translated from (Grabmann 1941, p. 61)) complained that many Christians (and even monks and canons) endangered their salvation by studying

> Poetical figments, [Aristotle's] philosophical opinions, the [grammatical] rules of Priscian, the laws of Justinian [Roman Law], the doctrine of Galen, the speeches of the rhetors, the logical ambiguities of Aristotle, the theorems of Euclid and the conjectures of Ptolemy. Indeed, the so-called liberal arts are valuable for sharpening the genius and for understanding the Scriptures; but together with the philosopher [i.e. Aristotle] they are to be saluted only from the doorstep.

As we see, the "new learning" of the twelfth century encompassed a new level of literary, grammatical and rhetorical studies; Roman Law; Galenic rational medicine; Aristotelian (natural) philosophy and advanced logic; and finally the planetary hypotheses of Ptolemy and the *Elements*. Hindu-Arabic numerals go unmentioned – they were probably seen only as a tedious tool by those who used them, hardly something that could call forth undue enthusiasm. Unmentioned are also other mathematical topics to which the translations had given access (geometrical optics, spherics and algebra), as well as such that were mere continuations of the previous age – neither computus nor the abacus had been raised to a new level as had the study of Latin poetry, nor had they been linked to "the Philosopher". However, the *Elements* are there, and Étienne may even have meant them as *pars pro toto*, as a stand-in for mathematical studies in unspecific general.

5 The Era of Universities

Étienne's complaint is located at a watershed. As he was writing, the number of teachers and students had reached the level in some towns where the mutual protection provided by a guild could serve. Since neither teachers nor students were normally citizens of the town where they stayed, the need for juridical protection was obvious. Such guilds – in Latin *universitates* – are first attested around 1200 in Paris, Oxford and Bologna (in the latter town, the guild was for students only, the masters of Roman Law being ordinary citizens of the town and possessing their own organizations).[16]

One of the weapons possessed by such a guild was emigration – students often brought money with them from home, and if they left a town, its commercial life might suffer severely. Even when an agreement was reached, some masters might stay together with their students. In this way, an emigration from Oxford produced the university of Cambridge in 1209, while that of Padua resulted from an emigration from Bologna in 1222.

[16] The medical schools of Salerno and Montpellier were older but only came to be characterized as "universities" at a moment when this term had acquired new meanings.

The whole process by which the universities emerged is much too complex to be treated justly in the present context. A recent fairly detailed description is Pedersen (1998, pp. 138–188).

The northern universities grew out of the cathedral school system and thus had as their original core the liberal arts as these had been shaped from the Carolingian age onwards; they can thus be expected to be relevant for discussions of mathematics teaching. Those of Bologna and Padua were initially schools of law, later also of medicine; in this context, mathematics was an auxiliary discipline for astronomy, itself an auxiliary discipline for astrology, a tool for medicine.

Since Paris eventually came to serve as a general model, we may look at what we know about its mathematics. At least when the structure crystallized during the earlier decades of the thirteenth century, the university was divided into *faculties*. A student (always a boy) first entered the Arts Faculty around the age of 14 or so, studying there for at least 6 years (unless part of the corresponding studies had been achieved elsewhere, as was gradually becoming possible); during the final 2 years, when he had acquired the degree of a *baccalaureus*, he was allowed to make his own "cursory" lectures under supervision. The name of the faculty refers to the hypothesis that it taught the liberal arts (whereas the later alias "faculty of philosophy" refers to the sway which Aristotelian philosophy possessed from the mid-thirteenth century onwards). Most students left after having finished the arts study, if not on the way (even less than the full curriculum might serve to obtain a post in the Church or in secular administration); some of those who graduated and got the *licentia docendi* stayed as masters at the faculty while normally pursuing studies at the "lucrative faculties" – the faculties of Medicine and Canon Law. Having graduated from one of these, they had the possibility to teach there, perhaps pursuing studies at the Theological Faculty. Mathematics was taught at the Arts Faculty.

The first approach to the definition of a curriculum is found in a Papal decree (issued on the Pope's behalf by Robert de Courçon, a local theologian) from 1215.[17] It rules in a few lines what should be taught in Aristotelian logic and grammar and what must not be presented in cursory lectures. All that is said about mathematics is that the masters.

> shall not lecture on feast days except on philosophers and rhetoric and the quadrivium and *Barbarismus* [a section of Priscian's grammar dealing with stylistic and rhetorical topics] and ethics, if it please them, and the fourth book of the *Topics*.

That mathematics was not compulsory seems to be confirmed in a decree from 1252 (trans. Thorndike 1944, pp. 53–56): an arts student presenting himself for the disputation leading to the bachelor's degree should at least be in his twentieth year; he shall have followed lectures on advanced grammatical and logical subjects (including Aristotle's *Prior* and *Posterior Analytics* – not easy stuff) and on Aristotle's *On the Soul* – about things mathematical not a word. However, since all of this belongs on the advanced level, the student *may* have been supposed to have pursued quadrivial studies along with elementary grammar and logic. A new decree from 1255 (trans. Thorndike 1944, pp. 64–66), famous as the demarcation of the complete Aristotelization of the faculty, leads to the same conclusions.

However, other kinds of evidence are at hand. One is a satirical poem "The Battle of the Seven Arts" (ed. trans. Paetow 1914), describing the fight between Orléans, a representative of twelfth-century learning at its literary best, and the University of Paris, where "the arts students, they care for naught except to read the books of nature" (that is, Aristotle's natural philosophy), but which nonetheless starts by loading "the trivium and the quadrivium in a tub on a large cart" as its arms. Among the warriors are necromancy, coming from Toledo and Naples (where translations from the Arabic had been made), together with her accomplice "the daughter of Madam Astronomy" (that is, astrology). Further, we encounter among the Parisian warriors Arithmetic, who counts and calculates (and thus appears to have nothing to do with the Boethian tradition), Geometry drawing a circle, and Madam music, presented in a way which suggests *musica practica* rather than Boethius. Astronomy herself also turns up repeatedly on the Parisian side.

[17] trans. Thorndike 1944, pp. 27–30.

This actual presence of astronomy and of what it presupposed is confirmed by some famous pedagogical treatises. One was written by Alexandre de Villedieu (c. 1175–1240), who directed a (presumably pre-university) school in Paris in 1209: the *Carmen de algorismo*, a versified introduction to the Hindu-Arabic numerals and their use. It became very popular but was soon followed by Sacrobosco's *Algorismus vulgaris*, a prose work which may have been meant as an explanatory commentary to the *Carmen* but soon became the foundation on which most subsequent expositions of the Hindu-Arabic system built (Sacrobosco may have been taught in Oxford, but he was a Paris master from 1221 until his death in 1244 or 1256). Sacrobosco also wrote an introductory treatise *De sphaera*, whose use in certain universities lasted until the seventeenth century. There is thus no doubt that a fair number of students were interested in these two subjects, both supports for astronomy. Alexandre also wrote a versified introduction to computus, and Sacrobosco an advanced treatise on the same topic; in particular the former was widely used for a long time in universities.

As to the *Elements*, two apparently contradictory statements confirm that they were read at the university in the 1240s. One was made by the ever-polemical Roger Bacon (ed. Brewer 1859), according to whom the *philosophantes* of his time – probably those whom he had met when in Paris in the 1240s – ran away after the fifth proposition of book I. The other is a collection of *quaestiones* (Grabmann 1934) – a specific university genre emulating the university disputation, raising a question, giving arguments in favour of one answer, formulating the counter-arguments, refuting these, etc. The collection was made in Paris in the 1240s and deals (so it says) with matters that can be discussed at examinations (thus reflecting the advanced level of the whole curriculum). Concerning mathematics, the contents of all 15 books of the *Elements* are analyzed. Since nothing promises that students were supposed to know them in detail, we may perhaps conclude that a commentary possibly written by Albert the Great (Tummers 1980) reflects better what a teacher would go through – namely, the first four books.

Jordanus de Nemore, competing with Fibonacci for the honour of being the best thirteenth-century Latin mathematician, probably taught in Paris somewhere between 1215 and 1240. According to its style and contents, an anonymous *Liber de triangulis Jordani* is a student *reportatio* of a lecture series held over one of Jordanus's works while it was still in process and thus plausibly by Jordanus himself (Høyrup 1988, pp. 343–351); however, his teaching appears to have influenced the happy few only.

One of these few – and one who certainly learned from Jordanus, in person or from his writings – was Campanus of Novara. He wrote a *Theorica planetarum*, which (together with a namesake) served university teaching for a couple of centuries and which was certainly much more accessible than the *Almagest* (even though the namesake, wrongly attributed to Gerard of Cremona, became the favourite scapegoat of the famous Vienna astronomers Peurbach and Regiomontanus in the fifteenth century). More influential, however, was his version of the *Elements*,[18] written around 1259, which replaced the preceding versions and was only itself replaced as the standard version by that of Clavius in the later sixteenth century. Like the Clavius version in later times, it owed its success to its accommodation to the pedagogical contexts in which it served. As the equally didactic Version II, Campanus thus points out the parallels between geometric and arithmetical propositions – but in agreement with the philosophical mood of the thirteenth-century university, Campanus discusses *why* apparently identical matters are treated twice.[19]

The situation of mathematics at the Paris Arts Faculty seems not to have changed much during the later thirteenth or the fourteenth century. Some scholars connected to the university were certainly interested in mathematics – some, like Nicole Oresme (c. 1320–1382), even made impressive contributions to the field. Nonetheless, the statutes of 1366 (ed. Denifle and Châtelain 1889, vol. III, p. 143) only require that students admitted to the licence should have "heard some mathematical works"

[18] Now available in critical edition (Busard 2005).

[19] Cf. the discussion in Murdoch (1968).

along with a specified list of Aristotelian books on natural philosophy; it is not excluded, given the language of the time, that some of works thought of would actually have dealt with the astrological "daughter of Madam Astronomy". A document antedating 1350 explains that the minimal requirement was that bachelors had "heard" *De sphaera* and were following lectures on another work with intention to finish them (Denifle and Châtelain 1889, vol. II, p. 678).

In any case, astrological chairs were established at the same time at the Paris Faculty of Medicine (Lemay 1976, pp. 200–204), inaugurating a local alliance between medicine and astrology which was to last until the 1530s. Ideally, according to a fourteenth-century list (Lemay 1976, pp. 210ff), the fundament for astrology included algorism (Sacrobosco's or a later work on the topic), *De sphaera*, computus, Boethius's *Arithmetica* and *De musica*, Euclid's geometry, Ptolemy's book on the astrolabe and *Almagest*, Theodosius's and Menelaus's treatises on spherical geometry, Jābir ibn Aflaḥ's and al-Bitrujī's works on planetary astronomy and finally a number of works on the principles of judicial astrology – all considered as "mathematics". How much of this was really taught to the medical students in Paris remains a guess.

Fourteenth-century Oxford is somewhat more explicit than the Arts Faculty of Paris. In the statutes from 1340 (Gibson 1931, p. 33), students passing the baccalaureate were requested to have heard six books of Euclid, Boethius's *Arithmetica*, computus with algorism and *De sphaera*. It is even stated that geometry was to be heard for five whole weeks, Boethius for three whole weeks, and algorism, sphere and computus each during 8 days (not counting feasts). Since Oxford was the home of the mathematically innovative "Merton College group" (Thomas Bradwardine, Richard Swineshead, etc.), we may safely assume that lectures were held on more advanced topics (proportion theory and its new links to natural philosophy and theology) without being part of the compulsory curriculum – cf. also (Weisheipl 1964, p. 149). In later statutes, Euclid may be replaced by Witelo's *Perspectiva*, book I of which is indeed an introduction to geometrical theory.

As we have seen, mathematics belonged with medicine in Bologna and other Italian universities (for Padua, cf. Siraisi 1973, pp. 67ff, 77); so did natural philosophy. From Bologna we have a list of the compulsory mathematical readings for the medical students (Rashdall 1936, vol. I, p. 248; cf. Thorndike 1944, pp. 281ff) (undated, but almost certainly fourteenth century): an algorism for integers and fractions (namely, the sexagesimal fractions used in astronomical calculation); the astronomical tables of Alfonso X (the "Alfonsine tables"), with rules for using them; the Campanus version of *Elements* I–III; treatises on the use of the astrolabe and the quadrant (another instrument for measuring angles); a *Theorica planetarum*; and book III (the theory of the sun) of the *Almagest*. Boethian quadrivial works are absent, in good agreement with the frequent employment of qualified abbacus masters (see below) as mathematics teachers.

On the whole, the northern universities that were established during the fourteenth and fifteenth centuries emulated Paris. However, Vienna at least was more explicit than Paris about mathematics in its regulations from 1389. Before the baccalaureate, the student should have followed lectures about "the sphere, algorism, the first book of Euclid, or other equivalent books" (Kink 1854, vol. II, p. 180); for the *licentia*, they should have followed "*Theorica planetarum*, five books of Euclid, [Pecham's] *Perspectiva communis*, some treatise about proportions, and one on the latitude of forms [the innovations of the fourteenth century, in which Bradwardine and Oresme had been involved], some book on music and some on arithmetic" (Kink 1854, vol. II, p. 199). A roughly contemporary document from the newly founded Heidelberg University (closer to the Paris model) requires a student who is examined for the *licentia* to have followed lectures on "several mathematical books in their entirety", and further *De sphaera*; another one fixes the fees for lectures on a variety of books, including *De perspectiva*, *Elements* I–IV, *De sphaera*, algorism, computus and *Theorica planetarum*, which must thus have been lectured on regularly (Winkelmann 1886, pp. 38, 42). The list of books that were printed time and again in university towns between 1450 and 1500 (Klebs 1938) indicates the Vienna and Heidelberg documents reflect widespread interests in the late medieval university environment.

We should take note, on the other hand, that even in Vienna mathematics was considered more a pastime than a really serious matter by teaching authorities. In the statutes from 1389, we read

(Kink 1854, p. 196) that since it is better that students "visit the schools than the taverns on feast days, fighting with the tongue rather than with the sword", afternoon at such days the bachelors of the arts faculty "should dispute and read gratuitously and for the sake of God on computus and other *mathematicalia*".[20]

We may perhaps wonder why the medieval university, with all its success in the domains of logic and natural philosophy, and in spite of the activity of several noteworthy mathematicians, never brought it far in the domain of mathematics education. At least a partial answer can be derived from its favourite teaching methods. Lectures alone, of course, do not give much, neither in philosophy nor in mathematics (in particular not when students do not have the textbook allowing them to reflect on their own – and we are pre-Gutenberg). However, combined with intensive discussion, they are an ideal means for furthering philosophical perspicacity. As far as mathematics is concerned, lectures combined with discussion favour the development of *metamathematics* – that is, also philosophy. But in order to become creative in mathematics itself, and possibly to enjoy it, one has to *do* mathematics, not only to speak about it. Inside the curriculum of the learned schools and the universities, the areas where one could do mathematics were few. Computus was one such area – but its mathematics did not go beyond simple arithmetical computation. *Rithmomachia* was another one, and the game indeed remained popular until the sixteenth century. The third was computation with Hindu-Arabic numerals in the use of astronomical tables – perhaps not too inspiring either, but nonetheless a domain that was practised assiduously well into the Renaissance, whether for its own sake or (rather) because it was a *sine qua non* for simple astrological prediction.

6 Lay Schooling

We know very little about the education of burghers' children after the twelfth-century revival of city life. A few institutions like the Saint Victor school in Paris admitted them, but what they offered seems to have been badly adapted to a future in commercial life (future artisans were in any case taught as apprentices); Pirenne (1929, p. 20) relates that a Flemish merchant's son was put into a monastic school around 1200 in order to learn what was needed in trade – but then became a monk. Some clerks served as house teachers in wealthy families (Pirenne 1929, 21ff), and some probably held private schools. That Italian merchants had been taught by Latin-writing clerics is illustrated by Boncompagno da Signa's description (1215) of their letters as written in a mixture of corrupt Latin and vernacular.[21] Computation was presumably learned on the job, during apprenticeship – but even this is nothing but an educated guess built on what we know from later times.

The region which provides us with the earliest detailed information is northern to central Italy. In his *Cronica*, the former Florentine banker Giovanni Villani (1823, vol. VI, 184ff) states about Florence in 1336–1338 that the children that were baptized.

> numbered every year by then 5,500–6,000, the boys exceeding the girls by 300–500 per year. We find that the boys and girls that were learning to read numbered from 8,000 to 10,000; the boys that were learning the abbacus[22] and the algorism in six schools, from 1,000 to 1,200; and those who were learning grammar and logic in four higher schools, from 550 to 600.

[20] More information on mathematics teaching at medieval universities (despite various imprecisions) in the first chapter of Schöner (1994).

[21] *Rhetorica antiqua* (ed. Rockinger 1863, p. 173). In any case, since notarial documents were written in Latin, merchants needed to understand the rudiments of that language.

[22] The "abbacus" is not, as one might believe, the calculation board; the word (mostly in this spelling) had come to designate practical mathematics – thus already in Leonardo Fibonacci's *Liber abbaci*.

Allowing for a pre-school mortality of c. 50 % (which seems reasonable from what we know about wealthy families[23]), we see that the majority of all children (within the city, *not* the surrounding countryside) learned to read and write (for the reliability of this information, cf. (Goldthwaite 2009, p. 354)). At least one third of the boys went to the 2-year abbacus school learning practical arithmetic, and perhaps one out of ten went to a grammar school (which lasted longer).

What interests us here is the abbacus school. From around 1260 onwards, such schools were created in the commercial towns between Genova, Milan and Venice to the north and Umbria to the south.[24] It was attended in particular by merchants' and artisans' sons, but patricians like Machiavelli and even Medici sons also visited it.

Two documents inform us about the curriculum, one from the 1420s (ed. Arrighi 1967), the other from 1519 (ed. Goldthwaite 1972, pp. 421–425). Scattered remarks in some of the texts written by abbacus masters confirm their general validity.

At first, the boys learned how to write numbers with Hindu-Arabic numerals. Then they were taught the multiplication tables and their application; the sources do not speak about addition and subtraction, perhaps because these techniques were implicit in the learning of the number system. Division came next, first with divisors known from the multiplication tables, then by multi-digit divisors. Then came calculation with fractions.

After this followed commercial mathematics (in varying order): the rule of three, monetary and metrological conversions, simple and composite interest and reduction to interest per day, partnership, simple and composite discounting, alloying, the technique of a "single false position" and area measurement. All teaching from the multiplication tables onwards was accompanied by problems to be solved as homework. More complex matters, like the use of a double false position and algebra, are amply treated in many abbacus books but seem not to have been part of the curriculum. They may have been part of the training of assistant apprentices, but this is another speculation with no support in the sources; what we do know is that proficiency in such difficult matters played a role in the competition for employment (smaller towns often employed abbacus teachers) or for pupils.

Strikingly, the accounting techniques of the great commercial firms were *not* taught in abbacus school and are not described in the abbacus books. These do not even mention the abacus boards used in the counting houses (*Cambridge Economic History of Europe,* vol. III, p. 90) nor double-entry bookkeeping before Luca Pacioli borrowed a whole *Libro di mercatantie et usanze de' paesi* (already printed in 1481) and inserted it into his *Summa* (1494) (Travaini 2003, p. 164). These techniques were assimilated on the job by apprentices who had already visited an abbacus school (Goldthwaite 2009, pp. 83ff, pp. 91ff, p. 354).

Flanders, also home to a wealthy merchant class already in the twelfth century, offers information about the effort of burghers to create their own schools at least from c. 1150 onwards; Pirenne (1929, pp. 24–28) shows how the effort was mostly successful, even though church and feudal princes often did their best to keep control and sometimes monopoly. It appears that the schools, like those in Renaissance Germany, taught reading, writing and calculation together (from the thirteenth century onwards basic Latin as well as vernacular literacy).

[23] In Fiesole outside Florence, in the relatively benign years 1621–1626, 20 % died with the first year of life; later in the century, this rate doubled, with peaks above 50 % (Cipolla 1993, p. 221).

[24] Recent discussions of the social history of this institution are Ulivi (2002a) and (dealing particularly with Florence) Ulivi (2002b).

Contrary to what is often claimed (also repeatedly by Ulivi), the abbacus school does *not* descend from Fibonacci's *Liber abbaci* – cf. Høyrup (2005). There is some (mostly indirect) evidence that the Italian tradition (as already Fibonacci) was inspired from what went on in the Iberian region, but we have no information of how teaching was organized there before the fifteenth century.

A mercantile arithmetic inserted in the thirteenth-century Picardian *Pratike de geometrie* (ed. Victor 1979, pp. 550–601) probably reflects the kind of computation the Flanders merchants made use of when visiting the fairs of Champagne. Here they met the merchants from Italy, and the arithmetic in question also meets what we know from Italy in some of the problem types it deals with. However, the contact is no more intimate in one than in the other case. The abbacus books generally offer methods that can be justified theoretically and do not excel in unexplained shortcuts, as could be expected from books written by professional teachers of (elementary) mathematics; the Picardian treatise is much closer to what appears to have been tricks developed and used by practical traders, and it was hardly based in a school tradition. We may surmise that there was no mathematics teaching in Flanders similar to that of the Italian abbacus school. That this was indeed so seems to be confirmed by the purely Italian inspiration of the German *Rechenmeister* tradition and the German *Schreib- und Rechenschulen* that emerge in the sixteenth century: they appear to have found nothing of interest in Flanders.

Apart from Iberia and Provence (similar in this respect to Italy), other European regions probably had even less lay teaching of mathematics than Flanders. The Norwegian *Speculum regale* (written perhaps c. 1195) may illustrate this common situation. It contains a long section where a father advises his son, a merchant *in spe*. All it says about mathematics is "practise [*gerðu*/"do"] number skill [*tölvisan*] well, that is much needed by merchants" (ed. Keyser et al. 1848, p. 7). No school is certainly implied.

7 Master Builders and Other Artisans

It cannot be excluded that some abbot or bishop asked a master builder to put some sacred number into a sacred building to be constructed – for instance, *three* for Trinity. However, this does not mean that the geometrical knowledge needed for the actual construction had any scholarly or sacred origin. Many artisans may certainly have been taught at the workshops connected to ecclesiastic building activities, but they were taught by more experienced artisans working there, not by monks or priests (*Cambridge Economic History of Europe*, vol. II, p. 772).

The best evidence we have for the actual type of geometric training received by master builders comes from the writings of Mathes Roriczer, himself an experienced master mason – in particular from his *Geometria deutsch* (between 1486 and 1490) (ed. trans. Shelby 1977). Roriczer is not ignorant of the way more scholarly geometry was written – he uses lettered diagrams and only uses each letter once; his way to explain the diagrams, however, is not in the scholarly tradition. Some of his constructions are exact and might for that matter come from the scholarly tradition. In their totality, however, they belong to a tradition that had been handed down within the craft since Antiquity and even longer. This tradition, moreover, was wholly separate from that of "practical geometry", which dealt with geometric calculation (and thus with scribal/administrative practice) and not with construction – cf. (Høyrup 2009).[25] Like the geometry of shipbuilders, it was never taught in any school of the abbacus type but only in apprenticeship – until, with the emergence of the engineering profession, the profession-specific and largely oral tradition was crowded out by scholarly mathematics more or less adapted to practitioners needs. That, however, was long after the end of the Middle Ages.

[25] Shelby (1970) reaches similar conclusions concerning late medieval English masons.

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Part III
Mathematics Education in Different Epochs and in Different Regions: Premodern Period

Introduction

The great civilizations and states of Antiquity and the Middle Ages discussed in the previous part of this Handbook were very different from each other. Still, some common patterns are visible in the emerging forms of education. Whether in Mesopotamia, Egypt, or China, the goal of the state in organizing education was to address its immediate needs, typically the preparation of administration; in other words, teaching served clear professional aims. Specifically, mathematics was an essential aspect of training scribes in the scribal schools of Mesopotamia and Egypt, and it was one of the disciplines examined to enter state services in China. Additionally, mathematical knowledge used to be an auxiliary branch of practicing astrology and astronomy, mainly exercised for religious aims; India provides a paradigm of the development of mathematics education for these purposes.

In the Greco-Roman world, a new pattern emerged: the first liberal education for the class of free citizens. Of the two directions this education took – rhetorical and philosophical – mathematics was part of the latter. In Late Antiquity, this kind of education became systematized as the *septem artes liberales*, with the two components *trivium* and *quadrivium*. This split within liberal education remained characteristic not only until the Premodern period but even into Modern times. Typically, in Antiquity, this teaching was organized without any state participation. Likewise, formation in the *artes mechanicae*, for professional aims, was organized by the practitioners themselves.

In countries of Islamic civilization during the Middle Ages, education and training were also practiced without state intervention. Institutions for advanced learning like the *madrasa* were financed by private pious foundations, *waqf*. With their principal aim being religious and their main disciplines *fiqh* and *hadith*, the religious and juridical sciences, respectively, mathematics could be taught as an auxiliary discipline.

In this sense, the Middle Ages in Western Europe followed similar patterns. While the Carolingian Empire had launched first initiatives for schooling after the darkest times of the Dark Ages, the ensuing development was promoted and effected without any participation of the state. The cathedral schools, the earliest forms of organized learning, were run by the Catholic Church. From the twelfth century on, universities became established as new forms of general and higher education. With their functioning largely comparable to the *madrasa* (see Makdisi 1981), they constituted closed corporations, based on a system of clerical prebends (the Paris model) or on forms of the self-organization of students and masters (the Bologna model). They were universal in the sense that their necessary legitimation by a papal bull assured the validity of deferred degrees throughout the entire Christian region. Mathematics, as part of the *quadrivium*, was taught as a secondary subject within the arts faculty or as an auxiliary discipline for the medicine faculty (in the Bologna model). The guilds themselves organized the preparation for the crafts as an apprenticeship. An innovative approach was provided by the group of *maestri d'abaco*, or reckoning masters, who taught practical and commercial arithmetic first in Italian cities where capitalism began to emerge.

By the end of the Middle Ages, the Renaissance prepared important structural changes, which developed more systematically later. As a part of the Humanism movement which fostered the cultivation of the arts and sciences, numerous sovereigns not only became proud to act as sponsors for artists and scientists, but also increasingly interfered with universities on their territories, in particular, imposing new professorships of Humanism disciplines – including mathematics – on them.

Premodern times (or the Early Modern period) is a term coined to indicate the period spanning roughly 1500 to 1800 in Europe. This period differed from previous eras in many ways. First and most obvious, the world had now changed geographically, with great geographical discoveries connecting vastly different lands, which, therefore, started exerting more influences on each other. Simultaneously, the national states emerged as new economic, political, and cultural realms. If the Western Europeans of the Middle Ages had usually identified themselves as residents of their relatively small localities as well as Catholics (and, therefore, members of a very large community opposed to non-Christian or Orthodox members but not to the citizens of other countries), now they considered themselves citizens of their national states. One example of this new understanding arises from the changes in the position of universities: influenced by the Reformation, many universities no longer needed papal privilege; they needed instead authorization of the territorial sovereign. (In the case of Germany additionally the Emperor's authorization was needed, and in Catholic regions throughout Europe in general, universities needed the joint authorization of the Pope and the King.) Universities thus became national and increasingly state-run institutions.

The society changed both socially and ideologically in Premodern times. During this period, a new social stratum of bourgeoisie emerged step by step from its medieval origins. Most importantly, that was a period of the Protestant reform and Catholic counterreforms, which affected the entire belief system of those who lived at that time. These changes influenced the system of education in general and mathematics education in particular. National states, while being obviously concerned with their own needs as were ancient states, understood these needs more broadly. Differently from previous eras, economy now entered a zone of government activity and control, but changes were not limited to economy. Education was now viewed as an instrument that could establish greater integrity across the country and thus help to gain more control of the country. Additionally, during Premodern times, the need for educated people substantially increased. The new influential social group – the bourgeoisie – emerged, and this group needed more education and specifically more mathematics education. The most important role played during this time, however, probably was that of religion and the Church. A new and more rational understanding of the world, which surfaced in this era, was beneficial for the development of mathematics; the religious controversies which also surfaced helped to establish new organizational, and methodological, frameworks in education (including mathematical education).

During the Protestant reform and the Catholic counterreforms in Western Europe, important educational changes and innovative ideas occurred. They served as models or triggers of change in other regions, including some Orthodox countries. Non-European regions also had their own specific evolutions of thought and change and experienced similar social processes, albeit with their own unique identity and not necessarily simultaneously with Europe. Given that the Americas became the new object of colonization, the European impact on American education was very forceful, sometimes leading to the elimination of originally existing structures. Asia remained basically independent in Premodern times, and its original systems of mathematics education underwent European influences differently while also integrating them into their systems.

This part contains three chapters. The first is devoted to Europe – its Catholic/Protestant and Orthodox regions. The second focuses on the Americas, with two sections on North America and Latin America. The third chapter discusses Asia, and given the enormous diversity of all the countries of that region, specifically concentrating on China and Japan.

A comment on the chronological borders is necessary. The specifics of the historical processes in different regions and countries do not permit each chapter to cover exactly the same period. For example, readers will see that the section on China extends up to the end of the nineteenth century. At the same time, following the Russian tradition of periodization, the history of Russian mathematics education in the eighteenth and nineteenth centuries is discussed together in the chapter devoted specifically to Russia in the part on Modern times. More generally, additional information on the period in question can in some cases be found in chapters in other parts of the Handbook.

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Chapter 7
Mathematics Education in Europe in the Premodern Times

Alexander Karp and Gert Schubring

This chapter is devoted to mathematics education in Europe in the premodern times. Looking at a map of Europe from the beginning of the period in question, the reader will see that the borders at that time were considerably different from the borders of today. In the East, along with the Grand Duchy of Moscow, an enormous region was occupied by the Grand Duchy of Lithuania, which contained many lands that later became parts of the Russian Empire. The Southeast of Europe was occupied by the Turkish Empire. In Western and Central Europe, nation-states were only beginning to emerge – the Holy Roman Empire, which formally included all the territory that today belongs to many countries (Germany, Italy, Austria, Czech Republic, and others) and which lasted nominally until 1806, was in effect a union of almost independent states.

Europe, of course, was not unified either politically or culturally, but the division into Western and Eastern Europe, which became customary and politically meaningful in the twentieth century, does not quite correspond to the reality of premodern times. For the spread of cultural influence, geographical factors proved less important than religious ones, particularly during the first part of the period under investigation. Consequently, this chapter consists of two parts. The first, written by Gert Schubring, is devoted to Catholic and Protestant countries, that is, first and foremost, the countries of Western and Central Europe, but with certain additions. The second part, written by Alexander Karp, is devoted to Orthodox countries, which geographically belong roughly to Eastern and Southeastern Europe.

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Map of Europe at the beginning of 16 century

1 Mathematics Education in Catholic and Protestant Europe

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1.1 A New School Type Emerging

Premodern times saw one major institutional and structural innovation, the introduction of secondary schooling. This was an effect of two movements, albeit of different dimensions: the establishment of schools with a graded structure of forms and the Protestant Reform, with the Catholic Counterreform as sequel.

The predominant pattern of organized teaching at the end of the Middle Ages had been that of higher education. Secondary schooling would be differentiated from it during the first half of the sixteenth century. In fact, the cathedral schools had lost their function with the advent of the universities and only a few of them survived in the Nordic countries where there were no universities yet. At the same time, the universities also covered what would later become secondary schooling because students used to enroll in their arts faculty as youngsters between the ages of 10 and 15.

By the end of the Middle Ages, a number of schools emerged providing basic qualifications in reading and writing and basic knowledge of Latin, which complemented private tutoring at home and preparation for entering the arts faculty. These prospered in major cities in Middle Europe and were controlled by the Magistrate, but functioned attached to a local church. Clearly, these schools used to be of a highly diverse and unequal status, as their names reflect: in the case of Germany, *Ratsschulen* (schools of the Magistrate), *Lateinschulen* (Latin schools), *Pfarrschulen* (parish schools), *Trivialschulen* (schools of the *trivium*), and so on (Seifert 1996, p. 223).

Two distinct developments led to the emergence of a somewhat homogeneous structure of a new school type. In the classical literature, this emergence was associated with the organization of the Gymnasium in Strasbourg by Johannes Sturm (1507–1589), but more recent studies show that the new school type had already been prepared earlier in two different settings.

The first development occurred within the arts faculty of the universities. The youngsters enrolled there created a great deal of unrest within the universities; the classical type of lectures was not adapted for their age, and their status of student gave them rights, which they abused due to missing discipline. A structural change helped to remedy this disturbing situation. By the end of the Middle Ages, the university transformed into the "collegiate university" (McConica 1986) in England and Paris, this process resulted in eliminating the lectures of the arts faculty and replacing them with instruction within the *collegia* (colleges), which were endowed student residences where instruction was provided by contracted *regentes* and *legentes*, the two traditional types of lecturers in the medieval arts faculty. As it proves particularly practiced at some of these Parisian colleges, the traditional type of lectures was increasingly substituted by a school-like manner of instruction: students were grouped into an ascending series of forms, each form taught by only one *magister* (literally: master). This structure applied even to the philosophy course of the arts faculty, which was now taught as an upper course in the principal colleges. Thus, student discipline and systematic learning could be enhanced. By the end of the Middle Ages, general lecture courses in Paris were no longer offered: the arts faculty was transformed into an umbrella association of the colleges and remained so until the French Revolution (Seifert 1996, p. 258). Ignatius of Loyola (1491–1556), the founder of the Jesuit order, had studied at these Parisian colleges; their structure of graded forms became the model for the Jesuit colleges in the Catholic countries.

It should be mentioned at this point that the Paris University constituted a rare case of the refusal to modernize according to the Humanist Movement, as Paris was a stronghold of Aristotelism. Since King François I failed in his attempt to introduce the humanist disciplines, he founded in 1530 the *Collège Royal*, a state institution for the free teaching of humanist disciplines, which also included a specialized chair for mathematics.

The second setting where a new type of secondary schools emerged was several municipal schools not only in the (present-day) Netherlands and Northern Germany but also in other regions of Middle Europe like Alsatia. There, the traditional way of grouping the pupils into "Haufen" (cohorts) by young, middle, and elder who were taught simultaneously by one schoolmaster, or even at times together with an assistant, was replaced by a series of up to eight ascending forms. These forms were counted from high to low (*Prima*, *Secunda*, etc.). Two outstanding examples of such a school are in Deventer (today in the Netherlands) and in Liège (Lüttich, today in Belgium). Johannes Sturm, originating from Liège, transposed his own school experience to the Protestant Strasbourg, where he was called in 1537 to be lector of philosophy. According to Sturm's proposals, the Magistrate established a *Gymnasium* in 1539 with nine ascending forms, each one conceived of as 1 year long, where the age group would ascend. The number of forms increased to ten; this large number is explained by the fact that – contrary to the original plan – the school had to also provide primary school functions such as teaching reading and writing. Other than that, the curriculum was focused on *eloquentia*, Latin and Greek being key subjects, and rhetoric and dialectic in the upper forms. In fact, the Strasbourg school was not only a secondary school, but it aimed also to provide elements of the courses of the arts faculty through so-called *lectiones publicae* (public lectures) on logic and geometry. This type became known as *Gymnasium illustre* or *Akademisches Gymnasium* and was copied by various cities because Magistrates wanted to retain the sons as long as possible in the proper city before they left for universities in other cities. The kernel of Sturm's Gymnasium as a secondary school became the model for the *Gelehrtenschulen* (scholarly schools, since preparing for university studies) or *Gymnasien* in the Protestant regions.

1.2 The Reform and Counterreform and Their Impact on Mathematics Teaching

The Protestant Reform, initiated by Martin Luther (1483–1546) in 1517, soon gained significant momentum. Not only did Northern Germany adhere within a short time to Protestant faith, but Southern Germany, in particular Bavaria, and Austria also converted in large part. Remarkably, contrary to previous reform tentatives within the Catholic Church, the political support of numerous sovereigns was so strong that the German Emperor was not able to suppress the movement. In Bavaria, for example, the *Landstände*, the feudal regional powers, promoted Protestantism. But Protestantism did not remain restricted to Germany and Switzerland. In its two forms, the Lutheran and the Calvinist, the reform movement gained all of Scandinavia and even parts of Spain and France until the second half of the sixteenth century.

Yet, Catholic rulers did not accept this expansion; they not only initiated a rollback, particularly in Spain, Bavaria, and Austria, but also launched religious wars. The first such bloody war occurred in France, during the second half of the sixteenth century, and ended eventually with the famous tolerance edict of Nantes, decreed by Henri IV in 1598 (later revoked in 1685 by Louis XIV). The most devastating war was the Thirty Years' War in Germany, 1618–1648, which included almost all of Europe and eventually ended with the acceptance of Christian religious plurality; at that point, even Calvinism was accepted. Thus, essentially a North–south delimitation in Western Europe marked the regions of Catholic and Protestant faith: Portugal, Spain, Italy, France, Ireland, Southern Germany, Austria, and Poland were Catholic, while Switzerland, Northern Germany, the Netherlands, Scotland, Estonia, and Scandinavia were Protestant.

1.2.1 Protestant Schooling

Because of the basic conviction of Protestantism – that each believer should be able to gain his faith by the proper reading of the Bible – education became the primary field to develop and promote for both directions. In fact, each town that converted to Protestantism would open schools. Philipp Melanchthon (1497–1560), called in 1518 to Wittenberg University as professor of Greek, not only became Luther's principal advisor in educational matters, but he was also the principal constructor of the Protestant educational system. He designed numerous regulations ("Schulordnungen") for new or reformed *Gymnasien*; he also organized the new Protestant universities. Melanchthon is therefore called the *Praeceptor Germaniae* (the Master of Germany). Johannes Bugenhagen (1485–1558) was another pedagogue cooperating with Luther who elaborated many regulations for *Gymnasien* and for school systems in various German states.

Melanchthon had studied mathematics himself; he was convinced of its educational value and promoted it in the curricula of the *Gymnasien* and universities. He edited mathematical publications and wrote prefaces for mathematical textbooks. Because the Protestant *Gymnasien* continued to focus on a classical program of studies, mathematics played only a marginal role. At first, it used to be taught only as arithmetic. The first Protestant founding of a school occurred in Magdeburg in 1524.

Here, the teaching of mathematics was declared desirable, but maybe not realizable (Paulsen 1919, p. 277). One of the next foundations was the Gymnasium in Nürnberg, one of the most important German towns. Here, J. Schoner was called as a mathematics teacher – likely the first such specialized teacher at a secondary school. Yet, the Nürnberg conception proved to be too ambitious; the school did not succeed (Paulsen 1919, p. 278). In 1549, it was reported as a remarkable innovation: that reckoning, until then taught privately, and arithmetic were introduced into the curriculum of the Gymnasium St. Elisabeth, a well-equipped municipal Gymnasium in Breslau in Protestant Silesia (Absmeier 2011, p. 164). Bugenhagen had demanded the "rudiments of mathematics" in his *Schulordnungen* (Paulsen 1919, p. 283). In the curriculum of 1564 for the state of Brandenburg, arithmetic was prescribed for the next-to-last form and astronomy for the last. The same was demanded in 1580 for the Saxon *Fürstenschulen* (Seifert 1996, p. 308; see below for this school type).

The Protestant *Gymnasien* had been founded first as municipal schools; a consequence of this status was the schools' generally precarious financial situation, which differed significantly from their competitors, the Jesuit colleges. Given chronic financial problems of most of the towns, the *Gymnasien* often suffered difficulties with their physical buildings as well as with paying teachers and contracting adapted teaching personnel. Each Gymnasium, depending on its Magistrate, would differ by the structure of their forms, their number, and their curricula and possibly even more so by the concrete local situation. From the second half of the sixteenth century on, however, *Schulordnungen* (school study regulations) were established for various German states, which sought to enhance a certain uniformity for the respective territory, for example, Württemberg, Kursachsen, Kurpfalz, Brandenburg, and Mecklenburg (Seifert 1996, p. 305). Furthermore, in several states, the sovereigns founded state-run schools, which were thus better endowed and able to realize a more coherent and uniform curriculum; a prominent case were the three *Fürstenschulen* (princely schools) in Saxony (Grimma, Pforte, and Meißen).

The specific feature of the Protestant educational system – and this applied to the Lutheran as well as the Calvinist system – was that the arts faculty had not been dissolved in favor of the colleges; in fact, they had not only been maintained but even upgraded to what was now called Philosophical Faculty. This new status was somewhat elevated and featured elder students, while still being a propaedeutic for the three professional faculties. In this consecutive system, beginning with the Gymnasium as secondary school, the Philosophical Faculty enjoyed a firmly established professorship for mathematics, sometimes combined with a professorship in physics. Thus, in the Protestant system, mathematics used to be marginally taught in the *Gymnasien* but enjoyed a stable status as one of the main teaching disciplines within the Philosophical Faculty. Because of the lack of a professional perspective, the courses were rather encyclopedic.

No genuine teacher training or formal requirements and procedures existed to be elected as a teacher; in general, one could become a Gymnasium teacher after some university studies, typically in the Philosophical Faculty and sometimes in theology.

Education in Calvinist territories was organized in a similar way to Lutheran education. *Gymnasien* there were often called *Pädagogium*.

1.2.2 Catholic Schooling: The Jesuits

The considerable success of the Catholic rollback, the Counterreform, was primarily due to the Jesuit order. Founded in 1534 as a militant order to defend and expand the Catholic faith particularly through education, the Jesuits soon began to establish colleges. In the first few decades, their curriculum was not strictly standardized, and several colleges included a considerable amount of mathematics teaching, especially in Sicily. Debates about a *Ratio Studiorum*, the document regulating all details of school life and of the curriculum, starting in 1580, led eventually to its final version of 1599, including a minimal role for mathematics.

A Jesuit college used to be a boarding school for the "internals," the future novices of the order who would continue to study theology, but "externals" were also accepted. An attractive aspect for parents was that externals could also be admitted to the internal boarding part. The boarding school character ensured that strict discipline exerted on the youngsters, including the surveillance of private reading. The curriculum was uniform in all of the colleges, independent of country, because of strict observance of the *Ratio Studiorum*, which regulated all details. This uniformity and the overall high quality of the teaching – combined with being gratuitous for the students – assured the enormous success of the Jesuit colleges over an extended time period. Moreover, there was no formal teacher education in the Jesuit system; the order chose among its members those who showed abilities for teaching. One exception, however, was the *Collegio Romano* in Rome, a higher education institution of the order that trained particularly qualified Jesuits.

Within a few decades, Jesuits were called to run colleges in all the Catholic states, from Spain and Portugal to France and Italy, from Bavaria and Austria to Poland. Catholic education for a long time was synonymous with Jesuit education. Since the colonial powers were. At the outset, primarily Catholic, Jesuit colleges blossomed as well in Spanish Latin America, Portuguese Brazil, and French North America. Later on, other orders also organized colleges, but initially following the Jesuit curriculum. Running a Jesuit college was a costly enterprise for the sovereign or Magistrate who called on the order to establish a college. A fully equipped college, including the theological formation, afforded a staff of 70 order members, among them 15 teachers. A college equivalent to a Gymnasium required still a staff of 30 persons (Seifert 1996, p. 319). All had to be paid by the sponsor. Also in the budget were the necessary school building and the church, which all had to be of impressive dimensions.

Given the uniform application of the *Ratio Studiorum*, one can already determine the curriculum from this document. Elementary qualifications – the three R's, reading, writing, and reckoning – had not to be given by the Jesuits. Their college consisted of seven yearly forms. In the lower part, the *scholae inferiores*, each year was devoted to only one discipline. The first 3 years, all devoted to grammar (*infima*, *secunda*, and *syntaxeos*) were followed by 2 years of *humaniora*: *poetica* and *rhetorica*. According to the Jesuit notion of learning psychology, one was best able to learn one subject when not mixing it up with other ones; thus, each year would focus only on one discipline – therefore, mathematics already for this reason could not be taught in the lower forms. The upper part of the college was the course of philosophy, realized in 2 years.[1] The Jesuit philosophical conception borrowed from their understanding of Aristotle; the teaching of philosophy was conducted according to Aristotle's conceptions and preserved writings, using either his writings or modern compendiums: the first year of logic featured dialectic, logic, and ethics and in the second year, physics. Based on Aristotle's systemization of knowledge, mathematics teaching was prescribed in the last year. Given these upper forms, the Jesuit colleges functioned somewhat analogously to the Protestant *Gymnasium illustre*. But while there they were preparing for studies at the (transformed) arts faculties, the philosophy college course substituted the former lectures of the arts faculties. When the Jesuits took over existing universities, which had been reformed according to Humanism principles, the arts faculty became dissolved and mathematics chairs were suppressed. The essential part of university policy was to establish a coherent formation from the college to the theology faculty; they did not interfere with the law or medicine faculty. Only the examination procedures of the former arts faculties were maintained for conferring the degrees necessary to enter one of the faculties (Schubring 2002, 2008).

Contrary to a widespread assumption, mathematics was not a strong part of the Jesuit colleges. Christopher Clavius (1537–1612), professor of mathematics at the *Collegio Romano* since 1565, had tried, during the elaboration of the *Ratio Studiorum*, to give mathematics the character of a major discipline. In the version of 1586, Clavius had managed to achieve a good representation of

[1] Actually, the *Ratio Studiorum* prescribed 3 years for the teaching of philosophy, but in practice it was generally 2 years.

mathematics, extending its teaching over 3 years and even prescribing a specific formation of teachers of mathematics via an Academy of Mathematics (Krayer 1991; Paradinas 2012a, b). Yet, this part of the version raised adamant objections from various provinces of the order, negating the value of mathematics for the goal of the colleges. In the final version of 1599, mathematics teaching was drastically reduced and its teaching mode was determined in only two paragraphs. In paragraph no. 20, the amount is described: All students of philosophy have to attend in its last year a lecture on mathematics of about three quarters of an hour. A remnant of Clavius's proposals was that particularly adapted students can receive private tutoring in mathematics; yet, it is not reported in the literature that such private tutoring was ever realized somewhere. Moreover, in paragraph no. 40, the contents of mathematics teaching are defined:

– The teacher explains to the students of the physics class the Elements of Euclid, in about three quarters of an hour; if they were sufficiently progressed after 2 months, he should add something from geography and from *Sphaera* or whatever else they would like to hear, and this besides Euclid either the same day or each second day (Ratio Studiorum 1997).
– As pedagogical hints, it was recommended that some student should explain to his classmates a mathematical problem and that regular repetitions should take place.

There were no teachers specialized in mathematics, and the subject was taught according to a rotating principle by the philosophy or physics teacher. Rivard, a successful French textbook author from the first half of the eighteenth century, reports that 4 months were given to teaching mathematics according to the *Ratio Studiorum* (Rivard 1744, Préface). Moreover, research on the history of education in France revealed that many parents made their sons exit the college before the philosophy years so that they would get no mathematics at all (Dainville 1986, p. 61). It is noteworthy that the emphasis of Jesuit mathematics teaching was on popular astronomy; this, together with the first books of Euclid's *Elements*, basically continued the medieval practice of the *quadrivium*. For their ultimate goal, the formation of faithful priests, mathematics clearly played no integral role for the Jesuits.

Textbooks to be used were likewise defined in the *Ratio Studiorum*. For mathematics it was Euclid's *Elements* in addition to Sacrobosco's *Sphaera*: this was first taught in an edition by Clavius and later on in several other adaptations, even in vernacular translations. Particularly prominent was an edition by André Tacquet (1612–1660), a Belgian Jesuit, first published in 1654: *Elementa geometriae planae ac solidae. Quibus accedunt selecta ex Archimede theoremata.*

Despite uniformity across all Catholic countries, there were two particular developments. In Italy, the Jesuits' success was remarkably not as complete as in other regions: for example, the Republic of Venice prohibited the entrance of Jesuits on its territory. Moreover, universities, like Padua and Pisa, which continued to practice the Bologna model (i.e., not having a faculty of theology, among other special features), were not of interest to the Jesuits and continued with specialized mathematics professors (Schubring 2002). For example, Galileo was a mathematics professor for some years in Pisa and Padua.

And in France, where in 1594 the Jesuits had been expelled for the first time after an assassination attempt on Henri IV, they were readmitted in 1704 after becoming "nationalized." That is, only Frenchmen could act as Jesuits in France and all had to swear an oath of loyalty to the king (Romano 1999, p. 357).

1.2.3 Special Developments in England

At first, developments in England followed those on the continent: university reforms initiated in 1535 and continued in 1549 by King Edward were inspired by Humanism and Lutheran Protestantism; scholastic traditions were abandoned and Humanism disciplines like classical philology, mathematics, and natural sciences were introduced. After the intermezzo with Queen Maria Tudor, who returned the country from the Anglican Church (established by Henry VIII in 1531) to the Catholic faith, Queen Elizabeth I decreed the statutes of 1570. The two universities Oxford and Cambridge were now subjected to the Anglican religion; mathematics lost its former status, and logic and rhetoric became the main disciplines within the colleges. The university now being a "collegiate" university as the association of all its (endowed) colleges, all teaching was provided by tutors within the colleges. "Professors" were now endowed positions outside the colleges for some additional, non-compulsory courses, increasingly established since the seventeenth century, for example, the Savilian Professor of Geometry at Oxford and the Lucasian Professor of Mathematics at Cambridge (Isaac Barrow and Isaac Newton).

Contrary to the meaning of "college" within the Jesuit system, English colleges represented what one today calls an undergraduate college; there was no differentiation between a type of secondary school and higher education, as in the Protestant and Catholic cases. This was the college model that was transferred to the British colonies in North America. It was only from the beginning of the eighteenth century, because of the introduction of the Tripos exam in Cambridge, that mathematics achieved a higher status.

1.3 The Transitional Seventeenth Century

The seventeenth century can be characterized as a transitional period: basically, the educational structure in the various Western European regions remained stable and there were no decisive dynamics for promoting the status of mathematics teaching. Given the devastating effects of the Thirty Years' War, education was not considered a primary concern.

Yet, there was some expansion at the periphery, particularly in Scandinavia, which had not been really affected by this war. In Denmark, after converting to Protestantism, Latin schools were founded based on a classical profile. In Norway, only four cathedral schools existed from the Middle Ages and they had turned to Protestantism, too: in Christiania (Oslo), Bergen, Trondheim, and Christiansand. In the seventeenth century, King Christian IV founded five *Gymnasien* in Denmark and Norway because the quality of the existing schools was so poor. The one Gymnasium in Norway, in Christiania, taught subjects beyond the curriculum of the *Cathedralskole*, such as philosophy, astronomy, physics, and metaphysics, to prepare students for the university, in Copenhagen. German Rhodius acted in Christiania as mathematics professor from 1637 to 1660 (Brun 1962, p. 118). The Gymnasium was closed down in 1660 after it lost its land properties in Bohuslän, which were occupied by Sweden. The cathedral schools were modernized only in the second half of the eighteenth century. In Sweden, Gustav II Adolf (1594–1632) established various *Gymnasien*, the oldest of which was founded in 1620 in Stockholm (Göransson 1911, p. 29).

By the second half of the century, however, initiatives by Catholic states to promote applied mathematics became visible. The first such case was France. Due to extensive losses of ships caused by their captains' negligence or inadequate training, the French king established, from 1669, the so-called *chaires des mathématiques* or *chaires de hydrographie* for the formation of naval officers in mathematics and hydrography (Russo 1986, p. 423). These teaching positions became attached to 12 Jesuit colleges. This effected that at these colleges the fathers who taught mathematics would no longer do this in short rotations; instead, the superiors chose a more specialized father who would teach, besides his normal teaching duties for the college students, additional courses for external aspirants for naval services. These fathers remained in local service to that given college for an extended period (Schubring 2002, p. 375).

An analogous structural development occurred in Northern Italy, a region under permanent danger of devastating inundations by the river Pò, in the former Duchy of Ferrara, since 1602 part of the Papal

State, the governor from 1675 attached a mathematical lecture to the Jesuit college at Ferrara to train engineers in applied mathematics and particularly in hydrography and river regulations. The superiors of the college charged one father to specialize in these subjects; he thus became a permanent teacher of the mathematics course, which continued for students in the traditional way while adding on the attached course (Fiocca and Pepe 1985a, b).

In Portugal, which was in need of formation in navigation, such lectures became attached to the Jesuit college Santo Antão in Lisbon, as "aula da esfera," from the seventeenth century on (Leitão 2007).

Another noteworthy development proved to be the foundation of colleges by the *Oratoire* order in France. From the early seventeenth century on, this order founded numerous colleges, mainly in middle and minor towns because the principal cities had already been provided with Jesuit colleges. These colleges are noteworthy because the Oratorians were highly dedicated to the sciences, and therefore mathematics and the sciences were taught in their colleges with considerably more extension. Unfortunately, almost no documents about their curriculum have been preserved (Costabel 1986, p. 82), but the textbooks published by prominent Oratorian mathematics teachers document an impressive quality:

– Jean Prestet (1648–1691): *Elémens des mathématiques*, 1675 and 1689
– Bernard Lamy (1640–1715): a textbook on algebra, at least nine editions between 1680 and 1765, and a geometry textbook, at least seven editions between 1685 and 1758
– Charles-René Reyneau (1656–1728): two textbooks, one on arithmetic and elementary algebra and the other on algebra and differential and integral calculus (Schubring 2005, 74 ff.)

The self-confidence of Prestet in claiming superiority for the "moderns" over the "ancients" ("les anciens") proved to be a bold modernizing approach, disseminating Cartesian conceptions and preparing the way for rationalism in France (Schubring 2005, p. 67).

1.4 Dynamics of Changes During the Eighteenth Century

Contrary to the seventeenth century, the eighteenth boasts dynamism in structural and conceptual changes and in new mentalities. It is a period of strong economic development and industrial expansion and of the definitive rise of the "tiers état": the bourgeois class. All these developments became expressed in a series of intellectual and social movements, which culminated by the end of this century in the French Revolution. Mathematics proved to constitute a key element of the changes made in the educational system of this time.

While the states had now developed their own economic policy, known as mercantilism, the paradigmatic intellectual and social movement became the Enlightenment in France based on Cartesian rationalism. But there were also specific German forms of the Enlightenment, particularly Philanthropinism (see below).

1.4.1 Nobility and Military Schools

The first institutional means to organize the training of engineering in the seventeenth century became largely expanded in the new century. Because the major focus had been on military engineers and positions as army officers had been reserved during the *Ancien Régime* exclusively to the nobility, the typical form of such formation became special colleges for the nobility – and this was the case for both Protestant and Catholic countries. For all these institutions, mathematics was a central part of the curriculum, together with the sciences and modern foreign languages (mainly French). First founded in Southern Europe, for example, the *Collegium Nobilium* in Parma, the *Collegio dei Nobili* in Turin, the *Real Seminario de Nobles* (1725) in Madrid, they were likewise founded in Middle Europe. Besides the Netherlands (Ridder-School in Utrecht) and the Cadets' Knight School in Poland (Pardała 2010, p. 329), numerous *Ritter-Akademien* (knights academies) were mainly established in Prussia (Berlin, Brandenburg, Frankfurt/Oder, Liegnitz, etc.) and Austria. In Portugal, founding a *Real Colegio dos Nobres* was one of the first reform activities of Pombal in 1761. Most of these colleges were dissolved by the end of the eighteenth century because of social changes and more fundamental measures of educational reform.

In France, the formation of civil and military engineers became a much more systematic occupation of the government. A net of military schools was established through which mathematics ascended as the major discipline. For the young noblemen who prepared for careers as military officers, some mathematics teachers had already been attached to regiments in the seventeenth century. Professional training was first institutionalized for future artillery officers. In 1720, the Crown founded *écoles régimentaires d'artillerie* for five garrisons of artillery regiments. Each of these schools employed a mathematics teacher. Among these were also known mathematicians such as Bernard Forest de Bélidor (1693–1761), Sylvestre-François Lacroix (1765–1843), and Louis François Antoine Arbogast (1759–1803). Mathematics formed the central part of theoretical training and shortly afterward of the entrance examination that the aristocratic candidates had to pass after 1755. For this exam, a new function was created, that of the *examinateur permanent*, held by members of the *Académie*. Other military schools for young noblemen, but at a lower level (i.e., for officers' careers in the infantry and cavalry requiring less mathematical knowledge), were likewise founded in this period.

For future naval officers, there were, besides the royal *chaires* in Jesuit *collèges*, naval schools employing teachers for training in mathematics and engineering. The formalization of these officers' training led to the establishment of private preparatory schools, particularly because the entrance examinations increasingly focused on mathematics. All this contributed to an ever stronger presence of mathematics in the general culture.

The teaching of mathematics undoubtedly attained its highest and most innovative level in this period at the *École du Génie* in Mézières, the school founded in 1748 for training military engineers, in particular in fortification technology which was valued as especially *savant*. Both the entrance and the final examinations at this school were assigned to the same *examinateurs permanents*: a mathematician and member of the Paris Academy. The strong function of mathematics in the system of military schools served as model for the main role of mathematics in the public school systems after the French Revolution.

Equally influential were the textbooks published by Étienne Bézout (1730–1783), first for the marine schools and then for the artillery schools, but they became the general textbooks for all military schools; they were reedited and translated many times, even after the French Revolution: *cours de mathématiques*, in six volumes for the marine (1764–1769) and in four volumes for the artillery (1770–1772).

1.4.2 The "Pedagogical Century" in Germany

The eighteenth century has been called the "pedagogical century" for the German-Protestant educational system. Based on the reception of the ideas of John Locke and Jean-Jacques Rousseau, pedagogical movements arose challenging the one-sided classical nature of the *Gelehrtenschul-Wesen* and propagating a new programmatic realization of secondary schooling. The first important movement was that of Pietism, enhanced by Hermann August Francke (1663–1727) and realized in his educational institutions, the *Franckesche Stiftungen* in Halle (Prussia), from 1698. The second great movement occurred in the second half of the eighteenth century, Philanthropinism, which was propagated by Johann Bernhard Basedow (1724–1790) and realized in his institution *Philanthropin* in Dessau (Duchy of Anhalt-Dessau).

The *Franckesche Stiftungen* consisted of a net of schools and various economic enterprises, like a library, a printing press, a pharmacy, and a bookbinder's shop, whose incomes would finance the educational institutions, but also the missionaries who were sent abroad. In fact, Pietism was a movement for the reformation of Protestantism, directed against its orthodoxy – but not for a quixotic piety; rather, it aimed to improve life in the real world. Thus, education of youth became its primary goal, and the orientation towards the real world worked to focus education on "Realien" or realist knowledge. Hence, disciplines like history, geography, arithmetic, geometry, and natural history were introduced. Even at the *Latina*, the Gymnasium-like top school of the net founded in 1697, the focus continued on ancient languages but also evidenced a reformed curriculum with history, geography, mathematics, music, and botany (Bruning 2005, p. 284). This realism-minded Pietism influenced a considerable number of schools, in particular in Prussia and Württemberg.

Philanthropinism meant an educational reform movement aiming at educating youth towards philanthropy. It was the expression of a German-Protestant form of Enlightenment. At its main school, the *Philanthropin* in Dessau, founded in 1774, education was considered entirely different from all traditional schooling: learning should be fun! Religious tolerance was another educational aim, and knowledge should be useful, serving to prepare for such professions as merchant, jurist, physician, official, officer, and architect. Guided by useful knowledge and public utility, true to life, applicable, and cosmopolitan became the keywords for the reform of the curriculum. Modern foreign languages, drawing, mathematics, and natural history, together with manual-practical instruction, gardening, hiking, and gymnastics, became the focus of instruction (Schmitt 2005, p. 262). A practice was established to individualize instruction, according to the envisaged future profession of each student. Although short-lived – the *Philanthropin* closed in 1793 due to management problems – the pedagogical conceptions of Philanthropinism proved to be a matrix for profound educational reforms.

1.4.3 Realist Schools

One of the most long-standing effects of Pietism became the foundation of *Realschulen*, of realist-oriented schools, which for the first time did not intend to improve the traditional types of Latin schools or *Gymnasien* in some way, but rather consciously aimed at liberal education independent of university studies. A disciple of Francke, Johann Julius Hecker (1707–1768), first established a *Realschule*, after studying at the University of Halle and becoming shaped by Francke and his conceptions. After having been a teacher at his schools from 1729 to 1735, Hecker worked as a pastor in Potsdam and Berlin. There in 1747, he founded the "ökonomisch-mathematische Realschule," supported by the pietism-minded Prussian King. In his application of 1746 for approval to establish a "mechanische Real-Classe" (mechanical realist school), Hecker argued self-confidently that knowledge of mechanical operations is of greater utility than knowledge of the Latin gerund, of grammar, and of the logical form *Barbara Celarent* (Schubring 1991, p. 224). At this school, mathematics became a major teaching subject. In 1784, the school turned into a state school, now named *Königliche Realschule*. The *Realschule* proved to be a challenge for the traditional curriculum of the *Gymnasien*: in 1765, the school was inspected to investigate the accusations levied upon it and why it had effected the decay of the municipal *Gymnasien* (ibid., 31). The realist schools exerted a powerful pressure on the *Gymnasien* to reform their one-sidedly literary curriculum.

Yet, this emergence of a competitive type of secondary school did not remain restricted to German-Protestant territories. In Catholic Bavaria, the school reformer Johann Adam von Ickstatt (1702–1776), influenced by Francke and Hecker, proposed the founding of *Realschulen* (Seifert 1996, p.247).

In the Netherlands, there existed an analogous type of school which challenged the classical Latin schools from the sixteenth century. It is revealing that this type was called *Franse Scholen*, or French schools, since they would not teach Latin but French as a modern language. The first type of such schools were the "Nederdeuts-Franse scholen" (Low-German-French schools), with Dutch and French as instruction languages. They served in particular to train steersmen, and the mathematics teaching provided for them focused on geometry, trigonometry, and logarithms (Smid 1997, p. 189). Another type was the *Franse scholen*, which originally served to prepare for commercial professions, with the mathematics syllabus containing calculating and bookkeeping (ibid., p. 194). During the eighteenth century, these *Franse scholen* increasingly turned towards liberal education, no longer training exclusively for merchants. Then, the syllabus provided a relatively broad program ranging from arithmetic over algebra to geometry (ibid., p. 198).

1.4.4 Changes in Protestant *Gymnasien*

It was pressure by the competitive *Realschulen*, the influence of the focus on utility as enhanced by the Enlightenment, and the rise of the middle classes that occasioned a steady reform of the *Gymnasien* in the second half of the eighteenth century. Since most of these schools were municipal and since they had to serve not only the few who would continue with university studies but also the many who would become merchants or craftsmen, the lower forms of the schools turned out to be a combination of Gymnasium and *Realschule* and only the upper forms to be a genuine Gymnasium (Bruning 2005, p. 288). The teaching of mathematics became even more strengthened during the eighteenth century. It became typical to teach arithmetic in the lower forms and geometry in the upper forms. By the end of the eighteenth century, a few *Gymnasien* in Prussia already had mathematics as a major discipline. For instance, at the *Friedrich-Wilhelm-Gymnasium*, a prestigious school in Berlin, combined with the *Hecker-Realschule*, the program in 1802 reported of 6 weekly hours of mathematics in all forms, and the curriculum ranged from arithmetic to "higher geometry," that is, conic sections, and differential and integral calculus (Schubring 2010, p. 6). Protestant schools had proved capable of reforming from within, even without state intervention (Bruning 2005, p. 313).

Within the Protestant states, it was Prussia in particular that, in the last third of the eighteenth century, began a stringent policy to organize education as a task of the state.[2] By the middle of the eighteenth century, between 70 and 80 fully equipped *Gymnasien* were in existence; combined with the minor Latin schools, there were about 400 secondary schools (Bruning 2005, p. 292). In 1787, the Prussian government established the *Oberschulkollegium*, the initial germ of a ministry of education. Thanks to the reports it demanded from all secondary schools, the central state administration became informed for the first time about the schools on its territory (Bruning 2005, p. 293). A key measure, which the *Oberschulkollegium* established and implemented in 1788 as a sequel to the reports, was the introduction of the *Abitur*, the leaving exam of the *Gymnasien*. The *Abitur* was instituted as a sovereign act and delimited now between secondary schools and university. About 40 schools were conferred the right to execute the *Abitur* exam. Although the *Abitur* was not yet compulsory for matriculating at the university (as it became in the nineteenth century), the state was now able to determine which school was entitled to confer the *Abitur*. Thus, mixed types like the *Gymnasium illustre* were no longer admitted. Mathematics constituted one of the subjects of the final exam and had by now decidedly ascended in status (Bruning, ibid.).

A highly noteworthy innovation for mathematics teaching occurred in the electorate state of Saxony in the 1720s. At the three state schools, the *Fürstenschulen*, the government decreed to introduce for the first time specialized teachers of mathematics. As teachers of extraordinary lessons and

[2] For state administration of secondary schools in other Protestant territories, see Bruning 2005, pp. 296–305.

only attached to the school, the first contracted teachers had to fight vigorously for their position. Actually, the budget had not been increased because of the additional teachers, and thus the staff at each school protested against the reduced share of the income. Moreover, one of the first teachers at Meissen had been a practitioner. Dating from this situation in Saxony is the ominous statement "mathematicus non est college,"[3] which is thus not generally true. One of the Saxon teachers, Klimm at Meissen, even published a widely used textbook (Schubring 1991, p. 29).

1.4.5 The Senate House Examination in England

The literature always mentions the extraordinary position of mathematics at Cambridge University: it was the almost exclusive exam subject for *undergraduate* students. Gascoigne emphasizes the often overlooked development that a basic curriculum reform had been effected in Cambridge about 1700 by accepting Newtonism (Gascoigne 1989, p. 7). The ruptures caused by Humanism and religious schisms are thus essential for understanding the position of mathematics at the English universities, just as they are for Germany and France. Developments in England after the Elizabethan reforms had confined mathematics to marginality for a long time in an analogous way to that in the Catholic states.

The eminent role that mathematics eventually attained in Cambridge after the mid-eighteenth century due to the *Senate House Examination* was not really conducive to a progressive development of mathematics.[4] While the subject was indeed intended only for a minority of students who strove for an "honors" degree, it determined the style of the entire university studies. Obviously, mathematics served as a substitute for logic, a subject prescribed since 1570 but since considered outdated. In line with this one-sided function, the study of mathematics was primarily restricted to geometry, in its Euclidean version (Gascoigne 1989, p. 270). Actually, mathematics here meant geometry, in Euclid's traditional version. It is because of this examination that Euclid became the proverbial "very English subject."

1.4.6 Changes in Catholic Colleges

Also in some Catholic countries, the state had initiated reforms even before the dissolution of the Jesuit order. Most noteworthy were the radical reforms occurring in Austria, enacted by the Empress Maria Theresa from the 1750s on, and the Josephine reforms, by her successor Joseph II since the 1770s. Already in 1735, the Austrian state reformed the curriculum of the *Gymnasien* through a governmental regulation of the courses: the teaching of German language and history was decreed, Greek was intensified, the method of extensive memorizing was restricted, and the textbooks were admitted and controlled by the government. Empress Maria Theresa continued these reforms in 1752: realist disciplines like arithmetic, geometry, and geography were introduced now and history and German language were extended. In 1760, a "Studienhofkommission" was established, the first state administration of education controlling all the *Gymnasien*. This was not only a step towards separation of state and Church but also the institution of school superintendence by the state. An even more radical decision was decreed in 1764: by an "Instructio pro Scholis humanioribus" (Regulation for the Humanities Schools), Austria simply abolished the *Ratio Studiorum*! The new decree was valid for all secondary schools, thus also for the *Gymnasien* run by the Benedictine and Piarist orders. Teaching of Latin was reduced again, German became the fundament of teaching in the lower forms, and mathematics and Greek were enhanced in status again. A curriculum decreed in 1775 by the *Studienhofkommission*

[3] "The mathematics teacher is not a colleague."

[4] In the nineteenth century, the name was changed to *Mathematical Tripos*.

remained valid until 1804, continuing the emphasis on the realist disciplines and the devaluation of Latin. After the dissolution of the Jesuit order, some of their colleges were closed and others were assigned to the Benedictines and Piarists. Under Josephinism, state control of schools, examinations, and textbooks as well as secularization increased. Yet, there were problems finding adapted teaching personnel and obtaining sufficient funds (Bruning 2005, p. 346). In the reactionary period at the beginning of the nineteenth century, many of the Theresian and Josephine reforms were revoked.

School reforms were also realized in other Catholic states. Those in minor or middle countries proved more successful in the long run. A remarkable case is presented by the prince-bishopric Münster, a clerical middle state in Westphalia (Germany), usually governed by the elector and archbishop of Cologne. From 1763 on, his minister Franz von Fürstenberg (1729–1810), the real governor in Münster, drove an energetic reform of primary and secondary schools, based on Enlightenment principles. In his Gymnasium reforms of 1770 and 1776, he replaced the traditional dominance of classical languages as the only major discipline with a multiplicity of major disciplines, introducing as new ones such as the German language, psychology, and mathematics. Mathematics became a teaching subject in all forms. The motivation for this new function of mathematics in his *Schulordnung* of 1770 was not, as in many Protestant motivations, via utility but via mental discipline: mathematics, he argued, presents "the most suitable way to instruct the youth in the capability to think correctly" (quoted from Schubring 2010, p. 36). The problem to find adapted teachers was solvable in this state with a small number of colleges by an extraordinary measure: Fürstenberg himself trained them. The first such teacher, Caspar Zumkley (1733–1794), proved to be a good teacher and a successful textbook author. He even became the director of the Gymnasium at Münster. The reforms in this Catholic state prepared for the Prussian reform of the *Gymnasien* after 1815 (Schubring 2010).

In addition to these reforms in some states, the expulsion of Jesuits respectively the dissolution of the Jesuit order in 1773 constituted the most profound break in the development of education in the Catholic states. At first, the Enlightenment movement in some states had resulted in the expulsion of the Jesuits from their territory. The first such state was Portugal, where the minister Pombal succeeded in decreeing this measure in 1759. This applied also to the colonies, too, hence also for Brazil. The next to follow was France in 1762 and Spain in 1667. The official dissolution by the Pope occurred in 1773. Despite the chance to organize new systems of education based on considerable now-disposable funds, most of the states were not prepared to assume these responsibilities.

In Bavaria, for instance, the elector misused the funds to endow his illegitimate sons; colleges were just entrusted to other orders without enduring or significant reforms (Hammerstein and Müller 2005, p. 348). In France, several plans were discussed, but no reform decisions taken; the major project, "Essai d'Éducation nationale" (1763) by la Chalotais, focused on the elaboration of textbooks. Here, too, other orders took over the colleges. Portugal proved to be particularly radical: Pombal realized not only the profound reform of the only university of the state at Coimbra, by – among other acts – establishing the first Faculty of Mathematics with its own professional perspective for its students, but he also dissolved the colleges and replaced them with a system of *aulas regias*, royal lectures, actually independent courses in various school disciplines.[5] In Poland, after 1773, the parliament founded the National Committee of Education (KEN) to administer the schools. The regulations issued by this Committee for provincial schools elevated mathematics to constitute one of the major disciplines, the Polish language became the language of instruction, measures for teacher education were undertaken, and a competition for composing new textbooks, in particular for arithmetic, algebra, and geometry, was organized. The competition was won by Simon L'Huilier, whose textbook was translated into Polish (Pardała 2010).

[5] See the chapter on Portugal in the next part.

1.4.7 The French Revolution

The French Revolution from 1789 constituted the culmination of the Enlightenment reform movements for education. In the following years, all colleges and universities were dissolved as hotbeds of obscurantism and corporatism. The most famous project for establishing from scratch a public education system for all was the Plan Condorcet of 1792, which provided a consecutive structure from primary schools to secondary schools and eventually to higher education and academies. Condorcet resumed the Enlightenment evaluation of mathematics by several reasons to support the "preference attributed to the mathematical and physical sciences":

- "even elementary studies of these sciences are the surest means to develop the intellectual capacities of the students", while the elementary knowledge of the other disciplines "applies the reasoning but does not form it,"
- "the literature has limits, but the sciences of observation and calculation have none,"
- they are privileged means to disseminate Enlightenment, as remedy "against prejudices, against narrow-mindedness of thinking"?
- "to bring the social order to perfection," for realizing social equality: for "substituting the ambition to dominate people by the ambition to enlighten them." (quoted from Guillaume 1889, pp. 197; my transl., G. S.)

Yet, the Plan Condorcet was never decided and then it was realized only in fragments. Establishing primary schooling proved much more complex than imagined; thus, the first ladder realized were secondary schools in 1795 in the form of *écoles centrales* – a loose collection of courses freely offered featuring a strong role for mathematics. Napoleon replaced this revolutionary approach in 1802 with a centralized system of *lycées*, based on Latin and mathematics as the two major disciplines. Teacher education was not established, for not creating corporations again (Schubring 1984).[6] The best qualified persons would be chosen in *concours* for concrete teacher positions. This new structure of public education (not accepting, however, in general the exclusion of teacher formation) served as model for other European states to be adapted in some form during the first half of the nineteenth century.

2 Mathematics Education in the Orthodox Europe

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2.1 Introduction

Countries whose dominant religion was Orthodox Christianity received this religion, along with much else of cultural value, from Byzantium; therefore, we must begin with a few words about the development of mathematics there. Our knowledge concerning this topic is rather limited, if only because many sources have not reached us. Nonetheless, despite definite achievements and certain important contributions (recall Psellos or Planudes; see Yushkevich 1970), we can assert that mathematics in the East as in the West occupied quite a small place in culture and education during the Middle Ages, including even the thirteenth to fifteenth centuries. Kapterev (2004) not without venom quotes the biographers of Theodore the Studite who claimed that his students became learned scribes, singers,

[6] See the chapter on teacher formation.

poets, and experts in melodies (p. 46). At the same time, although the number of Byzantine achievements in mathematics of any significance is quite small, the Byzantine focus on the achievements of Antiquity in itself spurred educators to include the mathematical disciplines within the orbit of studied subjects – in the schools of Maximus Planudes or George Akropolites, for example, mathematics was studied using Euclid and Nicomachus (Skazkin 1967).

Mathematical knowledge in the countries of the Orthodox world, beginning with the use of letters for indicating numbers, came from Byzantium. After the fall of the Byzantine Empire, the cultural ties within the Orthodox world did not disappear, although they sometimes did become weaker. The direction in which cultural influence flowed also changed over time: if during the fourteenth to fifteenth centuries, South Slavic and Greek influences were very important in Russia (Meyendorff 2007; Obolensky 2007), then later on – by the seventeenth to eighteenth centuries – Russian cultural influence, on the contrary, became more important (Dostyan 1998).

The causes of this reversal lay first and foremost in the fact that Russia was becoming a great power, while the other Orthodox countries were conquered by the Ottoman Empire, which undoubtedly had a negative effect on their cultural development. In the Orthodox world, the sixteenth to seventeenth centuries were not a period of bourgeois development, as in many countries in the West. Even Russia's growing military and economic strength relied only to a very limited extent on those circles which may be characterized as bourgeois. Nothing analogous to the popularity in the West of "Rechenhaftigkeit" and the corresponding publication of numerous commercial arithmetical manuals existed in the Orthodox world at this time.

Below, our main focus will be on Russia and on countries which later became parts of the Russian state but which prior to that had been parts of the Lithuanian state and the Polish-Lithuanian Commonwealth (these countries to some extent correspond to present-day Ukraine and Belarus). Therefore, below we will briefly describe the state of mathematics education in Kievan Rus', which antedated the period examined here. Since we are unable here to describe what happened in all other countries in the group being examined, we hope that what follows will help to convey at least some impression of mathematics education during this time in these countries as well, since it was sufficiently similar in character to what is described below.

2.2 Mathematics Education in Old Rus' Before the Sixteenth Century

After the Christianization of Rus' by Prince Vladimir (in 988), educational institutions for children of priests and the lay nobility were founded in Kiev, Novgorod, and other major cities. *Russkaya Pravda*, the legal code compiled under Vladimir's son Yaroslav the Wise, gives a sense of the kind of mathematics that was needed and, consequently, studied. The *Pravda* included a set of arithmetic problems – connected with assessing increases in animal stock or agricultural production over a certain period. Simonov (2007), supporting the hypothesis that the problem book was meant to be used in the preparation of clerks, expresses the opinion that students were supposed to solve these problems using an abacus. As in other countries, mathematics was used for determining calendar dates for church services. Evidence of these uses of mathematics is found in a composition by the monk Kirik Novgorodets, "Instruction: How a Man Can Learn to Reckon Years" (Polyakova 1997).

The Mongol invasion dealt a blow to education in ancient Rus'. However, in Novgorod, which was untouched by the invasion, and which had commercial ties with Western Europe, elementary mathematics continued to be taught quite widely, for example, students were taught numbering and learned to write numbers (Simonov 1974, p. 80).

At the same time, Kapterev (2004), noting the influence of Byzantium, Bulgaria, and in part Serbia on the course of Russian education, has pointed out the weakness of secular education in these countries and the preference given to spiritual-didactic instruction, which was adopted in ancient Rus'.

2.3 Mathematics Education in the Balkans During the Sixteenth to Eighteenth Centuries

Lawrence (2005) remarks that "Mathematics in Serbia was not a well-developed subject of study until the second half of the 19th century" (p. 28). Probably the same can be said about other Balkan countries, which during these years were under Ottoman rule.

Papadopoulos (2008) notes that after the fall of Constantinople, there was a "restriction (or even the elimination) of every remarkable intellectual activity in the occupied areas" (p. 118). Under these conditions, the role of intellectual leadership, including intellectual leadership in the field of education, was assumed by the Orthodox Church. As early as 1593, the Synod resolved to establish schools in the provinces (Terdimou 2003). In Greek schools, which appeared later, students systematically studied the trivium and had books for doing so (some of these schools were located outside the borders of modern Greece as well – Kastanis and Kastanis 2006). However, the church in no way supported the teaching of the exact sciences, fearing the dissemination of the "Western European spirit" (Terdimou 2003).

Only in the eighteenth century did the situation began to change somewhat. A significant role in this respect was played by Greek communities in Europe. Thus, for example, Papadopoulos (2008) speaks of Greek mathematics books printed in Venice or Leipzig during the eighteenth century. Translations of Western books were also in use. Iosipos Moisiodax (1730–1800) and Spyridon Asanis (1749–1833), both of whom were educated in Padua and Vienna, taught courses that were already quite advanced, using translations of books by the Frenchman Lacaille (Kastanis and Kastanis 2006). Subsequently, Prussian influence played an increasingly greater role, which may be explained not only by commercial ties with Germany but also by the fact that the church had a more neutral attitude towards "Protestant education [than] to that offered by Jesuits and by Catholics in general" (Kastanis and Kastanis 2006, p. 519). Fundamental changes, however, took place only in the nineteenth century.

Since it is impossible to address all of the Balkan countries in any degree of detail here, we will confine ourselves to a few remarks concerning Serbia (what occurred there during a large part of the period under discussion was similar to what could be observed in other Balkan countries, while Serbia's transformations during the second half of the eighteenth century subsequently exerted an influence on other Orthodox countries as well). An altogether elementary mathematics education likely existed in Serbia in some degree during the sixteenth to seventeenth centuries. Monasteries, as natural locuses of cultural resistance to Ottoman rule, constituted important centers. One researcher of Serbian education (Borbeviħ 1935) has written that these monasteries were not only places for public worship but places for public learning, too. Schools also appeared, in towns and cities, based on agreements with local communities. Kulakovsky (1903) sternly points out, however, that schools were practically nonexistent in Serbia, since such a label cannot be applied to temporary learning institutions that taught rudimentary skills for reading religious texts and writing (he makes no mention of counting, which clearly was not taught everywhere).

More fundamental education, including mathematics education, appeared in Serbia only in the eighteenth century (Kastanis and Lawrence 2005) under the influence of other countries. It is noteworthy that the influence of Austria, which at one time ruled a considerable portion of the country, was more beneficial to the development of mathematics education than the influence of Russia – the Serbian schools founded by the Russian Maxim Suvorov clearly focused more on teaching language (Kulakovsky 1903).

Nikolić (2010) lists three key elements in the development of mathematics education in Serbian lands that lay under Habsburg rule: the issuing of laws by the Empress Maria Theresa, which opened up possibilities for secular education; the publication of the first mathematics book in Serbian, *Nova Serbskaja Aritmetika (The New Serbian Arithmetic)* by Vasilije Damjanović (1734–1793), in 1767; and the establishment of a teacher training school named *Norma* in the free royal town of Sombor.

2.4 Mathematics Education in Russia During the Sixteenth to Seventeenth Centuries

In the course of history, the principalities of ancient Rus' became parts of different states during the sixteenth to seventeenth centuries – some of them were unified by the Moscow czardom while others, which will be discussed in a separate section, were first part of the Grand Duchy of Lithuania and then the Polish-Lithuanian Commonwealth. Much evidence has survived of the extreme paucity of education that existed in Moscow czardom, including education of a religious nature. It is telling that, as late as 1719, in the introduction to a reissue of a grammar book, the director of a sinodal printing house, writing about the state of school education and clearly well informed about it, does not even mention that students were taught to count (Kapterev 2004). Kapterev notes that "the main source of public education were… masters of letters" (p. 57), that is, literate peasants and church scribes who earned extra money by teaching people how to read (a similar state of affairs existed in the Balkans). Several times the government planned to open schools in which foreigners would teach their languages and subsequently possibly elementary science as well. Such plans never came to fruition, however: the authorities feared the possibility of social unrest that could arise from multilingualism (today we would say: multiculturalism) (Soloviev 1989, p. 373).

Even so, in certain cases elementary mathematics was taught. A manuscript from the sixteenth century entitled "Greek Merchant Accounts, Teaching Youngsters to Count," and representing effectively a multiplication table (Simonov 2007, p. 414), offers evidence of this, as well as evidence of the cultural influence of Balkan countries even on the teaching of arithmetic. Other mathematical manuscripts from the sixteenth century are known as well (Polyakova 1997).

We also have records of Russians – albeit very few of them – who received a mathematics education that was quite comprehensive by the standards of the time: one such person, for example, was Nikolay Bulev, a German by origin, a physician, and an astrologer, who got his master's degree from Rostock University (Simonov 2007). It is noteworthy that interest in astrology, and in mathematics as well, was categorically censured by one of the most learned Russians of the age, Maximus the Greek, who wrote that "those who follow mathematics philosophize that celestial bodies rule over all creatures" (quoted in Simonov 2007), subsequently accusing astrologers of demonic influences and those who follow mathematics of iniquity. Simonov (2007), however, interprets Maximus the Greek as censuring specifically astrology, while considering the study of mathematics in itself (or more precisely, the subjects of the quadrivium) as even useful. Russian *Azbukovniki* (something halfway between a textbook, an encyclopedia, and a collection of moral precepts) from the end of the sixteenth to seventeenth centuries, however, regarded the subjects of the quadrivium as parts of astrology and therefore "cursed by the holy fathers" (Simonov 2007, p. 264).

A monk from Mount Athos who had studied in Italy, Maximus the Greek was invited to Russia to translate and correct sacred texts. The learned scribe did not fail to become embroiled in the ideological discussions of the time, however, including discussions concerning the right of monasteries to landownership, and spent a large part of his life in Russia in imprisonment. Subsequently, however, his teaching and students turned out to be influential, exerting an influence – although not a direct one – on mathematics education as well.

The seventeenth century began in Muscovite Rus' with social unrest, when foreign troops took over the Kremlin in Moscow. Ties to the West over the course of the whole century were closer than before – foreign experts were invited, and young Russians were sent abroad to study as well (thus, in 1603, Hanseatic merchants took five boys away with them, pledging to teach them Latin, German, and other languages – Soloviev 1989, p. 373). Demand for mathematics education grew, as evidenced by the existence of several dozen surviving manuscripts of mathematics textbooks from the seventeenth century. In what circumstances they were used, however, can only be conjectured (Polyakova 1997; Yushkevich 1968).

Russian textbooks in arithmetic, like Western ones, taught numbers and operations involving integers and regular fractions, the rule of three, and the false position method and contained some propositions in geometry and word problems. The exposition had a clear dogmatic character – justifications were not given. Problems were usually classified not in terms of solution methods, but based on their application. Polyakova (1997) notes the careful adherence to national traditions found in these textbooks, as manifested first and foremost in their use of Russian measures and stories. At the same time, their ties to Western textbooks are easy to trace.

The textbooks used Russian-letter numerals, which made comprehension difficult – printers in Moscow started using Arabic numerals only in the middle of the seventeenth century. Hence, as Kapterev (2004) points out, it is not surprising that arithmetic was presented in the seventeenth century as "the free wisdom of farseeing and perspicacious reason, coming from God" (p. 104).

While books on arithmetic were comparatively numerous – as they were also used, as might be supposed, as reference manuals – matters stood worse with geometry. The geometric knowledge contained in arithmetic manuscripts, and in the few surviving books that are devoted to geometry exclusively, is often simply wrong, not to mention incomplete (Polyakova 1997). A manuscript that bears the title "Synodal 42" is considered an exception to this rule. Its own author describes it as a translation from English, but it is actually based on several books. The manuscript constitutes a sufficiently complete textbook in plane geometry, containing discussions of, and to a certain extent even proofs of, numerous theorems. The book was written on orders from Czar Mikhail Romanov, and it might be supposed that it was intended to be published as a textbook perhaps for some school; that was planned – but never materialized. The manuscript was not published (Polyakova 1997).

In general, schools of any level higher than the very basic started to appear in Rus' only in the second half of the seventeenth century. The most famous of these schools was the so-called Slavic-Greek-Latin Academy in Moscow, which officially opened in 1687. This school, which was supported by the Moscow government, grew out of another school that had been founded several years earlier by the Likhud brothers, Greek monks who had been educated in Greece and Italy. Mathematics, however, had no place of any significance at the Academy (and possibly was not studied at all). The curriculum did include physics, which was mostly based on Aristotle. In one way or another, the Slavic-Greek-Latin Academy, as well as other schools that appeared in Rus', and as well as all educated Russians, all came under the influence of the education that was developing in "Lithuanian Rus'" (or "Western Rus'"), which to a large extent corresponds to present-day Ukraine and Belarus.

2.5 Mathematics Education in Lithuanian Rus' During the Sixteenth to Seventeenth Centuries

While Orthodox lands in the Grand Duchy of Lithuania belonged to regions with the highest levels of culture, the situation changed radically after the formation of a unified Polish-Lithuanian Commonwealth. Titov (2003), a historian of Orthodox education in Ukraine, has complained about persistent and systematic Lithuanian-Polish propaganda. In parallel with Catholic propaganda, Protestant propaganda began to appear in the middle of the sixteenth century, and "one of [its] most important tools was the school" (p. 20). Under these circumstances, the Orthodox community found it important to increase attention to education, and here an important role was played, as in other European countries, by princes, on the one hand, and by the urban population, on the other.

Prince Konstanty Ostrogski is considered the founder of the first Orthodox school, which opened in Ostroh during the 1570s. The well-known Russian theologian and historian A. V. Kartashev (1991), however, wrote that Ostrogski was "restored" to his Orthodoxy by Prince Andrey Kurbsky (p. 600). A pupil of Maximus the Greek and one of the most notable military commanders of Czar Ivan the

Terrible, Kurbsky fled to the Polish-Lithuanian Commonwealth from the czar's wrath and became his fervent opponent. According to Soloviev (1989, p. 157), Kurbsky was "the most zealous defender of Orthodox Christianity" in the commonwealth. He bemoaned the lack of education among the Orthodox, the lack of sacred books in a language accessible to them, and in his old age began to study Latin himself in order to read these books and to translate them. It has been suggested that Kurbsky opened his own school (Kartashev 1991, p. 603), but this was likely a home school and unlikely offered instruction in mathematics.

Far from all figures in Orthodox education considered mathematics education desirable. The Ostroh priest Vasily in 1588 published a composition in which he characterized the study of geometry as "the raving of human fantasy" and ridiculed the attempt to represent the mystery of the Father, the Son, and the Holy Ghost by means of "triangular land-surveying formations" (quoted in Senchenko and Ter-Grigoryan 1998, p. 18). The programs of the Ostroh school have not survived and they have been a subject of debate. Kartashev (1991) believes that the school was organized like a Jesuit college. Kharlampovich (1897), a historian who conducted a thorough study of the school's history, considered it a type of secondary educational institution whose curriculum included a course in the liberal arts and thus mathematics. It is known, in particular, that one of its teachers was a non-Orthodox doctor of medicine and philosophy from Cracow University, Ivan Lyatos, who was not a stranger to mathematics. It is unlikely that his work at the school was connected only with languages and the Orthodox disciplines and nothing else.

From the end of the sixteenth century, schools in Orthodox regions also began to be founded by Orthodox brotherhoods, which were formed in many cities of the Eastern part of the Polish-Lithuanian Commonwealth. These schools were structured along the lines of Greek schools of the Turkish period – "Greece gave brotherhood schools their first teachers, and with them a whole basic foundation for an Orthodox upbringing and education of young people" (Linchevsky 1870, p. 108). This basic foundation, however, was no more than that – a basic foundation for learning and learning of a religious-humanistic character. In some schools, especially at first, education was limited to reading and writing. We know, however, that the charters of the Lviv and Lutsk schools required students on Saturdays to study "the Easter Computus and the lunar cycle, calendar, and reckoning" (p. 135).

Linchevsky (1870) goes even further, expressing the view that at least in some brotherhood schools students were taught arithmetic, geometry, astronomy, and other liberal arts. In support of this view, he refers to a poetic album that was presented in 1632 by the students of a Kiev school to Metropolitan Peter Mogila, in which the students briefly described all of the liberal arts (p. 129). This evidence can hardly be regarded as a conclusive proof, considering that Mogila always supported education and that it was precisely he who, shortly after receiving the aforementioned album, substantially reformed the Kievan brotherhood school on the model of the Polish Jesuit colleges, so that it subsequently became the famous Kiev-Mogila Academy.

Mathematics undoubtedly did not occupy a prominent position at the reformed school (Academy) but was still covered quite fully. In the lowest grade students were taught basic arithmetic – numbers and operations involving them. Then for several years they merely repeated what they had learned, but after that they were introduced to fractions and squares and basic algebra, with the option of covering a more challenging course that included geometry. Finally, in the upper grades, which were the ones that made the school an Academy, students were taught mixed mathematics – architecture, geometry, hydraulics, optics, and so on. It is known that by the beginning of the eighteenth century, Feofan Prokopovich himself, who later became the effective head of the Russian Orthodox Church, "lovingly taught geometry." There were other defenders of mathematics as well, who devoted 12 classes a week to mixed mathematics, in other words, as many classes as to philosophy and theology (Linchevsky 1870, p. 548).

The Kiev Academy became a key center of learning for the whole Orthodox world. Out of it emerged Moscow's most influential hierarchs and teachers (who were still sometimes accused of having become “Latinized,” by contrast with those who came from Greece). This school and others, which opened later and were modeled on it, were attended by many students from the Balkan countries (Kolybanova 1997).

Throughout its history, the Kiev Academy was supported by Orthodox princes and potentates (e.g., it received important support from the hetman Petro Sahaidachny, who graduated from the Ostroh school – Titov 2003). However, no general organized system developed around it. Such systems appeared only in the Russian Empire and at a later period.

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Chapter 8
History of Mathematics Education in East Asia in Premodern Times

Andrea Bréard and Annick Horiuchi

1 Introduction

Although the title of this chapter refers to mathematical education in East Asia, only the cases of China and Japan will be addressed here in details. This choice is due to the extent of the area concerned and the difficulty of taking into account its wide diversity. However, in this introductory note, some important features most sinicized states of the region had in common will be recalled as well as the major events they experienced during this period (fifteenth to nineteenth century).

For neighboring countries of China such as Korea, Vietnam, and Japan, which were exposed to Chinese influence for many centuries and adopted the Chinese model of bureaucracy and conception of learning, the early modern period is synonymous with maturity and autonomy. By this time, the Chinese model was no longer a simply foreign system. Each country has successfully developed a model of state that integrated Chinese elements into ancient social and political structures. However, this does not suggest that relations with China had stopped. Tributes to China's Emperor continued to be paid by most countries, China continued to be regarded as the source of innovation in most disciplines, and Chinese books were steadily purchased and studied. In all these countries, state administration leaned heavily on a stratum of literate men, trained in Confucian Classics and viewing service to the state as their mission. These men had been educated according to Chinese standards and generally used Classical Chinese as the language to produce their scholarly work. Education for them meant chiefly reading and commenting on Chinese Classics under the guidance of a teacher, following traditional commentaries. These teachers who were generally officials rarely took the risk of breaking molds and seeking new approaches. But this established order could be threatened by wars and social upheavals. Japan, for example, went through a long period of civil wars from the mid-fifteenth century up to the seventeenth century, which transformed the ancient order and propelled warriors in the foreground of the political and cultural stage. The Japanese invasion of Korea at the end of the sixteenth century was also a major event that led to a significant breakthrough in the field of education and the technical training of the literati. Wars as well as social upheavals led to an important loss of cultural legacy. But most often the greater the loss, the stronger the desire in the following period to rediscover traces of past knowledge was.

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At the same time, major events and transformations took place in East Asia during these centuries, widening the scope of knowledge and deeply transforming mind-sets. On the one hand, the mid-sixteenth century presence of Portuguese merchants in the South China Sea provided an international dimension and a significant impetus to regional trade. International trade stimulated the circulation of people and goods throughout East Asia, but measures were taken soon by most states to control circulation and exchanges. The Portuguese introduced new knowledge and new practices, particularly in the field of calculation and measurement techniques.

In many cities located in coastal zones, the development of trade led to the emergence of a vigorous and wealthy merchant class, strongly engaged in cultural and intellectual activity. The commercial printing that developed in China and Japan during this period more specifically addressed these new social strata.

Lastly, the Jesuits who reached the Japanese coast by the mid-sixteenth century soon extended their activity in China. They took an active part in conducting calendar calculations at the Imperial Board of Astronomy and inspired a wide range of scientific books, mainly devoted to astronomy and calendar science. Those books would have a significant impact on neighboring countries, especially Korea and Japan, and their influence would last until the nineteenth century. However, in the case of Japan, scholars will also seek knowledge about Western science in Dutch imported books, and Dutch studies will gain from the late eighteenth century a prominent place in many disciplines, such as astronomy.

The first section of this chapter devoted to China and written by Andrea Bréard focuses on the period covering the late sixteenth to the early twentieth century. The lower limit corresponds with the downfall of the Ming dynasty (1368–1644) and the beginning of the Jesuit influence in China. The upper limit is determined by the educational reforms that took place at the turn of the twentieth century, ending the more than 1,000-year-old traditional state examinations and introducing a modern school system based on the Western model. Confucian learning was no longer privileged, and new fields of learning were introduced with a compulsory mathematical curriculum at all levels of education.

The impact of Jesuit science on the Chinese literati can only be fully understood in light of the cultural background of the late sixteenth century. Chinese society had reached a high level of literacy by this period. Among the factors that favored this phenomenon were expansion of international trade, increasing social mobility, the wealth accumulated by local gentries, as well as the flourishing of wood-block printing and commercial publishing (see Chow 2004). In the wake of this growing aspiration for education, a new kind of textbooks emerged, intended for a wider audience. These textbooks were classical in the sense that they covered the whole range of technical knowledge literati used to be exposed. However, they differed from ancient books in that they took into account the needs of society and more specifically of the merchant class. Manuals on abacus computation are a good example of the new books that appeared during this period and constitute the bulk of materials for our study of mathematics education of this time.

The diffusion of mathematical practice throughout the society did not mean that mathematical research was prospering. On the contrary, the Ming dynasty is generally considered a time in which higher mathematics declined. Few original treatises were produced in mathematics and in calendar computations, and specialists were no longer able to understand the algebraic methods contained in the works of the mathematicians of the previous period. Whatever the reason for this decline, both China and Japan were deeply affected by the rapid growth of globalized trade and warfare which took place by the end of the sixteenth century.

During the seventeenth century, Jesuit missionaries in China introduced large portions of astronomical and mathematical knowledge and translated them into Chinese with the help of the Chinese literati. Together, they produced a number of important collections of books devoted to Western science that transformed the literati's educational training. The Jesuits' influence reached its height under the reign of Kangxi (1661–1722), whose interest in mathematics was so keen that he took private lessons from the French mathematicians who were sent to China by the King of France. However, after the mid-eighteenth century, the Chinese literati's attention progressively shifted to the cultural heritage of antiquity and philological issues. Mathematics was considered part of ancient learning and was studied with this view. Alternative systems of mathematical education emerged before the fall of the Qing dynasty in 1911. In particular, private academies and missionary schools or educational societies schools created after 1860, the end of the Opium Wars, are worth mentioning.

The second section of this chapter, written by Annick Horiuchi and dealing with Japan, focuses on the Tokugawa period (1600–1868). Except for medicine, there is no trace of original scientific production work before the seventeenth century. From the mid-sixteenth century onwards, East Asia became the stage of international trade, and Japanese society was significantly transformed by this change. Within a few decades, Japan came to occupy a leading position in the regional economy. New leaders of the archipelago, who were of warrior origin, engaged in a harsh competition to develop their domain economically and militarily. This was followed by many important technological improvements in the fields of shipbuilding, castle construction, military technologies, land surveying, navigation, farming, mining, and so on. This finally led to the unification of the Japanese territory on one hand and to a war against Korea and China on the other. Militarily, the Japanese lost the war, but on the ground they took great advantage of it, plundering the treasures of the Korean imperial library and bringing back to Japan skillful Korean craftsmen.

Mathematical knowledge benefited greatly from the development of trade in the sixteenth century. As exemplified by the *Jinkôki*, the merchant class of Kyoto had played a leading role in the rise of the mathematical tradition in Japan. At this time, Kyoto gathered into a single place the intellectual elite of the imperial aristocracy and the nascent bourgeoisie.

The lasting peace the Tokugawa brought to Japan led the military to find new occupations. A number chose intellectual activity, and Japan was soon involved in the large-scale and comprehensive enterprise of introducing continental science and technologies. From the early seventeenth century, massive numbers of Chinese scholarly books were imported and a significant portion of them were reprinted in Japan. However, due to Tokugawa's anti-Christian policy, collections of scientific treatises produced by Jesuits and Chinese literati were banned until the 1720s.

Consequently, the samurai along with the society in general were strongly sinicized and influenced by Confucian values. In Japan as well, calendar computations attracted much attention and stimulated mathematical research. From the mid-seventeenth century on, a significant number of experts in calendar computations considered it their duty to improve mathematical learning and train young talent in this science. Seki Takakazu (?–1708) and Takebe Katahiro (1664–1739) were two such experts. With these two samurai mathematicians, mathematical science reached a high standard and, as a consequence, private academies flourished throughout Japan. One important issue examined in this chapter is how this specialized knowledge was transmitted in these schools.

2 History of Mathematics Education in China

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This section outlines the different institutional frameworks within which mathematics was taught roughly from the late sixteenth to the early twentieth century in China, a time frame that covers the premodern and early modern era of Chinese history. This story cannot be written without taking into account the history of mathematical texts, instruments, and teachers. Organizing the narrative around diverse institutions permits a better understanding of the extent to which mathematical education was primarily a local affair until 1905, when the first Ministry of Education was established and centralized the nationwide curriculum, the compilation of textbooks and dictionaries, and a budget and a legal framework for teachers and the teaching of mathematics (Woodside 1983; Schneewind 2006).

In particular during the seventeenth century, mathematics brought to China and translated by European Jesuit missionaries was taught at the court. Private academies, missionary schools, and educational society schools created after 1860, the end of the Opium Wars, along with government arsenals played an equally important role in transmitting mathematical knowledge, but many mathematicians were also self-trained, relying solely on the long commentarial tradition of the mathematical classics and a flourishing book production, especially after the late Ming dynasty (1368–1644).

2.1 Teaching the Emperor: Jesuit Mathematical Instruction

As is the case in many other fields of Chinese history, archival and other sources are more complete and better preserved the closer we get to the Emperor. Thus, the Jesuit mission to China and related mathematical activities under imperial patronage are richly documented (Hart 1997). The personal teaching of Jesuit mathematics received by the Kangxi Emperor (r. 1662–1722) has been particularly well studied by Jami (1994, 2002, 2012). Following the arrival of the first Jesuit missionaries at the court at the end of the sixteenth century, some elements of European science were introduced to Chinese scholars to arouse their interest in Christianity. Mathematics gradually came to be considered not only as foundational within the intellectual movement of "concrete learning" concerned with statecraft but as a geometric model of the universe because accurate calculations were desperately needed to improve predictions of eclipses and develop a precise calendar. Mathematical education was thus highly emphasized among the literati engaged in calendar making. But for official astronomers, mathematical education remained a matter of family occupation conducted within informal modes of instruction at least until the early eighteenth century, when the Kangxi Emperor created an *Academy of Mathematics* in 1713:

> In the fifty-second year of the Kangxi reign, an Academy of Mathematics was established in the Studio for the Cultivation of the Youth.... A grand minister well versed in mathematics was put in charge of its affairs. The emperor's third son was specially designated to supervise it. Young men from the honorable families of the eight banners were chosen to study mathematics there. In addition, grand ministers and Hanlin academicians, both Manchus and Hans, were appointed to compile the *Exact meaning of the pitch-pipes* and the *Collected basic principles of mathematics*. At the end of the previous dynasty, those who were in charge of astronomy had failed in their task. They had made hardly any verifications, and there were so many mistakes that calculations and measurements went wrong and did not coincide with observations…. In the banner schools it is necessary to add sixteen instructors in mathematics. They should teach the students mathematics; from each banner more than thirty gifted students should be put to study mathematics from 1 to 5 p.m. (Quoted in Jami 1994, p. 238)

The compilations mentioned in the above quote were intended to become part of a textbook for the *Academy*, but no Jesuits were among the compilers. Their influence was more directly related to the Emperor and his personal curiosity, mainly for geometry, astronomy, and cartography. For example, symbolic algebra, taught to Kangxi by the French Jesuit Jean-François Foucquet, was not understood by the Emperor and thus was not included in the curriculum of the *Academy*, where "lines, surfaces, and volumes should each be studied for one year. The Sun, Moon, and planets should be studied for two years" (Quoted in Jami 1994, p. 231).

2.2 Education as a Private Enterprise

Outside the imperial institutions of the Ming and Qing courts, higher mathematical education played a less prominent role, although in the late eighteenth century, the mathematical sciences had begun to

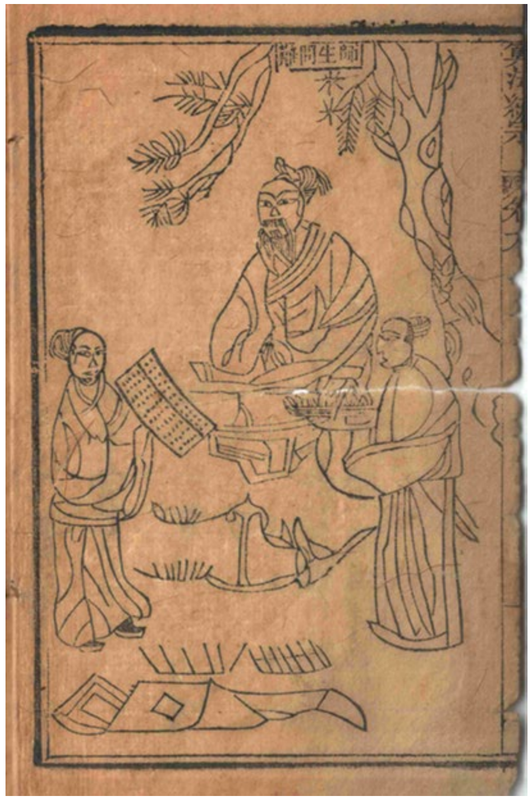


Fig. 8.1 "The Master's students asking about difficult [problems]," in (Cheng 1882)

grow in importance among the literati beyond the reach of the imperial court. Government and private educational institutions were mainly preparation grounds for the candidates in the examination system, with the private academies (*shuyuan*) gradually taking over the role of the government schools by the middle of the sixteenth century "as centers for preparing for the examinations for the higher degrees" (Hagman 2005, p. 495). To enter civil service, students had to familiarize themselves for many years with the canonical *Four Books* and *Five Classics* and their commentaries, considered as the orthodox Confucian doctrine of the time. But during the Ming dynasty, astronomy, mathematics, calendrical science, and musical harmonics, as fields of knowledge allowing an understanding of the role of natural phenomena in governance, "successfully penetrated the Ming imperial service in that they were required periodically as policy questions…of the provincial and metropolitan examinations" (quoted from Elman 2000, p. 466; see also pp. 468–481 for some examples of such policy questions and answers) (Fig. 8.1).

Unfortunately, we know little about the teaching of mathematics in government and private educational institutions before the nineteenth century, apart from what we can glean from surviving mathematical manuals. The most complete and the most popular manual, judging from the number of editions and reprints, was Cheng Dawei's (1533–1606) *Unified Lineage of Mathematical Methods*, but many other primers providing mainly memorization verses for calculations on the abacus were printed in late imperial China. As in traditional Confucian education, learning to read characters was followed by rote memorization, before finally reaching an understanding of what had been recited out loud – a three-step program corresponding with certain stages of development from early childhood to the age of around 25.

The opening paragraph of "Methods of Learning Mathematics" (*Xisuan zhi fa* 習算之法) in the first scroll of Cheng Dawei's manual outlines such a three-stage process in a mathematical curriculum, starting with the recognition of characters for number words and leading *in fine* to the comprehension of more complicated algorithms for root extraction (translated from Cheng 1990, *juan* 1, pp. 1A–1B):

1. First, one has to recite fluently the nine numbers[1];
2. One has to learn by heart the verses for the division methods;
3. One has to know how to determine the positions in addition and subtraction;
4. One has to know measures of content, length, width and surface;
5. One has to know anything concerning the numerator and the denominator in fractions;
6. One has to know length, width, heaps and volumes[2];
7. One has to know excess, deficit, mutual elimination[3];
8. One has to know positive and negative numbers, rows and columns[4];
9. One has to know the numbers for the three sides of rectangular triangles;
10. One has to know all the colors in the extraction of square roots.[5]

The multiplication table and rhymes for division thus seem to have had a special, foundational status within content to learn: they are mentioned in the first two lines of the verse as topics. Basic methods are methods one has to master fluently and know by heart, whereas more advanced mathematical methods need to be "known" or understood. In the section of his book where Cheng actually gives division verses, he underscores again the fact that these as well as memorization verses for multiplication "shall thoroughly be committed to every student's memory." The sets of concise lines for division and multiplication contain only few characters beyond the ten numerals and do not necessarily represent entire grammatically correct phrases (see Bréard 2013). Their spread and survival until nowadays seems intimately linked with the use of the abacus (*suanpan* 算盤). But jingles such as the "nine nine" multiplication verse and the "nine returns" for division were both available in earlier manuscript or printed sources. Recipes for the abacus, an instrument widely used among scholars and merchants under the Ming dynasty, were exclusively written in concise jingles and do not give any further explanation about the operations to be followed mechanically.

Xu Xinlu 徐心魯 (Xu 1573) indicates in the complete title of his *Methods for Calculating on an Abacus* (*Panzhu suanfa* 盤珠算法) that the "secret" jingles contained in his book were "handed down through family instruction" (*jiazhuan mijue* 家傳秘訣). It is doubtful that the title refers to any precise reality in the teaching of arithmetic through "secret verses for memorization." A vast number of commercial imprints from the Song dynasty and beyond carried such expressions in their titles to attract curious book buyers and readers, persuading them to believe that they will acquire knowledge hitherto reserved for a limited social circle. Family instruction, by contrast, certainly played an important role in early childhood education, as is well illustrated in Fig. 8.3 from Yu (1599), a popular household encyclopedia printed during the late Ming (reprint in Sakade 2000).

During the second half of the nineteenth century, after the Chinese defeat of the Opium Wars, the political climate for institutionalizing the learning of mathematics had considerably improved. Even then, though, about one third of the well-known late Qing dynasty mathematicians and mathematics teachers were self- or family-trained (see Table 5-3-1 in Li 2005, pp. 177–180, for the educational

[1] The expression "nine numbers" probably refers to the multiplication table up to nine times nine.

[2] The Chinese term for "heaps" refers to an accumulation of discrete objects in a certain geometric shape, whereas "volumes" refer to a continuous geometric solid.

[3] The terminology here refers to the method of double false position.

[4] Necessary to solve systems of linear equations.

[5] Liu Hui's commentary (dated 263) to the algorithm for square root extraction in the *Nine Chapters of Mathematical Procedures* makes use of colors to refer to the geometric entities in a square corresponding with the terms in the algorithm. See Chemla and Guo 2004, pp. 322–329).

background of late Qing mathematics teachers). Others followed mathematics courses in newly created missionary and government schools, supported by the Chinese government in the name of "self-strengthening," suggesting that knowledge of the sciences that underpinned Western military technology was essential if China were to defend itself against foreign steamships and guns. Thus, after the Opium Wars, foreign powers benefited from a right to create confessional schools and Westerners were allowed to teach science in government schools, even in the imperial capital. A second wave of foreign missionaries then came to China and taught Western mathematics in the Chinese language in sometimes rather isolated contexts. Edward Moncrieffe (d. 1857) was among the first Protestant missionaries to do so. Teaching at St Paul's College in Hong Kong in 1850, he found no suitable textbook, a common problem tackled later by the *School and Text-books Series Committee*, a united effort of Protestant and Catholic missionaries. Moncrieffe was probably even unaware of a scholarly translation of the first six books of Euclid's *Elements* by the Jesuit Matteo Ricci and his Chinese literati co-translator and convert Xu Guangqi, published in 1609; as Moncrieffe noted:

> I adopted the following means to supply the defect: Each day, I wrote out in English a short lesson on one or two of the subjects, which I frequently explained as well as I could in my broken Chinese. One of the boys who understood English rendered this into Chinese as well as he was able, my own teacher then corrected this and wrote it out in a book for us, after which each of the boys copied it out and learned it for the day appointed. In this way we now have in progress a kind of textbooks [sic] on Geography, English History, an abridgement of Genesis, and as far as the 8th proposition of the first book of Euclid. The latter was peculiarly difficult, as my teacher had no idea whatever of it, & the greater part of the burden fell upon myself, but I hope I have succeeded in some degree, as a few of the boys appear to have understood the subject. (Quoted in Wright 1998, pp. 659–660) (Figs. 8.2 and 8.3)

As for private academies, it was after 1870 that reform-minded officials started to modify their curricula by introducing mathematics and Western sciences into their courses. Mathematics questions were even allowed in civil examinations by an 1887 imperial edict (Barber 1888). Examination scrolls from more than 20 academies are still preserved, which allow us to analyze the questions and solutions given by the students (Li 2005, pp. 90–103). It is interesting to note that they contain parallel traditional Chinese mathematical methods as, for example, in algebra or indeterminate analysis, as well as problems on geometry, plane and spherical trigonometry, or logarithms imported from the West.

2.3 Public Institutions of Mathematical Learning

The first government institution employing a mathematician to teach his field was the Interpreters' College (*Tongwenguan* 同文館) in Beijing, founded in 1862, where science teaching was included from 1867 on. But it seems that its graduates did not face a good outlook working in the government service as interpreters or in the fields of science and technology. Li Shanlan 李善蘭, one of the self-trained mathematicians of the Qing, taught at Interpreters' College until 1882. He can be considered the first professional teacher of mathematics in China, one with a position in the Chinese government for transmitting mathematical knowledge to students, but he also contributed actively to mathematical research and the translation of Western mathematical writings. In 1880 Li Shanlan edited a collection of mathematical exercises (*Suanxue keyi* 算學課藝) together with test papers from eight students. Some exercises in Book 2, which contained problems in geometry, trigonometry, continued proportion, and summation of finite series and infinite series, were solved by the traditional algebraic *tianyuan* method. Yet this was clearly the concern of Book 3, which collected exercises from Li Ye's Song dynasty *Sea-mirror of the circle measurements* (*Ceyuan haijing* 1247). Li Shanlan used this thirteenth-century work as a textbook in one of the two courses he taught on traditional Chinese mathematics:

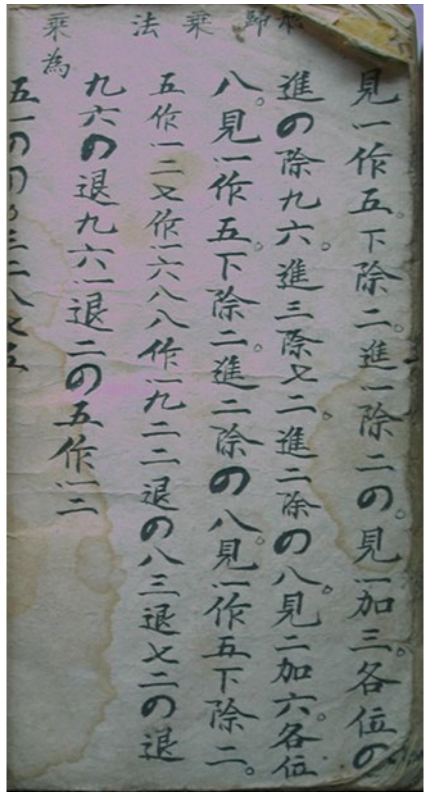


Fig. 8.2 A student's handwriting on the back cover of an eighteenth-century mathematical primer, reciting the basic jingles for multiplication and division (private collection of the author)

Fig. 8.3 Women teaching the six arts: rites, music, archery, charioteering, calligraphy, and mathematics

"Studying the *Sea-mirror of the circle measurements.*"[6] Li Shanlan clearly acknowledged his debt to the Yuan dynasty algebraic *tianyuan* method, a field which he developed further in traditional style through his mathematical writings and hoped would ensure its survival by merging Chinese algebra with Western techniques:

> Now I am teaching at the Interpreters' College using this book as a textbook. My major concern is to **synthesize** Chinese and Western methods into one method by the way of demonstrating [the Western] algebra in the *tianyuan*-method. (Quoted from the preface to Li 1880, translated in Horng 1991, p. 61)

But the courses he taught on (Western) algebra and calculus also seemed to have influenced the mathematical style adopted by his students in their published examination papers, as illustrated by an exercise asking to find the binomial expansion of $(a+x)^{\frac{1}{3}}$.

The only solution here is given in Western algebraic style using the syncretistic symbolism that Li Shanlan had developed as a translator of Western mathematical books earlier in his life (for a discussion of late Qing Chinese mathematical symbolism, see Bréard 2001).

For other kinds of problems, particularly those that appeared in Song and Yuan dynasty texts, two solutions – one using the *tianyuan* method (Hoe 1997), the other using algebra – were published. The following example stems from Zhu Shijie's *Jade Mirror of Four Unknowns* (1303):

> Now we have a General, who calls for soldiers according to a cube. On the first day each side of the cube measures 3 *chi.* Afterwards day by day the side of the cube grows by one *chi.* Each of the soldiers is accorded a daily amount of 250 *wen* cash. 23.400 men have already been called; they have obtained 23.462 *qian* [= 23.462.000 *wen*] cash. Deduce for how many days [soldiers] were called.

The answer says 15 days (translated from Li 1880, *juan* 2, p. 32a).

In Zhu Shijie's text, this originally was problem II(10)5 in the middle scroll (*juan* 中), a chapter entitled "Number of calls according to images" (*Ru xiangzhao shu* 如象招數). The problem here asks the student to establish the "celestial unknown" as the number *n* corresponding to the unknowns in the following two finite series (Figs. 8.4 and 8.5):

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left[3+(k-1)\right]^3 = 3^3+4^3+\ldots+\left[3+(n-1)\right]^3 = 23{,}400[\text{men}]$$

$$250[wen]\cdot\sum_{k=1}^{n}\sum_{i=1}^{k} a_i = 250[wen]\cdot\left\{3^3+\left(3^3+4^3\right)+\ldots+\left(3^3+\ldots+\left[3+(n-1)\right]^3\right)\right\} = 23{,}462[guan]$$

Both solution methods outlined by the students led in the end to a tabular layout of the coefficients of the fifth-order equation $12n^5+180n^4+1060n^3+3060n^2+2168n-22523520=0$, which then needed to be solved (on solution techniques of such equations see Hoe 1977).

The coexistence of teaching two mathematical styles during the Qing dynasty was not only methodological and cultural issues but also a political issue. The decision on "the difference or similarity, the superiority or inferiority of [Western] algebra and the *tianyuan* method" (Hua Hengfang's preface to the translated *Britannica* article on *Algebra* in Fryer and Hua 1873, p. 2b) reflected a rising competition between the nations and became embedded in a framework of neo-traditionalist discourse, particularly after the Nationalist Revolution of 1911. But even earlier, debates about which mathematical symbols and numerals should be used in education acquired a political dimension.

Li Shanlan's proto-grammatical system of notation was used for approximately half a century in all science texts of the Jiangnan Arsenal, "one of the earliest projects of the Chinese government to introduce Western machinery, armament manufacturing, and modern shipbuilding" (Biggerstaff 1961, chapter III, p. 166). In 1880, John Fryer (1839–1928), a secular science missionary and then head of

[6]The other course was "Studying the *Nine Chapters*"; modern mathematics was taught in Arithmetic, Algebra, and Differential and Integral Calculus.

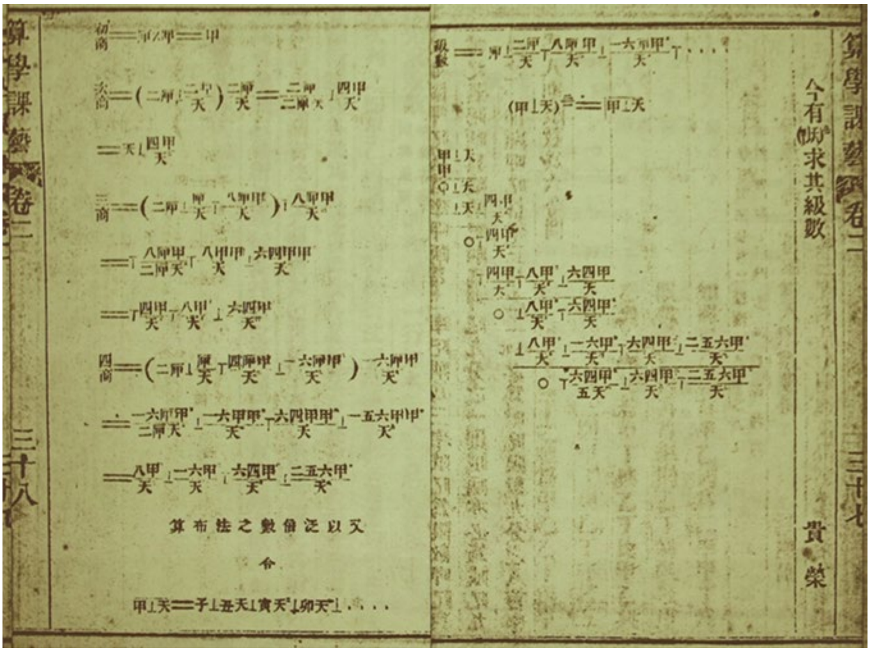


Fig. 8.4 Binomial expansion using a syncretistic algebraic symbolism (Li 1880 *juan* 2, pp. 37b–38a)

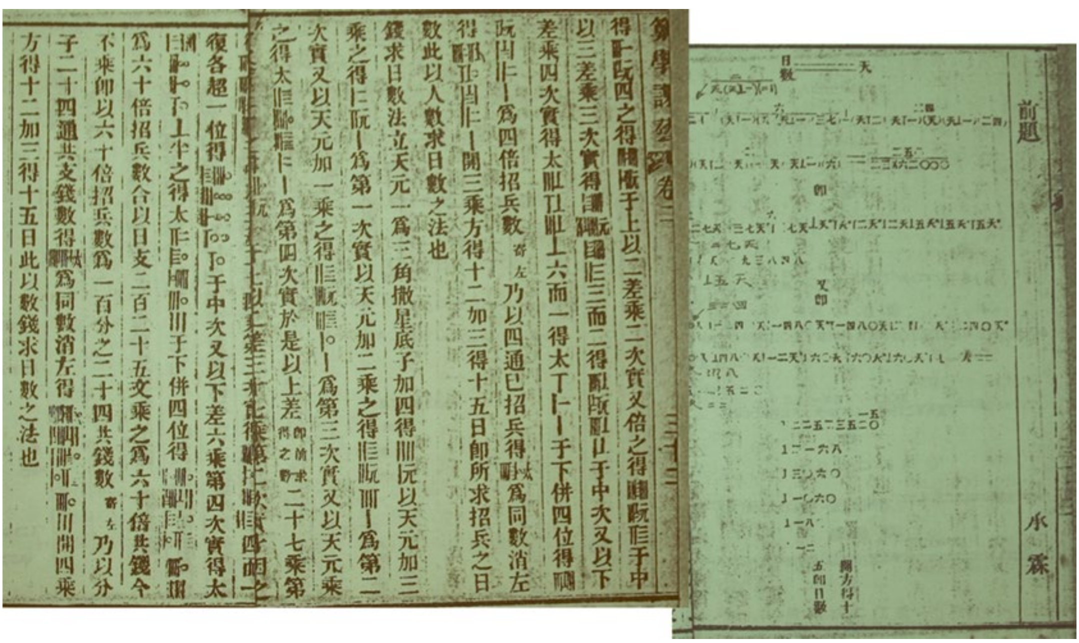


Fig. 8.5 Solution with the traditional *tianyuan* method (*left*) and the algebraic method (*right*) (Li 1880 *juan* 2, pp. 32b–33b)

the Translation Department of the Jiangnan Arsenal, defended Li's notation as being more likely to elicit approval among the supposedly conservative *literati*:

> What shall we say of those teachers of mathematics who insist on substituting the Arabic numerals for the Chinese throughout their text books ? Is not any Chinese figure, ≡ "three," for instance, every bit as easy to read, write or print as the Arabic numeral 3 ? Is there any magic charm in the Arabic figures that we must drag them into our Chinese books to suit our hobbies, and to the perplexity of annoyance of the conservative Celestial mind? (see Fryer's essay on *Scientific terminology: present discrepancies and means of securing uniformity* in Fryer 1890, p. 543)

Fryer even justified the inversion of the Western convention for fractions:

> Or still worse, what shall be said of those who would turn the mathematical world upside down, as far as lies in their power, by changing the time-honored and rational system of writing vulgar fractions with the denominator above and the numerator below? Is there any valid reason why a Chinese mathematician's mind should have to bear such a shock as to see fractions turned "topsy-turvy," just to suit the whim of a foreign professor? Is the practice of ages to be upset in this arbitrary manner? As well might all mathematical books be made to read from left to right because ours do, or to read from the bottom of the page to the top, so as to come to the numerators of Chinese fractions first! (Fryer 1890, p. 543)

Overall, Fryer at that time, and in front of missionaries, defended a diplomatic strategy that respected Chinese cultural traditions:

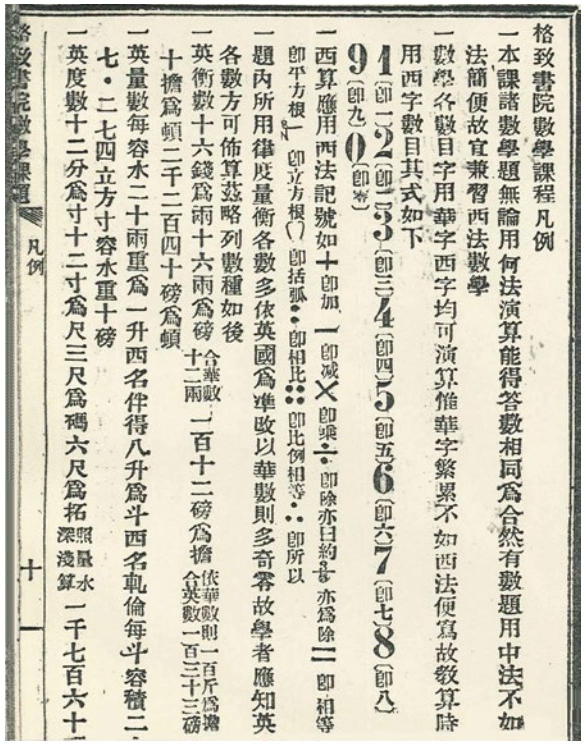
> The fact is that in all such trivial points we must be willing to sink our distinctive and conventional Western practices. We must carefully avoid standing in our own light if we want the Chinese to respect our Western learning. Our systems have no more right to universal use than the Chinese. Their ancient and wonderful language, which for some reasons is more suited to become the universal language of the world than any other, must not be tampered or trifled with by those who wish to introduce Western sciences. (Lewis et al. 1890, p. 543)

But the prevailing view among translators was that China should not be deliberately isolated by a system of notation which was not used elsewhere. The missionary educator Calvin Mateer (1836–1908), who ran the Hall of the Culture Society (*Wenhuiguan*) school in Shandong where science and advanced mathematics courses were taught in Chinese, stated:

> I differ in toto from Mr. Fryer in regard to the Arabic numerals and mathematical nomenclature. I consider that the effort to propagate in China a system of mathematical nomenclature, different from that which prevails in the whole civilized world outside of China, is to put a block in the way of progress, and greatly to retard the advancement of modern science in China. Those who invert fractions and introduce new signs into Chinese mathematics, are the theoretical men who sit in their studies and make books. They are not the practical men who are teaching mathematics in the school room. I have yet to hear that this system is used in a single school in China.[…]
>
> Why perpetuate the barrier dividing the mental life of China from that of the rest of the world? If China is to join the intellectual comity of nations, let the junction be complete, and let a Chinaman, who sees a foreign mathematical book, whose words he cannot understand, yet understand without confusion the symbols common to the race. (Cited from Mateer 1890, p. 550)

Other arguments, specifically pragmatic arguments, were brought against Fryer's position: Chinese boys learn Arabic numerals more quickly, it takes fewer strokes to write numerals, and so on. Fryer indeed seemed to have slightly shifted from his position. While directing the Shanghai Polytechnic which opened in 1876 (Wright 1996), Fryer prescribed Arabic numerals and mathematical symbols for their practicality in teaching in the *Curriculum of Western Studies in the Shanghai Polytechnic Institute.* As he stated (Fu 1895, p. 10a):

- But there are several problems which, when solved with Chinese methods, are not as simple as with Western methods. That is why it is appropriate to study jointly Western mathematical methods.
- In mathematics, for writing numbers one can either use Western or Chinese characters. But for calculations, only Chinese characters are complicated, they do not equal the simple writing of Western methods. That is why when teaching mathematics we still use Western characters for the numbers. Their pattern is as below.... (See Fig. 8.6)


格致書院數學課程凡例

一本課諸數學題無論用何法演算能得答數相同為合然有數題用中法不如法簡便故宜兼習西法數學

一數學各數目字用華字西字均可演算惟華字繁累不如西法便寫故教算時用西字數目其式如下

1(即一)2(即二)3(即三)4(即四)5(即五)6(即六)7(即七)8(即八)9(即九)0(即零)

一西算應用西法記號如 + 即加 − 即減 × 即乘 ÷ 即除亦曰約 亦為除 = 即相等 √ 即平方根 即立方根 ( ) 即括弧 : 即相比 :: 即比例相等 ∴ 即所以

一題內所用律度量衡各數多依英國為準改以華數則多奇零故學者應知英各數方可佈算茲略列數種如後

一英衡數十六錢為兩十六兩為磅(合華數十二兩)一百十二磅為擔(依華數則一百斤為擔合英數一百三十三磅)十擔為頓二千二百四十磅為頓

一英量數每容水二十兩重為一升西名伴得八升為斗西名軋倫每斗容積二七・二七四立方寸容水重十磅

一英度數十二分為寸十二寸為尺三尺為碼六尺為拓(照量水深淺算)一千七百六十

格致書院數學課程 凡例 十一

Fig. 8.6 Introducing Arab numerals in Fu (1895), the mathematical manual for the Shanghai Polytechnic Institute

2.4 Learning Mathematics After the Educational Reforms

China's examination system and lack of universal education has been viewed by Chinese reformers and foreign observers alike as the cause of China's backwardness and defeat in the Sino-Japanese war: there is no, and in reality never was, a school system in China, although a characteristic method of instruction has prevailed for ages, which because of its imitative and servile nature has repressed originality and drilled the nation into a slavish adherence to venerated usage and dictation without supplying practical or useful knowledge. Without doubt, the heart of China's nationhood to this point was her system of literary examinations, and the place given to scholars in all phases of the nation's life combined with the inefficient character of the learning they possessed were the primary causes of the nation's peculiar course and its present weak condition. Fortunately, its vitality was beginning to be reenergized by adjusting the nature of education to meeting the needs of life (*The Popular Science Monthly,* February 1906, p. 99).

Educational reforms, which ultimately abolished the examination system, established mathematics as a compulsory subject not only in the New Schools for primary and secondary education and Teachers' Schools (Cong 2007) but also in the curricula of new fields of higher learning, such as political science, economics, or law (Fig. 8.7).

In 1906, the newly created *Ministry of Education* for the first time compiled a booklist indicating which manuals were to be used nationwide at the elementary level of schooling. For mathematics, interestingly the Ministry prescribed for the first 5 years specific manuals teaching written calculation

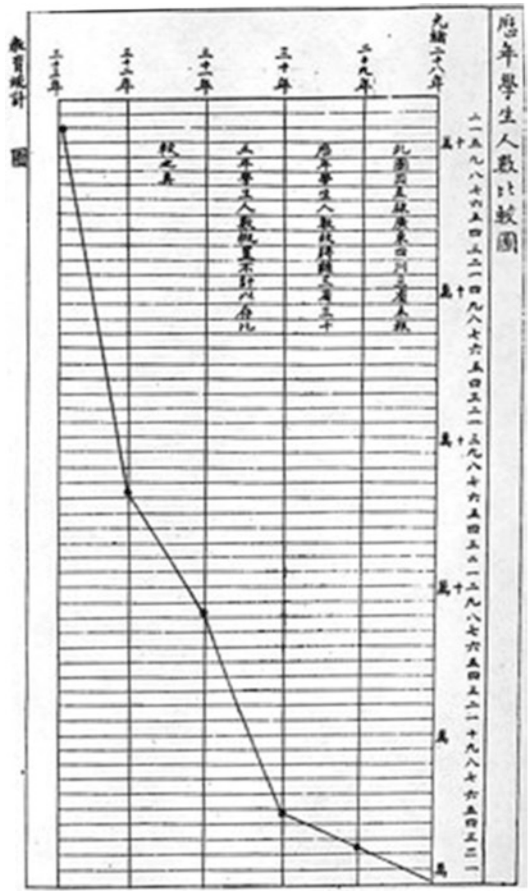


Fig. 8.7 First statistical tables of the Ministry of Education (1907), showing the rapidly growing number of students in the New Schools (from right to left)

(初等小學筆算教科書) as well as manuals on abacus (初等小學珠算入門) and mental calculation (心算教授) (Ministry of Education 1906, table 乙三). Most of the elementary education manuals were compiled by Chinese authors and provincial *Departments of Education*; whereas for secondary and higher education, teachers relied more on translations from Japanese and English-language mathematical textbooks.

2.5 Higher Mathematics Education Overseas

Since the end of the Opium Wars, opportunities to be educated abroad gradually appeared. In particular, after the Boxer Rebellion, 15 million US dollars became available for educational purposes from the remissions of the Boxer Indemnity Fund. These funds were used to send Chinese students to study in the United States after 1908. Although by then many more students went to Japanese universities with government or personal funding, the United States, followed by Europe, was the preferred destination for pursuing advanced degrees in pure and applied mathematics, science, and engineering, while many Chinese students obtained PhDs from the 1920s on. Jiang Lifu (1890–1978), for example, left China in 1910 to study mathematics at the University of California, Berkeley, and obtained his PhD from Harvard in 1919 on *The Geometry of Non-Euclidean Line Sphere Transformation*. Upon his return to China, he was the only mathematics professor in Nankai University in Tianjin, and among his many students was the later-famous geometer Chern Shiing-shen. Through this channel of transmission, students returning to China brought with them many translations of Western mathematical books that changed the content of mathematical education in modern China.

3 History of Mathematics Education in Japan

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3.1 An Example of an Elementary Textbook

There is no better representative of an elementary mathematical textbook than Yoshida Mitsuyoshi's (1598–1672) *Jinkōki* [*Inexhaustible Treatise*]. The first edition of *Jinkōki* was published in Kyoto in 1627. It was such a great success that within 20 years, Yoshida produced five substantial revisions of the book. After his death, the name and text of *Jinkōki* came to be used and reproduced freely by commercial publishers in the main cities of Japan. The popularity of *Jinkōki* never waned and revised editions of the book appeared until the end of the nineteenth century. By this time, the name *Jinkōki* was used as a synonym for "introductory manual to calculation."

In the 1620s, both commercial publishing and mathematical education were in their infancy. The *Jinkōki* achieved a breakthrough in both domains. As a mathematical textbook, it covered a whole range of problems that could be met in everyday life. As a printed book, it could hardly be surpassed. Movable-type printing had been used and talented craftsmen produced calligraphy and illustrations. The whole book was designed with meticulous care and nothing was left to chance. The author belonged to a wealthy clan of merchants and entrepreneurs of Kyoto, formerly involved in the international trade. The clan was also renowned for its printing activity and for its highly sophisticated production in Japanese literature. This favorable environment explained why *Jinkōki* contained so many qualities. This favorable environment and the book's many qualities explain why *Jinkōki* was so popular for so long (Yoshida 1977, pp. 255–271).

The 1643 edition, which was also the last edition Yoshida Mitsuyoshi supervised by himself, was composed of three chapters, divided into 19, 16, and 21 sections, respectively. The first chapter of the *Jinkōki* started with the rules of numeration and the units of weight and measure (volume, length, and area). It continued with the multiplication table, the division tables of *hassan* ("eight operations") used when dividing by a one-digit number (i.e., by 1, 2, 3, etc.), and the division tables of *ken.ichi* ("see one") used when dividing by a two-digit number (beginning, respectively, by 1, 2, etc.). These tables constituted an important component of the book. In China, the division tables were used long before the diffusion of the abacus, but in Japan where the instrument was recently introduced, they came to be closely linked to its use. To make memorization and understanding easier, Yoshida took great care with the layout of these tables. In each page, an elegant abacus with a black frame was depicted and the description of the computation rule relied heavily on these visual aids (see Fig. 8.8). As can be seen by many other examples in the book, Yoshida was well aware of the role that illustrations could play in the process of learning and memorizing, and he used them thoroughly.

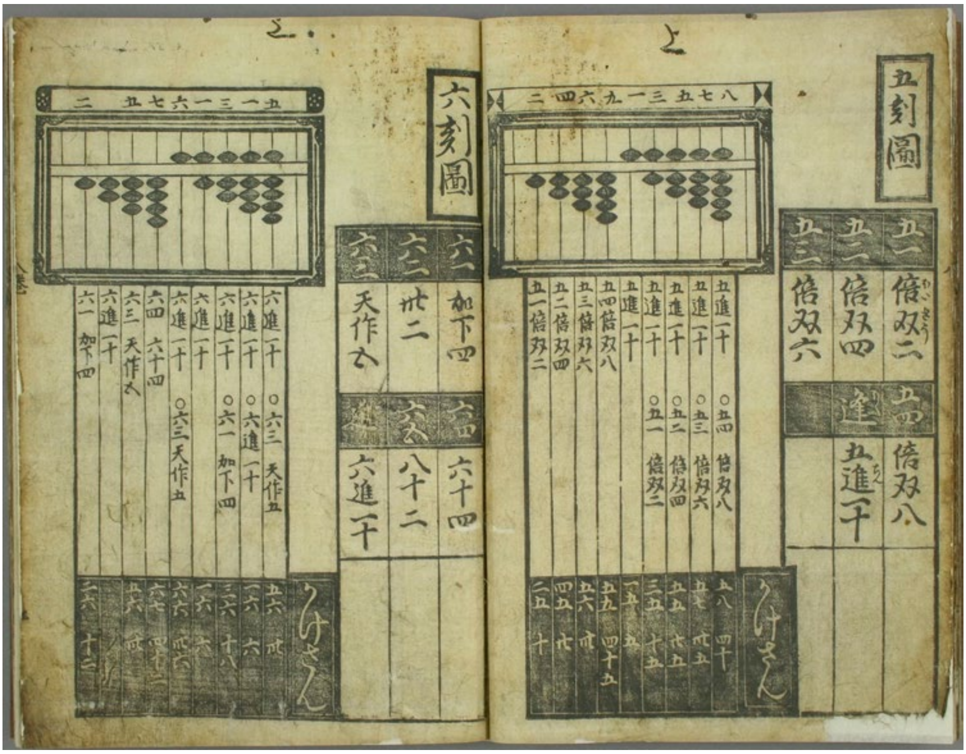

Fig. 8.8 Division tables by 5 (on the *right* page) and by 6 (on the *left*). *Jinkōki*, 1627 edition. Waseda University Library

Chapter one ended with a set of problems concerning merchant life: commercial problems, currency conversions, sums of series, measurement conversion, interest calculation, proportion calculation, land surveying, estimation of rice production, sharing out taxes, and so on (Wasan Institute 2000, pp. 65–85).

The problems in the second chapter were more directly related to the daily lives of craftsmen and officials: calculation of bundles of bamboo, estimation of the number of tiles required to cover a roof, estimation of strips of gold required to cover folding screens, calculation of the capacity of various containers given in standard units, problems of constructing new shipping routes in the countryside, and so on (Wasan Institute 2000, pp. 86–137).

The problems in the last chapter were recreational and of no immediate utility. Yoshida did not explain why he included them, but these problems had a strong visual impact on the readership and came to be intimately associated with depictions of this textbook. For example, the problem called *mamakodate* ("disposing of offspring from a previous marriage") – a variant of the problem known in the West as the "Josephus problem" which appeared in the second edition of *Jinkōki* of 1631 – took on an almost emblematic value. The picture of 30 children disposed in a circle, with 15 children born from one marriage and 15 kids from another, could be found in all subsequent editions of the book (Fig. 8.9). Likewise, *irekozan* (arithmetic of a nest of containers), *karasuzan* (arithmetic of crows: "We have 999 crows and each caws 999 times on 999 beaches. The total number of cawings is asked."), *nezumi zan* (arithmetic of mice, see Fig. 8.10), and *hini hini ichibai* ("the number which doubles every day") all had strong visual effects on the public. These problems were not Yoshida's invention. The circulation of most of these problems has been documented since the Muromachi period (1337–1573), but this was the first time they had been included in a mathematical textbook (Wasan Institute 2000, pp. 138–172).

Fig. 8.9 Mamakodate or Japanese version of Josephus problem, polychromatic impression. *Jinkōki*, 1631 edition, Waseda University Library

The use of Japanese for writing the *Jinkōki* was meaningful. The seventeenth century was a period when scholarly attention was mainly focused on ancient or contemporary Chinese learning. A great majority of scholarly publications were reprints of Chinese Classics, and many experts in medicine, pharmacopeia, or calendar science wrote their treatises in Chinese. The *Jinkōki*, as mentioned earlier, was a work by educated people, and while it could have been written in Chinese as well, the choice of the vernacular language was important for the future. The book provided a common Japanese lexicon and a common ground for computation practice. It allowed for a rapid diffusion of mathematical knowledge throughout the society and significantly widened the circle of people involved in its practice.

Because of its success, the *Jinkōki* became a model for all elementary textbooks published afterwards. The curriculum of elementary education was thus definitely established in the early seventeenth century and did not change to any degree until the mid-nineteenth century. Almost all Japanese mathematicians working after the mid-seventeenth century did their basic training with the *Jinkōki*.

The *Jinkōki* has often been described as a "popular" textbook of Tokugawa period, even as a "bestseller." But one should bear in mind that readership for this kind of technical book remained primarily confined to specialists and well-educated people, and for the great majority, learning mathematics meant learning with a master. It may thus be erroneous to consider that any commoner at this time could read and understand the *Jinkōki*. But it is also true that, as time went by, pocket editions or shortened versions of the *Jinkōki* grew in number, demonstrating that this book was reaching all segments of society.

Fig. 8.10 At the New Year, father and mother mouse give birth to 12 baby mice. It makes 14 mice including the parents. These mice, including the babies, give birth at the second moon, to 12 little mice each. It makes 98 mice including the parents. This way, once a month, parents and babies give birth to 12 little mice each. How many mice does it make at the 12th moon, is it asked. *Jinkōki*, 1631 edition, Waseda University Library

3.2 Schools of Mathematics and Communication Between Mathematicians

In the 1641 edition of the *Jinkōki*, Yoshida introduced a new practice that consisted of challenging his contemporaries with a set of 12 unsolved problems. This practice became very popular in the following decades, and mathematicians used to include a set of unsolved problems at the end of their publications. The "bequeathed problems" (*idai*), as they were called, encouraged mathematicians to enhance the difficulties of their problems and explore new fields of research.

At least five publications contained the solutions to Yoshida's problems: the *Sanryōroku* (1653), the *Enpō shikanki* (1657), the *Kaisanki* (1659), the *Sanpō ketsugishō* (1661), and the *Sanpō shigenki* (1673).

The *Kaisanki*, for instance, boasted about correcting the errors of the *Jinkōki*, the *Sanryōroku*, and the *Shosanki*. The author explained his aim as follows:

> When examining the bestselling treatise of *Jinkōki*, one notes that solutions are unnecessarily intricate and that it contains mistakes in great number. After the *Jinkōki*, books such as *Kameizan* or *Sanryōroku* have been published. They are of a better quality than the former but they still retain many errors. […] It is therefore my intention to correct the abovementioned books, to propose written solutions of the bequeathed problems and to name it «Kaisanki», 'the treatise which rectifies mathematical books'. (Nihon Gakushiin 2008, vol. 1, pp. 274–275)

This passage shows how the practice of bequeathing problems was used as a means of communication among mathematicians and as an advertising tool by each school. Mathematical schools experienced a strong growth in the seventeenth century after the publication of the *Jinkōki.* All these schools were involved in a harsh competition and each mathematician was eager to prove he was the best expert in the city. In this context, printed mathematical materials fulfilled an advertising function that could overtake the didactic one. As time went by, the space devoted to the solution of bequeathed problems increased. Thus, the single book published by leading mathematician Seki Takakazu (?–1708) contained only the solutions of 15 unsolved problems proffered by a mathematician in Kyoto. It was a very efficient way to demonstrate one's superiority over rival schools. A good mathematician did not have to publish many books. One reason Yoshida published so many revised editions of *Jinkōki* might have been that his authority was continually being challenged by other mathematicians.

The *sangaku* or "mathematical tablet" was another means of communication that encountered great success during the early modern period. The *sangaku* were painted wooden tablets containing mathematical problems that were hung in the precincts of temples and shrines. Contrary to a widespread view, these tablets were not offerings and had no religious character. Temples and shrines were simply crowded places perfectly fit for a mathematician to advertise his achievement or the achievements of his school. These tablets exhibited under the eaves of the halls could be commented on, corrected, and completed by others (Horiuchi 1998a).

The way mathematics was taught within private academies resembled the way craftsmen or artists transmitted their art, and for this reason, Japanese mathematics has often been compared to an art form. The master was regarded with great respect and his teaching was followed scrupulously. The master supervised his pupil's advance at each step of his learning. Sophisticated methods for solving problems were only transmitted to the most deserving pupils and were barely mentioned in printed books. Books were not designed as tools for teaching, but more as display windows for the school.

The contents of the teaching depended on the master's personality and skill. As far as elementary education was concerned, the *Jinkōki* curriculum was generally followed. Divergences appeared when one entered the domain of higher mathematics.

3.3 Higher Mathematics

The success of bequeathed problems cannot be isolated from the ancient Chinese mathematical texts that were being discovered at the same time. At the beginning of the seventeenth century, manuals of abacus arithmetic from the Ming period were already known to the Japanese elite. For instance, Yoshida explicitly mentioned the *Suanfa tongzong* (*Unified Lineage of Mathematical Methods*, 1592) as a source of inspiration. But the impact of Chinese manuals on Japanese mathematics became more significant after the publication in 1658 of a Japanese reprint of Zhu Shijie's *Suanxue qimeng* (*Introduction to the mathematical science*). The *Suanxue qimeng* was a relatively ancient book, published in China in 1299. It was transmitted to Japan through Korea where the book had been adopted as an official textbook of mathematics under the Chosŏn dynasty. The choice was appropriate. The book covered a wide range of topics ranging from elementary arithmetic to algebraic calculations. It was written by a mathematician of great stature, renowned for his important contribution to the development of algebraic calculation in China. The last two chapters of the *Suanxue qimeng* were, respectively, devoted to the *fangcheng* and *tianyuan* methods. The former was a general method for solving systems of linear equations, and the latter a method for solving problems using the algebraic calculation with one unknown (Kawahara 2010, pp. 61–66; Hoe 1977, pp. 41–91). These methods, which were carried out using counting rods, were immediately identified as essential tools of calculation and used for solving bequeathed problems.

Other Chinese treatises are known for having attracted attention from Japanese mathematicians. For example, the *Yang Hui suanfa* (Yang Hui's mathematical textbook), another textbook used by Korean officials, had probably played an important role in Seki's reflection on algebra. The treatises gathering principles and methods of computation for the Shoushili calendar were also intensively studied because of their interesting geometric issues. These new discoveries convinced mathematicians, especially those of warrior origin, that mathematics, especially when calculations were carried out with counting rods, was a noble activity that should be assessed separately from merchant arithmetic. These mathematicians willingly wrote their works in Chinese and boasted about following the path of their continental elders.

Seki Takakazu (?–1708) was one such mathematician. He contributed much to developing Japanese mathematical tradition and enhancing the prestige of mathematics through his deep knowledge of Chinese books and his good command of Classical Chinese. His great merit lay, on one hand, in having enriched the mathematical field with geometric issues stemming from calendar computation and, on the other hand, in having introduced the algebraic tool (*tianyuan* method) into all traditional domains of Chinese mathematics. He created an algebraic calculation called *tenzanjutsu* that transformed the *tianyuan* method into a symbolic notation that could be used in any situation, without limiting the number of unknowns. His use of the algebraic calculation was particularly successful in the domain of trigonometry. His disciple, Takebe Katahiro (1664–1739), finished his work in which trigonometric lengths were expressed as infinite series for the first time in Japan (Horiuchi 2010, pp. 241–303).

After Seki's death, a school called *Seki ryū* was established by Seki's disciple in the city of Edo. As the affiliation to this school was indispensable to access Seki's achievements, the school attracted many young talents and its reputation grew steadily over the years. It was a school of higher mathematics, probably the first one in Japan, that had the largest network of disciples throughout Japan, but it was not the only one. For example, a very famous controversy pitted Aida Yasuaki (1747–1817) against Seki's school in the late eighteenth century. Each party accused the other of publishing incorrect or roundabout solutions. Aida Yasuaki was a mathematician who had begun learning mathematics in a remote village and came to the capital to have his talent widely recognized and earn a living from mathematical teaching. His decision to found a rival school called Saijōryū ("the best school") stemmed from the humiliation he felt when the head of Seki's school at the time proposed to correct one of his solutions.

Mathematical education in Seki's school was based on his own and Takebe's achievements but also included the achievements of the successive heads of the school; for example, Yamaji Nushizumi (1704–1773) contributed the most to organize and fix the curriculum. The system of five-graded degrees that he established remained almost unchanged until the Meiji period. The highest degree called *inka* proved that this disciple possessed complete knowledge of the school to be appointed as the next head. The *inka* was only delivered to one or two disciples. The first three degrees were named *kendai*, *indai*, and *fukudai*, in reference to Seki's treatises: *Kaikendai no hō* (method for solving visible problems), *Kaiindai no hō* (method for solving hidden problems), and *Kaifukudai no hō* (method for solving concealed problems). Seki's main achievements in the fields of algebra, arithmetic, geometry, and trigonometry were thus introduced step by step, and when the third level of *fukudai* was reached, the student could use the whole range of algebraic tools: algebraic notation, transformation of an intricate problem into a system of equations, simplification of higher-degree equations, use of determinants for eliminating unknowns, solving higher-degree equations, and so on. The fourth degree called *betsuden* (special transmission) was an introduction to methods like *senkanjutsu* (Chinese remainder theorem), *shōsahō* (interpolation method), or *dajutsu* (calculation of sums of series), which were all associated with calendar computations. But the disciple had to reach the highest level of *inka* to learn the essence of all methods, especially those related to the calculation of trigonometric lengths (Mikami 1931).

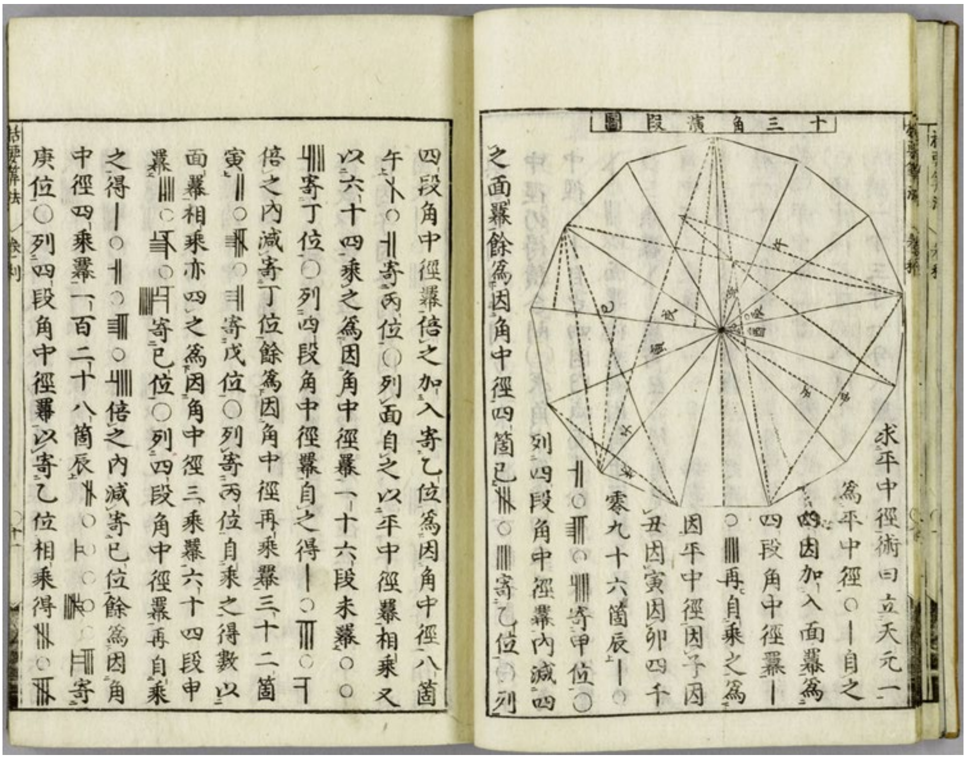

Fig. 8.11 Seki Takakazu's *Katsuyô sanpō*, 1712. Wasan Collection, National Diet Library

According to this system, the highest knowledge was supposed to be kept secret and transmitted only to a few students. But many signs indicate that as time passed secrecy was less and less respected. At the beginning of the eighteenth century, shortly after Seki's death, his major achievements were printed in a treatise called *Katsuyō sanpō* (*Compendium of mathematical art*) (Fig. 8.11). Although a novice could hardly understand its contents, the book offered a good introduction to higher mathematics. We also know that mathematicians liked to challenge each other by means of *sangaku*. Among the pupils were also mathematicians who did not wish to keep their work secret. In the early nineteenth century, Hasegawa's school, which was a branch of Seki's school, started publishing a series of textbooks that explained in detail the main chapters of higher mathematics. These didactic books contributed much to homogenize terminology and prepare for the great leap into modernity (Horiuchi 1998b).

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For Korea

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Chapter 9
Mathematics Education in America in the Premodern Period

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This chapter is devoted to the history of mathematics education in America and consists of two parts, which deal with North America (the USA and Canada) and the countries of Latin America, written respectively by Joseph W. Dauben and Karen Hunger Parshall and by Ubiratan D'Ambrosio.

1 Mathematics Education in North America to 1800

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1.1 Introduction

Although the Spanish and Portuguese were the first to colonize the New World in the early sixteenth century, French and then British efforts to settle in North America began in the sixteenth century. Successful, permanent establishment in the new lands to the north – of the British, French, and Dutch – came only after the turn of the seventeenth century. In the more southerly climes of North America, the Virginia Company of London established James Fort (later Jamestown) on an island in the James River in 1607. To the north, Samuel de Champlain founded Québec City in 1608 on the same site on

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the Saint Lawrence River where Jacques Cartier had constructed a fort as early as 1535, and Henry Hudson, working in the employ of the Dutch West India Company, claimed territory in 1609 in present-day New York State that ultimately fell to the English in 1664. These settlements were joined in 1620 by the English Plymouth Colony centered on present-day Plymouth, Massachusetts, and comprising the southeastern part of the modern state of Massachusetts. The Massachusetts Bay Colony located to the north and west followed in 1628 and encompassed the cities of Salem and Boston as well as territories that form much of the modern-day New England states of Massachusetts, Maine, New Hampshire, Rhode Island, and Connecticut.

This very brief synopsis of the early history of the colonization of the more northerly reaches of the New World suffices to make clear the diversity of nationalities of the European settlers, but they were also diverse in their motivations for leaving Europe behind. Some came as soldiers of fortune; others to avoid religious persecution; still others to proselytize and convert the indigenous peoples to Christianity. In every case, however, the religious sentiments of the colonists – Catholic in Québec and later in Louisiana, Protestant Congregationalist and Baptist in New England, Dutch Reformed in New York, Quaker and Presbyterian in the middle colonies, and Anglican in Virginia and elsewhere – affected their attitudes towards education. Although as historian of American education Lawrence Cremin (1970, p. 176) has put it, "preaching and catechizing were the forms of education most widely practiced in each of the North American colonies planted during the first half of the seventeenth century," "schooling was rarely far behind, since it was viewed by the colonists as the most important bulwark after religion in their incessant struggle against the satanic barbarism of the wilderness." The support that the colonists gave to education in general and to mathematics in particular was thus influenced by their deeper religious convictions.[1]

1.2 Elementary Education in the North American Colonies

Provisions for the education of colonial children, nevertheless, came only gradually in North America. While the Collège de Québec was founded by Jesuit missionaries in Nouvelle-France as early as 1635 and Harvard College began a year later as a Congregationalist educational initiative of the Puritans in the Massachusetts Bay Colony (see the next section), elementary education tended to be less institutionalized and more dependent on the scant intellectual and financial resources available in the far-flung colonial outposts. Where grammar schools existed, their curriculum focused on reading, writing, and spelling. Arithmetic was often omitted completely. In this, the British colonies, at least, followed the example of their homeland, where as late as 1677, one John Newton, an Oxford-educated mathematical practitioner, bemoaned the fact that he had "never yet heard of any Grammar School in England in which [mathematics] is taught" (Taylor 1954, p. 5). Whether in England or in the colonies, arithmetic was, until the end of the eighteenth century, deemed a skill needed by and so learned by those engaged in commerce. Neither the lower laboring classes nor those higher up on the socioeconomic scale required arithmetic prowess (Cohen 1982, pp. 26, 117).

Among the earliest schools in the colonies, the Collegiate School of New Amsterdam (later named New York City) traces its origins to 1628. This was a day school for boys supported by the Dutch West India Company and overseen by the Dutch Reformed Church. Its originator, the Reverend Jonas Michaelius, was "the only Dutch university graduate to arrive in New Amsterdam" and the first minister

[1] For an analysis that highlights the role of different religions in the transmission of mathematics to the Americas, see Schubring (2002).

of the Dutch Reformed Church in New Amsterdam from 1628 to 1632.[2] Although it is not clear how much, if any, mathematics may have been included in the earliest curriculum, by 1639 an official program for their colonies was promulgated by Dutch authorities back in Europe that made clear the responsibilities of the East or West Indian schoolmaster: "He is to instruct the youth—reading, writing, ciphering, and arithmetic, with all zeal and diligence; he is also to implant the fundamental principles of the true Christian religion and salvation, by means of catechizing" (Mutchler and Craig 1912, p. 220). By the end of the seventeenth century, some six schools had been established in Virginia, at least one in Maryland, 11 in New York, and over 20 in Massachusetts, all providing basic instruction in reading, writing, and ciphering (Cremin 1970, pp. 183–185). Elsewhere in the colonies, the Friends' Public School (now known as the William Penn Charter School) began operation in Philadelphia in 1689 and emphasized the *four* R's of "religion, arithmetic, writing, and reading," despite the Quakers' reputation for downplaying the importance of (especially higher) education (see below) (Woody 1920, p. 190).

During and immediately after the Revolutionary War (1776–1783), new academies such as Phillips Andover (1778), Phillips Exeter (1781), and the Deerfield Academy (1797) also sprang up in New England. Other regions of the country witnessed widespread educational developments only later, into the nineteenth century. The Transylvanian West, that is, the frontier beyond the Appalachian range, was only sparsely inhabited wilderness, outside of small population centers such as Lexington (now in Kentucky), while the largely rural South supported few schools outside of Louisiana and towns such as Charleston in South Carolina until after the Civil War in the 1860s.

1.3 Colonial Mathematics: Ciphering and Practical Arithmetic

"Ciphering," as arithmetic was termed, was a topic for the secondary schools and consisted of drilling in the basic manipulation of integers. The teachers – sometimes recent college graduates on their way to some other career, sometimes itinerant foreign scholars who earned their living while moving from school to school year after year, and sometimes the local minister – often knew little more than this. Instruction in the manipulation of fractions was rare, and the so-called rule of three – that is, the rule for how to solve for x the equation we would write as $\frac{a}{b} = \frac{c}{x}$, given integers a, b, and c – was viewed as the pinnacle of mathematics education. As historian of mathematics Florian Cajori (1890, p. 9) put it, "[h]e was an exceptional teacher who possessed a fair knowledge of 'fractions' and the 'rule of three,' and if some pupil of rare genius managed to master fractions, or even pass beyond the 'rule of three,' then he was judged a finished mathematician."

In the seventeenth century, moreover, teachers only rarely had an actual printed text at their disposal for teaching "ciphering." Rather, they tended to employ what were called "manuscript" or "ciphering" books with strings of problems worked out in detail and by hand but with no additional explanation. The teacher sets the problems to the students, who then worked them out as best they could (often on scraps of paper), checked them with the teacher against the solutions in the teacher's cipher book, and copied those solutions into their own evolving books. From a pedagogical point of view, it was a system that sought to foster an understanding of the underlying arithmetic fundamentals through a series of examples arranged according to topics of increasing levels of difficulty (Ellerton and Clements 2012, pp. 3–4). Furthermore, "[i]t was assumed that the act of writing down correct solutions to problems compelled students to 'reflect' on the structures of the problems, and made them

[2] This date is not without controversy, but, if accepted, places the founding of the Collegiate School at a date slightly earlier than that of either the Boston Latin School (founded in 1635) or Roxbury Latin School (founded in 1645). It is based on the premise that, in this period, schools were not so much founded as they founded themselves. See Cremin 1970, p. 197, and Frost 1991, p. 183.

more likely, in the future, to be able to identify problems embodying such structures" (Ellerton and Clements 2012, p. 33). Their carefully prepared cipher books were thus intended to serve as ready references for a lifetime of future use. It is important to note, however, that not all students were taught in this way. Boys were instructed in reading, writing, and arithmetic, whereas girls, when taught at all, tended to study reading, writing, and sewing.[3]

The eighteenth century witnessed the actual printing of arithmetic texts in North America (Karpinski 1940).[4] The first, *Hodder's Arithmetick: Or That Necessary Art Made Most Easy*, was published in Boston in 1719 and based on a 1661 English arithmetic by writing master, James Hodder. This and several other foreign texts had likely been the basis of "ciphering" lessons in the rare seventeenth-century colonial school, but they gradually served as texts in the eighteenth century complementary to the ciphering pedagogical tradition (Ellerton and Clements 2012, pp. 23–38).

The first arithmetic actually written by a North American author and published in the colonies was Isaac Greenwood's 1729 *Arithmetick Vulgar and Decimal: With the Application Thereof, to a Variety of Cases in Trade, and Commerce*.[5] Greenwood, the first Hollis Professor of Mathematics at Harvard College, likely composed this work for his own Harvard classroom, an indication of the mathematical expectations for eighteenth-century college students. Whereas Greenwood's arithmetic was essentially unknown to all but Greenwood's students, the same cannot be said of the popular English work by Thomas Dilworth, *The Schoolmaster's Assistant: Being a Compendium of Arithmetic Both Practical and Theoretical*. First published in London in the mid-1740s, it went through dozens of American editions beginning in 1781, making it easily the most popular arithmetic published in North America up to the early decades of the nineteenth century. Dilworth, like other writers of the day, cast his book in terms of questions and answers. For example, his treatment of the "rule of three" opens this way (quoted in Cajori 1890, p. 15):

Q. By what is the single Rule of Three known?
A. By three Terms, which are always given in the Question, to find the fourth.
Q. Are any of the terms given to be reduced from one Denomination to another?
A. If any of the given terms be of several denominations, they must be reduced into the lowest Denomination mentioned.
Q. What do you observe of concerning the first and third Terms?
A. They must be of the same Name and Kind.
Q. What do you observe concerning the fourth Term?
A. It must be of the same Name and Kind as the second. …

As Cajori (1890, p. 15) wryly remarked, "[i]t is not easy to see how a pupil beginning the subject of proportion could get clear notions from reading the above."

Not all students in eighteenth-century North America learned their mathematics in the context of more traditional school rooms, however. Following the seventeenth-century English example of mathematical practitioners who set up private schools for instruction in the mathematical arts of navigating, mapmaking, and surveying, some three dozen teachers had established such schools in the colonies in the 100 years preceding the American Revolution. Students in those schools became practitioners skilled in the requisite algebra, trigonometry, and geometry needed to ply their trades

[3] The Ursulines founded a convent and the first school for girls in North America in Québec in 1639. By the early eighteenth century, the order had succeeded in establishing schools across Canada and as far south as New Orleans on the Gulf of Mexico. Apparently, the assumption was that boys needed to study a trade, whereas girls would ultimately civilize the colony by teaching their children. An exception to the usual rule, the Ursulines taught basic reading, writing, and mathematics along with religious instruction and such domestic skills as sewing and knitting (Robenstein 1992). Girls also studied ciphering in at least four schools in Bruton Parish, Virginia, in 1724 (Ellerton and Clements 2012, p. 38).

[4] The first arithmetic to appear in French Canada was Jean-Antoine Bouthillier's 1809 *Traité d'arithmétique pour l'usage des écoles*. See Karpinski (1940, p. 174) and compare Archibald and Charbonneau (2005, pp. 150–151).

[5] Also printed in 1729 was Pieter Venema's Dutch *Arithmetica of Cyffer-konst, volgens de munten en gewigten, to Nieu-York, gebruykelyk als mede een kort ontwerp van de algebra* (Arithmetic or the Art of Ciphering, According to the Coins, Measures, and Weights Used at New York, Together with a Short Treatise on Algebra). Printed in New York, it would have served mainly as a text in private schools emphasizing commerce (see below) (Mutchler and Craig 1912, p. 226).

and, in so doing, gained a level of mathematical expertise that far exceeded what could be acquired in the regular grammar or secondary schools. As a result, "the existence of these few specialized centers for mathematical training perpetuated the segregation of the mathematical arts from general education and in that way presented an obstacle to the diffusion of numeracy" (Cohen 1982, p. 85).

This situation began to change after the American Revolution when men like Thomas Jefferson began to argue for educational reforms in the new nation. In his *Notes on the State of Virginia* first printed in 1787, Jefferson envisioned in his home state the diffusion of "knowledge more generally through the mass of the people" (p. 146). He advocated the creation of a statewide system of schools that would provide education at all levels. First, all boys would attend local schools for three years to learn reading, writing, and arithmetic. The brightest boy in each of those schools would then be selected to attend one of the twenty grammar schools that would be established throughout the state. There, the best of the best, as determined at the end of one or two years of study, would spend six more years learning "Greek, Latin, geography, and the higher branches of numerical arithmetic" (p. 146). Finally, the top half of the remaining students would proceed for three years to the College of William and Mary to engage "in the study of such sciences as they chuse [sic]" (p. 146). As Jefferson put it, "[t]he ultimate result of the whole scheme of education would be … to provide an education adapted to the years, to the capacity, and the condition of every one, and directed to their freedom and happiness" (pp. 146–147). Jefferson's plan, the germ of which may be found in the educational reforms he proposed as early as 1779 while a member of the Virginia Assembly, was never implemented in Virginia. An Education Act passed in Boston in 1789, however, required boys between the ages of 11 and 14 "to learn a standardized course of arithmetic through fractions" (Cohen 1982, p. 131). A solid grounding in elementary mathematics became increasingly central to North American educational objectives as the eighteenth century closed.

1.4 Mathematics in French and British "Colleges" in North America

In French-colonized regions, colleges functioned as *collèges* in the sense of the Jesuit curriculum formalized in the *Ratio* studiorum of 1599, that is, they were effectively secondary schools that prepared students for their further study of theology. Colleges founded by immigrants from Great Britain, on the other hand, were modeled on the Cambridge or Oxford colleges, although they first functioned as secondary schools and only later rose to higher educational levels. The modern differentiation between secondary and higher education emerged only during the nineteenth century. Both the *collèges* and the colleges in pre-nineteenth-century North America represented intermediate stages in that evolutionary process.

1.4.1 Harvard College

Although the Jesuit Collège de Québec was founded in 1635, it only began offering a full classical curriculum, including mathematics, in 1659. Congregationalist Harvard College was founded one year later on the banks of the Charles River and in the midst of cow pastures in then-rural Cambridge, Massachusetts (Morison 1936, pp. 6–7). Modeled on the English universities, especially Cambridge, Harvard had mathematics as a component of its curriculum from 1638 on. There, all students took a prescribed course in six of the traditional seven liberal arts, which included arithmetic, geometry, and astronomy; they also studied philosophy, ancient history, Hebrew, and Greek. Latin, the language of instruction, was assumed. Thus, while mathematics was not a prerequisite for entering Harvard, Latin was. As historian Samuel Morison (1936, p. 30) characterized it, mathematics education at Harvard "was distinctly weak," as was the case at "the English universities … until the age of Newton."

The situation improved in 1727 when the Englishman Thomas Hollis established an endowment of £1200 for the creation of the Hollis Professorship of Mathematics and Natural Philosophy. Isaac

Greenwood, the first incumbent in the chair and a 1721 Harvard graduate, thus "[s]ettled at last in the only position in America where a man could live by science" (Shipton 1963, p. 176). In addition to his 1729 arithmetic, Greenwood published a series of his lectures on natural philosophy between 1726 and 1734, but his productive career effectively ended in 1738 with his dismissal from the Harvard faculty for habitual drunkenness. Although he met a sad end, Greenwood was nevertheless among the first contributors to a fledgling tradition of mathematics and science in New England.

One of Greenwood's students, the 23-year-old John Winthrop, succeeded him in the Hollis chair in 1739 and continued his legacy of teaching and natural philosophical research. Observations from the Boston Common that Winthrop made of sunspots in April of 1739 and of a transit of Mercury and a lunar eclipse in 1740 comprised the first of eleven papers he would ultimately publish in the *Philosophical Transactions* of the Royal Society of London.[6] Like Greenwood, he was one of only eighteen colonial men of science to be received as a fellow in that society (Bedini 1975, p. 74). In the Harvard classroom, Winthrop, again like Greenwood, taught the elementary mathematics in which his students were, by and large, otherwise deficient, but he also introduced lectures based on Euclid's *Geometry*, John Ward's *The Young Mathematician's Guide: Being a Plain and Easie Introduction to the Mathematicks* (first published in London in 1707), and the first volume of Willem's Gravesande's *Mathematical Elements of Natural Philosophy, Confirm'd by Experiments: or, An introduction to Sir Isaac Newton's Philosophy* (published in London in 1747). Cajori (1890, p. 25) noted that it was probable that Winthrop introduced instruction in algebra at Harvard for the first time through Ward's text. It is certain that he taught the Newtonian fluxional calculus there as early as 1751.

Average students, like Timothy Pickering, a 1763 Harvard graduate and the future Secretary of State under Presidents George Washington and John Adams, found this elevated mathematical curriculum "all Greek" (Shipton 1963, p. 355). Others, like 1740 graduate Samuel Langdon, discovered that Winthrop "had the happy talent of communicating his ideas in the easiest and most elegant manner, and making the most difficult matters plain to the youths which he instructed" (Shipton 1963, p. 355). When he died in 1779, Winthrop had held the Hollis chair for 40 years. He had introduced at Harvard a mathematical course of study that can truly be considered "higher mathematics" for its day, treating as it did not only Newtonian mechanics but also the Newtonian fluxional calculus. In eighteenth-century North America, this curriculum was rivaled only by that put into place at Yale by Thomas Clap between 1743 and 1766 (see below).

Samuel Williams succeeded Winthrop in the Hollis chair. In 1787, the year before he was forced to resign in the wake of a financial scandal, the course of study at Harvard was revised to emphasize the classics more strongly. In the wake of this reform, students were still exposed to mathematics but at nowhere near the level of sophistication provided under Winthrop's tutelage. First-year students studied arithmetic, while those in their second year took algebra and other branches of mathematics. There were no mathematical requirements during the final two years of the four-year program (Cajori 1890, p. 28; Anonymous. n.d.).

While this curriculum remained in place at the Harvard of the fourth and final eighteenth-century incumbent in the Hollis chair, Samuel Webber, at least some students apparently had the opportunity to encounter the fluxional calculus under his guidance. John Pickering, son of the same Timothy Pickering who had studied mathematics under John Winthrop a generation earlier, entered Harvard in 1792 and earned his A.B. there in 1796. As he explained in a letter to his father dated 21 March 1796, he had been assigned "a part at another Exhibition … I am to exhibit solutions of certain mathematical problems, but my manner of solving them will be partly by algebra, and partly by fluxions" (Pickering 1887, p. 78). The presentation of his results a month later apparently proved gratifying. As he told his father, he "had the satisfaction to be informed that my solution of one of them was more elegant than the solution of the great Mr. Simpson, who wrote a treatise on Fluxions, in which the same problem

[6] For these details on Winthrop's life and work, see Shipton 1963, pp. 349–373.

is solved by him. I mention this that you may determine whether I have derived any advantage from going through college" (Pickering 1887, pp. 78–79). That the young Pickering's experience was the exception rather than the rule, however, is clear from the fact that in 1801 Webber published a two-volume compendium entitled *Mathematics: Compiled from the Best Authors and Intended to Be a Text-book of the Course of Private Lectures on These Sciences in the University at Cambridge*. This included algebra, plane and solid geometry, surveying, navigation, conic sections, and both spherical geometry and trigonometry, but no calculus. In the very next year, Harvard raised its admissions standards requiring, for the first time, knowledge of arithmetic.

1.4.2 The Collège de Québec and Its "Successors"

Like Harvard in New England, the Collège de Québec in Nouvelle-France served as the colonial seat of learning.[7] In 1635 when the Collège was founded, the French had only been back in the wilderness outpost on the banks of the Saint Lawrence River for three years, but they had already begun to build a "ville" there in the seventeenth-century French style. By 1663, that "ville" was home to some 500 people in 100 homes (Vallières 2011, p. 29). During the college's first two decades, the course of study lasted five years and exposed the few (scarcely more than a dozen) young boys in attendance to that part of the classical curriculum centered on grammar, reading (in Latin), and writing. Mathematical training entailed elementary arithmetic and its commercial applications. By 1659, the curriculum had fully expanded into the curriculum laid out in the *Ratio* studiorum and thus deepened to include the humanities and rhetoric as well as the intended final two years of "philosophy," that is, logic, metaphysics, physics, and mathematics (Audet 1971, 1, pp. 173–176; Archibald and Charbonneau 2005, p. 145). A century later, the mathematical coursework included instruction in the "pure" mathematical subjects of arithmetic, algebra, geometry, and plane trigonometry, in addition to such "mixed" mathematical topics as the measurement of length, area, and volume, mechanics, hydrostatics, and spherical astronomy. As early as 1661, in fact, Martin Boutet had introduced mathematical courses at the Collège geared towards surveying and navigation, skills necessary in a colony defined by the Saint Lawrence and the lands surrounding it.

As in France, where the crown had annexed "royal chairs" to selected Jesuit colleges for the instruction of engineers (see the chapter on Europe), so too in Québec with Boutet's official appointment as royal engineer in 1678. Charged with giving instruction in hydrography and piloting, he provided his students with the requisite training in geometry, trigonometry, and physics. Following Boutet's death in 1683, however, colonial authorities experienced difficulties in keeping this critical position filled. To insure the needed instructional continuity, a chair in hydrography was officially granted to the Collège de Québec in 1708 (see O'Malley 1993, 1999). It, like the Collège de Québec itself, lasted until 1759. The defeat of French General Louis-Joseph de Montcalm's forces at the hands of English Major General James Wolfe in the French and Indian War resulted in the annexation of Nouvelle-France by English Canada. As an indication of the level of scientific expertise achieved in Nouvelle-France before its fall to the British, however, the final holder of the hydrography chair, the Jesuit Joseph-Pierre Bonnécamps, like his Harvard contemporary John Winthrop, made scientific contributions recognized by the standard-bearers back in the metropole. Astronomer Jean-Nicolas Delisle presented to the Paris Académie des Sciences new and much improved calculations of the longitude of Québec made by Bonnécamps and Michel Chartier de Lotbinière, calculations critical to establishing the physical geography of the colony (Archibald and Charbonneau 2005, pp. 144–145).

[7] The Jesuits and Franciscans also established educational outposts in what would only much later become the state of California. The Jesuits founded the first permanent mission in Baja, California, in 1697 and some seventeen additional missions over the course of the next seventy years along the so-called Camino Real. The Franciscans followed beginning in 1769 and took over the work of the Jesuits when that order was suppressed in 1773 (Butler 2000, pp. 36–41).

Even after the British takeover of Nouvelle-France, Québec dominated Canada relative to instruction in mathematics. English-speaking Canada would only begin to develop in that regard at the end of the eighteenth century.[8] Following the closure of the Collège de Québec and the English ban on the recruitment of Jesuits to Canada, the Séminaire de Québec (founded in 1668 as a purely religious seminary) began offering a course of study in 1765 in Latin, French, English, and Greek. Five years later, it supplemented this instruction with coursework in "philosophy" in the classical sense, with mathematics added to the curriculum in 1773.[9]

Several theses that survive from the final quarter of the eighteenth century provide insight into the actual content of the mathematical course of study at the Séminaire. One, dated 1775, involves "material from elementary arithmetic, algebra, and the calculation of proportions" as well as "the solutions of equations in one to four variables, theorems on arithmetic and geometric progressions, and quadratic equations. There are also propositions of elementary geometry, practical geometry, and trigonometry" (Archibald and Charbonneau 2005, p. 153). By 1790, the mathematical curriculum had expanded further to include "conic sections, spherical trigonometry, [and] propositions on differential and integral calculus" (p. 153). This level of mathematical sophistication compared favorably to that of the colleges back on the French mainland.

1.4.3 William and Mary: From Grammar School to College

The second college to be established in the British colonies, the College of William and Mary, was located in Williamsburg, Virginia, the capital of the colony beginning in 1699. The first college set up outside of New England, William and Mary was, unlike its New England counterparts, chartered by the English crown and therefore strongly aligned with the Church of England. Although the charter was awarded in 1693, William and Mary only opened in 1699 and then only as a grammar school. As conceived, the college was to include the grammar school as well as schools of divinity and philosophy, that is, moral and natural philosophy. In all, the collegiate portion would be staffed by a President and six professors, among them a professor of mathematics and natural philosophy. Envisioned as "a certain Place of universal Study, a perpetual College of Divinity, Philosophy, Languages, and other good Arts and Sciences," William and Mary struggled in the opening two decades of the eighteenth century to establish the envisioned, higher courses of study (Tate 1993, p. 13).

It was not until 1711, for example, that Tanaquil Le Fevre was hired as the first professor of mathematics and natural philosophy, and he was fired for moral turpitude in 1712 after only a few months on the job. The chair remained unfilled until the Rev. Hugh Jones, who had earned his M.A. at Jesus College, Oxford, in 1714, arrived in Williamsburg in 1717 at the age of 24. Before embarking for Virginia, Jones had further prepared himself for his new post by supplementing his Oxford education with private lessons in algebra. Unfortunately, Jones's tenure at William and Mary was brief. After only four years in Williamsburg, he returned to England in 1721, again leaving the college without a professor of mathematics and natural philosophy (Tate 1993, pp. 53, 56).

Six years after Jones's departure, William and Mary adopted a set of statutes that aimed further to define the school's structure and curriculum. After successfully passing through the grammar school by the age of 15, boys would enter the school of philosophy, where they "would study rhetoric, logic,

[8] Some 40,000 loyalists to the British crown left the lower British colonies for Nova Scotia, New Brunswick, and Québec following the outbreak of the American Revolution in 1776. They transplanted their Anglican educational ideals to new institutions such as King's College founded in Windsor, Nova Scotia, in 1789 and the College of New Brunswick established in Fredericton in 1800 (Axelrod 1997, p. 6).

[9] The Collège de Saint-Raphaël (now the Collège de Montréal, a private secondary school), founded in Montréal in 1767, initiated a humanities curriculum in 1773 as well. It only rounded that curriculum out with "philosophy," ultimately including arithmetic and other basic mathematics, in the early 1790s.

and ethics under the professor of moral philosophy and study physics, metaphysics, and mathematics under the professor of natural philosophy" (Tate 1993, p. 65). The entire collegiate course of study was only two years.

The courses in mathematics and natural philosophy passed through the hands of Scottish- or Oxford-trained scholars from the late 1720s through the end of the century. Most notable among these, William Small had earned an M.A. from Marischal College, Aberdeen, in 1755 and had taken the professorship of mathematics and natural philosophy in 1758. Two years later, a young Thomas Jefferson arrived from rural Virginia to attend the college in a Williamsburg that doubled or tripled its population of two thousand during the "public times" "[i]n the fall and spring, when the courts sat and the Assembly was generally in session" (Malone 1948, p. 63). Before Jefferson's first year was completed, and through a bizarre sequence of events, Small was instructing the young Virginian not only in his designated subjects of mathematics, Newtonian physics, and metaphysics but also in ethics, logic, and rhetoric (Hull 1997). Jefferson later credited Small with giving "to his studies enlightened and affectionate guidance" (Malone 1948, p. 54). Those studies fundamentally shaped not only Jefferson's political philosophy but also the views on educational reform he expressed in the late 1770s and 1780s.

1.4.4 Yale College

Yale College was established in 1701, at least in part, as a reaction to dissatisfaction with education at Harvard. When Harvard's sixth President, Increase Mather, found himself increasingly at odds with the other Harvard clergy, whom he regarded as excessively liberal and by no means sufficiently strict ecclesiastically, he resigned in 1701. Twice more during the first quarter of the eighteenth century, the presidency of Harvard fell vacant, and twice Mather's son, Cotton, failed to get the nod. The Mathers of Boston, thus snubbed, directed their support to what they deemed the more conservative and religiously orthodox Collegiate School in Connecticut (Morison 1956, pp. 45–77).

The Collegiate School was founded by a group of Congregationalist ministers, all but one of whom was a Harvard alumnus. On 9 October 1701, they met in the seaport town of New Haven on the occasion of a meeting of the General Assembly of Connecticut and drafted what became "An Act for Liberty to Erect a Collegiate School" (Kelley 1974, p. 6). Fifteen years later, after the college had moved from Saybrook to New Haven, Cotton Mather was responsible for arranging a contribution for the construction of the first building from wealthy New Haven resident, Elihu Yale. Yale donated nine bales of goods worth £560 along with 417 books and a portrait of King George I (comparable to what John Harvard had left his namesake institution, except for the portrait). It was also Cotton Mather who suggested the change of name to Yale College.

According to Yale's charter, its purpose was to constitute a collegiate school "wherein Youth may be instructed in the Arts and Sciences who thorough [sic] the blessing of Almighty God may be fitted for Publick employment both in Church and Civil State" (Kelley 1974, p. 7). To the clergy who comprised Yale's early faculty, this seems to have meant primarily theology – construed in narrowly Calvinist terms – as the basis of all the arts and sciences. In 1722 Yale's Rector, Timothy Cutler, was fired for being too Episcopal, and tests, not abandoned until 1778, were initiated to insure that the school's instructors adhered to Calvinist orthodoxy. As late as 1818, the President and tutors of Yale College still had to affirm their basic adherence to Calvinist beliefs.

The first to leave his mark on mathematics at Yale was Thomas Clap, a 1722 Harvard graduate and Congregationalist minister who was formally installed as Rector of Yale College in 1740 and who served in that post until 1766. Clap built the first orrery in the North American colonies and advocated the "new sciences," in particular Newtonianism. Considered along with John Winthrop, Benjamin Franklin, and others as a founding father of American science, Clap reportedly underwent "a transformation of his personality" when a quadrant was substituted for the Bible in his hand (Tucker 1961, p. 57).

Clap had learned his science at Harvard from tutor Thomas Robie. A devout Puritan, Robie "carefully fed" his students "Newton in a scriptural spoon," making sure that they understood science as forging a path towards Christian truth and not towards materialism and atheism (Tucker 1961, p. 57). As a result, Clap came to revere Newton and sought to build a foundation for an appreciation of Newtonian thought in the Yale curriculum. Beginning in 1743, freshmen studied arithmetic and algebra; sophomores, geometry; and third-year students, mathematics and natural philosophy with the mathematics likely comprising trigonometry and other topics. In 1758, problems on fluxions began to appear in commencement examinations, becoming increasingly difficult over subsequent years, and in 1766 Clap was able to boast that in the second year, some of the students "make good proficiency in trigonometry and algebra," while in the third year, "some of them are considerable [sic] proficient in conic sections and fluxions" (Cajori 1890, p. 31). Clap was the only member of the staff at Yale who could teach the fluxional calculus, however, so when he left in 1766 in the wake of student uprisings over "his unswerving desire to impose old-style religious values upon a society no longer responsive to the orthodox creed," the subject disappeared from the curriculum (Tucker 1961, p. 57).

Clap's successor in the Yale presidency, Ezra Stiles, was less concerned about religion and believed that Yale should be not "a narrowly sectarian institution" but rather "one where knowledge could be pursued for itself" (Kelley 1974, p. 111). To this end, he established the college's first professorship of mathematics and natural philosophy in 1770. Nehemiah Strong and Josias Meigs were the eighteenth-century incumbents, while Jeremiah Day, who had earned his Yale A.B. in 1795, was appointed in 1801 and continued in the chair until he became the President of Yale in 1817. The curriculum he oversaw at the turn of the nineteenth century had freshmen studying from Webber's *Mathematics*, sophomores continuing with Webber and picking up Euclid's *Elements*, juniors reading astronomy and William Enfield's *Institutes of Natural Philosophy: Theoretical and Experimental* (first published in 1783 but with numerous subsequent editions into the nineteenth century), and seniors continuing the latter courses. Yale's mathematical curriculum at the turn of the nineteenth century was thus on a par with what contemporaneous English students would have encountered at Oxbridge.

1.4.5 The Academy, Then College, and Finally University in Philadelphia

The Academy of Philadelphia was founded in 1740 in the Quaker colony of Pennsylvania. Of a very different philosophy from the Puritans in New England or the Anglicans in Virginia, the Quakers in Pennsylvania believed in salvation through introspection of one's "inward light," which largely implied that the usual trappings of learning, "Latin, Greek, mathematics, and natural philosophy" were all regarded as useless (Fisher 1896, p. 60). Although Quakers appreciated the necessity of a basic education, instruction in reading, writing, and elementary arithmetic was usually deemed sufficient. They basically saw no need for higher education and distrusted the study of mathematics beyond simple arithmetic as an unnecessary distraction from what should be the focus of one's life, namely, salvation and good works. In pressing for the establishment of the Academy of Philadelphia, then, Benjamin Franklin actually insisted on a secular institution. The Academy thus began with a Board of Trustees comprised of three-fourths Anglicans and only two Quakers in a Philadelphia in the process of transforming itself into the most important city in British North America. The capital of Pennsylvania, Philadelphia boasted, a major library, botanical gardens, and, after 1743, the American Philosophical Society, the first scientific society founded in the British colonies. Franklin's Academy, which "became the College of Philadelphia in 1755 and granted its first degree two years later," thus took its place beside a number of other key educational and cultural institutions in the city on the banks of the Schuylkill River (Baltzell 1979, p. 163).

According to the school's first Provost, William Smith, the mathematical course of study in which he led those first students was ambitious:

> First year – Common and decimal arithmetic reviewed, including fractions and the extraction of roots; algebra through simple and quadratic equations and logarithmical arithmetic; first six books of Euclid
> Second year – Plane and spherical trigonometry; surveying, dialing, and navigation; eleventh and twelfth books of Euclid; conic sections; fluxions; architecture with fortification; physics
> Third year – Light and color, optics, perspective, astronomy (Cajori 1890, p. 36)
> The texts recommended to accompany this curriculum were equally impressive:
> First year – Barrow's Lectures, Pardie's Geometry, Maclaurin's Algebra, Ward's Mathematics, Keil's Trigonometry
> Second year – Patonn's Navigation, Gregory's Geometry and Fortification; Simson's Conic Sections; Maclaurin's and Emerson's Fluxions
> Third year – Helsham's Lectures, Gravesande; Cote's Hydrostatics; Desaguliers; Musschenbroek; Keil's Introduction; Martin's Philosophy, Maclaurin's View of Sir Isaac Newton's Philosophy, Rohault per Clarke (Cajori 1890, p. 37)

Although it is not clear to what extent this curriculum was actually followed, it certainly represented the most ambitious mathematical training envisioned up to that time in colonial North America. A reorganization in 1779 during the height of the Revolutionary War – and owing to Smith's outspokenness as a loyalist – dealt a serious blow to the college from which it would only begin to recover in the nineteenth century under the name of the University of Pennsylvania.

1.5 The Contours of Mathematical Education in North America By 1800

By the outbreak of the American Revolution, the colonies south of the Saint Lawrence could boast nine colleges. In addition to Harvard, William and Mary, Yale, and the University of Pennsylvania, the College of New Jersey (now Princeton University), King's College (now Columbia University), the College of Rhode Island (now Brown University), Queen's College (now Rutgers University), and Dartmouth were all founded with specific denominational roots. Princeton, like the University of Pennsylvania, was located in the middle colonies but was of Protestant Presbyterian leanings; King's, like William and Mary, was Church of England; the College of Rhode Island had Protestant Baptist leanings; Queen's was Dutch Reformed; and Dartmouth, like Harvard and Yale, was Congregational. By the time of the founding of these colleges during the middle third of the century, the die had largely been cast relative to the motivations, structures, and mathematical curricula of North American institutions of higher education.

Puritan and other Protestant stripes of colonies in New England and the middle colonies, Catholic settlements first in Québec and later in Louisiana on the Gulf of Mexico and on the West Coast, Anglican enclaves in Virginia and New York, all "depended on an educated ministry for the preservation of their religion. Learned ministers were necessary for the interpretation of the Bible. Without them error would creep in and soon the pure religion would cease to exist. This was the prime reason for their desire to found a college. But there were other motives as well" (Kelley 1974, p. 3). New colonies "could not exist … without a continuous supply of educated laymen," and, regardless of religious affiliation, mathematics was deemed, to a greater or lesser extent, a component of that education (Kelley 1974, p. 3).

After the turn of the nineteenth century, but at different rates in Canada and the evolving regions of the United States, mathematical education more distinctly differentiated into elementary, secondary, and postsecondary levels. By the last quarter of the nineteenth century, secondary and higher education had distinctly separated, and true universities that emphasized teaching, the production of original research, and the training of future researchers had emerged (see Archibald and Charbonneau 2005; Parshall and Rowe 1994). Those nineteenth-century developments built (see Parshall and Rowe 1994; Dauben and Parshall 2007a, b), however, on the educational foundation laid prior to 1800.

2 Mathematics Education in Latin America, in the Premodern Period

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2.1 Introduction

This section covers both secondary and higher education in the period preceding the independence of Spanish and Portuguese colonies in Latin America. This period has been very poorly researched for all school subjects and particularly for mathematics. Brazil has been relatively better studied than all Spanish America. While some sources and studies about colonial education (mainly religious sources) are available, they generally contain little or no reference to mathematics education. Some sources on higher and informal education which are also available will be reviewed below.[10]

The conquest, colonization, and early independence of the Americas had an enormous influence on the development of the civilization. The cultural encounters between the Old European World and the New American World profoundly affected each. Chroniclers of the conquest offered absolutely different explanations for the cosmos and the creation and for coping with the surrounding environment. The religious systems, political structures, architecture and urban arrangements, sciences, and values of the native peoples in the American World were, within a few decades, suppressed and replaced by those of the conqueror. Much of the remaining original behavior and knowledge of the conquered cultures were outlawed, and today remain treated as folklore only, but they clearly reflect the cultural memory of the peoples descending from the conquered. Many of these vestiges of behavior and knowledge are easily recognized in everyday life. Hence, a new approach to historiography is fundamental for the World History of Science and Technology by recognizing the contributions of non-Western cultures.[11]

The early conquests of Latin America paved the way to colonization. Spanish and Portuguese colonizers were soon challenged by the French, the English, and the Dutch, who also established colonies in the New World. Soon, the colonies received immigrants from Africa in the form of slaves as well as some Asians and many Europeans. They brought new ways of coping with the environment, of dealing with daily life, and of explaining it and learning. The result was an emerging synthesis of different forms of knowing, doing, and explaining. These new forms were generated in different communities and were available to workers and to the general populace. This is clearly seen in the emergence of modified religions, cuisine, music, arts, and languages. All new forms of knowledge and behavior were intrinsically interrelated and synthesized the cultural forms brought by the outsiders and those retained by the natives.

Particularly in the Americas, the variety and peculiarity of the cultures and the specificity of the migrations reveal the colonizers' efforts to transfer, with minor adaptations, the forms of social, economic, and political organization and administration prevailing in the metropolises, including schooling and scholarship (academies, universities, monasteries). The new institutions in the Americas were based on these dominant metropolitan styles, which were mostly under the influence, and even control, of religious orders.

[10] This essay is based on my paper D'Ambrosio (2006). I recently published a book that, although focusing Brazil, has references to the transmission, acquisition and diffusion of mathematics in Latin America: D'Ambrosio (2008).

[11] See a discussion on these issues in D'Ambrosio (2000).

All these changes, spanning the sixteenth, seventeenth, and eighteenth centuries, occurred while new philosophical ideas, new sciences, new means of production, and new political arrangements were flourishing in Europe. Cultural facts produced in Europe were assimilated in the colonies under specific, mostly precarious, conditions. Indeed, the colonies were the consumers of some of these new cultural facts. There is a clear coexistence of cultural goods, particularly knowledge, produced locally as well as abroad.

Although this transmission is a question affecting relations between academia and society in general – hence between the ruling elites and the population as a whole – the dynamics of these relations between social strata are particularly important for understanding the role of intellectualism in the colonial era. Thus, ethnomathematics becomes as a fundamental instrument for historical analysis.[12]

2.2 The Preconquest Situation

Information about pre-Columbian civilizations has mostly relied on chronicles of the conquest, which describe many specific facets of the conquered peoples, such as Mayan stelae and writing, Peruvian quipus, Aztec daily life, and, in general, their vision of the universe. These chronicles were interpreted to justify the so-called civilizatory mission during the establishment and consolidation of the colonies. Such interpretations were biased and understandably failed to identify and recognize any form of mathematical knowledge in these cultures. The emergence of new scholarship permits a new reading of the chronicles and a new understanding of the complexity of knowledge systems and societies of pre-Columbian civilizations.

Scholarship in the history of mathematics in the traditional cultures of the New World is intensifying. Although current mathematics is fully integrated with World mathematics in all countries, demands of a social and psychological nature justify greater attention to the traditional roots of native mathematics. For the pre-Columbian period, sources are available primarily for the Aztec, Mayan, and Incan civilizations. To look at other civilizations, mainly of the Northern and Southern prairies and tropical cultures in the Amazon Basin, an enlarged concept of sources is needed, mainly drawn from anthropologists. Much finer divisions of these civilizations, taking into account both political and cultural specificities, and new methodologies are needed for a special study of pre-Columbian mathematics. This would resemble the process of studying traditional African cultures.

Education in pre-Columbian cultures differs throughout the various cultures of the continent. The conquerors' imposition of their priorities for production and services profoundly affected preexisting models of education. The heterogeneity of cultures makes it practically impossible to refer to one pre-Columbian educational model. For example, in the lowland cultures of hunter-gatherers, initiation rites were central in the preparation of new generations, compared with complex cultures in the highlands. In the former, education focuses on clarifying the roles of women in caring for children and plantations and of men in hunting and warfare. Other strategies would identify those individuals who would comprise the ruling class of chiefs and shamans. In the highland cultures, such as the Aztecs and the Incas, education was far more elaborated.

The colonial regimes made a clear effort to ignore or obliterate any sense of the history and achievements of these native cultures. Today, we are faced with the difficult task of reconstructing the histories of these cultures, both through examining the chronology of the events and understanding the important migratory currents that shaped their development. But our knowledge of the pre-Columbian period is still very incomplete. Of course, this leads us to look at the conceptual framework of what might be considered in today's categories of Science, Technology, and Mathematics in pre-Columbian times.[13]

[12] See D'Ambrosio (1995, 1996).

[13] For a brief introduction to this theme, see D'Ambrosio (1977).

Fig. 9.1 Accountant/treasurer, in charge of the quipus, from the book of Poma de Ayala (Image 360)

The Incas – whose capital was Cuzco, now a city of modern Peru – manipulated a complex texture of knotted strings called *quipus* either to record narratives or to conduct accounting. Its actual use in the Incan civilization has still not been clarified, but the *quipu* clearly contained qualitative and quantitative information. For calculation, the Incas also used a device, which was a combination of arithmetical table and abacus, called the *yupana*. Both are clearly illustrated in the well-known picture of Poma de Ayala (1987), a chronicler of the sixteenth century (Fig. 9.1).

A detailed analysis and classification of Peruvian *quipus* were accomplished by mathematician Marcia Ascher and anthropologist Robert Ascher, with the 1981 publication of the *Code of the Quipu*, which became a classic in the field (Ascher and Ascher 1997). An important source for understanding pre-Columbian cultures is the current practices of extant native communities. Of course, these practices are the result of the cultural dynamics of the conqueror/conquered encounters but nonetheless reveal much of the knowledge and behavior of the ancestors.[14]

2.3 The Conquest

Latin America was colonized starting in the sixteenth century, mainly by Spain and Portugal, as part of the expansion of mercantilism and commercial capitalism (Baumont 1949; Dobb 1954; Heckscher 1955). This explains why Spanish and Portuguese are the main languages spoken in Latin America, except for a few surviving native languages, particularly those spoken by the Incas and Mayas, as well as French in French Guiana, English in Guyana, and Dutch in Suriname.

[14] An important survey of pre-Columbian Mathematics of extant cultures is the book by Closs (1986).

The Spanish conquest of the Americas began in 1492 with Columbus. In the period 1519–1521, Cortéz conquered the Aztec Empire, followed by the much longer and harsher conquest of the Yucatan Peninsula. In 1540, the viceroyalty of New Spain was established, followed 2 years later by the creation of the viceroyalty of Perú, which had been triggered by the difficult conquest of the Inca Empire by Pizarro in 1529. Some 200 years later, Spain established the viceroyalties of New Granada and Rio de la Plata, with capitals in Bogotá and Buenos Ayres, respectively.

Brazil, with an area approximately equal to 47 % of the total area of South America, was discovered in 1500 by the Portuguese. Contrary to the situation in Spanish America, with its initial bounty of silver and gold, Brazil was at first a disappointment to its "mother country," which decided to settle the colony mainly to avoid its takeover by France, Spain, England, and Holland. Brazil was a relatively poor colony till 1659, when gold and diamonds were discovered (Boxer 1962).

The colonial times that followed involved the exploitation of the lands, resources, and peoples of the conquered regions. The colonizers brought with them traditional European agricultural and mining techniques for which they exploited the native production, mainly in metallurgy. The colonizers changed the means of production, to a large extent indirectly. The native religions were simply destroyed and food habits were considerably modified. Wheat, rice, coffee, citrus, sugarcane, and bananas were all grown for export.

In the sciences in general, Latin America was the recipient of scientific advances. In the process of cultural dynamics, this knowledge was modified and adapted to a new reality. Such peripheral science was maintained throughout independence. Indeed, the colonial style and submission of the native population, with land distribution determined by the conquerors and colonizers, was retained after independence. Independence was a movement impregnated with republican ideas led by *creoles*, not necessarily an aspiration of the native populations, throughout the three Americas. Even after independence, education was modeled on the former imperial system. Colonial science, which contributed to the mainstream of scientific development, was at best modest.

2.4 Early Colonial Times

Spanish America and Portuguese America followed parallel but distinct historical paths. While Portugal never allowed the establishment of universities in its colony so that the prevailing form of education consisted of *colégios* (secondary schools organized by the Jesuits), numerous universities were founded in the Spanish colonies. Gregorio Weinberg (n/d) explains the reason for the different educational structures as the different ethos of the colonial enterprise. While the Portuguese had exclusively economic interests, the Spanish aimed, from early periods of conquest, to establish complex societies in the New World, with stable social and cultural Christian bases. Some historians refer to this as the concept of the "reconquest" of the New World for Christianity, similar to interpreting the end of the Islamic rule in Spain as the reconquest of the Iberian Peninsula for Christianity.

Publications on the universities used to be of a general historical character, without any details on mathematics. Despite some studies on the Brazilian *colégios*, few materials are available on schools in the Spanish colonies.

Since Latin America was colonized by two Catholic countries, strongholds of the Catholic counter-reformation, the Church had an important role in the educational systems of the two empires. In Spanish America during the colonial period, several religious orders (Franciscans, Augustinians, Dominicans, among others) maintained reading and writing schools, in which rudiments of arithmetic were also taught. In their monasteries or mission houses, they provided "secondary" schools for their prospective monks or priests as well as for laymen who could pay for this education.

The first Dominican missionaries reached the West Indies in 1510 and founded the first American province in 1530. They were also the first to establish a university in the Americas: in Santo Domingo, on the island of Haiti, in 1538. This initiative stemmed from their express intention to treat the subjected indigenous population as having equal rights with the conquerors. In 1562, they set up a mission in the *Nuevo Reino de Granada* (New Kingdom of Granada, roughly present-day Colombia and parts of Venezuela), which later became the viceroyalty of New Granada. The Franciscans are reported in the Spanish domains in North America starting in 1519, with Cortéz, the Spanish conqueror of Mexico (Habig 1944a, b, 1945). According to Muñoz and José (1918, pp. 64–67), the Dominicans and the Jesuits were the most important religious orders in Chile, where they provided, as in other regions of Latin America, secondary and the equivalent of postsecondary education starting from the end of the sixteenth century.

In Mexico, as early as 1523 – 2 years after the fall of the Aztec Empire – the Franciscan monks founded the first school on the continent, with the purpose of teaching the Indians. In 1527, they also founded the *Capilla de San José de los Naturales*, which provided primary education for children and courses in the arts and professions for adults. In addition, very early on in Portuguese America, the Jesuits immediately upon their arrival in 1549 took up the task of opening schools for the Indians.

Schools were not open to every boy; students were children of European immigrants and of the indigenous elite. The main objective of education was recruitment for the religious orders and for public services. Students learned reading, writing, and counting and became fluent in Spanish or Portuguese, Latin, and the native languages.

Identifying books or notebooks of early colonial times has proved impossible so far. Some documents refer to the qualifications of being a teacher. For example, instructions from around 1600 read: "the one who will be a teacher will not be a negro, mulatto nor Indian, but only an old Spanish Christian, [...] must know the five rules of counting, algorisms, adding, resting, multiplying, half break, and break by integer, and all the other necessary counting, and adding count in Castilian, as algorisms" (Weinberg n.d., p. 64).

Significantly, some instructions caution that the education of natives should be restricted to religious matters because all the revolutions have been led by instructed Indians. In other words, educated Indians might be dangerous.

After the first university of the Dominicans, more universities were founded as were Royal Spanish ones: in 1551, in Lima (Peru) the *Universidad Nacional Mayor de San Marcos*, which originated as a school run by the Dominicans, and in 1553, in Mexico the *Real y Pontificia Universidad de México*. Its school of arts was supposed to have a chair of mathematics, which was actually established only in 1637 as a chair for astrology and mathematics, in the charge of Diego Rodríguez of the order of the Blessed Virgin Mary of Mercy (Alejandre 2005, pp. 12; 60; 71–72).

Among further foundations were those at Bogotá (1576) and Quito (1586). It is said that all these foundations adhered to the structure of the major Spanish university in Salamanca. Even newer histories of this university do not investigate the role and history of mathematics there; thus, one has to infer that the Spanish American universities applied a largely medieval model of collegiate studies with its predominance of propaedeutical functions and only a marginal role for mathematics and the sciences (Steger 1965, p. 37 ff.).

According to Jilek (1984) and Roberts, Rodriguez, and Herbst (1996), 6, 12, and 11 universities were created in Spanish America in the sixteenth, seventeenth, and eighteenth centuries, respectively. The purpose of these institutions was "the education of the clerical and secular colonial elite" (Roberts et al. 1996, pp. 231f). The universities offered theology, law, and medicine courses.

One important difference between Portuguese and Spanish America was the overwhelming dominance of education in Brazil by the Jesuits. They first arrived in 1549, in Portugal's first serious attempt to establish itself in the new colony. In Chile, by contrast, the Jesuits arrived only in 1593, at a time when the order had already opted to concentrate its efforts in secondary and postsecondary education. In Brazil, until their expulsion from the Portuguese empire by Pombal in 1759, the Jesuits had the best organized schools.

Fig. 9.2 Cover of the book *Sumario Compendioso*

Originally, the Jesuits aimed to establish reading and writing schools to convert the Indians to the Catholic religion. Soon, the schools were opened for both Europeans and native non-Indian students. However, by the end of the sixteenth century, the Jesuits in Brazil were obligated to obey the general direction as decided by the *Ratio Studiorum* of 1599, which provided an option for secondary and postsecondary education. With schools restructured along the lines of the *Ratio Studiorum*, the Jesuits created schools (*colégios*) to teach the "liberal arts" to the sons of the landed gentry, thus changing the social focus. After having passed such schools in Spanish America, the future elite went on to one of the local universities or to Salamanca. In Portuguese America where there were no universities, the students went to Coimbra in Portugal to study law and theology or to Montpellier, in France, to study medicine (Carvalho and Dassie 2012, p. 500).

Although not much is known about the concrete functioning of the Jesuit colleges, one can safely assume that they applied the *Ratio Studiorum* as their curriculum; thus, mathematics was taught as a marginal subject only, in the last year of attendance, as a part of the teaching of physics. This included the beginnings of Euclid's geometry and some popular astronomy, instructed generally by padres who were not specifically trained in mathematics.[15]

Given the marginal role of mathematics within general education then, practical and professional needs typically demanded and entailed a more intense cultivation of certain aspects of mathematics. At first, the needs were mainly commercial, which affected publications on mathematics. For example, the first nonreligious book published in the Americas, which was very useful for the colonizers, was a book on arithmetic related to mineration, the *Sumario compendioso de las quentas de plata y oro que en los reinos del Pirú son necessarias a los mercaderes y todo genero de tratantes. Con*

[15] See the chapter on premodern times in Europe. Due to the Jesuits' strong interest in astronomy, several works and publications were written by Jesuits in Latin America on astronomical issues, particularly comets.

algunas reglas tocantes al arithmética, by Juan Diez Freyle, printed in Mexico, New Spain, in 1556 (see Smith 1921) (Fig. 9.2).

More such practical books were published during the first 100 years after the conquest. Mexico was a pioneer in printing in the New World, and not only for Bibles and religious books. Yet in 1554 Alonso de la Vera Cruz published his *Recognitio, Summularum* (Burdick 2009). Later, in the last third of the sixteenth century, Juan de Porres Osorio wrote about the division of the circumference and the construction of polygons of 36 sides. Pedro Paz wrote the *Arte menor de arithmetica* (1632), and Atanasio Reaton wrote *Arte menor de arithmética y modo de formar campos* (1649). Diego Rodriguez studied equations and imaginary numbers and applied logarithms to astronomical calculations. These studies were not published and remained manuscripts (Trabulse 1994, pp. 187f.; Alejandre 2005, pp. 60–66). Many more mathematicians revealed the vitality of the mathematical sciences in Mexico (*Nueva España*). These and other authors were studied by Elias Trabulse in a series of books and papers (see Trabulse 1985).

Diego Rodriguez was the first professor with chair of astrology and mathematics at Mexico University. There is much information about this chair and its functioning, and all of its subsequent professors are known. It in fact constituted the kernel for the evolution of mathematics in Mexico. The professorship was lifelong; when the position was vacant, a successor was determined by a public *concurso*; its exam, however, had to be on the *Sphera* of Sacrobosco – a classic of astronomy since the thirteenth century. The lectures of the chair were compulsive for students of the arts and of medicine. In fact, the name of the chair is explained by the fact that physicians should be able to calculate astrologically the adapted dates for their medical practices. Lectures were first given as dictates in Latin and were later on changed to Spanish. In 1668, Rodriguez's successor was Carlos de Sigüenza y Góngora (1645–1700), who excelled in astronomy and later became Royal *Cosmográfo* (Alejandre 2005, pp. 71 ff.). Other *catedráticos* in the eighteenth century were Joaquin Velazquez de León (1732–1786) and, as his substitute several times, José Ignacio de Bartolache (1739–1790) (p. 96).

As in several European countries, the practical needs of the state promoted at least applied aspects of mathematics. In Brazil, for instance, to defend the long coast against other colonialist rivals, numerous fortifications had to been undertaken. Given the lack of educational infrastructure, the Portuguese government initially sent engineers from the mainland to the colony. Eventually, it proved to be more effective to train such staff in Brazil itself, and from 1699 on, schools for training experts in fortification were established in Salvador, Maranhão, and Rio de Janeiro. This was followed, in 1738, by the creation of an artillery and fortification school (*Aula de artilharia e fortificações*) in Rio de Janeiro¸ which proved to be the basis for the later military academy.

For the training of civil and military engineers in Brazil, two important textbooks were published (albeit still in Lisbon): *Exame de Artilheiros* (1744) and *Exame de Bombeiros* (1748), both written by José Fernandes Pinto Alpoim (1695–1765) and both focused on what might be called Military Mathematics, for use in non-formal schooling in preparation of a military career. Both books revealed the strong influence of the catechetic style in the form of questions and answers (Fig. 9.3).

2.5 Beginning of Changes in the Colonies

By the middle of the eighteenth century, the situation, which had been stable for almost two centuries, began to change, basically as an effect of the Enlightenment movement, which affected not only metropolis Spain and Portugal, but also the colonies. While there were, in Spanish America, a good number of expatriates and *creoles* that played an important role in creating a scientific atmosphere in the colonies, the comprehensive reforms instigated by the Marquis of Pombal in Portugal (see Maxwell 1995) effectively brought about change to Brazil as well.


EXAME
DE
ARTILHEIROS
QUE
COMPREHENDE ARITHMETICA, GEOMETRIA, E
DEDICADO
GOMES FREIRE
DE ANDRADE,
DO CONSELHO DE SUA MAGESTADE,
POR
JOZE' FERNANDES
PINTO ALPOYM,
LISBOA,
Na nova Officina de JOZE' ANTONIO PLATES,

Fig. 9.3 Cover of the book *Exame de Artilheiro*

The most dramatic reform was the expulsion of the Jesuit order, first for Portugal and its colonies in 1759 and for Spain earlier than the official dissolution in 1772 – namely, in 1767. While its impact for Spanish America has not been well studied, sometimes, one identifies it as the instigation of the complete collapse of the Latin American educational systems. This view does not take into account the presence of other religious orders and, to a minor degree, privately owned institutions.

Moreover, Brazil also applied the reform policy of Pombal, who had instituted *aulas régias* as new forms of secondary schooling offering courses in various disciplines. At first established for Greek, Latin, and rhetoric, the courses also included mathematics, modern languages, and drawing. These *aulas* are seen as the first public education schools in Brazil (Carvalho and Dassie 2012, pp. 500f.).

In Spanish America, one notes in the same period university reforms which improved the role of mathematics and the sciences. In Chile, for instance, the Universidad Real de San Felipe, inaugurated in 1747 in Santiago, was provided with a "catedra" of Mathematics. Fray Ignacio León de Garavito, a self-instructed *creole* mathematician, was responsible for this chair. In Guatemala, at the *Universidad de San Carlos de Guatemala*, reforms of 1785 instigated a modernization of the studies of mathematics, thanks to José Antonio Liendo y Goicoechea (1735–1814), who introduced experimental physics and used more modern mathematics textbooks like those written by Christian Wolff. At the University in Mexico, the name of the chair changed in 1778 to be solely of "mathematics" (Alejandre 2005, p. 101).

Wolff's textbooks also proved to be a means of modernizing the teaching of mathematics in Colombia, thanks to José Celestino Mutis (1732–1808), of Colombia, who founded the Observatory at Bogotá, in 1803.

Textbooks used in the University of Mexico and published by its professors testify to the level of lectures there. Rodriguez' manuscripts show that his part on arithmetic began with four operations with whole numbers (Alejandre 2005, p. 61 ff.). Moreover, Bartolaches's *Lecciones matemáticas*, published in 1769, while him being a substitute for the chair, was comprised of arithmetic, geometry, and mechanics – but – no advanced mathematics (p. 123). Yet, only its first part was published and again restricted to arithmetic with numbers (p. 125). One can thus doubt how much mathematics students who were entering the university at that time actually knew.

Not only did mathematics in general education eventually improve, but also its functioning in professional training and applications. In Mexico, in 1781, the *Real Academia de San Carlos de las Nobles Artes de la Nueva España* had been established originally to "improve the quality of the money coined in the viceroyalty" but, in practice, with a much wider purpose. It offered courses in architecture, painting, and sculpture and lectures in mathematics, which were needed for its architecture course. Also in Mexico, the *Real Seminário de Minas* was created in 1792, to prepare professionals for mining. The first year of the curriculum was dedicated to mathematics (Garcia 1998, pp. 4 and 9). In this engineering institution, infinitesimal calculus was taught for the first time, using Bails's textbook (Alejandre 2005, p.105).

In 1772, Benito Bails published *Elementos de matemáticas*, which treated infinitesimal calculus and analytic geometry. A special mention should be given to Benito Bails (1731–1797), a major figure of the Spanish Enlightenment (Saavedra Alías 2002). Although Bails seemingly never visited México, it is important to refer to him as an example of how Mexican mathematicians were aware of progress in Spain. Bails's principal mathematical work, *Elementos de matemáticas*, was very influential in the colony. His influence in Mexico was well studied in a recent paper by Martínez Reyes (2006).

Also revealing was the development of a special kind of applied mathematics, stimulated by the complexity of problems related to hydraulics and to mining. These are two of the most important problems in the technological development of the country. "Subterraneous geometry" became a major theme in Mexican science. Particularly important were efforts for urbanization which took place in all the colonies (Catalá 1994). The book *Comentarios a las Ordenanzas de Minas* by Francisco Javier Gamboa, published in 1761, is most representative of these developments.

2.6 Independence

The tides of independence, greatly influenced and supported by the recently independent United States of America, succeeded in all the Spanish and Portuguese colonies. The Spanish colonies obtained their independence in the early nineteenth century, and Spain retained sovereignty only in Puerto Rico and Cuba. Brazil obtained independence from Portugal in 1822.

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Part IV
Mathematics Education in Different Epochs and in Different Regions: Modern Period

Introduction

This part is devoted to mathematics education in Modern times. For Western Europe, the French Revolution and the Napoleonic wars make a natural starting point for this period. These events, indeed, transformed not only the system of socioeconomic relations but also the system of education, including, and perhaps even first and foremost, mathematics education. The *École polytechnique*, which was established in Paris on the crest of the French Revolution, became a model for international imitation, while the decrees of Napoleon, a former artillery officer who had studied mathematics seriously, opened up for it a far more respectable place than it had hitherto occupied in school curricula, and not only in France but gradually also in the rest of what was then Europe.

The change in the role of mathematics in education reflected a change in its role in life. The nineteenth and twentieth centuries witnessed a rapid development of science and technology, for which mathematics was indispensable. As the need for mathematics grew, so did the opportunities for teaching it – technology, in the broad sense, acquired an ever broader presence in mathematics classes, from mass-produced textbooks and blackboards to computers and the Internet.

Already in the eighteenth century, enlightened absolutism had begun restructuring the system of school education, taking a more and more active part in its organization: a number of countries witnessed the appearance of the beginnings of a system of government-run schools and government-run teacher preparation. In the nineteenth century (and even more so in the twentieth), such systems began operating far more broadly and comprehensively. The government, naturally, still needed to have qualified personnel prepared for its own operation, but the government's objectives also came to include a broader preparation of educated specialists, including preparation that to some extent provided an opportunity for upward mobility, attracting people from formerly poorly educated sectors of the population (although restrictions based on economic and social status continued to exist and can hardly be said to have disappeared completely even now). The increase in the number of students from mathematics for the few to *mathematics for all* – which took place alongside of changes in the contents of education and teaching style – is probably the most important aspect in the development of mathematics education in the modern period.

While the contents of mathematics teaching had been restricted during premodern times to some rudiments of arithmetic and of geometry, these contents evolved considerably over this period and came to give students decent access to key concepts and branches of mathematics. Some topics and some approaches became widely used and may be characterized as traditional in mathematics education. As a counterweight to it, the modern period witnessed two waves of reform movements: the first in the late nineteenth and early twentieth centuries, which it is natural to link first and foremost with the name of Felix Klein, and the second from the late 1950s until the early 1970s, which, again with certain qualifications, may be called, using its American name, New Math. Both movements were international, and the emergence of international movements in mathematics education is also one of the distinctive features of the modern period (or more precisely, of its last century), along with the recognition of mathematics education as a field of international scientific research.

The world of the modern period was very heterogeneous – during the nineteenth century, a number of countries established colonial empires; other countries, first and foremost the countries of Europe and North America, were independent, but many others were either colonies or semicolonies, or had only relatively recently become independent. The construction of a system of mathematics education in these countries, far from complete, is also a subject of the present part of the book.

The system of mathematics education in the colonies and semicolonies usually consisted of two parallel subsystems, which virtually did not interact with one another: on the one hand, education for the colonizers and a very small number of representatives of the local elite who were given access to it, and on the other hand, rudimentary education for the rest (or more precisely, for the relatively very few individuals who were given access even to it, since most of the population received no formal education at all). The developmental processes of this system are interesting not least because they vividly expose the socioeconomic and political roots and causes of successes and failures in mathematics education. In addition, in studying these processes, researchers can observe the interaction of different sources of influence within the bounds of the same region – on the one hand, approaches to education, usually informal ones, that are traditional for the given region, and on the other hand, methods introduced from elsewhere, which were themselves varied, if only because during the period of the Cold War, education too became a part of that conflict, and thus local education came to be modeled sometimes on Soviet mathematics education and at other times, say, on American or British education.

Several chapters in this part are devoted to the history of mathematics education in specific countries, which allows their authors to connect the history of mathematics education more closely with the socioeconomic and political history of these nations. Other chapters – sometimes due to the greater interdependence of particular countries and the greater homogeneity of the regions to which they belong and sometimes simply because they have been studied less and fewer materials about them are available – are devoted to whole regions.

As in the other parts of the book, the temporal borders of the period under examination in this part are not everywhere the same. As has already been said, the traditional periodization of Russian history suggests that the discussion should begin with Peter I and the intensive use of the Western European experience, in other words, 100 years before the French Revolution. In other countries, the period when such reliance on Western European models begins is the second half of the nineteenth century, and consequently, this is the most convenient starting point for the analysis of their mathematics education histories. Sometimes it turns out to be natural to begin even later, as practically no evidence from earlier times has survived. In this way, one can speak only of an approximate frame for the period under examination. On the other hand, the events of modernity, that is, the most recent decades as a whole, are not part of the subject of this book.

Chapter 10
Secondary School Mathematics Teaching from the Early Nineteenth Century to the Mid-Twentieth Century in Italy

Livia Giacardi and Roberto Scoth

Abbreviations

APN	*Atti del Parlamento Nazionale*
BUMPI	*Bollettino Ufficiale del Ministero della Pubblica Istruzione*
CAI	Commissione Alleata in Italia (Sotto-Commissione dell'Educazione). 1947. *La politica e la legislazione scolastica in Italia dal 1922 al 1943 con cenni introduttivi sui periodi precedenti e una parte conclusiva sul periodo postfascista*. Milano: Garzanti
GU	*Gazzetta Ufficiale del Regno d'Italia*
MAIC	*Ministero di Agricoltura, Industria e Commercio*
RARS	*Raccolta degli Atti del Governo di Sua Maestà il Re di Sardegna*
RU	*Raccolta Ufficiale delle leggi e dei decreti del Regno d'Italia*

1 Secondary School Instruction in the First Half of the 1800s: The Role of Mathematics

In the Italian states at the end of the eighteenth century, education was still in large measure controlled by the clergy and centred around humanistic studies. The new educational policies established by Napoleon laid the foundation for an education that was state controlled and secular and affirmed the importance of educating citizens who were responsible and aware of their place in society. By contrast the Restoration period generally implied a return to the past for the educational systems, a greater involvement of ecclesiastic authorities, and strict control over teachers and students, fundamentally dictated by the desire to quell any revolutionary spirit. It was the small Kingdom of Sardinia which, in the mid-1800s, provided the impetus for renewal and laid the foundation for a general reform of the school system, placing it entirely under the control of the state.

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1.1 French Domination in Italy and Its Influence on Instruction: The Napoleonic Liceo

The years from the end of the eighteenth century through the beginning of the nineteenth were a time of great political instability. Following the invasion of Napoleon in the period 1796–1799 and a brief restoration of Austrian-Russian dominion, there was a new French occupation, during which the Italian territory was partly annexed to France and partly divided into satellite states, each with its own laws but without any real political autonomy. The two most extensive territorial states, as well as the most strategic for French politics, were the *Italian Republic* in the north (transformed into the Kingdom of Italy in 1805) and the *Kingdom of Naples* in the south. The former state was ruled over by Eugène de Beauharnais, Napoleon's stepson, and the latter by Joseph Bonaparte, his brother.

In the area of education as in other sectors of public administration, the organisational models applied in these states were inspired by those created in France, where the law of 1 May 1802 had redrawn the secondary school system. As a result, new *lycées* were created to replace the central schools and *écoles spéciales* replaced the dissolved universities, later becoming the (isolated) faculties.

1.1.1 Italian Republic

On 4 September 1802, the legislative council of the Italian Republic approved a systematic law on instruction that divided public education into *superior* (*sublime*), *middle*, and *elementary*.[1] For higher education, only two universities were recognised, those of Pavia and Bologna, while for middle instruction, *licei*[2] were introduced, although their characteristics differed from the French. In many cases, they were created to convert the numerous smaller universities present in many urban centres into schools for non-resident students with additional university-level school subjects. For example, the university in Ferrara was closed and turned into a departmental *liceo*; the study plan regarding mathematics was rather rich, calling for plane and solid geometry with a selection of the principal theorems of Archimedes; the theory of equations up to the third degree; the arithmetic and geometric series; and the use of logarithmic tables (Fiocca and Pepe 1989, pp. 17–23). The *licei* were also flanked by *ginnasi*, schools which arose mainly in smaller towns, on the same level but with a more limited curriculum than the *licei*. Neither of these schools was directly administered by the state, but rather by the individual municipalities in the case of *licei* and by the departments in the case of *ginnasi*; for this reason, they had programmes that varied from city to city. A subsequent decree established durations of 4 years for *ginnasi* and 2 years for *licei* and made arithmetic a mandatory subject in the first 3 years of *ginnasio* and algebra and geometry mandatory in the 2 years of *liceo*. More advanced topics in mathematics could be dealt with in the context of additional courses, which were by law made equivalent to those given in the two universities of Pavia and Bologna. For example, in the central *liceo* in the territory of the Adige River, located in Verona since 1804, four of the seven courses activated were in the sciences, including the teaching of mathematics, with an introduction to calculus and its applications to mechanics and hydraulics.

This system of secondary teaching was modified between 1807 and 1808 in the context of a new, more systematic educational itinerary. This provided for the sequence of *ginnasio-liceo* university, with the institution of new residential *licei* similar to the French ones and the standardisation of

[1] The text can be found in Bucci 1976, pp. 253–261.

[2] To date, there has been no systematic investigation of Napoleonic *licei*, in Italian states but studies are available on local situations; see, for example, Assirelli 1984; Fiocca and Pepe 1989, pp. 17–23; Naggi 1994–1995; Piaia 1995; Scirocco 1996; Pagano 2000, pp. 69–125; Patrucco 2003, pp. 18–24.

teacher positions. The course was made triennial and the number of teachers was fixed at nine: one teacher was provided for mathematics and three for the sciences.

In 1809 Giovanni Scopoli – who had a background in scientific studies and pedagogical knowledge – was nominated General Director of Public Instruction in the Kingdom of Italy (Pepe 1995). On 15 November 1811, he passed regulations regarding *ginnasi* and *licei*, which returned the *liceo* to a 2-year course and reduced the number of its teachers to five – one for mathematics and one for the sciences. Textbooks were standardised throughout the kingdom. Further, the teaching of arithmetic in the first 3 years of *ginnasio* was confirmed, but algebra and geometry were limited to only the first year of *liceo*. *Ginnasio* and *liceo,* which now constituted a consecutive structure, divided the secondary school into two phases – deviating hence from the French model and following the educational structure of Bavaria and Austria.

1.1.2 The Kingdom of Naples

In the Kingdom of Naples, *licei* were introduced later under the government of Joachim Murat, the successor to Joseph Bonaparte. The overall reform of the educational system was enacted with the *Decreto organico per l'istruzione pubblica* of 29 November 1811 (Collezione 1861, pp. 230–239). Colleges and *licei* were both framed in the context of secondary instruction, but *licei* were on a higher level, provided with university professorships and created to favour the diffusion of advanced instruction in a territory which had only one university, that of the capital. In the *licei* in the Kingdom of Naples, examination commissions were also instituted for conferring lower academic degrees that permitted the exercise of some professions. The university professorships aggregated into *licei* were thus divided among four disciplinary areas: art, medicine, jurisprudence, and the sciences. At the secondary level, 'pure and mixed' mathematics were taught in both colleges and *licei*, although in the latter they were combined with philosophy. There was also a *liceo* for the introduction to studies of mathematical sciences and physics, with an additional course in higher mathematics.

One extremely significant phenomenon in the Kingdom of Naples, unseen in any of the other Italian states of the period, was that of the massive presence of schools run by private entities. These schools were opened by scholars, often laymen, offering instruction that was equivalent to an entire course in the *liceo* or special training to prepare for academic degrees or entrance to schools of engineering or military academies. There were schools of mathematics, mathematics and physics, mathematics and philosophy, and mathematics and architecture, all of which were good alternatives to public schools.

1.1.3 Tuscany

Tuscany, annexed to the French Empire in 1807 and, beginning in 1809, a Grand Duchy administrated by Elisa Bonaparte in the name of her brother the Emperor, deserves separate mention not so much for the two *licei* instituted in Pisa and Siena, as for the creation in 1810 in Pisa of a 'branch of the *École Normale Superieure*' in Paris, with the aim of training teachers in the humanities and sciences for secondary schools. The school opened in 1813, and the scientific disciplines were especially emphasised. Closed soon after the fall of Napoleon, the *Scuola Normale Superiore* in Pisa would be reopened only in 1846.

The *liceo* was without a doubt one of the most important legacies of the Napoleonic domination of Italy, especially with regard to two aspects: the establishment of a secular, state education and the important role attributed to scientific disciplines. The emerging division of the secondary school into the two subsequent types – *ginnasio* and *liceo* – eventually became generally used in Italy.

1.2 From the Restoration to the Eve of Italian Unification

1.2.1 The Return to the Dominance of the Clergy in Instruction

The Congress of Vienna held in 1814–1815 decided to divide the Italian peninsula into seven principal states: the Kingdom of Lombardy-Venetia; the Duchy of Parma and Piacenza; the Duchy of Modena and Reggio; the Grand Duchy of Tuscany, which were either directly or indirectly under Austrian hegemony; the Kingdom of the Two Sicilies – the union of the former Kingdom of Naples and the Kingdom of Sicily – which was returned to the Bourbons; and the Papal State and the Kingdom of Sardinia, ruled respectively by the Pope and the Savoy, which retained some degree of autonomy from the Hapsburg Empire. In general, the Restoration period in Italy provoked a return to the past for educational systems and a greater involvement of ecclesiastic authorities.

One clear example is given by the Kingdom of the Two Sicilies. In the first phase, the Bourbons seemed to carry on with the school policies of the French,[3] but after the uprisings of 1821, they allowed public instruction to be managed almost entirely by ecclesiastic and private entities. The most important secondary schools and colleges were entrusted to religious orders: Jesuits, Piarists, and Barnabites. The situation in secondary schools is clearly depicted, for example, in the testimony of the writer Luigi Settembrini who affirmed that students 'learn useless things' and 'leave the college ignorant' (Settembrini 1934, p. 8). The inefficiencies and deficiencies of public schools resulted in a further development of the private schools, which numbered 800 in the capital alone by the beginning of the 1830s (Lupo 2005, p. 172) and offered courses more closely reflecting the needs of society. Following several uprisings, on 10 February 1848, Ferdinand II signed the constitution. The following month, the Ministry for Public Instruction was established and a provisional commission was nominated to present a project for reforming public education in the Kingdom (Collezione 1861, II, Appendice, pp. 1–102). The counterrevolution of 1849 led to abolishing the constitution, and the reform project was shelved.

In the Papal State, a general plan for reform begun by Pope Pius VII was approved by his successor, Leo XII, with the papal bull *Quod divina sapientia* of 28 August 1824.[4] The institution of the *Congregatio studiorum*, with the functions of direction and supervision, testifies to the willingness to centralise and standardise the educational system, but this objective was not achieved. Secondary education was neglected: the colleges directed by religious orders were allowed to continue their work without any effort towards standardisation and had a free hand with regard to curricula and methods. Often the teachers themselves introduced some cautious modifications into the programmes and textbooks in order to adapt the students' education to the needs of the times. For example, in 1832, the Jesuits of the Roman College introduced several modifications into the *Ratio Studiorum* of 1599, some of which concerned mathematics and the natural sciences which were supposed to be taught by taking recent discoveries into account. The traditional system, however, was not in question nor were new textbooks introduced (Sani 1994, pp. 734–735).

[3] See regulations for *licei* of 1816 (Collezione 1861, I, pp. 366–420), which confirm and integrate the pre-existing system of conferring doctoral degrees. The regulations also established the books to be used; for mathematics, they recommended books by mathematicians of the *scuola sintetica* of Nicola Fergola, but teachers were also invited to refer to the texts of Lagrange, Euler, Biot, Hachette, and Bossut (p. 368). See § 1.3.

[4] The complete text can be found in Venzo 2009, pp. 493–536.

In the Grand Duchy of Tuscany, the Restoration period was marked by a more moderate tendency and the essential indifference of the government to public education. The position of *Consultore Sovrintendente agli Studi* (supervising consultant for education) was instituted in October 1816 and the position entrusted to mathematician Pietro Paoli, a professor of algebra at the University of Pisa, but the results were insignificant: an enquiry conducted in the 1830s by the Tuscan government made evident the dismal conditions of the public schools. Only in 1852 did Grand Duke Leopold II issue a law regarding primary and secondary teaching. The public schools were divided into lower (*minori*), secondary, and higher (*maggiori*). Secondary schools, mandatory in the most important cities, provided instruction for those who intended to pursue agriculture, commerce, and craftsmanship, while the higher schools (*ginnasi* and *licei*) were intended to prepare students for university studies.

In the Kingdom of Lombardy-Venetia, the educational system underwent a process of harmonising it with that of Austria. In the spring of 1817, the General Direction for Public Instruction was suppressed and Scopoli was removed from his office. The following year, the *Codice ginnasiale*, an Italian translation of the German text approved by Franz I, was printed with only minimal modifications and additions. The courses consisted of 4 years of grammar and 2 years of the humanities. Compared with the strong emphasis on Latin, Italian, and Greek, the scientific disciplines played a marginal role; instead, religious instruction became central. New regulations for *licei* were introduced in 1817–1818 (*Istruzione per l'attuazione degli studi ne' Reali licei*). The educational itinerary lasted 3 years: the first two were common to all students, while the third year was divided into a number of specialties (law, theology, medicine) with different mandatory subjects. Mathematics was only offered in the first year for 7 h per week and physics in the second year for 8 h. As far as the sciences were concerned, this system was distinguished from that of the Napoleonic government by teaching which was strictly theoretical and paid no attention to applications (Ciprandi et al. 1978, II, pp. 125–165).

In the Savoy domains, the definitive fall of Napoleon and the re-establishment of the monarchy meant the cancellation of French innovations for public instruction and the restoration of the Royal Constitution of 1771. The royal policies were intended to give the clergy back their lost influence: the process of clericalising the schools was especially evident in primary schools, but involved secondary schools as well, as shown by the return of colleges to religious orders. It was only in the 1840s that a system of instruction marked by greater centralisation and secularisation would begin to take shape in Piedmont.

1.2.2 The First Significant Examples of Technical-Professional Education

In Italy, the creation of technical-professional education in the modern sense of the term can be dated back to the first decades of the nineteenth century, when the onset of a phase of industrialisation brought with it a substantial demand for skilled labour for employment in new processes of production. Initially this new kind of education was regarded with only scant interest on the part of the state and was primarily concerned with the needs of the economic entities involved. In this first phase, technical-professional education generally consisted of a wide variety of individual local initiatives that were derived from the requirements of the community in which they were located.

Shortly after, in the 1830s and 1840s, state authorities became more interested in this new educational model. *Schools for crafts and trades*, run by private entities and aimed at artisans and labourers, were soon flanked by *state technical schools*, which provided alternatives to lower and upper secondary school (*ginnasi-licei*) programmes and were aimed at training specialised workers for employment in public administration.[5]

[5] At present there are no overall studies of technical-professional education in Italy in the first half of the nineteenth century. For information regarding education in individual regions or single institutions, see the bibliography in Hazon 1991, pp. 59–60.

In the private sector, the most significant initiatives were those in Piedmont and Lombardy related to processes of industrialisation and those for the modernisation of agriculture in Tuscany and the Veneto. A good example is provided by the schools of the *Società d'incoraggiamento d'arti e mestieri*, founded in 1842 in Milan, which offered four laboratories: chemistry, geometry and mechanics, industrial physics, and silk manufacturing. Courses in theoretical geometry and mechanics were established beginning in 1845; these were later reinforced by new courses in industrial mechanics and descriptive geometry applied to crafts. Other examples, in Turin, Biella, and Novara, date from about the same period.

In the public sector, one of the first initiatives was carried out by the Austrian authorities, which in 1838 founded state technical schools in Venice and Milan. Technical instruction in Lombardy-Venetia was successively arranged into two levels of schools denominated *reali* (realist), in keeping with the model then in use in Prussia. In 1841 and 1843, the technical schools in Venice and Milan were transformed into *Scuole reali inferiori*, and in 1851 a *Scuola reale superiore* was established in Milan. These realist schools represented an alternative to programmes for classical studies, and the upper-level diploma permitted enrolment in the faculty of mathematics at the university. In addition to arithmetic and algebra, the teaching programme for mathematics was comprised of plane and solid geometry, with an emphasis on practical geometry, conic sections, plane and spherical trigonometry, analytical geometry, and architecture. The introduction of descriptive geometry illustrates this more modern programme (MAIC 1862, pp. 238–258).

It is worth mentioning among the other public initiatives in the area of technical instruction, the creation in 1852 of the *Regio Istituto tecnico* in Turin – the embryo of the future *Scuola di applicazione per ingegneri*. The professor of practical geometry was the engineer and mineralogist Quintino Sella, one of the most important statesmen of the Italian Risorgimento (Ferraresi 2004, pp. 298–315). In the same period the *Regio istituto tecnico* was established in Florence. In 1857 the institute was subdivided into two 3-year sections – one dedicated to physics/mechanics and the other to physics/chemistry – and offered a good level formation (Bacci and Zampoli 1977; Patergnani and Pepe 2011).

1.3 Mathematics Textbooks for Teaching: Translations of French Texts and Italian Texts

The widespread circulation of French textbooks and the flourishing of translations characterised the entire first half of the nineteenth century and made it clear that France was the principal point of reference for mathematics teaching on the Italian peninsula. Works by Bossut, Clairaut, Bézout, Dupin, Lacroix, Francoeur, and Lefebure de Fourcy, to name the most well known, were translated throughout Italy. Most importantly, numerous translations, often annotated, appeared of Legendre's elements of geometry: the first of these was published in Pisa in 1802, followed by that of Gaetano Cellai (Florence 1809–1810), reprinted several times, and still more often in later years (Schubring 2004).

The translations of French books flourished most of all in Naples, the city which had a particular place in school publishing in the first half of the nineteenth century, thanks to the stimulus of a large number of private schools. Moreover, here, the debate between the supporters of the synthetic method and the defenders of the analytical methods also favoured the publication of textbooks by Italian authors (Ferraro 2008).

In the other states, proper Italian textbooks production was generally based on French models. One of the most widespread books was written by Vincenzo Brunacci, *Elementi di algebra e geometria ricavati dai migliori scrittori di matematica* (Milan 1908), decreed in 1811 as the official textbook for mathematics in the Kingdom of Italy. Brunacci largely referred to nineteenth-century French textbooks, but among his sources were also the works of Guido Grandi, Antonio Cagnoli, and Paolo Ruffini (Pepe, in Giacardi 2006a).

In the years immediately before and after Italian Unification, there was a new flourishing of translations of elementary French treatises into Italian, in the attempt to modernise mathematics teaching, thanks in particular to the efforts of the Florentine publisher Le Monnier and to the involvement of a new generation of Italian mathematicians. The year 1856 saw the publication of three texts: *Trattato di aritmetica di Giuseppe Bertrand*, a translation by Giovanni Novi; *Trattato di trigonometria di Alfredo Serret* by Antonio Ferrucci; and *Trattato d'algebra elementare* also by Bertrand and translated by Enrico Betti. In 1858, *Trattato di geometria elementare di A. Amiot* was published, again as a translation by Novi.

1.4 Pre-Unification Piedmont

1.4.1 The Importance of Good Teachers: The Creation of the *Scuole di Metodo*

The first signs of the new climate following the reactionary phase of the Restoration period were seen in the mid-1830s in the Kingdom of Sardinia. From among the ruling class of the Savoy State emerged enlightened aristocrats and intellectuals who urged the king towards reform policies. The interest of this group centred on the themes of instruction and education, to which was assigned the task of training a ruling class capable of modernising the nation as well as educating average citizens.

Remarkably, the focus of their reform initiatives was schools for training primary school teachers. First attempts in the early 1840s to create such *Scuole di metodo* failed, but eventually King Carlo Alberto approved the opening of such a school at the University of Turin in 1844. The priest and educator Ferrante Aporti was called to be the director, although he was opposed by the exponents of the extreme right, by the Jesuits, and by ecclesiastical authorities. The creator of the first nursery schools in Lombardy-Venetia, Aporti maintained the importance of the intuitive methods and the usefulness of basing teaching on the interests of young people rather than the catechistic and rote method normally used up to then. In 1845, the school was promoted to form the *Scuola superiore di metodo* at the University of Turin, specifically to train professors of *metodo* (pedagogy); also, it was planned to open provincial schools for primary teacher training. In 1853, this system of teacher training institutions became more firmly established as *Scuole magistrali*, separated for men and female teachers. A later legislative measure, of 20 June 1858, definitively formalised the institution in the Kingdom of Sardinia of state schools for primary school teachers, now called *Scuole normali*. The courses of study lasted 2 years for future teachers of the first 2 years of primary school and a further year for teachers of the third and fourth years. Attendance was open to those who had passed an entrance examination and were at least 15 years old (for women) and 16 years old (for men). Among the principal subjects were arithmetic and accounting and elements of geometry.

1.4.2 The Enactment of the Boncompagni Law of 1848 and the Proposal of Experimental 'Special Courses' for Technical Education

The creation in 1847 of the Ministry for Public Instruction, the granting of the constitution in March 1848, and the formation of a constitutional government paved the way for further progress of educational reforms, in particular for the three branches of pre-university education: elementary school, classical secondary school, and technical instruction. The Law of 4 October 1848, named after Carlo Boncompagni who was one of the leading intellectuals in Piedmont (R.D. no. 818 and no. 819: RARS, XVI bis, pp. 939–967 and pp. 969–978), placed the entire educational system – public and private – under the authority of the Ministry of Public Instruction. Schools were redistributed into three levels: *elementary* (*lower* and *upper*), *secondary*, and *university*. Alongside these, post-elementary schools were created for technical instruction, called *special schools*. Secondary schools were meant to prepare students for university studies, while special schools were postprimary schools that prepared students to exercise a profession that did not require university training.

Boncompagni launched a significant experiment creating six *Collegi-convitti Nazionali* (state residential colleges), in Turin, Genoa, Chambery, Novara, Nice, and Voghera, with the aim to test the new programmes. They were housed in the buildings that had previously been colleges of the Jesuits, who had been expelled from the kingdom in August of that year, and were conceived of as experimental schools to develop new programmes and teaching methods. The state colleges offered a full itinerary of studies, from elementary schooling (4 years) up to entrance to university in the case of the classical branch and to entrance to the professional world in the case of special schools. Interestingly enough, the classical branch followed the old Jesuit curriculum: 3 years of grammar, two of rhetoric, and two of philosophy. More modern teaching was given, as it was in France then, in 'accessory' courses, emphasising arithmetic, geometry, sciences, and history. Special schools, instituted only in the residential colleges of Turin, Genoa, and Nice, lasted 5 years. The study programme called for classes in Italian, foreign languages, statistical and commercial geography, physics and chemistry, mechanics applied to the arts, drawing, and elementary mathematics (arithmetic, algebra, trigonometry, geometry), with particular emphasis on applications to commerce and the arts.

The Boncompagni Law was not sufficiently innovative for the needs of the new ruling class (intellectuals, skilled workers) because it adhered essentially to the old traditional classical model. In the following decade, several attempts were made to revise it, but all of them bogged down in Parliament.

The so-called *special schools* were reformed in 1856 by a decree issued by Minister Lanza, marking the end of the phase of experimentation: they could now also be instituted outside of the state residential colleges and were divided into a 3-year lower course called *primary* and a 2-year upper course called *secondary*, which in turn was divided into *commercial* and *industrial* sections. The mathematical content in the lower level focused on arithmetic, the metric system, domestic and commercial accounting, and principles of algebra and geometry; in the upper level, the content focused on commercial accounting and elements of statistics in the commercial section and on applied mathematics in the industrial section. This new system of special schools in the Kingdom of Sardinia became similar to that of the *Scuole reali* in Lombardy.

The educational structure furnished by the Kingdom of Sardinia would come later to constitute the model for the united Italy. This small kingdom, over the course of about a decade, would succeed in reforming a school system which had, up to then, been based on eighteenth-century models under the control of the state. The area of secondary education, still prevalently aimed at humanistic instruction, opened the door to several important innovations regarding the technical branch of public instruction and the normal schools for training elementary school teachers.

2 From the Casati Law to the End of the 1800s

2.1 Merits and Limits of the Casati Law with Respect to Secondary Instruction

On 17 March 1861, the Kingdom of Italy was proclaimed, with Turin as its capital. The process of the unification of the Italian states was completed 10 years later with the conquest of Rome, which would become the capital of the new nation.

The new kingdom inherited the school system introduced in the Kingdom of Sardinia by the law of 13 November 1859, enacted by the Minister of Public Instruction Gabrio Casati (RARS 1859, pp. 1903–1998). This legislation, whose distinctive characteristics were the dominant role of the university, centralisation and bureaucratisation, and the concern for educating a ruling class firmly anchored to the values embodied by humanistic culture, would govern public instruction in Italy for more than 60 years.

The educational system was divided into *primary*, *secondary*, *technical*, *normal*, and *superior* (university). Primary education, entrusted to municipalities, was divided into two levels, lower and upper, each lasting 2 years. Secondary instruction consisted of the *ginnasio-liceo*. Studies in the *ginnasio*, which followed primary school, lasted 5 years, and constituted the lower level of secondary education, were financed by municipalities. Studies in the *liceo* constituted the upper level of secondary education, lasted for 3 years, and were partially financed by the national government. Technical instruction constituted a separate stream and occupied a position that was clearly subordinate, so much so that in Art. 1 of the law, it was inserted in the third branch of instruction, along with primary school. It was divided into two levels, one called *technical school* and the other *technical institute*. The technical school, which lasted 3 years, started after primary school and was also financed by municipalities; the technical institute, which lasted 2 or 3 years, was funded partly by the national government and partly by the provinces. Finally, normal instruction was designated by the Piedmont decree of 1858 and broadened with the institution of nine government schools for men and nine for women.

The *ginnasio-liceo*, which was a refinement of the former system of secondary studies established by the Boncompagni Law, constituted the real pillar of the Italian school system: it provided access to all university faculties and differed from the other postprimary forms of instruction in having more years of study, a wide range of fundamental subjects, teachers who were more qualified, and direct control by the state over school activities and functions. It was aimed to form the future ruling class and had a clearly humanistic stamp. It is symptomatic that teachers of mathematics earned less than those of the humanities; not until 1914 did they achieve an equal footing with teachers of Italian, rectifying this inequality (RU 1914, III, Tab. A, p. 2418).

Instead, technical-professional training, which the Casati Law placed alongside humanistic education, allowed almost no entry into university faculties and was controlled by norms that were often contradictory and difficult to interpret. Casati had to make an enormous effort to create a broad-based model for technical instruction: this model had to respond to the needs of the middle class for employment and commerce, and it had to fend off opposition by the upper classes and a sector of intellectuals firmly against any school system other than humanistic. The result was a technical education that, along with subjects of general culture such as that offered in *ginnasi* and *licei*, also offered new technical-scientific disciplines. Technical institutes were divided into four sections, called *commercial-administrative*, *chemistry*, *agronomics* (which lasted 2 years each), and *physics-mathematics* (which lasted 3 years and was the only section that permitted entrance into the university faculty of mathematical, physical, and natural sciences).

In the years following its publication, the Casati Law was applied only in part and with great difficulty. The old school regulations in effect in the pre-Unification states were substituted by transitory norms that regarded specific regions and in some cases remained in force for many years. Another serious problem was that teachers were insufficiently prepared. The Casati Law called for three kinds of teachers: *titolari*, *reggenti*, and *incaricati*; only *titolari* were appointed by the King consequent to a competitive examination. In the other cases, the recruitment often degenerated into various kinds of abuse (see Furinghetti and Giacardi 2012).

In order to appreciate and evaluate the legislative measures adopted after the Casati Law, the choices made, and their consequences for mathematics teaching, it is essential to understand the situation of Italian schools post-Unification. First, there was an alarmingly high rate of illiteracy.[6] Second, in 1864, the number of students who attended secondary school was 18,627, equal to 0.7 per 1,000 inhabitants (Talamo 1960, pp. 61–62).

[6] *Sommario di statistiche storiche italiane 1861–1955,* Istituto Centrale di Statistica, Roma.

The actual condition of secondary schools emerges clearly from the inquiry undertaken by the Higher Council for Public Instruction in 1864. In his well-documented report, Giovanni Bertini, professor of philosophy at the University of Turin, underlined the following points: the inadequate recruitment of teachers; the poor quality of textbooks; the 'premature bifurcation' into classic and technical courses, which excluded from the *ginnasio* all disciplines useful to everyday life; and the low level of rigour and strictness in the final exams for the diploma (Bertini 1865). Among the remedies that he proposed to address this situation was the institution of a single kind of middle school for all students, without Latin – a proposal that was destined to remain on paper until 1962.

In 1876 when the left came to power, the need to transform the ignorant masses into citizens aware of their rights and responsibilities received more attention. In 1877 the minister Coppino enacted legislation which introduced mandatory education. The legislation was significant, because it affirmed the state's control over individuals and communities, although only to a certain extent; in fact, compulsory education extended only to children up to 9 years of age. It was a purely secular measure, and this aspect was reinforced by the fact that religious instruction was not one of the mandatory subjects; it was instead replaced by 'the elementary notions of the duties of man and citizen'.

At the end of the nineteenth century, the distribution of classical and technical schools varied widely throughout the Italian peninsula, meaning that while in some cases the classes were overflowing, in others, they were practically empty. Similarly variable was the relation between the number of students enrolled in state schools and that in the numerous private schools. The number of students in private *ginnasi* used to be nearly double that in state *ginnasi* from 1861; their number began to fall in 1896 and ended with just one third of state schools in 1916. On the other hand, the number of students in private *licei* used to be only one half of that in state schools, began to rise in relation until reaching two thirds by 1896, and then also fell to just one sixth (Lacaita 1973, p. 61). For both technical schools and technical institutes, almost the same process can be observed: in the end, state schools were by far dominating (p. 56).

2.2 The Programmes for Ginnasi and Licei of 1860 and 1867: Luigi Cremona's Contribution

Teaching regulations and programmes[7] for the various types of schools were decreed between September and November 1860 and those for normal schools in November 1861 (Table 10.1).

The programmes of the classical school show substantial differences from the technical courses. While several novel subjects were proposed for the classical school, the programmes for the technical school remained tied to the usual subjects and assigned a single year each to the teaching of arithmetic, geometry, and algebra, thus penalising continuity of teaching.

It is not known exactly who devised the mathematics programme for technical schools and institutes and normal schools of 1860 and 1861, whereas those for *ginnasi* and *licei* were designed by Luigi Cremona,[8] without a doubt the mathematician of the second half of the nineteenth century who was most involved in constructing the new school system of the united Italy. The programmes were innovative and marked a break with the educational praxis of the day. Cremona took the French textbooks published by Le Monnier (see § 1.3) as his starting point (Cremona 1860a, b) and explained his reasons underlying the choice of topics in an article in which he also provided instructions for teachers (Cremona 1861). This article makes evident the fundamental points of his project for education: broadening and

[7] All legislative measures cited herein are available at http://www.subalpinamathesis.unito.it/storiains/uk/excerpts.php.

[8] See his correspondence with Genocchi (Carbone et al. 2001, pp. 152–154).

Table 10.1 Mathematics in the Casati law: subjects and number of class hours per week

School/year	1st	2nd	3rd	4th	5th
Ginnasio	1[a]	1[a]	1[a]	3[a]	3[a]
Liceo	8[b]	0	3[c]		
Technical school	5[a]	5[d]	5[e]		
Normal school for men	3[f]	3[g]	2[h]		
Normal school for women	3[f]	3[h]	2[h]		
Technical institute (physics-mathematics)	4[i]	4[j]	0		
Technical institute (commercial-administrative)	0	0			
Technical institute (chemistry)	0	0			
Technical institute (agronomics)	0	0			

[a]Arithmetic
[b]Algebra, plane and solid geometry, and trigonometry
[c]Algebra and plane and solid geometry
[d]Plane and solid geometry
[e]Algebra
[f]Arithmetic and accounting
[g]Arithmetic, accounting, and plane and solid geometry
[h]Arithmetic and plane and solid geometry
[i]Solid geometry, algebra, and trigonometry
[j]Topography and descriptive geometry

modernising programmes, establishing an interdisciplinary approach, creating stimulus for research, and setting the rigour of method (Scoth 2010). He wrote:

> The new program […] embraces a great number of topics, quite new to our schools […]; these are the theory of inequalities, the problems of maxima and minima, notions of limits, the harmonic division of line, the symmetry of polyhedra, etc. Thanks to the addition of these extremely important topics, and the reasonable rearrangement of the others, it seems to us that the program corresponds to the current state of science. In geometry as well it is helpful to accustom the students to solving problems and proving theorems. (Cremona 1861, p. 299; transl. by the Authors)

For the first time, topics related to projective geometry were inserted in the mathematics programmes of the *licei*. Yet, Cremona's ideas necessitated a corps of teachers who were capable of developing them in classrooms and, as mentioned, Italian schools of the day lacked qualified teachers. This was probably one of the reasons why topics related to projective geometry, and indeed a large part of the innovations, were eliminated in the next programmes decreed in 1863.

Important changes in the teaching of mathematics resulted from the Act of Parliament issued by Minister of Public Instruction Michele Coppino on 10 October 1867 (RU 1867, VII, parte supplementare, pp. 256–410). The mathematics curricula and instructions on teaching methods were actually the brainchild of Cremona, who succeeded in prescribing the original method of Euclid's *Elements* – 'the most perfect model of rigorous reasoning':

> In the classical secondary schools, mathematics should not be seen merely as a set of propositions or theories having their own intrinsic value, which young people are required to learn in order to apply them in real life; rather, it is principally a means to develop general knowledge, a kind of mental gymnastics aimed at exercising the faculty of reason. (RU 1867, p. 310; transl. by the Authors)

The timetable, issued together with the new curricula, shows a structure then unique in Europe and regarded as 'strange' later on in Italy (see Scarpis 1911, p. 26):

	Ginnasio					*Liceo*		
Year	1	2	3	4	5	I	II	III
Hours	0	0	0	0	5	6	7½	0

Of the 8 years of secondary schools, mathematics was taught only in three of these years and beginning only in the fifth grade with arithmetic – without prior teaching: moreover, the last grade preparing for the final exams was also without mathematics. For both the *ginnasio* and *liceo*, Cremona extolled the deductive method: arithmetic and geometry (book I of Euclid) – for the last grade of the *ginnasio* – were to be taught using the deductive and demonstrative method. For geometry, he suggested following the Euclidean method because 'this is the most appropriate for creating in young minds the habit of inflexible rigour in reasoning'; he exhorted teachers not to contaminate 'the purity of ancient geometry, transforming geometric theorems into algebraic formulas, that is, substituting concrete magnitudes […] for their measurements' (RU 1867, p. 314).[9] Besides some algebra, the other books of Euclid, traditionally taught, figured in the syllabus: II to VI and XI, and XII, with the addition of the treatment of the circle, cylinder, cone, and sphere, according to Archimedes.[10] There was also the recommendation – only hinted at – that in the algebra curriculum students should be guided towards the fruitful method of limits and the concept of function; however, this suggestion remained vague and isolated (RU 1867, pp. 316–317).

Immediately following the Coppino Act, an Italian translation of Euclid's *Elements*, based on Viviani's edition of 1690, with supplementary notes and exercises was published by Betti and Brioschi[11] to serve as a textbook for students. The real author, however, was Cremona, with the help of Giacomo Platner, a professor at the *liceo* in Pavia (Gatto 1996, pp. 36–49). Cremona's aims were various: to do away with myriad 'worthless' books, compiled merely to make a profit; to foster the publishing of good Italian textbooks; to oppose Legendre's approach to geometry; and to 'coordinate' mathematics teaching with the formative orientation of the 'classical studies', as Betti and Brioschi stress in the introduction to their textbook (p. IV). The text presents Euclid's *Elements* without any didactic mediation regarding either the language adopted or the content. The language is purely Euclidean, with no concession made to algebraic symbolism, sometimes at the expense of clarity. It was natural that when the book was published, it aroused a lively debate: teachers did not like its complicated language, while mathematicians saw in it an unwelcome return to the past.

The debate surrounding the publication of the Betti-Brioschi text broadened when Giuseppe Battaglini published the translation of an article by an English mathematician, J. M. Wilson, in his journal, *Giornale di matematiche*. In direct opposition to the prevailing opinion in England, Wilson severely criticised Euclid's *Elements* from both scientific and didactic points of view, concluding categorically: 'Euclid is antiquated, artificial, unscientific and ill-adapted for a textbook'.[12]

The reply by Cremona and Brioschi was immediate, but the two authors were finally forced to admit 'that in various points it is to be desired that the *Elements* be emended and simplified' as long as they were not misinterpreted and 'as long as what is done is true geometry, and not arithmetic'.[13] Other opinions either for or against Euclid's *Elements* as a textbook appeared in successive issues of *Giornale di matematiche* and the debate remained a source of interest for years to come among both academics and teachers.

[9] For a more detailed treatment of the Coppino Act and the debates that followed it, see, for example, Giacardi 1995; and Schubring 1997, pp. 81–90. See also Schubring 2003.

[10] This corresponds neatly to the contents of Tacquet's edition of Euclid's *Elements* which had served from the seventeenth century on as standard textbook in Jesuit colleges.

[11] E. Betti, F. Brioschi. *Gli Elementi di Euclide con note aggiunte ed esercizi ad uso de' ginnasi e de' licei* (Florence, 1867).

[12] See J. M. Wilson, "Euclid as a textbook of elementary geometry," *Educational Times*, 1868, pp. 125–128, translated by R. Rubini as "Euclide come testo di geometria elementare," *Giornale di matematiche*, 6, 1868, pp. 361–368.

[13] See Brioschi and Cremona 1869, p. 54. An extract of this article was translated into French by J. Hoüel and published in the *Nouvelles Annales de Mathématiques*, 2, 8, 1869, pp. 278–283.

For Cremona, the return to Euclid was an interim solution for improving mathematics teaching:

> I am convinced that modern methods, especially those of Steiner and Staudt, are destined to renovate the whole of geometric knowledge, from the elements; with those methods even the most elementary things can be treated in a way that is simpler, more original, more fertile. But such methods cannot be introduced in schools until there exists an elementary textbook written especially for that purpose.[…] Until the day, still far-off, in which such a radical reform can be set into action, I believe that Euclid will always remain the best guide for teaching geometry in the classical schools. (quoted from Gatto 1996, pp. 52–53; transl. by the Authors)

His real hope was for a much deeper reform, as is clearly shown by the programmes of 1860, discussed earlier, by his correspondence, and by his article (Cremona 1860a) in which he set forth some of his ideas about the teaching of elementary geometry, outlining, among other things, a 'dynamic teaching' method based on the concept of transformation (Brigaglia, in Giacardi 2006a). Further, in 1865, Cremona produced an Italian translation (Genoa 1865–1868) of *Die Elemente der Mathematik* by the German mathematics teacher Richard Baltzer. This text was well suited to the innovations Cremona had proposed in 1860 but was not approved by the Ministry for use in *licei* because it was not held to be in keeping with the school programmes.

Worthy of mention is the fact that teachers chose the textbooks, but the minister oversaw these choices by means of quality checks and held competitions for textbooks from time to time. See, for example, the competition of 1874, in which all 23 of the textbooks submitted were declared unsuitable by the commission, which included Betti and Cremona (GU 1874, n. 191, n. 192, n. 193).

2.3 The Castagnola Programmes of 1871 for Technical Institutes and Later Modifications

In 1861, the *istituti tecnici* were placed under the supervision of the Ministry for Agriculture, Industry and Commerce, thus documenting their different aims and character of professional schools. In 1871, the minister approved a new decree (MAIC 1871)[14] that divided technical institutes into four 4-year sections (*physics-mathematics*, *commercial*, *agronomics*, and *industrial*), increased the number of weekly hours, broadened the subjects studied, and improved the didactic programmes. The aim was to create what at the time was called a 'school of culture': a kind of technical school in which theoretical subjects and general culture prevailed. It was also suggested that the physics-mathematics section might be transformed into a preparatory school for university studies in engineering. Brioschi and Cremona were part of the commission charged with planning the regulations; Cremona was responsible for the mathematics programme, for which he reformulated and developed some ideas of his 1860 project regarding technical instruction as the most well-suited context for implementing it.

In fact, the topics in algebra ranged from financial mathematics to combinatorics and probability, from the theory of complex numbers to methods of approximate solving algebraic and transcendent equations, and from the theory of determinants to indeterminate analysis. The geometry programme for the final 2 years of the physics-mathematics section was centred on a topic Cremona considered highly prominent for teaching: the projective theory of the conics. Moreover, the first elements of graphic calculus were included. In the introduction to the programmes, Cremona stressed that in addition to aiming to transmit 'a good fund of real knowledge […] in a way as to allow them [the students] to profit openly in their later studies and the exercise of professions', the technical schools should foster 'the other aim, which is also shared by classical schools, that is to reinforce the faculty of reason' (MAIC 1871, p. 52).

[14] Also available at: http://www.subalpinamathesis.unito.it/storiains/uk/istitut.pdf

Because there were no textbooks for projective geometry and graphic calculus, Cremona published his *Elementi di geometria projettiva* (Turin 1873)[15] and *Elementi di calcolo grafico* (Turin 1874). These textbooks, too, were not successful in secondary teaching. It had not been considered that projective geometry became mandatory in the university courses only in 1875.

In 1876, a new ordinance went into effect (MAIC 1876), scaling down the 1871 project. Projective geometry was reduced to mere principles and deprived of the whole theory of conics. In December of that same year, technical institutes were returned to the control of the Ministry of Education. Over the course of the ensuing years, the study of projective geometry was progressively reduced and finally eliminated altogether in the programmes decreed in 1891. The structure of these programmes remained substantially unchanged until the Gentile reform of 1923.

The failure of the 1871 proposal was due to various factors, both internal and external to the scholastic world (Scoth 2011). Actually, one part of the intellectual class believed that technical instruction should aim exclusively at training qualified workers and employees and oppose any developments that might compete with classical instruction. Nevertheless, the physics-mathematics section remained the school sector that provided the best scientific preparation; it is worthwhile recalling that many famous mathematicians attended these schools, including Corrado Segre, Francesco Severi, Gino Loria, Gino Fano, Giuseppe Veronese, and Vito Volterra.

2.4 The Flourishing of Textbooks for Secondary Schools

The reintroduction of Euclid's *Elements* as a textbook and the lively debate that followed acted as a catalyst in breaking the stagnant situation in which mathematics teaching in Italy was mired. As Enrico D'Ovidio and Achille Sannia wrote, 'it was like a surgical operation, extremely painful, but healing' (D'Ovidio and Sannia 1895, p. V). On one hand, the discussions helped focus attention on several important questions regarding geometry teaching, which were debated at length during the congresses of the Mathesis Association, an association of teachers of mathematics founded in Turin in 1895–1896. These questions involved the need for an in-depth examination of the foundations; the role rigid motions should play in the study of geometric problems; whether or not the treatment of geometry should be independent of a previous theory of real numbers; and, finally, the relation between rigour and intuition. On the other hand, as Cremona had hoped, the debate stimulated the publication of textbooks on elementary geometry for secondary schools: in fact, in the following 40 years, there would appear a large number of high-level textbooks written by some of the greatest Italian mathematicians of the day who, comparing and contrasting different methodological approaches, provided further stimulus for debates on mathematics teaching (Giacardi 2003).

We mention here some characteristic examples. *Elementi di Geometria* (Naples 1868–1869) by Sannia and D'Ovidio follows the Euclidean method while remedying its weaknesses and adding supplementary topics and exercises to prepare students for advanced levels of geometrical study. *Elementi di geometria ad uso dei licei* by Aureliano Faifofer (Venice 1878) is noteworthy for the method adopted in treating the theory of equivalent figures, following the guidelines laid down by Duhamel. Riccardo De Paolis's *Elementi di geometria* (Turin 1884) marks the beginning in Italy of 'fusionism', the name given to a teaching method where the related subjects of plane and solid geometry are studied together, with the properties of the latter applied to the former in order to gain the maximum benefit (Borgato, in Giacardi 2006a). De Paolis's book was too difficult and unsuitable for elementary teaching, so fusionism spread in Italy thanks to *Elementi di geometria* (Livorno 1891) by Giulio Lazzeri and Anselmo Bassani, which was more attentive to didactic requirements.

[15] This was translated into French, German, and English; for more information, see works of Di Sieno, in Giacardi 2006a and Menghini 2006.

Some of the textbooks were strongly influenced by the study of the foundations of geometry, which at the end of the nineteenth century – thanks also to the contributions of Moritz Pasch, Giuseppe Peano, Mario Pieri, Federigo Enriques, and Giuseppe Veronese – would culminate in David Hilbert's *Grundlagen der Geometrie* (1899). The textbook by Michele De Franchis, *Geometria elementare ad uso dei Licei e dei Ginnasi superior*i (Palermo 1909), is notable for its rigorous approach to the theory of congruence, an approach which introduced the 'group of motions'. *Elementi di geometria ad uso dei ginnasi e licei e istituti tecnici* (Padua 1909, with later adaptations for various kinds of schools) by Veronese and Paolo Gazzaniga (professor at the *liceo* in Padua) was clearly influenced by Veronese's famous *Fondamenti di geometria* (1891) but shows an effort to take into account the didactic requirements of secondary teaching, although this was not always effective (Klein 1925, pp. 247–248).

In the successful textbook *Elementi di geometria ad uso delle scuole secondarie superiori* (Bologna 1903, adapted for various kinds of schools and republished several times up to 1992), the two authors – Enriques, an eminent figure in the Italian school of algebraic geometry, and Ugo Amaldi – attempted to reconcile scientific rigour with the demands of teaching. In fact, in Enriques and Amaldi's *Elementi*, the subject is approached through the *rational-inductive* method in an attempt to overcome the defect typical of Euclidean exposition: on the basis of a series of observations, the authors enunciate certain postulates from which the theorems depending on them are developed by logical reasoning; then they are continuously related back to observations or explanations of an intuitive nature.

Among the algebra textbooks, two – one by Cesare Arzelà and the other by Peano – deserve mention because of their differing methodological approaches, which influenced subsequent mathematical literature. *Trattato di algebra elementare* (Florence 1880) by Arzelà was one of the most widely adopted textbooks in secondary schools. Written for the physics-mathematics section of technical institutes, it featured a methodological approach that differed from that of Bertrand, whose textbook (translated by Betti) was at that time very widely used in Italy: actually the core concept behind the presentation of the subject was not the equation, but rather the function. Peano's *Aritmetica generale e algebra elementare* (Turin 1902) reproduced whole sections of his *Formulaire Mathématique* and featured the systematic use of logical symbols.

2.5 Training of Teachers for Classical and Technical Secondary Schools and Related Debates

A major problem faced by the Italian political class was to establish a corps of adequately qualified secondary teachers, given that at the time anyone even without a degree was permitted to teach. Although the problem was urgent, it was not until 1906 that legislation was approved regarding the legal status of teachers, making it mandatory that only those who had passed a competition (*concorso*) could teach in the various kinds of schools and that a degree was required for admission to the competition (GU 1906, 106, p. 2085).

The *Scuola Normale Superiore* (SNS) of Pisa was, at least on paper, the only genuine institution for training secondary school teachers. From 1862 on, students from all over Italy were allowed to participate in the competitive examination for admission to the SNS. The final examination to obtain the qualification for teaching was comprised of a dissertation on a topic selected from among the subjects studied during the course of study at the SNS, a public lecture on a mathematical subject from the programmes for *licei*, and demonstration of the candidate's ability to use the instruments of physics and geodesy. But when Betti became director of the school in 1865, he gradually transformed the SNS from an institute for teacher training mainly into an institute for advanced research.

Eventually, in 1875, the ministry established *Scuole di Magistero*, or teacher training schools, to respond to the need to train future teachers and thus guarantee a higher level of secondary schools. In 1875–1876, out of twenty-one universities, only eight established *Scuole di Magistero*; among these, only three (Pavia, Pisa, and Rome) opened courses in mathematics. The initial purpose of the *Scuole di Magistero* was ambiguous, emphasising both an introduction to research and to professional training. This ambiguity was eliminated by the Royal Decree of 30 December 1888, which stated that 'the aim of the Scuola di Magistero is the practical preparation for secondary teaching' (GU 1889, 19, p. 219). It also placed a certain emphasis on mathematics by assigning 4 years to it but only 2 years to other scientific disciplines.

Notwithstanding the numerous successive decrees that concerned them,[16] *Scuole di Magistero* in many cases were completely inadequate for reliably addressing the problem of teacher training. There were many reasons for this, the main one being that the professors who taught there were the same ones who taught institutional courses at university because they had, with rare exceptions, no experience in secondary teaching. Other reasons were that supporting structures (libraries, laboratories, etc.) and teaching materials were practically non-existent; the number of course hours assigned was inadequate and there was scant funding. Therefore, it is not surprising that the problem of the professional training of teachers was one of the most hotly debated topics in the Mathesis Association congresses (see Furinghetti and Giacardi 2012) from the very beginning to 1920, when *Scuole di Magistero* were abolished by Minister Benedetto Croce.

2.6 Mathematics Teaching Till the End of the Nineteenth Century

The years from Italian Unification up to the beginning of the twentieth century were a period of great political and social ferment. In addition to the renovation of the universities, the mathematicians of the day focused their attention on secondary schools, writing new textbooks, creating associations and new journals concerning mathematics teaching, trying to improve programmes, and discussing methodological problems. Moreover, Italian research in mathematics, which was experiencing a moment of extreme vitality, stimulated the attention of mathematicians for education: in particular, studies on the foundations of mathematics constituted a common area of interest between elementary mathematics and advanced research. In addition, the mutual interchange between universities and secondary schools, when it existed, was a further source of enrichment: numerous university teachers had begun their careers as secondary school teachers (Cremona, Betti, D'Ovidio, De Paolis, etc.), while some of the most distinguished secondary school teachers also taught courses at university (Lazzeri, Faifofer, Bettazzi, Vailati, etc.).

However, there was no significant improvement in the quality of mathematics teaching during the last 30 years of the nineteenth century. Official reports on the examinations for the technical diploma in the 1870s and the *liceo* diploma in the 1880s show that mathematics teaching (as well as the teaching of other subjects) was considered in many cases to be inadequate.

In fact, the status of mathematics as a teaching subject continued to be fragile. Although the anomaly of the 1867 curriculum was remedied after some years, the structural effect remained. In 1870, the mathematics teaching in the *ginnasio* was again introduced in all its grades, but with just 1 h in the first three grades and 3 h in the last two. It is clear that mathematics was not a major subject and student preparation for the *liceo* was inadequate. Afterwards, the timetables changed very often, and the number of hours devoted to mathematics varied from a minimum of 8 (2, 2, 2, 1, 1) in 1882 to a maximum of 13 (3, 3, 3, 2, 2) in 1891.[17]

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[16] All of the legislative measures cited here can be consulted in the section *Teacher Training* of the site http://www.subalpinamathesis.unito.it/storiains/uk/training.php. More details on this subject can be found in Furinghetti and Giacardi 2012.

[17] These and the following data can be found in http://www.subalpinamathesis.unito.it/storiains/uk/timetable.php.

Likewise in the *licei*, mathematics teaching was introduced again in the last grade from 1870 to prepare for the final exams. In the same year, a ministerial circular addressed to the *licei* permitted the treatment of solid geometry with methods other than the Euclidean one. Over the years, the timetable varied from a maximum of 14 h (7½, 4½, 2) in 1876 to a minimum of 9 h in 1888, 1892, 1901, and 1911. With the distribution of 3, 3, and 3 or 4, 3, and 2 h, often in vigour from 1888 on, mathematics did not have an important status in the *licei* either; by the end of the nineteenth century, public opinion considered it among the minor teaching subjects (Scarpis 1911, p. 31). An international comparison of timetables for all secondary school subjects in various European states carried out in 1887 confirmed the weak position of mathematics teaching in Italy (see Giacardi 2006b, p. 597). Only in physics-mathematics at the technical institutes, there was a reasonable amount of mathematics teaching from 1871 on.

Yet, the public became ever more concerned with the feeble results of mathematics teaching. Various administrative and legislative measures from the 1870s to 1904 reflect these concerns, but the remedies weakened the status of mathematics teaching even further. The attention was drawn especially to the yearly final leaving exams of the *licei*. In 1878, the task assigned for these exams was a problem concerning quadratic equations: the results constituted a 'true debacle' – in fact, in many places, not a single student passed. This failure showed the lack of preparation in the *ginnasio* (Scarpis 1911, pp. 28–29). Various palliative decisions did not improve the results. Thus, in 1881, a decree abolished the written test in mathematics from the examinations for the diploma for the *ginnasi,* and in 1884, the mandatory character of this test was abolished for the *licei*, introducing the choice between mathematics, physics, and another scientific discipline. In 1888, students were allowed to choose between either an exam in one of the scientific disciplines or in Greek, with the result that only a minority opted for mathematics. The written exams in mathematics for the diplomas from the *ginnasi* and *licei* were made obligatory again in 1891, only to be abolished definitively in 1893 (Vita 1986, pp. 19–20).

In 1904, the decree by the minister Orlando gave second-year *liceo* students the option of choosing between Greek and mathematics, 'relieving congenitally incapable students of a useless burden' (BUMPI 1904, XXXI, p. 2851). Orlando's criticisms of the quality of classical secondary schools were correct, but they relegated mathematics even more to a role of secondary importance (see Scarpis 1911; Fazzari 1911). Despite protests from the Mathesis Association, the possibility to choose was abolished only in 1911.

Technical instruction also suffered from deficiencies in both structure and teaching methods. Further, this branch of education was penalised by the lack of coordination between technical schools and technical institutes that resulted from the separation of the ministries responsible for them until 1877. When the left came to power, greater attention was paid to the technical schools which were seen as a means of perfecting the elementary education received by the children of the lower classes. This attention resulted almost exclusively in the modification of regulations and programmes (there were five such modifications in the final two decades of the nineteenth century), without, however, producing actual improvement in teaching. On the other hand, the structure of technical institutes was not further modified after 1885, but the difficulties experienced by these schools were documented by reports on the diploma examinations for the years 1885–1888, which indicate that slightly over four out of ten managed to pass the exams (Scoth 2010b). It should be observed that in the diploma examinations for technical schools and the physics-mathematics section of technical institutes, the written test in mathematics remained mandatory.

The dysfunction of the Italian school system was, in part, due to the fragmentation of national politics: from the proclamation of the Kingdom of Italy in 1861 to the end of the nineteenth century, the country was guided by 36 different governments, and the ministers of public instruction changed 33 times. Moreover, by identifying 'secondary school' exclusively with *ginnasi-licei*, on which a rigorously humanistic imprint had been imposed, and relegating technical studies to an inferior level, the Casati Law expressed a rigidly elitist vision that was inclined to maintain a strict division of social classes. The consequence of this attitude was the tendency to keep humanistic knowledge separate from technical-scientific knowledge. This fact certainly did not foster a kind of mathematics teaching, which was at the same time related to everyday life and closely connected with both scientific and non-scientific disciplines.

3 From the Reform Project of the Royal Commission to the Eve of the Gentile Reform

3.1 The Reform Project of the Royal Commission

Eventually, in 1905, a Royal Commission was appointed by the Ministry of Education to elaborate on a reform for the secondary school system. This commission conducted a comprehensive inquiry into secondary schools and faculties and, in 1908, presented a draft law (*Commissione Reale* 1909) proposing two educational itineraries. The first was a professional technical school with 3-year courses enabling entry to the technical institutes. The second was a 3-year course for the lower secondary school common to all schools (*scuola media unica*), which did not include Latin as a subject and which would grant students access to the three different branches of upper secondary school: *liceo classico* (with Latin and Greek), *liceo scientifico* (with two modern languages and an expanded science syllabus), and *liceo moderno* (with Latin and two modern languages) – thus a threefold structure as in German secondary schools.

The syllabi for mathematics and instructions regarding teaching methods were written by Giovanni Vailati, a mathematician, teacher, and philosopher who belonged to the Peano School. He proposed not only methods emphasising experimental and active approaches – a ‘school as laboratory’ – but introduced, following Klein’s reform programmes, the concepts of function and derivative in all three branches of upper secondary school, the concept of integral in the *liceo scientifico*, and probability theory and its applications in the *liceo moderno* (see Vailati 1910; Giacardi 2006b, pp. 598–602).

This reform, and especially the unification of the lower secondary schools, was considered too radical not only by conservative thinkers but also by the majority of the members of the National Federation of Secondary School Teachers. The proposed mathematics curricula also attracted criticisms from both mathematics teachers and mathematicians. By contrast, Vailati’s plan was favourably viewed outside of Italy as innovative and following German and French reform movements (Giacardi 2009). In any case, given the many forms of resistance, the Ministry of Education never approved the proposed reform.

3.2 Castelnuovo and the Liceo Moderno: The Introduction of Infinitesimal Calculus in Secondary Schools

Thanks to Guido Castelnuovo, some elements of the reform proposed by the Royal Commission became implemented in 1911 when Minister Luigi Credaro established a *liceo moderno*, in which Greek was replaced by a modern language (German or English), greater attention was paid to scientific subjects, and elements of economics and law were added. Castelnuovo was given the task of preparing the mathematics syllabi and instructions for the teaching methods to be adopted for the new courses.[18]

[18] See Castelnuovo 1913; BUMPI 1913, XL, 45, pp. 2791–2795.

His interest in the school system was motivated by social factors (Castelnuovo 1914, p. 191), and unlike Cremona, he believed that the main aim of the secondary school was 'to educate the future citizen', not the elite (see Brigaglia and Gario, in Giacardi 2006a).

Castelnuovo had been one of the supporters of Vailati's proposals to reform mathematics teaching; he even had urged teachers to introduce some of these proposals without waiting for the Ministry to approve the reform (Castelnuovo 1909, p. 3). So it is no surprise that he adopted some elements of Vailati's reform project in developing the syllabi for the *liceo moderno*: he introduced the concepts of function, derivative, and integral, suggesting that they be illustrated by applications to the experimental sciences, and he attached great importance to numerical approximations. Nevertheless, the number of teaching hours for mathematics over the 3 years increased only from 9 to 10, compared to that of the *liceo classico*.

The syllabus for the *liceo moderno* was introduced into the schools from 1914–1915, despite difficulties caused by the lack of trained teachers, by the hostility of the teachers in the *liceo classico*, and by the absence of funds, which made it difficult to provide science laboratories.[19] Various textbooks were written for the new kind of school, which sought to apply the guidelines laid down by Castelnuovo: the most noteworthy of them was *Nozioni di matematica da uso dei licei moderni* (Bologna 1914) by Enriques and Amaldi.

3.3 Harmonising the Programmes of the Annexed Provinces of Trento and Trieste with Those of the Kingdom of Italy

After World War I, mathematics teaching in Italy confronted a revealing issue: how to harmonise the mathematics programmes in Italian schools with those in the provinces of Trento and Trieste (recently annexed from Austria), where the syllabi had been based on Klein's ideas since 1908–1909 and thus included the teaching of the elements of the calculus (Beke 1914). The Mathesis Association asked the Minister of Public Education to hear teacher representatives from the former Austrian provinces before making any decisions about curricula. This Association also submitted some proposals, which took into account that the content of the Austrian syllabus was, in some ways, more extensive and detailed, with more teaching hours allocated; in particular, it was suggested that the introduction to concepts of function and graphic representation begin in the *ginnasi* and that analytical plane geometry, which was introduced in the last year of secondary school in the annexed provinces, be retained as an experiment (*Bollettino della Mathesis* 1920, XII, pp. 55).

Discussions held during the many meetings of the various local chapters of the Mathesis Association generally showed appreciation for the methods and organisation adopted in the former Austrian provinces; several papers also stressed how similar the method and content were to those of the *liceo moderno* (Zuccheri and Zudini 2010). The proposals, put forward by the Mathesis Association concerning the harmonisation of the two mathematics curricula, were accepted, up to a point, by the Ministry (Giacardi 2006b, pp. 607–610). By late spring of 1922, the Ministry had prepared a draft law for the New Provinces, but in the following autumn, Mussolini became head of government and the Fascist dictatorship started.

[19] Cf. for example the Mathesis inquiry, *Bollettino della Mathesis* 1916, VIII, pp. 94–96.

3.4 The Lack of Training for Secondary School Teachers and the 'Combined Degree'

The existing inadequate institutions for teacher education had already been suppressed before Fascism: Minister Croce abolished the *Scuole di Magistero* with the Royal Decree of 8 October 1920. After intense protest by the Mathesis Association, the Ministry in 1921 substituted the *Scuole di Magistero* with the 'combined degree' (*laurea mista*) in physical and mathematical sciences, aimed at qualifying young people to teach scientific subjects in secondary schools. In addition, in 1922, a course in complementary mathematics (*matematiche complementari*) was instituted, accompanied by didactic and methodological exercises (BUMPI 1920, p. 2064, 1922, pp. 22 and 349). In this course, elementary mathematics from an advanced standpoint was to have been taught placing an emphasis on the historical, critical, methodological, and didactical aspects.

Contrary to other countries where mathematics used to be taught by the same teachers as physics, the combined degree in mathematics and physics was unpopular with many Italian mathematicians, even those who had always been in favour of a special degree for teachers. For example, Castelnuovo was pessimistic, predicting that universities offering this special degree would produce 'mathematicians lacking in culture and physicists lacking the skills for experimentation, thus turning out to be mediocre teachers in both disciplines' (*Relazione* 1922, in Gario 2004, p. 119). This judgment was shared by Volterra. By contrast, Enriques held that the combined degree had to be maintained and experimented with, 'in the conviction that bringing together mathematics and physics constituted an advantage for scientific and professional purposes' (Ibidem). He also appreciated the institution of the course of complementary mathematics.

4 From the Gentile Reform to the Mid-Twentieth Century

4.1 The Features of the Gentile Reform and the Protests of Teachers and Mathematicians

In 1923, the neo-idealist philosopher and Minister of Education, Giovanni Gentile, taking advantage of the full powers given to him by the first Mussolini government, carried into effect in a single year a complete and systematic reform of the Italian educational system according to the pedagogical and philosophical lines he himself had begun to develop in the early twentieth century. The decree relating to secondary schools was issued in May 1923, and the curricula and timetables were approved in October (BUMPI 1923, 50, pp. 4413–4510).[20]

Gentile rejected the democratic concept of a lower secondary school common to all students, designing instead a new programme of studies subdivided into two levels. Several kinds of schools comprised the first level, which followed immediately after primary school, including the *Scuola complementare* (a course that was an end in itself), the *Ginnasio inferiore* and *superiore*, the lower level of *Istituto tecnico*, and the lower level of the *Istituto magistrale* for primary teacher training. The second level was comprised of the *Liceo classico*, the *Liceo scientifico*, the upper level of the *Istituto magistrale*, and the upper level of the *Istituto tecnico* divided into two sections only – for 'commerce and accounting' and 'land surveying'. In addition to the *licei* mentioned above, a 3-year *Liceo femminile* (for girls) was also established. This was clearly a lower-quality school: it provided no

[20] Now in http://www.subalpinamathesis.unito.it/storiains/uk/rifgent.pdf.

teaching of the sciences whatsoever, it resulted in a diploma that had no professional value, and it did not permit entrance to university. Within this fragmented scheme of secondary education, three main courses of study can be identified: classical education, still made up of the *ginnasio-liceo*; primary teacher training; and technical instruction, in which the lower level of the technical institute took the place of the former 'technical school', while the upper level took the place of the former 'technical institute' created by the Casati Law.

The classical-humanistic line of study was intended to train the elite and was considered overwhelmingly superior to the scientific-technical line. The teaching of Latin was introduced in all the lower levels of the secondary school system. Thus, Gentile eliminated the physics-mathematics section of the former technical institutes as well as the *liceo moderno* and replaced them with a watered-down *liceo scientifico* with no specific lower level to prepare its students. In the preceding *ginnasio superiore*, mathematics remained a secondary subject with two weekly hours; moreover, this *liceo scientifico* provided only limited access to university faculties. In addition, mathematics was to be taught together with physics, and the lesson hours allocated to this combined course were generally fewer than those previously assigned to the two disciplines in the physics-mathematics section,[21] even if more than the hours in the *liceo moderno*. In addition, the reform established an official public examination (*esame di stato*) at the end of every school cycle and put public and private schools on the same footing. The final examination of the *liceo classico* assumed the character of an entrance examination to university and had to provide proof that the candidate possessed a broad humanistic culture.

The deeply humanistic basis of this reform is evident from the preface to the syllabi of the *ginnasio-liceo*:

> The *liceo-ginnasio* must be an institute of historical humanistic culture; it must prepare (students) for the high offices of civil life, for the professions, for political careers; it must prepare from the roots, preparing the man: the moral man, who occupies his place in history, and who is therefore aware of the arduous progress of humanity […], which consists not in the technical perfections so vaunted in our modern life, to the point of appearing ends not means, but rather consists in the more profound sense of human liberty and duty, in the more profound awareness of one's own personality. (BUMPI 1923, 50, II, p. 4435; transl. by the Authors)

Despite protests from mathematicians and mathematics teachers of the Mathesis Association, the curricula and timetables for the secondary school system were passed as law and none of their requests were granted.

The assistant to the Minister for the mathematics curricula was Gaetano Scorza, who was also a member of Mathesis and one of the Italian delegates to ICMI. Scorza succeeded in introducing elements of infinitesimal calculus and number theory in the *liceo scientifico* programmes, but in designing new curricula, he was strongly conditioned by the general framework of the reform and by the neo-idealistic epistemological view that the sciences can and must find their meaning and educational value only within the great Italian philosophical tradition. Moreover, the old curricula, which provided the teacher with an outline of how the subject matter was to be distributed over the course of the years and with helpful instructions on methodology, had been replaced by examination syllabi indicating the objectives to be reached but not the path to achieve them. The reform thus presumed that teachers were capable of developing a teaching plan on their own, but paid no attention to the question of their professional training.

The new curricula and the Gentile reform were also officially opposed by the Accademia dei Lincei, the most prestigious society of Italian scholars. Opposition reached its zenith in 1925 when faculties of the sciences of Naples, Bologna, Milan, Rome, Genoa, Florence, and Pavia were joined by other science associations to show their dissent, albeit in different ways (*Periodico di matematiche* (IV) 5, 1925, pp. 202–204). But while mathematicians like Volterra and Castelnuovo absolutely

[21] See the timetables in http://www.subalpinamathesis.unito.it/storiains/uk/doc/9.pdf.

opposed the Gentile Reform, others, like Enriques, assumed a conciliatory position. In fact, Enriques agreed with Gentile that among the various kinds of secondary schools, those which best performed the function of education were the *ginnasi-licei* (classical schools) and that encyclopaedism had to be fought. His ideal was to achieve a fusion between 'scientific knowledge' and 'humanistic idealism'. Enriques's position emerges clearly from his correspondence with Gentile (Guerraggio and Nastasi 1993) as well as from the report on the reform, which he prepared for ICMI in 1929 (Enriques 1929).

Moreover, when the racial laws against Jews were enacted in 1938, many of the talented university and secondary teachers had to leave their posts. For its part, the Mathesis Association was forced to change its by-laws on 3 November 1939. Art. 5 of the new by-laws gave the National Minister of Instruction the right to revoke memberships at will and to nominate the president and vice-president; and Art. 8 imposed an oath of loyalty to the Fascist regime on the president and vice-president.

4.2 Initiatives for Teacher Training by Individual Mathematicians

Gentile, identifying 'knowing' with 'knowing how to teach', believed that teacher training consisted of nothing more than 'genuine, profound and authentic scientific preparation' (Gentile 1907, pp. 178–179) and did nothing for the professional development of teachers. The course in complementary mathematics thus constituted the only aid given to future mathematics teachers. The sole measure he carried out was the decree of March 1923, which made the existing *Istituti Superiori di Magistero* for women[22] equal in status to university faculties; he also decreed that they then had to admit male students as well. However, these institutes were to limit themselves to training students to teach philosophy and pedagogy or to obtain the positions of school principal or school inspector; no training for future teachers of scientific subjects was to be provided.

Some initiatives promoted by individual mathematicians tried to fill these institutional gaps and improve teacher education. Two publishing enterprises by Enriques are noteworthy: the third enlarged edition of the 1900 collective work entitled *Questioni riguardanti la geometria elementare*, released in the 1920s with the new title *Questioni riguardanti le matematiche elementari* (Bologna 1924–1927), and the book series *Per la storia e la filosofia delle matematiche* (1925–1938) (Giacardi 2012). Another remarkable initiative was that of Peano in Turin: the *Conferenze Matematiche Torinesi*, a lecture series for teachers that started in 1915 (Luciano and Roero 2010).

Despite the institution of the 'combined degree' and the commendable initiatives of individual mathematicians, the level of teacher preparation was rather low. In 1938, among the competitive examinations for teaching positions in the various subjects in secondary schools, those of mathematics and physics registered the poorest results, with only eleven candidates out of a total of 381 declared suitable; three positions remained vacant (Vignola 1938, pp. 40–41).

The overall situation in Italy regarding teacher training can be seen in a general report on the theoretical and practical training of mathematics teachers for secondary schools presented by Loria for ICMI (Loria 1933), but is above all evident in the national report prepared by the ministerial supervisor Alfredo Perna (Perna 1933), which lists the institutional shortcomings in this area: there were no institutions for professional teacher training; there were no courses on methodology and pedagogy at the universities; there were no scholarships designated for teacher training; life-long learning was not compulsory; and training courses for in-service teachers were left up to individuals.

[22] These institutes had been established in 1878 in Florence and Rome to broaden women's cultural preparation and prepare teachers to teach the humanities in schools for women. Scientific subjects, while present in the didactic programmes, were considered secondary. See Di Bello et al. 1980.

Yet, mathematicians were divided between those who thought that university courses ought to provide training that was exclusively scientific in order not to lower the quality of the teaching and those who maintained the necessity of having professional training. It was only in 1961 that the combined degree was abolished and a special section of the degree programmes for mathematics devoted to future teachers was created.

4.3 New Textbooks

The publication of texts for secondary schools between the two wars was still flourishing, and the contribution of university mathematicians was significant. Besides the textbooks by Enriques and Amaldi, the two series – one directed by Roberto Marcolongo and Onorato Nicoletti (Perrella, Naples) and the other by Francesco Severi (Vallecchi, Florence) – are noteworthy. In particular, *Geometria* (1923) by Marcolongo and Cesare Burali-Forti and Severi's *Elementi di geometria* (1926–1927) merit special mention.

Other textbooks were written in response to the programmes prescribed by the Gentile reform. Remarkable among these for their special characteristics are *Lezioni elementari di Analisi Matematica* (Turin 1924) by Guido Ascoli and *Aritmetica intuitiva* (Milan 1923–1924) by Alessandro Padoa. Ascoli's textbook is formulated for *licei scientifici* and its main feature is the introduction of the concept of the definite integral before that of the derivative, because the treatment is simpler and more intuitive when the point of departure is the calculation of areas. Padoa's *Aritmetica intuitiva* is directed to lower-level middle schools, and its distinctive trait is the narrative method used; games and short stories are introduced to arouse curiosity and hold attention. The approach is gradual, featuring even graphic representations whenever possible.

4.4 Giuseppe Bottai's School Charter

The Gentile reform was unpopular with both Fascist leaders and mathematicians, and immediately after Gentile was dismissed as Minister of Education in 1924, various revisions were put into place in an attempt to relieve the rigorous standards for selection in order to favour the lower and middle classes that were seen as supporters of Fascism. The programmes for secondary schools were revised first in 1936 and again in 1937; in both instances, as far as mathematics was concerned, the revisions were mainly limited to rearranging the distribution of the subjects across the various years of study. In the 1937 programmes for the *maturità* (diploma) in classical and scientific *licei*, an emphasis was placed on the connections among the various theories as well as on applications to physics. In the final examinations in the primary teacher training institutes, greater attention was paid to methods of teaching arithmetic and geometry. Instead, mathematics was not one of the subjects studied in the final year of technical institutes, according to the regulations that had been in effect since 1933 (RU 1933, pp. 1492–2305).

The 'policy of retouching' (*politica dei ritocchi*) progressively altered the structure of Gentile's scholastic system and betrayed its spirit. The desire to impose a Fascist stamp on schools led to the reform called *Carta della Scuola* (School Charter), which was approved by Minister Giuseppe Bottai in 1939 at a time when war was imminent. The aim was 'to put Italian schools, all schools, from nursery to university, on another plane: that is on the plane of Fascism, of its doctrines' (Charnitzky 2001, pp. 454–455).

With the introduction of hours of work, the final 2 years of the 5-year primary school were renamed the *scuola del lavoro* (school of work), while the programmes remained substantially those laid out by Gentile, with the integration of elements of Fascist ideology. Primary school became a genuine vehicle for spreading Fascist propaganda. This school cycle, which was the same for everyone, was followed by three different kinds of 3-year schools: *scuola artigiana*, which prepared students for traditional jobs in rural or outlying areas – an end in itself; *scuola professionale*, which gave access to a 2-year technical school and provided training for low-level employees and specialised jobs as required in large cities; and the *scuola media unica*, which brought under a single umbrella the former lower-level secondary courses (*ginnasio*, *magistrali,* and *tecnici*) and gave access to upper-level secondary education. This was divided into five kinds of 5-year courses (*liceo classico*, *scientifico*, *artistico*, *istituto magistrale*, and *istituto tecnico commerciale*) and four other lines of study, each lasting 4 years, at the *istituto professionale* (Bottai 1939, plate I).

The beginning of World War II in September 1939, and Italy's entrance into the conflict in June 1940, prevented the new ordinances from going into effect; in contrast to the Gentile reform, the new regulations were meant to be introduced gradually. Of the new types of schools called for, only the *scuola media unica* was instituted in the 1940–1941 academic year.[23] From the remarks that prefaced the new programmes, what clearly emerged above all was the tendency to emphasise intuition along the path to abstraction and to use the history of mathematics to enliven the lessons and make them more interesting (RU 1940, pp. 4835–4836).

The *Carta della Scuola* also established a plan to ensure that in-service teachers remained up to date, with the creation of experimental *Centri didattici* (didactic centres) annexed to the principal universities. The first of these arose in Milan and Padua, and in 1941 in Florence, Bottai inaugurated the National Didactic Centre, annexed to the existing National Museum for Didactics. In his inaugural lecture, Bottai outlined the main objectives: to revive in in-service teachers an interest in studies concerning teaching methods, to initiate and stimulate specialised research in pedagogy and education, to create experimental classes to favour new teaching methods, to promote special courses for families and teachers, and to provide future teachers with the opportunity to practise teaching.

4.5 A Short Account on the Period from the Allied Commission (1943–1946) Actions Up to the End of the Century

After the fall of Fascism in summer 1943, Italy was split into two distinct governing bodies at war with each other. The monarchy survived in the south and exercised its government under the control of the Allied Military Government of Occupied Territory (AMGOT), while in the north Mussolini created the Italian Social Republic. The Allied Commission in Italy established a sub-commission for education, coordinated by the educator Carleton Washburne. Its aims were to reorganise instruction, eliminate all traces of Fascist propaganda from schools, and begin the process of democratising the country: 'Mussolini did his best to transform the school system in Italy into a machine to produce Fascists.... The school system that we found in Italy truly resembled the Roman ruins' (CAI 1947, pp. 370–371). Assisted by a sub-commission of Italian experts, Washburne, a follower of John Dewey, well known for the creation of the 'Winnetka Schools' and a supporter of the active method of teaching, designed new programmes for primary schools (BUMPI Part I, 1945.1, pp. 265–313), secondary schools, and teacher training institutes (Washburne 1970; CAI 1947, pp. 382–386).[24]

[23] Teaching programmes were enacted with the Royal Decree of 30 July 1940, no. 1174 (RU 1940, pp. 4824–4840).

[24] See Piano 1945, and *Ginnasio superiore, Liceo classico, Liceo scientifico, Orari e programmi d'insegnamento*, Milan, 1969.

With few modifications, lower- and upper-level secondary schools remained substantially as they had been under the Fascist regime. The mathematics programmes proposed by the Allied Commission for *ginnasi* and *licei* and for the teacher training institutes contained several innovations with regard to the past, above all from the point of view of methodology. According to the brief introductory chapter to the programmes:

> It is also useful, in order to keep interest in successive developments alive, to provide ample space for intuition, common sense, psychological and historical origins of the theories, physical reality, the developments that lead to immediately practical statements, leaving static and purely logical notions aside, which exclude all intuitive impulses. (*Piano* 1945, p. 13; transl. by the Authors)

The Allied Commission undoubtedly deserves credit for eliminating all reference to the Fascist regime from the Italian educational system and substituting nationalistic ideals with democratic ideals. However, its actions were not so effective in renovating national education, both because of the difficult situation in Italy and because of various political reasons.

When World War II ended, conditions in Italy were dire, but there was a strong impetus to rebuild, although the country's political and economic rebirth was not accompanied by an actual reorganisation of the educational system. Nevertheless, several significant modifications were made in lower-level secondary schools. The common middle school was introduced in 1962, with mathematics teaching combined with that of natural sciences (1963). In 1979, new programmes, to which Emma Castelnuovo (daughter of Guido Castelnuovo) contributed, were approved; these were very innovative from the point of view of content and methodology. Various reforms and new programmes for upper-level secondary schools were proposed (the Gonella Reform of 1952, the Frascati Programmes of 1966 and 1967), which, although not carried out, stimulated debate, as did the influence of international organisations such as the CIEAEM (Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques), the ICMI, and the OECE (Organisation Européenne pour la Coopération Economique).

Beginning in the 1980s, experimentation was carried out to test innovative mathematics programmes for upper secondary schools (PNI, *Piano Nazionale per l'Informatica* from 1985, the Brocca Programmes from 1988). These programmes proposed teaching by problems and introduced computer science in upper secondary schools, together with new subjects such as logic, statistics, probability, and geometric transformations.

Various projects were designed for teacher training until finally in 1990, the *Scuole di Specializzazione per l'Insegnamento Secondario* (SSIS, schools of specialisation for secondary teaching) were instituted, becoming operative in 1999–2000. These schools were closed by Minister of Instruction Mariastella Gelmini in 2008 as a consequence of her reforms, which replaced them by special teaching degrees accompanied by a practice teaching programme.

5 Conclusion

The political events related to Italy's unification established the premise for the emergence of a nationwide educational system. For more than 60 years, the Casati Law, which was the first law to provide an overall system for Italian instruction, formed the basis of all future legislation regarding education. The particular emphasis on university education, bureaucratic centralisation, and concern about preparing a ruling class firmly anchored in the values of humanistic culture were the Law's distinguishing features. Various degrees of importance were attributed to mathematics teaching over the following decades. During the Risorgimento phase, mathematicians, despite their efforts and political engagement, were not able to ensure an important role in education for their discipline. At the turn of the century, the devaluation of mathematics, especially in the classical schools, reached its high point.

Only after the first decade of the twentieth century, thanks to the commitment of a new generation of mathematicians and to the involvement of outstanding teachers, was the role of mathematics in education improved in certain sectors.

The outbreak of World War I interrupted further developments, and in 1923, the improvement of the role of mathematics was swept aside by the Gentile reform. This was the product of a philosophical and epistemological context, in which the meaning and educational value of the sciences had to be sought exclusively within the great Italian philosophical tradition. The reform was carried out during the Fascist period, which opposed the broad diffusion of scientific culture. The legacy of Gentile's imprint on culture weighed on scientific teaching for the entire twentieth century (Barra et al. 1992).

Despite many efforts, recent attempts at reform (from 2001 to 2011) have made clear that some perennial questions about mathematics teaching, which had been raised by mathematicians and teachers at the beginning of the twentieth century, remain partly unanswered: the best choice of didactic method, the scope of subjects to be studied, the connection between secondary schools and universities, and the scientific and professional training of teachers.

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Chapter 11
Mathematics Education in France: 1800–1980

Hélène Gispert

1 The Marginalization of Mathematics in Elite Secondary Education in the Nineteenth Century

The political principles that governed the reconstruction of the education system in France at the beginning of the nineteenth century after its destruction by the Revolution were, first, that education is a field of power that has to belong to the prerogatives of the State and second, that the official education system contributes to the consolidation of class structures in society. Therefore, under all the political regimes of the nineteenth century – the First Empire (1804–1815), the Restoration (1815–1830), the July Monarchy (1830–1848), the Second Republic, followed by the Second Empire (1848–1870), and finally the Third Republic (since 1870) – there were two distinct types of instruction, depending on the social status of the students ("on their natural destiny," as was the expression then in use). These education systems were opposites of each other in all the institutions, the curricula, and the teachers, with the barriers separating them written into law.

In 1808, Napoleon created the *Université impériale* (Imperial University), which had a monopoly on administrating education in France and encompassed all teaching institutions within a strict hierarchy. At the peak were the *facultés* (corresponding to higher education), followed by the *lycées* (7-year secondary schools from sixth form to the *terminale* in preparation for the *baccalauréat*[1]), and the collèges (secondary schools, run by municipalities, and of less importance than the *lycées*, which were run by the State) and at the bottom of the scale, the primary schools. The *lycées*, in which the State was interested almost exclusively, were the masterpiece of the institutional edifice and were intended to educate its future leading administrative and military officers. Throughout the nineteenth century, this purpose defined what teaching the *lycées* offered in mathematics as well as in all other subjects.

The first part of this text relies on "L'enseignement mathématique dans le primaire et le secondaire" written with Renaud d'Enfert (D'Enfert and Gispert 2010, pp. 333–341).

[1] The structuring of the *lycée* in the nineteenth century is described in Schubring 2003, pp. 50–52.

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1.1 *1802–1850: From a Foundational to a Secondary Discipline in the* Lycées

"The *lycées* will essentially teach Latin and mathematics." In placing mathematics at the same level as Latin in the male secondary curriculum, Napoleon's ordinance of 19 *frimaire* of Year XI (December 10, 1802) took into account the new situation following the French Revolution, in which mathematics had become a core aspect of an intellectual education combining theory and practice. But the organization of secondary schooling since the Empire and under the different monarchies until the 1840s grew to favor Latin and the classical humanities, which dominated the actual education given to future intellectual and administrative elites who went to the Napoleonic *lycées* (called "*collèges royaux*" with the return of the monarchy). Mathematics remained molded by its central role in the examinations for entering the professions of military engineers under the *Ancien Régime.*[2] The subject was considered "special" since (compared to subjects considered "general") it prepared for a professional specialization, and based on that fact, mathematics was not seen as qualified to contribute to the classical education of the *lycées* and *collèges*. As well as the physical and natural sciences, mathematics was not covered until the two highest classes of the *lycée*, which were called "humanities classes," the name itself suggesting the supremacy of humanities and classics over the sciences; taught as an incidental ("accessoire") discipline, in these classes, mathematics was deprived of its practical applications, focusing on abstraction and rigorous reasoning of a hypothetic-deductive character. However, mathematics occupied an important place in the specialized courses that were offered, often in private institutions, starting with third form – after 4 years of *lycée* – and that prepared for the *écoles speciales* of the government,[3] of which the *École Polytechnique* held the highest rank. The admission program to the *École*, drawn up by Monge and Lacroix in 1800 and changing little over the course of the next half-century, to a large extent directed teaching mathematics at the secondary level and included geometry and algebra (in particular, the theory of equations); calculus and its applications were, on the other hand, completely left to be taught at the École.[4]

1.2 *To Each Social Class "Its" Mathematics: Formation of the Mind Versus Training for the Practice*

The question of the character of school mathematics and the opposition between education for the practice and teaching of culture and the intellectual formation of the mind take on a new dimension once one considers the education system in general and looks beyond the education of elites in the *lycées* (which contained no more than 3 % of each year's cohort of boys). Since the early 1830s, the education system was organized into two separate systems of teaching: the secondary system of the *lycées* reserved for a narrow social elite and a primary system for the people. The latter was composed of lower primary schools, which gave a minimum of schooling for all and which could be extended by the upper primary school intended for the elite of the lower social classes.[5] We note first that the word "mathematics" hardly belonged to the official vocabulary of the nineteenth-century primary system. The terminology used was much less ambitious. In the lower primary school, which increasingly focused on the simultaneous (rather than sequential) teaching of "reading, 'writing, and 'arithmetic," the terms were

[2] "Old Regime," i.e., the political situation of France, with its feudal structures, before the Revolution.

[3] These "special schools" are also known as the *Grandes Écoles*.

[4] On the mathematical curricula of the *collèges* and *lycées* in the nineteenth century, see Belhoste (1989, 1995). On mathematics teaching in the *lycées* in the 1830s, see Ehrhardt (2008).

[5] This is the result of the law on primary schooling of 1833, presented by François Guizot (Chervel 1992).

"calculation" and "the metric system." As for upper primary school, the introduction of geometry by the Guizot law symbolized the challenges of the dual education system that prevailed at the time and acted as an indicator of educational aspirations and limits that had to enclose the curriculum for the common people, including its elite. If the teaching of the humanities, inconceivable in the primary system, paramountly embodied the social distinction of those with a secondary education, it was not the unique such subject. The way geometry was taught – though it was taught in both systems – was an equally strong marker. In the upper primary school, geometry could only be practical, taught through "its applications," line drawing and surveying, and removed from all abstract reasoning, which was a priori reserved for secondary schooling. As an inspector expressed it to upper primary school teachers in 1835: "You cannot develop geometry in the same way as it can be developed in secondary school…for your students, just the statement of the theorem will often be sufficient to apply it."

Under the weight of its image as a theoretical science liable to lead to a social disorder, geometry was even excluded from the primary school program in the Falloux law of 1850 and then again authorized as an elective under the Duruy ministry in 1865. We note that, however, a remarkable effect of this divide between cultural and practical learning, among young men still in school after 12 years of age (an extreme minority), only those who attended upper primary schools (or the institutions that succeeded them after 1850) had, in contrast to their peers from secondary school where the discipline was marginalized, a real practice of mathematics, studied from the viewpoint of its applications, as part of their normal course of study.

This duality in mathematics teaching between the school systems was encountered again exactly in the modes of teacher training. Regarding the secondary system (Belhoste 1995, pp. 30–32), mathematics teachers holding a chair of elementary or special mathematics in the *lycées* (or *collèges royaux*) had themselves gone to secondary school at the *lycée*, then were students in the science section of the *École Normale Supérieure*, and had passed the national competitive recruitment, the *Agrégation* in the sciences, and after 1841 the *Agrégation* in mathematics.[6] The teachers of mathematics in upper primary schools, however, were themselves from the primary system.[7] They attended neither the *lycée* nor any other secondary school, but after upper primary schools had studied in the primary normal schools (*écoles normales primaires*), whose programs – which had to be, above all, practical – were enclosed in tight limits and supervised by the State. As a reaction to the ambitions of certain schools, the directors of these schools were asked in 1843 to make sure that teaching mathematics never rise above elements and that neither algebra nor logarithms be introduced (D'Enfert 2003, p. 114).

A problem spanning the two school systems was the school education of girls. The Napoleonic secondary school system, which lasted up to the Third Republic, as well as the Guizot law of 1833 for the upper primary schools, concerned itself only with boys and ignored girls. It wasn't until the 1880s, during the Third Republic, that public secondary education of girls was created, but the girls received a different mathematics training: conceived with a practical orientation adapted "to the qualities and destinies of young girls," the programs gave lesser place to rigor and abstraction than those in *lycées* for boys. One had to wait until the mid-1920s that girls' mathematics instruction became aligned with that of boys.

1.3 Parentheses: The Reform of the Bifurcated System (1850s)

A reform of the *lycées* carried out in the first decade of the Second Empire, called the reform of the bifurcation, attempted to correct the untrammeled domination of the classical humanities by enlarging

[6] In many municipal *collèges*, however, the "*regents,*" who had neither studied at the ENS nor obtained the *agrégation*, but simply hold a license, oversaw all teaching in the sciences.

[7] On primary schools, see D'Enfert (2003).

the horizons of the education of social elites. With France amidst the Industrial Revolution, the reform created, alongside the classical track, another track that was better adapted for the education of scientific and technical, as well as industrial and commercial, elites. This new purpose revived both the contents and methods of mathematics teaching in the *lycées* and *collèges*, prioritizing utility and simplicity, giving instruction a practical character, and emphasizing the concrete rather than appealing to abstract reasoning. This attack on the pedagogical model of classical mathematics, and more generally on the aims of general education, was in part responsible for the failure of the reform, which the teachers, including those of mathematics and the sciences, vigorously opposed. The reform was abandoned about 10 years later; the classical *lycée* and with it, "the admirable concatenation of Euclid's propositions[8]" restored to the place of honor. "Special secondary schooling," deliberately practical and vocational, was however created. Its concentric organization of studies stood in contrast to that of classic secondary schooling, which assumed a "slow satiation" of concepts. Teaching of mathematics, very ambitious but anchored in the realities of life, was closely linked to practical applications: its objective was to "allow the understanding of the truth that the teacher will convey by citing numerous examples drawn from industry or the crafts" and in geometry, to "rely on evidence whenever possible."[9]

2 Reform of the *Lycées*: New Ambitions for Mathematics (1902–1914)

At the end of the nineteenth century, a major structural problem affected the teaching of mathematics. There were three different types of mathematics curricula, which were intended for different social classes and which reflected different statuses of mathematics. The first type, reserved for the intellectual and social elite and included in the domain of the sciences, was that of the classic *lycées*, which provided first and foremost a classical humanist education. Teaching of mathematics, still restricted to the margin of this secondary curriculum, was thrown back to the last year of the *lycée*. This was in particular the case for the scientific elite, destined to study in the *grandes écoles* such as the *École Polytechnique*, where mathematics was essential. The second and third types of schools were intended for training future managers in industrial and commercial areas. The one and the other, the upper primary schools for the lower classes and the *collèges modernes*, which succeeded the special secondary schools for the higher classes, played a key role in mathematical and scientific education, pursued for practical goals and oriented toward applications.

2.1 Mathematics and Modernities

The dichotomy between the purposes of teaching and the monopoly on classics and humanities by the *lycées* became more and more unsustainable for the political and economic elites of the Third Republic, when France was entering the Second Industrial Revolution. In 1899, the Chamber (French Parliament) launched a great survey across the whole country to debate *the* education question of the time: what education for what elite does a modern country need? What modernity and what humanities does it need?

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[8] Circular of September 22, 1863 (Belhoste 1995, pp. 384–388, (citation p. 384).

[9] Curriculum of this "enseignement special," communicated to the headmasters, April 6, 1866 (Belhoste 1995, pp. 413–442, citations p. 418 and 422).

Concerning mathematics and the sciences, one can remark different answers to these questions, and the debates highlighted different educative values, sometimes complementary: cultural values relating to "scientific humanities," which, together with the living languages and the modern study of French and literature, constituted the new modern humanities, but equally, practical values, with mathematics and sciences considered in this case as applied subjects, the applications being another part of modernity. Thus was questioned the unique function of study in the *lycées* up to that point, namely, to prepare an aristocracy of the mind for liberal careers, for the *grandes écoles* and for the teaching profession. This course of study now had to also prepare young men to participate in the economy and for active work.[10] Following the survey, the State undertook a profound reorganization of the structure and content of secondary schooling in the name of this new modernity. The reform of 1902 thus took account of a new goal and a new public.

The changes brought about by the reform of 1902 were very important. It was, for a time, the end of the monopoly on classical humanities by the *lycées*, through the creation of a modern curriculum that was on par – at least in theory – with the classical curriculum (Gispert et al. 2007, p. 41 and Hulin 2007, pp. 119–130). It also furthered the development of new disciplines such as the living languages, sciences, and mathematics. It effected a change in the profile of the elite, which still remained a very narrow social group but was now able to receive the *baccalauréat* in sciences and not only the *baccalauréat* in classics.

Whatever the arguments advanced, the inclusion of mathematics in modernity was done in company with the other sciences. Mathematics, in their own way, had to support the "scientific approach" that was explicitly promoted by the supporters of the reform of 1902 in their defense of a new place for the sciences (including mathematics); it had to allow to educate the future elite to perceive things exactly, to distinguish the real from the unreal. The challenge of the reform, as stated by the mathematician Emile Borel, was thus to "introduce more of life and of the sense of the real into our mathematical teaching" so that the students "realize for themselves that mathematics is not pure abstraction" (Borel 1904).[11] It is with that goal that mathematics was included in a modern humanist education and acquired a recognized place in the general education and formation of the mind.

Insofar as the structure of the mathematical curriculum was affected by the reform of 1902, one can note several convergent factors: first, the growing place of mathematics teaching, especially of geometry, in the first years of the *lycée*; second, the effects of the diversification of goals for secondary schooling; and finally, the effects of a third factor outside the world of education – the new concepts of geometry developed by mathematicians at that time. All these factors led to a renovation of the contents and the methods in all mathematics curricula. Thus, it was now necessary to teach mathematics and geometry – and in considerable extension – at the beginning of the first cycle: it was recommended to use the concrete, experience, and induction as the first step necessary before the transition to deductive reasoning. In the second cycle, it was necessary to introduce the new and connected notions of functions and their variations: teaching has to be now linked to physics and its needs.

This reference of the concrete related to epistemological and pedagogical arguments. Thus, the recommendations of Henri Poincaré to make use of "*instruments mobiles*" [i.e., drawing instruments like ruler and compass] to establish definitions in the teaching of geometry (Poincaré 1904)[12] were legitimized by his conceptions of what geometry was. Breaking explicitly with the Euclidian approach characteristic for teaching until then, Poincaré declared that it is "the consideration of motions of solid bodies that is … the true source of geometry" (Poincaré 1904, pp. 18–19). "What is geometry…?" he continued. "It is the study of a group, and of what group? of that of the motions of solid objects." Hence, his conclusion: "How then to define this group without making some solid objects move?" (Poincaré 1904, p. 19).

[10] See "séance des débats à la chambre, les 12 et 14 février 1902," in: *Journal Officiel* p. 666.

[11] Concerning Borel's position, see Gispert 2007, pp. 212–218.

[12] Concerning Poincaré's position, see Gispert 2007, pp. 205–212.

But this relating to the concrete and the experimental also traces to pedagogical principles that were totally new at the time for the *lycée* and the education of elites. They came out of the primary system, conceived initially for the teaching of the popular classes, with practical goals and centered around applications as we have seen. At the turn of the century, this reference to the practical changed its meaning and function: instead of restricting the ambition to learn beyond the acquisition of usual knowledge, it gained pedagogical value as a mode of teaching complementary to theoretical instruction, which up to this point was the sole method of teaching in secondary education (Gispert 2007, pp. 216–218).

2.2 Crisis of the Lycée*? Preserving the Systems of Instruction*

These ambitions to reform were very rapidly challenged. They threw themselves at the political and social reality of the systems of instruction. Their principles – reliance on the concrete and the practical – were challenged on an ideological base as irreconcilable values with the traditional secondary model – theoretical and disinterested – of the education of the intellectual and social elite. This challenge was one of the parts of a vast campaign orchestrated in the press of the time against the reform of 1902 and with which certain mathematicians associated themselves.[13] It protested against the "abandonment" of the classical humanities at the expense of a modernity that privileged the sciences, but especially modern literature and living languages. Among the teachers of mathematics and their newly created association of specialists, the APMEP (*Association des Professeurs de Mathématiques de l'Enseignement Public* [The Association of Mathematics Public School Teachers]), the reactions were mixed (Barbazo 2010; Barbazo and Pombourcq 2010). In response to a survey initiated in 1912 by the Chamber on the implementation of the reform of 1902 in each discipline, the Association expressed reservations concerning the method of relying on the concrete in mathematics. It highlighted the potential dangers of such a method and the harm that it could do if it was used to substitute experience for proof more generally.[14] Students should not be deprived of the advantages they could obtain from the study of mathematics, which had always been a school of logic.[15]

Over the course of these decades shaken by soul-searching in the secondary system, the upper primary schools developed with great success, featuring a method of instruction "clearly and boldly utilitarian and practical" [16] that advanced "the scientific and theoretical culture of [their] students by stressing mathematical knowledge, physics and mechanics that will allow them to understand and master tools" (D'Enfert 2003). For that, the pedagogical methods were clear: it was necessary to "supplement rigorous and abstract reasoning, for which time and attention spans are lacking, by applications and repeated experiences" (D'Enfert 2003). Its success did not abate until the interwar period, and its competition with secondary schools in small or medium towns after the First World War worried even the local leaders, who feared that attendance in the *collèges* would fall.

[13] Among others, one can mention here Emile Picard and Henri Poincaré (Rollet 2000, pp. 289–291).

[14] Declaration of the APMEP to the Parliamentary Commission of 1913, Bulletin no 14, février 1914, and Barbazo and Pombourcq 2010, pp. 29–37.

[15] See Barbazo (2010) on the views of APMEP, founded in 1910, regarding the methodology of teaching and their evolution.

[16] Projet de programmes de l'enseignement primaire supérieur – Exposé des motifs de juillet 1908 (D'Enfert 2003, pp. 312–315).

3 From the 1920s to the 1950s: Mathematics as a Discipline of General Education?

3.1 The Interwar Period: Return to the Education and Culture of the Mind

The challenges to the reform of 1902, which expressed themselves in the years following the reform, amplified after the First World War. During these years heavily marked by nationalism, a manifest consequence of this conflict, a nationalist dimension was added to the social dimension of the prewar critique. The reform of the beginning of the century was accused of being inspired by the German model of the *Realschule* to the detriment of the specificity of a "French spirit" based on Latin and the classical humanities. In 1923, the Chamber, strongly dominated by conservatives, voted for a new reform that revoked the 1902 programs and principles. Secondary instruction, including mathematics, was again dominated for decades by a theoretical and abstract conception.

The new reform removed modern secondary education from the *lycée*: Latin again became compulsory in the first years, and the monopoly of the values of the classical humanities was affirmed.[17] The foremost goal of the *lycée* once again became the education and formation of minds and hearts of the intellectual and social elite. Finally, the reform imposed "scientific equality," that is to say the same curriculum in mathematics and the sciences for all students up to their last years of the *lycée*, which had important consequences. There was thus a decrease in the number of hours devoted to science and mathematics as compared with the period after the 1902 reform and the postponement to the last year (for students concentrating in science) of teaching of almost all mathematical concepts (as before 1902). The affirmation of the general purpose of mathematics revealed itself in the theoretical and "disinterested" character with which it was taught believed to promote the awakening of critical awareness and access to general ideas.

The reform of scientific equality probably was a turning point in the arguments legitimizing mathematics that were given not only by ministerial authority but also by the teachers of secondary school. Certainly, the latter denounced the reform as likely, in their eyes, to lower the standards in science through a lightening of the schedules and the content. But at the same time, they adopted the arguments of the "classicists" to claim that mathematical teaching "aimed to educate and form the culture of the mind.[18]" Secondary school instruction in mathematics, as in other disciplines, was thus characterized as cultural, liberal, and disinterested instruction that excluded any practical and concrete goal, and this held throughout the interwar period.

This is all the more remarkable given that, at the same time, the idea of a "comprehensive school" (*école unique*) was making headway, and at the end of the 1930s, under the left-wing regime of the Popular Front, there was a circumspect reorganization of middle schools (corresponding to 12–15-year-olds) and a progressive harmonization of upper primary school programs and the first cycles of the *lycées*. Mathematics teachers in the *lycées* stood against any sign of applied studies and links to the real world, which were identified with the primary system, and therefore wanted to reject anything that could resemble a "primarization" of secondary education.

Another consequence of the First World War was linked to the role that women had played in society during the war by replacing the mobilized men. In 1924, the Chamber voted to align male and female education. Girls now had the same mathematics programs as did boys.

[17] Latin as a requirement was abolished in 1925, with the reintroduction of modern secondary education into the *lycées*.

[18] *Bulletin de l'Association des professeurs de mathématiques de l'enseignement secondaire public*, no 104, May 1938, p. 132.

While no significant changes affected mathematics teaching during the years of the Vichy regime in the Second World War, there were nevertheless structural measures that had an impact on the French educational system after the war. For political reasons, Vichy tried to destroy the primary school system (the primary schools, the upper primary schools, and the normal schools that future teachers attended when they were between 15 and 18 years old), which was known for its independence, its consistency, and its sturdiness. The primary school system was wedded to republican ideals and hostile to Vichy's collaboration with German occupying authorities. At the outset, Vichy abolished the "upper primary" schools in order to integrate them into secondary teaching and created the *collèges modernes*, less prestigious than the re-classicised *lycées*, where Latin had again become compulsory. Then, Vichy abolished the normal schools, and, consequently, future teachers now had to study in the new *collèges modernes* and take the *baccalauréat*, formerly reserved for the secondary system.

3.2 The 1950s: What Mathematics? What Teaching?

At the beginning of the 1950s, French mathematicians and teachers of mathematics individually and collectively deliberated anew about the teaching of mathematics. In 1950, a new commission called "Axiomatic and Rediscovery" was created under the auspices of the APMEP, in order to deliberate, experiment, and promote a reform of the content and methods of mathematics instruction (D'Enfert 2010). Such a title showed certain epistemological and pedagogical choices. "Axiomatic" refers to a conception of mathematics developed after the Second World War by the Bourbaki group of mathematicians, who wrote in 1947 the manifesto "The Architecture of Mathematics," where they characterized mathematics as follows: "In the axiomatic conception, mathematics appears all in all as a reservoir of abstract forms – the structures of mathematics; and one finds – without knowing well why – that certain aspects of experimental reality mold themselves in some of the forms, as a sort of pre-adaption." (Bourbaki 1962, p. 46). They put forward the new central role of structure in mathematics, which became the nucleus of what has since been called "modern mathematics." "Rediscovery" refers to another context, that of education. The teacher no longer must be the person who proves theorems; he must cultivate in his students the spirit of initiative and free inquiry, making them do mathematics as the mathematicians do it. At the same time, the APMEP, together with the *Société Mathématique de France*, organized lectures for secondary school teachers on the notions of structures that, for the most part, they had not seen in their studies. In fact, only in the 1950s did such ideas enter university instruction.

The new mathematics and its structures were recognized not only by mathematicians but even by scholars in other fields, in particular in the humanities, as a language and scientific tool that were essential for having access to any knowledge. In the area of education, one of the consequences was the convergence between mathematicians agreeing with Bourbaki and psychologists and philosophers such as Piaget and Gonseth. Conferences were organized since the early 1950s by the newly created international organization CIEAEM (*Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques* [International Commission for the Study of and Improvement of Teaching Mathematics]), in which French mathematicians played an important role (Gispert 2010, pp. 131–137).

The strategic importance given to such mathematics, and the need of a consequent reform of mathematics teaching, was similarly recognized by the agents of economic development. The OEEC (which became OECD in 1961) took a series of initiatives starting with the end of the 1950s. In 1958, it opened an office in Paris to "improve the efficiency of teaching mathematics and the sciences" and promoted a reform of the content and the methods of teaching mathematics for students 12–19 years old. At the end of the 1950s and the beginning of the 1960s, the OECD organized and financed multiple conferences of experts dedicated to secondary mathematical education for future engineers and scientists, to which French mathematicians were invited as experts. But the goal of these conferences was, in part, different from the previous initiatives. What was at stake was the adaptation of teaching – secondary and higher – to new developments in mathematics that have come with and after the Second World War. The stress was, therefore, put not only on sets and structures, algebra, and vectors but also on optimization, statistics, and numerical analysis – all subjects declared necessary for the modern education of a scientist or engineer.

4 The 1960s–1970s: Mathematics for All? The Reform of "Modern Mathematics"

Two important institutional reforms (the Berthoin reform of 1959 and the Fouchet reform of 1963) came in force during these years in France, establishing compulsory schooling until age 16 in a system of more or less complex institutions,[19] which all belonged to the secondary level, which constituted an essential innovation. It meant two structural effects: first, primary school became for all children the first step of prolonged schooling to prepare for secondary school; second, this "middle schooling" (given in a variety of institutional settings) had from now on new goals and new publics that differed from those of the preceding periods, delivering teaching to children whose academic and social future was to be very diverse, from extended general education to professional training and apprenticeship. Teaching of mathematics, conceived and delivered up to that point within the framework of the dual schooling system, also was directly impacted by these reforms.[20]

4.1 Place and Role of Mathematics in the Schooling System: Profound Upheavals

Starting in the 1960s, teaching of mathematics was impacted by profound upheavals that affected its content and its goals, but also its place and role in the school system. The changing definition of institutional frameworks was a prime factor in the evolution. Democratization of access to secondary schooling effectively increased the weight of modern tracks ("sections") in the secondary curriculum. The adaptation of the model of education to the needs, real or supposed, that were spawned by industrial modernization – it had to train the scientific and technical specialists needed by the country – reinforced the place of the sciences and particularly that of mathematics, in the ranking of school disciplines, to the detriment of Latin. The rising weight of the above disciplines in higher education (hence, in medicine or the human sciences) reversed the hierarchy of sections (constituting the last forms of the *lycées*) so effectively that starting in 1965, in the second cycles of the *lycées*, section C (specifically dedicated to mathematics and science) and to a lesser extent section S that followed it became the track for the elite. Mathematics, replacing Latin as the decisive discipline discriminating between student academic orientations, became a true subject of selection.

[19] "Collège d'enseignement général" (CEG), "Collège d'enseignement technique" (CET), and the first cycle of the *lycées*.

[20] These institutional reforms could also be considered either as a factor in the democratization of the educational system or as a factor of "massification", that is to say quantitative growth without real change in the social order. One of the arguments for this second interpretation lies in the planning by the State that more than 60 % of the workforce will be without a secondary education or diploma in the 1970s.

With the sixth form (the first of the forms of secondary school) opened to all students, mathematics teaching in elementary school also experienced important changes (D'Enfert 2011). There was no longer the goal of preparing students for active life by giving them techniques to solve real-life problems but "to assure a correct approach and a real understanding of the mathematical concepts linked to these techniques".[21]

Added to this extension of studies, the baby boom that followed the years of war caused an enormous increase in the number of students in secondary school and created a need for teachers. Thus, in middle school, less than 20 % of teachers of mathematics were certified or had obtained the *agrégation*, which could not miss to constitute a serious handicap at the point of launching the reform of "modern mathematics." The remaining 80 % of mathematics teachers were either auxiliary teachers, often without license in mathematics, or former teachers of higher primary schools, also without higher certifications in mathematics.

4.2 The "Modern Mathematics" Reform

In December 1966, in this context of profound institutional changes, the National Education Ministry gave in to the demands of mathematics teachers and created a commission for the study of teaching mathematics, led by André Lichnérowicz. The reform project presented by that commission was inspired by the reflections of the 1950s and was largely supported by the APMEP, which wanted a renewal "from kindergarden to *facultés*." More broadly, the reform was desired at the outset and unanimously supported in France. The program of the Commission was clear. It had to first work out new guidelines for teaching mathematics in primary and secondary school and assess their viability by pilot experiments. It also had to develop a training program for teachers and create new institutions for that – later named the IREMs. In force as of the 1969 school year, the reform based itself on a critique of traditional mathematics teaching (symbolized by classical geometry), considered too far removed from living mathematics, that is to say mathematics as taught and done in universities since the mid-1950s, with algebra of sets, probability theory, and statistics. Euclidian geometry and calculus were no longer taught as such. Convinced that mathematics has to act as a driving force in the development of hard sciences and of human and social sciences as well, in citizens' daily lives, and, beyond that, in the modernization of society, the proponents of the reform saw in mathematics above all a new language that allowed all citizens to understand its functioning. One of the principal challenges of these reformers was to offer to all children, no matter what their academic future, the most modern mathematics. That was the path taken by the democratization of mathematical education.

The new programs, which integrated the concepts, the vocabulary, and the symbolism of modern algebra, favored the introduction to mathematical structures and their axiomatic construction. In middle school, for example in geometry, the formulation of the programs favored a logical presentation of the different concepts in order to eliminate any appeal to sense-intuition. The programs for forms with students of 13–14 years of age (third and fourth form) also indicated that one needs to employ distinct terms for concrete objects and their mathematical models each time that the risk of confusion between them appeared. The break was radical (D'Enfert and Gispert 2012). In the second cycle, where a renewal of the program, together with the introduction of structures and the language of sets began in the early 1960s, the changes were less brutal.

[21] Circular of January 2, 1970 on teaching mathematics in elementary school, *Bulletin officiel de l'éducation nationale* (*BOEN*) no 5 du 29 janvier 1970, p. 349.

4.3 Obstacles and Protests

At the beginning of the 1970s, dissent erupted in the heart of the Commission and the unanimity of the beginning collapsed. The middle-school programs, in classes at the end of the compulsory schooling period, in which the teachers were, for the most part, poorly trained, posed particularly sharp problems both on the mathematical and the social levels. Mathematicians and physicists inside and outside the Commission began to criticize the predominance of formal and abstract approaches in the mathematical programs. Formalism and abstraction were not beneficial for the large majority of students and teachers, who were poorly prepared for this. But neither were they beneficial, said the critics, for the formation of future physicists and engineers. Equally severe criticism came from professional associations such as the APMEP or even the IREM, from the universities, from the Academy of Sciences, and from industrial and economic environments.

The ambitions of the "modern mathematics" reform broke, in part, against the rock of the new social reality of the middle school, given the end of the former school structure, with which the reform found itself confronted. The reformers from the Lichnérowicz Commission, all coming from the elite secondary system, could not and did not know how to take into account the specifics of this new school structure in which the majority of the students and teachers had inherited the "practical" tradition of the primary system, and these teachers had no familiarity whatsoever with the "modern mathematics." The temporal coincidence between the increase in the number of students caused by the institutional reforms and the mathematical reform created the fact that, for the first time in France, the same mathematics curriculum was presented to students going into the workforce as to students who had to follow a longer course of study. However, the conception of democratization of teaching, inherited from the interwar period and taken for granted, saw the model for the elite as the best possible and as applicable to all. And it was so for mathematics, where the mathematical and pedagogical traditions from the primary system were abandoned in favor of those from the secondary system.

Moreover, challenged since the early 1970s, including by its supporters who believed the reform did not correspond to their recommendations, the "modern mathematics" reform was abandoned in the early 1980s in favor of a teaching method that, envisioning mathematics in the diversity of its applications, placed the accent on problem solving and favored "applied" components of the discipline. These two aspects now occupy a central place in mathematics teaching. At the same time, since the early 2000s, there has developed the ambition to make mathematics into a subject that allows students to throw themselves into a true research program, capable of developing their abilities to reason and argue but also to experiment and imagine.

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Chapter 12
Mathematics Education in Germany (Modern Times)

Gert Schubring

1 The Napoleonic Period

Given the key role of mathematics in the new French public school system and the dominance of Napoleonic France over great parts of continental Europe, the first task here is to reveal how far this French dominance influenced reforms of mathematics teaching in German states. To do this, we first need to report on the political geography in Germany; in fact, the effects of the French Revolution changed its structure completely. In 1803, the venerable but apparently politically incapable of acting Holy Roman Empire of the German Nation had completed a major reversal: it decided to dissolve its considerable number of ecclesiastical states. These territories were simply divided among the greater of the remaining states, thereby reducing the originally more than 300 sovereign units. In 1806, two major events followed. The Emperor of the Hapsburg Dynasty resigned from his crown, becoming Emperor of Austria-Hungary; the German Empire thus ceased to exist. At the same time, Napoleon had forged an alliance of the majority of German states, the *Rheinbund* (alliance of the Rhine), Saxony, and Bavaria being among its greater members. Prussia and Austria did not adhere.

Thus, on one hand, there were the Rhineland territories – at the left side of the Rhine, with Cologne and Mayence – as direct parts of France since 1795 and hence applying all measures of its centralistic school policy, particularly for mathematics, too. A great part of *Rheinland* became Prussian after 1814, with its schools proving to be well prepared for the Prussian Gymnasium reforms. On the other hand, there were the states of the *Rheinbund,* some relatively independent like Bavaria and others satellite states. In fact, two member states were governed by relatives of Napoleon: the kingdom Westphalia, with Kassel as capital, and the Grand Duchy Berg, with Düsseldorf as capital.

Bavaria proved to lead the educational reforms of this period, especially in the enhancement of teaching mathematics. Promoted by the energetic Minister Maximilian von Montgelas, extensive social and political reforms were enhanced, and school reforms were part of this comprehensive program. From 1804, a new Gymnasium was established combining classical studies with a minor amount of realist disciplines. A profound reform followed in 1808, instituting a bipartite system of secondary schools and prefiguring the later characteristic split between *trivium* and *quadrivium*. Both systems consisted of two ladders: on the classical side were the *Lateinschulen* for the lower grades and the *Gymnasial-Institute* for the upper grades; on the realist side were the *Realschulen*, followed

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by the *Real-Institute*.[1] However, this did not mean a restriction of mathematics in the realist system. To the contrary, mathematics was well represented on both sides, being a major discipline taught in all grades. In the *Real-Institute*, mathematics was even taught between 6 and 8 hours every week (Wieleitner 1913).

Regarding the two satellite states, one would expect the introduction of *lycées*-like *Gymnasien*, with mathematics playing a strong role. Yet, this was not the case. In the Grand Duchy Berg, created in 1806, no initiatives were taken until Napoleon's personal intervention in 1811. According to the literature, nothing happened even afterwards. In fact, however, a law was passed in December 1811 introducing an education system analogous to the French one. Few activities to implement it were observable. The Gymnasium in the capital Düsseldorf was reorganized, however, by the proper initiative of its director in 1813, almost in agreement with Prussian neo-humanist conceptions (Schubring 2010, p. 52). In the Kingdom of Westphalia, reforms also began only shortly before the end of the Napoleonic period. The decree of 28 June 1812 reformed the Gymnasium in Kassel, serving as a potential model for the other secondary schools but was never carried through. Mathematics was organized as a main teaching subject, and a teacher of mathematics was required (Schubring 2010, p. 63).

2 Development from 1815

After the definitive defeat of Napoleon, the Vienna Congress of 1815 reorganized the European political map. Germany became a collection of 39 independent states, still including Austria, all loosely connected by the *Deutscher Bund* (German Union): one Empire, five kingdoms, one electorate, seven grand duchies, ten duchies, eleven princedoms, and four *Freie Städte* (independent towns like Hamburg). The two dominant – and rivalling – states were the Kingdom of Prussia and the Empire of Austria (and Hungary). The development of the education system in these states occurred in diverse ways, even with regard to the status of mathematics. Prussian neo-humanism was confronted by classical humanism.

2.1 Prussia

Prussian reforms had already begun during the Napoleonic period because of the devastating defeat of 1806. Various initiatives for educational reforms were systematized from 1810 by Minister Wilhelm von Humboldt (1767–1835), as an integral part of the profound social and political reforms of this state. The specific measures taken were the foundation of the Berlin University in 1810 and the edict of 1810 establishing a state examination for prospective Gymnasium teachers, thus enacting the reform of the Philosophical Faculty, and the edict for the *Abitur* exam of 1812. The comprehensive study plan for the Gymnasium, also elaborated in 1810 by the counselling body of the ministry, is known as the Süvern-Plan of 1816, given that Johann Wilhelm Süvern, leading official of the Ministry, edited it as part of a school law.[2]

The conception underlying the reforms was neo-humanism, which understood human knowledge as an interconnected organism so that intellectual development afforded an all-sided incitement of cognitive abilities in the students. One of the reformers and members of Humboldt's 1810 counselling body, August Ferdinand Bernhardi, for instance, affirmed the complementarity of learning both

[1] For the emergence of the school type devoted to the realist disciplines of *Realschulen*, see the chapter on Western Europe.

[2] The complete text of the Süvern-Plan is published in Schweim (1966).

languages and mathematics. He attributed educational values to mathematics for inciting an "understanding for ideas" and to languages for inciting an "understanding for phenomena." Regarding formal education, mathematics incites "the cognizing abilities of the mind, profoundly and thoroughly," while languages incite "the purely perceiving abilities of the mind, of memory" and also comprise "the formation of sentiment and of ethos" (quoted from Schubring 1991, p. 76; my transl., G. S.).

Based on this concept of an organism, the curriculum prescribed three main disciplines: (classical) languages, history and geography, and mathematics and the sciences; all three constitute the three pillars of instruction in each grade. For the 9 years of this neo-humanist Gymnasium, two mathematics teachers should act at each school; the Süvern-Plan required 6 weekly hours for mathematics in each grade. The syllabus itself – elaborated by Johann Georg Tralles (1763–1822) who was the first mathematics professor at Berlin University – was very demanding and followed the analytic approach which was prominent in education since the French Revolution.

Presupposing the foregoing elementary schooling, the syllabus began with arithmetic, including the decimal system and decimal fractions, and continued with non-decimal systems, elementary geometry; algebra (equations of first and second degree); operating with powers and binomials, logarithms and their applications, elements of analytic geometry, and angular functions; algebraic equations and numerical solutions, analytic geometry of two and three dimensions, and conic sections; elements of the theory of series; theory of combinations, plane and spherical trigonometry; equations of the third and fourth degree; arithmetical series, the theorem of Taylor and developing in series; probability theory; and a branch of applied mathematics (Schubring 1991, p. 44).

Although the Süvern-Plan was not obligatory because of the neo-humanist concept of teacher autonomy and was only communicated to regional authorities and schools as a recommendation, this demanding program was taught in a number of *Gymnasien* in the 1810s and 1820s. Given the reform of the Philosophical Faculty, mathematics became instituted there as a study course leading graduates to a real profession: as mathematics teachers at a Gymnasium. The considerable number of *Gymnasien* in Prussia prompted a steady demand for mathematics teachers; ever more students inscribed themselves as students of mathematics and a steadily growing number graduated, thanks to the state examination, as mathematics teachers (Schubring 1991, pp. 126 ff.; Schubring 1992).

From the 1830s, the analytic orientation of the Tralles-Süvern-Plan came under pressure. Since there was no officially prescribed curriculum, the exigencies for the *Abitur* exam increasingly determined what should be taught during the Gymnasium years. Here, a contradiction in the reform programs was revealed as decisive: the *Abitur*-Edict of 1812 had postulated much less knowledge than the curriculum – only up to its middle grades, probably expressing the consciousness that in those early years the exigencies of the curriculum in general could not be achieved. But in 1834, when an updated regulation for the *Abitur* exam was issued for the by now consolidated system of Prussian *Gymnasien*, the exigencies for mathematics had basically not improved. Consequently, in a decree of December 1834, the Ministry excluded two subjects in mathematics from the Gymnasium curriculum that had been favored by the teachers: conic sections and spherical trigonometry. Since conic sections had been the paradigmatic subject for practicing analytic geometry, its exclusion introduced the emphasis on synthetic geometry (Schubring 1991, p. 60). It became now a keyword for school authorities to demand that mathematics teachers not exceed the "limits" of the *Abitur* exam.

The Humboldtian Gymnasium reform has been characterized as an "utraquist" solution to the tension between the *trivium* and *quadrivium*, between a liberal education leading to university studies and the transmission of knowledge useful for middle-class professions. In Humboldt's view, *Realschulen* thus should not exist. The Gymnasium should serve both aims – in modern terms as a comprehensive school. This idea came under pressure from two sides: from philologists on the one hand who did not agree with the neo-humanist vision of knowledge as an organism and tried to reinstall the supremacy of classical languages and from fervent adherents of *Realschulen* on the other hand who thought only narrowly of immediate professional training. This *Realschulen* movement was evidently much stronger in the other German states where classical studies still dominated, but some activities for founding such schools emerged even in Prussia. It is revealing that in Berlin, another variant of an utraquist solution was founded in 1829: a *Realgymnasium*, a school with less classical studies – namely, with Latin, but without Greek – and with extended teaching of mathematics and the sciences. Mathematics teachers at the *Realschulen* in Prussia did not propagate a special vision of mathematics, i.e., an applied one; as graduates of the universities, they shared the same visions as their colleagues at *Gymnasien* – as can be judged from their publications, particularly the *Schulprogramme*, a special German type of yearly school report, accompanied by the dissertation of one of the school's teachers (Schubring 1986).

2.2 Different Status in Other German States

Following Prussia's example, most of the other German states also introduced the *Abitur* exam, mainly in the period between 1815 and 1830 – a period named *Vormärz*, or before the revolutionary movements of March 1830 – so that in most cases, mathematics obtained the status as a main teaching subject. Actually, only in a formal manner since the real status could be much lower. A telling example is presented by Bavaria.

2.2.1 Bavaria

After 1815, due to strong political reaction against the earlier commitment to Napoleon, a drastic setback occurred in 1816, when the reform achievements of the Napoleonic period were renounced. On one hand, the realist branch of secondary schools became suppressed completely, while on the other hand, the classical part resumed the earlier dominant position of Latin. Mathematics teachers were dismissed, and the teaching of mathematics was reduced to only 1weekly hour – now to be taught by the single generalist teacher, usually without specific mathematics training, in his respective grade. In fact, mathematics instruction did not remain on this super-marginal level; in 1822, the 1 weekly hour doubled and mathematics teachers were again admitted to the upper form after passing centralized exams for this instruction. However, they had to serve solely as specialist teachers – all other subjects continued to be taught by the generalist teacher. For the lower school type, the *Lateinschulen*, mathematics continued to be taught by the generalist who focused on teaching Latin even until 1864. Given the lack of teaching of the sciences at the Bavarian secondary schools until 1854, mathematics was conceived of there in a rather one-sided orientation to the humanities, focusing on Euclidean – hence synthetic – geometry. The absence of science teaching was substituted by lower-ranking commercial schools (*Gewerbeschulen*) where mathematics played a strong role but was tied to applications.

2.2.2 Kurhessen

Another revealing case is presented by Kurhessen, a minor among the medium-sized German states, with a mainly Protestant population, situated in the middle of Germany, with Kassel as capital. The country was particularly backward in its political and social orientation. For instance, the guild system continued in full practice here, with a strict separation of classes of craftsmen. The author of a history of instruction in Hessen-Kassel stated that essentially "theologians," i.e., graduates of Protestant theology studies, had served as teachers at *Gelehrtenschulen*; only its annexation by Prussia in 1866 brought about the change. The author, even in 1911, deplored all tendencies towards secularization of schooling (Wolff 1911, p. 492).

Regarding education, the Gymnasia were still understood as *Gelehrtenschulen*. The draft version of an edict of 1835, “Verordnung über die Gelehrtenschulen,” declared classical languages as the major disciplines. Accordingly, the role of mathematics teaching was rather ambiguous. On one hand, it had been a subject of the *Abitur* exam since 1820; on the other hand, the number of weekly hours attributed to it changed several times between 2 and 4 h. In 1843, however, it became generally to at most 3 h as a part of a radical reduction of the role of mathematics teaching – unique among the German states.

A decree by the Ministry of the Interior of 28 February 1843 declared that equations of second degree already transcended the “limits” of school mathematics and geometry should be restricted to the elements:

> that the teaching of mathematics in Gymnasia should be extended, regarding its coverage but until the equations of first degree, while those of the second degree will be omitted; furthermore that the teaching of plane trigonometry will be restricted to its elements, and that consequently the demands in mathematics for the Abitur-exams will be reduced. (quoted from Schubring 2012, p. 531)

The decree further underscored that the methodology of teaching should avoid abstraction and enhance concrete and intuitive approaches. This reduction of teaching contents was accompanied by another reduction to at most three teaching hours. One of the authors of this decree revealed in an expert opinion of 1846 the motivation for the reduction: low results in mathematics should not endanger students achieving well in classical languages from passing the *Abitur* exam (Schubring 2012, p. 531). Only after the annexation of Kurhessen by Prussia in 1866 was this extreme reduction overcome.

2.2.3 The Non-neo-humanist Gymnasium Conceptions in Germany

One often reads in the literature on German school history that neo-humanism focused on classical studies and did not enhance mathematics. Since Prussia established neo-humanism, the contrary, in fact, is true. The origin of this widespread conviction is confusion with old humanism, or a moderated form of it, prevailing in other German states, as well explained by Friedrich Paulsen (1846–1908), the author of the classical study on the history of the German *Gelehrtenschulen*. As he pointed out, the Bavarian-Saxon-Schleswig-Holsteinian Gymnasia perpetuated the dominance of the humanities until the second half of the nineteenth century. He characterized their “moderate” forms of humanism regarding mathematics teaching by the practice of the teaching staff in these countries: if a student succeeded in mathematics, it was fine; if not, it did not constitute a problem (Paulsen 1909, p. 114).

Paulsen, who had studied from 1863 at the highly traditional Gymnasium *Christianeum* in Altona (duchy Holstein) and graduated there in 1866 with the *Abitur*, vividly revealed how he and his comrades had internalized the role of mathematics prevailing there. The *Gymnasien* in the Holstein and Schleswig duchies had been moderately reformed in the 1830s by a philologist from Saxony, implementing from there the conception that realist disciplines had to be contented with being minor subjects. The students completely disregarded their mathematics teacher – Paulsen called him the most sorrowful figure among all his teachers (Paulsen 1909, p. 120). By contrast, they had completely assimilated the preeminence of Latin. He described their devotion to attain a true Latinity:

> The precondition for the success of this treatment of Latin consists evidently in the need that writing Latin is reputed by the students as the first and foremost skill of the scholar. For us, it still completely applied; primarily it was just Dr. Henrichsen who was able to maintain us this conviction: Latin characterizes the scholar—with a good Latin style one will succeed throughout the entire world, he used to say. Nothing occurred, which could have made us doubting in this. We confidently disprized the modern languages: they are for the merchant, not for the scholar; we disdained mathematics and we knew nothing of the sciences; and of technique and industry nobody in our context ever spoke. […we regarded] the Antiquity as the courtly world in which to be at home constitutes the privilege of the scholar; the Latin style being the common shibboleth and practically the legitimation of scholarly education. (quoted from Schubring 2012, p. 532)

Here, too, the annexation by Prussia in 1866 eventually brought about changes. Paulsen emphasized the earlier and more thorough secularization in Prussia as the major reason for the social, cultural, and curricular divergences with the other German states (Paulsen 1909, p. 114).

3 Improvements from About 1848

While the political movements of 1830 had not brought about an improvement in the teaching of mathematics and the sciences in non-Prussian countries, the revolutionary movements of 1848 had such effects, in particular in Austria and Bavaria. In 1854, the teaching of mathematics became consolidated there with 3 weekly hours in the three grades of the *Lateinschulen* and 4 h in the ensuing Gymnasium. This weekly share was still less than that in use in Prussian Gymnasia, in particular in the lower grades, but still somewhat comparable to the official share for the Prussian Gymnasia of the upper grades, meanwhile. From 1864, with the foundation of the "Realgymnasium" in Bavaria, a secondary school teaching the sciences and more mathematics, the classical school assumed the name "humanistisches Gymnasium" – hence the term usually thought of as characteristic of the neohumanist conception of schooling is in reality characteristic of the deviant Bavarian conception.

A new program of 1864 had as a result to pass mathematics teaching at the *Lateinschulen* from generalists to specialist teachers, too. The new program of 1866, replacing those of 1864, already aspiring a reasonable level, defined teaching topics corresponding to what used to be taught in Prussian Gymnasia: in the last two grades, the notions of permutations, variations, and combinations (the traditional foundations in Germany for teaching and proving the binomial theorem); applications of the theory of combinations to probability theory; some applications of algebra to geometry; and trigonometry. These topics could only be covered superficially, however, since half of the 4 h in both grades had to be devoted to mechanics (because of the lack of proper teaching time for physics) as well as to popular astronomy – a traditional teaching subject in Bavaria, inherited from the former Jesuit program (Schubring 2012, p. 528).

Saxony, since 1815 a kingdom but with reduced territory, had introduced the *Abitur* exam in 1830. When the government tried in 1834 to pass a law reinforcing the teaching of mathematics and the sciences in its two state schools (the *Fürstenschulen* or *Landesschulen* in Meissen and Grimma) and in the municipal *Gymnasien*, the rare situation occurred that the Parliament refused the law. The main reason for the refusal was the applied aspect of mathematics as well as teaching the natural sciences, both of which were fiercely criticized by radical philologists. The mere introduction of natural sciences in the three first classes was condemned on the ground that "one cannot unite two directions, and one should not be allowed to serve two masters at the same time. Classical education should always stay the foundation" of the classical schools in Saxony (Morel 2013, p. 51). Only in 1846 was it possible for the government to decree a curriculum with the amount of teaching hours comparable to Prussia. In 1848 followed regulations for teacher training.

The kingdom of Württemberg in Southern Germany showed particular reluctance to modernize. It continued to adhere strictly to the principles of the *Schulordnung* (School Study Regulation) that was introduced in the sixteenth century after converting to Protestantism. Secularization had been delayed in this agrarian country. The curriculum of its state schools, here named *Klosterschulen* since using the dissolved monasteries, was essentially determined by preparing the students until the end of the middle grades for the so-called *Landexamen*, the entry examination for the "Protestant-theological seminaries" which were the training institutions for pastors in Württemberg. Since this theological career still constituted the major orientation of the students, subjects that were not relevant for the *Landexamen* were neglected in the secondary schools. Hence, as was traditionally the case, some arithmetic relevant for pastors was taught, but not the otherwise predominant geometry (Schubring 2007, p. 2).

4 A Certain Harmonization from 1871

The rivalry between Prussia and Austria over dominance in Germany ended with an Austrian defeat in 1866 and a territorial expansion of Prussia by annexing various allies of Austria. Eventually, Prussia succeeded in 1871 to forge the *Deutsches Reich* (German Empire), which despite its name was just a confederation of now 25 states remaining independent particularly in all educational matters. Austria being excluded, the Prussian king became the Emperor of this confederation. In fact, Prussia, with two thirds of the territory and the population, became the leading German state.

Until then citizens of one German state were foreigners in all the other ones and therefore their diplomas were of no value in them, but it was now necessary to establish a procedure for mutual acknowledgment of diplomas, most notably of the *Abitur* and teacher examinations. This entailed that the curricula needed to be sufficiently harmonized to be acknowledged as equivalent. The Prussian government actively initiated evaluations for establishing automatic equivalence statements from 1866 because of the incorporation of a great number of hitherto "foreign" states. The government pursued this policy at a new level since the formation of the *Reich* as a federal state. A Conference held in 1874 in Dresden, the capital of Saxony, meticulously studied school structures and curricula for all relevant disciplines of all federal states to verify compatibility of the criteria for the *Abitur* at the *Gymnasien*. The non-Prussian states were measured in comparison with the regulations in vigor in Prussia.

The President of the Conference was Ludwig (von) Wiese who had been, since the 1850s, the main person responsible in the Prussian Ministry of Education and the representative of changed politics about mathematics teaching, which no longer corresponded with its original neo-humanist key role. It was him who had declared that one line of the classical Latin author Cornelius Nepos presented more educational impact than 20 mathematical formulae (Schubring 1991, p. 84). In fact, Wiese's list of criteria for the *Abitur* diploma was just a rephrasing of the qualifications in the Prussian regulations of 1834 for this exam (Wiese 1867, p. 219). Wiese thus represents the new tendency of the Prussian administration in the second half of the nineteenth century to no longer adhere strictly to the original neo-humanist conception. As the criteria for the level and extension of the exams, it was again the Prussian regulations, according to page 219 of Wiese's 1867 collection and hence the 1834 Prussian *Abitur* exigencies, that were accepted as the common demand (Schubring 2012, p. 529). Basically, through this measure, the *Abitur* diploma from all member states became accepted as equivalent. Harmonization thus occurred for mathematics at a low common denominator.

Yet, this rather elementary status by now corresponded with a consensus among the mathematics teachers. At the first meeting of mathematics teachers from various German states in 1864, on the occasion of the traditional yearly meeting of the "Philologen" (the general term for all teachers at the *Gymnasien*), the participants tried, for the first time, to define what constitutes "Elementar-mathematik" – elementary mathematics, with the meaning of school mathematics. The following definition was eventually voted on unanimously: starting from the notion of quantities as the basic concept of all mathematics, elementary mathematics is constituted of quantities when regarded as fixed, limited, and finite ones, while that mathematics which regards quantities as variable ones constitutes higher mathematics and hence is beyond the limits of school mathematics (Schubring 1991, p. 187). This static notion of school mathematics corresponded with the revived dominance of Euclidean, analytic geometry being excluded.

In particular since the formation of the *Deutsches Reich*, the economy and industry had experienced an enormous takeoff. Despite the needs of the by now industrialized country for an adapted modern education, mathematics instruction everywhere was dominated by elementary teaching goals, focusing on classical, Euclidean geometry and enhancing the formation of logical thinking as a key function. The teaching of variables was banned as not being elementary in that sense, and therefore the teaching of functions was also banned. Consequently, conic sections were to be taught only via the synthetic method, i.e., as geometrical loci, but not by means of the analytical method. In 1891, an association of mathematics and science teachers had been founded – the *Förderverein*: association for the promotion of teaching mathematics and sciences. It did not yet initiate actions for modernizing teaching mathematics and changing the structural problems.

In the second half of the nineteenth century, the traditional tension and conflict between orientation towards classical humanities and orientation towards modernity had effected eventually the establishment of three different types of secondary schools. Complementing the Gymnasium, now called *humanistisches Gymnasium*, according to the Bavarian model, separate school types providing more practical or more "modern" teaching lasted a shorter time than the Gymnasien and did not provide the *Abitur*, the university entrance degree. Actually, the social classes promoting these schools claimed this more practical orientation to be "modern" in the sense of preparing for life in the present world. By the end of the nineteenth century, these originally complementary schools had been expanded in duration and qualifications offered so that marked conflicts arose among the different competing partial systems. Eventually, in 1900, these conflicts were resolved for the entire *Reich* by what one might call a historical compromise. All three types of secondary schools were granted the right to hold the *Abitur* exam and thus give access to higher education. The three types were defined by the kind of classical learning they provided: the classical school type, *humanistisches Gymnasium*, with Greek and Latin, and the two realist school types, *Realgymnasium*, with Latin, and *Oberrealschule*, with none of these languages.

An essential feature of this compromise was that mathematics constituted a major teaching subject in any of these three types, albeit according to different views of mathematics.

5 The Reform Movement Initiated by Felix Klein

The historical compromise at the level of secondary schooling had been complemented that same year by an analogous one for higher education: the two likewise competing types of higher education, the universities, and the technical colleges (*Technische Hochschulen*). These newcomers – as polytechnic schools originally of a secondary school status – were granted the same academic status by the same year, 1900. Before, there had been no free choice between these two types for graduates of the three school types. Originally, *Oberrealschulen* graduates were restricted to technical colleges and a few disciplines of the university. Hence, there was the danger of a culturally segregated guidance cementing barriers between classicality and modernity. Specifically, mathematics had been affected by this split.

Thus, at stake for mathematics was a problem of transition from secondary schooling to higher education. The problem was all the more acute as the technical colleges, due to their origin as polytechnic schools, provided a large portion of basically elementary mathematics. When young mathematics professors, formed in the spirit of the new Weierstrassian rigor in analysis, used them to present rigorous foundations of mathematics, this not only annoyed their students but even provoked the emergence of an anti-mathematical movement among engineers (see Schubring 1989, p. 180).

A university mathematician, Felix Klein (1849–1925), however, understood the challenge of the anti-mathematical movement and realized that the implied problem of transition from secondary schools to higher education simultaneously afforded a profound reform of mathematics teaching. Klein, who was not an exponent of the Weierstraß school but of the rivalling Königsberg-Clebsch school, which was dedicated to a broader vision of mathematics including its applications, had always been highly engaged in all educational aspects of mathematics. Having familiarized himself with some of the main problems facing mathematics teachers in the schools, Klein proceeded to coin the key phrase that would hereafter serve as the slogan for his reform program. This was the famous notion of *functional reasoning* or the idea that the function concept should pervade all parts of the mathematics curriculum. With this slogan of functional reasoning in hand, Klein began in 1902 to gather support for this reform movement from below. He succeeded in forging an extraordinarily broad and powerful alliance of teachers, scientists, and engineers that was to advocate a series of reforms for the mathematics and sciences curricula.

In fact, at the basis in the schools, mathematics teachers were enormously active in realizing the program of functional reasoning, including the elements of the calculus, at all three school types. Moreover, there was a "modern" textbook, published by two teachers from Göttingen, which corresponded to Klein's program: Otto Behrendsen and Eduard Götting, *Lehrbuch der Mathematik nach modernen Grundsätzen* (Teubner, Leipzig).

A committee established in 1904 in Breslau, reflecting this broad movement in its composition – the so-called *Breslauer Unterrichtskommission* – presented at the annual meeting of the association of German mathematicians a year later, in 1905 at Meran, a profoundly revised syllabus for a modernized course, based in fact on that idea of functional reasoning and ending with the elements of calculus. This later became the famous Meran program. The Meran text contained but one shadow: because of the resistance of some functionaries, particularly from the *Förderverein*, calculus was recommended for both realist school types, but was optional for the *humanistisches Gymnasium*. Klein's conception of free transition should likewise apply to the realist and to the classical school types and, hence, contribute to overcoming – at least for mathematics – the split along contrasting views of culture or cultures. This reform movement enabled the curricular basis for mathematics teaching, which ensured the realization of the historical compromise at the secondary and higher education levels for mathematics (Schubring 2007).

Demands for modernizing mathematics teaching had emerged in other European countries as well, in particular in France and in Great Britain. As a consequence, at the fourth International Congress of Mathematicians in Rome (1908), a committee was created to report on the state of mathematics teaching in the "civilized" countries: *Internationale Mathematische Unterrichtskommission* (IMUK) or *Commission Internationale de l'Enseignement Mathématique* (CIEM). Klein, elected president of its *Comité Central*, developed an indefatigable energy for internationalizing the reform movement. Well beyond the original purely compiling task, he succeeded in having the IMUK accept a reformist agenda and disseminate the conception of curricular change internationally. The *Comité Central* took as decisive step to complement the (more or less descriptive) national reports submitted by the national subcommittees with international comparative reports on a few key topics representing the major reform concerns. The best prepared such report with significant results proved to be the one on the introduction of the first elements of the differential and integral calculus in the upper secondary school grades. The presentation and discussion of this report constituted the major topic of the international meeting of IMUK in 1914 at Paris, shortly before the outbreak of World War I. Klein, also president of the German national subcommittee of IMUK, organized a voluminous series of highly qualitative reports on the state of mathematics teaching in the German states, even giving due attention to aspects of teaching in the primary schools.

6 The Interwar Period: The Weimar Republic

World War I ended these activities, in particular international cooperation. Due to the November Revolution of 1918, Germany became a republic. Still named *Deutsches Reich*, unofficially called the Weimar Republic (according to the town where the constitution was deliberated), the republic maintained the character of a federal state. The 18 federal states thus continued to be autonomous about education, issuing proper syllabi and textbooks. The history of mathematics teaching in these states is poorly researched, except for Prussia.

Despite the competence of each state for educational policy, a decisive law for the entire Reich had been enacted: the *Reichsschulgesetz* of 1920 abolished the social segregation between a primary school system for the lower classes and a secondary school system with separate preparatory schooling for the higher classes and established a consecutive system of primary school for all, followed by a streaming into diverse types of secondary schools. Moreover, the formation of teachers for these new primary schools became attributed to institutions belonging to higher education – the Pedagogical Academies,[3] admitting only students provided with an *Abitur* – whereas the earlier seminaries had as students graduates of the *Volksschulen* so that the formation of these instructors and the *Volksschulen* had constituted a closed system. The professorships established there for the methodology of teaching reckoning and geometry (*Raumlehre*) paved the way for seriously studying pedagogical conditions for learning – and teaching – (elementary) mathematical knowledge, the first step towards what would later become didactics of mathematics in West Germany.

After World War I, the entire political, social, cultural, and economic situation had changed. Given the horrors of this war, exacerbated by the application of science, mathematics and the sciences had lost their legitimacy to a considerable extent and had to act defensively. A cultural crisis of mathematics and the sciences arose. Subjects now valued in the school context were of a different, nationalist character: *kulturkundliche* subjects, i.e., German language and literature, geography, and history, were favored, to the disadvantage of mathematics and the sciences. Weekly hours for the latter subjects were reduced in all types of secondary schools. Nevertheless, a considerable progress was achieved. In the new curriculum for all secondary schools in Prussia of 1925, the so-called *Richertsche Richtlinien* now officially enacted what had for a long time been practiced by mathematics teachers: the Klein program with the elements of calculus in all three types of secondary schools and thus also in the *humanistisches Gymnasium*.

The restructuring of the school system was accompanied by new pedagogical methods, called *Reformpädagogik,* and specifically featuring the so-called *Arbeitsunterricht* (work instruction), i.e., replacing old formalist teaching – which addressed memory only – via active methods, claiming proper activities by the students themselves, and emphasizing in fact manual occupations. In a number of textbooks during the Weimar period, one finds examples of nationalistic content in exercises given to the students.[4]

7 Nazi Germany from 1933 to 1945

It is remarkable and characteristic that the nationalist overtones during the Weimar period were directly transformed in Nazi times into militaristic, anti-Semitic, and eugenic indoctrination. Immediately after the seizure of power by the Nazi Party, the two organizations for mathematics teaching – the *Förderverein* and the *Reichsverband deutscher mathematischer Gesellschaften* (union of German mathematical associations) – decided for themselves their *Gleichschaltung* or adoption of key principles of the Nazi system: replacing elections for the presidency by the *Führerprinzip* (i.e., a permanent authoritarian presidency) and changing their statutes by adopting the so-called Aryan paragraph (i.e., excluding the so-called Jews from membership). This *Reichsverband* decided to compose a handbook for mathematics teachers to help or guide them to accommodate their teaching to the Nazi system, entitled "Mathematics in the service of national political education, with examples of application to sociology, topography, and natural sciences," edited by Adolf Dorner.[5] The handbook, published in 1935, was recommended for use in schools by the federal ministries and reedited several times. It contained a collection of ideologizing, indoctrinating, and discriminating exercises. It did not meet refusal.

[3] This applies in particular to Prussia; other states maintained the seminary structure.

[4] See the analysis of German mathematics schoolbooks for the Weimar and Nazi periods in Ullmann (2008).

[5] Mathematik im Dienste nationalpolitischer Erziehung mit Anwendungsbeispielen aus Volkswissenschaft, Geländekunde und Naturwissenschaften.

The schoolbooks even for primary grades are full of examples of such self-mobilization. The illustrations feature a militarist context for playing youngsters; exercises for multiplication are visualized by showing SA troops marching in groups of four, six, etc. (Gispert & Schubring 2011, p. 85). Mathematics teaching in general was put into the service of Nazi ideology. For instance, mathematics teaching in girls' secondary schools was reduced and oriented towards concrete geometry – specifically to the study of geometrical figures represented by objects of use in the household (Erziehung 1938, p. 202).

The Nazi state was apparently the first German one to establish a centralized government, abolishing federal structures. For instance, a central ministry of education was founded, in 1934, the *Reichsministerium für Wissenschaft und Erziehung*. Typically for the Nazi system, although the federal states had thus lost, de jure, their competences for the educational sector, they not only continued to exert them in the practice, but a multitude of instances of the Nazi party also competed to influence decisions about school issues (Zymek 1989, p. 191).

In a decisive measure for modernizing the traditional structures, the differentiated system of secondary schools was reduced to only two starting in 1937: *Oberschulen* (high schools) and *Gymnasien*. The school attendance of both types was restricted from 9 years to only 8 years, and the number of *Gymnasien* (with Latin and Greek) considerably reduced – thus, unintendedly, increasing their elite character. The curriculum in mathematics did not differ essentially between the Oberschule and the Gymnasium and corresponded largely to the 1925 syllabus for Prussia (Erziehung 1938, p. 194); yet, there was a differentiation into a mathematics and sciences stream and a (modern) languages stream in the last two forms of the Oberschule, where the mathematics hours were halved, to just 2 weekly hours (Erziehung 1938, p. 26).

The ministry planned, among other tasks on its agenda, to publish an arithmetic textbook for all German primary schools. In reality, the federal structures never became eradicated and the unitary schoolbook project did not succeed.

8 Afterwar Developments

Ironically, it was in the first post-World War II years, from 1945 to 1948, that a unitary "Deutsches Rechenbuch" was published as an eight-grade series for eight grades of reestablished schools by the Allied Forces in Germany. Germany was divided into four zones, each one controlled by one of the four allies. Soon, the American, British, and French zones would cooperate, while the Soviet zone remained separated. From this split originated the constitution of two German states in 1949: the capitalism-oriented Federal Republic of Germany (FRG) and the socialist German Democratic Republic (GDR).

8.1 The Federal Republic of Germany

After World War II, a conservative stabilization was effected by a return to the pre-Nazi period; in particular, the segregated school structure was reinforced. Ideologically, a backward-oriented conservatism ruled and emphasized the values of an allegedly "Christian West," thus establishing a cultural distance from an allegedly barbaric East.

The conservatism of West German society directly affected the teaching of mathematics and the sciences. This was illustrated by a fact unique to Western countries. In 1960, while other Western countries had already been profoundly affected by the Sputnik shock and had reinforced mathematics and science teaching and while the OECE was strongly active in modernizing mathematics teaching, the KMK (*Kultusministerkonferenz*), the body of federal education ministers, decided to reduce the weekly hours for mathematics and the sciences in the secondary schools, in favor of the humanities, convinced that they would thus be able to save the *Abendland*, the Occident. This reduction document was the so-called *Saarbrücker Rahmenvereinbarung*, the framework agreement of Saarbrücken.

Regarding curricular change, only a few relatively isolated discussions were held, since 1953, and these exclusively concerned the Gymnasium. One of the exponents of this group was Hermann Athen, director of a Gymnasium and an influential schoolbook writer; moreover, he was one of the three West German participants of the OECE Seminar 1959 at Royaumont. When the group presented its proposals for a moderate reform within the Gymnasium, the *Nürnberger Lehrplan*, to the 1965 *Förderverein* annual meeting, it met with flat refusal by the mathematics teachers.[6] Criticizing the 1960 *Saarbrücker Rahmenvereinbarung*, the proposal alluded to the reform intentions of OECD and UNESCO and to the structuralist conception, based on set theory. It also emphasized specific German mathematical conceptions in the spirit of Felix Klein: *Abbildungsgeometrie* (geometry of applications) and the principle of application in geometry and algebra, vector and topological spaces, and even probability theory (see Athen 1966).

The overly long stagnation of West German society and its educational system built pressure for modernization to such an extent that eventually an even more radical explosion of modernizing policy at all levels burst through. Prepared in the early 1960s by apocalyptic prophecies about an imminent catastrophe of education and society if the country did not change its policy (*Bildungskatastrophe*), as depicted by Georg Picht, the second half of the 1960s brought a profound turnabout: the educational system became massively expanded; hitherto underdeveloped regions and marginalized sections of the population received largely improved access to education; and the institutional system was improved and upgraded. For instance, the Pedagogical Academies that had been inherited from the Weimar period as institutions for teacher training for primary and middle school types were "upgraded" to Pedagogical Colleges (*Pädagogische Hochschulen*), raising former lectureships for the methodology of teaching subjects to the level of professorships for the "Didaktik" of these subjects. This thus gave scientific status to the discipline of mathematics teacher training. As a result of many innovations enacted now, the world of schooling for lower classes became definitely united with the world of schooling for middle and upper classes.

The by now peremptory curricular reforms were likewise radical and followed the universal structuralist spirit of the epoch. Moreover, for mathematics, due to the refusal of an internal reform in 1965 by the teachers (at least for parts of the Gymnasium), all reform initiatives were imported from abroad and not grown within their respective communities. Thus, the decisive document became a text voted in 1968 from above, by the KMK, without any contact or consultation with teachers and mathematics educators, decreeing a profound reform which was to be enacted from 1972 on. Primary and secondary schooling were now seen as a unity, subject to a common curriculum developing the key thematic issues of mathematics over the school years, from the first grade on – despite the maintained system of streaming within the secondary schools. These issues, organized in thematic areas, ranged from sets, magnitudes, and positional systems to congruencies, real numbers, and trigonometry. It was, in fact, a less revolutionary reform than it later became accused of being (see Keitel 1980).

These reform decisions from above fell on teachers and educators who were entirely unprepared for them. There were didacticians (teacher educators) for primary teaching, but they had never been involved in the preparation by the KMK. For secondary teaching, there were only practitioners, rather than didacticians, of teacher training. Execution of the reform decision was thus realized by the textbook industry, which quickly produced numerous but poor textbooks, which grossly exaggerated the importance of the set language.

[6] See "Nürnberger Lehrpläne," *Der mathematische und naturwissenschaftliche Unterricht*, *17* (1965/1966), 433–439.

Soon, organized public resistance concentrated on the alleged set theoretical nonsense. The public uproar led in 1975 to a backlash in which the syllabi were replaced by new ones free of sets. This was then perceived as a return to basics. In the long run, however, this was not confirmed. Rather, the main effect of a common curricular structure of the entire school mathematics, developing the fundamental concepts of mathematics, was maintained. As a result, a consensus emerged on all syllabi of the federal states, which established a few conceptual fields as constituting school mathematics for primary and lower secondary grades: number, figure and form, magnitudes, functions, and data.

A decisive structural reform was enacted in 1972: the three traditionally different types of secondary schools entitled to confer the *Abitur* diploma – *humanistisches Gymnasium*, *Realgymnasium*, and *Oberrealschule* – all merged into one single Gymnasium. Finally, its last three grades were completely reorganized as *reformierte Oberstufe*: as a system of courses in which students were entitled to choose according to definite specializations. Mathematics remained an obligatory choice: either as a *Grundkurs*, basic course, or as a *Leistungskurs*, advanced course, with more demanding teaching.

8.2 The German Democratic Republic

Contrary to the FRG, the GDR abolished federal structures – gradually and definitely in 1952 – and established a centralized system of education, realizing a unitary school for all. The first structure of 1946, an 8-year basic school for all (*Grundschule*), followed by a 4-year high school (*Oberschule*) preparing for university studies, changed from 1951 to a 10-year school for all, the *polytechnische Oberschule* (POS) or polytechnic high school. After the obligatory 10 years of school attendance focused on general education, there was a differentiation: into 3 years of professional schooling and into 2 years of university-preparation schooling, the *Erweiterte Oberschule* (EOS, extended high school) (Birnbaum 2003).

In the first years, *Reformpädagogik* and *Arbeitsunterricht* were preferred as teaching methods because of the reorientation to conceptions before the Nazi period. From 1949 on, they became increasingly criticized, particularly in reference to their slogan "vom Kinde aus" (starting from the child), for not considering the primacy of social demands; these approaches were substituted with an orientation towards Soviet pedagogy (Borneleit 2003, pp. 31 and 33). After the June 1953 revolt, this orientation was reduced; from then on, curricular and structural developments occurred rather independently from Soviet models (Borneleit 2003, p. 34).

Mathematics and the sciences were key and highly valued teaching subjects. Mathematics enjoyed, relatively of all the subjects, the highest share of teaching hours, 17.7 % for the POS in 1988 (Birnbaum 2003, p.18). There was one special school for mathematics, comprised of grades 8–12; moreover, there existed various mathematical-technical special schools. Highly revealing was the so-called mathematics decision (*Mathematik-Beschluß*) of 17 December 1962 by the politburo of the Central Committee of the governing Socialist Party, together with the Council of Ministers, to improve and develop mathematics teaching in the GDR.[7] This document showed a comprehensive regulation for curricula, schoolbooks, teacher training, in-service training, mathematical Olympiads, creation of a series of booklets for students and a journal for school mathematics, measures for research on mathematics teaching, and the like. In a certain sense, it initiated a modernization of the curriculum in a manner somewhat analogous to the new math.

[7] Zur Verbesserung und weiteren Entwicklung des Mathematikunterrichts an den allgemeinbildenden polytechnischen Oberschulen der DDR"; published among others in: Mathematik und Physik in der Schule, 2 (1962), 141–150.

Since the first curricula of 1946, a stronger orientation towards mathematical science had prevailed. In the curricula of the 1950s, the traditionally segregated knowledge and terminology of the old primary school (*Volksschule*), emphasizing reckoning and *Raumlehre* (a characteristic conception of geometry, not aspiring academic, criteria) were replaced by "arithmetic" and "geometry" (Borneleit 2003, p. 33). The new curriculum of 1963, based on the *Mathematik-Beschluß* of 1962, was now conceived of as an integrated learning of mathematics, no longer perceiving the first grades as those of "Rechenunterricht" (teaching of arithmetic), but developing school mathematics systematically and coherently from the first grade on. Thus, already in the first grade, x was introduced as "sign for an unknown number"; the teaching topics were organized according to thematic principal lines: development of the number concept, development of functional thinking, justifying and proving, and geometry teaching according to *Abbildungsgeometrie* (Borneleit 2003, p. 35).

The elaboration of always more improved curricula proved to constitute the key issue of the central educational policy; systematic evaluation of their implementation was constitutive as well. Steady revisions of the mathematics curriculum for the POS between 1963 and 1971 led eventually to a consolidated, modernized syllabus, influenced by structuralism, too: it was characterized by being founded on set theoretical and logical conceptual structures. Pervasive guidelines were the early use of variables, the use of the set theoretic notion of function, a proper geometry already in the first grades, the fusion of planimetry and stereometry, and the capability to prove and to define (Borneleit 2003, p. 38).[8] This modernization was not as radical as new math in other countries; for instance, geometry was not transformed into linear algebra and vectors were not introduced in the POS. Research on mathematics education –here called "Methodik des Mathematikunterrichts" – became considerably influenced since the 1970s by the reception of A. N. Leontjev's activity theory and its adaptation by P. J. Galperin.

After the end of the GDR in 1990, when it entered the Federal Republic as five new federal states, the Western system had to be accepted with its lesser standards in mathematics teaching.

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[8] It should be emphasized that these curricular reforms occurred without direct influence by the Kolmogorov reform in the Soviet Union.

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Chapter 13
Mathematics Education in the United Kingdom

Geoffrey Howson and Leo Rogers

1 Introduction

The United Kingdom comprises four countries: England, Wales, Scotland and Northern Ireland. Until 1922 (i.e. for much of the period with which this chapter is concerned) the whole of Ireland was included in the United Kingdom. The educational histories of these four countries differ, but the most significant differences, throughout history, have been between the educational systems of England and Scotland. Scotland has been an independent country since the Middle Ages and has maintained religious, legal and educational systems of its own. The Act of Union in 1707 brought Scotland and England under one king but maintained the Scottish traditions: the Scottish Education System has retained its independence and high quality status. Accordingly, the histories of these two countries are treated separately in this chapter. Some differences relating to the Irish and Welsh systems can be found in a footnote,[1] but, otherwise, little specific attention will be paid to these two countries.

[1] The population sizes of the four countries are given, in millions, in the following table.

	England	Wales	Scotland	Ireland	Northern Ireland
1800	8.3	0.6	1.6	5	
1901	30.5	2.0	4.5	4.5	(1.2)
2001	49.1	2.9	5.1		1.7

These data show the greater proportional population growth in the more prosperous England and, in fact, conceal that in Ireland the population dropped significantly from 1845, when it was in excess of eight millions, because of deaths due to a great food famine between 1845 and 1848 (not eased by its partner countries) and large-scale emigration. The poverty which gave rise to this fall in population was reflected in the educational opportunities offered to the Irish people Thus, in Ireland (and later Northern Ireland) state education, as opposed to that provided by religious bodies, did not really commence until about 20 years after the 1922 partition of Ireland. The provision of education in Wales and Ireland was also complicated by religious issues: in Wales between those belonging to the Church of England and the Nonconformists and in Ireland between the Catholics and the Protestants. Certain differences still exist between the educational systems of England and those of Wales and Northern Ireland, for example, in the national curricula (for provision is made in the latter two countries for the teaching of national languages) and in the way that the system of frequent testing introduced in the Education Act of 1988 operates. The national curricula of the four constituent countries of the United Kingdom can be found on the ICMI web site.

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It will be noted that Scottish governments have traditionally tended to play a greater role in influencing education than their English counterparts. This difference and, in particular, the great diversity of provision in England both in the forms that school education took and also the curricula the schools followed make the description of the history of mathematics education in that country particularly complex. For, although in the nineteenth century, the United Kingdom occupied a leading position in industry and even in science and technology and possessed the largest world empire, it did not follow that it occupied a leading position in mathematics education. Nonetheless, the interconnection of the various aspects of the life of a society is evident. Below, much will be said about the legal and administrative sides of the changes that took place in the country – in particular, about various educational acts. They reflected the social processes unfolding in the United Kingdom and, in particular, the gradual governmental moves from a *laissez-faire* stance to a dictatorial one and, accordingly, from an educational system of great diversity to a more uniform one: moves with great implications for mathematics education at all levels.

The section on England of this chapter is written by Geoffrey Howson, and the section on Scotland is authored by Leo Rogers.

2 England

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2.1 Prelude

As in many other countries, the first schools in England were provided by religious bodies and were primarily concerned with religious matters and needs (Howson 1982; Orme 2006). However, considerable consequences for education followed Henry VIII's break with the Church of Rome which led by 1540 to the dissolution of the monasteries and friaries. This had an immediate effect on the number and governance of schools. New grammar schools were established by Henry and his son, Edward VI, and throughout the following centuries by trade guilds, merchants and other wealthy persons (Watson 1908). Yet, there was little place for mathematics in any of them; their curricula reflected humanist thought. Pressures for change did, however, lead some schools in the seventeenth century, particularly in the seaports, to introduce a little mathematics, beyond basic arithmetic, into their curricula (Howson 1982). If one wished to learn mathematics, however, it was usually necessary to turn to a private tutor or, so-called mathematical practitioner (Taylor 1954, 1966; Howson 2011). The only two universities, Oxford and Cambridge, placed no pressure on schools to teach mathematics. For although both appointed professors of mathematics in the early seventeenth century, neither paid much attention to the teaching of the subject before the nineteenth (Warwick 2003; Howson 2010). Indeed, although the period up to 1800 witnessed a marked growth in the applications of mathematics and a need for knowledge of the subject, it did not lead to changes in the curricula of the vast majority of English schools and neither did it cause any government to demonstrate a marked interest in public education. The activity of mathematicians such as Briggs, Wallis, Gregory and Newton had a minimal impact on mathematics education in schools, and in general what was accomplished was due to the work of individuals rather than to that of authorities (Howson 1982, 2011).

2.2 1800–1902 Elementary Education

2.2.1 First Moves

By the beginning of the nineteenth century, England was clearly lagging behind several continental countries in its provision of elementary education and there were appeals for the state to help remedy this. However, the government did not wish to become involved and left education to other bodies. As a result, the Church of England in 1811 gathered together schools under its care in the so-called National Society, while the nonconformists in 1814 offered their own alternative, the schools of the British and Foreign School Society (Wardle 1976; Howson 1982). These offered a higher level of education than did the 'dame' schools, which had low educational aims, were run by ill-educated persons and were attended mainly by children of the working class. Some of the private schools for middle-class children run by clergymen or similarly well-educated ladies and gentlemen reached a much higher level. The government, however, still had worries concerning the education of the working class – would it lead them to be dissatisfied with their lot? Memories of the French revolution were still strong. Yet despite considerable opposition, Parliament agreed in 1833 to make a grant to further the work of these two societies. In 1839 an inspectorate was formed to ensure that the grant was being well spent. However, mathematics in such schools still meant, at best, arithmetic (see Yeldham 1936).

It is not known exactly what proportion of the population attended schools, but in 1840 it was believed that working class children attended school on average between 1 and 2 years, frequently interrupted, and that few stayed after the age of 10. By 1852, 33 % of pupils in inspected schools had been at school for less than 1 year and 73 % for less than two; by 1861 the respective percentages were 38 and 61 (quoted in Wardle 1976, p. 65). At both dates over half the pupils were under eight and less than 15 % over 11 (of whom many were 'pupil teachers'). Although some pupils progressed to become teachers through classroom experience, from the 1830s teacher training colleges began to be established by the two societies including one in 1842 specifically for women (Rich 1933; Howson 1982, 2010).

2.2.2 Growing Government Involvement

In 1858 the government established a public commission, chaired by the Duke of Newcastle, 'to inquire into the State of Popular Education in England and to consider and report what measures if any are required for the extension of sound and cheap elementary instruction to all classes of people'. The picture of mathematics education revealed in the Commission's Report is bleak indeed. Of the children attending the 1824 public weekday schools[2] visited, only 69.3 % were taught arithmetic, 0.6 % mechanics, 0.8 % algebra and 0.8 % Euclid. The corresponding data for private schools were 33.8 %, 1.29 %, 1.35 % and 1.15 %, respectively. However, such arithmetical instruction as was given appeared totally inadequate: 'in working sums explicitly stated the children were often successful enough, but they were usually quite ignorant of anything that required the simplest knowledge of a principle' (p. 47).

The Commission's recommendations were broadly to leave the system in place but to award grants to schools based on the pupils' attainments. This led to the system known as 'payment by results': the establishment of national 'standards' on which pupils would be examined and on the results of which payments would be made to their schools. The curriculum was set out for arithmetic (and there was no other mathematics) and the six standards consisted of no more than the four operations (up to short division) plus sums (exercises) on money, weights and common measures (reprinted in Howson 1982). By 1871 changes were made and a new Standard VI, 'proportion and vulgar and decimal fractions',

[2] Sunday schools were first created in the late eighteenth century, but apart from some attempts to teach literacy, usually concentrated on religious ends.

introduced. Teaching restricted to this unappetising diet and solely with the passing of tests in mind gave what the leading inspector, Matthew Arnold, called ‘a mechanical turn to school teaching’ that must be ‘trying to the intellectual life of the school’. (*Minutes of the Committee of Council* 1869–1870, p. 291). Yet even these limited objectives were not attained, in 1873 only 15 % of pupils passed the examinations at the standard appropriate for their age, 57 % were a year, and 26 % 2 years, too old. This system¸ which helped raise standards in the weaker schools, but at the cost of inhibiting excellence and initiative among the better teachers, was abandoned in 1897 (Wardle 1976; Horn 2010).

2.2.3 State Elementary Education Introduced

Demands for greater, state-directed provision of elementary education grew, however, responding to data such as those disclosed by Arnold in 1868 (Howson 2010, pp. 49–51), for example, the rates of illiteracy in the English army (57 %) compared with those in the armies of France (27 %) and Prussia (2 %), and comparisons of economic growth. The result was the Education Act of 1870 which established school boards throughout the country with the duty to build schools where denominational schools did not already exist and to provide education up to the age of 13. In 1880 attendance became compulsory for all children, although many denominational schools still charged for attendance: not until 1918 was elementary education free for all.

2.3 1800–1902 Secondary Education

2.3.1 Early Provision

By 1800 secondary education (i.e. post-elementary education, although that adjective was never used at the time) was supplied by three bodies. First were the nine great ‘public schools’, including Eton, Harrow and Winchester – largely attended only by the children of upper-class parents; second were the endowed grammar schools; and third were private schools both for boys and for girls (see De Bellaigue 2007 for the latter). Although the private schools might teach some mathematics other than arithmetic, the subject did not feature significantly in the curricula of the public and endowed schools although, for example, Winchester and Eton offered optional lectures in mathematics from visiting lecturers for additional fees (Howson 1974). The first move to change the *status quo* failed, for an appeal in 1805 by Leeds Grammar School to abandon the humanist curriculum set out by its founder and to allow it to teach mathematics and modern languages was rejected by the Lord Chancellor. Not until 1840 were schools given complete freedom to revise their curricula, although concessions were made which resulted in Winchester and Eton Colleges appointing their first mathematics masters in 1834 – several centuries after their foundation.

The curricula in those schools that opted to teach mathematics soon settled into a stable pattern of arithmetic, algebra and, most importantly, Euclid. The last was not only classical, and so more aligned to humanist aims than vocational ones, but, to quote De Morgan (1831), better served ‘the great end [of mathematics teaching], the improvement of reasoning powers’ (see Howson 1984). Nevertheless mathematics still did not enjoy esteem within the curriculum comparable to the classics (Howson 1974).

2.3.2 New Schools and New Demands

Matters changed, though, with the creation of a new type of school and with new demands from the armed services for would-be officers to demonstrate some mathematical ability. Upper middle-class parents who were unable to send their children to the great public schools wanted something better than was being provided by the endowed grammar schools. As a result, from the 1820s, ‘proprietary’ schools, funded, organised and managed by a committee, began to be established throughout the country. These were seen as worthy alternatives to the older public schools and many provided ‘modern’ curricular alternatives for older pupils, specifically designed to meet the demands of the armed forces. Their popularity led to a government commission being established in 1861, chaired by the Earl of Clarendon, to investigate the management and curricula of the nine public schools. The resulting report (1864) precipitated great changes in their curricula and in the way that mathematics was taught – changes that because of the prestige of the schools, the high quality of the mathematics teachers they attracted and the textbooks they produced had considerable effects upon mathematics teaching throughout the secondary school sector (Howson 1982, 2010).

2.3.3 More General Secondary Education

A further government commission, the Taunton Commission, was established in 1864 to consider the endowed grammar, private and proprietary schools. Its recommendations, published in 1868, were that three grades of secondary education should be established, one for each of the three social classes: thus emphasising the established view that nothing should be done that might disturb the prevalent social class system. Its report provided information concerning the varied treatment of mathematics in the grammar schools. There was a great divergence in aims: some schools offered wide-ranging mathematics courses, others aspired only to Euclid Book 1. While some schools studied Euclid, others used the practical geometry books written by practitioners rather than university fellows.

Yet already moves had begun that initiated a move towards a more standardised secondary mathematics school syllabus. Prompted by a local competition held in Devon in 1856, the universities of Oxford and Cambridge established school examinations set on syllabuses laid down by the two universities and awarded certificates to successful candidates.[3] The newer universities followed suit, and once entry to university began to depend upon success in such examinations, rather than recommendation by graduates, a rough (for there were differences in the syllabuses offered) national curriculum began to be formed. Euclid was still a major feature, but other topics were logarithms, algebra to simple equations (or quadratic for the more advanced), sequences, trigonometry, mensuration (calculation of lengths, areas, volumes) and some applied mathematics (mechanics, statics and dynamics) (Howson 1982, 2010).

2.3.4 The First Mathematics Teachers Association

The emphasis on Euclid, and a growing feeling that it was outdated and inappropriate, led in 1871 to the creation of the Association for the Improvement of Geometrical Teaching (AIGT), probably the world’s first subject teachers association. As its name suggests, the AIGT argued for a replacement for Euclid, an aim already dismissed by a committee of the British Association set up in 1869 (including Cayley, Clifford and Sylvester) which thought nothing so far produced ‘is fit to succeed Euclid’. The AIGT produced its own course, but it was not to prove a success or be widely welcomed by universities. Undeterred, the AIGT broadened its interests to other branches of mathematics teaching and in 1894 published the first number of its *Mathematics Gazette*, before changing its name to ‘The Mathematical Association’ (MA) in 1897 (see Price 1994).

[3] London University had, in 1838, shortly after its creation, established a matriculation examination that could be taken by students other than those attending the university or those who did not wish to proceed to a degree. The 1838 syllabus for mathematics and a specimen examination paper set that year can be found in Howson 1982, pp. 214–216.

2.3.5 A New Curriculum and Perry

An alternative to the classically dominated curriculum was offered by the Department of Science and Art (DSA), a governmental body established following the 1851 Great Exhibition. Grants were given to any school that would teach the more scientifically and technically oriented DSA curriculum, and further payments were made according to its pupils' results. Such classes were given mainly in private schools and in endowed schools experiencing financial difficulties. After 1870, publicly funded 'central' or 'higher grade' schools were also created in the larger cities intended to follow on from the elementary schools and with the children of the working class in mind (although they came to be dominated by those of the lower middle class) (Wardle 1976). They too made use of DSA funding but their number was small and their distribution throughout the country far from uniform (Horn 2010). Even more significantly for mathematics education, the DSA encouraged experimentation. In particular, in 1899 a new syllabus for evening courses was introduced, largely based on the work of John Perry who in 1900 published proposals for a new school syllabus in mathematics (reprinted in full in Howson 1982, pp. 222–224), the content of which was, in time, to be absorbed into school syllabuses. It included topics such as the slide rule and its underlying principles, the use of Simpson's and other rules for estimating the area of irregular figures, volumes of three-dimensional bodies, practical means of finding areas and volumes, using squared paper (displaying data, interpolation, Cartesian and polar coordinates, solving equations, finding maxima and minima, etc.) (see Brock and Price 1980), scalars and vectors, and differentiation. (See also Perry 1902; Price 1986)

2.3.6 Teacher Education

The last quarter of the century saw a growth in the provision of training and the establishment of Diplomas in Education for those who wished to become teachers in secondary schools (see Rich 1933; Howson 2010).

2.3.7 The Public Schools Set a New Pattern

By the late 1890s some students entering Cambridge had already covered so much mathematics at school (in G. H. Hardy's case, Winchester College) that they were allowed to sit the Mathematical Tripos (Cambridge examinations) at the end of six terms rather than the usual nine. Indeed the 1893 Cambridge *Student's Guide* warned prospective mathematics students that most students now arrived with 'some knowledge of Co-ordinate Geometry, Differential and Integral Calculus and Mechanics' (Warwick 2003, p. 260). This gives a clear indication of how the mathematics curriculum had developed in the public and proprietary schools from which, at that time, the vast majority of Cambridge students were drawn.

2.3.8 The Government Forced into Action

In 1899, an auditor disallowed public expenditure on 'higher elementary' classes on the grounds that such had never been approved by Parliament. This led to a court case that in 1902 effectively forced the government to take action: the outcome was the Education Act of 1902 abolishing school boards and replacing them with local education authorities (LEAs) which were given authority to establish new publicly funded secondary and technical schools and also to further the provision of elementary education.

2.4 1902–1988: Changes in Elementary/Primary Education and Some General Patterns of Development

2.4.1 Until the 1944 Education Act

Between 1902 and 1944, there were no major changes in the form of what was then called in England elementary education (other than the school leaving age being raised to 14 in 1918). The distinction between that and secondary education became much clearer – even in the salaries paid to teachers and the social status they enjoyed (Wardle 1976). Elementary school teachers were now better qualified and trained. Indeed, in 1905 the Board of Education issued a *Handbook of Suggestions for Teachers* containing in its Preface:

> The only uniformity of practice [sought] in Public Elementary Schools is that each teacher shall think for himself, and work out for himself such methods of teaching as may use his powers to advantage and be best suited to the particular needs and conditions of the school.

This rather remarkable tolerance of diversity is something which distinguished mathematics education in England, at both primary and secondary levels, from that in many other countries. Not until 1988 did a government take firm steps to change it. As a consequence the history of mathematics education in this period is one in which many individuals were able to have a considerable influence on teaching, although their names are not so well known internationally as those of, say, Rousseau, Pestalozzi, Montessori or Steiner.

Yet despite this apparent freedom, the tradition of payment by results was so deeply entrenched that many schools continued to be organised by standards and the curriculum, especially for the younger pupils, did not change significantly. (The Board of Education's Standards for 1905 are reprinted in Howson 1982, pp. 224–225.) The curriculum varied greatly in the senior grades, and in some parts of the country secondary education (post age 11), other than in grammar schools, became available to pupils. Some newly built elementary schools were also able to add subjects such as woodwork and domestic economy to the older pupils' curriculum but in country areas, however, such facilities were unlikely to exist and the curriculum remained very restricted. Teaching to the age of 11 became geared to the passing of the 'scholarship examination' that would provide pupils with free secondary education in a grammar school – for many such had been created, post 1902, by the new LEAs. The examination included a paper on 'Arithmetic' and so mathematical work was directed towards the attainment of a good mark in this paper. A typical paper from 1920 is reproduced in Howson 1982, pp. 229–230. Considerable technical facility is demanded and the questions are by no means straightforward. Drawing to scale (probably only with the use of squared paper), proportion and mensuration were expected. Relevance to practical matters is not very evident in questions such as 'Find the value of 13 2/3 – ¾ of 5 1/6′ and 'An iron article weighing 4 cwt. 7 lb is 16 times as heavy as a wooden model of it. The model consists of five parts, four of which weigh 8 lb. 7 oz., 5 lb. 3 oz., 6 lb. 11 oz. and 4 lb. 8 oz. respectively. Find the weight of the fifth part.' (For readers unfamiliar with imperial measures: 1 cwt. = 112 lb. and 1 lb = 16 oz. – a reason to be grateful for metric measures!). Some geometry was taught, but this related mainly to the names and properties of common two- and three-dimensional objects. This meagre diet led the Mathematical Association in 1938 to establish a committee 'to consider the teaching of mathematics at primary level'.

2.4.2 The 1944 Education Act and Primary Education

The war interrupted the work of the MA committee and when in 1946 a new committee was appointed it soon set aside the substantial pre-war work. By then, also, the 1944 Education Act had provided state secondary education for all children (see below) and so established a clear divide between primary (5–11) and secondary education (post-11): the term 'elementary education' fell into disuse. The new MA committee was much influenced by one of its members, Caleb Gattegno, who had just returned from working with Piaget and accordingly in its report (MA 1955) decided to deal with *mathematics*, not simply *arithmetic*, and to concentrate on the ways children think and the activities they find satisfying.

> Practice without the power of mathematical thinking leads nowhere; the power of mathematical thinking without practice is like knowing what to do but not having the skill or tools to do it; but the power of mathematical thinking supported by practice and rote learning will give the best opportunity for all children to enjoy and pursue mathematics as far as their individual abilities allow. (p. 4)

The document continues:

> We plead … for attempts to develop mathematical ideas through the study of broad environmental topics and through the investigation of situations and phenomena at first hand. (p. 20)

2.4.3 Change and Development

The MA report, but more effectively the many courses for teachers based on it, began to have great effect in primary schools and on teacher training. The 1950s were also to see the introduction of new materials into schools, Cuisenaire rods, Dienes' Multibase Arithmetical Blocks – and later his logic blocks, amongst others.

Even greater changes arose from the work of the Nuffield Primary Mathematics Project established in 1964 and the publication in 1965 by the newly established Schools Council for the Curriculum and Examinations[4] of *Mathematics in Primary Schools*, written by an outstanding schools inspector, Edith Biggs. Their treatment of certain mathematical topics differed, but both, along with the MA report and the Association of Teachers of Mathematics' (ATM)[5] *Notes on Mathematics in Primary Schools* (1967), only gave teachers advice, examples of lessons and guidance (in the form of multitudinous volumes in the case of the Nuffield Project). The design of an actual course was left to the teachers or, in practice, to textbook authors who supplied series 'based on' the advice and guidance offered and which, in general, left much to be desired. Later a scheme based on individual learning was produced for pupils aged 7–13 by the School Mathematics Project, but these left the teacher with too little to contribute to lesson planning as well as providing students with too unvaried a diet. Great improvements in the professional development of teachers came with the establishment by the Nuffield Project of teachers' centres (Corston 1969) and, following their success, the LEA local centres. Unfortunately, these, along with many mathematics adviser posts, were to disappear in the 1980s as LEA responsibilities and funds were cut. Television lessons, at primary and secondary level, did prove a continuing aid to teachers, particularly once they could be recorded and used when desired, rather than when classes had to be planned around the time of their transmission.

The effects of these initiatives on the actual curriculum were varied. Sets and multibase arithmetic came into many schools, but then gradually disappeared. Data gathering and display came in and stayed. More emphasis came to be placed on geometry and on number patterns in the hope that the

[4] A governmentally financed body, the Schools Council was established in 1964 as a consortium of interested bodies in which school teachers were dominant, mounted for 10 years various projects within the field of mathematics education. It was later reconstituted as a result of financial stringency and criticism before it was replaced in 1982 by separate examination and curriculum councils. These were later amalgamated and have since changed names and exact purposes at regular intervals.

[5] The ATM, created in 1962, evolved from the Association for Teaching Aids in Mathematics (ATAM 1952). Its journal *Mathematics Teaching* first appeared in 1955. The ATM grew rapidly in the 1960s and its appeal to primary and modern-school teachers led to its soon having more members than the older MA. The ATM web site contains a fascinating history of its early years and the mathematical and teaching concerns of its early leaders.

latter would facilitate the later learning of algebra. This naturally meant that less time was spent on the learning of arithmetic with the expected results and public reaction. Often, and particularly after the National Curriculum was established following the 1988 Education Act, primary school children in England tended to be introduced to concepts far earlier than were children in other countries (Howson et al. 1999).

2.4.4 Teacher Training and the Beginnings of Research in Mathematics Education

The training of primary school teachers entered a new era in the mid-1960s: ‘Training Colleges’ became ‘Colleges of Education’, a 2-year course was replaced by a 3-year one, and gradually ‘certificate’ courses were phased out as students aspired to a B.Ed. degree and the teaching profession moved to becoming all graduate. Again, Open University television programmes also proved valuable in-service training. Primary education was the focus for much early research in mathematics education, and the need for, and the value of, educational research was recognised in 1946 by the establishment of the National Foundation for Educational Research. It was not, however, until the creation of the first two university chairs in mathematics education in the late 1960s that research in mathematics education really became established (Howson 2009).

2.5 1902–1988 Secondary Education

2.5.1 1902–1944

The establishment of state secondary education immediately raised the question of what form it should take. The answer was to base the new grammar school curriculum firmly on that of the old public and proprietary schools. The mathematics curriculum was still effectively determined by university entrance requirements and by the syllabuses of the various university examining boards. This meant that the differential calculus came to be taught in most grammar schools and high standards were expected of students. A typical 1910 ‘Analytical Geometry and Differential Calculus’ paper for 18-year-olds is reprinted in Howson 1982, pp. 225–226. The questions on conic sections would be meaningless to most 18-year-old students these days, and the technical demands of the calculus and the ability to ‘explain a method’ would also be excluded.[6] However, much has entered the mathematics syllabus of all 16–18 schools in the meantime, most obviously integral calculus, and, even more importantly, the examination is now taken by a far greater percentage of the population.

The mention of ‘analytical geometry’ indicates that the rule of Euclid had finally come to an end. This happened when Cambridge, in 1903, decided that it would accept any proof in geometry that appeared to form part of a systematic approach to the subject. Various courses in geometry were to appear in the next decade: the most influential being a rethought and pupil-centred, ‘watered-down’ Euclid by C. Godfrey and A. W. Siddons which continued to be used in some schools for the next 60 years. More surprising was a transformation geometry textbook published by W. J. Dobbs in 1913. An MA report on the teaching of geometry, published in 1923, was to consolidate the teaching of this topic in a ‘Godfrey and Siddons’ form for some 40 years. Indeed, as in other countries, the years after the First World War were times of retrenchment rather than innovation.

A report prepared by Godfrey for CIEM (ICMI) in 1912, showed that much new work had been introduced into schools in the past 10 years: about half of the 370 schools he surveyed (probably with

[6] The broadening of the curriculum at the loss of gaining technical fluency in a limited number of areas raises interesting questions as to which is easier to gain away from the classroom: knowledge of new areas or technical fluency and confidence?

the non-state schools over-represented) undertook out-of-door practical work on surveying and over half descriptive geometry (in the sense of Monge). Elementary statistical work was a common feature and about half claimed to use vectors in the teaching of mechanics and/or complex numbers. Godfrey had himself established a laboratory for practical mathematics at Winchester College where he taught, which was copied in several other public schools and which included many experiments on applied mathematics (Board of Education 1912, vol. 1, pp. 393–428 – a publication which together with Wolff 1915 are two key references for this period). By the 1920s, however, such work moved to the domain of the physics teachers. In fact, grammar school mathematics syllabuses were to show little change until the reform period of the 1960s. Students up to the age of 16 (when many in the grammar schools left after taking an examination known as the School Certificate) studied arithmetic, algebra up to quadratic and simultaneous equations, a geometry course loosely based on Euclid, logarithms and trigonometry. (A typical 1934 School Certificate paper is reprinted in Howson 1982 (pp. 232–233), along with proposals made in 1944 for a new School Certificate syllabus (pp. 234–237).

Beyond School Certificate (16–18), students began to specialise, taking usually no more than four subjects on which they were examined when seeking the Higher School Certificate. Those continuing to study mathematics, for its study was no longer compulsory, could take it as two separate subjects 'Pure' and 'Applied', or as one combined subject (but some universities demanded that students wanting to study for an honours degree in mathematics had to take 'double'-subject mathematics). The Pure syllabus typically comprised further algebra (e.g. properties of roots of polynomials), co-ordinate geometry, conic sections, the theorems of Ceva and Menelaus, further trigonometry and differential and integral calculus. The Applied papers tested dynamics, statics and, to a limited extent, hydrostatics (topics which in many countries were considered part of physics). For those wishing to obtain scholarships from universities, state or county, there were 'Scholarship Papers' of a more testing kind.

2.5.2 Three Government Committees and the Form of State Secondary Education

A government committee (Hadow) had reported in 1926 that some form of post-primary education should be provided for all children and suggested that three types of secondary schools were needed: grammar, 'modern' and technical (Trade) schools. Some local authorities attempted to put this plan into practice, but little was done to make such a scheme universal for the resources needed to put the proposals into practice were non-existent. A later committee (Spens) reported in 1938 on secondary education in grammar and technical schools and its report included some interesting remarks concerning mathematics teaching: 'The content of school mathematics should be reduced', its teaching suffered 'from the tendency to stress secondary rather than primary aims', it concentrated too much on 'tricky problem solving' rather than giving a 'broad view', the type and 'rigour' of the logic it presented had 'not been properly adjusted to the natural growth of young minds', … (pp. 235–242). The virtues of Hadow's 'tripartite' system were upheld by the Norwood Committee when it reported in 1943. As one critic, S. J. Curtis, wrote, according to Norwood the Almighty had benevolently created three types of children in just those proportions which would gratify educational administrators and, moreover, which class a child belonged to was clearly to be observed by the age of 11 (quoted in Howson 1982, p. 277).

[7] The move to comprehensive schools within the state system has been a long one and is by no means complete. London had 11 by 1957 and there were others in rural areas such as Cumberland. However, in 1965 the new Labour Government circulated all LEAs requiring them to draw up plans to convert to comprehensive education. This circular was rescinded by the Conservative government when it took office in 1970, but by then, the plans were so far advanced that, in fact, more comprehensives were created under that government than under any other. Yet some English (but no Welsh) authorities still retain grammar schools, as does Northern Ireland. Since 1988 various new types of schools have arisen and the situation is now extremely complex, but outside the remit of this chapter.

2.5.3 The 1944 Education Act

This important act, produced in the closing years of the Second World War, followed the lead given by Hadow and Norwood and established secondary school education for all based on a tripartite system with entry to state grammar schools being dependent on passing an examination taken at 11. The school leaving age was raised to 15 and when possible (in the event 1972) to 16. Entry to state grammar schools was now to be solely on merit. Previously parents could purchase a state grammar school education for their children at a relatively small cost, provided the school would accept them. Now, middle-class parents of children who failed the '11+' and who could not afford to send their children to a non-state school were often dissatisfied by the education provided, and the status enjoyed, by the new secondary modern schools. They were to play a part in replacing the tripartite system, in the 1970s, by local comprehensive schools: as was, more importantly, the growing belief that valid decisions concerning a child's future could not be taken at the age of 11.[7] (Nevertheless, selection had thrown an academic lifeline to able children from poor working class districts.)

The percentages of children who attended a grammar school varied from 10 % to 30 % depending on their local authority, but in 1961, according to official figures, only 22.1 % of pupils in England and Wales were in maintained (state) grammar schools. The technical schools intended to supply pupils with a specialist form of practical education had a mere 3.1 % of pupils, and 10.4 % were in independent or 'direct grant' schools (the latter occupying a middle position between state and independent schools: a position that ceased to exist when, later, such schools had to choose between becoming comprehensive or independent – the vast majority choosing the latter option). The majority of pupils were, then, in the secondary modern schools created following the 1944 Education Act.

In the late 1940s revisions had been made to the school examination system. School and Higher School Certificates were replaced by Ordinary Level and Advanced Level General Certificates of Education (O-level GCE and A-level GCE), usually taken, respectively, at age 16 and 18. In theory, these were qualifications intended for grammar school pupils. However, increasingly secondary modern schools began to enter their more able students (some of whom, had they lived elsewhere, would have qualified for a grammar school education) for some O-level GCEs. This was thought to be distorting the general curriculum of such schools, and so in 1965 a new Certificate of Secondary Education (CSE) was created intended to satisfy their particular needs. Again, diversity was actively encouraged, for schools were offered three options: to be examined externally on a board's syllabus, to be examined externally on the school's own syllabus and to set their own examinations, with external moderation, on their own syllabus (examples of papers set are reprinted in Griffiths and Howson 1974). This did not deter schools from entering 'border-line' pupils for both GCE and CSE examinations (with the result that the latter became, in practice, merely watered-down GCE ones), and in 1986 the difficulties caused by running GCE and CSE courses in parallel in comprehensive schools led to their abandonment and replacement by a General Certificate of Secondary Education intended to serve the needs of all types of student.[8]

2.5.4 Curriculum Reform

By the 1960s dissatisfaction was beginning to grow both with the curriculum that had not changed significantly for many years and the way that mathematics was being taught. An MA report, published in 1959, *Mathematics in Secondary Modern Schools*, made many useful suggestions, but was

[8] The problems this caused, for it did nothing to solve the great problem of varying mathematical abilities in pupils, and the subsequent claims of a consequent 'dumbing down' of the GCE have meant many independent and, more recently, a few state schools opting for other international qualifications.

criticised by Cyril Hope, a leading figure in the ATAM and ATM, for the backward-looking nature of the mathematical content (Secondary School Mathematics 1959, pp. 37–38).

However, encouraged by the publication of a radically new text by D. E. Mansfield and D. Thompson, and by what was happening in other countries – particularly the United States, a variety of projects sprang up from 1962, of which the most important were the School Mathematics Project (SMP) (ages 11–18), Mathematics in Education and Industry (MEI) (ages 16–18) and the Midland Mathematics Experiment (MME) (ages 11–16). SMP's early work (including syllabuses and examination papers) is documented in Thwaites (1972) and Howson (1987, 2009), and an account of MEI's history can be found on its web site, http://www.mei.org.uk. Howson (1978) and Watson (1976) provide general surveys of these and other projects, and their social and educational contexts are described in Cooper (1985); a brief account of a variety of textbooks used in English schools (primary and secondary) in the 1950s and 1960s is in Breakell (n.d). MME differed from SMP and MEI in that, from its initiation, it directed its work to secondary modern schools in addition to grammar schools. It failed to make a lasting impact, not on mathematical grounds but because it lacked the money that SMP and MEI were able to attract and because the schools attached to it did not have the prestige and status of those connected with those two projects.

The reforms did not emphasise sets as much as, say, the SMSG in the United States or abstract algebra as in France. Much new material such as co-ordinate geometry, probability and statistics entered the 11–16 curriculum and has stayed there. Other innovations, such as transformation geometry, stay only in an emaciated form, and other topics simply disappeared – sometimes because of the changed nature of schools and examinations. (The SMP's transformation geometry soon lost its initial goals – including a first introduction to matrices and, from ages 16–18, group theory and linear algebra. Providing a sound geometry course has remained a major problem (Royal Society 2001).)

2.5.5 Increased Diversity and Attempts to Bring Unity

These curricular innovations were made possible because of the freedom given to schools, or groups of schools, to create their own syllabus, provided that an examination board would agree to set examinations on it. This led to a great number of syllabuses on offer to schools with very different mathematical demands – and schools were free to select the examination board (now with greatly diminished university influence) and syllabus most suited to their own goals. (See, e.g. the examination papers reproduced in Thwaites 1972 and Griffiths and Howson 1974.) In the 1970s it was estimated that about a third of secondary schools were still following a traditional-style syllabus, a third modern ones, and the remaining third hybrids. It was a time of curricular chaos and the first attempt to rectify this came when governmentally established bodies over-seeing examinations drew up lists of 'core' items that had to be present in all curricula. Such restrictions made innovation increasingly difficult. Moreover differences began to grow in what was taught, and how, to more able pupils in the better independent, fee-paying schools and those in state comprehensive ones (see, e.g. the two 1980s SMP series: the New Books 1–5 used almost only in independent schools and the 11–16 series used in the majority of comprehensive ones).

Many questions came to be raised about the mathematical standards of school leavers and this resulted in the establishment of a government enquiry which led in 1982 to the publication of the Cockcroft Report. This set out many ideas for improving the teaching of mathematics at all levels, but, like so many other reports of its type did little to solve any problems. More significant were the projects for low attainers that were established in its wake: the Low Attainers Mathematics Project (LAMP), Raising Achievement in Mathematics Project (RAMP) and the SMP Graduated Achievement Project. Basically, the problems consequent upon the establishment of comprehensive schools and the greater number of students aspiring to 18+ qualifications and university entrance have still to be solved in a satisfactory manner.

It was in an attempt to end such diversity and ensure that all schools had the same goals, which in 1988 an Education Act was passed which established a National Curriculum in Mathematics for students aged 5–16 to be followed in all state-funded schools (but not necessarily in independent ones). The hastily assembled curriculum (see Howson 1991) was designed to fit into a controversial and untried scheme for testing students at various attainment levels at ages 7, 11, 14 and 16. The years since then have seen continuous attempts to solve the problems created by the poorly designed curriculum, the testing proposals (used in incompatible ways reminiscent of 'payment by results', i.e. to assess pupils' progress and to rank schools for accountability) and more general social changes. At the time of writing, yet more changes – and even some diversity – are promised!

3 Scotland

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3.1 Prelude

In the past, Scotland suffered from periods of instability, and incursions from England, but developed independent trading and political links with many countries (Mitchison 2002) For more details on the general situation described here, the reader is referred to the works of Knox (1953), Hunter (1968) and Rogers (2012). These connections as well as economic contexts and differences in religion influenced the country's culture and its legal and education system. General histories of both education systems can be found in Gillard (2011) and Scotland (1969).

Scotland followed the typical pattern of European education with the Church organising elementary schools in the main Burghs (local administrative regions) and parishes where writing and basic elements of arithmetic were taught in the vernacular and by the end of the fifteenth century some of these schools admitted girls (Wright 1898, pp. 244–246). A few grammar schools were attached to cathedrals or abbeys, to train boys for the priesthood. In 1496 a Scottish Parliament Act required all sons of landowners to attend grammar schools and the landowners to provide funds for the upkeep of the schools and a salary for the teachers (Wright 1898, pp. 47–48).

The University of St Andrews was founded in 1413, followed by the Universities of Glasgow (1451) and Aberdeen (1495). In 1574 the humanist Andrew Melville (1545–1622) was appointed to Glasgow University and began a liberal arts programme with a 4-year curriculum (for his biography see University of Glasgow n.d.). In 1580 he moved to St Andrews and continued his curriculum there (Wright 1898, pp. 70–81; Wilson 1935, pp. 7–19). In 1582 Edinburgh University was founded and followed the same tradition.

By the end of the sixteenth century, Scotland had free primary and secondary education and a university system with an established humanist curriculum of 4 years and these remain as a significant feature of Scottish Education. Following Henry VIII's Reformation, John Knox (1514–1572) reformed the Church of Scotland on Presbyterian principles. The Scottish Parliament renounced the authority of the Pope and redistributed the wealth of the Church to the ministry, the schools and the poor. John Knox, with five colleagues, wrote *The First Book of Discipline* (1560) that required all citizens to read the Bible, so literacy became important (Laing 1895, vol. 2, pp. 183–260). Free education for the poor was established, so Scotland had a public education system committed to widespread literacy and funded in part, by both Church and State (Wright 1898, pp. 82–92).

Education Acts of 1633 and 1646 extended taxes to municipalities, providing a better foundation for schools, and another Act of 1696 regulating elementary education continued in force until 1872.

Mathematics books written in English appeared in the sixteenth century, and *Abacus*[9] mathematics was taught in the vernacular. Parish and Burgh schools provided elementary and some secondary education. In spite of the emphasis on the 'three Rs' (reading, writing and arithmetic), the erratic attendance of children from an agrarian society meant that numeracy and literacy were poor. By the end of the seventeenth century, practical arithmetic consisted of the four rules in integers, vulgar fractions and the rule of three, but extraction of roots was often left until the university (Yealdham 1936, pp. 75–87). Other aspects of mathematics were not taught in most Scottish schools until the end of the century (Wright 1898). However, state schools were not providing the mathematics people needed for commerce or practical occupations, so private schools for writing and arithmetic and some 'commercial academies' were set up by independent teachers from the early seventeenth century.

By the end of the century, mathematics was becoming well established in the universities, including the teaching of Euclid's Elements, plane and spherical trigonometry, astronomy and navigation and algebra up to quadratic equations (Wilson 1935, pp. 23–26). The contribution of well-known individuals as John Napier and James and David Gregory to the development of mathematics is described in detail elsewhere (see, e.g. The MacTutor History of Mathematics archive is a website at St Andrews University: http://www-history.mcs.st-and.ac.uk/history/).

3.2 The Eighteenth Century

In 1707 the English and the Scottish Parliaments signed the Treaty of Union by which the two kingdoms became the Kingdom of Great Britain with one Parliament based at Westminster, London. This treaty maintained the religious, legal and educational independence of Scotland.

3.2.1 The Universities

The work of a number of well-known mathematicians (particularly Simson, Stirling, Maclaurin and Playfair) who all taught at one or other of the universities began to develop university courses. Only two of these figures had any direct impact on the teaching of mathematics in school.

Robert Simson (1687–1768) became professor of mathematics at Glasgow in 1711. His best-known work, *The elements of Euclid*, was published in 1756, is the translation of Euclid (Books I–VI and XI and XIII). Separate editions in Latin and in English were printed, and the English version appeared in many schools, becoming the model for editions by later authors (Simson 1756).

John Playfair (1748–1819) appointed professor of mathematics at Edinburgh in 1765 was also involved in the foundation of the Royal Society of Edinburgh. In 1795 he published an edition of the *Elements* for use by his students. His work began to standardise the notation for points and sides of figures and introduced his alternative to Euclid's parallel axiom (Playfair 1795, pp. 295–296).

3.2.2 Writing Schools and Commercial Academies

Frequently in competition with the state system, writing schools taught arithmetic and practical mathematics, surveying, navigation, basic trigonometry and occasionally algebra. Writing schools, their teachers and some academies were often subsidised by town authorities to encourage business development. More commercial academies appeared that taught practical arithmetic and geometry to those who needed it for business; for surveying, gauging and astronomy. Navigation was a practical art, part learned at sea, and boys were trained from quite young. Typical textbooks included the use of

[9] The term *Abacus mathematics* indicates the traditional European body of mathematics for business and commerce.

the globe and the celestial sphere, with measure in degrees for astronomical calculations. (A similar development appeared in England (Rogers 1993).)

Some academies had a wider curriculum including foreign languages and took pupils from elementary and grammar schools for professional training. This influenced some grammar schools, forcing changes in their mathematics curriculum, but others resisted this attack on their monopoly of classical secondary education.

Founded in 1760, Perth Academy provided an ambitious programme in the higher branches of arithmetic; mathematical, physical and political geography; algebra, including the theory of equations; differential calculus; geometry, consisting of the first six books of Euclid; plane and spherical trigonometry; mensuration of surfaces and solids; navigation and fortification; analytical geometry and conic sections; and natural philosophy, consisting of statics, dynamics, hydrostatics, pneumatics, optics and astronomy. Later, this curriculum included the binomial theorem for a positive power and evolution of algebraic expressions up to the sixth root and some infinite series; arithmetic and geometric progressions, permutations, combinations and compound interest; probability, co-ordinate geometry, graphs and fluxions. Wilson (1935, pp. 75–79), so some academies, provided a range of knowledge that began to rival the universities.

3.2.3 Secondary Schools

At this time a confusing mixture of institutions provided some form of secondary education: traditional grammar schools, academies and writing schools, and even some primary schools whose teachers were sufficiently knowledgeable. There were no formal qualifications or age limits for pupils to progress from primary to secondary schools, nor to university. Lucky pupils progressed if they had the knowledge or ability, or if they were able to pay for private tutors as the growing middle class was able to afford to send its children to the better establishments. However, many pupils left education without any academic or technical qualification.

Algebra as 'generalised arithmetic' became accepted in school mathematics; arithmetic and geometric progressions, permutations and combinations, probability, co-ordinate geometry, graphs and elementary differential and integral calculus were commonly taught. Simultaneous linear equations in two variables were usually solved by substitution and quadratic and cubic solutions were limited to finding one positive root. Geometrical constructions and surveying instruments of the period were in common use.

Arithmetic books developed beyond explaining the basic operations to provide more sophisticated mathematics and many of the practical applications for commercial use. Algebra was finding its way into the general curriculum; both John Mair's *Arithmetic, Rational and Practical* (1766) and Hamilton's *Introduction to Merchandise* (1788) included sections on algebra. Cocker's *Arithmetic* (see Hawkins 1667) was produced in Scottish editions from 1748, and other texts from England were used to prepare pupils for commercial life and entry to the universities. Wilson (1935, pp. 89–91) has a list of over 60 mathematics books of the period showing two thirds were published in Scotland. Many of these were used well into the next century. Trigonometry was included in most books on pure and applied geometry, Wilson's *Trigonometry, with an Introduction to the Use of both Globes and Projection of the Sphere* (1714), became the standard reference work for more than a century, and most books had tables of trigonometrical functions and logarithms.

3.2.4 Progression from Primary Schools

Towards the end of this period, girls and boys in primary schools received the same teaching in basic arithmetic as in the previous century. Boys who achieved reasonable success in the 3Rs, proceeded to secondary education, while some girls became bookkeepers in family businesses.

Sufficient knowledge, or the ability to pay fees, enabled progress to secondary school at age 8 or 9 and to enter university at about 12, but ages of transfer began to increase as the secondary and university curriculum expanded. Bright pupils from parish schools were lucky if they met a primary teacher who knew enough to get them into university. For the majority who continued in primary school, their mathematical diet was basic, repetitive and arithmetic.

In the closing years of the eighteenth century, Edinburgh was the centre for the *Royal Society of Scotland* and maintained active and enlightened interest in science. The university chairs were occupied by men who had a competent knowledge of mathematics and who conducted courses efficiently; at the same time advances in knowledge were being made and the level of attainment was high; in the country, some schools included conics and elementary calculus, indicating a provision of teachers and an outlook for their pupils that could only be met by the universities.

3.3 The Industrial Revolution and Education of the Workers

In the 1750s Scotland was a poor, agricultural society with a population of about 1.3 million. But Glasgow and Edinburgh grew, and by the 1850s the total population was some 2.6 million. The Treaty of Union had increased access to the British Empire, and Glasgow developed as a centre of trade, while Edinburgh became the administrative and intellectual centre of the Scottish Enlightenment where the views of David Hume, Adam Smith, Thomas Muir and Adam Ferguson influenced the development of modern social and political theory. These people contributed to the intellectual and scientific life of the whole United Kingdom.

With the development of steam power and the change to manufacturing, entrepreneurs, like James Watt and Thomas Telford, developed engineering and infrastructure; James Young set up the first oil refinery; James Brown founded the shipbuilding industry; these and many others led industrial development in Scotland through the late eighteenth and nineteenth centuries (Whyte 1995).

There was enormous social upheaval. The mechanisation of weaving destroyed the cottage-based industries and drove people into factory work; two famines in Ireland, in 1739–1741 and 1845–1852, drove many people to Liverpool and Glasgow; and the Clearances (the forced displacement of the population of the Scottish Highlands during the eighteenth and nineteenth centuries due to agricultural reform) destroyed the life of the Highlands so that thousands moved to urban centres or emigrated to the colonies.

From the latter part of the eighteenth century, liberal-minded people and factory owners became concerned about the state of the poor and established public parks, libraries and new educational institutions. The improvement in technology provided cheaper books and printed material, improving access to information. During the early nineteenth century, the Westminster Parliament passed a series of acts that restricted child labour and laid down a maximum number of hours of work for adults.

Education for working people began when John Anderson and George Birkbeck at Glasgow University started giving free lectures to workers in the 1780s. Anderson died in 1796, leaving money to provide education for the ‘un-academic classes’, leading to the foundation of Anderson’s College and the beginning of the ‘Mechanics Institute Movement’. Institutes were founded to provide technical education for working people and to incorporate fundamental scientific thinking and research into engineering. They revolutionised education in science and technology for ordinary working people. Birkbeck later moved to London and became the first president of the London Mechanics Institute in 1824. Many industrialists supported this movement, and by the mid-nineteenth century, there were institutes in towns and cities across the United Kingdom that provided access to scientific and practical knowledge for working people.

3.4 *Education in the Nineteenth Century*

By 1800 the education system had failed to cope with the upheaval of the industrial revolution and was showing signs of strain. In 1834 it was estimated that less than 8 % of the school-age populations of Glasgow and Dundee attended educational institutions, and in the Highlands many inhabitants could neither read nor write. In spite of the emphasis on the 3Rs, erratic attendance meant that the standards were poor. Institutions, like the Church and the landowners, found it increasingly difficult to provide funds for educating a growing population swollen by Irish immigrants. The serious Disruption in the Church in 1843[10] and the need to provide for Roman Catholic pupils increased the problems.

3.4.1 Schools and Pupils

By now, the role of Scottish schools had significantly changed. The curricula of the Burgh and grammar schools (the original secondary schools) had met the entry requirements for the universities. However, owing to the competition from the parish and writing schools (originally the primary sector) who were taking subjects to a higher level, the Burgh and grammar schools began to admit primary pupils. Thus, both types of schools became providers of general education. The better primary schools had 'upper classes' with pupils aged 11 or 12 who were achieving a standard equal to some secondary schools (see Scottish School Reforms of the 1870s at The MacTutor History of Mathematics archive is a website at St Andrews University: http://www-history.mcs.st-and.ac.uk/history/).

In 1868 the Argyll Commission showed that over 50 % of students attending Scottish universities had come directly from parish schools (Cruickshank 1967). An education (Scotland) act of 1872 legislated for inspections of parish schools, and in 1882 inspections were extended to all Burgh and grammar schools. This applied also to the academies, which had gained independent secondary status or merged with Burgh or grammar schools. Attendance was low, and finding a job was the real priority for children from working-class backgrounds (Knox 1953) so that under 5 % of pupils attended.

3.4.2 Nineteenth-Century Secondary School Mathematics

The mathematics taught at this time can be judged by many surviving documents. Madras College was founded in 1832 in St Andrews by Andrew Bell[11] and gained a high reputation. An account of the classes of one of its headmasters, W. O. Lonie, tells us that from 1846 to 1894, 'their programme exhibited an extensive course of teaching in geometry, practical mathematics, algebra, and geography' (see Lonie's biography at the MacTutor History of Mathematics archive). James Walker's *Fair Book* of 1852, from a pupil at the college, (also found on the MacTutor site) contains pages of typical exercises covering a variety of applications of mathematics; the four rules in arithmetic (with quite large numbers and six decimal places), use of six-figure logarithms; common and vulgar fractions, ratio and proportion, conversion of money and mensuration; capacity and gauging; measurement of areas and other geometric and trigonometric exercises using identities; exercises on the circle, parabola and ellipse; numerical solution of elementary quadratic equations; and questions on 'artillery'. Apart from the explicit mention of some trigonometric formulae and Heron's formula for the area of a triangle, it seems that pupils were taught procedures for solving standard problems.

[10] The Disruption of 1843 was a schism within the established Church of Scotland where 450 ministers broke away over the issue of the relationship with the State to form the Free Church of Scotland. It had a serious effect not only on the Church but also on Scottish civic life.

[11] Andrew Bell was one of the founders of the 'Pupil-Teacher' or 'Monitorial' system which he introduced in the late eighteenth century. The method became very popular and was based on the abler pupils being used as 'helpers' to the teacher, passing on the information they had learned to other students.

Finding 'artillery' in a school exercise is not unusual; John Davidson's *System of Practical Mathematics* (1832) is a typical compendium of the 'mixed mathematics'. As well as the usual sections on algebra, geometry, plane and spherical trigonometry, it deals with the measurement of heights, distances, surfaces, and solids, specific gravity, conic sections, land and wood measurement, and artificers (craftsmen) works, gauging, gunnery, geographical and astronomical problems and navigation. Books like this provided an important social function; apart from their use in schools, property owners used them for checking craftsmen's work and calculating payment.

During the nineteenth century there were considerable advances in university mathematics, but in most secondary schools, arithmetic and geometric series and quadratic equations were the limits of algebra, Euclidean geometry was quite formal, and enough plane and spherical trigonometry were taught for practical surveying and navigation.

Important and influential changes in teaching in the second half of the century can be exemplified by George Chrystal's textbooks. Chrystal graduated from Aberdeen University in 1871 and became second wrangler at Cambridge, in 1875. He was professor of mathematics at St Andrews and then in 1879 took the chair of mathematics at Edinburgh University. Chrystal was concerned about standards in schools and in 1883 he supported the foundation of the Edinburgh Mathematical Society by two schoolteachers, demanding better trained schoolmasters and higher teaching standards. Chrystal's lasting contribution to education was his concern with the teaching of mathematics, particularly algebra.

He published the first volume of *Algebra: An Elementary Text-Book for the Higher Classes of Secondary Schools and for Colleges* in 1886. In his preface he warns that the book is not for beginners and says,

> [It] becomes necessary if Algebra is to be anything more than a mere bundle of unconnected rules, to lay down generally the fundamental laws of the subject and to proceed deductively—in short to introduce the idea of *Algebraic Form* which is the foundation of all modern developments of Algebra….

Volume 2 Algebra: An elementary text-book for the higher classes of secondary schools and for colleges followed in 1889.

This 'elementary' textbook was for the 'higher classes', who would have already studied algebra, as far as quadratic equations during their first 3 years of secondary education. The expected level for the higher classes was considerable, and the results of the first leaving certificate examinations were very poor.

Probably the most significant publication that would influence school mathematics throughout the United Kingdom for the next half century was Chrystal's *Introduction to Algebra for the Use of Secondary Schools and Technical Colleges* published in 1898. In his Preface (1898, pp. vii–xii), Chrystal is admitting that his earlier work contained too little practical application, and not enough *graphical illustration* for students in technical colleges. His motivation is to reform the 'English text-books in vogue during the latter part of this century [that] have tended to denigrate [algebra] into a mere farrago of rules and artifices, directed to the solution of examination puzzles of a somewhat stereotyped character having little visible relation to one another and still less bearing on practice' (1866, p. viii).

He wanted students to be aware of *algebraic form* and demonstrated the commutative, associative and distributive properties of arithmetic operations applied in a generalised form and introduced graphical methods. The contents of the book display a progressive demonstration of structure and technique in a rational reorganisation of the material found in earlier texts, and his examples and exercises are deliberately chosen to achieve a standard that set a clear example for many subsequent algebra textbooks.

3.4.3 The Leaving Certificate

In 1885, Henry Craik, secretary of the Scottish Education Department (SED), appointed a committee to examine the state of education. Following its report, it was decided to conduct an inspection of all schools teaching higher classes (for pupils aged 15–17). Thus began the first move to achieve common standards for secondary education. Chrystal took part in the inspections, and the report showed many school staff were inadequate and underpaid, the curricula inappropriate, and teaching methods ineffective.

It was decided that inspections would monitor school programmes and certificates should be awarded to pupils on the result of an examination. This provided a link to the universities and became an important factor in improving teaching methods and results in secondary schools.

Chrystal examined pupils in the higher classes of 12 secondary schools. He designed questions so that pupils who knew their ‘bookwork (routine procedures and basic facts) could just pass. The mathematics covered was geometry, algebra, trigonometry and arithmetic, and special papers were offered in differential calculus, analytical geometry and geometrical conics (Philip 1992). Mathematics was a compulsory subject in this examination, and all professions required some mathematical competence.

A leaving examination would show that pupils were academically equipped to enter university. However, Chrystal realised that many who ‘failed’ this new examination had reached the level demanded by the General Medical Council and other professional bodies. So it was decided that two levels of certificate (so-called higher and honours higher) would be issued and the first leaving certificate examination was sat in June 1888.

In mathematics, the higher and honours levels consisted of three separate papers, each requiring answers to seven compulsory questions: arithmetic (1½ hrs.) standard questions plus optional questions using logarithms; geometry (2 h) standard questions on Euclidean geometry using memorised proofs and optional questions on solid geometry that might need the use of logarithms and trigonometry; and in algebra (2 h) with standard questions on polynomials, permutations, complex numbers and trigonometry, using the cosine rule, trigonometric expansions and inverse ratios. Since different institutions catered for different professions, questions were written with options to meet their needs. For those who chose honours grade, candidates were encouraged to complete as many questions as possible. Mathematics was the only one of the six leaving certificate subjects to publish notes for guidance from the beginning.

3.5 The Scottish Mathematics Education System 1900 to the 1980s

The first half of the twentieth century saw many changes in educational administration, with increased state intervention as schools were seen as an important agency for social welfare. In 1901 the school leaving age was raised to 14, and in the 1908 Education Act, parents were made responsible for their children’s attendance, medical inspection was introduced, and free meals are given for needy children. After the First World War, the 1918 Education Act created new educational authorities and brought Catholic schools into the state system (Humes and Paterson 1983; Finn 1983). The leaving certificate also underwent many changes in detail, and it played a vital role in the development of secondary education. It raised the standard in secondary schools so that they began to cover work previously studied in the universities.

In the 1920s and 1930s policymakers faced a growing demand for secondary education and divided children into two types: the ‘academic’ and the ‘nonacademic’. To tackle this problem, ‘advanced divisions’ were created in elementary schools to provide post-primary education. In 1936, maintaining the meritocratic system from the nineteenth century, the secondary schools were divided into 3-year junior secondary schools and 5-year senior secondary schools.

By this time, the post-primary education offered consisted of the following:

- Primary schools with advanced division classes, where it was still possible for some pupils to proceed to a junior secondary school or, very exceptionally, to gain a leaving certificate
- Junior secondary schools with a 3-year curriculum that later ran more ‘academic’ classes to enable a few pupils to transfer into the higher class schools, but the majority received no formal qualifications
- Senior secondary (higher class) schools with a 5-year curriculum and the expectation of entry into university or the professions

However, in spite of more reforms in the 1940s, 87 % of young adults in the age group 20–24 in 1951 had left school at age 15 or younger. It was not until the introduction of Comprehensive schools[12] that the inequality between senior and junior secondary schools was finally addressed. In 1947 raising the school leaving age to 15 increased the school population by 40,000. The system required more teachers and serious thought about teaching larger groups of pupils of a wide range of ability for longer than before (Cruickshank 1970).

To describe the problems and development of mathematics education during a relatively prolonged period of time is difficult. Below an attempt is made to do so, relying on figures from certain official reports. For a more detailed analysis of them, see Rogers (2012).

3.5.1 The Hamilton Fyfe Report 1947

This report, Secondary education: A report of the advisory council on education in Scotland, chaired by William Hamilton Fyfe (1947) considered compulsory secondary schooling from age 12 to 15 and was an opportunity to reconsider the curriculum and the majority of pupils unable to achieve the leaving certificate because they were judged only suitable for vocational occupations.

The authors thought that different aspects of mathematics should be relevant to the real world and not independent from each other. These views follow the Spens Report where treating aspects of mathematics as if they were different subjects is described as 'partial, unhistorical and un-philosophical' (Spens 1938, p. 236).

The report presented recommendations for curriculum reform; re-thinking attitudes, connecting subjects to the culture, teaching methods and administrative organisation. They confronted the system where a junior secondary school pupil had virtually no possibility of transfer to a senior school at age 14 or 15. Proposals were made to develop apprenticeships and semi-professional work and to allow transfer to senior school.

According to the report, mathematics in Scottish schools needed a 'drastic overhaul' (paragraphs 441–478); teaching was unsatisfactory and the majority of pupils of average ability suffered from 'the dullness and futility of much school teaching'. Mathematics was 'devoid of appeal to ordinary youngsters' and the nature of the Leaving Certificate in Mathematics made it difficult for all schools to break with the formal academic nature of the subject. Provision for the 'lower ability' pupils should be simple everyday arithmetic, easy mensuration and the elements of graph work, with clear practical applications. Diagnostic tests would be more suitable than formal examinations (p. 445).

The report made a number of critical comments and suggestions about teaching specific topics in mathematics and took a practical, visual, tactile approach to the curriculum, encouraging engagement of the senses and making a cultural and aesthetic appeal to the historical development of concepts, art, architecture and the technical and social applications of mathematics.

Finally, a special note was made that girls should have access to a similar course, and since the school leaving age had been raised to 15, the report challenged Scottish teachers to address the problem of suitable practical activities for girls.

3.5.2 The Revised Mathematics Syllabuses of 1950 and the Arrival of the Comprehensive System

In 1950 the SED published a series of five new mathematics syllabuses for junior secondary schools, together with some brief teaching notes, and in the same document, syllabuses for Ordinary and Higher courses for Senior Schools. It continued the social stratification; the language of 'ability' and

[12] 'Comprehensive' in the United Kingdom means that a state school does not select its intake on the basis of academic achievement or aptitude.

division of the sexes was maintained, and pupils would be destined for low-level occupations. Three-year courses had no provisions that would enable transfer to a year 4 or 5, nor ideas that would integrate mathematical topics into a meaningful technical programme.

The SED acknowledged pure mathematics as an abstract subject suited the ablest pupils and recognised that 'for the less well-endowed the academic rigours must be tempered'. The *'need for a greater differentiation of courses has become still more pronounced with the general acceptance of the principle that promotion to the secondary stage should in the main be determined by age rather than by attainment. The raising of the school leaving age has at the same time made it necessary to reconsider existing schemes of work'* (SED 1950 p. 5).

The five syllabuses were intended to suit pupils' abilities, with content progressively modified to relate them to the 'needs of life' as perceived by the Department. The most challenging Syllabus 1 was described as a general course in mathematics for pupils capable of passing the leaving certificate. This contained arithmetic with a commercial bias; algebra with solution of linear and quadratic equations and inverse proportion, including graph work; and geometry with practical knowledge and use of instruments and selected theorems and proofs including some circle geometry, mensuration and elementary trigonometry. Contrarily, Syllabus V was described briefly as a very simple course for pupils of low ability with a limited list of topics concentrating on everyday quantities, units and measurements, probably repeating what they had failed to learn in primary school.

In spite of the clear intentions of the 1947 report (SED 1947), *these syllabuses were all content driven and fundamentally unchanged since the previous century*; the teaching notes emphasised speed and accuracy in calculation and precision with language. *With these attitudes and no change in educational philosophy, schools were unprepared for Comprehensive Education in 1965 when the state grammar and secondary schools were combined and teachers had to deal with teaching the 'academic' and 'less academic' pupils in the same school and often in the same class.* In 1967 the Ruthven Committee report on the Organisation of Secondary Certificate Courses recommended that all pupils should follow a common course (which included mathematics) in their first 2 years of secondary school. This was not intended to be taught at the same rate, use the same methods as 'normal' classes or share a common syllabus. When the school leaving age was raised to 16, a report by inspectors (SED 1972) found that just under half of secondary schools had no clearly defined policy for the first 2 years of secondary education.

3.5.3 The Scottish Mathematics Group (SMG) 1965–1975 and Other Important Projects

Facing the challenge of mixed ability classes and the 'new mathematics',[13] the SED established[14] a committee of inspectors, lecturers and teachers in 1964 to prepare a new syllabus and textbooks for an Alternative Ordinary Grade examination for the new Scottish Certificate of Education.

SMG mathematics was, to a large extent, promoted by the work of Geoffrey Sillito, a lecturer at Jordanhill College of Education and a key member of the *Association of Teachers of Mathematics* (ATM), who had arranged conferences where teachers and administrators were challenged to think differently about school mathematics, its content and its teaching. The original Scottish Mathematics Group, who wrote the first texts for *Modern Mathematics for Schools*[15] (SMG 1965a, b) had 16

[13] This term is only very roughly comparable to the 'New Math' of the Americas. The United Kingdom was not driven by government bodies and committees of mathematicians but influenced more by European sources and our own *laissez faire* local and 'grass roots' curriculum development.

[14] This government initiative contrasted with curriculum development in England where individual local authorities, teachers' organisations and publishers were encouraged to develop their own schemes.

[15] This series 'For pupils taking the General Certificate of Education Ordinary Level, or an equivalent examination' was also clearly intended for a wider market and was used by some Secondary schools in England and other countries.

members whose names appeared in the front of each book. The series was published in Glasgow from 1965 onwards.

Some 7,000 pupils took part in piloting in 1964, and the first textbooks appeared in 1965. Since the SED was the only authority responsible for both examinations and curriculum development, all agencies concerned cooperated to train teachers, and the new syllabuses and books were quickly adopted (the Scottish Government does not supply or recommend all books for use in school). The authors claimed that the most appropriate areas of the 'new mathematics' were 'blended with the necessary elements of the traditional'.

The course emphasised the relevance of mathematics in the world today, offering pupils material for developing mathematical ideas, with a balance of exploration and computation. The conflict of the new and old ideologies is clear in the approach towards pupil-centred activities of investigation and discovery, stressing the learning situation and at the same time supplying adequate material 'in a form which could easily be adapted to suit individual schemes and teaching methods'. The textbooks emphasised the structurally integrating areas of algebra, geometry and arithmetic (Trigonometry was introduced in books 5 and 6 and some elementary Calculus in book 7), and in each book the content was organised under these headings, while topics were developed within each section.

Six subthemes are found throughout: the language of sets; number systems; relations, mappings and functions; coordinates and graphs; logic; and deduction. About one third of the algebra and the arithmetic was new, including iterative solution of equations, linear programming, elementary probability and more statistics. The remaining material was approached 'from a new angle or set in a new context'. For teachers brought up on a traditional syllabus, with fixed views about pupils' abilities, this was highly radical. It was not surprising to find strong conservatism among teachers and, over time, much of the original content was modified, and the geometry chapters in particular were largely rewritten (SMG 1967).

Among other attempts to solve the problems of mixed ability classes was the Fife[16] Mathematics Project which started in 1970 and lasted until 1977. The leader of the project Geoffrey Giles described the project as 'an open-ended experiment aimed at developing *teaching methods* for mixed-ability classes in the first 2 years of a comprehensive system'. The educational aims encouraged pupils to take responsibility for their own learning, stimulate understanding through activity and develop self-confidence. A number of evaluations of the project were undertaken including Giles (1977), Crawford (1975) and Morgan (1977). A critique and extended quotations from Crawford and Morgan can be found in Howson et al. (1981) (pp. 217–225).

3.5.4 Changes in the Primary Curriculum 1946–1980

Hamilton Fyfe also chaired a report on primary school education, *Primary education: A report of the advisory council on education in Scotland*, published in 1946. It criticised the system that still relied on old traditions and unfounded beliefs; arithmetic was condemned as outdated, and the new proposals for education were radical. Classrooms needed displays and equipment, children should be more active, their senses developed, and subjects no longer taught separately. Teachers should become aware of recent research and educational theory.[17] The report shows concern with transfer from

[16]The County of Fife is a peninsula on the East of Scotland. The county government supported this project. Many other authorities in the United Kingdom supported local curriculum projects at this time.

[17]The Scottish Council for Research in Education (SCRE) was set up by the Scottish Teachers' union (The Educational Institute of Scotland) in 1928. It was the first institution of its kind in the world. It now forms part of the Faculty of Education at the University of Glasgow. (The National Foundation for Educational Research (NFER) in England was founded in 1946.)

primary to secondary education and addresses the means of selection for junior or senior secondary schools, discussing intelligence, attainment and aptitude tests with sensitivity. Numerical test results cannot be 'final verdicts' on pupils, and personal and educational development will change possibilities for a child. Unfortunately, teachers were slow to change and primary mathematics remained as training in arithmetic with some simple mensuration (SED 1946).

In 1965, the SED published the 'Primary Memorandum' (SED 1965) that set out a philosophy of education starting with the needs and interests of the child, was appropriate to age, aptitude and ability and saw pupils as active in their own learning. This was followed by a progress report (SED 1971) on guidance for teachers offered by education authorities, colleges of education and head teachers. However, progress was still slow. A further report SED (1973) *Primary Education in Scotland: Mathematics* was issued updating the Memorandum, providing national guidelines for the Primary mathematics curriculum. However, a research report (SED 1980) on pupils aged 9 and 12 found only formal aspects of mathematics being taught. Teachers favoured closed responses, and there was very little discovery learning, open-ended questioning and discussion (Hartley 1987).

4 Conclusion

Two reports commissioned by the SED published in 1977 were to illustrate again the fundamental conflict in the Scottish education system brought on by raising the school leaving age and comprehensive education (Ewan 1978). The Munn report of the Committee to Review the Secondary Curriculum (SED 1977) recommended some measures in the third and fourth years of secondary school to meet the needs of pupils of all abilities. The Dunning Committee (SED 1977) worked on the aims and purposes of assessment in the higher grade Scottish Certificate of Education and the Certificate of Sixth Year Studies.

However, that the real issues were avoided again and the problem about producing an integrated system that could meaningfully and fairly assess the 'less able' remained. In 1983 new National Qualifications were introduced where the Scottish Government (SED 1983) declared its intention to reduce content and place more emphasis on the learning process and skills of a generic nature, and while the content changed as pupils progressed through the secondary school, the three-level certificate remained, leading to Highers and Advanced Highers for university entrance (see Learning and Teaching Scotland (n.d.) for more details).

Since then, we have seen more centralisation of the curriculum and more government intervention in England. Scotland has been able to resist this to some extent, but has been influenced by events over the border, and in 2008 launched a new 'Curriculum for Excellence'.

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Chapter 14
Mathematics Education in Spain and Portugal

Elena Ausejo and José Manuel Matos

Once a great nation of Europe that aspired to global supremacy under Charles V and Philip II, Spain in later times went through a prolonged crisis that was both political and economic. Portugal which was 60 years under the rule of Spanish king became again completely independent from Spain in 1640. By the beginning of the eighteenth century, following the so-called War of the Spanish Succession, the new Bourbon dynasty became established in Spain, remaining in power for succeeding centuries, with only certain interruptions. This war marked the definitive end of Spain's prominent role in international politics. During the first third of the nineteenth century, both Spain and Portugal lost many of their colonies. In both countries, the influence of the bourgeoisie was very limited – the dominant powers were the nobility and the Catholic Church. A very large part of the population remained illiterate.

The struggle for liberal reforms gave rise to numerous revolutions – reforms were implemented and subsequently abolished, and constitutions were passed only to give way to new constitutions. People and armed forces from many different countries became involved in the Spanish Civil War of 1936–1939, at the end of which, with support from Mussolini's Italy and Hitler's Germany, power fell into the hands of General Franco, who ruled Spain for decades. Portugal also became a dictatorship for decades under Salazar.

The situation began to change during the 1970s. Democratic governments were reestablished. Economic growth had actually begun even earlier (during the 1960s) and continued. The political-economic and social trends and changes that had taken place over the previous centuries could hardly fail to influence the development of education in general and mathematics education in particular.

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1 Spain

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1.1 Introduction

The following section offers a brief overview of Spanish mathematics education over the last two centuries. Mathematics education is viewed in the context of the educational system as a whole, which in turn depended to a very large extent on the country's political and economic circumstances. In order to understand how education developed in more recent times, we must begin by examining (albeit cursorily) education as it existed during the premodern era, since its legacy continued to manifest itself during much later periods. Many details, including relatively significant ones, will necessarily be omitted. The discussion will focus on the general process of transformations and on a few of the most important aspects and events.

1.2 Mathematics Education Before Modern Times

On acceding to the Spanish crown in 1700, the Bourbons found a scholastic university system with three major universities (Salamanca, Valladolid, and Alcalá) and a raft of minor universities. Each university had a maximum of four faculties, Theology, Law, Medicine, and Philosophy (or Arts), the first three being major faculties. The Faculty of Philosophy gave the Bachelor's Degree in Arts required to access the major faculties (Theology, Law, and Medicine). In this system, there was no place for modern science – on the contrary, intellectuals were openly hostile towards it (Peset and Peset 1974).

Within the framework of the universities were colleges, the original purpose of which had been to help poor students attend the universities. Usually, each college had a regional character and was restricted to students from certain cities and provinces. Kagan (1974) stresses the differences between Spanish colleges and British or French colleges:

> while many of the colleges in the north of Europe had been designed for graduate students who later took on undergraduate as boarders, pupils and wards, all but six of Spain's colleges were for undergraduates who were expected to attend university lectures. (p. 65)

The colleges were of major importance as they controlled the academic posts and the chair system. Even more, they acted within the whole state administration, building networks of influence and lobbies.

These university colleges should be distinguished from other educational institutions that were also called colleges but which served a different function and existed outside the universities. Such were the Jesuit colleges, which played an important role in the education of the Spanish nobility. Their college in Madrid, operating since 1572 and refounded as *Colegio Imperial* in 1603, was the leading teaching institution of the Habsburg period, especially after 1625, when it was chosen to host the new *Reales Estudios de San Isidro*. Later, the Bourbons chose the Jesuits to rule the new *Real Seminario de Nobles,* founded in Madrid in 1725.

While these and other educational institutions of a relatively high level existed in the country, it is impossible to speak of an educational system during this time or even during a slightly later period. While there were city schools, church schools, schools of religious orders, and so on, "it was rather a patchwork of provision of teaching at different levels, and varying from region to region" (McNair 1984, p. 17).

The Bourbons' attempt to reform Spanish university did not occur until the end of the reign of Charles III (1759–1788), with Minister Campomanes' Royal Charter of 1786. The reform tried to put an end to the power of the colleges, placing the Spanish university under state control. At the same time, it attempted to rationalize the teaching structure by systematizing degrees, syllabi, and textbooks and by arranging for the provision of chairs. The reform put an end to the profusion of minor universities, which were left void of content because they did not comply with the minimum requirements to grant official professionally recognized academic titles. As a whole, however, the reform failed, mainly because of fierce corporate resistance by the major universities but also because of the lack of financial resources.

With regard to mathematics education, it is worth mentioning the new regulation of 1771 at the University of Salamanca (*Plan general de estudios dirigido a la Universidad de Salamanca por el Real y Supremo Consejo de Castilla*) that replaced the chair of logic by the chair of geometry – also devoted to arithmetic and algebra – and created a new chair of mathematics specifically addressed to those committed to mathematics.

In this context, Juan Justo García, professor of Arithmetic, Geometry, and Algebra at the University of Salamanca between 1777 and 1824, succeeded in modernizing the content of mathematics courses and was the first to teach differential and integral calculus at a Spanish university. His *Elements of Arithmetic, Algebra, and Geometry* (1782), containing infinitesimal calculus, was widely used as textbook, both in secondary and higher education, during the first half of the nineteenth century (Cobos and Fernández-Daza 1997).

Not unusually, efforts for change and development made in institutions parallel to the university, such as military schools (engineering, artillery), navigation schools, and schools for the education of the nobility (*Real Seminario de Nobles* and *Reales Estudios de San Isidro*), turned out to be more efficient than in universities. Noticeable progress had also been made in newly created institutions, such as the so-called Economic Societies of Friends of the Country (*Sociedades Económicas de Amigos del País*), the Spanish institutional invention that attempted to attain a transformation of the society by developing productivity. Within the general wave of utilitarian progress that ran through Europe in the eighteenth century, the Economic Societies were concerned with issues of agriculture, industry, trade, and political economy, and in their attempt to instruct artisans, they created schools with a significant level of development of scientific disciplines such as mathematics or chemistry (Arenzana 1987).

Setting up the colleges and seminars abandoned after the Jesuits' expulsion in 1767 eventually facilitated the emergence of teachers and institutions that gave new impetus to mathematics teaching in Spain. In 1787, the academic validity of the courses of the most important former Jesuit colleges (*Reales Estudios de San Isidro, Real Seminario de Nobles* of Madrid, Bergara, and Valencia) was officially recognized for accessing the major university faculties. The old *Compendio Mathematico* in nine volumes (1707–1715) by the Jesuit Tomás Vicente Tosca – reprinted for the last time in 1760 – was replaced by Benito Bails's works: *Elements of Mathematics,* a complete mathematical treatise in ten volumes (1772–1783), and *Principles of Mathematics,* a more elementary and concise treatise in three volumes (1776). Both works were produced at the *Real Academia de Bellas Artes de San Fernando* (Royal Academy of Fine Arts of San Fernando), an institution devoted to the teaching of painting, sculpture, and architecture, where two new chairs of mathematics were created in 1768. Bails's *Elements* and especially his *Principles* were widely circulated as textbooks during the first half of the nineteenth century.

1.3 The Nineteenth Century: Establishing the System of Teaching Mathematics

Readers should bear in mind that the gradually developing system of Spanish education, as well as the terminology it employed, differed substantially from the educational systems that developed in other countries (above all, the United Kingdom and the United States). The universities were under strict government control, but at the same time, they were regional centers of education. Within an educational district (starting at a certain point, Spain was divided into educational districts), the rector of the university, appointed by the king, supervised all education (in this respect, the similarity with the French system, e.g., is obvious). The meanings of such terms as "bachelor" or "master" differed from the meanings they had in English-speaking countries.

In the nineteenth century, the Spanish legislation on teaching and curricula is complicated and even controversial; for example, there were more than 25 mathematical syllabi in less than three quarters of a century. This variety was the consequence of political instability, different liberal and conservative viewpoints – the former promoting science, the latter humanities – and the influence of academic lobbies. Nevertheless, the syllabi of 1836, 1845, and 1857 set up the basic structure of secondary education throughout the nineteenth century.

1.3.1 Regulation of 1807 and Some Further Regulations

The new regulations of 1807 extended the regulation of 1771 of the University of Salamanca to all Spanish universities and defined courses more precisely. The Bachelor's Degree in Philosophy was to be obtained in 3 years, but different courses could be taken in the third year depending on the destination Faculty (Law, Theology, or Medicine).

Two specific mathematical courses were established, namely, *Elements of Arithmetic, Algebra and Geometry* and *Application of Algebra to Geometry*, both with Juan Justo García's *Elements of Geometry, Arithmetic and Algebra* as textbook. Classes were held for an hour and a half in the morning and 1 h in the afternoon and followed the order as defined in the course title: arithmetic, algebra, and geometry. The professor had to illustrate his explanations with proofs and promote student participation in exercises on the blackboard. Moreover, all students at the Faculty of Philosophy had to attend a Sunday *academy* for 3 h, the first one devoted to a particular item on arithmetic, algebra, or geometry proposed the previous Sunday. The *Application of Algebra to Geometry* was an hour-and-a-half morning course to be taken in the third year at the Faculty of Philosophy before Physics and Chemistry, only by students willing to enter the Medicine Faculty.

These regulations were not implemented until the end of the so-called War of Independence (1808–1814) against Napoleon, which stopped the enlightenment reform program: the French period in Spain was basically just a war. This institutional paralysis continued during the Bourbon Restoration until Ferdinand VII's death (1833), except during the Constitutional Triennium (1820–1823).

The first regulation of secondary education in Spain appeared in this short period, with the *Reglamento General de Instrucción Pública* (General Regulation of Public Instruction, 1821). It established two chairs devoted to pure mathematics at the *Universidades de provincia* (province universities), the new institutions that were to host the new level prior to higher education. This regulation never came into practice because of the invasion of the Holy Alliance that restored absolutism.

1.3.2 The Further Development of Secondary School Mathematics

Important attempts to organize the system of education occurred after Fernando VII's death. From 1836 on, institutions that may be regarded as secondary educational institutions (*Institutos)* began to appear in the provincial capitals. They were organized according to the plans of Gaspar Melchor de Jovellanos, a major figure of the Age of Enlightenment in Spain. Education in these institutions lasted for 3 years; the first one included *Logic and Grammar* (1 h a day), *Elements of Mathematics* (1 h a day), and *Geometry Applied to Drawing* (3 h a week). During the second year, mathematics went on (1 h a day) together with *Physics and Chemistry* (1½h a day) and *Mathematical and Physical Geography* (3 h a week).

The system of education represented by the new regulations of 1845 included three stages: primary education (usually up to age ten), secondary education, and university education. The same regulations established three degrees: bachelor (*bachiller*), master (*licenciado*), and doctor.[1]

Basic secondary education lasted 5 years and led to the Bachelor's Degree in Philosophy. This course included *Arithmetic and Algebra* in the third year of studies and *Geometry, Trigonometry, and Topography* in the fourth (1 h a day in both cases).

The *Secondary Education Expansion* (*Segunda Enseñanza de Ampliación*) that lasted 2 years and gave the Master's Degree in Arts or Science was now required not only to access the major faculties (Law, Theology, Medicine, and Pharmacy) but also to become a teacher at an *Instituto*. The necessary degree courses could be completed at the Faculty of Philosophy or at the *Institutos* (when no university was available in the province). The Master in Science syllabus included Greek, Higher Mathematics, Chemistry, Mineralogy, Zoology, Botany, Physical Astronomy, and Physics. In order to become teachers at an *Instituto*, graduates had to pass a tough state competitive examination and could choose their place of work depending on their score.

To include professional training, the Law of Public Education of 1857 divided secondary education into two branches: *general studies*, giving the Bachelor's Degree in Arts, and *applied studies to industrial professions*, granting proficiency in agriculture, arts, industry, trade, or navigation. General studies were also divided into two periods of two and 4 years; the first included arithmetic, the second arithmetic, algebra, and geometry. A general examination had to be passed between the first and second periods and also to obtain the bachelor degree which was required to enter a university.

No substantial changes were introduced in mathematical content at this time, but a new Bachelor's Degree in Science (in 3 years) was established. Prospective teachers at the *Institutos* were required to have this degree. The course of studies leading to this degree included *Complements of Algebra, Geometry, Rectilinear and Spherical Trigonometry,* and *Analytic Geometry in Two and Three Dimensions* as mathematical content (3 h a week).

It should be noted that the state competitive examination for prospective teachers included discussion, requiring candidates to solve problems and questions posed by their opponents (other candidates). This procedure stimulated high preparation in content to the detriment of teaching methods. As the century went by, an increasing number of mathematics secondary school teachers had not only a Bachelor's Degree in Science but also a Master's Degree in Mathematics. In the last quarter of the century, many mathematics university professors had started their careers as secondary education teachers. As a result, throughout the nineteenth century, secondary education teachers (Vea 1995) gradually formed the Spanish mathematical community (Hormigón 1995) together with military men (Velamazán 1994), civil engineers (Martínez García 2004), and university professors (Ausejo 2007).

1.3.3 On Mathematics Textbooks

Mathematics teachers were involved in translating foreign textbooks (the level of difficulty of which was sometimes beyond the level of these teachers' own practice). Especially popular were French textbooks – the first German work translated was Baltzer's *Elements of Mathematics* (1879–1881). As in other European countries, the most popular textbooks in Spain in the first half of the nineteenth century were Lacroix' *Cours élémentaire de Mathématiques pures* (Lacroix 1807–1808) (translated by José Rebollo Morales); Francoeur's *Cours complet de Mathématiques pures* (not translated, but published by Alberto Lista as his own course *Elementos de Matemáticas puras y mixtas* (Lista y Aragón 1822), based on Francoeur); Bourdon's *Éléments d'arithmétique* (Bourdon 1843) (first translation by Calisto Fernández

[1] These degrees basically correspond to the French pattern (*Baccalauréat, Licence,* and *Doctorat*).

Formentany, second by Agustín Gómez Santa María as part of his *Tratado completo de Matemáticas*, third by E.A., and fourth by Lope Gisbert); *Éléments d'algèbre* (Bourdon 1849) (first translation by Agustín Gómez Santa María as part of his *Tratado completo de Matemáticas*, second by Lope Gisbert); Legendre's *Éléments de Géométrie* (Legendre 1807) (translated by Antonio Gilmán); Boucharlat's *Éléments de calcul différentiel et de calcul intégral* (Boucharlat 1834) (translated by Jerónimo del Campo); Navier's *Résumé des leçons d'analyse données à l'École Polytechnique* (Navier 1850) (translated by Constantino de Ardanaz and Agustín Gómez Santa María); Poisson's *Traité de Mécanique* (Poisson 1845) (translated by Jerónimo del Campo); and Olivier's *Cours de géométrie descriptive* (Olivier 1879) (translated by U. Mas Abad).

The government published official lists of textbooks (Vea 1996) to be followed in secondary education between 1846 and 1852. The chosen authors were Lacroix, Bourdon, and Legendre, together with the Spanish José Mariano Vallejo (*Tratado elemental de Matemáticas)* (Vallejo 1812–1817), José de Odriozola (*Curso completo de Matemáticas puras*), and Juan Cortázar (*Tratado de Aritmética, Tratado de Álgebra elemental, Tratado de Geometría elemental, Tratado de Trigonometría rectilínea y esférica y de Topografía)*. Since 1850 only the three Spanish authors were listed, together with Acisclo Fernández Vallín's *Tratado elemental de Matemáticas* (Fernández Vallín 1851) in 1852.[2]

In 1858, the official list of textbooks excluded Odriozola (Odriozola 1827–1829), which reduced the level of mathematical content. The lists of 1861, 1864, and 1867–1868 included Cortázar's *Treatises* (Cortázar 1846, 1847, 1848a, b) and Fernández Vallín's *Elementos de Matemáticas* (Fernández Vallín 1852) and replaced Vallejo with Joaquín María Fernández Cardín's *Elementos de Matemáticas* (Fernández Cardín 1858–1859). As a result, Cortázar's treatises had multiple editions (minimum of 24 and maximum of 45 editions). The number of editions for the books of Fernández Cardín varies between 16 and 25, while the number of editions for the books of Fernández Vallín varies between 12 and 15.

It is worth mentioning that Fernández Cardín's works were specifically written to be used as textbooks in secondary education. Moreover, their content was precisely adapted for the mathematics level required in syllabi, and from a methodology point of view, they introduced concepts starting with examples and particular cases and moving up to abstract notions and general rules.

The lists of 1861 and 1864 included for the first time a workbook, namely, Felipe Picatoste's *Principios y ejercicios de Aritmética y Geometría* (Picatoste 1861).

As for foreign authors, the key author in the two first decades of the second half of the nineteenth century was Cirodde. His *Lecciones de Aritmética* (Cirodde 1857a) (translated by Francisco Zoleo, 37 reprints), *Lecciones de Álgebra* (Cirodde 1857b) (first translation by Bartolomé Peregrín, 28 reprints; second translation by Lope Gisbert, 7 editions), *Lecciones de Geometría con unas nociones de la descriptiva* (Cirodde 1857c), and *Elementos de Geometría rectilínea y esférica* (Cirodde 1860) (both translated by Manuel María Barbery, the first one 22 reprints and the second one 20 reprints) were listed among official textbooks for military engineers and science faculties (1864 official lists) but were also used in secondary education.

Other French authors' works translated in the last quarter of the nineteenth century were Briot's *Lecciones de Álgebra elemental y superior* (Briot 1880) (translated by C. Sebastián), Rouché and Comberousse's *Tratado de Geometría elemental* (Rouché and Comberousse 1878–1879) (translated by A. and J. Portuondo), and Serret's *Tratado de Aritmética* and *Tratado de Trigonometría* (Serret 1879a, b) (the first translated by T. Monterde, the second by F. Pignatelli).

Among mathematics university professors who started their careers in secondary education was Zoel García de Galdeano, professor of Mathematical Analysis at the University of Zaragoza. Thanks to his deep understanding of current mathematical developments, he was influential in

[2] There are no comparative studies on these French and Spanish authors except for Vallejo and Odriozola in *Differential Calculus* (Medrano 2005). Medrano's study shows that the books by Spanish authors were basically original works based on a careful selection of different French sources.

importing modern mathematics to Spain and also connected the secondary and higher education perspectives with research agenda (Ausejo 1995, 2010a). In 1899, *L'Enseignement Mathématique*, the soon to be most prestigious international journal on mathematics education, opened with a paper by García de Galdeano (1899), who became a member of this journal's Comité de Patronage.[3] Later on, in 1909, he was appointed Spanish Delegate of the International Commission on Mathematical Instruction (ICMI).[4]

1.4 The Twentieth Century: Mathematics Education Before and After the Spanish Civil War (1936–1939)

1.4.1 Mathematics Education Up to the Spanish Civil War (1936–1939)

In 1907, a new institution was created to support scientific research, the *Junta para Ampliación de Estudios e Investigaciones Científicas* (Board for Advanced Studies and Scientific Research). This institution was instrumental in bringing the spirit of reform to secondary education. With regard to mathematics, between 1910 and 1915, the Junta awarded grants to three mathematicians to study teaching methods and syllabi in Berlin, Geneva, London, and Cambridge and gave financial support to Galdeano's participation in the International Commission on Mathematical Instruction (Ausejo and Millán 1989, pp. 264–265).

The Junta, a well-financed institution, was independent from the underfinanced university; its new laboratories were ruled by young professors instead of the senior faculty members of the University of Madrid. This resulted in a strained relationship between the *Junta* and the university to the detriment of the young researchers of the *Junta*: despite their high qualifications, many of them did not get a post as university professors, but became secondary teachers instead.

Soon after, when the Spanish Mathematical Society was founded (1911), 38 % of its members were teachers (65 secondary education teachers, 39 university professors, 27 high technical schools teachers, 2 school teachers, 3 mercantile teachers). Moreover, among 54 institutional members, 20 were secondary education centers, 5 science faculties, and 2 high technical schools. These data speak for themselves about the importance of secondary education in the Spanish mathematical community (Ausejo and Hormigón 2002).

In this context, it is not surprising that the Mathematical Laboratory and Seminar, which was founded in 1915 as part of the *Junta*, developed a line of work parallel to research in the field of mathematics teaching and preparation for secondary education (Ausejo and Millán 1989, pp. 269 and 271). It is also worth mentioning that in 1927, Julio Rey Pastor, the first director of this laboratory and the most famous Spanish mathematician at that time, started a series of mathematics textbooks for secondary education significantly titled *Intuitive Collection* (in collaboration with his disciple Pedro Puig Adam).

These initiatives earned proper political attention during the Spanish Second Republic (1931–1936), when a remarkable effort was made in secondary education, not only in terms of financial investment but also by improving scientific contents. As for mathematics, the syllabus of 1934 not only included infinitesimal calculus and complex numbers in secondary education for the first time in Spain but also presented mathematics cyclically and progressively throughout the seven courses. That is, content was distributed in seven progressively difficult academic years (3 h a week each year except for the 4 h a week in the third year). The first two courses were devoted to arithmetic and geometry; the third and fourth to rational arithmetic and plane geometry; the fifth to algebra and space geometry; the sixth to real numbers, trigonometry, and complex numbers; and the seventh to infinitesimal analysis and analytic geometry (Vea 2008).

[3] *L'Enseignement Mathématique,* 1 (1899): front cover.

[4] *L'Enseignement Mathématique,* 11 (1909): 194 (http://www.icmihistory.unito.it/19081910.php)

1.4.2 The Introduction of "Modern Mathematics" in Secondary Education

Regarding secondary and primary education, the worldwide wave of mathematics reform of the 1960s and 1970s also reached Spain under Franco's fascist dictatorship that, since the early 1950s, had been trying to overcome international isolation (Ausejo 2010b).

In fact, the Spanish reforms of secondary education in the 1950s were conducted prior to the international mathematics reform movement. Regarding mathematics, they were motivated by Puig Adam's connection with the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM). An applied mathematician, Puig Adam was deeply connected with the educational reforms of the Spanish Second Republic. His above-mentioned collaboration with Rey Pastor continued from 1944 onward in a new series of mathematics textbooks, the *Rational Collection*. They both produced the complete collection of mathematics textbooks for the 1954 and 1957 syllabi.

When Puig Adam died prematurely at age 59 in 1960, he was replaced by Pedro Abellanas, professor of projective geometry at the University of Madrid and a strong supporter of the need to update secondary school mathematics syllabi and establish more connections between them and modern mathematics. In 1961, he was appointed President of a Commission for Experiments on Teaching Modern Mathematics that was officially created in the Ministry of Education to study the possibility of introducing ideas of modern mathematics in upper secondary education (5th–6th course, 14–16 years old) to improve courses without changing the overall syllabi. The Commission published two pilot textbooks entitled *Modern Mathematics* (Abellanas et al. 1967, 1969) that were intended to establish new and "effective" methods and structures, as well as new trends in modern mathematics teaching, in order to provide mathematical training for high scientific education and professional awareness.

This new approach was visible from the very beginning, especially in coursework for 15-year-olds: elements of set theory and algebraic structures were introduced, classical geometric and algebraic approaches disappeared, and the level of required knowledge increased.

This trend also appeared in the reform of the so-called Elementary Baccalaureate (1st–4th grades, 10–14 years old) in 1967. According to the new regulation, students would be provided with the possibility of acquiring the concepts and working methods of contemporary mathematics, and the new syllabus would be arranged based on set theory and fundamental algebraic structures (semigroups in the first grade, groups and rings in the second, fields in the third, and revision and consolidation of these concepts in the fourth).

Regarding organizational changes, the new Education Act of 1970 established primary education for students ages 6–14 and secondary education for students ages 14–17. It was immediately applied to primary education but only applied to secondary education from 1975 onward – that is, in democratic Spain. As for mathematics education, the modern structural style remained until the early 1980s, when the first adjustments and corrections appeared in response to severe criticism from teachers and parents.

1.5 Conclusion

Political instability delayed the regulation of secondary education in Spain until 1836. The same instability also caused a certain lag in the development of the Spanish educational system, including its mathematics education. Excessively frequent changes were one way in which this instability manifested itself. Nonetheless, it may be said that the syllabi of 1836, 1845, and 1857 established the basic structure of secondary education for the rest of the nineteenth century. Despite its difficulty, the country developed a system for preparing mathematics teachers that was relatively demanding (above all with respect to mathematical knowledge).

Spain absorbed international experience – above all during the nineteenth century through translations of textbooks. Later, Spain took part in the principal international movements in mathematics education, becoming involved in the ICMI and the reform movements of the 1950s–1970s. At all stages, Spain had its own distinctive characteristics, which owed no less to its distinctive history and culture than to its peculiar political and economic features.

2 Portugal

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This text will center on the understanding of how school mathematics has developed in Portugal, focusing on the changes that occurred from the eighteenth century. Four of the most relevant dimensions of these changes will be considered. First is the emergence of a consolidated secondary school system that slowly bred during the nineteenth century with its institutions, values, professionals, and school disciplines. I will discuss this dimension in two parts, the first focusing on the differentiation of school mathematics and the second on its consolidation. Second is a shift in perspective triggered by the movement *Escola Nova* – in Portugal, it occurred essentially from the beginning of the twentieth century – that placed the pupil in the center of the educational enterprise. Third is the Modern Mathematics movement in the middle of the twentieth century that recomposed school mathematics curricula at all levels. Finally, a reflection of today's school mathematics field with the emergence of specialists researching all the dimensions associated with teaching and learning mathematics will conclude this discussion.

2.1 The Differentiation of Secondary School Mathematics

From the second half of the eighteenth century, following European trends, school education gradually acquired a fundamental role in Portuguese society. A proto-public educational system – beginning with the so-called First Letters (reading, writing, counting, and praying) and ending with the university that aimed to provide a comprehensive view of humanistic, scientific, technical, and professional education – was gradually put in place. This was paired with changing perspectives on the mission of the state and the emergence of a citizenship that transformed the roles of man and society.

At this time, the education in mathematics at the primary school consisted of learning to write numbers and perform the four arithmetic operations. More advanced topics such as Euclidean geometry, rules of proportions, and elementary algebra were taught in some colleges using Portuguese textbooks. Military schools included the study of arithmetic (the four operations, fractions, and proportions), some algebra, and practical geometry. Euclidean geometry, including the study of conics and mechanics with many applications to artillery, was also studied. Portuguese translations of excerpts from Euclid's *Elements* were used, some circulating in manuscript form. Textbooks were framed into a "question-and-answer" format using very little mathematics notation (Valente 2002).

Social needs for broader mathematics education were recognized in the middle of the eighteenth century. The initiative for conducting political reforms, which were aimed at reinforcing the role of the state under an absolutist regime that viewed the King as the top of the social order, is usually attributed to the Marquês de Pombal, the Secretary of State of the Kingdom – a position equivalent to a prime minister. His reforms strengthened industry and commerce. In addition, a reorganization of the military forces was undertaken.

These political changes had consequences for education, especially beyond the elementary. Professional schools geared towards commercial, nautical, and artistic activities were created from 1759 to address the need for qualified personnel in those fields, and all included mathematics in their curricula. *Aula do Comércio* in Lisbon, for example, was created in 1759 and admitted students with an elementary education in reading, writing, and counting. The King sometimes attended classes, giving a political sign of the relevance attributed to the school. The course, which lasted for 3 years, included elementary arithmetic, fractions, proportions, roots, currency conversion, insurance, and accounting. The number of certified accountants in commerce grew, and its diploma became a requirement for public officers.

Two years later in 1761 and in the context of a strong attack on the power of aristocracy, the *Real Colégio dos Nobres* was created by the state; it was dedicated to the education of the people of noble descent. Its curricula stressed the importance of a scientific background for the nobility, and consequently, mathematics, experimental physics, and astronomy had a prominent role. Euclid's books were translated (Euclides 1768) and didactic material for teaching science was purchased. Unlike teaching in the professional (and the military) schools, scientific teaching at the *Colégio* rapidly declined.[5]

The military was among the specific professions in preparation for which mathematics education was improved. Mandatory classes in the regiments (1763) and several military academies (the first was the *Academia Real da Marinha*, 1779) were established, and the *Cours de Mathématique* written by Bellidor was translated for use in these courses (in 1764).[6] The algebraic treatment of mathematical questions was now pervasive, and the study of geometry included trigonometry. Later, as the Napoleon War forced the establishment of the Portuguese court in Brazil in 1810, the newly founded *Academia Real Militar* (located in Rio de Janeiro) adopted translations of Lacroix's works (Lacroix 1810, 1811, 1812a, b; see Valente 2002).

In 1759, the Jesuit order that owned the large majority of colleges and that had strong influence on the University was banned from the Portuguese kingdom and its domains. Alternatively, a system of *Aulas* of Latin, Greek, and Rhetoric, supported by public funding, was created and rapidly expanded under the pressure of local forces. These classes were not aggregated in colleges and fell under the responsibility of individual teachers who had the status of public servants. In that same year, a central organism for controlling access to the teaching profession, teaching methodologies, and textbooks was put in place. *Aulas* related to mathematics were virtually nonexistent but there were still some exceptions. These exceptions were located in major cities and established by professional corporations or private colleges (usually owned by the remaining religious orders).

After the expulsion of the Jesuits, the University of Coimbra, now the sole university in Portugal, underwent a profound reform in 1772, guided by illuminist and rationalistic approaches. For a number of years, mathematics had not been taught at the university, and reformers sought to create a Faculty of Mathematics (and another of Philosophy). Teachers were appointed almost immediately, and in the next year, translations of Bézout's *Cours de Mathématique* supported by the university were available.[7] Later, translations of Carnot's *Métaphysique du Calcul Infinitesimal* (Carnot 1798) and Lagrange's *Theorie des Fonctions Analytiques* were also translated (Lagrange 1798) – intended specifically for students who wanted to deepen their mathematics studies. Other French authors' works (La Caille, 1800, 1801; Cousin, 1802; Legendre, 1809) were translated for use either in the university, the military academies, or religious colleges.

A major political change in the country occurred in 1820, when a liberal system was established that was strongly influenced by the ideals of citizenship professed by the French Revolution. An elected parliament had now been created. The King lost his absolutist powers and signed a Constitution, assuming the role of an overseer of the government. This new regime maintained the commitment to education, and in 1836 following a bloody civil war, it adopted a proposal to establish new secondary

[5] As scientific teaching faded, the large collection of scientific didactic apparatuses was sent to the University of Coimbra and is now on display at the University museum.

[6] The second edition of Belidor's course was translated (Bellidor 1764/1765). By the end of the century, Bézout's works began to be used in military academies.

[7] The first publications of these translations were in 1773 (Bezout 1773, 1774a, b) many editions followed until 1879.

Table 14.1 Denomination of mathematics disciplines in legislation of reforms for the *Liceus*

Date of reform	Denomination
1836	Arithmetic, algebra, geometry, trigonometry, and drawing
1844	Arithmetic and geometry with applications to the arts, and first notions of algebra
1860	Elementary mathematics, comprising arithmetic, algebra until second-degree equations, synthetic geometry, or principles of plane trigonometry – mathematical geography
1868	Mathematics
1880	General course: arithmetic, plane geometry, principles of algebra and bookkeeping
	Complementary course: algebra, space geometry, and trigonometry
1883	Arithmetic, algebra, geometry, trigonometry and principles of bookkeeping and accounting
1892	Mathematics

schools[8] the *Liceus*, across the entire country. Instead of dispersed *Aulas* where isolated teachers worked, the intention was to aggregate secondary education under a sole public institution. Instruction would have clear goals of not only offering a preparation for higher education but also providing a general scientific and technical education needed for life for the great masses of citizens who did not desire a higher education. Although these intentions took several years to complete, a broad network of public secondary schools was already in place by 1850. Also in 1836, primary education was advocated as a mandatory requirement; both parents and the central government were considered responsible for its accomplishment.

From the start, the *Liceus* were created with the intention of countering the prevalent classic education centered on Latin by providing a scientific education which was considered a necessary component of modern citizenship. Their curricula included physics, chemistry, mechanics, political economy, public administration, French, and English. Mathematical courses included arithmetic, algebra, geometry, trigonometry, and drawing. The curricula of the *Liceus* fluctuated considerably until 1895, but a mathematics discipline was always included.

In 1837, two polytechnic schools were created in Lisbon and Oporto. Although their original purpose was to prepare students for the courses of the military academies, they in fact also provided a general scientific formation necessary for scientific professions (e.g., civil engineers) (Papança 2010). The monopoly of higher education held by the University of Coimbra began to be broken.

In these early years, the distinction between the *Liceus* and other institutions was not always clear. The mathematics courses, in particular, were given either in the university (of Coimbra) or the polytechnic schools (of Lisbon and Oporto) and shared the same teachers and textbooks, usually Bézout's works. Nor was there a clear professional identity for the teacher: the same teacher could lecture mathematics and philosophy or drawing (however, the term *mathematics teacher* was occasionally used). Even the denominations of the mathematics courses in the *Liceus* were lengthy names, evoking several contents of the discipline and not simply "mathematics" (Table 14.1).

Gradually Portuguese textbooks of mathematics emerged and replaced the French translations. The first, intended for the military academies, was written by Villela Barbosa in Brazil and published in Lisbon (Barbosa 1815) and closely followed Bézout's approach (Valente 2002). Other teachers either from military institutions, polytechnic schools, or the *Liceus* were engaged in the production of textbooks. They gradually incorporated didactic techniques (e.g., the use of exercises) and started to develop a series of textbooks aimed at the different grade levels. For the early years, a typical series would include two books of arithmetic and geometry; for the next years, two books on algebra and plane geometry; and for the final years, one book on advanced algebra and another on trigonometry. Occasionally books on rational arithmetic and cosmography would also be available for the last years.

[8] For the first time, the term "secondary school" was used in the legislation.

The best example may be found in the works of José Adelino Serrasqueiro[9] who published a whole series of books for the entire mathematics curricula for the *Liceus* (1876, 1877, 1878, 1879).

2.2 The Consolidation of Secondary School Mathematics

By the end of the nineteenth century, the network of public secondary schools had encompassed the major cities of the country and its colonies; however, major problems began to emerge as contradictory reforms were issued and quality of teaching declined. Consequently, students preferred to enroll in private colleges or have tutors in specific disciplines and attend the *Liceus* only for final examinations. Moreover, given that international comparisons became possible, the weakness of the Portuguese educational system became even more evident.

The centralized nature of the Portuguese educational system was consolidated in 1894 as a major educational reform lead by Jaime Moniz, who was strongly influenced by the German system. He established the secondary school system in a form that is still the model for today's secondary schools.[10] First, children would enter mandatory primary school by the age of 6. After 4 years of education, they could attend the *Liceus* where the secondary curriculum was organized into two sequential courses: a general course of 5 years (two+three) and a complementary course of 2 years, considered preparatory for higher education. With minor fluctuations, this structure still remains today. Second, and as a consequence, a regime of classes was adopted, and children were expected to attend the same class for every discipline during the school year. Students of the same age would now be in the same school year, making classes more homogeneous. The option of obtaining a degree by taking exams in several disciplines without taking classes was cancelled. Third, requirements for the teaching profession were enforced and included an examination. In mathematics, even before this reform, from the foundation of the *Liceus* in 1837, the university course of mathematics had been a requirement, but it had not been strongly enforced.

The 1895 programs[11] for the *Liceus* were the first to provide universal political control of the educational system. They contained a detailed list of mandatory contents to be enacted in class, together with observations on the importance of mathematics teaching and methodological considerations. Their meticulous approach to many details of school life, paired with the stated intention of a re-foundation of secondary schools, made them an important instrument for consolidating the curricula of secondary school mathematics at the end of the nineteenth century.

Access to the teaching profession in public secondary schools was made through examinations. In 1895, influenced by a growing understanding of the importance of teacher quality, examination procedures were more clearly stated and centralized. For example, all candidates had to pass a written 1 h test on Portuguese and another on psychology or logic, followed by 30-min oral debates on these subjects. Later, the candidates for each discipline were examined (again by written and oral assessments) on the specific topics from the program. Candidates for mathematics, for example, had to solve problems in algebra, three-dimensional geometry, and differential and integral calculus (4 h) and write a dissertation on physics in 90 min. The oral examination was composed of a 1 h presentation of a topic in pure mathematics and another in physics. Then, the jury would question the candidate for 90 min.

[9] His series of textbooks had numerous editions until 1936. His book of algebra was adopted as a de facto national mathematics curriculum in Brazil in the beginning of the twentieth century (Valente 2002).

[10] He also formatted the primary school system, but its previous organization was not as chaotic as the system of the *Liceus*.

[11] In the Portuguese context, the term *program* – a list of topics to be taught – appears in the last quarter of the nineteenth century.

The secondary program was structured around the following topics: arithmetic, geometry, algebra, and cosmography. In the first 3 years, the program expanded primary school arithmetic and included powers, multiples and divisors, remainders, fractions and decimals, square roots, proportions, and the rule of three. Geometric figures were also considered, including properties of straight lines, circles, and triangles. In general, these topics were associated with "theorems" or "theories." For example, in the third year, students studied a "theory" of the maximum common divisor and minimum common multiple and that every non-prime number has a prime divisor.

Algebra started in the fourth year with a discussion of "formulas," monomials, polynomials, and operations with them, including division. First-degree equations, systems of equations, inequalities, and integer solutions of first-degree equations were also mentioned. Geometry included a detailed study of properties and constructions of polygons as well as circles.

By the fifth year, arithmetic and geometric progressions were studied, including their sums and a brief discussion of limits. Logarithms were applied to the computation of interests. Geometry focused on ellipses, parabolas, and hyperbolas. Cartesian coordinates were applied to the study of the equation of a straight line.

During the sixth year, systems of equations and second-degree equations were discussed, imaginary numbers were introduced, and space geometry was studied. The last year included continuous fractions, trigonometry, and cosmography.

Mathematics was viewed as a particularly valuable discipline for introducing students to a specific kind of reasoning and emphasizing the importance of simplicity, clarity, exactitude, and the possibility of complex intellectual constructions. The course of mathematics contained a large body of knowledge essential for life and as preparation for science. As for teaching methods, the use of intuitive and practical approaches was recommended in the early years of education; that led to establishing rules that would be later supported by frequent revisions and exercises.

By 1905, there was a growing discomfort with the reform, particularly with the requirement of single textbooks, the excessive role of Latin, and the length of the programs. The main change in the new programs was the introduction of the study of limits, functions, continuity, and derivatives (Aires 2006). The introduction of the study of calculus in secondary schools was a considerable innovation (Andrea 1905), and although limits had already been studied for some time in association with sequences, their comprehensive study was new. Minor changes included the elimination of cosmography and the inclusion of commercial applications to some topics. The 1895 and 1905 programs constituted the fundamental matrix of secondary mathematics curricula that lasted until the second half of the 1960s.

Until the late 1960s, mathematics secondary programs underwent some changes but retained most of the organization of the 1895 and 1905 curricula. One change related to the emphasis on drill and practice. Between 1930 and 1947, the programs, especially in the first five grades of secondary schools, repeatedly stressed the importance of solving numerous exercises. This emphasis contrasted with previous and later programs that valued mental and written computations but warned teachers against overdoing them. The other programs simultaneously emphasized the importance of intuition and practice.

The major fluctuations in content, however, were connected with the study of calculus. A brief introduction of the study of integrals was included in the course in 1918 but removed in 1926. The study of derivatives was eliminated between 1936 and 1948 in the wake of the dictatorial regime's steps to lower the quality of education at all levels.[12] Only after the Second World War were derivatives studied again. Its introduction prompted debates about the quality of mathematics terminology in the programs and the use of only one textbook, as well as ways in which its study should be articulated with the study of limits (Almeida 2012).

[12] Other measures taken during the 1930s included closing schools for the preparation of teachers, lowering mandatory schooling, and reducing the curricular years of the *Liceus* by 1 year. Teachers associations were also closed.

2.3 *The New Educational Goals of* Escola Nova

While secondary schools were slowly construed only from the nineteenth century, mention of schools for reading, teaching, counting, and praying can be found as early as the fifteenth century. During the nineteenth century, however, as requirements for the knowledge of arithmetic operations were gradually fostered, a major change occurred. In 1862, the teaching of the metric system was introduced in primary schools. This reform prompted the development of specific didactic materials and the renewal of textbooks. This reform had much social relevance because the adoption of new ways to measure goods was vital to the economy (and to tax collection). A special inspectorate was established, and primary teacher practices were thoroughly scrutinized to ascertain they conformed to the new rules.

The republican regime that began in 1910 attributed great importance to primary education, which was seen as the basis for the formation of the new citizen – a republican, educated, active participant in the politic life that was considered a laic democracy. The formation of primary teachers was then of paramount importance, and schools for the education of primary and secondary teachers (*Escolas Normais*) were created in 1911.[13] Prospective teachers would now follow courses in pedagogical and methodological topics. Simultaneously, teacher examinations were terminated.

The spread of the new pedagogical proposals of the *Escola Nova* (New School or Active School) movement, usually following francophone influence, played a major role in the culture of these new institutions[14] (Palma 2008). Ideas of student-centered education, learning by discovery, fostering active participation of students, connecting school to daily life, and supporting and developing intuition became popular and caused major changes in the rhetoric of educators. At the same time, as schools for teacher education developed and teaching of pedagogy and methodology emerged, the field of education started to develop separately from the fields of psychology or medicine. From the 1920s, public primary schools and some prominent private primary schools extensively sought new teaching forms and were deeply influenced by the ideas of Decrolly, Montessori, Fröebel, and Pestalozzi.

By 1930, however, following the change to the dictatorial regime that lasted until 1974, most of the relevant figures of the movement were either imprisoned or forced into exile.[15] *Escolas Normais* were replaced by other kinds of schools that were eventually closed in 1936, even at the cost of having no institution for primary teacher preparation.[16] The need for primary teachers was then fulfilled by attracting individuals who had only completed primary school. However, by the end of the 1930s, the regime itself integrated the ideas of the *Escola Nova* movement giving it a nationalist and religious tone.

From 1930 on, following the closure of Escolas Normais, prospective teachers for the *Liceus* were required to attend an in-service course of 2 years at two *Liceus Normais* in Lisbon and Coimbra taught by a special professor (*metodólogo*). This system for teacher education lasted until the end of the 1960s, when under the rapid expansion of the educational system it could not produce the required number of secondary school teachers.

Traces of the ideals of the New School movement can be found in the documents that remain from this epoch. After the 1940s, the discourse of legal documents and the representations and practices of teacher education at *Liceus Normais*, as well as writings by teachers and other mathematics educators, emphasized the importance for the quality of learning of intuition, heuristics, and the active participation of students (Almeida 2012).

[13] The first schools for the formation of teachers were founded in 1839.

[14] These ideas first appear in Portugal in the previous century but only now began to be widely disseminated.

[15] The dictatorship was always particularly suspicious of teachers. As late as 1947 and following a political uprising against the regime, nearly all of the mathematical elites were expelled from the universities (among them Bento de Jesus Caraça), and the majority chose exile (to Brazil, Argentina, France, and the United States).

[16] Some of these new schools, *Escolas do Magistério*, were opened in 1942.

2.4 The Modern Mathematics Movement

At the outset of the 1950s, Portuguese society was adjusting to the postwar period. Although political leadership remained authoritarian, the drive to strengthen the industry, further economic integration with other countries, and participate in international (economic, political, and military) organizations took place. In education, a program to diminish illiteracy by fostering adult primary education was organized. Also a coherent system for technical education was developed, in which some existing institutions were transformed into technical schools of very good quality. The impetus to keep the pace with other European countries especially intensified since 1956: mandatory schooling was expanded, more centers for teacher certification were opened, and programs at the universities were updated.

The reform of Modern Mathematics occurred in this context (Matos 1989) and affected mathematics teaching across the educational system in Portugal, as it did in many countries. From the early 1950s and despite the restrictions imposed on foreign contacts,[17] Portuguese educators were aware of international trends via ICM, ICMI, and CIEAEM and essentially through the work of Sebastião e Silva (1914–1972), a prominent mathematician keenly interested in educational issues who had special relations with Italian educators. In 1962, he proposed a program for the introduction of the reform (Silva 1962), stressing the extreme care with which changes should occur. Minor changes in language could be slowly introduced in the first two cycles of secondary education and topics of modern mathematics (he specifically mentioned mathematical logic, set theory, abstract algebra, probabilities, and statistics) were to be introduced only in the last cycle of the *Liceus*.

By 1963, an experimental project supported by the Organization for Economic Cooperation and Development (OECD) was put in place to teach modern mathematics to three special classes during the last 2 years of the *Liceus*. The project was proposed by Sebastião e Silva, who in 1964 wrote textbooks for these courses (Matos 2009). In the following years, these classes gradually spread throughout the country. An overview of the new approach can be found in Sebastião e Silva's textbooks (1964a, b, 1965, 1966). Algebra and analysis were the core topics, each accounting for about one third of intended course. Logic and set theory were the next important topics (about one fifth). Analysis was now permeated by logic and set theory. Other topics were also added, specifically integral calculus, probability, and statistics (Almeida 2007). Silva's books together with accompanying teacher guides went well beyond these topics. Mathematics content included discussions on applications, philosophy and history of mathematics, and other aspects. The texts also presented a comprehensive humanistic perspective.

The influence of the Modern Mathematics movement in school culture can be observed in four aspects: understanding the nature of learning (following Piaget's proposals), school mathematical content, methods of teaching mathematics (building on perspectives already laid out by the *Escola Nova* movement), and understanding the social role of mathematics (now perceived as a major driving force for human development). Changes in school mathematics curricula were one of the main alterations of this time. The Euclidean approach to geometry and rational arithmetic that relied on proof was eliminated. It was believed that the Modern Mathematics movement would clarify, simplify, and unify school mathematics. The new perspective valued rephrasing mathematical language in terms of "relations," "structures," or "operations," centering mathematics in the study of algebraic structures.

Other branches of the Portuguese school system also experimented with new ideas. *Telescola* – a national network of schools complemented by classes on television – was created. In 1968, the first cycles of the *Liceus* and technical schools were merged into a new autonomous cycle for 10-year-old/11-year-old children (Almeida 2012). From that date until 1974, secondary programs were sequentially modified to incorporate these new ideas. These programs were in use (although with gradual changes) until early 1990.

[17] Traveling was strictly controlled and mail surveyed.

From 1968, experiments were also conducted in technical schools. Here, new perspectives collided with a consolidated applied (and practical) approach deemed necessary for the specific professions. The penetration of these new ideas, however, was not limited to the secondary school. Although state primary schools were left out of reforms until 1974, the new perspectives had actually started experimentation since 1961 in private primary school classes, and many short-term courses on the use of manipulative materials (essentially Cuisenaire rods, Dienes blocks, and other structured material) became available (Candeias 2008).

2.5 Emergence of the Field of Mathematics Education

The actual introduction of modern mathematics was controversial. Although there are indications that it was a success for many students, in other cases it prompted many concerns. As early as 1969 and as the reform was generalized to all grades, many problems emerged. On the one hand, the didactic use of sets raised many issues related to linguistic coherence. On the other hand, it is likely that many teachers felt very uncomfortable with a mathematical content they had not previously experienced[18] and simply tried to reproduce in class the theoretical approaches of set theory that they knew from their in-service short-term courses (Matos 2009).

During the second half of the 1970s, the entire educational system (and the whole society) was in turmoil adjusting to new democratic values. In 1981, a series of meetings organized by the newly founded Portuguese Society of Mathematicians discussed the programs of senior secondary schools (Matos 2011). Modern Mathematics school practice was criticized because it rendered mathematics hermetic and formalized, with its great emphasis on symbols and neglect of reality and applications. It was demanded that programs should change to integrate problem-solving, calculators, and computers and pay special attention to applications. These three dimensions (problems, technology, and applications) that were a departure from official curricular options at the time were influenced by NCTM's proposals.

This movement led to the foundation of the *Associação de Professores de Matemática* (APM, *Association of Teachers of Mathematics*) in 1986 that congregated virtually all small groups of innovators interested in teaching and learning mathematics (Matos 2011). Gradually a functional and complex network and exchange among their members was established through national (or regional) meetings or the association's journal. For some, a regular presence in international meetings diversified sources for the circulation of ideas.

APM played a prominent role in shaping the new programs that by the early 1990s had replaced the Modern Mathematics movement. In order to influence the new curricula, the Association produced a position book *Renovação do currículo de Matemática* (Renewal of mathematics curriculum, APM 1988) containing a coherent perspective on learning and teaching mathematics, again favoring problems, technology, and applications and heavily influenced by NCTM's positions.

Given ongoing reflections on successful mathematics teaching and learning, new master's courses on education were created to promote the development of research. In fact, from 1987 on, teaching experiments were conducted that centered on four axes for innovation – technology, problem-solving, and, to a smaller degree, applications and manipulatives. Moreover, specific research groups appeared in teacher education, problem-solving, and learning. In short, the beginning of the 1990s witnessed the emergence of a community focused on issues relating to teaching and learning mathematics composed of a very large group of teachers and researchers.

[18] By the 1970s (and well into the 1980s), the majority of teachers of mathematics were either not certified or did not have sufficient mathematics background.

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Chapter 15
Mathematics Education in Russia

Alexander Karp

1 Introduction

It is natural to begin a history of mathematics education in Russia in modern times with the reign of Peter I. The changes that took place during this period were recognized even by contemporaries. "What is there to say about arithmetic, geometry, and other mathematical arts, which Russian children today learn with zeal and acquire with joy, displaying what they have acquired in a praiseworthy manner: was it ever so in the past?" exclaimed Feofan Prokopovich, a prominent writer of the era (cited in Polyakova 1997, p. 83). It had, indeed, never been so in the past: the role of government in education and particularly in mathematics education had grown immeasurably. However, while in the countries of Western Europe similar processes were taking place against a background of solidifying and developing capitalist relations, Peter's modernization accompanied a hardening absolutism. Both at that time and in subsequent periods, social and political differences from the countries of Western Europe were responsible for the distinctive characteristics of the development of mathematics education in Russia, which will be the focus of the discussion below.

No sufficiently systematic and complete history of the development of mathematics education in Russia since Peter's time has yet been written, in our view, although prominent Russian historians have paid attention to it including S. M. Soloviev (1993a, b) and P. N. Miliukov (1994). Among the not very numerous works entirely devoted to the history of mathematics and mathematics education in Russia, we would mention old studies by V. V. Bobynin (1899), A. P. Yushkevich (1947–1948), and V. E. Prudnikov (1956), as well as more recent work by T. S. Polyakova (1997, 2002, 2010). The Soviet period (following the Revolution of 1917) has been addressed, for example, in studies by Karp (2009, 2010, 2012) and Abramov (2010). More detailed information concerning certain events and materials examined in this chapter, along with more detailed bibliographies, may be found in the sources listed above.

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2 The Formation of the Russian Mathematics Education System: Eighteenth Century

As Miliukov (1994) wrote: "Because it was difficult to master, mathematical knowledge was among the least widespread kinds of knowledge in ancient Rus'; it was acquired only out of necessity, and specialists themselves had only a very imperfect command of it" (p. 223). No organized system for acquiring mathematical knowledge existed – "knowledge was acquired on an individual basis, through reading or learning from master specialists" (p. 223). This knowledge, moreover, was extremely scanty, and not infrequently techniques were employed that were so imprecise that they may be considered simply wrong.

2.1 Peter's Reforms

Striving to strengthen the armed forces and to build a fleet, Peter I invariably needed "professionals" with a much better command of mathematics than anything required in the past. Undoubtedly, one option was simply to invite learned foreigners, and another was to educate Russians abroad. Both of these methods were widely used by Peter. He recognized, however, that this was not enough and spoke about the need to organize a school that would produce "people for all kinds of purposes" (Soloviev 1993a, p. 74). However, there were no teachers, no textbooks, and often even no potential students for such a school.

Peter hired his first teachers abroad. Thus, Aberdeen University professor Henry Fargwarson was invited to teach at the Mathematical and Navigational School, which was founded in 1701. Two other British teachers, who had come with Fargwarson, also taught at this school, as well as the Russian Leonty Magnitsky. It is not clear where Magnitsky himself was educated – perhaps he did receive some kind of schooling in Moscow at the Slavic-Greek-Latin Academy, which had been founded in 1687, although its curriculum devoted only cursory attention to mathematics even in the best of times; most likely, however, he was an autodidact (Polyakova 1997).

The first textbooks published in Russia were written by these teachers themselves. Among them, a special place has traditionally been allotted to Magnitsky's "Arithmetic." This famous Russian textbook was a kind of compilation of Western and old Russian books that contained expositions of elementary arithmetic. Other textbooks that appeared around the same time were also largely repetitions or simply translations of Western textbooks. Such, for example, were the geometry textbooks prepared by Fargwarson and Peter's general Jacob Bruce, who had a British background; these textbooks effectively established the teaching of this subject in Russia.

Finding students was also not easy, although we do possess a letter written in 1703 by the effective head of the education system, Alexey Kurbatov, to his superior Admiral Golovin, in which the former optimistically reports that "many people of all ranks… have discerned the sweetness of that science, send their children to those schools, and some minors and Reiters' children and young clerks from administrative offices even come on their own with no little enthusiasm" (Soloviev 1993a, p. 75). Kurbatov even wondered what should be done if the number of students should exceed 200. And yet in 1722, for example, 127 students ran away from the Navigational School in Moscow – and by that time, senior navigational classes had been transferred to St. Petersburg, where they became a foundation for the Naval Academy, founded in 1715 (Polyakova 1997). Teaching and living conditions at the school were difficult, while the benefit that students themselves derived from the school were not all that clear.

At the Mathematical and Navigational School, students methodically studied reading and writing, arithmetic, and later also geometry, trigonometry, geography, and navigation. Some graduates were subsequently sent to England or the Netherlands for further studies. Sometimes students who had not yet graduated were sent on specific assignments.

Table 15.1 Social background of students

Clergy	931 (45.4 %)
Soldiers' children	402 (19.6 %)
Administrative officials (clerks)	374 (18.2 %)
Townspeople (merchants, craftsmen, etc.)	93 (4.5 %)
Nobility	53 (2.5 %)

Miliukov (1994) notes that a fundamental feature of the system created by Peter was that his schools were organized as professional schools. General education and development was the least of Peter's concerns. The medieval system of liberal arts had penetrated into Russia only to a small degree. And in the system created by Peter, there was nothing "liberal" in any sense of the word. The state needed specific professional experts and it undertook to prepare them. Schooling was not limited to training navigational experts, and soon artillery schools and mining schools, which prepared experts for metallurgical factories, appeared as well. Education in school was seen as a kind of civil service, difficult and to a large extent compulsory.

Life, however, introduced its own correctives: it often turned out to be necessary to employ students who had graduated, say, from the Navigational School not as navigation experts but simply as educated people. This was the case because there were not enough people who possessed even elementary knowledge, and therefore, those who were capable of imparting such knowledge were in demand.

In 1714, a number of so-called arithmetic (*tsifirnye*) schools were founded in provincial cities in Russia's gubernias. The teachers in these schools were ordered to teach arithmetic and elementary geometry, and two Navigational School students were sent to each gubernia for teaching in these schools. By the early 1720s, several dozens of such schools had opened across Russia. By 1727, they had over 2,000 students. In terms of their social backgrounds, these students were distributed as follows (Table 15.1).

This distribution did not last long, however; merchants and craftsmen tended to keep their children at home, in order to help out with the family business, while the clergy (both of their own free will and because they were compelled by their superiors) began to send their children to religious educational establishments. Gradually, arithmetic schools started to close, and by 1744 they were abolished by being merged with so-called garrison schools, which had been attached to regiments since 1732.

There were few children of the gentry among arithmetic school students, nor is this surprising: starting at a certain point, these schools were formally prohibited from accepting noblemen's children – they were intended to serve a different purpose (although teaching noblemen "arithmetic and geometry" was considered so important that a decree of 1714 prohibited noblemen from getting married "before they learn these" – cited in Soloviev 1993a, p. 445). But in "professional" schools in the capitals, noblemen gradually began to predominate – initially, children not from noble families, who had "discerned the sweetness of the sciences," had to be accepted, since there really were no other students available, but even then they often began encountering difficulties as they advanced to higher grades. Subsequently, it became even more difficult for non-nobles to enter professional schools, and most importantly, educational institutions (military schools) began to appear whose names explicitly included the word "noble."

Here, we should also mention the appearance, during Peter's reign, of secondary educational institutions of a completely different character: general educational schools. The first actual gymnasium – Pastor Glück's – appeared in Moscow in 1703, but it met with no success and finally closed in 1715. A different fate awaited the so-called St. Petersburg Academic Gymnasium, established in 1724 in conjunction with the founding of the St. Petersburg Academy of Sciences and its affiliated university. This gymnasium was conceived as a general educational institution to prepare students for the university, which in its turn was supposed to prepare scholars above all "for glory among foreigners" (quoted in Miliukov 1994, p. 258); these goals were obviously different from the other objectives set by Peter in education, and in order to realize them, many outstanding scientists were invited from abroad, including Leonhard Euler. The gymnasium was to provide a thorough grounding in mathematics – arithmetic, geometry, and trigonometry. The school opened in 1726, already after Peter's death.

2.2 The Post-Petrine Period

The system created by Peter functioned poorly or did not function at all. Miliukov considered Peter's attempts to create in Russia elementary schools for low classes fruitless. One may form an impression of how nobles studied mathematics during the period following Peter from the memoirs of M. V. Danilov (1913), who was born in 1722 and ended his career as an artillery major. This man, who had been sent by a decree of 1735 to an artillery school and had obtained, by the standards of the age, quite a full education in mathematics, writes that initially the school had two apprentices only for teaching arithmetic and that subsequently a certain Alabushev – who was then under suspicion for murder – was also sent to teach at the school. Alabushev was "a man who, although he knew something, went over Magnitsky's printed arithmetic and demonstrated some of the geometrical figures to the students, and for this reason presented himself as a learned man, was yet a quarrelsome drunkard" (p. 23). However, later one more teacher appeared at the school, who finally "put things at the school into better order." Danilov, undoubtedly, was lucky – he himself relates how he taught arithmetic to those who had not been taught anything in school.

At the Szlachta Land-Forces Cadet Corps, which opened in 1732, only arithmetic was mandatory; geometry, despite all of Peter's injunctions, was studied in 1733 by only 36 students, while the German language was studied by 237 (Soloviev 1993b, p. 502). Even at the Academic Gymnasium, instruction clearly left much to be desired. Simply finding a teacher who was not a drunkard was not easy. Reports have survived from an even later period in the gymnasium's history, 1767, which contain requests for a new teacher of arithmetic who "is capable of carrying out his duties more assiduously and diligently" than the existing teacher, who has "so fallen into drunkenness [that he] is rarely seen sober" (Tishkin 2001, p. 143).

In addition to the shortage of teachers, there was also a shortage of students. When the Academic Gymnasium first opened, 120 students enrolled in it during its first year, 58 during its second year, 26 during its third year, 74 during its fourth year, and many of these students were foreigners (Miliukov 1994, p. 259). But afterward, the inflow of new students dried up. The school began actively enrolling children of soldiers and workers and offering state stipends, but even these failed to attract a sufficient number of new students.

No functioning system existed, nor could any functioning system probably have emerged all at once – too much had to be done at the same time. And yet, important steps had been taken. Garrison schools and the army in general started producing individuals who, if only in an elementary sense, had command of arithmetic and sometimes even of geometry. Fonvizin's famous play "The Minor," written in 1782, depicts a landowner's son who is taught by three teachers, only one of whom, the teacher of arithmetic, a retired soldier named Tsifirkin ("*tsifra*" being the Russian for "numeral" or "number"), evokes the author's sympathy. Very many children had such Tsifirkins. The great poet Derzhavin (1743–1816) recalled that his mother gave her children

> to be educated, for lack of better teachers of arithmetic and geometry, first to a garrison schoolboy, Lebedev, and then to a bayonet-junker of the artillery, Poletayev; but since they themselves had little knowledge of these sciences… they [taught] arithmetic and geometry without proofs and rules, contented themselves in arithmetic with the first five parts only, and in geometry with the drawing of figures, with no understanding of what purpose anything served. (Derzhavin 2000, p. 11)

The education offered by these Tsifirkins was, admittedly, not brilliant; still, it was better than nothing. As a result, there gradually began appearing in Russia increasing numbers of people who were ready for a more advanced education and people who were capable of bringing elementary education to a more large-scale population. Perhaps even more importantly, through the efforts of the Tsifirkins, although fundamentally as a consequence of the government's will, the idea that the study of mathematics was something useful and even necessary began taking root in the country.

The country also saw the appearance of people who (although not many in number) had a more thorough grounding in mathematics. And this pertained not just to foreigners, whose contribution, of course, was very significant. The educational institutions that had appeared, although they lacked the means to educate even large numbers of students, let alone everyone, nonetheless produced a certain number of quite well-prepared specialists, who were in a position to carry their knowledge further. The man who might be described as the founding father of Russian science, Lomonosov (1711–1765), studied initially at the Slavic-Greek-Latin Academy in Moscow but then became a student at the St. Petersburg Gymnasium and the University. Nikolay Kurganov (1726–1796), the author of very important textbooks in mathematics, himself attended the Navigational School and the Naval Academy. Notable figures in Russian education, including such mathematics educators as Semyon Kotel'nikov (1723–1806) and Stepan Rumovsky (1734–1812), studied in seminaries and then at the Academic University in St. Petersburg (Kotel'nikov attended the gymnasium as well); another famous textbook author, Mikhail Golovin (1756–1790), was schooled at the Academic Gymnasium; and so on. Thus, from a certain point of view, the Academic Gymnasium, which by 1804 had been shut down mainly due to the fact that by this time it had clearly become useless, had justified its existence.

The Academic Gymnasium was also important as a kind of center for the organization and methodology of the teaching of mathematics. It was precisely for the Academic Gymnasium that Euler wrote his so-called *Universal Arithmetic* (it came out initially in German, but in 1740 was published in Russian as well, in a translation by V. Ye. Adodurov, a former student at the gymnasium). Even before this, Euler prepared plans for a reorganization of the gymnasium, in which, among other concerns, he wrote about standards for mathematics textbooks. The development of the aforementioned Russian mathematics educators and the preparation of new textbooks took place under Euler's influence. Polyakova (2010) speaks of the existence of an "Euler school" in methodology.

In 1755, Moscow University was founded, and two gymnasia opened in Moscow shortly thereafter – for nobles and for the *raznochintsy* (a social estate consisting of "people of miscellaneous ranks") – and then also one in Kazan. Various boarding schools and schools for children of nobles began to appear. As a result, the possibility of obtaining a mathematics education of a relatively high level became formally accessible to a larger and larger number of people. Bobynin (1899) noted, however, that even though improvements were made in certain academic curricula, they were not necessarily accompanied by improvements in teaching methodology. Students were often taught, as before, using foreign textbooks and often in German, not in Russian, with instruction that consisted essentially of dictation. Derzhavin (2000), who attended the Kazan gymnasium, wrote that students there were "taught the study of languages: Latin, French, German; arithmetic, geometry, dancing, music, drawing, and fencing…; however, for want of good teachers, with hardly better rules, much as before" (p. 12).

Instruction at the gymnasium, in Derzhavin's opinion, "although it did not make pupils skilled in the sciences, nonetheless gave them humaneness and a certain freedom in conversation" (p. 12). This was, in fact, what most of the noble pupils were looking for, not necessarily having any special need for the sciences – especially after a decree of 1762 freed them from the obligation to enter government service (indeed, education at a school, as opposed to at home with a personal Tsifirkin, was often not necessary at all). On the other hand, the career of a nobleman was usually linked to military service; consequently, general educational institutions, particularly in St. Petersburg, lost out in attracting nobles to privileged military educational institutions.

As for these military educational institutions, they could not do without mathematics. Therefore, both the Naval Cadet Corps, which grew out of the Navigational School and the Naval Academy, and the Szlachta Land-Forces Cadet Corps (subsequently the First Cadet Corps) were and remained important methodological and mathematical centers. The teachers at these schools included the authors of important textbooks, and the courses taught at these schools were therefore models for other educational institutions. One of the most influential figures in mathematics education there was Euler's assistant and relative Nicolas Fuss (1755–1825), permanent secretary to the St. Petersburg Academy of Sciences, who taught at both corps and wrote for their students.

2.3 The Reforms of Catherine II

Catherine II corresponded with French Enlightenment thinkers, and her views on education were far more sophisticated than Peter's. Miliukov identified two periods in her activities in this sphere – initially, in keeping with the spirit of her age, she dreamed of a system that would create a "new man," but eventually she came to the conclusion that what had to be supported was precisely a system of education with a strong emphasis on the word "education." As we have already noted, however, at the beginning of her reign, no such system existed. At a comparatively high level, there were only isolated educational institutions, while at the lowest level, outside of military service and military schools, there were no state institutions at all. The man who founded Moscow University, Shuvalov, had plans to cover the country with a network of schools – elementary schools where students would be taught reading, writing, and arithmetic in small towns and gymnasia in large cities – but these plans came to nothing.

A system for providing people around the entire country with at least some kind of education may be said, with various qualifications, to have been created by Catherine, who relied on Austrian ideas, which in turn drew on Prussian sources. The implementer of the reforms was a Serb, Fyodor Yankovich de Mirijevo, whose plans were adopted in 1786.

These plans established schools of three types: "small" schools with two grades, "middle" schools with three, and "chief" schools with four grades. Mathematics (arithmetic) was to be taught in the second grade, but in addition elementary geometry and certain "mathematical" subjects (something like mathematical geography) were to be taught in the fourth grade in the chief schools.

A Teachers' Seminary was founded for preparing teachers. In general, the role of the teacher in Yankovich de Mirijevo's conception was supposed to consist not so much in giving and checking assignments, as in actually teaching. The idea of "collective instruction" appeared at this time, that is, the idea of teaching the entire class (rather than having students work individually on their own assignments). In his "Instructions for Teachers," Yankovich de Mirijevo noted that, in presenting a rule, teachers must elucidate it using an example and even explain why they proceeded as they did and not differently. Teachers had to be prepared for such work. Thus, albeit in embryonic form, mathematics teacher education appeared in Russia. Textbooks for schools for low classes (*narodnye uchilischa*) appeared as well – translated textbooks and probably some that were written in Russia.

For all these schools, too, students were not easy to find. The same Derzhavin, while serving as governor of Tambov, brought children to schools using the police. The population, particularly in small towns that were supposed to have "small schools," saw no need for them. In general, by 1800 no more than 20,000 students were enrolled in all of these schools around the entire country (Russia's population at that time was of the order of 26 million). It should be noted, too, that although the need for village schools was discussed, practically no village schools appeared at that time, despite the fact that the overwhelming majority of the population were peasants. In general, Catherine's reforms began much more energetically than they were subsequently implemented. Nonetheless, her importance is evident.

3 The Classical Russian System of Mathematics Education: Nineteenth Century

The development of mathematics education in Russia was largely determined by the development of the education system as a whole, and from Catherine's times to the beginning of the twentieth century, the education system as a whole went through major transformations ("four times Russian schools have been subjected to radical restructuring," Miliukov wrote), becoming incomparably more organized and coming to encompass a far greater, although still only a comparatively small, part of the population.

3.1 On the Organization of Education in the Nineteenth Century

In 1802, the Ministry of National Education was founded in Russia. In the course of a series of organizational reforms, the Ministry of National Education founded new universities, each of which was supposed to oversee the less advanced educational institutions that were located in its district. A statute of 1804 established these educational institutions of three sequential types: parish schools, uyezd (district) schools, and gymnasia.

Education in parish schools (including education in elementary arithmetic) was to last 1 year; education in uyezd schools (for which students were prepared by parish schools) was to last 2 years, and students would be taught arithmetic and elementary geometry. Students were to be admitted to the (four-year) gymnasia after uyezd schools or after home schooling, and the curriculum here was to include comparatively elaborate courses in both pure and applied mathematics (Nikoltseva 2000). The reorganization proceeded slowly, however, and old forms coexisted with new ones for some time.

In 1828, a statute introduced a number of changes, some of them in response to a social issue that was perhaps the most important problem in the development of Russian education: in a country that was divided rigidly into social classes, the schools, too, remained in general based on the same principle. The role of social class here, however, was not always the same and not always equally prominent; in the early nineteenth century, gymnasia were at least nominally open to virtually everyone aside from serfs (although along with gymnasia, the so-called boarding schools for nobles, and a few years later "lycees," which gave their graduates considerably greater privileges, enjoyed popularity at this time). The 1828 statute changed this state of affairs: it explicitly referred to an "education appropriate to rank" (Nikoltseva 2000, p. 91). The gymnasium program was now divided into 7 years, and although the new rules allowed for the possibility of entering a gymnasium not just in first grade but even in fourth grade, they did not in any way provide for the possibility of entering gymnasia after graduation from uyezd schools. It is noteworthy that the government was not satisfied by what it had achieved and strove to reduce access to gymnasia for non-nobles, which was small enough to begin with; the Emperor Nikolay I wrote explicitly in 1845: "determine whether there are ways of making it difficult for the *raznochintsy* to enter gymnasia" (Kapterev 2004, p. 249). Concurrently, the government also attacked private educational institutions, making it more difficult to open them and curtailing their rights.

A statute on secondary school passed in 1864, during the period of great reforms under Alexander II (1861 – the abolition of serfdom in Russia), again changed the situation, stating explicitly that "gymnasia and pre-gymnasia accept children from all backgrounds without distinction of the profession or religion of their parents" (Dzhurinsky 2004, p. 257); still, the parents' economic level turned out to play an important role. This statute provided for the establishment of gymnasia of different types: gymnasia with two classical languages, with one classical language, and without any classical languages at all – "real schools."

The opposition between "classical" education and "real" education – which was offered in the so-called real schools (or real gymnasia), which were largely modeled on the German *Realschulen* and which focused on technical and natural scientific disciplines instead of classical languages – endured in one way or another for many years. Champions of classical languages included both Sergey Uvarov, minister under Nikolay I during the years 1833–1849, and Dmitry Tolstoy, who became head of the Ministry in 1866. At certain times, the opposition between classical and real education possessed an openly political character: the "new people" of the 1860s were "natural scientists," such as Bazarov, the hero of Turgenev's famous novel *Fathers and Sons*, and among the revolutionaries of 1860–1890, there were many natural scientists and technical experts as well (although not infrequently, they still had diplomas from classical gymnasia, without which one could not enter the university). On the other hand, classical languages were seen as a defense against liberal influences – classical languages were no longer learned in order to attend lectures at the university in Latin, but for general educational purposes.

A statute passed under Dmitry Tolstoy in 1871 and programs approved in 1872 marked a victory for the opponents of real education. The privileges of graduates from real schools were reduced by comparison with those of gymnasium graduates. A memo that came out in 1887 and subsequently gained notoriety ("On Kitchen Maids' Children") again introduced measures that limited access to education in gymnasia for representatives of the lower classes, as a result of which during the next decade the fraction of gymnasium students whose parents were not nobles or civil servants dropped from 53 % to 44 % (Dzhurinsky 2004, p. 262) – although this figure was still incomparably higher than what it had been at the beginning of the century.

On the other hand, for example, in 1897 in the country as a whole, 58,092 students attended classical gymnasia and 24,279 students attended real schools. This was approximately 15 times greater than the number of students attending gymnasia in 1809 (Miliukov 1994), and yet for a country that at this point had a population of 125 million, these figures were negligibly small. The overwhelming majority of the population – peasants – remained as before without any secondary education and very often without any education at all.

In examining changes in the Russian education system, it is important to note that mathematics played an important role in virtually every type of educational institution, and their variety prior to the Revolution of 1917 was considerable. In classical gymnasia, it was viewed as a formal discipline that facilitated mental development and was not subject to short-term political influences, while in real schools it was a necessity for future technical experts and natural scientists. In elementary public schools, mathematics was a required subject – no competent worker, let alone merchant, could do without it. And finally and most importantly, mathematics was practically the cornerstone of military education, which for many years was the most attractive form of education for those Russian noblemen who wished to pursue a career.

3.2 How Mathematics Was Taught: Contemporaries' Accounts

Many people have left memoirs that describe how mathematics was taught to young noblemen (Karp 2007c). For example, Alexey Galakhov (1999), man of letters and author of school textbooks in literature, recalled how "at first our father himself taught us arithmetic, but then he found somewhere a retired navy officer, who added grammar to arithmetic" (p. 29). Subsequently, Galakhov attended an uyezd school, where virtually no other students came from noble families (since noble families that were better off than his own did not send their children to study at the uyezd school) and where he once again studied arithmetic, but did not study elementary geometry – although he was supposed to study it – since it was not feasible to cover all of the 15 subjects included in the curriculum. After that, he attended a gymnasium, from which he graduated in 1822 at the age of 15 and a half. Here, the mathematics was somewhat more serious: Galakhov recalled how the "algebra teacher, after explaining to us, albeit very hazily, the theory of first-degree equations, posed several problems" (p. 65). The memoirist was unable to figure out these problems and therefore sent for his former arithmetic teacher, who arrived from his village "fortunately sober" and explained everything at once, so that Galakhov "solved all of the problems himself in front of [his teacher]." Although this intellectual triumph did bring him joy, as he himself explained "a gymnasium student, while succeeding in some subjects, could have unsatisfactory results in others" (p. 64) without any administrative repercussions.

Such freedom was even more evident in "institutions for nobles" and in the education of children from noble families. Nikolay Markevich, a future historian and man of letters who attended the St. Petersburg Boarding School for Nobles at around this time, recalled not without pride how he rebuffed his mathematics teacher, Dmitry Chizhov, a future distinguished professor at St. Petersburg University, who demanded that he pay attention in class; and when Chizhov threw Markevich out of the classroom in anger, Uvarov, the supervisor of the St. Petersburg school region and future minister, took Markevich's side and admonished Chizhov (Karp 2007c).

Such practices, however, were impossible in military and military-engineering schools. Here, on the contrary, mathematics found itself in a privileged position. As one graduate from such a school recalled, "For laziness in mathematics, students were punished with the rod; for laziness in other subjects, they were given bread and soup for supper" (Miturich 1888, p. 526). Graduates from military schools did not go on to serve only in the army; for example, among ten ministers of national education from 1802 until the freeing of the serfs in 1861, three had graduated from the Naval Corps, which provided perhaps the finest mathematics education in the country at that time (AS. Shishkov, P.A. Shirinsky-Shikhmatov, Ye.V. Putyatin), and another (Ye.P. Kovalevsky) had graduated from a specialized mining corps. As a result, recognition of the value of mathematics education was sufficiently widespread, penetrating even into women's education and home schooling (Karp 2007c).

Memoirists often recall meaningless rote memorization and a general absence of connection and structure. The historian Pogodin (1868), for example, recalled that "in algebra, we solved various problems well, and in geometry and trigonometry, we proved all theorems clearly, but separately from one another, so that their connections, applications, and significance were not explained" (p. 620). Educators did struggle against rote memorization, however. As early as 1810, Minister Alexey Razumovsky advised the Main School Directorate that "teachers must be required to know not any mechanical methodology, but one that can facilitate a genuine enrichment of the mind with useful and necessary truths" (Nikoltseva 2000, p. 37). And indeed, there are accounts of how, as early as the first third of the nineteenth century, teachers tried, for example, to use letters differently, in order to encourage students to think and not merely to memorize by heart (Karp 2007c).

The stiffening of the rules that took place under Nikolay I to some degree forced even those representatives of the ruling class who had no interest in mathematics to study the discipline. But the overall level of organization should not be overestimated. Sergey Vitte (1960), the future prime minister of Russia, who attended a gymnasium in the 1860s, recalled how nothing was demanded of him initially, since he was the son of an important civil servant. Subsequently, however, he decided to hire a teacher of mathematics and to study with him independently day and night in order to pass his gymnasium exams. Interestingly, because the hired teacher had confirmed that Vitte had considerable mathematical abilities, the director of the gymnasium agreed that if Vitte got perfect scores on his exams in physical and mathematical subjects (arithmetic, geometry, algebra, physics, including geography), he would be given credit for all the other subjects as well.

The goal and structure of the mathematics lesson gradually became more complex. For example, V. Omel'chenko-Pavlenko, who worked as a teacher of mathematics in the 1870s, believed that it was important at the beginning of a lesson through questioning to establish a connection with what has already been covered and subsequently to make sure that the teaching of new material rested on the conscious assimilation of old material. It was customary for teachers to pose leading questions and questions that tested students' understanding, to discuss the steps that had to be taken to solve a problem with the class, to analyze examples and models, and to have the students practice solving problems based on these examples (Ganelin 1954). Such practices can be said to have been fairly widespread in gymnasia.

At the same time, already at the beginning of the twentieth century, the outstanding mathematics educator Andrey Kiselev, defending the importance of exams, complained that at times teachers had no chance to hear a student's voice even once over an entire school year (Karp 2002). And the questions that were posed to test students' understanding – even those offered as examples by the same Omel'chenko-Pavlenko – seem at times somewhat pointless and scholastic. In a humorous story written, again, at the beginning of the twentieth century, a teacher of mathematics, requesting that students define a fraction, insists that the definition begin by indicating that the fraction in question is specifically an "arithmetical fraction," since the Russian word for "fraction" also has other meanings that have no relation to mathematics (Averchenko 1990). Such excessive refinement, although exaggerated in the short story, was sufficiently widespread, which fueled society's negative view of gymnasia as schools for "drills and rote memorization."

3.3 Mathematics Educators

Discussing the position of the teacher during the eighteenth century, Miliukov (1994) wrote that he "could neither move up the social ladder nor leave his job other than by becoming a soldier – for drunkenness and 'unethical conduct'" (p. 273). A grievance letter written in 1806 by Semyon Gur'ev, an academician from the St. Petersburg Academy of Sciences, indicates just how simple ethical norms in the academic world were at the time. Invited to observe an exam at the St. Petersburg Pedagogical Institute, Gur'ev noted that the proofs given by the students were poor. In response to this, as Gur'ev writes, Novoseltsev, the president of the Academy of Sciences and supervisor of the St. Petersburg educational region, who was also present, "remarked that he knows mathematics and, understanding all that is being asked, finds it to be optimal… [and further] berated me as a fool and poorly brought-up person" (p. 1).

Teachers were indeed far from the top of the social ladder, but the differences between its rungs in general were extremely significant. And although between a teacher, particularly one from a non-noble background, and a "person from the first four ranks," that is, a general, there lay an abyss, no less an abyss lay between, say, an officer or any civil servant possessing a rank (the system introduced by Peter I provided for 14 grades of ranks, with the fourteenth at the bottom and the first at the top) and a person who may have even been free but possessed no rank.

The career of a teacher was not an attractive one for a nobleman, but a teacher of mathematics simply by virtue of his connection with the world of the military found himself in a somewhat better position than other teachers. Polyakova (2010) notes that a mathematics teacher at a gymnasium at the beginning of the nineteenth century usually possessed a rank of Grade 9, whereas by the end of the century, Andrey Kiselev, say, had been elevated for his pedagogical work to a rank of Grade 5. In addition, in certain cases former students could become their teachers' patrons, and the students of the same Gur'ev, for example, included Arakcheyev, at one time a virtually all-powerful figure in the Russian government (Prudnikov 1956).

Gur'ev (1766–1813) came from a poor, noble family, received his mathematical education at the Artillery and Engineering Cadet Corps, and subsequently was associated as a teacher with various engineering and naval educational institutions, although not only with them – in particular, he taught mathematics at the St. Petersburg Religious Academy (Prudnikov 1956). Another outstanding mathematics educator and author of numerous textbooks, Fyodor Busse (1794–1859), was not a nobleman by birth – his father was a Lutheran minister. Busse studied at first in a gymnasium and then at the St. Petersburg Pedagogical Institute (which in 1819 was transformed into the University) and then was sent abroad (the aforementioned Dmitry Chizhov (1785–1853) had a similar biography, with the exception that the latter spent some time studying at a seminary prior to enrolling at the Pedagogical Institute). Subsequently, he taught at the Pedagogical Institute and various gymnasia. Vasily Evtushevsky (1836–1888), a mathematics educator active at a later period, who wrote textbooks in arithmetic that were used in many elementary schools, came from a poor noble family; attended a gymnasium, the Pedagogical Institute, and St. Petersburg University; studied abroad; and taught in gymnasia and pedagogical programs (Prudnikov 1956). Andrey Kiselev (1852–1940), who eventually became a kind of icon, came from a merchant background, attended a gymnasium and then St. Petersburg University, and taught at a real school and in the Cadet Corps (Karp 2002).

The family backgrounds and education of these mathematics educators were quite typical. Mathematics teachers initially came from among graduates of military and military-engineering schools and later from among graduates of pedagogical institutes and universities. The first independent pedagogical institute opened in 1804 on the foundation of the aforementioned Teachers' Seminary; in 1819, it was transformed into the University; almost 10 years later, it was reopened once more as a special educational institution; and in 1859 it was again shut down. From that time on, pedagogical programs affiliated with universities began to appear. Teachers for public schools were also prepared in special gymnasium classes. However, the teachers who came out of teacher preparation programs were by no means completely specialized: teachers for public schools were prepared without being divided according to subject, but even a university graduate had to be prepared to teach a rather broad range of disciplines at a gymnasium.

The mathematics education methodology that arose in the eighteenth century developed substantially during the nineteenth. Polyakova (2002) stresses the important role played by Gur'ev as the author of one of the first texts on teaching methodology, "Essays on Improving Elementary Geometry." Fyodor Busse was the author of a number of methodological manuals, the most important of which was probably his "Manual on Teaching Arithmetic for Teachers." Subsequently, numerous similar manuals and even more general works in methodology appeared. The first Russian mathematical-methodological journal – "The Educational Mathematics Journal" (*Uchebnyi matematicheskiy zhurnal*) – began to be published in 1833 in Derpt (now Tartu) by Karl Kupfer. It did not last long, and new mathematical-methodological journals appeared only after several decades, but pedagogical as well as general literary journals regularly devoted a certain amount of attention to issues in mathematics education (Depman 1951).

Foreign publications and sources had a significant influence on the development of methodological thought. At the same time, Russian scholarly literature gradually began to voice a desire to isolate itself from foreign influences and to counter them with Russian practices. A vivid example of this was the discussion about the teaching of arithmetic based on the methods of the German pedagogue Grube (Karp 2006).

Of course, the development of mathematics education in Russia did indeed have its own distinctive features (as was true of other countries). One of them was a very great degree of involvement in education by professional research mathematicians. Mathematicians in other countries were also quite actively involved in school education (e.g., recall French Poisson or Italian Betti), but the Russian system of government control over education assigned a dominant role to universities and hence to research mathematicians, who not infrequently became involved in school administrations and school inspections, which inevitably led also to their participation in solving methodological and substantive problems. Nor must we forget about the initially professional nature of education in Russia, which fostered the involvement of professional mathematicians, and about the tradition that gradually grew out of the foundation laid down already by Euler.

About Perevoshchikov (1788–1880), a professor and at one-time rector of Moscow University, the already-cited Galakhov wrote that, together with his colleagues, he set "the teaching of mathematics at Moscow University, and through it also in the gymnasia of the Moscow educational district, on a rational path. [Previously, teachers] would say what had to be done in proving one or another theorem, but did not explain why this in particular had to be done, as opposed to something else.…

Perevoshchikov dispelled this fog" (p. 78). In the Kazan Gubernia, a significant role was played by Nikolay Lobachevsky (1792–1856). Mikhail Ostrogradsky (1801–1861) wrote and reviewed teaching materials. Pafnuty Chebyshev (1821–1894) supervised the adoption of textbooks for the whole country. Another Moscow University professor, Avgust Davidov (1823–1885), was the author of one of the most popular gymnasium textbooks and so on.

3.4 The Contents of Mathematics Education: The Classical System

One can form an impression of what was studied in a comparatively full course in mathematics in a gymnasium or military school during the first quarter of the nineteenth century by looking at the textbooks of Nicolas Fuss (1755–1825). His "Elementary Foundations of Pure Mathematics" consists of three parts. The first part, "containing the elementary foundations of algebra, extracted from the foundations of this science by the famous Euler" (quoting from the title page of the 1820 edition), consisted of four sections: the first dealt with fractions, roots, exponents, and logarithms; the second addressed operations involving letters along with such topics as the "representation of unextractable roots as infinite series." The third section was devoted to algebraic equations, and the fourth section to relations, proportions, and progressions. The second part of the "Foundations" was geometric and included a course in plane and solid geometry that followed Euclid pretty closely (in the opinion of some, however, this course was not sufficiently rigorous – Polyakova 2002). Finally, the third part contained: "(1) Applications of Algebra to Geometry, (2) Plane Trigonometry, (3) Conic Sections, and (4) Basic Differential and Integral Calculus" (quoting from the 1823 edition). The third part was intended to be used for rather highly specialized preparation.

The material in the textbook was broken up into paragraphs, which contained expositions of theoretical issues. There were also problems – "questions" or "examples" – the solutions to which were immediately analyzed. For example, one question from the second part read as follows: "Find the side of a [regular] tetrahedron inscribed in a sphere with radius R." The end of the book contained drawings to accompany the theoretical material and the problems.

It is worth noting that no less popularity – indeed, as far as we can judge, greater popularity – was enjoyed at the time by foreign textbooks, including those translated into Russian. For example, many memoirists recall how they studied using the textbook of the Frenchman L. B. Francoeur (1809). Also popular were textbooks by A.-M. Legendre, S. Lacroix, and E. Bézout.

Gradually, Russian teaching manuals improved, above all from the methodological point of view. Polyakova (2002), for example, especially notes the complex-based character of Fyodor Busse's approach to the school textbook – his textbook in arithmetic for gymnasia was published along with a special manual for teachers and a collection of problems in arithmetic. Naturally, the gymnasium geometry textbooks of Avgust Davidov, first published in the 1860s, or Andrey Kiselev, first published in the early 1890s, already contained diagrams in the text, included problems for students to solve on their own in each section, and in general organized material incomparably more clearly and conveniently for both student and teacher. There were also differences in the content of the course, of course, but they were not so substantial and conspicuous.

The reviewers of Kiselev's textbook noted that his elementary geometry "relies on the views on the exposition of this subject expressed by the authors of the latest French and German manuals, the former in particular.... It contains nothing that might reveal the author's desire to show off his originality. Nonetheless, it contains much that is new, intended to satisfy existing demands, both theoretical and practical" (Nasha uchebnaya 1893, pp. 26–27). Similar observations may be made about other textbooks of the end of the nineteenth century. Russia became a part of what is called (e.g., by Polyakova 2010) the international classical education system. The basic school mathematics subjects became defined, corresponding to mathematics from the sixteenth to seventeenth century or earlier, but presented in a more modern fashion; the contents of the mathematics curriculum became distributed over different years of schooling in a standard way; and methodological principles were formulated. The teaching of mathematical subjects was no longer conducted in a few isolated institutions: the network of schools, although still far from encompassing all potential students, was nonetheless broad by the end of the century; there were many highly qualified teachers; and a system for the preparation and further training of teachers was in place. The influence of what was happening abroad (above all, in Western Europe) and the resemblance to it was considerable, but Russia no longer simply copied and borrowed foreign findings; rather it created and accumulated its own distinctive manuals, techniques, and textbooks. Arguably, however, the country's most distinctive feature at this time lay not in mathematical-methodological details, but in the position that the mathematical-methodological aspect of education occupied in Russian life and how it developed and penetrated into that life.

4 The Period of Reforms

The need for reforms in mathematics education had been actively discussed in Russian society since the end of the nineteenth century. It was evident that the curriculum was overloaded and that many of its sections were excessively artificial, and it was also clear that greater attention needed to be devoted to issues that had not been included in the course. Reformist ideas in mathematics education were in many ways consonant with what was being developed in the West; indeed, to a certain extent, they were stimulated by these developments.

Probably the greatest evidence of the support for methodological reforms at this time is provided by the proceedings of the All-Russian Congresses of Mathematics Teachers that took place in 1911–1912 and 1913–1914 (Trudy 1913; Doklady 1915). At these congresses, special emphasis was placed on the role of the international movement and ICMI in what was happening in Russia. The range of problems discussed at the congresses was in line with the issues being addressed in international discussions: the need to introduce the study of functions and basic calculus into the school curriculum, the role of visual geometry and laboratory work in mathematics, the importance of a propaedeutic course in geometry, and so on.

Discussions about transforming the mathematics curriculum were part of far broader discussions, which involved a far greater number of participants, about the need to reform education in general and in particular to reorganize or, more precisely, to create a large-scale system of public schools and also to transform the nature of secondary and above all gymnasium education. After the two revolutions of 1917, which successively abolished the monarchy in Russia and handed the government to the Communist Party, education was indeed radically restructured.

In place of all existing types of educational institutions, a statute of 1918 established the so-called unified labor schools. These schools were divided into two stages, and the network of first-stage schools, which were far more numerous to begin with, continued to be intensively developed. (According to the official figures, the number of students in elementary and secondary schools rose from 7,800,000 in 1914 to 20 million in 1931 – Abakumov et al. 1974, p. 156.) The goal was to eliminate from schools anything reminiscent of former discipline and drills, including exams, textbooks, and even separate subjects (including mathematics). The ideas of American progressive educators were taken up and developed in Russia (Soviet Union); schools made use of projects, laboratory work, group work, and, above all, "complexes" (Karp 2012).

"Complexes" had to link through one overarching theme topics that had previously been studied in different subject classes. For example, teachers could use a theme such as "The Post Office" to get their students to do some writing, to perform some computations, to talk about geography, and even to discuss the difficult position of the working class in other countries (Karp 2010, 2012). The themes were varied.

It is now difficult to judge to what degree "complex-based education" and other innovations were actually applied in practice and to what extent they remained mere wishful thinking, while teachers continued to teach as they always had. By all appearances, in first-stage schools, mathematics education was indeed complex-based: as evidence of this, we have the testimony of students who attended these schools as well as a large number of textbooks oriented toward "complex-based" teaching (Karp 2012). In higher grades, however, such an approach became completely unworkable, and educators were consequently willing to regard the mere establishment of links between subjects as a form of "complex-based education," while very frequently, and perhaps usually, using old textbooks, sometimes somewhat updated, to teach their classes. At the same time, it is clear that many reformist ideas, such as increased attention to the study of functions or the visual element in geometry, were indeed widely applied in teaching.

The ideas promoted in schools by the state – for example, the notion that knowledge should always be derived from experience – were consonant with the views held by many thinking teachers before the Revolution. But now, all teachers in all schools were requested to work in this new manner. In addition, it was constantly explained that "mathematics in itself has no educational value in schools; mathematics is important only insofar as it helps to solve practical problems, since students become aware that command of mathematical methods facilitates their participation in the struggle of life and construction" (GUS 1925, p. 134). The directness and haste with which the reforms were implemented, and their deliberate rejection of existing traditions, clearly did not help their popularity among mathematics educators.

5 Soviet Mathematics Education: After 1931

The period of reforms ended as decisively as it began. Between 1931 and 1936, the Central Committee of the All-Soviet Communist Party (Bolsheviks) issued a series of resolutions that fundamentally transformed the school system.

5.1 Schools Under Stalin

A resolution passed in 1931 stated that the principal shortcoming of school education consisted in the fact that "teaching in the schools does not provide students with a sufficient breadth of general knowledge and does not satisfactorily solve the problem of preparing for vocational schools and colleges sufficiently competent individuals with a sound grasp of the fundamentals of science" (Abakumov et al. 1974, p. 157). Consequently, former innovative techniques were declared to have been left-wing distortions, and they were replaced by a gradual revival of the style and substance of pre-Revolutionary education, often in their more conservative versions; for example, analyzing the curricula for the 1937–1938 school year, Sakharov (1938) wrote: "With a single stroke of the pen, the propaedeutic course in geometry for the fifth grade has been eliminated – a course for which more than one generation of mathematicians had fought" (p. 78).

Much else disappeared as well. This was motivated by the argument that students were overburdened with work and thus failed to assimilate the basic topics in the course (the Central Committee's resolution of 1932 explicitly singled out the propaedeutic study of three-dimensional geometry in seventh grade as an example of the fact that "a number of subjects are covered hastily, and children fail to acquire a sound grasp of the relevant knowledge and skills" (Abakumov et al. 1974, p. 161).

The changes did not take place overnight, but every year more and more aspects of education became more and more rigid. Standard mandatory textbooks were introduced across the country; after a few trials, the textbooks that became established in this position were textbooks from the pre-Revolutionary period by Andrey Kiselev. Exams returned; gradually, exams started being composed not in individual schools and not even in regional centers, but in Moscow – identical exams for the whole country – something unheard-of prior to the Revolution (Karp 2007a).

Curricula – for example, the ninth grade curriculum for 1935 – included such topics as progressions, the generalization of the notion of an exponent, exponential functions, inscribed and circumscribed polygons, the concept of the limit, the length of a circumference and the area of a square, the relative positions of lines and planes in space, and basic trigonometry. The curricula during the period of reforms were somewhat different, which is partially explained by the fact that ninth grade was the highest grade during this period, while by 1935 a tenth grade had been created and certain topics were moved to the tenth grade curriculum. But more important than such changes was the fact that deviations from curricula, which had previously been viewed as inevitable, since teachers were instructed to take local conditions into account, were now practically prohibited. A teacher's very conduct in class and the structure of the lesson were strictly regulated. The instructions for 1933 stated:

> Homework must be checked during every class for 10-15 minutes.... The teacher must call the student up to the blackboard, take his notebook, and look through it quickly, pointing out mistakes to the student if they are minor. If the teacher sees that the student needs additional instruction, he should arrange a "working with failing students" session. (Berezanskaya 1933, pp. 11–12)

Normative pedagogy and methodology presupposed comprehensive monitoring and control over the work of the schools by government agencies and over the work of teachers by both school administrators and general and specialized subject supervisors. School principals and vice principals had to visit hundreds of classes per year and keep track of possible shortcomings.

The government's goal of preparing vast numbers of students for colleges and technical schools led to the broad recruitment and involvement of research mathematicians in the reorganization of mathematics education. In general, while in the years immediately following the Revolution the role of pedagogues and psychologists was very great, and both mathematicians and mathematics itself were considered to be not all that important, now the situation became completely reversed. Pedagogical psychology was demolished, but attention to teachers' mathematical preparedness conspicuously increased. Teachers were taught and retaught mathematics, and the authorities insisted that the teachers' own classes be substantive, active, and demanding. Inspectors criticized classes for insufficient mathematical content and the failure to use time productively for teaching students how to think:

> the exercises and problems that are given to students are very simple: there is nothing to think about. Such work cannot attain the main objective of mathematics education: the ability to think and reason correctly. Furthermore, it does not teach students to apply theoretical knowledge to solving exercises and problems, and reduces their interest in mathematics. (LenGorONO 1936, p. 31)

A lesson had to employ a variety of methodological techniques, challenging strong students but also teaching the weaker ones; it had to teach them to think and reason but also to provide them with a firm grasp of the basic knowledge and skills prescribed by the curriculum; it had to involve students in active and independent work, but this work had to be done under clear and even rigid supervision from the teacher. For the interested and gifted, the lesson could be extended through so-called extracurricular work, while for failing students as well as for teachers, supplementary lessons were practically mandatory.

We have already noted that, in many respects, school curricula and school practices returned to prerevolutionary models. But what had previously been offered merely to a comparatively small segment of the population now became accessible to an incomparably greater number of students. A Leningrad city school board report (LenGorONO 1938) for the 1937–1938 school year gives the following statistics about the number of students in the city's schools, by year and grade (Table 15.2).

The preliminary selection of students for a serious and challenging course in mathematics was practically eliminated. Meanwhile, requirements in mathematics (at least, official requirements)

Table 15.2 The number of students in Leningrad's schools, by year and grade

School year	Grade 7	Grade 8	Grade 9	Grade 10
1935–1936	22,997	10,993	6,211	2,500
1936–1937	26,984	14,328	7,342	4,533
1937–1938	34,074	19,411	11,360	6,461

constantly increased (Karp 2010). A crucial problem for schools was the struggle against the failing rate, which not infrequently was around 20 %. With no less pathos, however, educators at the same time struggled against "rotten liberalism" in handing out grades.

In reality, of course, things did not always run smoothly. The struggle against failure often turned into harassment of failing students. On the other hand, there are accounts not only of grade inflation, in order to avoid criticism for low pass rates, but also of straightforward violations of rules and instructions, for example, disclosing classified exam materials. Ensuring that students had full understanding of the material which they were learning was seen as a paramount objective, while meaningless drilling was considered unacceptable, but in reality examiners' reports are full of complaints about students' inability to respond to questions that sounded even slightly unfamiliar or to offer examples to illustrate general concepts that they could talk about quite fluently.

And yet, the study of mathematics, which was regarded as the cornerstone of technical (and hence also military-technical) education, was in an incomparably better state than the study of other arears, if only because it was much less subject to ideological pressures. Moreover, it was deliberately emphasized that in mathematics classes, students had to study specifically mathematics, and therefore, even the biggest ideological campaigns affected the teaching of mathematics only to a very limited degree (Karp 2007b).

5.2 More Reforms and Counterreforms

The liberalization of the regime that began in the 1950s after Stalin's death affected schools as well: certain practices that had been introduced by him during the last decade of his life, such as separate schooling for boys and girls, were abolished. It soon became evident that something also had to be done with mathematics, the main school subject. The prerevolutionary textbooks, even those that had been somewhat revised, were simply too poorly suited for the new generations and the new objectives, such as the implementation of universal eight-year schooling. The scientific-technological revolution and scientific-technological rivalry with the West spurred educators to seek new approaches to the preparation of future scientists. To this was added an awareness of the fact that the school curriculum in its existing form differed radically from the spirit, style, and language of contemporary mathematics – a similar line of thought led to New Math in the United States and related movements in other countries. Lastly, it became more and more apparent that a kind of corruption was creeping into the education system: it was simply impossible to administer punishments with the same viciousness as in former years – there were not enough resources – and under such circumstances, "liberalism" in grade-giving, which had been persecuted previously, turned out to be the most natural response to demands for improved student success rates.

The first attempts to reform the content of mathematics education took place in the late 1950s, and in early 1965 a Central Commission for Developing the Content of School Education was established under the aegis of the Academy of Sciences of the USSR and the Academy of Pedagogical Sciences of the USSR, under the chairmanship of A. I. Markushevich, a prominent mathematician and vice president of the Academy of Pedagogical Sciences. The subject commission in mathematics was headed by A. N. Kolmogorov, and although other outstanding mathematicians also took part in the reforms enacted during the 1960s and 1970s, these reforms are rightly often referred to as the Kolmogorov reforms, since the leading role of A. N. Kolmogorov in them was beyond dispute (Abramov 2010).

In substance, the reformers' proposals resembled what was happening in other countries: they built their course in mathematics on a set theoretical foundation, increased attention to functional-theoretical questions (basic calculus had already become a mandatory part of the curriculum in the upper grades somewhat earlier), in geometry offered many proofs that used geometric transformations, and introduced the study of combinatorics. On the other hand, far less attention was devoted to achieving a high level of technical skills, and the most complicated and intricate assignments aimed at honing such skills disappeared from the course.

New textbooks and teaching manuals were systematically prepared for all grade levels and for all mathematics subjects taught in schools. The time allotted for implementing the reforms, however, was very brief. In addition, the new textbooks were still conceived, as before, as being standard and universal for the whole country; meanwhile, in 1973 the government set a goal of making secondary education universal, which did not go well with the heightened scientific character of the mathematics curriculum. The consequence of all this was that already by the late 1970s, voluble objections to the reforms began to be voiced by prominent mathematicians – I. M. Vinogradov, L. S. Pontryagin, and others – who found support among many teachers and parents. In the end, a decision was made, nominally by the Ministry of Education but in reality by the Central Committee of the CPSU, to roll back the reforms. Support was given to new textbooks, which effectively revived the traditional approach, although with rather notable changes.

In the process, some of the textbooks written by Kolmogorov's group, on orders from the authorities, were removed from the schools. Others remained in schools in somewhat revised form, but no longer in their former capacity as the only sanctioned textbooks. This revolutionary change – the permission to use several different textbooks in the country – was the outcome of a countrywide mathematics textbooks competition conducted in 1987–1988, which to some degree took stock of and summed up the preceding two decades (Abramov 2010).

Far more successful was another reform, which was also largely associated with A.N. Kolmogorov: the establishment of schools with an advanced course of study in mathematics. The first such school appeared in Moscow at the end of the 1950s under the banner of the idea of "polytechnization" and the preparation of workers in schools, which was being promoted by the government at that time. The objective of this school, which was created by S.I. Shvartsburd, and of several others, which appeared shortly after it, was to prepare computer programmers. In 1963, physics-mathematics boarding schools appeared at four leading universities. Subsequently, the network of schools with an advanced course of study in mathematics first expanded somewhat, then – under pressure from the government, which saw these schools as hotbeds of dissent – contracted, and finally, during the years of Gorbachev's *perestroika*, expanded once more. Most Russian scientists who today work in the fields of mathematics and physics graduated from these schools. These schools evolved a special curriculum, which included sections not ordinarily studied either in schools or colleges, but even more importantly, which approached the study of relatively traditional material in a new way as well (Karp 2011).

6 Discussion and Conclusion

The changes that began occurring in the country in 1985 led to the collapse of the Soviet Union and dramatic transformations in its largest part, Russia. These changes could not but affect education in general and mathematics education in particular. We will not discuss them here, except to note that the textbooks that took part in and won the 1987–1988 competition are still used, although even they have to some degree been revised and new textbooks have been added to them. The traditional Soviet

(Russian) system is still regarded as a norm with which new developments are compared and in terms of which they are evaluated.

Russian mathematics education evolved as an offshoot of Western European education, but in the course of its development, it acquired its own distinctive traits, including traits that subsequently influenced many other countries. Thanks to Stalin's counterreforms, Soviet (Russian) education preserved many features that over the years disappeared from education in the West. Russian schoolchildren of the 1980s–1990s, and even the schoolchildren of today, spend much more time than their Western counterparts on algebraic transformations and proofs. One may wonder how many of them go on to use this knowledge and to what extent, but it cannot be denied that the Russian (Soviet) system at a certain stage offered the opportunity to obtain a high level of mathematics education if not to all children, then to very many. Gymnasium and real school education before the Revolution was elitist; it was made available to a vastly greater student population; and on the whole, at least for a certain time, it has been successful. By comparison, the designs of the reformers of the early twentieth century, which had also been nurtured in elitist institutions (Karp 2012), clearly proved incapable of making the transition to large-scale schools, despite decisive support from the government during the 1920s. The Kolmogorov reforms were also unquestionably connected with ideas formed in the process of elitist, intensive education, but in this instance, too, the transition from the elitist to the universal failed.

One can discuss the reasons for these failures and these successes – what is clear is that time was required in order to accumulate resources, above all, human resources. Peter I introduced new practices with no less decisiveness than the Soviet leaders, but almost a century passed until enough Tsifirkins had been prepared, who in their turn gave a preliminary education to those who were able to study (often with foreign teachers) and then to teach in gymnasia. From among the graduates of these gymnasia and with their assistance, it eventually became possible to prepare people who went on to teach using Kiselev's textbooks in schools "for the masses." The interrelations between education for the elite and education for the masses were not simple, and the one did not exist without the other.

The "anti-humanistic," professional character of education, which predominated for much of the time after Peter, and the fact that mathematics was almost always the queen among the subjects in the school curriculum were, of course, important for the development and success of mathematics education. The study of mathematics was always seen as necessary and important, and the voice of professional mathematicians was always very significant in its teaching. On the other hand, it may be argued that because the humanities were pushed to the periphery of Soviet consciousness, mathematics assumed many functions of the humanities (this development, too, was to some extent anticipated by the teaching of mathematics before the Revolution). The development of speech, thinking, the ability to construct arguments, and so on came to be seen in Soviet public school practice as objectives specifically for mathematics education. "Mathematics must be studied if only because it puts the mind in order": posters with these words of M.V. Lomonosov's hung in virtually all mathematics classrooms in the country.

On the other hand, it may be said that the deliberate conservatism of the system, typical by no means of the Soviet period alone, not only aided teaching, allowing teachers to accumulate experience with working with the same materials, but also hindered them, preserving many sections whose study could no longer be justified in any way and preventing new ideas from penetrating into the schools. The crisis that mathematics education went through during the post-*perestroika* period was, in part, the price for this conservatism. Preserving the best traditions while adopting and, above all, creating new ideas – that is, the task that Russian mathematics education faces.

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Chapter 16
Mathematics Education in the United States and Canada

Jeremy Kilpatrick

1 Beginnings

This chapter traces the history of mathematics education in the United States and Canada during and after the nineteenth century. It begins, however, with some comments about the origins of mathematics education there in previous centuries. Even before the two countries gained their independence and formed their own governments, they had begun to promote the establishment of schools and colleges. As of 1647, every town of at least 50 families in Massachusetts was required by the colony to have an elementary school teacher, and every town of at least 100 families had to have a Latin grammar school. Such schools gave no attention to mathematics, but a century later, they were being supplanted by academies, such as the one established by Benjamin Franklin in 1749, where mathematics was included, at least partly in response to pressure for a more practical education.

In 1784, each parish in New Brunswick reserved a plot of land to support a schoolmaster, who typically taught, as in the other provinces, in a one-room log schoolhouse built and maintained by local settlers. In 1790, a Canadian committee recommended the establishment of free elementary schools in all parishes and villages, as well as schools for older, more advanced students roughly equivalent to secondary schools. As a result of the Parish or Common Schools Act of 1816 in New Brunswick, the people of any town, village, or community might establish a common school by building a schoolhouse, supplying at least 20 pupils, and electing three trustees. They would then receive a grant from the governor to support much of the school's operation.

Before the 1840s, education in what would become the Province of Ontario was typically voluntary and informal; not until the province's 1871 School Act was each municipality required to provide free common schools, thereafter known as *public schools* (Prentice 1977, pp. 15–17). The high schools authorized by the School Act were to offer "Mathematics, so far as to prepare Students for University College, or any College affiliated to the University of Toronto" (Hodgins 1908, p. 98). Progress was slow in establishing schools in Canada, and as in the United States, mathematics was not

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always prominent in the curriculum. In New Brunswick in 1845, for example, a group of school inspectors reporting on the quality of the teaching force said,

> Some [of the teachers] did not profess to teach any Arithmetic, and many who did were very deficient. For instance, several were unable to make up the average attendance of their scholars. The number of those who claimed to give instruction in English Grammar, Geography, Bookkeeping, and Mathematics was great in proportion to the number actually capable of teaching these subjects. (MacNaughton 1947, p. 98)

Outside Quebec, the school systems of Canada developed much as in the United States, whereas in Quebec, the development was more top-down: Classical secondary schools began to be established in the early decades of the nineteenth century, and only later did many primary schools appear (Lavoie 2003).

The first secondary schools in North America were Latin grammar schools that prepared students for college entrance. The first secondary school, and the first public school, in North America was the Boston Latin School, which was established in 1635 and still exists today. Until late in the nineteenth century, Latin and Greek were college entrance requirements and therefore essential to the Latin grammar school curriculum. Only gradually, in contrast, did mathematics enter the curriculum of those schools. First to appear was arithmetic, which entered the curriculum in the middle of the eighteenth century and became a requirement for college entrance late in the century. Algebra was next (Kilpatrick and Izsák 2008), early in the nineteenth century, followed closely by geometry (Sinclair 2008). At the same time, the Latin grammar schools were losing ground to the more popular academies, such as Benjamin Franklin's, and the growing number of public high schools. In both of the latter types of institution, mathematics played a more central role than in the Latin grammar school, with the mathematics taught being shaped by its applications to practical matters such as navigation and surveying as well as by college entrance requirements. For the United States, Ontario, and Quebec, respectively, mathematics became a compulsory secondary school subject ca. 1730, 1816, and ca. 1840 (Kamens and Benavot 1991, p. 144).

As the eighteenth century began, state-supported primary schools were established in many states in what was known as the common school movement. The schools were called *common* because they were meant to educate the children of all the people, even though that was seldom the case, particularly for students of color. In these primary schools, the promotion of literacy occupied center stage, but arithmetic – as one of the three R's – was important. For the United States, Ontario, and Quebec, respectively, arithmetic was made a compulsory primary school subject in 1790, 1828, and ca. 1850 (Kamens and Benavot 1991, p. 144; Ellerton and Clements 2012, p. 40, question their claim regarding the United States but do not supply alternative evidence).

At the postsecondary level, religious groups began to establish colleges in the United States and Canada beginning after the first third of the seventeenth century (see Dauben and Parshall, this volume, for details on the earliest colleges in the colonies). The level of mathematics offered in those colleges was low; they concentrated on theology, and the faculty members were likely to have been rather ill-equipped mathematically:

> When, after 1600, the Atlantic coast of North America was being colonized, there were many settlers with a university education, something often meaning no more, mathematically speaking, than some knowledge of the rule of three. Few of these men ever crossed the *pons asinorum.* (Euclid, Elements 15: The angles at the base of an isosceles triangle are equal.) This also may have been true for the teachers at the two newly founded colleges, the Puritan one at Cambridge, and the Jesuit one at Quebec. (Struik 1977, p. 2)

Established in 1785, the University of New Brunswick is often considered the first public university in North America, but there are three other claimants for that title: the University of Georgia (also founded in 1785, but no students until 1801), the University of North Carolina (not chartered until 1789, but students admitted in 1795), and the College of William and Mary (founded in 1693, but not public until 1906). Regardless of priority, there were few public institutions of higher education in North America before the nineteenth century. Summarizing the mathematics offered in American higher education as of 1850, Grabiner (1977) observed, "The mathematics usually taught in the colleges [and universities] was arithmetic, elementary algebra, and the geometry of Euclid, with bits and pieces of surveying, trigonometry, or conic sections thrown in" (p. 16).

2 The Expansion of Public Education

Until late in the nineteenth century, the United States was much more rural than urban, and industrialization was limited largely to the northeastern states. Nonetheless, public education grew throughout the century and across the country so that, by 1870, across all states, almost 78 % of the eligible children – even in rural areas – were enrolled in a public primary school, and that number had risen to almost 90 % by 1900 (Lindert 2004, Vol. 1, p. 92; Meyer et al. 1979, p. 597, report somewhat lower percentages). After 1880 in the United States, "secondary enrollments doubled each decade until World War II" (Hammack and Greyson 2009, p. 832).

By 1862, Canada had a greater percentage of its population (21.8 %) in primary school than the United States did by 1870 (17.1 %; Peaslee 1967, p. 62). The percentage of eligible children enrolled in primary school in 1870 was also somewhat higher in Canada than in the United States (Lindert 2004, Vol. 1, p. 92). Nevertheless, attendance continued to be a problem throughout the century:

> In 1900 the average daily attendance rate (among those enrolled) for the whole of Canada was 61%. Moreover, most children received only a few years of schooling. Both boys and girls often left by age 9 or 10 to work in factories or at home. (Oreopoulos 2006, p. 25)

Potvin (1970) argues that in Quebec, after 1850 and until well into the twentieth century, there was an "almost total eclipse in the publication and teaching of mathematics" (p. 363) as employment opportunities for mathematicians evaporated and the belief arose that French Canadians were suited for the arts but not the sciences. Throughout the nineteenth century, universities were being established throughout the Canadian provinces, but they struggled with small enrollments owing to religious tensions, inadequate financial resources, and "the periodic problem of indifferent institutional leadership" (Axelrod 1997, p. 89).

Cohen (2003) has traced the journey taken by school mathematics as it developed from "an arcane, difficult, memory-based subject relegated to specialized classrooms for older boys bent on mercantile careers . . . into a standardized school subject in the early nineteenth century" (pp. 70–71). As public education expanded in North America during the century, textbooks began to play a larger role in mathematics instruction. Until well into the century, however, the teacher might be the only person in the classroom with a commercially printed book in hand. College students were being required by their instructors to purchase books for their mathematics classes (Kidwell et al. 2008, p. 5), but students in school were more likely to be using so-called cyphering [see Ellerton & Clements] books into which they carefully copied worked-out problems and solutions (Ellerton and Clements 2012, p. 107). Ciphering was not meant to be simply copying; instead, students were expected to have learned how to solve problems of a specific type, to have had their solution checked by the teacher, and to memorize the solution as they copied it into their book in their best calligraphy.

Early in the nineteenth century, textbook publishers were modest in size and scattered across North America. The distribution of textbooks in quantity was rare in the United States outside of New England and was especially infrequent in Canada through the middle of the century (Michalowicz and Howard 2003). As railroads came to span the continent, publishers built larger facilities. By midcentury, New York City had become the major center for American textbook publishing, and graded series of mathematics textbooks began to be published. In Canada, Quebec City, Montreal, and then Toronto became centers of textbook publication. The development of US national publishing is illustrated by Kidwell et al. (2008, pp. 10–20) through the chronicle of Charles Davies, a mathematics professor at the US Military Academy at West Point, who, beginning in 1826, wrote almost fifty mathematics textbooks for schools and colleges, publishing them in multiple editions with A. S. Barnes & Company. By 1860,

> the scale of Barnes's operation necessitated [a] shift from personal marketing to extensive print advertising and the use of regional sales agents. . . [By then], Americans were accustomed to the idea that relatively inexpensive, uniform, graduated textbooks were the cornerstone of mathematics instruction at all levels of education. Teachers looked to textbooks for the content of their courses, and students expected to study a book when they entered a classroom. Daily sessions were conducted as recitations from the first years of primary school through college. (pp. 18–19)

Early in the nineteenth century, most North American arithmetic textbooks followed the *rule* method – students memorized rules and then practiced them – but two alternatives that did not begin by stating rules, the *inductive* method and the *analytic* method, began to compete for attention with the rule method as the century went on (Michalowicz and Howard 2003). The inductive method arose from the theories of Johann Pestalozzi. The student was to be led to an unstated concept such as fraction by a graded series of questions, typically accompanied by illustrative objects or drawings. In the analytic method, the instruction used reason and logic to help students think through a problem, whose solution was followed by examples to provide practice in such thinking. Only then was a rule derived.

> Whereas rule books were more concerned with applications of arithmetic and algebra to business and daily life, the inductive and analytic texts, although not uninterested in the applicability of concepts taught, were more concerned with the understanding of processes than with their memorization. (p. 104)

The apparent popularity of the inductive and analytic textbooks "shows that teachers largely supported . . . methodologies based on conceptual understanding and real-world applications" (p. 106). School mathematics changed during the century in the United States and Canada as the schools reached a wider, more diverse population and instruction began to be more heavily influenced by progressive educational thought that emphasized learning by reflecting on one's experience rather than simply preparing for the university by studying classics in the field.

3 Mathematics Teacher Preparation and Certification

Normal schools for the training of teachers were first established in the United States in the 1820s. These schools were modeled on the French *école normale,* which attempted to provide models, or norms, for teaching. The first such schools were private. In September 1838, at a meeting in Hanover, Massachusetts, a convention chaired by Horace Mann, who was secretary of the state board of education, had met to discuss the advisability of establishing state normal schools. Speaking at the convention, the former US president John Quincy Adams noted that normal schools for educating teachers were flourishing in France and Germany. He said, "Monarchs are expending vast sums for normal schools throughout their realms and sparing no pains to convey knowledge to the children of the poorest subjects. Shall we be outdone by Kings?" ("The First Normal School" 1890). The next year, the first state-supported normal school (now known as Framingham State University) was established in Lexington, Massachusetts; by 1860, there were 12 such schools in the United States (Burns 1970, p. 430), and by 1867, there were 37 ("The First Normal School" 1890).

The Upper Canada School Act of 1846 created a provincial normal school that would train teachers as well as set uniform standards for certifying teachers throughout the province (Prentice 1977, p. 18). The next year, the first normal school in Canada was established in Toronto, followed during the next decade by similar schools in New Brunswick, Nova Scotia, Prince Edward Island, and Quebec (Axelrod 1997, p. 40).

The level of mathematics taught in the normal school of the late nineteenth century was not very high. "As late as 1900, the typical normal school provided only one or two years of work beyond high school" (Burns 1970, p. 430). The 1884 catalog of the West Chester (Pennsylvania) State Normal School (today West Chester University) demonstrates that the level of the mathematics taught was similar to what one might expect in an upper secondary school. It lists the content of two courses of instruction approved by the state authorities for normal schools: (a) in the Elementary Course [which led to a Bachelor of the Elements diploma and was designed to prepare primary school teachers], the mathematics consisted of "Arithmetic; Elementary Algebra; Plane Geometry, including the circle" (p. 20), and (b) in the Scientific Course [which led to a Bachelor of the Sciences diploma and was designed to prepare secondary school teachers], the mathematics included all that was in the Elementary Course together with "Geometry; Higher Algebra; Plane and Spherical Trigonometry; Surveying; Analytical Geometry, and Calculus" (p. 21). A note in the Scientific Course said, "Students may substitute for Spherical Trigonometry, Analytical Geometry, Calculus, the Mathematical parts of Natural Philosophy and Astronomy, and the latter third of Higher Algebra—an equivalent amount of Latin, Greek, French, or German" (p. 21). Apparently, students in the Scientific Course who found the mathematics they were studying too daunting could pursue language instead. The Elementary Course was for 3 years, and the Scientific Course was an additional 2 years. The great majority of the students took only the Elementary Course. On graduating from either course, students received a certificate to teach in Pennsylvania "common schools"; practicing teachers who had not attended the normal school but had taught at least 3 years could also obtain a diploma or teaching certificate by passing an examination given by the school.

The low level of the mathematics taught in the normal schools is not surprising, given that in a survey of state normal schools, fewer than a quarter of them required high school graduation for entrance; the great majority required only completion of the eighth grade (Taylor et al. 1912, p. 214). Because most normal school students were preparing to teach in the primary grades, arithmetic was seen as the most important topic in the normal school's mathematics curriculum. Nine out of ten normal schools surveyed offered a course in methods of teaching arithmetic, whereas fewer than one in ten had a course in methods of teaching algebra and geometry (p. 215).

During the final decades of the nineteenth century, more than 300 US "normal schools began slowly to grow from advanced secondary schools to ones of a higher education caliber. By 1898, twenty-eight states recognized graduation from normal schools and universities as evidence of qualification for certification" (Burns 1970, p. 430).

In a 1915 report on the training in mathematics of elementary school teachers in countries represented in the Commission internationale de l'enseignement mathématique (CIEM, anglicized as the International Commission on the Teaching of Mathematics), I. L. Kandel noted that in some states in the United States:

> The qualifications demanded of teachers in elementary schools are still no higher than the attainments of one or two years beyond the elementary schools, while at the other end of the scale teachers may have the qualifications of college graduates. The most general standard may, however, be said to be graduation from high school, with two years of normal school training. The mathematical attainments will thus have been acquired in the eight grades of an elementary school, four years of high school and two years of normal school. (p. 46)

Although by the 1870s, only a minority of Canadian teachers were receiving professional instruction in normal schools, "the general level of their educational backgrounds rose. Most elementary teachers, with or without normal-school credentials, had some high-school training, and secondary-school teachers increasingly were attending university" (Axelrod 1997, p. 47). In both countries, the twentieth century saw an evolution of the normal school into all or part of an institution of higher education, with many renamed teachers colleges. In the United States, such colleges often became part of the state system of higher education. Many an institution that had been known as a normal school or teachers college at the beginning of the century would by midcentury have been renamed as "[city or state name] State College" and by the end of the century as "[city or state name] State University." In Canada and many US states, a different pattern prevailed for some normal schools and teachers colleges: They were assimilated into the state or provincial university as its college, school, department, or faculty of education. When a teachers college became a regular college or university, the mathematics education program was usually housed in the mathematics department, whereas when it was made part of an existing college or university, the program typically remained in the education faculty. When a new university was founded, either alternative might be followed. For example, when Concordia University and the Université du Québec à Montréal were established in the late 1960s, "scholars who were interested in the teaching and learning of school mathematics were affiliated with mathematics departments rather than education departments" (Kieran 2003, p. 1708), contrary to the common practice elsewhere in Canada.

As normal schools were transformed or absorbed into universities, they became places where research in mathematics education was undertaken and promoted (Kilpatrick 1992). Wherever the mathematics education program was located, the new professors of mathematics education were expected to conduct research in their field as well as to train prospective teachers. Professorships in education in North American universities were not common 100 years ago, but since then, the growth of the field both within and outside of universities has been remarkable.

4 Mathematics Education Becomes an Academic Subject

In a 1912 survey, the United States was one of four countries (the others were Belgium, Germany, and Great Britain) in which university lectures in mathematics education were being offered to supplement mathematics lectures (Schubring 1988). The first Ph.D. degrees in mathematics education in the United States, and presumably anywhere in the world, were those of Lambert L. Jackson and Alva W. Stamper, completed under the supervision of David Eugene Smith at Teachers College, Columbia University, in 1906 (Donoghue 2001). Smith was the initiator of the CIEM, which had been organized at the 1908 International Congress of Mathematicians in Rome (Kilpatrick 2008, p. 26). He has been considered one of the first two US mathematics educators (Jones and Coxford 1970, p. 42), establishing Teachers College as the Eastern Seaboard center for the academic study of mathematics education. The other pioneering US mathematics educator was Jacob William Albert Young, who helped to establish the University of Chicago as the Midwest center for such study (see Donoghue (2003) for a detailed examination of the emergence of university programs in mathematics education in the United States).

The first Canadian doctorate (of pedagogy) that dealt with mathematics education was received by E. T. Seaton. It was awarded in 1924 by the University of Toronto, and the thesis title was *Practice in Arithmetic or the Arithmetic Scale for Ontario Public Schools* (Kieran 2003, p. 1706). Toronto was for some time the only center for Canadian research in mathematics education, partly owing to the groundbreaking work of James McLellan, who was a professor of pedagogy at the university and in 1895 had coauthored a book with John Dewey on the study and teaching of arithmetic (p. 1707). The development of university programs in mathematics education proceeded much more slowly in Canada than in the United States. Kieran notes that "in 1990, there were merely seventeen Canadian universities at which it was possible to obtain a doctoral degree involving research related to mathematics education" (p. 1710). In contrast, various summary reports from the US National Research Council indicate that some 126 US universities were awarding doctoral degrees in mathematics education as the major field in the years from 1980 to 1998 (Reys et al. 2001, p. 24).

5 The Role of Mathematicians

By the beginning of the twentieth century, the province of Quebec had few well-trained mathematicians and consequently almost no contributions of mathematicians to school mathematics. That situation changed somewhat in 1921 with the appointment of Adrien Pouliot, a Sorbonne graduate, as professor of mathematics at Laval University (Lavoie 2003, p. 309). Pouliot, who also established a mathematics society in Quebec City, launched a campaign to improve mathematics teaching in the secondary schools, arguing that graduates of those schools were poorly prepared and were not pursuing science, mathematics, or engineering at the university. The classical high schools (*collèges classiques*) in Quebec fought back, claiming that mathematics was neither one of the humanities nor a suitable subject for developing the mind. The conflict was not resolved until the new math reforms began in the 1960s.

Outside Quebec, mathematicians apparently influenced Canadian school mathematics in the early twentieth century primarily by authoring or coauthoring textbooks (Sigurdson et al. 2003, pp. 218–223). Some mathematicians contributed, along with mathematics teachers and other educators, to provincial curriculum development efforts, but before the 1960s, they had little involvement in issues of school mathematics.

In contrast, US mathematicians have a somewhat longer and more extensive history of addressing school mathematics. One of the first and most prominent mathematicians to weigh in on such matters was Eliakim Hastings Moore. In his 1902 address on retiring as president of the American Mathematical Society, Moore (1903/1967) called for a laboratory approach to mathematics and a school curriculum in which the various branches of mathematics would be unified. He called on professional mathematicians to contribute to the reform of school mathematics. The founding of the Mathematical Association of America in 1915 was an attempt to reconcile research in pure mathematics and mathematics education in the secondary school (Donoghue 2003, p. 183). It served as a venue in which mathematicians could address matters of school teaching. As in Canada, when the new math movement appeared on the scene, mathematicians began to play an even greater role than before.

6 The Progressive Era

As the twentieth century began, North American educators increasingly began to promote the so-called progressive pedagogical ideas of thinkers such as Jean Jacques Rousseau, Johann Pestalozzi, Friedrich Froebel, and John Dewey, encouraging teachers to take the interests of children into account and to serve as guides rather than taskmasters, letting children learn by doing. At the same time, the school population, especially in the secondary grades, was increasing:

> In 1900, only 11 percent of [U.S.] children attended secondary school; by 1930 more than half did and this period saw a fervent debate about the type of education American society required. Progressives believed that a curriculum of memorized classics, math, science, and history should be replaced and schools should consider the nature of the child as well as the needs of society in carrying out their mission. (Heilman 2005, p. 108)

School mathematics, in particular, came under various forms of attack (Kilpatrick 2009b). Arithmetic was viewed as dealing with too many irrelevant and uninteresting topics, taking too much instructional time, and forcing too many pupils to repeat a grade. The algebra and geometry of secondary school appeared unsuited to the growing population of students, many of whom were described by G. Stanley Hall (1904), founder of the child-study movement, as a "great army of incapables, shading down to these who should be in schools for dullards or subnormal children" (p. 510).

> Responding to the pressure of increased enrollments and rising dissatisfaction with the quality of instruction, states began to reduce their requirements in high school mathematics. In 1921, Ohio dropped its requirement that students take a year of mathematics, and other states soon followed suit, making mathematics an elective rather than required subject in Grades 9 to 12. (Kilpatrick 2009b, p. 113)

In Canada, the growing diversity of the population began to raise concerns that too many immigrants were coming from southern climates and other undesirable places. Educators responded by stressing aspects of British-Canadian citizenship. For example, "arithmetic courses stressed the virtues of 'growth, progress, and competitive business practices'" (Axelrod 1997, p. 85). In Ontario, mathematics had received reduced emphasis as a school subject during the closing years of the nineteenth century (Crawford 1970), and in response, efforts were made to introduce "more problems from physics, other sciences, and practical life" (p. 391).

The theme of making mathematics an optional school subject or even removing it altogether was promoted in Canada by William E. Blatz (1936), architect of child study in the province of Ontario who studied the Dionne Quintuplets. He advocated a progressive approach to education: "If all reading, grammar, mathematics and other academic subjects were removed from the school time-table, and drawing, modeling, craftwork, music and dancing substituted, we should have in the next generation not only more intelligent but happier adults" (p. 124).

Despite the best efforts of progressive educators, however, school mathematics in North America remained a stalwart of the curriculum and changed relatively little during the first half of the twentieth century. Progressive educators were always a minority in the profession, especially among mathematics educators, and their efforts became equated, in the public mind, with extreme permissiveness in the classroom and a general lowering of standards.

7 The New Math Era

The Second World War ushered in a "time of uncertainty and change" (Garrett and Davis 2003) for school mathematics in the United States and Canada. The war "proved to be a pivotal event that revived interest in school mathematics as an area of curricular concern following decades of decline" (p. 515). After the war, increased criticism of public education in general and mathematics education in particular, as well as the perception of a growing threat from Soviet technological prowess, ultimately gave rise to several projects to improve school mathematics, efforts that collectively became known as "the new math." These efforts included the curriculum developed by the University of Illinois Committee on School Mathematics beginning in 1951, two conferences of mathematicians in early 1958 that resulted in the establishment of the School Mathematics Study Group later that year, and the 1959 report of the US College Entrance Examination Board Commission on School Mathematics – all of which were aimed one way or another at improving the mathematical preparation of the scientific workforce (see Fey and Graeber 2003, and Kilpatrick 2012, for further discussion of these efforts). In every province of Canada by 1964, "curriculum committees [were] examining and revising curricula, [with] the most ambitious program in experimentation [being] carried out by the Ontario Mathematics Commission" (Fehr 1964, p. 325). Regarding the new math efforts in Canada, however, O'Shea (2003) noted:

> Implementation of mathematics content reforms took place later in Canada than in many areas of the United States, partly as a consequence of the centralized bureaucratic structure of Canadian provincial educational systems and partly because few Canadians had been involved in the development of experimental curricula in the United States or elsewhere. (pp. 846–847)

The new math movement in North America took a variety of forms, but essentially it was an effort to bring school mathematics closer to the mathematics being taught in the university by eliminating needless jargon and taking advantage of recent developments in the foundations of mathematics. In particular, "the language of sets, relations, and functions would provide not only a more coherent discourse in the mathematics classroom but also a more meaningful structure for learning. Students would be drawn to mathematics by seeing how it fit together" (Kilpatrick 1997/2009a, p. 87).

The common verdict on the new math in North America is that it failed in the effort to reform school mathematics. That verdict is disputed by Davis (2003), who argued that (a) the new math consisted of too many programs for it to be characterized as a single movement or reform effort; (b) those programs were implemented in relatively few schools; and (c) when they were implemented as designed, they were often dramatically successful. Fey and Graeber (2003) come to a somewhat different conclusion:

> A balanced assessment of the reform effort suggests that many positive changes occurred in the content of school curricula, but the emphasis on abstract unifying concepts and logical precision was not as effective as proponents had conjectured. Changing the day-to-day practices of American schools, teachers, and testing proved to be a much more complex problem than the new math developers imagined, and winning community support for programs that changed traditional content and teaching was also a difficult task. (pp. 537–538)

Whatever the effects of the new math efforts on the school curriculum, their most dramatic outcome was to stimulate the development of the US and Canadian community of mathematics educators, particularly those engaged in research on mathematics education (Kieran 2003; Stanic and Kilpatrick 1992).

8 The Standards Era

The new math era, which ended in the 1970s, was followed by efforts to bring school mathematics “back to basics” (Fey and Graeber 2003; McLeod 2003; O’Shea 2003). These efforts were stimulated in large part by the public perception that the new math had “failed” but also by declines in college admission test scores and an increased emphasis on accountability in education. In reaction, the National Council of Teachers of Mathematics (NCTM) undertook an effort to develop recommendations for the mathematics curriculum of the 1980s. Conference reports and position statements were collected, and several surveys were undertaken to learn the preferences and priorities of various groups, including teachers, supervisors, teacher educators, principals, school board presidents, and presidents of parent-teacher organizations (Fey and Graeber 2003). The resulting document, *An Agenda for Action* (NCTM 1980),

> recommended that problem solving should be the focus of school mathematics, that basic skills should be defined more broadly than simple arithmetic and algebraic calculation, that calculators and computers should be used at all grade levels, and that more mathematics should be required of students. (Fey and Graeber 2003, p. 553)

The *Agenda* also made recommendations regarding the improvement of assessment, teacher professionalism, and public support of mathematics education.

In April 1983, the report *A Nation at Risk* (National Commission on Excellence in Education 1983) was published in an effort to raise concern about the quality of US education. It contained apocalyptic claims and warnings such as the following: “If an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war” (p. 124). Responding to the concerns raised by the *Nation at Risk* report, the NCTM was able to draw on its *Agenda* report as “a useful weapon in the battle to update the curriculum” (McLeod 2003, p. 762), but more specificity was needed, given the brevity of the *Agenda* report (30 pages). Ultimately, the NCTM (1989) responded by producing *Curriculum and Evaluation Standards for School Mathematics*. That document was unique in several respects: It was the product of a teachers’ organization and not a governmental agency; it was produced with (initially) no outside funding; it attempted to go beyond local, state, and provincial boundaries in laying out recommendations for the curriculum and for evaluation; and its release was accompanied by an unprecedented public relations effort at an especially opportune time. McLeod (2003) relates in detail the story of how NCTM’s complicated effort to produce standards to improve school mathematics ultimately led to a number of related documents and to new visibility for the council on the national and international scene.

In Canada, education is left to the provinces, and in the United States, it is left to the states. Neither country has one ministry of education that sets policy for the entire nation. Further, both countries have high rates of geographic mobility, which means that children transferring from schools in one province or state to another can encounter a different mathematics program. During the twentieth century, despite the tradition of local control of education, geographic mobility together with textbooks published and available nationwide helped to create something of a de facto common school mathematics curriculum across each country. The standards promoted by the NCTM took that process further by leading states and provinces to adopt a common set of objectives for school mathematics.

9 Two Centuries of Progress

Since 1800, mathematics education has grown and developed in the United States and Canada as an arena of professional practice and a field of academic study. In the nineteenth century, the growth was mainly in the educational systems being established by states and provinces as frontiers moved west and north. Widespread public elementary education was achieved by midcentury, and the end of the century saw secondary school enrollments and graduation, especially in the United States, beginning to outstrip those of European countries. Elementary schools taught mathematics – mainly arithmetic – because it was seen as a vital part of every child's preparation for adult life. Secondary schools taught mathematics primarily to prepare students for college entrance.

During the twentieth century, the curriculum in North America slowly expanded so that elementary school mathematics began to include attention to the rudiments of geometry, algebra, and data analysis and secondary school mathematics began to focus on preparation for vocations as well as college. Further, mathematics education began to flourish as an academic field. Doctoral programs appeared in universities, and teachers' organizations were formed. Despite its name, the NCTM drew its membership from both Canada and the United States, as did the American Educational Research Association, whose Special Interest Group for Research in Mathematics Education became one of the largest divisions of the organization. US and Canadian mathematics educators have been active internationally, both in the International Commission on Mathematical Instruction (ICMI), established in 1952, and in its predecessor, the CIEM, organized in 1908 as noted above. ICMI officers have come from both countries, and both have hosted congresses of ICMI. Other organizations with active Canadian and US participation include the International Group for the Psychology of Mathematics Education (PME) and the North American Branch of PME.

Dozens of journals for mathematics teachers and mathematics teacher educators and researchers are published in the United States and Canada. Among the most prominent are *School Science and Mathematics,* the *Journal of Mathematical Behavior, For the Learning of Mathematics* (published under the auspices of the Canadian Mathematics Study Group), and the *Journal for Research in Mathematics Education* (published by the NCTM). The NCTM also publishes *Teaching Children Mathematics, Mathematics Teaching in the Middle School,* the *Mathematics Teacher,* and the *Mathematics Educator.* Many states and provinces have their own organizations of mathematics teachers and publish their own journals.

During the past two centuries, multiple overlapping groups concerned professionally with mathematics education developed in both Canada and the United States. People in those groups taught or studied mathematics, taught or studied mathematics education, or engaged in some combination of those activities. Collectively, those groups have formed an extensive, vibrant, productive mathematics education community in North America. Although that community remains active in promoting the cause of school mathematics, the history of mathematics education in the United States and Canada suggests that any change in the mathematics taught in school is likely to come rather slowly and that pressures on the public schools from politicians and the public may be as powerful in shaping such change as anything the professional mathematics education community proposes.

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Chapter 17
Mathematics Education in Latin America

João Bosco Pitombeira de Carvalho

1 Introduction

At the beginning of the nineteenth century, Spanish Latin America was divided into four viceroyalties, namely, New Spain, Peru, New Granada, and Rio de la Plata, with seats in Mexico City, Lima, Santa Fé de Bogotá, and Buenos Aires, respectively. Portuguese America, which was part of today's Brazil, was ruled by a viceroy. In the first half of the nineteenth century, both Spanish and Portuguese America became independent from their mother countries. In 1822, Brazil became an independent empire that lasted until 1889 and was able to preserve its territorial unity. Meanwhile, Spanish America underwent a fragmentation process, particularly in Central America, from which eventually emerged the Spanish American countries with their present boundaries. There were wars or treaties among the new Latin American countries, the upshot of which was changes in territories, and some of present-day countries were carved out from already independent ones.

Here, we will sketch the history of mathematics education in Latin America, stressing the creation and consolidation of its educational systems from independence through the middle of the twentieth century. If we take into account that we are dealing with a vast geographic region with great cultural differences,[1] it is clear that our goal cannot be completed in a single chapter, unless it reaches encyclopedic dimensions, and so our treatment will be unavoidably sketchy. In fact, this chapter could more aptly be named "Episodes in the History of Mathematics Education in Latin America," as we are only singling out a few representative occurrences showing similarities and differences in this history among the countries of the region.

The history of mathematics education in Latin America offers many instances of the transmission process, as viewed in Schubring (1987, 2000), not as a static concept but as a dynamic situation, in which ideas and institutions are transformed to adapt themselves to new cultural and social settings. A task that remains to be done is a study of the history of mathematics education in Latin America that takes the transmission process as its key element and unravels its several instantiations.[2]

[1] See Bethell (1998) for a good introduction to the cultural history of Latin America.

[2] The author of this chapter thanks the editors for their very useful comments.

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We divide this chapter into three sections, which correspond approximately to the following time periods:

- First Period: Creation and Attempts at Consolidation – 1800–1850
- Second Period: Consolidation and Expansion – 1850–1900
- Third Period: Challenges and Innovations – 1900–1960s

These time periods are rough estimates made to organize this chapter. Some overlap is unavoidable in that in one period sometimes the others are treated or that the time boundaries stipulated for each part are extended.

2 First Period: Creation and Attempts at Consolidation – 1800–1850

The independence movements in Spanish-speaking America had inherited the liberal ideas of the Enlightenment (Hobsbawm 1996), and so, after independence, most of the newly empowered governments believed that education would redeem their countries from ignorance and poverty. This can explain the widespread concern with preparing more and better teachers and establishing national school systems throughout Latin America (Garcia 2002, p. 37). Moreover, these governments realized that education systems were important in building and unifying the new nations (Garcia 2002, p. 38). In most cases, the legislations that created these systems were naïve in proposing actions completely out of the reach of the new countries that were often involved in liberation wars against Spain or foreign interventions of various kinds.

In Brazil, before the successful independence movement of 1822, Portugal was able to smash several attempts at self-rule, some of which also drew inspiration from the Enlightenment (Rodrigues 1975). After 1822, the Brazilian political elite was particularly afraid of the republican movements in the surrounding countries and worried about a process of fragmentation in Brazil. This conservative outlook was responsible for the sometimes divergent educational conceptions between Brazil and Spanish-speaking America. For example, interest in primary education was far more genuine and explicit in some countries other than Brazil, even though this country's first constitution (1824) stipulated that elementary education was free for all citizens and the first Brazilian law concerning education (1827) ordered that a public "school of first letters should be opened in all towns and villages." The law stipulated that boys would study the four elementary operations and practical geometry and the girls would study the same as the boys in arithmetic but with less geometry.

We organize this section by topic, not country, because this period witnessed considerable changes in the political division of Latin America. For example, after its independence from Spain, the viceroyalty of New Granada, with its seat in Bogotá, became (with some minor differences) Greater Colombia (1819–1830), which included several present-day countries in northern South America such as Venezuela and Ecuador and in southern Central America such as Costa Rica and Panama. As a general rule, for the sake of clarity, we refer to events in present-day Venezuela as Venezuelan, even if they occurred while Venezuela was part of Greater Colombia.

2.1 *Mutual Learning and Normal Schools*

What was the situation of education, particularly elementary mathematics education, in Latin America right after independence from Spain and Portugal? It seems safe to assume that Prieto's (2010, pp. 33–34) words about Chile can be generalized to Latin America:

> The few schools were run by persons whose intellectual preparation seldom went past reading and writing. Some of them were soldiers made prisoners during the wars of independence, and others came from less reputable

occupations. One can mention the case of a man arrested for stealing in a church and condemned to be an elementary school teacher.[3]

Who could be an elementary school teacher in the first half of the nineteenth century? In Brazil, practically anyone could set up a "reading and writing" classroom. In small towns and villages, sometimes the local priest took care of this. The large estates of the landed gentry often had a resident chaplain, who served as a tutor for the owner's children (Freyre 1946). To become a teacher paid by the government, one had to pass an exam and have a teaching permit. Salaries were very low and paid irregularly. Cardoso (2002) was able to collect actual sheets of these exams, for the period 1797–1807, which consisted of two tests: one in arithmetic and the other in spelling. Similarly, Soares (2007) studied the teachers of mathematics in Brazil from 1759 to 1879. She located many exam sheets of the candidates and studied their mathematical content, which was indeed very elementary (Carvalho and Dassie 2012, p. 502). Of course, all candidates had to present certificates of "good behavior."

The situation is similar in other Latin American countries. In Chile, for example, sometimes generous persons established schools in the villages, which painfully survived with only scant help from the municipal government. The teachers were priests or laypersons, with very little training (Labarca 1939, pp. 71–72). Confessional schools in monasteries provided a more systematic instruction. There was no separation between elementary and secondary education, and the emphasis was on the subjects that would help the learning of theology. By 1810, the Chilean government ordered schools to be established for boys and for girls in religious convents and monasteries (Labarca 1939, p. 85), but the fact that this order had to be repeated several times indicates that it was not obeyed. The priests and nuns who would teach in these schools had to pass examinations to verify their "ability to read, write and count." The exams included the use of "all kinds of letters" and "examples of the four arithmetic operations" (Labarca 1939, p. 85). When these schools were actually opened, "it would have been better if this had not happened" because of their very low quality (Labarca 1939, p. 87).

In Colombia, the education act of 1826 established that in all parishes of each province, at least one elementary school for boys should be created and, where possible, another one for girls. In these schools, the children were taught, among other subjects, how to write and read and the first rules of arithmetic (Zuluaga 1979, p. 17). In the same year, another law instituted schools for poor children, in which only Lancaster's method (described below) would be allowed. The three examples of Brazil, Chile, and Colombia are sufficient to show the difficulties the new independent nations faced in establishing a system of public elementary education.

How to establish educational systems with a small number of schools, few and poorly trained teachers, and scant resources was the main challenge facing these new nations. A widely adopted approach to these problems was Lancaster's method of mutual learning, which was used in many Latin American countries to teach children reading and elementary school mathematics, as well as in normal schools to prepare teachers who would propagate this method.

The ideological and pedagogical aspects of Lancaster's method can be found in Lancaster (1803), Hassard and Rowlinson (2002), Munévar (2010) and Neves (2000, 2007, 2009), while the mathematics prescribed for the classrooms is specified in Lancaster (1810): it was arithmetic up to the "rule of three." The method was used in many countries, including some in Latin America – namely, Argentina, Bolivia, Brazil, Chile, Colombia, Ecuador, El Salvador, Guatemala, Honduras, México, Peru, Uruguay, and Venezuela (Munévar 2010, p. 57). It was also introduced in Costa Rica in the late 1820s and used until the mid-1880s (Garcia 2002, p. 38; Barrantes and Ruiz 2000, p. 146). In fact, it was used in Latin America even before independence, for example, in Venezuela (Jáuregui 2003, p. 225), and it seems that Argentina was the first country to adopt it in 1819 when a Lancasterian school for teachers was opened in Buenos Aires (Muñoz 1918, p. 95) and the method was officially adopted by

[3] All translations from Portuguese and Spanish were made by the author of this chapter.

the recently created Universidad de Buenos Aires (Vidal and Ascolani 2009, p. 93). In 1819, 3 years before independence from Portugal, we find a Lancasterian school in Rio de Janeiro for the "disadvantaged" (Bastos 1997, p. 125) and one for the military (Neves 2007, p. 3). The method was officially adopted in 1827, but very soon there were strong objections to its use (Moacyr 1936, pp. 197, 205, 216, 252; Moises 2007, pp. 67–68).

The privately instituted *Compañia Lancasteriana* was created in Mexico in 1822 and had a great influence on Mexican education. From 1842 to 1845, the company headed the *Dirección General de Instrucción Primaria*, which was in charge of public primary education in Mexico. The company was officially extinguished only in 1890, ending its great influence on elementary education in Mexico (Solana et al. 1982).

Colombia officially adopted Lancaster's method in 1821, and the task of its diffusion fell to the normal schools, of which three were opened in 1822. In 1826, the government ordered that the method be adopted in all primary schools (Zuluaga 2001, pp. 42–43). The progressively increasing criticism of this method and discussions about the role of public education slowly eroded support for Lancaster's method; in 1844, when the reform instituted by Mariano Ospina Rodriguez became law, it was no longer used officially (Zuluaga 2001, p. 73).

In Chile, Lancaster's method was strongly supported by Bernardo O'Higgins, one of the leaders of the independence movement, and we find a Lancasterian normal school in 1821 in Santiago.

The purpose in using Lancaster's method differed between Europe and the Latin American countries. In Europe, it "consolidated itself as an autonomous and terminal instruction mode, whose purpose was the quick education of industrial workers," while in Latin America it became "the first step of the school system" (Vidal and Ascolani 2009, p. 93). This is a clear example of how ideas and concepts can be transformed in the transmission process.

The creation of normal schools was an essential part of efforts to set up public education systems in the new Latin American countries, thereby disseminating elementary education. There are conflicting claims for the creation of the first of these schools. The fact is that several, normal schools were established in the first half of the nineteenth century: 1821 in Santiago, Chile; 1822 in Lima, Peru; 1822, the *Escuela Nacional Lancasteriana* in Mexico City, operated by the *Compañia Lancasteriana*, the *Escuela Normal de Enseñanza Mutua de Oaxaca* (1824) at Oaxaca (which also used Lancaster's method), and the normal schools of Zacatecas (1825) and Guadalajara (1828); 1942, the *Escuela Normal de Preceptores* in Santiago, Chile; and 1849, the *Escuela Normal Mixta de San Luis de Potosí*; 1835, in Niterói, Brazil, which has remained continuously in operation until today, with only a few name changes.

2.2 Secondary and Postsecondary Education

We are turning now to post-elementary education; this level roughly corresponds with our present-day secondary and postsecondary education, the organization of which usually preceded that of elementary education in Latin America.

Despite many statements about the importance of elementary education for the new independent Latin American nations, these countries very soon realized the need for a professionally trained bureaucracy and military to run (and rule) them, and so, in practice, secondary and postsecondary education had priority in the distribution of scarce resources.

This section presents this story in two parts corresponding, respectively, to postsecondary and secondary education, even though it is sometimes difficult to distinguish between the two levels, as, for example, in the Jesuit *colégios* throughout Latin America. Also, some institutions like the *Instituto Nacional* in Chile changed their status over the years, initially providing several levels of instruction and, later, restricting their activities.

2.2.1 Postsecondary Education

While no university existed in Brazil at this time, 29 universities had been established in Spanish America by the beginning of the nineteenth century, according to Jílek (1984) and Roberts et al. (1996). Yet mathematics was not well implemented and cultivated there. However, mathematics obtained a stronger position in institutions for training civil and military engineers from the second half of the eighteenth century onward.

These institutions were improved and extended from the beginning of the nineteenth century. Mexico created its military school in 1823, followed the next year by the establishment of the *Colegio de Aspirantes de Marina* for the training of naval officers. In Chile, a military academy was created in 1817 but was closed a few times, and its continuous existence dates from 1842; the *Academia de Guardas Marinas* was established in 1818 (Campbell 1959, p. 355). In Colombia, the first military academy was a naval and mathematics school, opened in 1811. Argentina only established its *Colegio Militar de la Nación* in 1869, but this is beyond the scope of this section (1800–1850).

In 1808, the ruling prince of Portugal fled to Brazil because of the invasion of Portugal by Napoleon's armies, and there was the urgent need to prepare professionals both for defense and administration. The prince created schools of medicine in Salvador and in Rio de Janeiro (1808), a course on economic sciences in Rio de Janeiro (1808), and two military schools to prepare naval and army officers – the *Real Academia de Guardas-Marinha* and the *Academia Real Militar da Corte* (1810), respectively, whose mathematics courses were carefully studied by Saraiva (2007). Very soon, the military academy opened its doors to nonmilitary students. From 1858 on, it was comprised of three different courses of instruction: a theoretical course on mathematics, physics, and the natural sciences; a course on military engineering; and a course on civil engineering. The creation of the above-mentioned professional schools inaugurated the model of postsecondary education in Brazil up to the first half of the twentieth century: the establishment of isolated professional schools, following the French model. Thus, all these schools, both military and nonmilitary, were greatly influenced by the French, with its stress on the mathematical preparation of military officers or engineers. This fact had a positive effect on the dissemination of mathematics, given that some military officers and engineers, former students from these schools, became teachers in secondary or postsecondary institutions – a situation common to other Latin American countries.

The Portuguese rulers forbade the establishment of printing presses in Brazil and imposed censorship on imported books. Of course, presses were smuggled in any way and records show the existence of printing presses before 1808, the year the Portuguese ruling prince settled in Rio de Janeiro, and books were soon printed in Brazil (Silva 1981; Frieiro 1957). It seems safe to say that more mathematics books were printed in Spanish America than in Brazil (Hallewell 1982; Garcia 1998; Alcaráz 2007), even though starting in 1808, the ruling prince of Brazil made a great effort to translate mathematics and science books for use in the professional schools he had just established in the country. Saraiva (2007) informs us about the mathematics books used at the *Academia Militar* in Rio de Janeiro:

> In the first year: Lacroix's arithmetic translated by F.C.S.T. Alvim,[4] titled *Tratado de aritmetica*, and published in 1810. For algebra: Lacroix's algebra, also translated by F.C.S.T. Alvim as *Elementos de algebra* and published in 1812; Euler's *Vollständige Anleitung zur Algebra*, translated by M.F.A. Guimarães[5] under the title *Elementos de álgebra* and published in 1809; and Guimarães' *Complementos dos elementos de álgebra* de Lacroix, published in 1813. For geometry and trigonometry: Legendre's *Éléments de géométrie* translated by Guimarães and divided in two books, under the titles *Elementos de geometria* (1809) and *Tratado de trigonometria* (1809).

[4] Francisco Cordeiro da Silva Torres Alvim (1775–1856) was an engineer and a military man. He was born in Portugal and died in Brazil.

[5] Manuel Ferreira de Araújo Guimarães (1777–1838) was a Brazilian who received his postsecondary education in Portugal. He was remarkable as a translator of important mathematics works, both in Brazil and in Portugal.

In the second year: Lacroix's algebra again, in conjunction with J.V.S. Souza's[6] *Tratado Elementar de Applicação da Álgebra à Geometria*, published in 1812, and F.C.S.T. Alvim's translation of Lacroix: *Tratado Elementar de Cálculo Differencial e Cálculo Integral* in two volumes published, respectively, in 1812 and 1814. For geometry: Monge's descriptive geometry, with the Portuguese title of *Elementos de geometria descriptiva com applicações às artes* (1812).

Third year was dedicated to mechanics and hydrodynamics only. In the fourth year: for spherical trigonometry: the *Tratado de Trigonometria* of 1809, translated by Guimarães from Legendre.

At the end of the eighteenth century and the beginning of the nineteenth century, mathematics and the sciences in Chile did not fare well. *The Real Universidad de San Felipe*, created in 1747 in Santiago, had not turned out a single doctor in mathematics or the sciences until the beginning of the nineteenth century (Muñoz 1918, p. 72). In 1797, Manuel de Salas Corbalán established the *Academia de San Luis*, a fruit of the Enlightenment belief in the power of science, with the purpose of improving life in Chile through technical and scientific education (Academia de San Luis 1796, p. 596). In 1813, during the independence wars, it became the *Instituto Nacional*, initially an elementary, secondary, and postsecondary establishment until 1842, when the *Universidad de Chile* was created and the *Instituto* became strictly a preuniversity school. It was closed in 1814 during the Spanish "reconquista," part of the independence wars, and reopened in 1819. The *Instituto* was set up by joining the *Academia de San Luis* and a few religious educational institutions in Santiago with the faculty of the *Universidad de San Felipe*.

The curriculum proposed by Manuel de Salas for the *Instituto* included the following chairs: a reading and writing course, followed by Catholic religious doctrine and elementary arithmetic. It also included Latin, drawing, French and English, logic and metaphysics, and pure mathematics, but there were no professors for the created chairs, materials were not available, and "the light of illusion hid the reality of poverty" (Labarca 1939, p. 78). In fact, the chairs of higher mathematics, experimental physics, chemistry, and mineralogy were provided for on a regular basis only from 1847 onward (Labarca 1939, p. 121).

Starting with the 1820s, there was an effort to modernize education in Chile. In 1826, the French engineer Charles Ambroise Lozier,[7] hired to make a map of Chile, was chosen as director of the *Instituto*. He tried to model the school after a French *lycée* (Labarca 1939, p. 81) and designed a curriculum that emphasized modern languages and non-elementary mathematics. Unfortunately, Lozier was not successful with his reform attempts and was forced to resign. But,

> The jolt that his reforms gave the *Instituto Nacional* and the new methods that he tried to introduce made clear to the Chileans the defects that existed and indicated a course more in harmony with the educational aims that were to be sought later. (Campbell 1959, p. 359)

During his period as head of the *Instituto*, Lozier commissioned translations of the works of Lacroix, Biot, Francoeur's *Dessin linéaire et arpentage* and *Cours complet de mathématiques pures*,[8] and Leroy's *Traité de géométrie descriptive*. He also encouraged the staff of the *Instituto* to write their own textbooks (Labarca 1939, p. 82). Lozier made several important works known in Chile, among them Laplace's *Traité de Mécanique Celeste* and Lagrange's *Mécanique Analitique*, and greatly improved the Instituto's library (Maldonado 1999, p. 27).

The first attempts at the serious study of mathematics in Colombia were made by José Celestino Mutis (b. 1732, d. 1808) in the year 1761; he set up a mathematics chair in Bogotá, more precisely in the *Colegio Mayor del Rosario*, a university founded in 1653 (Sánchez 2012, pp. 209–210). He used Benito Bails' works *Elementos de Matemática* (1772–1783) and *Principios de Matemática* (1776) and Christian Wolff's *Elementa Matheseos Universae* (1731).

[6] José Victorino dos Santos e Souza (?-1852)

[7] Charles Ambroise Lozier (1784–1864) was a French engineer who led a very adventurous life (see Campbell 1959, pp. 357–359; Marcelin 2001).

[8] Translated by Andrés Antonio Gorbea (1792–1852) and published between 1833 and 1855

In 1819, after independence, the state took charge of all levels of instruction and issued in 1826 the first law regulating public instruction in the new republic; here, we see the establishment of three central and three regional universities. Even though the original plan for the universities contemplated the study of the natural sciences, this remained an ideal. The stress was on the humanities and law studies. In 1841, a new reform created a course for the physical and mathematical sciences, but, once more, this did not become a reality. The proposed curriculum included algebra, differential and integral calculus, applications of algebra to geometry, spherical trigonometry, descriptive geometry, and the study of conic sections. In 1848, the government set up the course for the natural and physical sciences and mathematics in the three regional universities. Foreign scientists were hired, among them the French mathematician Aimé Bergeron (Sánchez 2012, pp. 113–114).

2.2.2 Secondary Education

From 1759, when the Jesuits were expelled from the Portuguese empire, until the early 1830s, secondary education in Brazil was basically provided by *aulas régias*, which had been created by Pombal in 1759 and which were the first attempt at a public education system in the Portuguese empire, particularly in Brazil. An *aula régia* offered lectures on specific isolated subjects, mainly in the humanities, but a few of them taught drawing, mathematics, or modern languages. In Brazil, they lasted until 1834 (Cardoso 2002, 2004). Shortly before 1837, secondary-level institutions had been established in some of the provinces, followed by the creation of *Colégio Pedro II* in Rio de Janeiro in 1837, a very important event for education in Brazil. With this there was, for the first time in Brazil,

> (…) a gradual and integral plan of studies for secondary school, in which the students were promoted year by year, and not, as before, by subjects and earned, at the end of the course, a bachelor's degree, which enabled them [from 1843 on] to enroll in post secondary establishments, without the need to pass entrance examinations. In this plan of studies, that followed French models, the emphasis was on the classics and the humanities. (Miorim 1988, p. 87)

Colégio Pedro II was conceived as a model for secondary establishments in the provinces. As an inducement, students from schools that followed *Colégio Pedro II*'s curriculum were exempted from entrance examinations to postsecondary schools, which were regulated by the central government. In practice, excepting the military schools, these postsecondary establishments were law and medicine schools.

The mathematics programs of *Colégio Pedro* II from 1838 on have been analyzed in Beltrame 2000 and Gussi 2011. The length of the course varied from a minimum of five to a maximum of 8 years,[9] and mathematics was not always taught in each school year. Moreover, the teaching of mathematics was strictly divided: 1 year for arithmetic, another for geometry and trigonometry, and yet another for algebra. The distribution of mathematics in the curriculum varied throughout the years, as shown by Carvalho and Dassie (2012, p. 513). We can roughly describe the very stable corpus of secondary school mathematics which took shape in Brazil after the creation of *Colégio Pedro II* as follows (Beltrame 2000):

> Arithmetic – operations with numbers (including roots); *números complexos* ("compound numbers");[10] proportions; ratios;[11] arithmetic and geometric progressions; logarithms; commercial mathematics (simple and compound interests, "rule of three," etc.); the decimal metric system.[12]

[9] In the second case, the two or three last years were required only of students planning to enter engineering schools.

[10] For example, 1 h and 23 min is a *número complexo* (compound number) as opposed to 83 min, a *número incomplexo*. In nineteenth-century Brazil school mathematics, this term was used to name quantities with non-decimal subunits, like time (seconds, minutes, etc.), money (pence, shillings, pounds), and lengths (inches, feet, etc.). As long as the metrical system had not yet been generally accepted, operating with such quantities constituted a major teaching subject. There was no confusion with the numbers of the form $a + ib$ because they were then called *números imaginários*.

[11] The study of proportions and ratios was purely arithmetical and had nothing to do with Euclid's treatment of these concepts.

[12] The decimal metric system was officially adopted in Brazil by law in 1862, but only in 1872 did this law become effective.

Algebra – algebra as generalized arithmetic; linear and quadratic equations with applications.

Geometry – elementary plane and space geometry: relative position of straight lines; triangles; quadrilaterals; polygons; the circle; similarity of plane figures; relative position of straight lines and planes; polyhedra; spheres, cones, and cylinders; areas and volumes.

Trigonometry – proofs of formulae; construction of trigonometric tables; the theory of the triangles.

The establishment of this corpus of secondary school mathematics is clearly a transfer process from French models, the *lycées*.

In Chile, during 1832, at the government's request, Ventura Marin, Manuel Montt, and Ventura Godoy proposed a curricular reform for the *Instituto Nacional*. This reform deliberately adopted the conservative viewpoint of creating a national elite, "at first in Santiago and, later on, in the provinces. This elite should be educated by humanist and scientific studies" and its role in society was "to prepare [the] lawyers required to the country's development whenever needed" (Cruz 2003, p. 44). It exhibited the most conservative version of the Enlightenment, with the humanities at the very center of the curriculum and directly useful knowledge relegated to a secondary role (Cruz 2003, p. 45).

One important feature of the 1832 reform was its requirement that secondary school students follow a mandatory curriculum, progressing from year to year. Until then, at the *Instituto*, as in other Latin American countries – for example, Brazil (Carvalho and Dassie 2012, p. 504) – the student could enroll only for the specific subjects he wanted to master in order to be admitted to his intended postsecondary establishment. In the proposed curriculum, mathematics was taught only in the first 2 of the 6 years of study, and the same number of hours was given to Greek as to mathematics (Cruz 2003, p. 45). After 6 years in the humanities course, the student could proceed to one of the following courses: law, medicine, mathematics, and theology.

2.2.3 Some Elementary and Secondary-Level Mathematics Textbooks Used in Latin America

According to Soares (2011, p. 5), the first geometry textbook adopted at *Colégio Pedro II* was Lacroix's *Elementos de Geometria*. The first arithmetic book written in Brazil was the *Compêndio de Aritmética* by Cândido Baptista de Oliveira (1801–1865), published in 1832, just 5 years before the creation of *Colégio Pedro II*, while the first geometry textbook written by a Brazilian was Francisco Villela Barbosa's *Elementos de geometria* (1815) (Silva 2000, p. 124). A very successful geometry textbook was Christiano Ottoni's *Elementos de geometria e trigonometria rectilinea* (1842). The half century (1800–1850) closed with *Elementos de aritmética* by José Joaquim de Ávila (1850).

According to Moctezuma (n.d.), until 1850 there were no elementary arithmetic textbooks for children in Mexico. The teacher who knew some mathematics simply used Benito Bails' book,[13] the arithmetic of Thomas More, and *Tablas para los niños que empiezan a contar* by Rafael Ximeno. The textbooks used both at *Real Seminario de Minas* and *Real Academia de San Carlos* were at first the *Principios de Matemáticas* of Benito Bails and, later and less influential, José Mariano Vallejo y Ortega's *Compendio de Matemáticas*[14] (Garcia 1998, p. 32).

The first mathematics schoolbook published in Venezuela was *Lecciones de Aritmética* by Lucas María Romero y Serrano, printed in 1826 by Tomás Antero in Caracas (Beyer 2006). Starting in 1830, the process of dismembering Greater Colombia almost halted the production of textbooks. Nevertheless, in 1831, *Lecciones de aritmética razonada (…) para la enseñanza de los niños* by Domingo Navas Spínola was issued by the Imprenta de Fermín Romero (Beyer 2006, p. 85). A few years later, in 1839, we have the first edition in Venezuela of Rebollo y Morales' translation of Lacroix's arithmetic

[13] The title given by Moctezuma to Bails' book, *Compendio matemático*, does not correspond with any of Bails' published works.

[14] The *Compendio*, strongly influenced by Cauchy's modern ideas about analysis, was studied by Astudillo (2005).

(Beyer 2006, p. 90). Next, as Beyer (2006, p. 96) mentions, were a booklet published in 1840, *Conocimientos de las definiciones de las tablas de sumar, restar, multiplicar y partir*, author unknown; *Compendio de Aritmética Razonada following Lacroix,* by Martín Chiquito (1842)*;* Manuel María Echeandía's *Compendio de Aritmética Razonada* (1843), with many editions; and *Elementos de Aritmética Teórica y Práctica* (1844) by Juan Bautista Montenegro, also with many editions.

An interesting fact about elementary mathematics education in Spanish America were the mathematics *catechisms* written by Spanish liberals who went into self-exile in London. The mathematics *catechisms* were written for use in the newly liberated countries of Spanish America, in which there was a great scarcity of texts for elementary education (Ausejo and Hormigón 1999; Beyer 2009). The catechisms taught mathematics using questions and answers, just as the religious catechisms did very commonly in the nineteenth century. The two major writers of these mathematics catechisms were José de Urcullu and José Núñez de Arenas, who used Bails' *Princípios de Matemática*. Beyer (2009) describes 22 mathematics catechisms, nine published between 1825 and 1849, some of which had surprisingly long lives. *Compendio de aritmética razonada*, by Manuel Maria Echeandia, was still published in 1926, 82 years after its first edition. Beyer also mentions the famous *Aritmética* of Juan José de Padilla, published very early in 1732 in Guatemala, in the form of a catechism (Beyer 2006, p. 80). So far, the author of this chapter has not found any evidence of the use of mathematics catechisms in Brazil in the nineteenth century.

3 Second Period: Consolidation and Expansion – 1850–1900

By the 1850s, it was generally accepted that most of Latin America was composed of politically independent countries (notwithstanding a few European colonies); Latin America had experienced, however, short-lived European adventures, like the French occupation of Mexico (1861–1867) as well as repeated covert or open interventions by the United States. From 1850 to 1900, there were some regional wars, such as those involving Chile, Peru, and Bolivia (1879–1883) and Argentina, Brazil, and Uruguay against Paraguay (1879–1883). In this period, most Latin American countries had already established institutions that prepared the bureaucracy, administrators, and military needed to consolidate their rule and exert control over their countries. Moreover, they had secondary-level institutions to prepare students for further studies. In most cases, the elementary school systems were not the priority, even though they were slowly becoming more inclusive.

While the section of this chapter devoted to the first historical period was divided by themes, it seems more natural now in dealing with the second historical period to devote subsections to separate countries, which from 1850 on showed stronger national identities and had already established their education systems along more specific lines. A few themes common to the Latin American countries are also treated.

3.1 Chile

Around 1850, Chile shared a trait with the other Latin American countries: its education system still needed much improvement, relative to both inclusiveness and quality. For example, only between 1847 and 1850 were the chairs of higher mathematics (*matemáticas superiores*), experimental physics, chemistry, and mineralogy incorporated into the curriculum of the *Instituto Nacional* (Labarca 1939, p. 119).

In the preceding part, we discussed the 1832 reform which stressed the humanities and the classics. This emphasis was repeated in the reform decreed in 1843 (Cruz 2003, p. 46), which we now discuss.

Following a memoir written in 1842 by scientist and educator Ignacio Domeyko,[15] the government decreed in early 1843 a reform of Chilean secondary education, known as *Plan de estudios humanistas* (Humanistic studies plan), which remained in force until 1876.

Plan de estudios humanisticos was addressed to secondary and postsecondary education, with the clear purpose of preparing an intellectual elite for the country. It stressed the humanities and the classics, and Latin was the only subject studied in each school year. It also created, in a very ambitious way, secondary education establishments in all the provinces. According to Domeyko, one had to institute

> [A]n independent school education that should contain all elements necessary for the integral formation of a person. This instruction has value in itself, the benefits it provides, not as a preparation for jobs or for the professions. (Cruz 2003, p. 17)

Domeyko believed that mathematics should be present in secondary school education not for its helpfulness but for its logical thinking, useful in all walks of life, and because it promotes the discipline of the mind. The reform also stipulated that secondary school students follow a mandatory curriculum and attend classes regularly, progressing yearly in the curriculum.

Arithmetic and parts of algebra were taught in the first curriculum year. In the second year, students studied algebra, geometry, and trigonometry. The remaining years had no mathematics. Shortly thereafter, the government issued a new law, creating a secondary school mathematics course of 4 years for those planning to study topography or advanced mathematics, with mathematics in every school year. This course did not have many students because most young men preferred the humanities course, which gave access to jobs, and the law course. Also, the problem of the entering students' poor mathematical knowledge was ever present (Cruz 2003, p. 54). In 1865, the curriculum was modified to introduce a living modern language and more science and mathematics, the last two present in the first 3 of the 5 school years.

The reform of 1843 shaped secondary education in Chile. It did not concern itself with primary education, which was in a very bad state and constituted a major obstacle to good secondary education, but elementary education could hardly be improved anyway given the complete lack of qualified teachers. In fact, most elementary school teachers knew only how to read and write. In 1841, a law was passed providing funds to establish a normal school for prospective male teachers in Santiago, *Escuela Normal de Preceptores*. This was done in 1842 (Campbell 1959, p. 371), followed in 1854 by a normal school for prospective female teachers open only occasionally until 1880, when it was "recreated." The teachers formed by these normal schools spread out thinly all over the country, staffing the growing web of elementary schools.

Instituto Pedagogico was created in 1889 to prepare teachers for secondary school. Until then, these were improvised. Starting in that year, Chile received several missions of German teachers, who taught both at *Instituto Pedagogico* and at several normal schools. This German influence replaced part of the preceding French influence.

Chile distinguishes itself with the adoption of a national education act (*Lei organica de instrucción pública*) rather early in 1860. A few years later, in 1867, Mexico's president Benito Juárez decreed the *Ley de Educacion Pública*, which organized and consolidated a system of public education regulated by the state in Mexico. By contrast, nationwide educational regulations started rather late in Brazil in the 1930s and were completed only in 1961.

We will not follow the educational reforms in Chile after 1876, the year in which *Plan de estudios humanista* was substituted by another reform. Instead, we will briefly consider the liberal educational reform in Costa Rica in the 1880s.

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[15] Domeyko (1808–1889) was born in present-day Belarus, migrated to Chile in 1838, and died there in 1889. He studied engineering at *École des Mines* in Paris and had a strong interest in mathematics. In Chile, he made important contributions to mineralogy and played a major role as a teacher and educator. In 1847, he was made a professor at *Instituto Nacional*.

3.2 Costa Rica

In the 1880s, the educational situation in Costa Rica was poor. The central government had almost no control of education, in which the Catholic Church played a major role; moreover, the economic situation prevented any national investments in education. In elementary education, Lancaster's method was widely used till 1850 and classrooms grouped together students of different ages and levels, a common practice throughout Latin America at the time. There were only four secondary schools in the whole country and one university, *Universidad de Santo Tomás*, created in 1843 and at a very low point in its history (Ruiz 1995, p. 148).

One important educational reform was instituted in 1885–1886 by Mauro Fernández, minister for public instruction. It involved the reorganization of elementary education, the establishment of public secondary schools, the closing of *Universidad de Santo Tomás* in 1888, and the prohibition of religious instruction in all public schools. Fernández was familiar with the ideas of Horace Mann, Pestalozzi, Fröebel, Herbart, Jules Ferry, Andrés Bello, and Domingo Faustino Sarmiento.[16] He drew inspiration from Ferry's laws of 1881 and 1882 in France and was strongly influenced by the reform attempted – unsuccessfully – by Julián Volio in 1867 (Ruiz 1995, p. 148). Fernández's reform is noteworthy because it encompassed all levels of education.

The elementary school mathematics programs instituted by the reform were made up of arithmetic and geometry. In arithmetic, one studied the operations with natural numbers, divisibility, the greatest common divisor, fractions and measures of length, volume and time, and the "rule of three." Geometry included straight lines and curves, plane geometric figures, space figures, perimeters, and areas and volumes. Teaching was supposed to be intuitive, without overemphasis on formulas, and related to the other curriculum subjects. Elementary school lasted 4 years, and the different topics of the program were revisited several times, with progressive extensions (Ruiz 1995, pp. 39–40).

In 1892, secondary education was reorganized to consist of 6 years of study, divided into a preparatory course, followed by 5 years of secondary school. The course had two tracks, one directed to science and the other to the humanities (Ruiz 1995, p. 155). The mathematics program stipulated by the reform consisted of the following topics (Ruiz 1995, pp. 39–40):

> In the preparatory year:
>
> For arithmetic: various number systems, fractions, divisibility, the decimal system of measures and other measure systems used in Costa Rica, powers, square and cube roots, logarithms, arithmetic and geometric progressions, rule of three, and applications of arithmetic problems.
>
> For geometry: a review of the geometry taught in elementary school, congruence, similarity, parallel and perpendicular straight lines.
>
> First year: more rigorous arithmetic with theorems – elementary number theory, including the fundamental theorem of arithmetic, fractions, radicals, numerical proportionality, the metric decimal system, rule of three, arithmetic and geometric progressions, logarithms, elements of financial arithmetic.
>
> Second year: algebra – powers, radicals, algebraic expressions, integers, polynomials (operations, factorization, greatest common divisor, and least common multiple of polynomials), algebraic fractions, first and second degree equations, systems of first degree equations, combinatorial analysis, inequalities, logarithms, determinants, continuous fractions.
>
> Third year: deductive Euclidean geometry.
>
> Fourth year: trigonometry, more precisely the trigonometry of acute angles, trigonometric tables, resolution of triangles, applications, spherical trigonometry.
>
> Fifth year: projective geometry.

[16] Domingo Faustino Sarmiento (1811–1888) was a very influential Argentinean intellectual who was president of his country from 1868 till 1874.

Ruiz (1995, p. 40) states that there is not much information about the mathematics textbooks used in Costa Rica in the nineteenth century, only mentioning *Geometría para niños* written by Acisclo Fernandez Vallin y Bustillo and *Elementos de Geometría* by Giró y Miró, used in the first and last years of elementary school, respectively. In these last years, one also used *Aritmética Primaria* de Robinson and books called *Aritmética Comercial* written by one of the following authors: José Urcullú, González Lineros, or Molina Rojo. For geometry, one used, to a lesser extent, the geometry texts of Guim or López Catalán. Besides these, the following books were also used: *Curso superior de Aritmética y Geometría* by Vintéjoux; *Aritmética*, *Tratado de Geometría Elemental*, and *Trigonometría*, all three by Cortázar; *Tablas de logaritmos* and *Tablas Trigonométricas*, both by Vicente Vásquez Queipo; and *Eléments de Géométrie descriptive* by Dufailly. In addition to these books were others recommended as reference works: *Ejercicios prácticos de Aritmética y Geometría* by Terry; *Arithmétique* by Leyssenne; the arithmetic textbooks of Lacroix, Ferry, and Sánchez Vidal; the geometry books of Combette and Rouché; and *Cours de Trigonometrie* by Rebière and *Tratado de Geometría descriptiva* by Leroy. In Costa Rica, as in other Latin American countries, one needed books to teach the use of the decimal metric system. Among them was *Sistema Métrico, demostrado según el aparato del método Level*, translated by A. Quirós and published in 1886 (Ruiz 1995, p. 40).

3.3 Colombia

In Colombia, a decisive event for mathematics was the establishment of the military academy, *Colegio Militar*, in 1847, in which a leading role was played by Lino de Pombo (1797–1862). It was modeled after the French *École Polytechnique* and the American military academy at West Point, with mathematics taught in the first 3 years. It seems (Sánchez 2012, p. 115) that the most advanced mathematics course in the curriculum, differential and integral calculus, was taught for the first time in 1851 by Bergeron (Albis-González and Sánchez 1999). *Colegio Militar* was closed in 1854 and reopened in 1866, this time as *Colegio Militar y Escuela Politécnica*. The following year, the national university was reorganized and a school of engineering was established, with the faculty, students, and curriculum of the *Colegio* (Sánchez 2012, p. 116). "[T]he *Colegio* planted the seed for the formation of professional engineers with a firm grounding of mathematical knowledge" and from the *Colegio* came "the first Colombian mathematics textbooks for post secondary education" (Sánchez 2012, p. 116).

French influence at the *Colegio* and at other institutions is very clear:

> The institutions, supported by the government, hired professors in France and imported with them the curricula, the teaching methodology and French texts (particularly the representative texts for calculus: Lacroix, Boucharlat, Sonnet, Sturm, Bertrand, Jordan, Appell, Laurent and Humbert). The training of the teachers and mathematically minded engineers that leaded this process during four or five generations was basically French. The programs and the teaching of mathematics (at least in theory, maybe not in practice) were all French inspired. (Arboleda 2002, p. 6)

Following discussions on the position of mathematics in an engineering school, a mathematics institute, *Instituto de Matemáticas*, was created in 1888 with two different schools, one of mathematics and the other of engineering. If a student passed all the mathematics courses and presented a dissertation, he could graduate with a degree of *Profesor en Ciencias Matemáticas*.

The role played by the engineering school in Colombian mathematics may be summed up as follows:

> Even though our mathematical engineers had a great enthusiasm for mathematical studies, they were not aware of the great advances of this field in that century [19th century]. They knew well Euclidean and analytic geometry, elementary algebra and the elements of differential and integral calculus. (Sánchez 2012, p. 122)

That is, they were skilled in basic mathematics, not trained as researchers in the field, but could – and did – contribute to the development of mathematics through their teaching and positive attitude toward mathematics. This state of affairs coincides with what happened in other Latin American countries. For Brazil, this is evident by the mathematical production of "amateurs," usually engineers or military with a good basic mathematical knowledge, as shown by Dassie and Carvalho (2005), C.M. Silva (2006), and in the many papers on mathematical subjects published in *Revista da Escola Polytechnica* (Silva 2006).

The situation was similar in many Latin American countries during the nineteenth century. Most, if not all, of their engineering schools were modeled on French professional schools, which clearly emphasized mathematics and helped to promote mathematics, as in Colombia. Some of these engineering schools date from the first years of the nineteenth century, as in the case of Rio de Janeiro, but their number increased considerably in the second half of that century. In Chile, *Escuela de Minas de Copiacó*, a university-level school for mining engineers, was created in 1857. Its curriculum included, typically, the usual mathematics for an engineering school. It was much influenced by Domeyko's educational and scientific ideas. Later, in 1887, *Escuela de Minas de la Serena* was established, also influenced by Domeyko. In Brazil, the first institution specifically dedicated to mining engineering was established in 1876. Its first director and organizer was the French mineralogist Claude Henri Gorceix, a graduate of *École Normale Supérieure*, who modeled *Escola de Minas* after the equivalent French engineering schools, using a strong mathematical basis.

In Colombia, among the books used at *Colegio Militar*, we can mention Lino de Pombo's *Lecciones de Jeometría Analítica* (1850) and *Lecciones de aritmética y áljebra* (1858) and Indalécio Liévano's *Tratado de aritmética* (1856) and *Tratado de áljebra* (1875). The *Tratado de aritmética*, certainly the most important mathematical work conceived in Colombia during the nineteenth century, contains Liévano's

> [T]heory of the incommensurables which, though not a complete and polished theory like the ones of Weierstrass, Dedekind or Cantor has the merit of being published before them. (Sánchez 2012, pp. 117–118)

Among the textbooks used at *Instituto de matematica* were Sturm's *Cours d'analyse* and the Spanish translation of Sonnet et Frontera's *Éléments de géométrie analytique*.

3.4 Brazil

From 1850 on, the educational system of Brazil began to assume a more organized shape. We have the professional and military schools, *Colégio Pedro II* and similar establishments in the provinces, private schools for children of the rich (often run by foreigners), and growth in the number of normal schools. All of these contributed to improving the mathematical level of general education and to establishing a new profession or, at least, activity: the mathematics teacher (Soares 2007). During this period, the country followed the French model for secondary and postsecondary education, while English and American influences were progressively more prominent in primary education (Gomes 2011; Lorenz and Vechia 2005; Neves 2006, 2007, 2008, 2009; Valente 2012).

In the state of São Paulo, directly after the republic was proclaimed in 1889, a state educational reform augmented the curriculum of the city's normal school to 4 years and established a laboratory school. This reform was extended to the whole state in 1892 (Vidal and Ascolani 2009, p. 122). A very important development was the establishment of the *grupos escolares*, which were created first in the same state in 1893 and then slowly spread all over the country. Until then, Brazilian elementary school classrooms usually contained students of widely different ages and knowledge levels. A *grupo escolar* was a school with students grouped in classrooms by knowledge level who had to progress yearly through a regular curriculum. The classrooms – each one in the charge of a single teacher – and the administrative facilities of the school were housed in a specifically designed building. That is, *grupos escolares* were elementary schools as we conceive of them today – grade schools. They embodied the positivist and republican ideas prevailing at the time in Brazil and were an important means for making the public elementary school system more inclusive and for molding and disciplining citizens for the new modern republican society (Vidal 2006). They enforced a strict discipline that stressed punctuality, cleanness, "moral virtues," and "civic values" (Souza 2004, p. 127).

In 1889, after the republic was proclaimed, the government created the ministry of education. Its head, Benjamin Constant, a leading republican and a strong defender of Auguste Comte's ideas, decreed an educational reform which instituted a new encyclopedic curriculum for *Colégio Pedro II* following positivist lines and introduced the teaching of infinitesimal calculus in secondary education. With this reform, the curriculum of *Colégio Pedro* II reached truly encyclopedic dimensions. In mathematics, advanced subjects were introduced which, at least theoretically, would have received a slightly more than perfunctory treatment, like differential and integral calculus "restricted to the theories necessary for the study of general mechanics" (Carvalho 1996, p. 65), with six weekly hours in the third curriculum year. Calculus was revised in the fifth, sixth, and seventh years. In 1895,[17] calculus was studied in the fourth year, together with algebra as follows:

> Algebra: The study of polynomial equations $Ax^{m} + Bx^{(m-1)} + \dots Tx + U = 0$, including the number of roots; decomposition in first degree factors; relationship between the roots and the coefficients of the equations; limits of the roots.
>
> Differential and integral calculus: definitions of the derivative and of differentials; differentiation rules for functions of the form y=f(x); definition of the integral; tables of integrals; methods of integrations; easy applications. [The textbook mentioned is Sonnet's *Premiers éléments du calcul infinitesimal*].

There was widespread resistance to the inclusion of calculus in the programs because of the teachers' lack of preparation, and the ministry was forced to withdraw the subject from the curriculum in 1890. Only with the great secondary school reform of 1931 did calculus return to secondary schools in Brazil.

Auguste Comte's positivism had a considerable impact in Latin America, particularly among military personnel and engineers. Comte provided an alternative to the colonial heritage, a basis upon which to build national projects in the new nations. Comte's belief in progress without ruptures, through gradual changes, was very well received by the new elites who wanted this progress, but without the need for significant social or political changes. Positivism was particularly important among the military in Brazil, who fashioned their mathematics along Comte's ideas (Silva 1999), even though there were divergent views on the value of Comte's mathematics.[18]

A development specific to Brazil in the second half of the nineteenth century was the activity of the "brummers," Prussian mercenaries who were hired by the Brazilian empire to fight in the war against Rosas of Argentina (the Platine war, 1851–1852). After the war, around 1,800 brummers stayed in southernmost Brazil, immediately became leaders of the German immigrant community, and "stimulated material and cultural growth." By 1870, more than half of the teachers in the ethnic German schools in Brazil's southernmost states were brummers (Kreutz 2000, pp. 163–164). They were very active in the process of establishing the ethnic German school system in southern Brazil, the most organized of those created by German immigrants in Argentina, Brazil, and Chile. The school system produced extensive pedagogical publications, among them a unique periodical dedicated to elementary school textbooks (Kreutz 2007). This system was often in conflict with the church in Brazil because most of the teachers in the German schools were Protestants and the official religion of the Brazilian empire was Catholicism. After the republic, this system remained very active, but it eventually clashed with the unifying and nation-building drive of the central government and was dismantled in the late 1930s, when all German schools were required to adhere to the official curricula prescribed

[17] The program for 1895 has the first mention of the function concept in secondary school mathematics in Brazil.

[18] Otto de Alencar, a pioneer Brazilian mathematician, was the first in Brazil to criticize Comte's mathematics in two papers (Silva 1995).

by the ministry of education (Fonseca and Tambara 2012; Kreutz 2000, 2005; Marques 2010; Schubring 2003, 2004).

What mathematics books were published in Brazil in the second half of the nineteenth century? First were Christiano Ottoni's *Elementos de arithmetica* and *Elementos de algebra,* both in 1852. Other noteworthy textbooks published during this period were Antonio Trajano's *Arithmetica elementar ilustrada* (1879), which reached 136 editions by 1958, and his *Arithmetica progressiva* (1880). The last algebra textbook published in this period seems to be Antonio Gabriel de Moraes Rego's *Elementos de álgebra ou cálculo das funcções directas* (1886). In geometry, among others, we have Thimoteo Pereira's *Curso de geometria* (1888). Nineteenth-century Brazilian mathematics textbooks were also studied by Valente (1999), Beltrame (2000) and Costa (2000).

In Brazil, as stressed by Haidar (1972) and Zotti (2005), a great problem in secondary education in the nineteenth century – in fact, until 1931 – was the provision that students need not follow a regular secondary school course. They could present themselves at *Colégio Pedro II* or other accredited secondary schools in the provinces and take examinations in each subject until they had completed the requirements to enter postsecondary establishments. These "partial exams" foiled repeated attempts to make students complete the full secondary school course (Valente 2012, p. 61). Several reforms in the second half of the nineteenth century and the first half of the twentieth century attempted to make course attendance mandatory but failed systematically.

3.5 Winds of Change, Normal Schools, Intuitive Learning, and Object Lessons

Toward the end of the nineteenth century, new pedagogical ideas reached Latin America. Most often, they were first put into practice in the normal schools, which kept increasing in number, as several governments tried to consolidate and expand their respective school systems. Particularly influential were Pestalozzi's ideas which emphasized intuition mainly through the "object lessons" teaching methodology. In elementary school mathematics, "object lessons" were particularly innovative in geometry, stressing the handling of and interactions with actual solids.

Intuitive learning was influential in Europe (Shelbauer n.d.; Valdemarin 2000). For example, Felix Klein strongly advocated that the high school course on deductive geometry should be preceded by a course on "object lessons" (Carvalho 2006, pp. 74–75). We can surmise that ongoing discussions in Brazil's political and educational circles on the role of intuitive learning in school modernization (as reported by Schelbauer n.d.) also occurred in other Latin American countries. These discussions refer to several "object lessons" manuals, among them *Lições de cousas* by Charles Saffray, published in Portugal in 1908; *Plan d'études et leçons de choses* by Jules Paroz (1875); and *Exercises et travaux pour les enfants selon la méthode et les procedés de Pestalozzi et de Froebel* by Fanny Ch. Delon and M. Delon (1892 and 1913) (Valdemarin 2000).

In Latin America, two particularly influential books on object lessons were *Primary object lessons—manual for teachers and parents* by Norman Allison Calkins (1861) and *Lessons on objects, graduated series designed for children between the ages of six and fourteen years*: *containing also information on common objects* written by Edward Austin Sheldon (1863). Gvirtz (2000, pp. 182–183) mentions two editions of Calkins in Argentina, in 1871 and 1872, used officially in the city of Buenos Aires during the period 1872–1887 (Brafman 2000, p. 183). Calkins' book was translated into Portuguese in 1886 (Auras 2003; Gomes 2011) and was also used in Uruguay, where it was translated and published in 1872, and in Chile.

A very important event for education in Mexico was the creation, in 1867, of *Escuela Nacional Preparatoria*, the first senior high school in Mexico, with a curriculum set up following the ideas of Comte's positivism. It was free and open to all Mexicans. Its establishment was a result of the *Ley* *organica de instrucción pública del Distrito Federal* (Law of Public Instruction of the Federal District) of the same year, which had a strong anticlerical view. In 1887, *Escuela Normal para Profesores de Instrucción Primaria* was created to prepare teachers for elementary education.

As part of the increasing number of normal schools all over Latin America, the first normal school in Argentina was established in 1870, *Escuela Normal de Paraná*, as part of Sarmiento's political and educational ideas. Its first director and teachers were North Americans, and its textbooks were translations of American texts. It followed American pedagogical ideas with a very strict discipline. Students from all provinces in Argentina were sent to study at *Escuela Normal de Paraná*, which was indeed a model for other normal schools created in Argentina. With a course lasting 4 years, it accepted students of both sexes and boasted a laboratory elementary school (Puiggros 2006).

In Argentina, the school system was organized starting in the 1850s and 1860s, with four types of elementary education: private, municipal, rural, and that provided by *Sociedad de Beneficencia*, created in 1823 which provided elementary education for women (Vidal and Ascolani 2009, p. 118). After 1852, partially because of the ideas of Domingo F. Sarmiento, there was an attempt to unify the different types of education and make the elementary public school system more inclusive. As a result of these educational concerns, the first Argentinean pedagogical journal was created in 1858, *Anales de la educación común* (Vidal and Ascolani 2009, p. 118), which published Calkins' manual in a series from 1869 till 1871. Argentina, which by the end of the nineteenth century had 40 normal schools spread over the country (Vidal and Ascolani 2009, p. 122), was able to constitute a lay and modern normal school system which provided teachers who helped the country reach a high degree of literacy by the early twentieth century.

The role played by normal schools in modernizing school mathematics in Latin America cannot be overlooked. Their purpose was to replace improvised elementary school teachers, trained by practice, with academically trained elementary school teachers.[19] They were essential to the transition from an instrumental school based on Lancaster's ideas to one founded upon Pestalozzi's educational views, as shown by Zuluaga (2001) in the case of Colombia. In addition, the normal schools were important for the professionalization of elementary school teachers, while the corresponding professionalization of secondary school mathematics teachers usually came later, although in Brazil, it did not arrive until the twentieth century.

American influence on normal schools and elementary education was not restricted to Argentina, as shown by Valente (2012) for Brazil. As the nineteenth century ended, the ideas of Dewey, Francis Wayland Parker, and other American educators and psychologists became increasingly known in Latin America. Because of the influence of American elementary school mathematics textbooks, we see the appearance of books specifically conceived for the elementary school, not watered-down versions of secondary school texts (Valente 2012, p. 65).

4 Third Period: Challenges and Innovation – 1900–1960s

Starting in the early twentieth century, most Latin American countries underwent modernization of their education systems. The idea that elementary education should be made inclusive became more and more accepted, and there were genuine attempts to broaden the purposes of secondary education, taking into account that not all of its students would proceed to postsecondary academic training. The liberal nineteenth-century ideal of universal education was present in many reforms. The overall presence of French influence diminished as the nineteenth century ended, and American educational ideas came increasingly to the fore, particularly in elementary and secondary education. Of course, each nation found its own path, according to its educational tradition, history, and culture.

[19] In fact, the two types of teachers coexisted for a long time in Latin America.

The twentieth century saw the beginnings of industrialization in several Latin American countries, principally as a result of World War I. With this came an increase in urban populations, the growth of the middle classes, and, in some countries, mass immigration, particularly from Italy, Japan, and Portugal, but also from Germany and France. The transition from an agrarian society, based on the domination of the greater portion of the population by the landed gentry associated with traditional political power elites, to an urban diversified society in which the underprivileged struggled to have their voices heard and be granted rights was not easy. In many countries, this situation came to head in the 1920s and 1930s, with different results from country to country. We illustrate the events of the troubled 1930s with two examples of reforms imposed "from above" by the strong centralizing governments of Mexico and Brazil.

4.1 Mexico's "Socialist Education Reform"

Because of World War I (1914–1918), both Mexico and Brazil made significant steps toward industrialization, which were soon hampered by the Great Depression, during which Europe and the United States stopped buying goods from Mexico and Brazil and credit was very limited. Mexico was also suffering the aftermath of the Mexican Revolution, which began in 1910 to depose Porfirio Díaz and became a violent civil war, with many strands of anarchism, populism, and socialism opposing the conservatives and the church.[20] The Mexican Revolution had a major impact on Mexican society, and some historians claim it really ended only in 1940 (Knight 1986).

Lázaro Cárdenas was president of Mexico from 1934 until 1940, the final period of the Mexican Revolution. As part of his sociopolitical ideas to reform Mexican society, he promulgated an educational reform, the *Reforma de educación socialista*, which he viewed as essential to the success of his project for Mexico. The reform centered less on the specific contents of traditional school subjects and more on the ideological ideas schools ought to inculcate in its students (Niebla 1985).

This reform was polemic and divisive in Mexican society, but nevertheless vital to the construction of modern México, and it has great historical value (Guevara 1985, p. 9). It was definitely a secular reform, with the aim of expelling the church from the educational system. It subjected private schools to the officially decreed curriculum. As part of the reform movement, we can highlight the creation of *Instituto Politécnico Nacional* (1936) and the establishment of vocational schools, to incorporate young men and women into Mexico's workforce. However, the reform did not have a great impact on the mathematics content of elementary and secondary schools. In postsecondary education, the government stressed the sciences and technology, attempting to modify the predominantly humanistic profile of Mexican postsecondary institutions. Toward this end, in 1938, the mathematics and the physics institutes of UNAM (*Universidad Nacional Autónoma de México*) were created.

4.2 Educational Reforms in Brazil in the Period 1930–1940

In Brazil, at the turn of the twentieth century, most of the innovations introduced in the mathematics curriculum by the reform instituted directly after the victorious republican movement of 1889 had receded. Calculus was no longer a part of the secondary school curriculum. Indeed, from 1900 to 1920s, the curricula were relatively stable. The secondary educational reforms instituted by the central government concerned themselves with the roles of public and private education, lay and

[20] A strong portrait of Mexico in the 1930s with its anticlericalism is painted in Graham Greene's *The Power and the Glory* (1940).

confessional teaching, and the permanent problem of secondary education in Brazil since the 1820s, namely, that students did not need to attend a regular secondary school. They could present themselves to take, one by one and in any order, the examinations in the subjects required for admission to postsecondary establishments. The high school mathematics curricula underwent virtually no change. In elementary education, *grupos escolares* spread through many states, along with a growing number of normal schools that prepared teachers to staff the new schools. This made elementary education more formal, well structured, and inclusive, while secondary education remained less so and retained its almost exclusive propaedeutic function for postsecondary studies in the professional schools, mainly law, medicine, and engineering. A few states created job-oriented secondary schools to prepare students for the work market, but these lacked prestige because they did not prepare for further studies and were shunned by the middle classes.

In the period starting around 1920 and extending till the early 1940s, profound changes took place in education in Brazil as part of the major upheavals in that country. Brazilian educators discussed the ideas of Dewey and Montessori, among others, striving to establish an educational system that would meet the demands of a complex, changing society (Carvalho 2006, p. 71).

In 1929, there were major changes in the mathematics curriculum of *Colégio Pedro II*. Its director, Euclides Roxo, instituted a reform clearly influenced by the first international mathematics curriculum reform movement, which was strongly promoted by IMUK (Schubring 2000, p. 6, 2003; Carvalho 2006, p. 71). The reform was developed for *Colégio Pedro II* and was meant to be put in practice gradually as the first-year students progressed through the curriculum. The following year, 1930, Getulio Vargas' putsch overthrew the central government, created the Ministry of Education, and named as its head Francisco Campos, a very conservative and well-known educator. With Campos started a series of reforms which lasted even under his successor as head of the Ministry of Education, Gustavo Capanema. One of the main points of the reform was its mandatory regular attendance of secondary school; in addition, students had to pass examinations in all school subjects of 1 year to be allowed to progress to the next one. Moreover, changing the school for students during the academic year was banned. The reform also instituted compulsory school inspections by the ministry. Mathematics was present in all school years, and the strict division of algebra, geometry, and arithmetic taught separately in specific years was abolished. The function concept was introduced very early in the curriculum, one of the ideas of Felix Klein (Carvalho 2006). The reform was applied at once to all secondary school years and throughout the whole country (Carvalho 2006; Carvalho and Dassie 2012; Dassie 2001; Dassie 2008; Rocha 2001), and it did not touch elementary education. The reform was accompanied by a law that reorganized postsecondary education.

The secondary school mathematics curriculum stipulated in the 1931 reform was very ambitious. It reintroduced calculus in secondary school and introduced an intuitive geometry course before the formal course on deductive geometry.

Not surprisingly there was very strong opposition to the reform as a whole and to its mathematics curriculum specifically. For different reasons, some of the colleagues of Euclides Roxo at *Colégio Pedro* II in the military and the church joined ranks. It is interesting to note that the attack of the church on the reform was based on the same arguments used by proponents of Chile's *Plan de estudios humanistas*, namely, that the humanities should be the most important part of secondary education. The head of an elite Catholic school in Rio de Janeiro fought a long battle defending Latin and the classics as the real foundations of secondary education (Carvalho 2006). Also, the new mathematics programs were attacked because they stressed intuition and applications, forgetting that mathematics should be valued because it taught logical reasoning and was a means to "discipline the mind" (Carvalho 2006) – an argument also used by the defenders of *Plan de estudios humanistas*.

Roxo also played an important role in defining the mathematics curriculum of the next reform in 1942 by Campos' successor, Gustavo Capanema. Roxo was forced to abandon some features of his 1931 mathematics programs but was able to save some of his important ideas: an end to the division of school mathematics into the isolated parts of arithmetic, algebra, and geometry and the teaching of mathematics in every secondary school year. From 1942 until 1961, several topics were withdrawn from the mathematics program. Euclidean geometry steadily lost its position in the program and finally was replaced by analytic geometry in the last year of secondary school.

With Capanema's reform, secondary school was divided in two parts: the first was the *curso ginasial* with 4 years and the second one the *curso colegial* with 3 years. In the *curso collegial,* the student had three choices: *curso científico* (the science track), *curso classico* (the humanities track), and *curso normal* (normal school that prepared teachers for elementary schools). In both *curso clássico* and *curso normal,* the students had only 1 year of mathematics, a structure which lasted until 1967. *Curso ginasial* corresponded with middle school and *curso collegial* with high school.

4.3 The "New Math Movement" in Latin America

The "new math movement" (*Movimento da matemática moderna, Movimiento de las matemáticas modernas*) reached Latin American countries in the 1960s and strongly influenced elementary and middle school mathematics. It lasted roughly two decades, during which official programs underwent significant changes with the introduction of new topics and new textbooks. This movement left still-visible prints in school mathematics in Latin America: the use of the language of set theory, and formal introductions to the function concept and the algebraic structure of number sets of school mathematics.

At the height of the Cold War, there were strong ideological divisions in Latin America, one of the "battlegrounds" between the United States and the Soviet Union. The fear that Latin America would "go communist" had, as a result, considerable American investments in the region. As part of the "Alliance for Progress" (launched by President John F. Kennedy in 1961) or direct foreign aid programs, several cooperation agreements in the educational area were signed between the United States and specific Latin American countries. This helps to explain why, in Latin America, even though the European contribution to the new math movement was known, the major influence was American. For example, many publications of the School Mathematics Study Group (SMSG) were translated into Portuguese, including its complete secondary school mathematics textbooks (SMSG 1966), while non-American teaching materials were much less used. Exceptions were the use of Papy's textbooks in a very prestigious school in Rio de Janeiro (Costa 2012) and the translation of a textbook of the School Mathematics Project – from England – into Spanish in Venezuela. This textbook did not follow Venezuela's official mathematics curriculum, however, and had scant influence (Mosquera n.d.).

Decisive events for the consolidation of the movement in Latin America were the Interamerican Congresses of Mathematics Education. The first CIAEM (*Conferencia Interamericana de Educacíon Matemática*, *Conferência Interamericana de Educação Matemática*, Interamerican Conference on Mathematics Education [IACME]) was held in Bogotá, Colombia, in 1961, and was sponsored by ICMI and the Organization of American States. The purpose of the meeting

> [W]as to investigate the methods used in the teaching of secondary and post secondary mathematics and to approve resolutions towards future cooperation. More specifically, the purpose was to extend to Latin American countries the reforms in secondary school mathematics teaching that were happening in many countries, particularly in Europe and the United States. (Barrantes and Ruiz 1998, p. 20)

One of the main topics discussed was the need to replace the traditional approach to geometry, substituting Euclidean geometry with a geometry course based on linear algebra. Even though most participants agreed that the teaching of geometry called for modernization, some doubted the feasibility of the proposed approach (Barrantes and Ruiz 1998, pp. 21–22). For example, Omar Catunda from Brazil and A. John Coleman from Canada disagreed with Fehr's and Choquet's position. As stated by Catunda, "[T]he formula I would defend for Brazil is not down with Euclid, but at least Euclid!" (Barrantes and Ruiz 1998, pp. 26–27). Guillermo Torres from Mexico defended the position that "one should not discard whole topics from classical mathematics, because otherwise one risked being left only with definitions and concepts that have no meaning to the student." He claimed that the "new ideas acquired by the student should be accepted by him as natural and that mathematics should be taught following more or less the historical path of its development" (Barrantes and Ruiz 1998, pp. 25–26). Rafael Laguardia from Uruguay asked how the psychological development of the students would be taken into account, a subject completely sidestepped in the conference.

Choquet's ideas during the conference "took into account the needs of mathematicians, paying scant attention to psycho-pedagogical considerations." According to Choquet, mathematicians should have as their motto "algebra and the fundamental structures from elementary school through the university." He also asserted that teaching based on the historical method had become unthinkable (Barrantes and Ruiz 1998, p. 23).

The conference was attended by 48 participants from 24 countries (Barrantes and Ruiz 1998, p. 128). It was dominated by Marshall Stone, who was the driving force in establishing a permanent committee, *Comisión Interamericana de Educación Matemática* (CIAEM, *Interamerican Committee of Mathematics Education* [IACME]), whose task was to "assure the continuity of the projects and ideas discussed in the first conference and to promote actions to improve the level and efficiency of the teaching of secondary school and university mathematics" (Barrantes and Ruiz 1998, pp. 29, 46).

The second CIAEM was held in Lima, Peru, in December 1966. While the first conference had the purpose of promoting the reform ideas, the second one, as stated by its chairman Marshall Stone, was to survey the progress of the reform movement in Latin America. The participants were therefore asked to prepare reports on the progress being made in their countries. The report read by Luis Santaló from Argentina is particularly illustrative of the difficulties the reform movement faced (Barrantes and Ruiz 1998, pp. 31, 34–35).

In the first place, it was difficult to convince teachers of the need for reform and of its feasibility. The second challenge was to convince parents and, more generally, public opinion. The third difficulty was the teachers' inability to apply the reform in their classrooms and the lack of appropriate textbooks for the students. Santaló stressed that it was essential to introduce modern mathematics in the curricula of teacher preparation institutions and that in-service teacher training programs to prepare teachers for these new ideas should be instituted. As for textbooks, the only answer was to write texts following the reform proposals. As a last problem, Santaló mentioned the difficulty of changing the official mandatory curricula issued by the ministries of education of the Latin American countries. This made it very difficult to have experimental projects with the new programs.

Santaló did not consider it a major problem that many teachers did not understand what the reform meant and enthusiastically taught courses "full of trivialities, conceptual errors that sow general confusion" (Barrantes and Ruiz 1998, p. 35). In fact, this problem plagued much of the modern mathematics movement in Latin America.

Among the recommendations of the second conference, the following concrete curriculum proposal for middle and high school mathematics should be mentioned:

> Middle school (12-15 year old students): Sets, relations, integers, binary operations, introduction of the geometry axioms, introduction of rational and real numbers systems, two-dimensional vector spaces, functions and their representations, metric plane geometry, scalar product, analytic geometry in orthogonal bases, linear equations systems.
>
> High school (15-18 year old students): Real numbers, Euclidean spaces, orthogonal bases, the Cauchy-Schwarz inequality, linear transformations in the plane, complex numbers, trigonometry, combinatorial analysis, the Euclidean algorithm, polynomials, introduction to topology, continuous functions, limits, sequences, derivatives, integration, special elementary functions, determinants, space geometry, elementary probability and statistics.

Moreover, the recommendations stated that "it is necessary that primary schools prepare the students to follow these programs [for middle and high school mathematics]" (Barrantes and Ruiz 1998, p. 43). This recommendation might be viewed as an encouragement for the introduction of modern mathematics in elementary education, even though the conference wisely (we think) did not make specific proposals on this.

The modern math movement had varying degrees of success in Latin American countries. Perhaps its most important result was fostering the creation and development of the community of Latin American mathematics educators. The statement of Carvalho and Dassie about Brazil can be safely applied to many other Latin American countries: The "new math movement" promoted the creation of groups in which there were

> [R]eflections and research on the teaching and learning of mathematics; and in the second place, many teachers were motivated to study not only mathematics but also its learning and teaching. The ideas of Jean Piaget, George and Frédérique Papy, Hans Freudenthal, Tamás Varga, and Zoltan Dienes became known (…) mainly because of the interest aroused by the movement. (Carvalho and Dassie 2012, pp. 507–508)

5 Concluding Remarks

Much remains to be done in documenting the history of mathematics education in Latin America. The fact that the history of mathematics education in Latin America is being recognized as a field of study has promoted the appearance of master's theses and doctoral dissertations on the history of the subject on the continent, but the number of still unanswered questions is high. For example, to what extent did each Latin American country become aware of developments in mathematics education in other countries? For a long period, Brazil remained turned first toward Europe, later toward the United States, but other Latin American countries shared a common heritage, despite the eventual disputes and rivalries among them. To what extent did this bond favor exchange of experiences in mathematics education? Did the development of the respective systems of normal schools learn from each other? What were the exchanges of teachers, mathematicians, school administrators, and textbooks among these countries? We know that Sarmiento's educational ideas were influential in several countries, but were there other such cases? How did European and American teaching influence mathematics teaching at the engineering schools? Which were the first American mathematics books used in these schools? Besides Lozier, Domeyko, Bergeron, and Gorceix, who other Europeans or North Americans were important in mathematics education in Latin America? What was the result of several German missions sent to Latin America in the nineteenth century? How many countries or regions did Lancaster's method reach? Were there systems of ethnic schools besides the German ones in Brazil, Argentina, and Chile? How do the "descendents" of a European book published in several countries compare with each other? We have concrete evidence of North American influence at some normal schools as those in Paraná, Argentina, and São Paulo and Rio de Janeiro, both in Brazil. What other cases are known and how do they compare with the Argentinean and Brazilian cases? What was the influence in Latin America of confessional, usually Protestant, American, and British private secondary schools in the first half of the twentieth century? How extensive and influential were the networks of Catholic schools established by religious orders in Latin America? We hope the present work will motivate research on this wide and little explored field.

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Chapter 18
Mathematics Education in Modern Asia

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This chapter is devoted to the history of mathematics education in the countries of Asia. Since here it is impossible to analyze in detail what took place in each country, we limit ourselves to three parts devoted to China, India, and the countries of Southeast Asia, written respectively by Yibao Xu and Joseph W. Dauben, Dhruv Raina, and Lee Peng Yee. The chronological boundaries of these parts do not entirely coincide: the section devoted to India includes information about what happened in the nineteenth century, while the corresponding period in China is covered in the chapter on the premodern era in Asia (that chapter and the present chapter overlap in certain respects).

1 Mathematics Education in Twentieth-Century China

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1.1 Introduction

The twentieth century has been, without exception, the most traumatic in Chinese history, during which China has experienced dramatic political, social, and economic turmoil to an extent unprecedented at any other time in its history. At the very beginning of the century, the Manchus controlled 大清国 *Da qing guo*, or the Qing Empire, but in August of 1900, the capital Beijing was occupied by

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the allied forces of eight foreign nations in response to the Boxer Rebellion, a peasant uprising aimed largely at foreign Christians that had resulted in attacks on foreign missionaries and the deaths of many foreign and Chinese Christians alike. In 1901 the Qing court was forced to sign the Boxer Protocol, another in a series of unequal treaties imposed on the Chinese by foreign governments since the Opium Wars of the mid-nineteenth century. Ten years later, the Qing Empire finally collapsed under the persistent attacks of domestic and foreign forces. Replacing a dynasty having a 268-year long history, the Republic of China ushered in a new era for China in 1912. But fighting among factious warlords and frequent changes in leadership plunged the county into chaos for the next one-and-a-half decades. The central government was unable to exercise much power until the capital was relocated to Nanjing in 1928 under the control of the Nationalist Party headed by Chiang Kai-shek (1887–1975).

In the next decade, a civil war between the Nationalist Party and the Chinese Communist Party was fought mainly in Southern and Northwestern China. In 1937, Japan launched an all-out invasion of China and soon occupied major cities including Beijing, Shanghai, and Nanjing, as well as a corridor stretching from Manchuria to Canton. This necessitated a truce between the Nationalist and Communist Parties who joined forces with Allied support to resist the Japanese. But with the defeat of the Axis powers at the end of World War II, and following the surrender of Japan in 1945, the Nationalist and Communist Parties immediately resumed their civil war. Four years later, the Nationalists under Chiang Kai-shek retreated from the mainland to the island of Taiwan in 1949; simultaneously, the victorious Communists created the People's Republic of China, headed by Mao Zedong (Mao Tse-tung, 1893–1976) as Chairman of the Communist Party. In mainland China, the early five-year plans intended to create a socialist state included the so-called Great Leap Forward, but a series of natural disasters and widespread famine resulted in the deaths of millions of Chinese between 1959 and 1962. Following a period of reforms meant to increase agricultural productivity in particular, China embarked on its greatest disaster, the Great Proletarian Cultural Revolution, and from 1966 to 1976 the country lost a generation during which schools and universities were either closed or functioned along ideological rather than academic lines. Following the death of Mao Tse-tung in 1976, and thanks to the leadership of such politicians as Deng Xiaoping (1904–1997), the country looked in new directions and made education, economic reform, infrastructure, and technology national priorities. The "second generation" of Chinese leaders not only has pursued an "open door" policy with the West but has encouraged foreign investment and joint ventures to put the country on a solid economic and technological footing, which has led to considerable investment in both secondary and higher education, with large numbers of students encouraged to study abroad.

Mathematics education in particular, throughout China's modern history, has been directly affected by all of these political and social events. Given the complexity and interdependence of the country's political, social, and economic environments, and the changing fortunes of the political powers in control at different times, it is impossible to give a thorough account here of mathematics education in China in only a few pages. Instead, we sketch in broad outline the major factors and turning points in the history of mathematics education in China's schools, colleges, and universities under the legitimate *central* governments of China from the last decade of the Qing Dynasty to the 1990s. Unfortunately, it has not been possible to include mathematics education in schools, colleges, and universities sponsored by missionary or private groups or organizations. Likewise, we omit discussion of mathematics education in Hong Kong and Macao, in Yan'an under the Communists in the 1930s and 1940s, in the areas under control of the Japanese and their collaborators during the War of Resistance Against Japan, and in Taiwan since 1949.

1.2 Mathematics Education Under the Manchus: 1900–1911

When the Qing Empire, with its population of more than 415 million, was easily defeated by the Japanese in the First Sino-Japanese War in 1895, and again by the combined allied foreign forces that suppressed the Boxer Rebellion in 1900 with less than 55,000 combatants and mercenaries, the ruling Manchus realized that in addition to inadequate military conditions, the education system in China was indirectly at fault, with serious problems that needed to be reformed for the county's welfare. In 1902, the court enacted a new school system following a Japanese model. Known in Chinese as 壬寅学制 *renyin xuezhi* (the 1902 school system), this divided education into seven parts on three levels. The first part included kindergarten (4 years, with children starting at age 5), elementary school (3 years), and either senior elementary school (3 years) or vocational elementary school (3 years). The second part covered middle school (4 years), or middle school (1 or 2 years), and a trade school (2 or 3 years) or a vocational middle school (4 years) or normal school (4 years). The third and last part was comprised of advanced schools, which included college preparatory (3 years) or vocational high schools (3 years), normal schools (3 years), colleges (3 years), and advanced institutes (no fixed number of years). Mathematics education was required at all levels, except for the highest one at advanced institutes which might not involve any mathematical training (Li 2005, pp. 165–166).

Before the *renyin* system was actually instated, it was modified as the 癸卯学制 *kuimao xuezhi* (the 1904 school system). This revised system did not regulate the number of years for kindergarten, but elementary schools covered 5 years for children starting at the age of 6. It also added 1 year for the senior elementary and middle schools. For higher education, the *kuimao* system also increased the duration of college from 3 to 4 years and also set a 5-year maximum for advanced institutes (Li 2005, pp. 167–168). The *kuimao* system included specific regulations about the mathematical content of instruction at every level. For example, the regulations for middle schools were set as follows:

> First, teach arithmetic, covering how to use a brush on paper [笔算 *bi suan*] to handle addition, subtraction, multiplication and division, decimals, proportions, percentages, extraction of square and cube roots. Second, show how to use an abacus to calculate [珠算 *zhu suan*] addition, subtraction, multiplication and division. Book keeping should also be taught; let students know how to use different accounting methods and the formats of calculation tables. Plane and solid geometry, and algebra should be taught as well.
>
> When teaching arithmetic, mathematics teachers should explain in a very specific and clear way. The methods or rules introduced should be simple and quick. The principles behind the operations should also be provided in as much detail as possible. Students should master quick-calculations as well. When lecturing on algebra, mathematics teachers should explain clearly the mathematical principles of problems. When teaching geometry, teachers should pay more attention to reasoning, and let students know how to apply mathematics in surveying and in calculating areas and volumes. (Li 2005, p. 170)

For students in college preparatory courses majoring in science, engineering, and agriculture, they were required to take algebra and analytical geometry (first year, 5 h per week), analytical geometry and trigonometry (second year, 4 h per week), and calculus (third year, 6 h per week). For those who were studying medicine, they did not need to take trigonometry, and there was no mathematics course in the third and final year for medical students.

For college students majoring in the physical sciences, physics, astronomy, and chemistry, they were all required to take calculus. For the first two majors, students also had to take geometry, theory of functions, differential equations, theory of elliptical functions, and spherical functions (the last two were optional for astronomy majors). For those who majored in mathematics in college, the *kuimao* system required them to take calculus, geometry, algebra, theory of functions, partial differential equations, number theory, with options for study of spherical functions, surveys of advanced mathematics, or mathematical research (Li 2005, pp. 170–173).

The textbooks used were either approved or designated by the government. Those for elementary schools were primarily written by Chinese teachers. However, the majority of middle and higher school mathematics textbooks were translated either from Japanese, American, or English works, or were compilations based on foreign materials. Virtually all textbooks on higher mathematics were translated from books in English or Japanese. Some mathematics teachers even used English books directly as their textbooks (Wei *et al.* 1989, p. 99). The following is a short list of the most popular mathematical textbooks used during this period in China (Wei *et al*. 1989, pp. 46–51; Li 2005, pp. 189–194):

Arithmetic

- 中学适用算术教科书 *Zhongxue shiyong suanshu jiaokeshu* (Middle School Arithmetic Textbook), compiled by 陈文 Chen Wen. Shanghai Science Society, 1905
- 新数学教科书 *Xin shuxue jiaokeshu* (New Mathematics Textbook), 长泽龟之助 Nagasawa Kamenosuke, translated by 包荣爵 Bao Rongjue, 1905
- 中等算术教科书 *Zhongdeng suanshu jiaokeshu* (Intermediate Arithmetic Textbook), 田中德 Tanaka Toku, translated by 崔朝庆 Cui Chaoqing, 1906
- 中学算术教科书 *Zhongxue suanshu jiaokeshu* (Middle School Arithmetic Textbook), written by 陈幌 Chen Huan. Textbook Compilation and Translation Group, 1907

Algebra

- 最新中学教科书: 代数学 *Zuixin zhongxue jiaokeshu* (The Newest Middle School Textbook: Algebra), G. A. Miller, translated by 谢洪赉 Xie Honglai. Commercial Press, 1904
- 初等代数学讲义 *Chudeng daishuxue jiangyi* (Lecture Notes on Elementary Algebra), written by 丁福保 Ding Fubao. Shanghai Science Book Company, 1905
- 查理斯密大代数学 *Chali simi da daishuxue* (Charles Smith's *A Treatise on Algebra*), translated by 何崇礼 He Chongli, and 陈文 Chen Wen. Japan Tokyo Science Society, 1905
- 普通新代数教科书 *Putong xin daishu jiaokeshu* (Regular New Algebra Textbook) for 京师大学堂 *Jingshi daxuetang* (the Imperial College), 1905
- 查理斯密小代数学 *Chali simi xiao daishuxue* (Charles Smith's *Elementary Algebra*), translated by 陈文 Chen Wen. Commercial Press, 1906
- 新代数教科书 *Xin daishu jiaokeshu* (New Textbook for Algebra), 长泽龟之助 Nagasawa Kamenosuke, translated by 余恒 Yu Heng, 1908
- 中学校数学教科书: 代数之部 *Zhongxuexiao shuxue jiaokeshu: daishu zhibu* (Middle School Mathematics Textbook: Algebra), 桦正董, translated by 赵缭 Zhao Liao and 易应琨 Yi Yingkun. Shanghai Qunyi Book Company, 1908
- 温德华士初等代数学 *Wendehuashi chudeng daishuxue* ([George] Wentworth's *School Algebra*), translated by 屠坤华 Tu Kunhua. Commercial Press, 1910

Geometry

- 重译足本几何教科书 *Zhongyi zhuben jihe jiaokeshu* (A Re-translation of the Entire Geometry Textbook), 林鹤一 Tsuruichi Hayashi, translated by 彭清鹏 Peng Qingpeng. Shanghai Puji Book Company, 1906
- 中学教育几何教科书:平面之部 *Zhongxue jiaoyu jihe jiaokeshu*: *pingmian zhibu* (Middle School Geometry Textbook: Plane), written by 何崇礼 He Chongli. Shanghai Science Society, 1911
- 中学教育几何教科书:立体之部 *Zhongxue jiaoyu jihe jiaokeshu*: *liti zhibu* (Middle School Geometry Textbook: Solid), written by 何崇礼 He Chongli. Shanghai Science Society, 1911
- 温德渥斯平面几何学 *Wendewosi pingmian jihexue* ([George] Wentworth's *Plane Geometry*), translated by 马君武 Ma Junwu. Shanghai Science Society, 1911
- 温德渥斯立体几何学 *Wendewosi liti jihexue* ([George] Wentworth's *Solid Geometry*), translated by 马君武 Ma Junwu. Shanghai Science Society, 1911

Trigonometry

- 最新中学教科书三角术 *Zuixin zhongxue jiaokeshu sanjiao* (The Newest Middle School Textbook on Trigonometric Methods) translated by谢洪赉Xie Honglai, Commercial Press, 1907
- 平面三角法新教科书 *Pingmian sanjiaofa xin jiaokeshu* (New Textbook on Plane Trigonometry), 菊池大麓 Kikuchi Dairoku, 泽田吾一Sawada Goichi, translated by王永炅Wang Yongjiong. Commercial Press, 1909

- 平面三角法教科书 *Pingmian sanjiaofa jiaokeshu* (Plane Trigonometry Textbook), compiled by a mathematics study group. Shanghai Changming Company, 1909
- 高等数学平面三角法 *Gaodeng shuxue pingmian sanjiaofa* (A Treatise on Plane Trigonometry), Ernest W. Hobson, translated by 龚文凯 Gong Wenkai. Shanghai Science Society, 1911

Analytic Geometry

- 解析几何教科书 *Jiexi jihe jiaokeshu* (Analytic Geometry Textbook), Charles Smith, translated to Japanese by 宫本藤吉 Miyamoto Tokichi, and translated from the Japanese by 仇毅 Chou Yi. Shanghai Qunyi Book Company, 1910
- 温德渥斯解析几何学 *Wendewosi jiexi jihexue* ([George] Wentworth's *Analytic Geometry*), translated by 郑家斌 Zheng Jiabing. Shanghai Science Society, 1911

Calculus

- 微积阐详 *Weiji changxiang* (Illuminations on the Calculus), 陈志坚 Chen Zhijian, 1905
- 奥斯宾微分学 *Aosibin weifenxue* ([George A.] Osborn's *Differential Calculus*), translated by李德晋 Li Dejing, and 郑家斌 Zheng Jiabing. Shanghai Science Society, 1911
- 奥斯宾积分学 *Aosibin jifenxue* ([George A.] Osborn's *Integral Calculus*), translated by 郑家斌 Zheng Jiabing. Shanghai Science Society, 1911

The mathematics textbooks published in this period at the end of the Qing Dynasty, though still written in classical Chinese, were no longer printed in the traditional format of Chinese books which were read from top to bottom, left to right. Instead, the layout of mathematics textbooks followed the western style, adopting as well modern mathematical notation. No longer was the old system of labeling mathematical figures or of writing equations with the symbols of the 10 heavenly stems and 12 earthly branches followed, but instead letters of the Western alphabet were used. Symbols for mathematical operations were also systematically employed. Moreover, Chinese characters for numbers were replaced by Arabic numerals. Traditional Chinese mathematics, except for use of the abacus, was eliminated from school education, whereupon the westernization of mathematics was finally complete. In 1905, the Manchu court abandoned the civil service examination system, and this meant that the Confucian classics were no longer regarded as the basis for education. Instead, Western mathematics and the sciences became the focus of the new educational curriculum. For the first time, during the last years of Manchu rule, as Frank Swetz explains: "[I]t was in this era [1903–1912] that the concepts of mathematics education were put to work in China" (Swetz 1974, p. 68).

1.3 Mathematics Education in the Republic (Period I): 1912–1937

Although the *kuimao* school system initiated modern education in China, it was short lived and did not survive the demise of the imperial Qing Dynasty, which was overthrown in 1911. In the following year, with the founding of the Republic of China, a new era in the history of China began. A newly created Ministry of Education established a new school system on a German model, from which the Japanese system had been derived. The new system emphasized moral education as well as utilitarian education and military education. It also acknowledged for the first time the right of education for girls. The system shortened the number of years for primary and secondary education from 14 to 11 years (lower primary 4 years, upper primary 3 years, and middle school 4 years) (Dauben 2002, p. 271).

In order to reflect the new goals of the school system, Chinese mathematics educators wrote numerous new textbooks. The general level of these textbooks suggests that Chinese mathematics educators were able to write good surveys of the mathematics they taught (Wei *et al*. 1989, p. 179). According to a survey undertaken in 1920, the most popular mathematics textbooks used in middle schools were those in the series 共和国教科书 *Gongheguo jiaokeshu* (Textbooks for the Republic of China), or the series 民国新教科书 *Minguo xin jiaokeshu* (New Textbooks for the Republic of China), or the books on geometry and trigonometry based on textbooks written in English by the American educator George Albert Wentworth (Zhang 2000, p. 98).

In 1919, the 全国教育会联合会 *Quanguo jiaoyuhui lianghehui* (Alliance of Education Societies in China) initiated a movement for educational reform. The Alliance was led primarily by scholars who had returned from the United States, many of whom had been former students of John Dewey or Paul Monroe at Columbia University. Both had visited China, Dewey for 2 years from 1919 to 1921, whereas Monroe made more than a dozen trips to China in the 1920s and 1930s, and these visits served to encourage and reinforce the effectiveness of the Alliance (Wang Jessica 2007). In 1922 the Chinese central government enacted another "new" school system, 壬戌学制 *renxu xuezhi* (the 1922 school system), which was intended to address the following concerns: the needs of social evolution, education for ordinary people, development of individuality, financial burdens of the people, job training, and more freedom for local governments. The system focused on citizen education, and it resembled in many ways public education in the United States. Consequently, such concepts as "education for democracy" and "student individuality" were introduced into Chinese education (Swetz 1974, p. 72).

The Alliance set definite standards for the new system's curriculum. For mathematics in the middle schools, its general goals were to let students use mathematical reasoning in a wide variety of practical applications, to provide tools for studying the sciences, to meet the needs of daily life, and to develop students' reasoning ability using mathematical methods (for details, see Swetz 1974, pp. 79–80). High school students who wanted to study mathematics or science in a college or university were required to study trigonometry, algebra, geometry, and analytic geometry (Swetz 1974, pp. 81–85). In 1923, the 初级中学算学课程纲要 *chuji zhongxue suanxue kecheng gangyao* (outlines for the mathematical courses in middle schools) required that elementary algebra and geometry should be taught in an integrated way together with arithmetic and trigonometry. The integrated teaching method was initiated by Felix Klein in the late nineteenth century and promoted in the United States by E. H. Moore. Through the efforts of Ernst R. Breslich, Professor of Mathematics at the University of Chicago, the method became popular in the United States, along with a series of textbooks by Breslich that had a great impact on mathematics education in Chinese middle schools in the 1920s and 1930s.

The students took their mathematics courses based primarily on their grade levels. But attention was also paid to individual student abilities. Some exceptional students were encouraged to study by themselves, and teachers helped them individually. Students were also divided into different groups based on their actual levels of achievement (Wei *et al*. 1989, p. 221).

In March of 1929, the Nationalist Party enacted educational guidelines called the Three Principles of the People, that is, nationalism, welfare/livelihood, and democracy. The guidelines became law in June of 1931. However, the new guidelines had little direct impact on mathematics education, but instead prompted the establishment of standards for the mathematics curriculum in Chinese schools. In August of 1929, temporary standards for the middle schools mathematics curriculum were introduced, and a month later, temporary standards for high schools were announced. Both were revised and became standard curricula in November of 1932. These standards spelled out the goals, teaching times, and syllabi and also suggested ways of implementing them (Wei et al. 1989, pp. 232–254; see also Swetz 1974, pp. 94–104). Implementation of these standards varied from school to school and from place to place, but they indicate that secondary mathematics education in China had reached a new level.

Since the new guidelines emphasized nationalism, all textbooks used in classrooms had to be written in Chinese. As a result, a number of excellent textbook writers emerged, among them 傅种孙 Fu Zhongsun, Professor of Mathematics at Beijing Normal University, and 余介石 Yu Jieshi, Professor of Mathematics at Chongqing University. Fu's 高中平面几何教科书 *Gaozhong pingmian jihe jiaokeshu* (High School Plain Geometry Textbook, 1933) and a series of textbooks by Yu were highly praised by mathematics educators (Wei et al. 1989, p. 269).

Between September and November of 1931, four scholars were sent by the Institute of Intellectual Cooperation of the League of Nations at the request of the Nationalist Government to examine Chinese education. In their report, they strongly criticized China's "elitist" school system (Swetz 1974, p. 96; Becker 1974). Assessing mathematics education in China during this period, Frank Swetz argues that it was too academic and lacking in any practical substance:

> Education was elit[i]st and examination oriented. Curricula were formal and overburdening. Specifically, mathematical studies bore little relevance to the requirements of an agrarian society or industrial application. Methods of instruction encouraged rote memorization and provided little opportunity for individual inquiry. A student's accumulation of mathematical knowledge, while facilitating his ascension to a higher education, did little to acquaint him with the realities of his world. (Swetz 1974, p. 105)

In retrospect, however, the rigorous curriculum and the "academic" approach to mathematics education trained a generation of future first-rate mathematicians such as 许宝禄 Xu Baolu (Pao-Lu Hsu, 1910–1970), 华罗庚 Hua Luogeng (Loo-Keng Hua, 1910–1985), and 陈省身 Shiing-Shen Chern (1911–2004). If judged by the cultivation of such creative minds, Chinese mathematics education in this period may be considered fruitful and successful.

Indeed, the growing strength of China mathematically was reflected in new institutions throughout the country, but notably in Beijing where higher education underwent considerable expansion, with concomitant positive developments for mathematics. The former Capital University, or *Jingshi daxuetang* 京师大学堂, was renamed 北京大学 Peking University in 1912, and the very next year it created the first department of mathematics in China. Meanwhile, although the 1904 school system had spelled out specific requirements including courses for mathematics majors in college, as already mentioned, they were never implemented. In 1912 the Ministry of Education issued a new curriculum for mathematics students in higher education, which included calculus, differential equations, theory of functions, modern algebra, modern geometry, plain and solid analytic geometry, quaternions, statistics and the method of least squares, algebraic analysis and equations, calculus of variations, number theory, integral equations, theoretical physics, astronomy, and experimental physics (Ding et al. 1993, p. 75).

This curriculum, however, was never rigidly enforced due to either a lack of competent faculty members or issues of academic freedom. When Peking University admitted its first students majoring in mathematics, there were only two faculty members, one of whom was a graduate of Tokyo Imperial University, 冯祖荀 Feng Zuxun (1880–1940). Because Japanese mathematics then was heavily influenced by the Germans, the mathematics curriculum and the textbooks used in Peking University also reflected that. For instance, integral calculus was taught using the lecture notes on the subject by David Hilbert (1862–1943) which were used as the textbook (Ding et al. 1993, p. 75).

There were only two students who graduated in mathematics from Peking University in 1916. But when the famous educator 蔡元培 Cai Yuanpei (1868–1940) was appointed as President of Peking University in 1917, the mathematics department faculty was expanded to seven, including two graduates from Harvard University, 秦汾 Qin Fen (M.S. 1913) and 王仁辅 Wang Renfu (B.A. 1913) (Xu 2002, p. 294). The curriculum was revised and courses like differential equations and harmonic functions, group theory, and history of mathematics were all added as core courses (Ding et al. 1993, p. 75).

Before 1920, the Department of Mathematics at Peking University was the only one in all the national comprehensive universities in China. But when students who had studied mathematics in the United States, Germany, France, Great Britain, or Japan returned to China, mathematics departments were created one after another. Among those who returned, ten received their PhDs before 1930. The first was 胡明复 Hu Mingfu, who received his doctoral degree from Harvard University in 1917 (Zhang 2000, p.70). In the 1920s, more than twenty universities in China created mathematics departments, among them Nankai University in 1920 (for a detailed early history of the department, see http://sms.nankai.edu.cn/d/xygk/xyls), Southeast University in 1921, Xiamen University in 1924, Zhejiang University in 1928, Jiaotong University in 1928, and Jinan University in 1929. However, Beijing remained the center for higher mathematics education.

Besides Peking University, four other universities in Beijing created mathematics departments in the 1920s, namely, Beijing Normal University (1922), Tsinghua University (1927), Yenching University (1927), and Fu Jen Catholic University of Peking (1929). The first is a higher institution for training teachers, and the last two were missionary universities. Tsinghua University was developed from the Tsinghua School founded in 1911 with the American Boxer Indemnity Fund (Xu 2002, p. 289). The leading founding member of its mathematics department was 郑之蕃 Zheng Zhifan (1887–1963), who received his B.A. from Cornell University in 1910 and also studied for 1 year at Harvard. Zheng soon recruited as full professors 熊庆来 Xiong Qinglai (1893–1969) who had studied at the University of Paris and 杨武之 Yang Wuzhi (1896–1973) and 孙光远 Sun Guangyuan (1900–1979), both of whom studied at the University of Chicago (Xu 2002, p. 291). Given their Western backgrounds, the goal of the Tsinghua mathematics department was set to train future mathematicians (Guo 2003, pp. 44–55).

The mathematics curriculum at Tsinghua was designed to provide students with solid, rigorous, and balanced training in modern mathematics. In 1929 the 4-year curriculum required students to take calculus, modern geometry, advanced algebra, advanced geometry, differential equations, theoretical mechanics, modern algebra, a colloquium on general mathematical questions, advanced analysis, analytic functions, partial differential equations, elliptical functions, and differential geometry. They were also required to take several electives, among them spherical trigonometry, probability, projective geometry, elliptical integrals, number theory, theory of functions, group theory, integral equations, variations, hydrodynamics, celestial mechanics, and the history of mathematics. Moreover, a thesis was also required for graduation (Guo 2003, pp. 91–93).

Not many mathematics departments in China in this period could adopt Tsinghua's rigorous curriculum due to differences in the specialties of faculty members, funding, and differing educational goals. But in 1933, at a symposium sponsored by the Ministry of Education, the chairs of several leading mathematics departments proposed that at least the following 11 courses should be covered in all universities: introductory calculus, differential equations, analytic geometry, advanced calculus, infinite series, functions of complex variables, elementary algebraic equations, projective geometry, advanced algebra, differential equations, and theoretical mechanics (Guo 2003, p 99).

To enter a given department, students had to take an entrance examination given by the university to which the student was applying. The difficulty of the examinations varied, but all of them covered the basic high school curriculum. Graduation rates also varied from university to university, but generally the rates were low. For instance, only 20 students had graduated from the Mathematics Department at Tsinghua by 1937 (Zhang 2000, p. 85). According to the 教育年鉴 *Jiaoyu nianjian* (Education Yearbook of 1934), the number of students majoring in mathematics during the 1931–1932 academic year was about 500, and the total number of graduates for the entire country from 1916 to 1932 was about 300 (Zhang 2000, p. 103; Ren and Zhang 1995, pp. 22–23).

The year 1930 was a milestone for mathematics education in China. In that year, Tsinghua University founded its Institute of Mathematics, and of its first two graduate students, one was Shiing-Shen Chern, who completed his thesis on *Associate Contact Quadrants of a Rectilinear Congruence* in 1934 under the supervision of Sun Guangyuan, who had received his PhD from the University of Chicago in 1928. Because Tsinghua was especially selective in the admission of graduate students, by 1936 only six students had been enrolled. In addition to Chern, Xu Baolu was admitted in 1933. Both Chern and Xu later became renowned mathematicians and have made substantial contributions to differential geometry and statistics, respectively. Following Tsinghua, in 1934 Peking University admitted its first four graduate students, and 4 years later, Central University, which was later renamed Southeast University, began to admit its first graduate students in mathematics (Guo 2003, p. 55).

1.4 Mathematics Education in the Republic (Period II): 1938–1949

Many promising students after graduating from a university in China went to the United States or Europe to continue their studies. Between 1930 and June of 1937, more than 30 Chinese students earned PhDs in mathematics from Western institutions, for instance: 江泽涵 Jiang Zehan (Tsai-han Kiang) at Harvard University (1930), 胡坤陞 Hu Kunsheng (Kuen-sen Hu) at the University of Chicago (1932), 曾炯之Ceng Jiongzhi (Chiung-tze Tsen) at the University of Göttingen (1934), Shiing-Shen Chern at the University of Hamburg (1936), and 柯召Ke Zhao (Ko Chao) at the University of Manchester (1937). Meanwhile, the legendary mathematical genius Hua Luogeng studied for a year with G. H. Hardy at Cambridge University during the academic year 1936–1937. Although he did not have a PhD, he quickly established himself as a first-rate mathematician through an outstanding series of publications. All of these mathematicians returned to China with the ambition of making solid contributions to their own research fields, as well as to higher mathematics education in China. Unfortunately, the Second Sino-Japanese War that broke out in July of 1937 dashed any such hopes.

With the rapid advance of Japanese troops, the Ministry of Education issued an edict in September of 1937 directing the evacuation of the national universities in Peking and Tianjin to safer parts of China. Tsinghua University, Peking University, and Nankai University were ordered to move to Changsha, whereas Beijing Normal University, Beiping University, and Beiyang University all moved to Xi'an. Later, the universities moved again to Changsha where they merged to form Changsha Temporary University. It was not long, however, before Changsha became unsafe, and so the Temporary University was moved to Kunming in Yunnan province where it was renamed as National Southwestern Associated University in April of 1938. At the same time, the universities that had been moved to Xi'an were renamed as National Northwest Associated University.

During the War of Resistance Against Japan, mathematics education at all levels suffered a huge blow. But miraculously, the National Southwestern Associated University, or in Chinese, 联大 *Lianda* for short, the Department of Mathematics brought together the faculty members from three major universities, led by Shiing-Shen Chern, Hua Luogeng, and Xu Baolu (who had just received his PhD from the University of London in 1938). This easily constituted the best mathematics department in all of China, despite conditions of extreme adversity. Following Tsinghua's tradition, the goal of the associated department was to train future mathematicians, and thus standards were very high. If a first-year student could not get at least a 75 or higher score for the calculus course, he or she would be dismissed from the department. As a result, the number of students was very small. With a relatively large number of professors, a large number of courses were offered, and most professors gave advanced elective courses on their own research fields. For instance, Chern offered advanced geometry; Hua analytic number theory, distributions of prime numbers, and the zeta function; Xu Baolu statistics; Jiang Zehan topology; 王湘浩 Wang Xianghao set theory; and 刘晋年 Liu Jinnian ideals. Seminars on algebra, topology, analysis, group theory, analytic number theory, topological groups, and Lie groups were offered for seniors and graduate students (Ding et al. 1993, p. 78; Guo 2003, p. 109).

Despite many hardships, limited facilities, and personal insecurities, the dedication of faculty and students alike meant that the mathematics education at *Lianda* was very successful despite the adverse circumstances and represented a great achievement. A large portion of the graduates later became world-renowned scholars, including the logician 王浩 Wang Hao (1921–1995), the mathematicians 樊畿 Ky Fan (1914–2010) and 王宪钟 Wang Xianzhong (Hsien Chung Wang, 1918–1978), and the statistician 钟开莱 Kai-Lai Chung (1917–2009). John Israel, in his book *Lianda: A Chinese University in War and Revolution*, comments on the achievement of the mathematics department as follows:

> At times, students barely outnumbered their department's dozen or so professors. Between 1939 and 1945, Lianda and its constituent colleges granted a total of fifty-four mathematics degrees. There is no question, however, that theses graduates were qualitatively superior. Mathematics is one department in which Lianda had no trouble maintaining, if not exceeding, prewar standards. (Israel 1998, p. 205)

Another mathematics department that also made noticeable achievements during this same period was at Zhejiang University. The department was founded in 1928 under the leadership of 陈建功 Chen Jiangong (1893–1971) and 苏步青 Su Buqing (1902–2003), both of whom received their doctorates from the Imperial University of Tohoku (in 1929 and 1931, respectively). This department developed rapidly as a mathematics center, especially for studying Fourier series and affine geometry. After relocating several times from Hangzhou, in 1940 the university finally settled down in Meitang, Guizhou province. In Meitang, the department founded a mathematics institute and started to admit graduate students. One of the earliest was 程民德 Cheng Minde (1917–1998), who later received his PhD from Princeton University in 1949 and went on to become a renowned mathematician and mathematics educator in China.

When the War of Resistance Against Japan was finally over in August of 1945, the universities and colleges soon returned to their original homes, and mathematics education gradually returned to prewar levels. A notable advance was made in 1947 when the Institute of Mathematics was created as part of the national *Academia Sinica*. Shiing-Shen Chern, who had just returned from the Institute for Advanced Study at Princeton, served as Acting Director (Xu 2002, pp. 298–300). Chern immediately set about recruiting promising college graduates and trained them personally. Although such training lasted only a year without leading to any formal degree, most of the graduates later became world-renowned mathematicians, for example, 吴文俊 Wu Wenjun (Wu Wen-Tsün, 1919–) and 廖山涛 Liao Shantao (1920–1997). However, almost immediately, due to the civil war between the Nationalists and the Communists, Chern went back to the Institute for Advanced Study at Princeton in January of 1949, and the Institute of Mathematics was shortly thereafter relocated to Taiwan, where it has played a major role for mathematics education there since 1949.

1.5 Mathematics Education Under the Communists (Period I): 1949–1965

When the Peoples' Republic of China was created on October 1, 1949, one of the first major tasks facing the new government was the reform of education on a solidly Communist foundation. This time the immediate role model was the Soviet Union, the country's ally during World War II and, initially at least, a strong political supporter of the new government. The first 2 years were transitional. In areas liberated before 1949, the school systems and the curricula basically followed those in Yan'an (in Shaanxi province, where the Chinese Communist Party was headquartered during the War of Resistance Against Japan) or the Soviet model. Elsewhere schools were consolidated and reorganized. Major educational reforms did not begin until the fall of 1952, when the first five-year plan was issued calling for dramatic industrial growth and socialization in the years 1953–1957. For secondary mathematics education, in December of 1952 the Department of Education issued a draft mathematics syllabus which spelled out a number of specific goals: "to teach students basic mathematical knowledge and to provide necessary skills and techniques for solving all kinds of practical problems by applying that knowledge" (Cai 2002, p. 14). The middle schools had a 3-year curriculum, with 6 h of mathematics instruction in the first year and 5 in each of the following 2 years. The high school curriculum was also for 3 years, with 5 h of mathematics instruction in the first year and 6 in each of the last 2 years of high school. In July of 1953, the number of instructional hours for the first year of middle school was increased by 1 h per week. Mathematics textbooks published by the People's Education Press were mandatory and used throughout mainland China. These textbooks were basically either translations or slight modifications of Russian books. Soviet teaching plans and pedagogies were also adopted (Swetz 1974, pp. 136–150; Cai 2002, p. 14). The draft mathematics syllabus of 1952 was modified in October of 1954 and then again in May of 1956. Developing students' logical reasoning and improving their spatial intuition were added as new teaching goals. Textbooks were correspondingly revised, basically by deleting content that did not reflect Chinese culture and traditions. In 1956, mathematics competitions, also modeled on Soviet mathematical Olympiads, began to be held in major cities with the intention of stimulating high school students' interest in mathematics (Swetz 1974, p. 150).

For higher education, a major restructuring of existing institutions began in 1952. Within a year, the universities and colleges were consolidated into one of four categories: (1) comprehensive; (2) polytechnic; (3) specialized in engineering, agriculture, medicine, law, etc.; and (4) teachers colleges. Departments of mathematics were merged and mathematics professors were often relocated. For example, the mathematics departments at Peking University, Tsinghua University, and Yenching University were merged into a single department at Peking University, and some professors were sent to other universities or colleges. Although the mathematics curricula varied depending upon the institution, they all were heavily influenced by the Soviet Union. For instance, the newly merged department of mathematics at Peking University was modeled on Moscow University where students studied either mathematics or mechanics. The Moscow curriculum emphasizing mathematical analysis was also adopted. Integration of theory and practice was stressed. Following the Soviet Union, 教研室 *jiaoyan she* (Teaching and Research Groups) were also created at both advanced institutions and secondary schools. In 1955 the teaching and research group of computational mathematics was created at Peking University, with another group for statistics founded the following year (Ding et al. 1993, p. 80).

In 1958 China launched the "Great Leap Forward." In line with political calls for advancement, mathematics education was changed accordingly, and in colleges and universities, emphasis was given to applied mathematics. The integration of theory and practice was overstressed to the detriment of basic theoretical training, which was neglected. In secondary schools, arithmetic previously taught in middle schools was relegated to elementary schools, and middle schools were required to cover plane geometry and quadratic equations. At high schools, analytic geometry, variation methods, approximation, and derivatives were added to the curriculum (Cai 2002, no. 9, p. 15). The negative consequences of these changes were soon apparent. In 1962, the central government rescinded these drastic policies, giving appropriate attention once again to basic theories and also adjusting mathematical content to suit the appropriate levels and abilities of the students.

In this period, the two most important developments for mathematics education in China were the creation of the Institute of Mathematics at the Chinese Academy of Sciences in July of 1952 and the founding of the University of Science and Technology of China (USTC) in September of 1958. To head the Institute, Hua Luogeng returned to Beijing from the University of Illinois at Champaign and was appointed its first Director. The Institute has not only made solid contributions to the development of mathematics but also trained young research mathematicians ever since. As for USTC, its Department of Mathematics soon became a new center for mathematics in China. The University's Special Class for the Gifted Young (SCGY), initiated in 1978 at the suggestion of C. D. Lee (a Nobel Prize winner in physics at Columbia University), has proven to be a very successful experiment in educating precollege students with exceptional abilities. SCGY provides extraordinary opportunities for students from all parts of China who demonstrate both the ability and desire to take advantage of accelerated education (for details, see http://en.scgy.ustc.edu.cn/about/201107/t20110707_115587.html).

Since 1956, the leaders of the Chinese Communist Party and their counterparts in the Soviet Union had progressively divergent views of Marxist ideology. By 1961, their differences proved irreconcilable, and the two ruling parties formally denounced each other. Concomitantly, the Soviet influence on Chinese mathematics education at both the secondary and advanced levels was dramatically diminished. In response to the "Great Leap Forward," and in order to be politically correct, Chinese mathematics educators abandoned the former Soviet model in favor of their own progressive ideas. Experiments on shorter instructional hours were conducted, and textbooks were rewritten. In 1960, the Department of Education drafted new guidelines for compiling mathematics textbooks for a 10-year school system (5 years of elementary school, 3 years of middle school, and 2 years of high school). The guidelines divided mathematics into three independent subjects: arithmetic, algebra, and geometry. Trigonometry was not treated as an independent subject, but was incorporated into the teaching of both geometry and algebra. Analytic geometry was no longer part of the high school curriculum. Accompanying the new guidelines, new mathematics textbooks were written and published by the People's Education Press in 1961, specifically designed for the new 10-year system. In addition, Beijing Normal University, East China Normal University, and the Municipal Government of Shanghai City also compiled their own series of new mathematics textbooks. In May of 1963, the Department of Education issued yet another revised mathematics syllabus for regular full-time middle and high schools (Cai 2002, p. 15). The new syllabi clearly show that Chinese mathematics education was no longer following the former Soviet model, but was creating its own new model of mathematics education based on Chinese needs and traditions.

1.6 Mathematics Education Under the Communists (Period II): The Cultural Revolution, 1966–1976

The decade of the Cultural Revolution was generally a lost decade for mathematics in China. Many of those teaching mathematics in the major cities like Beijing and Shanghai were sent to the countryside to learn from the peasants, while those who remained in the cities spent much of their time in political indoctrination sessions. Leading mathematicians like Hua Luogeng turned their efforts to applied mathematics, and Hua, for example, sought to find applications of his work that would benefit factory production or agricultural output through knowledge of the methods of operations research, and that of the Critical Path Method (CPM) and the Program Evaluation and Review Technique (PERT).

When the Cultural Revolution began in 1966, many university students left to join the Red Guards, and across China universities closed their doors and did not reopen to accept students again until 1972. Admission to colleges and universities then was not based on scores of nationwide college entrance examinations, but on family background, political connections, and other nonintellectual criteria. Meanwhile, the anti-intellectual sentiments of the Cultural Revolution and the anti-rightist movement sought to "knock down the house of Euclid" in middle schools and "knock down the house of Cauchy and Galois" in the universities (Wang Yuan 1999, p. 267). In the schools, what mathematics was taught was directed heavily to presumed applications in factories and manufacturing centers and was in no way systematic but piecemeal, with constant interruptions dictated by political events.

Five months before the end of the Cultural Revolution, nine American mathematicians visited China under the joint sponsorship of the Committee on Scholarly Communication with the People's Republic of China and the National Academy of Sciences. About higher mathematics education then in China, they write:

> [T]he basic principles of the program [in mathematical education] seem to be fixed. Two dominant guidelines of the new system are (a) "combine theory with practice" and (b) "place politics in command"….To achieve the combination of theory with practice in mathematical education, the students and teachers engage in "productive labor" in a factory as an integral part of any mathematics course (Fitzgerald and Lane 1977, p. 60).
>
> It seems to us that there are two main goals of the Chinese educational system. The first and currently paramount concern is to promote egalitarianism and a socialist consciousness, preventing a sense of elitism among the educated. The second is to provide the training required to build up more technology in China. (Fitzgerald and Lane 1977, p. 63)

As for secondary mathematics education during the Cultural Revolution, schooling throughout the mainland was shortened from 12 to 10 years. Mathematical subjects regarded as being of little practical use – such as transformations of equations, construction of geometrical diagrams, Archimedes' axiom, and incommensurable magnitudes – were all removed from textbooks (Cai 2002, p. 16). But in overemphasizing utilitarian aspects of mathematics, the training of students' basic mathematical reasoning skills and development of their spatial intuition were severely lacking. Also, during the Cultural Revolution provincial governments enjoyed more freedom and control over designing and publishing their own textbooks.

1.7 Mathematics Education Under the Communists (Period III): The 1980s and 1990s

The Cultural Revolution ended in October of 1976, following which a new era in China was soon underway. In February of 1978, the Department of Education issued a new mathematics syllabus for middle and high schools. The guiding principle behind the syllabus was "to use advanced scientific knowledge to substantiate the teaching contents of elementary and secondary schools based on the ways in which students can understand and digest them" (Cai 2002, p. 16). Among the goals of teaching, the aim was to "gradually cultivate students' abilities of analyzing and solving problems." As for the content of instruction, the syllabus explained that subjects of little use in the traditional textbooks should be deleted. It suggested adding basic instruction on determinants, systems of equations, the differential and integral calculus, statistics, and Boolean logic. It also for the first time called for the introduction of the concepts of sets and correspondences. In line with this new syllabus, new textbooks were written and ready for use in fall of 1978. However, these proved to be difficult for some mathematics teachers. As a result, in 1983 the Department of Education decided to use two different levels of textbooks for high school students. One set high standards and included calculus, probability and statistics, determinants, and vectors. The other made statistics and determinants optional (Wei 2008, p. 8). These developments were surveyed by a visiting team of mathematics education experts from North America in 1983 (Steen 1984). In fact, the options for statistics and determinants were often skipped because the nationwide college entrance examinations did not cover them. Subsequently, the college entrance examinations have exerted a lasting influence on mathematics education since they were reinstated in 1977.

In 1978, a *baogao wenxue* (literary news item) about the mathematician 陈景润 Chen Jingrun (1933–1996) and his work of 1973 on the Goldbach conjecture created a national sensation (Wang Shan 2010). Although Chen's work neither proved nor disproved the conjecture, his result was the best obtained to date on the subject. Thanks to his successful story and probably to the powerful propaganda generated by the government, mathematics overnight became a very popular subject and suddenly attracted the best minds of college applicants from 1978 to the early 1980s. One such example is 田刚 Tian Gang (1958–), currently Higgins Professor of Mathematics at Princeton University as well as a Professor of Mathematics at Peking University and an Academician of the Chinese Academy of Sciences, who went to Nanjing University to study mathematics in 1978.

Following the Cultural Revolution, the mathematics curricula at colleges and universities no longer focused on integration of theory and practice, nor were they determined by politics. A standard academic degree system was quickly recreated in 1978, when a few higher institutions began to admit graduate students. On 25 May 1983, a special ceremony conferring doctoral degrees was held at the Great Hall of the People in Beijing. Among the first 18 PhDs awarded in China, ten were in mathematics. They were earned by the following: 谢惠民 Xie Huimin (1939–), 李绍宽 Li Shaokuan (1941–), 张荫南 Zhang Yinnan (1941–), 赵林城 Zhao Lincheng (1942–), 于秀源 Yu Xiuyuan (1942–), 白志东 Bai Zhidong (1943–), 单墫 Shan Zun (1943–), 苏淳 Su Chun (1945–), 李尚志 Li Shangzhi (1947–), and 王建磐 Wang Jianpan (1949–). Two years later, in 1985, a postdoctoral system was created allowing students to carry out advanced research after receiving their doctoral degrees.

In the two decades of the 1980s and 1990s, the old framework for education could no longer serve the needs of a country in which the economy had grown at phenomenal rates, social mobility brought many from the countryside to major urban centers, and the future of China clearly required substantial educational reforms to meet the expectations of the population in general and the growing needs in science, agriculture, and industry across the country in particular. In 1986, a "Compulsory Educational Act" was issued mandating 9 years of elementary and secondary education throughout the country. The last decade of the twentieth century saw adoption of a strategic plan to prepare China for the next century through innovations in education and technology. The new watchword was "ability training," and all subjects, including mathematics, were reformed to meet the demands of ability training (Zhang 2005, p. 4). As the idea of universal mathematical education was widely accepted, mathematics education was faced with new technologies. With computers and hand calculators increasingly available, especially in urban centers, teaching methodologies were revised to include instruction in the use and application of these devices.

Mathematics education since 1980 has been deeply influenced by the open door policy, which was initiated in 1978 (Hayhoe 1989). The numbers of foreign mathematicians who have been invited to give lectures or attend conferences in China have grown from hundreds in the early 1980s to countless thousands by the end of the twentieth century. Chinese mathematicians have also been invited or sent by the government to visit foreign institutions. A larger number of Chinese students either sponsored by the government or supported by themselves or their families have gone abroad to study in Western countries, primarily the United States, to do their graduate work. In the last two decades, some 3,000 Chinese students have received their PhD degree in mathematics, mathematics education, or mathematical sciences in the United States alone (Ding 2009).

On the eve of the new millennium, in 1999 the State Ministry of Education issued a pathbreaking document: The Education Development Program for the 21st Century, which provided a blueprint for education in China that would prepare its citizenry for the modern worlds of science and technology in which China would take its proper place. As part of the EDP, a set of Mathematics Curriculum Standards (MCS) was also promulgated. These were followed up in 2000 with yet another report: The National Curriculum Framework for Primary and Secondary Education. This report addressed, among other pressing issues, matters of national curriculum standards, didactical methodology, assessment, teacher training, and classroom materials, a major goal of which is "to develop students' life-long desire for learning and learning ability" (Sun 2008, p. 75):

> Mathematics education was the first field reformed as suggested by the guiding documents. The State Ministry of Education selected a research group in March of 1999. The research group activities led to the first national mathematics curriculum standards as required by the EDP and the NCF. Only a year later the group presented these standards for compulsory students, the Mathematics Curriculum Standard (experimental version) for compulsory education (MCS 6–15). It was the first new standard among all the subjects in the national educational development program in China. In 2003 the MCS for Senior Secondary Education (MCS 16–18) was finished. (Sun 2008, pp. 75–76)

Listed in the official report are a number of especially "innovative ideas" of the MCS Workgroup (Sun 2008, p. 76):

- All students should learn mathematics, and school mathematics should be essential and appropriate to students' needs.
- The content of school mathematics should be meaningful, realistic, and challenging.
- The content should explain the processes that produce it.
- Math teaching should take into consideration a student's personal knowledge and experiences.

Before implementation of the new MCS, extensive preparations were made in both teacher education and the materials to be used in mathematics classrooms across the country. Education journals carried details of the new curriculum; 10,000 copies of the new curriculum were distributed with questionnaires asking for specific feedback from teachers. The country was divided into nine sections, with mathematicians, mathematics educators, and teachers familiar with the new curriculum all asked to provide their response to the new curriculum and materials, including members of the Chinese Academy of Sciences, college and university presidents, and entrepreneurs in business and technology. Reflecting the major shift in mathematics education mentioned previously, that is, the use of handheld calculators and other new learning technologies that enhance classroom instruction (from Grade 3 on, i.e., 9-year-olds and above), the MCS 6–15 has placed special emphasis on the writing of new textbooks that explicitly make use of handheld calculators and information technology (Sun 2008, p. 80).

Finally, a nationally developed textbook series designed with the new curriculum in mind (six for the primary level, nine for the junior secondary level, and six for the senior secondary level) was upgraded from experimental to official status, with revisions ongoing in light of continuous comments from teachers using these curriculum materials:

> In April 2003, more than a third of the teachers who were using the new textbooks were surveyed and over 1,000 questionnaires collected. Between October of 2003 and June of 2004 the first revisions were carried out while the second ones started in April of 2005. Publication of the newest version is scheduled for the fall of 2006. Tenacious and meticulous efforts have been and will continue to be devoted to the perfection of the new textbooks. (Sun 2008, p. 82)

1.8 Conclusion

It would be difficult to exaggerate the extraordinary development of mathematics in China over the past century. At first taught only in a few government or missionary schools or to small groups working in the arsenals of China at the end of the Qing Dynasty, the first modern reforms of education on Western models occurred early in the formation of the new Republic of China, from 1911 on. With the success of the Boxer Indemnity Scholarship Program, many students trained abroad later returned to establish firm roots for mathematics in the new universities and departments of mathematics like those at Peking and Tsinghua Universities, as well as colleges and universities throughout China.

Abruptly, the advances made in the 1920s and 1930s through the serious efforts of mathematicians working in the universities and those teaching in the schools at all levels were interrupted by the Japanese-Sino War in 1937 and then by World War II. The several decades following creation of the People's Republic of China were hardly enough to reestablish mathematics before the disastrous Cultural Revolution put a hold on most serious mathematics research and teaching at virtually all levels throughout China.

However, with the end of the revolution, it is remarkable how quickly the country was able to recover. In fact, the solid recovery of China through practical mathematics taught at all levels, as well as the research mathematics produced in its universities, is a reflection, basically, of how sound mathematics teaching was prior to the Cultural Revolution. Today, Chinese students compete with outstanding results in the International Mathematical Olympiads, the first of which was held in Braşov, Romania, with only seven countries participating (at the most recent Olympiad, held in Mar del Plata, Argentina, 100 countries competed). China sent its first team of two students (吴思皓 Wu Sihao and 王锋 Wang Feng) to the International Mathematical Olympiad held in Joutsa, Finland, in 1985, where Wu won a bronze medal but the team ranked 34 out of 38 teams. However, just 4 years later, in Braunschweig, Germany, China fielded a full team of six students and placed number one (of fifty teams competing in 1989; among the six Chinese students, four won gold and two won silver medals). Since 1990, China has consistently placed not lower than among the top six teams and has won first place 16 times (out of the 23 Olympiads held since 1990) (IMO 2012).

If judged by the number of medals Chinese students have won in mathematical Olympiads, or the number of mathematics graduate students, or even the number of mathematical papers that have been published, mathematics education in the last two decades of the twentieth century has been impressive. If judged by whether the country's economic, industrial, and business needs are being met, mathematics education has clearly been a success. But if judged in terms of creativity, especially by the number of first-rate research mathematicians educated in China alone, rather than abroad, then mathematics education cannot be rated so highly. Shiing-Shen Chern once said that China will be a big country of mathematics in the twenty-first century. This is certainly a goal it is poised to achieve not only in terms of quantity but in terms of quality as well.

2 Mathematics Education in Modern India

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The mid- to late nineteenth century was a significant period of transition in the history of education in India and more specifically of mathematics education. Its significance arises from the fact that it marked the transition from one epistemic regime to another. While historians have differed in explaining the processes involved, the period is nevertheless marked by some very interesting dialogues and experiments in mathematics education. While pointing this out, one cannot overlook the history of epistemic violence that the colonial project entailed (Cohn 1997). The important concern is the nature of the relationship between the systems of education existing on the subcontinent and the system of education that was first established by the East India Company and later the colonial government. Histories of colonialism have nuanced our understanding of the process from a multiplicity of historiographical perspectives. Without delving too much into the politics of scientific or mathematical knowledge, we shall attempt to chronicle how through institutions of education a new mathematics curriculum was put in place.

2.1 The Encounter of Distinct Traditions

India's encounter with modern European mathematics predated the period of colonial rule. In the early eighteenth century, the astronomer King Jai Singh had established a school of observational astronomy that for later generations acquired an eponymous existence and was patronized by a Mughal emperor. In addition to this astronomical activity, he undertook an intense project to translate Ptolemy and Euclid into Sanskrit and Persian. Furthermore, Jai Singh actively sought European contacts and enrolled some of them into his project. His first exposure to new developments in the sciences was through the Jesuit Superior in Goa, the Portuguese Jesuit Emmanuel de Figueiredo, who visited his court in Jaipur in 1728. He succeeded in arousing Jai Singh's interest and led a delegation of Jai Singh's scholars to Lisbon (Forbes 1982, p. 237).

Towards the end of the eighteenth century as well as the early decades of the nineteenth century, as is known from reports of the East India Company employees, the subcontinent was populated with a large number of schools, naturally with variations across regions, and mathematics learning was an essential part of the curriculum (Adam 1868); Dharampal 1983). These reports often indicate that either the quality of education was in decline, the schools were understaffed, or the teachers were lacking in the intelligence of the methods underlying the knowledge they were imparting (Sen 1991). This trope is frequently employed to justify the reform of the system itself. In any case, with the East India Company's deepening interests first in the colonial city of Kolkata and then in what later came to be called the Presidency towns of Madras and Bombay, a new system of schools and colleges came to be established.

Analogous to other European colonial powers, the East India Company (EIC), a trading company, began to colonize India in the interest of Great Britain. It expanded the territories under its control from the Presidencies of Kolkata, Madras, and Bombay. From the last quarter of the eighteenth century, the EIC acquired more control over parts of India but was subjected since 1784 to the Crown's Board of Control. In 1858, after the 1857 uprising, the EIC was dissolved and British India became governed directly by the Crown.

Originally EIC had no intention to be active in education, but its policy began to change by the end of the eighteenth century. From the 1780s, English civilians had begun to establish schools in Kolkata and Benares. The EIC itself initiated the establishment of education, from 1805; in 1813, a budget of £10,000 per annum was provided for the introduction and promotion of British education in India. This budget was increased by £100,000 in 1833 (Aggarwal 2006, p. 87).

European missionaries and East India Company officials had a variety of motives for establishing educational institutions in the country. By the middle of the nineteenth century, two specific motives prevailed: the mission to civilize the presumably uncultured population and the attempt first by EIC and then the British colonial regime to prepare adequate personnel for the administration of the colony (Cohn 1971).

Before the 1830s, company officials were very careful not to encroach upon existing systems of education (Basu 1982). Thanks to the scholarly work of the early generations of British Orientalists, both the antiquity and deep knowledge of astronomy and mathematics on the subcontinent were highlighted within the Sanskritic and Indo-Persianate universes (Schwab 1984; Baber 1996). This appreciation structured the foundation of the colleges. The founded colleges were "Oriental Colleges," which meant that teaching occurred in the native languages. The Oriental Colleges were initially established as academies and institutions of learning in Oriental studies where company officials and later colonial officials were educated in local languages and customs by English and "native teachers." Gradually, local students were admitted to these colleges and introduced to the modern subjects as well as a Sanskrit or Persian learning.

Among these colleges were the Calcutta Sanskrit College and missionary colleges such as Bishop College in Calcutta, Delhi College, and Benares Sanskrit College. Similarly, at Poona Hindu College, established in 1827, subjects were taught along traditional lines in Sanskrit till 1837.

2.2 Translation and Engraftment: The First Period of British Education

Studies of the Orientalists on ancient and medieval Indian astronomy and mathematics had suggested to the first generation of educators in nineteenth-century India that Indian mathematics (arithmetic and algebra) was "grounded on the same principles as those of Europe" (Dodson 2010, p.25) and that the study of Sanskrit "was worthy in its own right." An educational methodology referred to as "engraftment" was adopted, according to which to "allure the learned natives of India to the study of European science and literature; we must…engraft this study upon their own established methods of scientific and literary instruction." Dodson writes that it was a "conciliatory policy" proceeding from an understanding that "a rational comparison of the contents of Indian and European science would always favour the latter" (Dodson 2010, p. 75).

For some decades thereafter, this resulted in a pedagogy drawing justification from the idea that it was possible to graft instruction in European mathematics onto a base of Sanskritic or Persian mathematical instruction. French and British Indologists soon recognized through their research that there were different mathematical traditions in India. In the Sanskrit tradition the *Bijaganita* and the *Lilavati* were considered the canon, while in the Arabic tradition the *Khulsat-ul-hisab* was seen as the most authoritative tradition on the Indian subcontinent (Hutton 1815, pp. 62–67). These texts were then employed by British educators as mediatory texts that facilitated the translation from local knowledge schemas to those favored by the East India Company. This gave rise to many translation projects in the first three decades of the nineteenth century. Yet, soon enough serious inconsistencies began to appear in the translations that British officials were concerned they would prove counterproductive and undermine respect for European knowledge (Dodson 2010, p. 130).

This resulted in the establishment of Vernacular Translation Societies and Native School Book Societies throughout the subcontinent. The Bombay Native School Book Society was established in 1823 by the new General Committee of Public Instruction, with the task of imparting a modern education in the vernacular (Aggarwal 2006, p. 90). The task of these societies was to identify textbooks used in English schools and colleges on mathematics, physics, and other subjects and initiate the process of their translation and adaptation. The Vernacular Translation Societies began to take up the translation of mathematical texts from English into such vernaculars as Hindi, Arabic, Urdu, Bengali, Marathi, and even Oriya. Felix Boutros would undertake this task for Urdu in Delhi (Raina 1992). Similarly, Lancelot Wilkinson (1801–1841), an official of the Bombay Civil Service, was very closely involved in school mathematics and astronomy education on many fronts and was the most significant force during the early decades when the modalities of engraftment were being worked out.

It is highly remarkable that these translation activities were not limited to Euclid, which characteristically used to be called "a very British subject."[1] EIC functionaries in the first decades of colonial education were typically graduates of two institutions that trained staff for EIC services in India, Addiscombe and Haileybury, and unlike many other British institutions, more modern textbooks were in use (Aggarwal 2006, p. 12 and passim). Correspondingly, these functionaries influenced the choice of textbooks: numerous translations of algebra textbooks can be found. Among others, Bridge's *Algebra*, Hutton's *Course of Mathematics*, and Bonnycastle's *Arithmetic* and *Algebra* were translated (Aggarwal 2006, p. 90). Two of Augustus de Morgan's books, *The Elements of Arithmetic* and *The Elements of Algebra*, were translated into the Marathi language in 1850 and 1848, respectively, by Colonel George Ritso Jervis with the assistance of Vishnoo Soonder Chutry, Gungadhur Shastri Phudkey, and Govind Gangadhar Phudkey.

This emphasis on algebra converges with the belief, established by the Orientalists, that the Indian mathematical tradition was largely algebraic and performative, and the boundary line between Western and Indian mathematical traditions was that algebra was cultivated more in the latter than geometry, which was dominant in England. The British Indologist Colebrooke who had assembled important Sanskrit mathematical and astronomical texts published his *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāscara* in 1817. The text was particularly influential in stabilizing a certain view of Indian mathematics. Colebrooke (1817) points out that in the Indian tradition: "[…] they cultivated Algebra much more, and with greater success than geometry; as is evident in the comparatively low state of their knowledge in the one, and the high pitch of their attainments in the other" (p. xv). This passage was frequently quoted in subsequent histories of science and in the writings of mathematicians as evidence of the algebraic nature of Indian mathematics.

But translation was not the sole problem. Within the traditional order, the *pandit* was endowed with authority derived from his knowledge; in a new order of higher learning, a transformation occurred in the system whereby the professor was endowed with authority, particularly at the Oriental Colleges (Deshpande 2001). Importantly, Wilkinson supported the career of Bāpu Deva Śāstri, a *pandit* who later went on to teach Indian and European astronomy at Benares Sanskrit College from 1841 and whose extensive publications included a work on the modernization of Indian astronomy. Among Bāpu Deva Śāstri's students was Sudhākar Dwivedi, who played an equally important role in mathematics education. In addition to authoring several books in Sanskrit, he wrote in Hindi and, till the end of the century, was closely involved in committees to stabilize scientific and technical vocabularies in the vernaculars (Minkowski 2001, p. 92).

Due to the high esteem of mathematics in both the Hindu and Arabic traditions, mathematics constituted one of the main teaching subjects in the Oriental Colleges, although the position of mathematics among other subjects in Great Britain at this time was relatively low.[2] Yet, despite this evident respect in the first decades, teachers of mathematics were in very short supply. In one case, a surgeon was eventually hired because he had at least some experience teaching mathematics. Thus, the level of teaching mathematics in the Oriental Colleges was very elementary (Aggarwal 2006, pp. 92, 98).

[1] See chapter on the United Kingdom in this handbook.

[2] See chapter on the United Kingdom in this handbook.

2.3 The Second Period from 1835

The first period of British education in India, from about 1800, came to an abrupt end in 1835, abolishing engraftment and substituting it with an Anglicization of the colleges. This rupture is linked with the name of Thomas Babington Macaulay (1800–1859), British historian and Whig politician. Initially secretary to the Board of Control, he served on the Supreme Council of India between 1834 and 1838. With his well-known "Macaulyan Minute" of 1835 (also known as "Bentinck's Resolution"), he convinced Parliament and the Governor-General of India to substitute teaching in native languages with teaching in English. Aiming at a class of anglicized Indians, Macaulay denounced the value of the traditional cultures in India, depreciating Arabic and Sanskrit. He called the local languages "poor and rude" and thought it not exaggerated "to say that all the historical information which has been collected from all the books written in the Sanscrit language is less valuable than what may be found in the most paltry abridgments used at preparatory schools in England." He referred to works on science as, among others, "astronomy which would move laughter in girls at an English boarding school" (Macaulay 1835).

Macaulay wanted to constitute an elite class of Indians and "form a class who may be interpreters between us and the millions whom we govern." It should be left to this class "to refine the vernacular dialects of the country, to enrich those dialects with terms of science borrowed from the Western nomenclature." Some of these translations were often compilations from several mathematical works and not of any one work. Thus in 1844 Yesudas Ramachandra, in addition to authoring two original works in mathemathies published in Urdu the *Musallas-o-Tarashai Makhrooti was Ilm—i-Hindsasah-b-Algebra* that was a translation compiled from portions of Hutton's *Trigonometry*, Boucharlat's *Conic Sections*, and Simon's *Analytical Geometry* (Habib and Raina 1989). In this sense, the elaboration of scientific terminology in the Indian languages underwent considerable development throughout the nineteenth century. Sanskrit pandits such as Sudhākar Dwivedi and mathematics teachers were producing new mathematical works which were not translations of English books but written with the express purpose of developing a new conceptual vocabulary and lexicon that the new mathematics would entail. Dwivedi's works like the *Samikarna Mimāmsa* would belong to this genre.

The first immediate effect of the new policy was that all government-funded activities of translating English textbooks were canceled. Instead, textbooks were now bought and imported from Britain. Among these were a number of moderately modern textbooks (at least regarding the situation of mathematics in British general education), like Bridge's *Algebra*, Hutton's *Logarithms*, de Morgan's *Arithmetic*, Euler's *Algebra*, Playfair's *Geometry*, and Herschel's *Astronomy and Natural Philosophy* (Aggarwal 2006, pp. 100ff.).

Regarding the education system in India, government policy was to substitute Oriental Colleges with English Colleges. This process required a certain time to become realized. Some colleges, particularly Calcutta Sanskrit College, were reluctant to follow the government policy; there, one continued to use *Lilavati* and *Bijaganita* for teaching (Aggarwal 2006, p. 109). These texts, as was the case with Ramchudra's book on calculus, were often used as starting points to develop an understanding of the new mathematics (Raina 1992; Raina and Habib 2004). Eventually, all Oriental Colleges came to use the English language for teaching, including the Hindu College. During this period, mathematics teachers began to arrive from England. Teaching now became more systematically organized, following a curriculum that ascended from algebra (solving equations) and plane geometry to "elements of natural philosophy (mechanics, hydrostatics, hydraulics and pneumatics), plane trigonometry and conic sections, practical surveying, and even to integral and differential calculus, spherical trigonometry, and astronomy" (Aggarwal 2006, p. 102). There is evidence that in 1847 students correctly answered questions on differential calculus (p. 111).

In 1840, the government took a further step to expand the educational structures: a normal school for training teachers was established at Calcutta (p. 114). A list of textbooks for mathematics teaching in the 1850s in the three Presidencies of Bengal, Bombay, and Madras shows a broad variety of mathematical subjects – including differential calculus – and of British textbooks, including Euclid which was less visible earlier (p. 132). Yet, some graduates of British universities began to insist on "the rigour of geometric demonstration" (p. 138).

2.4 Education in India Under British Government, to 1947

The year 1854 saw the establishment of a formalized administrative structure for education in British India. A "Department of Education" became inaugurated, with "Directors of Public Instruction" in each of the three Presidencies. The foundation of universities became established; on the lines of University College London. Universities at Madras, Calcutta, and Bombay were founded along these lines. All colleges, both government and private, had to be affiliated with the university in their presidency. This university would organize examination procedures and confer degrees. Thus, education in British India organized and administered by the state quite early.

In the decades to come, this development proved to be relatively stable. An increasing number of mathematics teachers were sent from the United Kingdom. At the same time, persons trained in Indian education became mathematics teachers as well. Yet, the shortage of mathematics teachers was a matter of concern and complaint. Given the growing number of teachers graduating from traditional British universities, the use of Euclid as a textbook, a methodology, and a subject for examination became even more enhanced. In the syllabi and lists of textbooks, one finds the term "Euclid" essentially synonymous with (plane) geometry (Aggarwal 2006, pp. 155ff., 168). English and mathematics were the two disciplines in which the greatest number of students failed.

Although at first mathematics was a compulsory teaching subject, it was later made optional, reflecting more the education system in Britain. Eventually, when the school system became more developed and extended in British India, mathematics became compulsory for students up to grade X; for girls, however, mathematics in grades IX and X was replaced by teaching "household accounts and domestic science" (Kapur 1988, p. 35).

The syllabi in schools were demanding, from arithmetic and algebra and geometry to trigonometry and logarithms and conic sections (Aggarwal 2006, pp. 155–158). The mathematics curriculum consisted of three branches: arithmetic, algebra, and geometry. While arithmetic greatly emphasized multiplication tables and mental arithmetic as well as commercial arithmetic, algebra centered on polynomials and their factorization. Geometry was "a watered-down version of Euclid with theorems and proofs but not much emphasis on axioms. Constructions with ruler and compass were quite popular, but proofs were seldom given" (Kapur 1988, p. 36). The upper grade course consisted of algebra (from quadratic equations to the binomial theorem), trigonometry, and conic sections, using both synthetic and analytic methods.

Textbooks were now being produced systematically: in 1873, the publisher Macmillan, London, launched a special branch of textbooks for India in English. One of Macmillan's bestsellers became Barnard Smith's *Arithmetic for Indian Schools* which included both Indian and English weights, measures, and currency. From 1876 to 1938, more than three million copies of this textbook were sold, and it was translated into Urdu and Hindi. In addition to versions of Euclid such as F. H. Stevens' *A Textbook of Euclid's Elements Books I-IV* (1889), reformers of geometry teaching in England became available in Macmillan's series, too: J. M. Wilson's *Elementary Geometry and Conic Sections* (1882). In the twentieth century, a second publisher entered this section of the schoolbook market: Longmans, Green, and Co (Aggarwal 2006, pp. 199ff.). Given that English was by now fully accepted as a teaching medium, schoolbooks in use in Great Britain at this time were imported or reprinted in India and often translated into Indian languages.

By contrast, in the appropriation of Western mathematics up to the early decades of the twentieth century, several lexicons in different vernaculars were prepared to arrive as the vernacular equivalents for corresponding English terms in higher algebra, trigonometry, conic sections, analytical calculus, and differential and integral calculus. Several attempts were premature, but lexicons were produced – for example, important lexicons in Hindi and Marathi were published (Ranade 1916, p. v). The lexicons were derived from the textbooks prepared in Hindi, Sanskrit, and Marathi and drew upon other publications such as a work on algebra by Oke, Sardesai, and Dwivedi and works on geometry by G. V. Karkare, practical geometry by Korgaonkar, and higher algebra by Gangadhar Shastri, Tilak, and Jambhekar (Ranade 1916).

2.5 Emergence of Research in Mathematics

As the twentieth century proceeded, more universities began to dot the cities of the subcontinent. But these universities were merely examining bodies and did not have any facilities for postgraduate teaching and research. While mathematics was taught in colleges affiliated with universities, little research was being done in mathematics in universities. The problematic in this new institutional phase shifted then from translation to the institutionalization of modern mathematics.

An important contributor to modern mathematics was the famous yet enigmatic Srinivasa Ramanujan (1887–1920), one of India's first modern mathematical researchers. Originally, his knowledge of mathematics was based on Carr's textbook of mathematics and did not extend much further. His style of doing mathematics and his mathematical creativity have often been the subject of much discussion among mathematicians, historians of mathematics, and psychologists studying scientific creativity and its relation to culture (Hardy 1940; Nandy 1980). An important issue worth flagging on this score is that in addition to his mathematical training at school and college, other mathematical influences on his work have not been well explored, in particular the influence of local mathematical knowledge that survived in orally transmitted traditions and perhaps still coexisted within the modernized mathematics school curriculum (Babu 2007). The ascent of Ramanujan is now the stuff of legend in the annals of the history of science in modern India. But it is equally important to note that within a few years after his death, a number of mathematicians – some of whom had been in Cambridge even before Ramanujan's arrival – returned to India and began to cobble together a school of mathematics.

From the beginning of the twentieth century, the differentiation between secondary schools and university-level undergraduate level became enhanced, following the British structures. In 1917, a Faculty of Sciences was established, integrating the former Faculty of Engineering; in 1927, a Department of Mathematics was created at the University of Madras.

2.6 Debates and Challenges for Mathematics Education at the School Level in Independent India, from 1947

Kapur (1978) noted:

> In India we have about 80 million children studying mathematics at any one time and we have at least two million teachers to teach them. There is a great diversity in the background of children. About 40–50 percent are first generation learners whose parents do not know any formal arithmetic, though it is possible that some one in the family may know some elements of it. (p. 245)

He continued: “After independence in 1947, enrolments in schools have increased 30–40 times.” In discussing other processes in mathematics education during the period in question, one cannot miss this major change. These processes were, however, of importance.

Once the modern system of education was well ensconced, most school textbooks on present-day mathematics were adaptations of British books. By the early twentieth century, through a century-long process of engraftment, the signature of the processes of adaptation was left only in the sections of arithmetic and mensuration rather than geometry, primarily because the measures of weights and areas remained local as a side effect of the metropolis not reforming to the decimal system of weights and measures. Students were expected to memorize multiplication tables, not just for the natural numbers up to 20 but also for fractions ¼, ½, ¾, 5/4, 3/2, 5/2, etc. This would facilitate the conversion from one set of units of measurement to another (Kapur 1991, p. 47). The central problem in the postcolonial period was that school learners came from widely diverse social, cultural, and linguistic, not to mention highly socially stratified, backgrounds; in addition, some came from tribal societies characterized by oral cultures (Kapur 1978, pp. 243–244). These concerns of social diversity were further intensified by diversity in the systems of schooling, ranging from elite schools, public schools, and central schools to schools lacking quality mathematics teachers and infrastructure. The immediate postindependence period was thus marked by the need for a qualitative and quantitative expansion of mathematics education at all school levels (Kapur 1991, p. 248).

While the program of National Education had its roots in the beginning of the twentieth century, in postindependence India (1947 on), emphasis was placed upon Indian contributions to world mathematics. The task was to overcome any residual alienation or cultural rejection as well as to create a sense of participation in a global community of mathematics. School mathematics textbooks began to include entries on the history of mathematics that were relevant to the accompanying pedagogic material. Textbook writers were encouraged to insert supplementary material on Indian mathematicians such as Aryābhatta, Brahmagupta, Bhāskarācharya, and Mādhavācharya (Kapur 1991, p. 47). Thirty years later at the International Workshop of Mathematics in Goa (1989), more than a dozen challenges were identified for mathematics education in India, but none concerned the cultural translation of science or the legitimation of modern mathematics (Kapur 1988).

However, the report of one of the most important commissions on education in postindependence India – the Kothari Commission (1966) – had reaffirmed the larger societal role of mathematics and science education; in particular, explanatory-deductive reasoning was seen to be symptomatic of the scientific temper. Thus, till the early 1970s, deductive reasoning served as the template for the axiomatic method, which in turn was exemplified by geometry; the curriculum devoted a substantial amount of time to this subject in the hope of developing logical thinking (Kapur 1991, p. 29). Till very late in the course of school education, in fact only in sophomore collegiate education were students informed about the axiomatization program in algebra – the latter was always introduced through geometry. Later, a discussion on “Vedic Mathematics” in mathematics education hardly challenged the foundation of mathematics teaching. The appeal of Indian mathematics was more related to new constructions of national identity rather than the development of an alternate mathematics curriculum. This in itself revealed the institutionally deep roots of the new curriculum.

The Kothari Commission must be seen as an attempt at stocktaking and reform in the educational sector a little over a decade after the achievement of independence. By this time, even researchers in mathematics such as A. Narasinga Rao began to feel that problems of mathematics teaching at the school level required the serious engagement of both mathematics researchers and teachers. To address some of these problems, the journal *Mathematics Teacher* was launched in 1966, and the Council of the Association of Mathematics Teachers was established in 1965. Similarly, the National Council of Educational Research and Training supported the applied mathematician J. N. Kapur in inaugurating the quarterly journal *School Teacher* (Kapur 1967, pp. 2–3). These networks of mathematics researchers stressed the urgency of school-level problems of mathematics education at the Mathematics Education Conference organized in 1966, but this time under the auspices of the Indian Mathematical

Society. One of the outcomes was the creation of five study groups distributed throughout the country to develop curricular material for school mathematics.

The National Council of Educational Research and Training played a leading role in preparing textbooks for higher secondary classes and for students preparing for the national school leaving examinations. Fortunately, a large enough resource of university professors and teachers were available to be called upon to work with school teachers to prepare the new textbooks on algebra, trigonometry, and probability. Later on, in addition to probability and statistics, these textbooks incorporated linear programming and integral and differential calculus. Geometry finally appeared in school textbooks as a mixture of intuitive arguments and heuristic and axiomatic proofs (Kapur 1990, p. 78), but for a long period the subject was at the center of debate as discussed below.

In the postindependence era, mathematics education at the school level was marked by a number of debates and controversies concerning what should and should not be included in the curriculum. Some of these debates overlapped with issues raging in other parts of the world, while others followed the cultural trajectories specific to the introduction of mathematics in South Asia. Furthermore, the pedagogy of mathematics education or teachers' responses to the pedagogy revealed a number of problematic premises and justifications for the curriculum. For example, one justification for extended instruction in Euclidean geometry was the cultivation of deductive reasoning and the introduction to axiomatic systems. The idea that mathematical systems also employ inductive reasoning and that physical systems and other mathematical systems could be axiomatically derived received short shrift at the school level. Formal aspects of mathematics were emphasized in school mathematics books during the 1950s, even though these aspects were not that important in the process of mathematical creation. The overemphasis on deductive reasoning in structuring the mathematics curriculum was sometimes even considered harmful for mathematics education (Kapur 1991, p. 22).

In effect, this produced a polarization of positions between those committed to removing defects in the rigorous teaching of geometry versus groups committed to teaching geometry as an experimental discipline. A third group wished to introduce transformation geometry that could lead to the integrated introduction of linear algebra, group theory and geometry, the concepts of transformation and invariance, and the idea of symmetries. These ideas could then illuminate similar ideas in other fields such as art and architecture, physics, chemistry, and crystallography (Kapur 1990, p. 48).

In global terms, however, the most important issue to emerge in mathematics teaching in the 1960s was the concern over the introduction of the so-called new mathematics and what this meant for traditional mathematics. In the mid-1960s, the Central Board of Secondary Education and the Indian Schools Council altered the curriculum for algebra by introducing sets, relations, functions, number systems, and the principle of mathematical induction employing set-theoretic and logical notations distributed through SMSG textbooks reprinted in India (Kapur 1978, p. 247). This was followed by the substitution of classical geometry with SMSG-type geometry. However, a few years later, the old geometry was reinstated within the curriculum and matrices, and differential equations were introduced at the secondary education level (Kapur 1978, p. 248). In short, the Association of Mathematics Teachers of India decided that features of the new mathematics such as the emphasis on concepts should be incorporated into mathematics teaching; discovery methods and transformation geometry needed to be retained, without fetishizing the use of set theory or the axiomatic approach to geometry. The strategy of reconciling new and traditional mathematics would build upon "relevant mathematics" (Kapur 1990, pp. 80–86).

In *Pakistan*, after 1947, the major landmarks in mathematics education were very similar to those in India (Alam 1980). The major difference, if any, was that the density of scientific institutions and universities at the time of the partition of the country was higher in India than in Pakistan (Hussain 2008). Relatively speaking, school education in India could draw upon a larger pool of university teachers and researchers to develop mathematics education.

In sum, in the twentieth century several phases in the history of mathematics pedagogy at the school level can be identified. During the first phase extending from the last decades of the nineteenth century to about 1950, the mathematics curriculum was inspired by the British model, and most mathematics textbooks were adaptations of British textbooks. From the 1940s and certainly in the 1950s, the winds of change began to blow. Mathematics researchers and teachers who had either undertaken their doctoral studies or spent time on the European continent became supporters of new ideas of mathematics education, which gradually echoed throughout the mathematics curriculum. The third phase commenced in the 1960s, during which new mathematics curricula were developed for schools as preparation for changes in the mathematics curriculum at the collegiate level. At this moment, global currents in pedagogy began to intersect with national developments. Unlike the colonial period, the 1960s was a time when dedicated educational societies, councils of mathematics education, and societies of teachers began to play a central role in these changes.

On the other hand, as Indian mathematicians joined the global community of mathematicians, the question of local conceptions of mathematical knowledge continued to haunt educators and historians of mathematics in the background. Institutes and social movements such as the Hoshangabad Science Teaching Programme and the Homi Bhabha Science Centre in Mumbai began to examine problems of mathematics teaching and inaugurated mathematics education research. By the end of the twentieth century, India returned to the translation problem at the level of pedagogy. Informed by the experiences of mathematics teachers, and certainly by the discourse of ethnomathematics and multicultural education, India's National Curriculum Framework Document of 2005 (NCF 2005) while addressing mathematics education highlighted the importance of everyday knowledge for children growing up in different communities and cultures that should be brought to bear in the classroom as scaffolding to impart knowledge of "academic mathematics." In a way at the beginning of the twenty-first century, the country has again returned to the problem of translation, but in a radically different historical era, where both global and local contexts have changed.

3 Mathematics Education in Southeast Asia

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3.1 Early Days

Southeast Asia refers to the land area east of India, south of China, west of Papua New Guinea, and north of Australia. It was called the South Seas by the Chinese. Now the countries in the region are known as ASEAN countries (Association of the Southeast Asian Nations) and consist of Brunei, Cambodia, Indonesia, Lao, Malaysia, Myanmar (Burma), the Philippines, Singapore, Thailand, and Vietnam. East Timor is not a member of ASEAN.

The kingdoms in the region before the colonization by European countries were very powerful. The Spanish came to Cebu, the Philippines, as early as 1521. The Portuguese arrived in Malacca, Malaysia, in 1511, followed by the Dutch, and later the English and the French. All countries in the region, except Thailand, were colonized gradually and successively from the sixteenth to the nineteenth centuries. The countries in Indochina, namely, Vietnam, Lao, and Cambodia, were colonies of France, while Myanmar (Burma), Malaysia, Singapore, and Brunei were colonies of England. Indonesia was Dutch, and the Philippines was Spanish and then American (United States). The boundaries of nations, partitioned by the European powers, were not necessarily the natural boundaries of the original states.

Before the Europeans came to this part of the world, the strongest influences came mainly from China and India. A great majority of the Chinese migrants, who traveled all over the world, settled in Southeast Asia. The Buddhist temple Borobudur in Central Java is evidence of the Indian influence in Indonesia. The Islamic faith spread to Indonesia via the Indian subcontinent. Arabs, being seafarers and traders, were also frequent visitors to the region. As physical evidence, both Chinese and Arabs left their imprint at Dragon's Teeth Gate, a natural harbor adjacent to Labrador Park in Singapore.

The colonial powers left behind their legacy. For example, the Vietnamese language was recorded in written form by Catholic priests from France. Tagalog, a common dialect in the Philippines, contains many Spanish words. Malay and Indonesian languages, originally written in Arabic script, were Romanized using English and Dutch alphabets, respectively, and their spelling was unified many years later. Many other examples can be found as well.

At the time, going to schools was a privilege; the boys outnumbered the girls in schools. In countries other than Thailand, education was provided on a small scale by the colonial government; missionaries and local communities provided the rest. After graduation, it was common for students to work for the government. The school language was the language used by the colonial government, except in Thailand. Mathematics was the next most important subject after language. As far as the educational system is concerned, there were two camps: English and continental European (although it is difficult to describe these differences in a few sentences, the difference between the Anglo-Saxon and Latin cultures can still be discerned in daily life and the development of education in the Southeast Asian countries). The Philippines belongs to neither camp. When the United States took over the Philippines in 1898, they established an extensive public elementary and high school system, beginning in 1901, and so the Philippines followed the American system.

Thailand followed a different path. King Chulalongkorn or King Rama 5 (1853–1910) had many sons whom he sent to be educated in England, France, Germany, and Russia. Subsequently, modern education was brought to Thailand as a mixture of English and continental European traditions. Following World War II (1941–1945), it became more Americanized, paralleling the system established in the Philippines (see above). The school language always was and remains Thai. The country produced its own teaching materials and employed its own people as teachers. Sometimes, however, they also used foreign textbooks, for example, *A School Geometry* by H. S. Hall and F. H. Steven (1917).

In those days throughout Southeast Asia, the syllabus was almost completely defined by the textbook, which was often foreign. A secondary school textbook normally consisted of examples and exercises: at first, teachers went through the examples and then students did the exercises. For example, in Singapore, the textbooks in use included *General Mathematics* by Durell (1946, first time published around 1910) and *College Algebra* by Fine (1904). The Fine book, published in the United States, was translated into Chinese and used by Chinese schools in Singapore for Grade 11.

Many details were adopted from abroad. For instance, the largest number in a multiplication table typically was 12 times 12, following the English, rather than 10 times 10. Also, teachers were often recruited from outside the region.

In short, the Western powers brought to the region trades and modern (Western) education. Importantly, the region was much more connected and united before the outsiders came than it was afterwards.

3.2 Becoming Independent Nations

After World War II, each country in the region became independent and the objective of education changed. Education was no longer for a few but for all. After independence, each country underwent a period of reconstruction and nation-building. Energy was spent on building more school premises. Initially, the attrition rate of students in schools was high. Many students did not complete the 10 years of schooling and left schools after Grade 8 or even earlier.

The age of entering primary school was between 5 and 7 years old, and it is fair to say that it gradually converged to 6 years old. The length of study before entering tertiary institutes varied from 10 to 13 years, with a norm of 12 years. Students who planned to take science in the university in Singapore would have typically spent more time doing mathematics during their school days than other students. This was accomplished by allowing this group of students to take two examination subjects in mathematics – a typically English practice. In general, some practices that had been developed in England but canceled there later continued to survive in Singapore. No drastic changes occurred in mathematics content and teaching during this period until the Math Reforms, discussed in the next section.

The medium of instruction in most countries reverted from foreign languages to their own languages. For example, in the Philippines, English was used. In Singapore, English was used in government schools, whereas other languages were used in the other schools. However, starting from 1976, science and mathematics were taught and examined in English in all schools in Singapore, although it was not fully implemented until 1984. In Thailand, English textbooks were used sometimes but instruction was conducted in Thai. In Indonesian universities after World War II, mathematics was taught initially in Dutch, then for about 2 years in English by American professors once the Dutch professors withdrew, and finally in Indonesian. However, schools in Indonesia used the Indonesian language. It is also relevant to mention that Myanmar in 1991 started to use English textbooks in science and mathematics for Grades 9 and 10 (see Khaing Khaing Aye 2010). Then subjects were taught in both Burmese and English. Around 2010, Indonesian and Malaysian also attempted to teach mathematics in English. Teachers were trained in their own countries, thereby leading to the establishment of their own teacher training colleges.

At first, no formal “streaming” or “tracking” was in place in schools, and weaker students benefited from peer learning. Later, streaming was introduced, however.

The spiral approach to teaching mathematics was implemented in Singapore as early as 1959 (spiral refers to teaching a topic at one grade level at first and then revisiting the same topic at higher grade levels again and again). The spiral approach was not fully implemented until many years later. The Philippines did not initially adopt the spiral approach but do so after the Math Reforms. It is interesting to note that this country reverted to the non-spiral approach recently, that is, teaching algebra in 1 year and geometry in another year, after it was found that it was easier to train teachers to teach topics separately than spirally.

In summary, countries in Southeast Asia either inherited or adopted Western education. After World War II when these countries became independent, the teaching of mathematics was not much different from the pre-independence days. However, colonization obviously influenced the growth of local culture in different ways. In a sense, at some point development was frozen under the umbrella of preserving indigenous culture and tradition.

3.3 Math Reforms

The most important event of the past 50 years, as far as mathematics education is concerned, was Math Reforms. It began in the 1960s or even earlier in the United States and Western Europe and in the 1970s in Southeast Asia. The mathematical communities in some countries in the region were dismayed by the reforms, although it made an indelible mark on mathematics curricula and mathematics teaching in the classroom.

That said, not all Southeast Asian countries were affected. When the Viet Cong (North Vietnamese) entered the city of Saigon on 12 April 1975 and renamed it Ho Chi Minh City, the Vietnam War and the 35-year struggle for independence by the Vietnamese ended. On the other side of Indochina, Myanmar (Burma) did not join the British Commonwealth after independence as did other British colonies (the British Commonwealth is an organization of 54 English-speaking nations, each of whom is associated in some way with Great Britain). Understandably, Vietnam, Myanmar, and neighboring countries in Indochina (except Thailand) were not affected at the time by the Math Reforms.

The Math Reforms came to Southeast Asia in the form of new syllabi, new textbooks, and massive retraining of teachers in order to teach new mathematics. A memorable event was the first Southeast Asian Conference on Mathematics Education held in Manila, the Philippines, in 1978 (see Lim-Teo 2008). The theme of this conference was the Math Reforms. Much effort was exerted into organizing the conference and many projects followed in the aftermath, including the production of new textbooks.

A major change occurred in content. The concept of "set" was introduced in Grade 7 in Singapore and in a Grade 1 textbook in Malaysia. In geometry, the term "a line segment" was used in addition to the term "a line," and "measure of an angle" was used in addition to "an angle." Different notations for "minus 3" (-3) and "negative 3" ($^{-}3$) were adopted. Commutative law, associative law, and distributive law were highlighted. Most of these terms disappeared afterwards, but for good reasons, the term "a line segment" stayed and did the Venn diagram. In short, mathematics became more structured and formal. Each country carried out the reforms in different ways and to various extents. Generally, one could say that the Philippines went the furthest and Singapore the least.

The major change in content was in geometry. For centuries, teaching geometry was the same as teaching proofs. This approach was abandoned to a great extent and a certain amount of transformation geometry was introduced. Consequently, school geometry became neither transformation geometry nor classical geometry, and this notion lasted till the end of the reforms. An attempt was made to introduce the concept of transformations into primary schools. Hence, tessellation was inserted in the primary school syllabus.

Mechanics was gradually replaced by statistics, and there was more statistics in the "English camp" than in the continental "European camp." In secondary schools, means and standard deviations were included and went as far as hypothesis testing. Statistics was also in the primary schools in the form of pictograms. Statistics was probably the only topic in the new syllabus that was not of the nineteenth century.

Linear programming was also included in the new syllabus as a way to show that mathematics was useful. To make it available to secondary school students, problems in linear programming were restricted to two variables so that they could be solved using the coordinate plane. It was fashionable at the time to teach advanced mathematics at the elementary level, even though the method used to solve problems in school was not the one used in real situations, as in the case of linear programming.

Binary numbers were not included. However, numerical analysis was a subject for the senior high school level, but was short lived. Cuisenaire rods (colored blocks) were used in primary schools to teach four operations, although they were not popular. At this time, algebra also first appeared in the upper primary schools.

While in many Western countries the new content came almost simultaneously with some new methodological ideas, not all of these ideas immediately became popular in Southeast Asia. Specifically, the so-called discovery method failed to be accepted (in simple terms, the discovery method is the approach by which teachers do not give students a formula but instead offer them the opportunity to discover the formula themselves). The discovery method, however, did influence further development in mathematics education. Problem solving was the center of a syllabus in the 1980s and beyond. In a way, problem solving is a kind of guided discovery.

Due to the new content, teacher training was an urgent matter. To train teachers quickly, special workshops were organized. Although a syllabus was designed and announced, it was not until 3 years later that all schools turned to teaching the so-called new mathematics.

Textbooks were produced either commercially or by the state. Some were written by local authors and some were adapted from foreign textbooks, although these textbooks had the shortest life span. Not only were they no longer used in the classroom, but also many of the newly introduced materials in these textbooks did not appear in subsequent textbooks.

There were, of course, some differences among Southeast Asian countries in the Math Reforms process (above, the experience of Singapore was mainly discussed), but generally the issues discussed were common for all countries involved in the reforms (see Lee 2008). At the International Congress on Mathematics Education (1980) held in Berkeley, California, there was a call for "back to basics" (see Zweng et al. 1983). In Southeast Asia and other countries undertaking the Math Reforms, the standard of mathematics declined globally. In particular, students were found to be weaker in algebraic manipulation. In general, while the verdict for the Math Reforms was negative, it was also true that the Math Reforms hastened the localization of syllabi, textbooks, and teacher training in those Southeast Asian countries undertaking the Math Reforms. It changed the way mathematics was taught in the classroom, and this remains an undeniably positive aspect of the Math Reforms for Southeast Asia.

3.4 Back to Basics and Thereafter

The Math Reforms lasted for 12 years, ending in the early 1980s, when it was realized they did not work and had to be stopped. Although many new topics introduced during the Math Reforms stayed on (e.g., Venn diagrams and statistics), the formal approach in teaching mathematics was replaced by the so-called problem-solving approach. In the years that followed, change in content was minor. The major change was in the teaching approach used in the classroom.

At this time, politically, the ASEAN countries were becoming more united. Within the ASEAN countries, mathematicians and mathematics educators were also having closer contact with each other through conferences and visits (see Lim-Teo 2008 for SEACME [Southeast Asian Conference on Mathematics Education] and EARCOME [East Asian Regional Conference on Mathematics Education]). SEACME was absorbed into EARCOME in 2002, and the latter is still ongoing. Educators in the region have started looking elsewhere for inspiration, including neighboring countries, beyond traditional sources. At the time of this writing (January 2012), many initiatives have been made by various countries in the region, three of which are identified below.

PMRI in Indonesia. PMRI stands for Pendidikan Matematika Realistik Indonesia or Realistic Mathematics Education in Indonesia. The project started in 1998 and received official funding in 2001 together with the help of Dutch educators. Its goal was to bring change to classroom teaching in Indonesia. For details, see Sembiring et al. (2008), Sembiring et al. (2010), and Lee (2010).

Benchmarking Singapore by the Philippines. The Philippines made a concerted effort to benchmark Singapore in teacher training in mathematics and science. For details, see Nebres (2008) and Lee (2011).

A new syllabus 2013. Singapore received a new syllabus in 2007. The syllabus was revised in 2013 to include more description in the teaching approach. One item is on the use of mathematical modeling. For details, see Ministry of Education (2013).

Clearly, the countries in the Southeast Asian region proceed to create their own opportunities and find their own ways to meet challenges in mathematics education.

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Chapter 19
Mathematics Education in Africa

Alexander Karp, Charles Opolot-Okurut, and Gert Schubring

1 Introduction

This chapter deals with mathematics education in Sub-Saharan Africa. The North African history of mathematics education is not discussed here.

During the Middle Ages, North Africa had become part of the countries of Islamic civilization; later, in modern times, as with other parts of Africa, it became an object of Imperialist European policy. Egypt, Libya, and the Maghreb – Tunisia, Algeria, and Morocco – came under British, Italian, French, and Spanish rule. The countries of this region have their specific histories. Chapter on the history of mathematics teaching in Tunisia represents a case study for North Africa.

The coasts of Sub-Saharan Africa were first cruised in the fifteenth century by Portuguese while exploring a way to reach India. These coasts at first served mainly to provide a staging post during the traveling. Later on, the coasts of West and East Africa proved convenient for European and Arab slave hunters and traders. For a long time, however, there was no colonizing policy: Europeans did not systematically enter the interior of the African continent, not to say established colonies. It was only during the period of Imperialist expansion in the second half of the nineteenth century that establishing colonies became a major concern for European powers, triggering a run to occupy as many colonies in Africa as possible. Latecomers like Germany wanted a share as well. At the Berlin Colonial Conference of 1885, the European Powers divided Sub-Saharan Africa among them: the main share for Great Britain and France and minor parts for Portugal and Germany. Belgium obtained the Congo with the unique agreement of being a private possession of the Belgian Crown. The only country remaining independent was Ethiopia, an old Christian Empire. As late as 1936, Fascist Italy attempted to transform Ethiopia into a colony (without any effective countermeasures from the League of Nations). Another special case is represented by South Africa: Cape Town was first a staging post for the Dutch East India Company and developed from the second half of the seventeenth century into the

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Cape Colony. Later, conflicts between the Dutch immigrants, the Burens, and the British determined the history of the South Africa; it became essentially independent in 1926.

After World War I, Germany lost its colonies (Deutsch-Südwest, today Namibia and Deutsch-Ostafrika, today Tanzania) and the British obtained them as a League of Nations mandate. Belgium, England, France, and Portugal remained colonial powers for decades more. It was only in the 1960s that the former colonies gained their independence.

The history of mathematics education in Sub-Saharan Africa is probably the least researched in international history; almost nothing is known about pre-colonial times. The influence of the metropolises on education generally and on mathematics education specifically has been very strong. This chapter contains a brief overview of the former French, British, and Portuguese colonies (the first is to some extent exemplified by Benin, and the last two are presented in a form of a case study, specifically on Uganda and Mozambique). Also, some information is provided on South Africa.

The section on Uganda was written by Charles Opolot-Okurut. Reports on other regions and general organization were completed by the editors of this volume. In the sources for the chapter, special note should be made of the works of Paulus Gerdes, which are often cited below.

2 Background of Modern Development

The modern development of African mathematics education has been largely shaped, on the one hand, by existing traditions which evolved during the pre-colonial era and continue to exert influence to this day, and on the other hand, by European and, subsequently, international influences.

As already noted, traditional education has not been sufficiently studied. Gerdes (1981) points out that traditional education in Mozambique was weakly institutionalized.

> Most training was informal: emulation of older children, listening to stories, watching and helping adults go about their daily tasks, singing and dancing, and games (incl. mathematical games such as *ntchuva* and a version of the ‘three-in-a-row’ game *muravarava*). In the initiation schools, the children were trained under strict discipline by little sleep, hard labour, long walks, cold, etc., for obedience to the rules of society. The aim of education was to bring the new generations to accept blindly the traditions of feudal society held up to them as dogma: the authority of the elders, tribal sentiments, contempt for women, and superstitions which reflect a superficial understanding of nature. In order to deal with problems related to production empiric-mathematical ideas were developed and transmitted to the young children. In the Nampula province, for example, children learn to use a stick to draw a circle in the sand to place fish equidistant from a fire for drying. In the Cabo Delgado province, children learn that the way to make sure that the base of a hut really is rectangular by making sure that the two diagonals are of equal length. (p. 456)

In her writings on other African regions, Zaslavsky (1973) studied the distinctive features of traditional African mathematical culture, as expressed first and foremost in art and architecture, identifying instances of attention to patterns and counting. Vogeli (1992) quotes from unpublished studies of South African traditional mathematics, which indicate that all available mathematical conceptions were employed in one form or another in everyday life and were studied within the context of everyday life. Mathematics as a distinct subject of study did not exist, which does not mean that traditional communities did not possess developed mathematical knowledge, including, for example, the ability to draw and make such shapes as circles, triangles, rectangles, parallelograms, stars, and so on or to estimate length, mass, heat, and volume. Importantly, as Gerdes (1981) emphasizes, “traditional education was imbued with a magic-religious view of the world, exemplified by the following taboo against the counting of men: ‘What? You are counting us? Whom do you want to see disappear?’” (p. 456).

Yet European missionaries brought some elements of European formal education into the continent. This education, however, was usually limited to religious propaganda with only very minor additions – even during the first half of the twentieth century. The ability to perform the four arithmetical operations was typically the summit of any education carried out on a large scale. More advanced educational practices were established if only to prepare functionaries for colonial administrations. The number of institutions providing such an education was very small – even during the last period of colonial power. It would be accurate to say that traditional mathematical culture existed side by side with formal mathematics; moreover, a clear majority of the population was exposed only to the former. As Gerdes (1981) writes:

> There existed two categories of the school system. Government schooling was reserved almost exclusively for the children of the settlers, particularly at the levels of secondary and higher education. The few Mozambicans who had access to schools were taught at the Roman Catholic mission schools.
>
> The black African children were taught the history and geography of Portugal. They were taught to despise their own culture, to submit themselves to the colonial and religious authorities, and to accept as valid the values of the colonial-capitalist society. The objective was to transform some Mozambicans into an elite—into *'assimilados'* or 'black Portuguese'—who could be docile servants of the interests of colonialism. Those Mozambicans were taught some mathematics to calculate better the compulsory quota of cotton production or to be more lucrative 'boss-boys' in South-African mines. (p. 457)

Such a picture was typical of other colonies as well. The colonies had schools for the colonizers which admitted only very few, if any, Africans, and education in these schools fully met European norms. Alongside them were schools for the local population, providing education at an elementary level.

When more advanced mathematics education was offered, it almost entirely copied education in the mother countries. Thus, schools in British colonies used British textbooks, curricula, and examination systems, including, for example, the division between the so-called O-level education (corresponding to general secondary education) and A-level education offered in the upper grades and more specialized (see, e.g., Doku 2003). Similarly, in the French colonies, both the textbooks and the curricula were French, while in the Portuguese colonies they were Portuguese. Colonialism left the African countries largely illiterate and without prepared teachers. Inevitably, the curricula and textbooks of the European-colonizing states continued to exert a significant influence even after the proclamation of independence, pulling only recently liberated countries into transformational changes taking place in mathematics education (say, in connection with New Math) in Europe and North America, for which they were unprepared.

3 Former British Colonies: Case Study of Uganda

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3.1 Introduction

In Uganda, mathematics education is less than 30 years old, but mathematics as a subject has always been in the school curriculum. To examine the historical and current development of mathematics education, this section identifies two eras – pre- and post-independence – by looking at the changes that have occurred in mathematics education. Most developing countries like Uganda still follow much the same curricula left over by colonial masters and have found it hard to start new innovations without external aid. The goal of mathematics education in Uganda is to develop the learners' mathematical concepts and computational skills that will enable them to live effectively in society and to contribute to the nation's social and economic development. Wilson (1992) argues that mathematics education in Africa is characterized both by diversity and similarity, and strategies for curricular changes are developed through both adopting or adapting an existing course from elsewhere and writing a new course.

3.2 Overview of Mathematics Education in Uganda

The education system in Uganda today results from the social, economic, and political events that have happened in the country over time. Mathematics education in Uganda follows the development and advanced experiences of mathematics education across the world. Mathematics education in the countries of Sub-Saharan Africa face similar challenges, including poor quality teaching and learning, inadequacy of mathematics learning materials, examination-oriented curricula and teaching, poor learning environment, and poorly motivated, trained, and updated teachers. Mathematics is compulsory for all primary- and ordinary-level students in the country (Opolot-Okurut and Opyene Eluk 2011, p. 16). Schools follow a national curriculum, syllabus, and textbooks recommended by the Ministry of Education and Sports. Also available are other textbook series written according to National Curriculum Development Centre (NCDC) requirements. The administrations of a few schools choose their own textbooks for their students.

Uganda has a pyramidal education structure with three levels of education: pre-primary and primary, secondary, and tertiary (Government of the Republic of Uganda 1992, p. 10). Mathematics is taught at all three levels. The pre-primary level is for pupils ages 3–5 years and lasts for 3 years, and the primary level is for pupils ages 6–13 years and lasts for 7 years (P1-P7). The secondary level is divided into two parts: (1) ordinary level for students ages 14–17 years and lasting 4 years (S1–S4) and (2) advanced level or technical or vocational education for students ages 18–19 years and lasting 2 years. The tertiary education level is for students ages 20–24 years and lasting between 2 and 5 years for the award of relevant certificates, diplomas, and degrees. Programs for Post Graduate Diploma in Education (PGDE), Master's and Doctoral Studies in Mathematics, and Mathematics Educations exist in several higher education institutions.

3.3 Pre-independence Period

In 1925 when Uganda was a British protectorate, the Phelps-Stoke Commission, a cornerstone in the development of education in Uganda, recommended adapting the British education system in Uganda (Quinn 1983, p. 10). This recommendation caused a rethinking of the country's educational policy. This period was characterized by:

1. The destruction of the African traditional education system that was based on the home setting and conducted through apprenticeship
2. The organization of a Western-style educational system that was formal, based on classroom teaching, and required trained teachers
3. The creation and acquisition of adequate resources in schools that supported the teaching and learning of mathematics (often supplied from abroad)
4. The presence of more expatriate teachers than local teachers
5. The plurality of the mathematics curricula

In 1952, African and European educators from several African countries attended the Cambridge Conference on African Education which was an important step in Uganda education.

3.3.1 Primary (Mathematics) Education

Africa had African indigenous/informal education before the arrival of the missionaries. In informal education, skills, knowledge, and other pertinent values were transmitted across the generations in the home setting. Education was based on clan and tribe values as the main political units for developing responsible members of society. Africans wanted to preserve their cultural heritage by transmitting their traditions, values, and norms to ensure the continuity of their culture. Practical skills were passed on to children to perform certain necessary tasks for the survival of individuals and the community through apprenticeships.

Formal education commenced in East Africa after the arrival of European missionaries from Britain, France, and others countries. But formal education destroyed indigenous education (Quinn 1983, p. 9). The mathematics curriculum in the primary schools was the "traditional" type concentrating on teaching a*R*ithmetical skills, *R*eading, and w*R*iting – the code named the basic 3Rs. Teachers for the schools were from religious leaders and vernacular teachers from the schools' locality.

3.3.2 Secondary (Mathematics) Education

The mathematics curriculum in Ugandan secondary schools was similar to that in Britain. It was the "traditional" mathematics syllabus A including separate papers in (1) Arithmetic and Trigonometry and (2) Algebra and Geometry, lasting 4 years to prepare for the Cambridge Overseas School Certificate. The textbooks used were similar to those in British schools like the *School Mathematics Series* by H.E Parr. These textbooks were not adapted to the Ugandan context, needs, and demands. Similarly, most schools were studying a combined mathematics syllabus B for examinations. While the schoolteachers were from Britain, many teachers of mathematics were non-mathematicians who had simply attended mathematics courses. At the very end of Colonial Era in 1962 appeared the innovative African Mathematics Program (AMP) (Entebbe Mathematics program) influenced by the School Mathematics Study Group (SMSG) in the USA. The AMP recommended writing a new set of mathematics textbooks combining existing textbooks with SMSG materials for schools in Africa.

3.3.3 Tertiary (Mathematics) Education

In 1922 Makerere College (now Makerere University since 1970) was established in Uganda for training teachers and artisans. By 1933 Makerere College introduced courses leading to the Cambridge University Overseas School Certificate as the Cambridge University (UK) award.

3.4 Post-independence Period

After independence on October 9, 1962, Uganda's education system continued to duplicate the education system of their colonial master (Britain), using the same syllabi, textbooks, and examinations. Between 1962 and 1972, mini-adjustments were made on syllabus content. Since 1983 the mathematics syllabus has remained virtually static at all levels, with minimal adjustments to the original syllabus. Curriculum development was still viewed as the production of new syllabi and textbooks without seriously addressing approaches and assessment procedures. In 1992 the Government White Paper on Education was issued (Government of Republic of Uganda 1992).

3.4.1 Primary (Mathematics) Education

In the 1990s, the Uganda National Examination Board (UNEB) offered a primary mathematics syllabus that aimed "to develop simple manipulative and computational skills, understand basic fundamental mathematical concepts and simple applications, and acquire clear mathematical expression and careful mathematical reasoning" (UNEB 1991, p. 69). The primary mathematics syllabus

Table 19.1 Examination registered candidates for mathematics by year and level

	Year				
Level	2007	2008	2009	2010	2011
Primary	419,206	463,658	488,745	490,374	514,916
Ordinary level	190,829	197,892	213,247	259,891	265,351
A-level	16,773	21,334	21,186	22,564	22,750

Source: UNEB Statistics (2011)

contained ten topics (NCDC 1999; UNEB 1991). The current primary mathematics syllabus is organized into six major themes, namely: Sets, Numeracy, Geometry, Interpretation of Graphs and Data, Measurement, and Algebra (NCDC 2010, p. 144) covering 12 topics. The syllabus suggests a problem-solving teaching approach based on Polya's (1957, pp. 36–37) stages of problem-solving. To teach in the primary schools, teachers need a 2-year Grade III teaching certificate or 2-year diploma in primary education or a 3-year education degree. The enrollment in mathematics continues to grow but the quality of teaching remains wanting.

Table 19.1 shows the numbers of candidates registered for primary, O-level, and A-level mathematics over the last 5 years that indicate annual growth. Primary, O-level, and A-level candidates increased from about 419,000, 191,000, and 16,700 in 2007 to over 500,000, 265,000, and 22,700 in 2011, respectively.

3.4.2 Secondary (Mathematics) Education

Ordinary Level

In the 1960s, there was plethora of mathematics curricula in the schools. In 1963 the School Mathematics Project for East Africa (SMPEA), a replica of the School Mathematics Project (SMP) in the UK, was adopted in East African countries. In 1967 Britain withdrew from the SMPEA project. In 1968 the East African Examinations Council (EAEC) was set up (East African Examinations Council 1968). The School Mathematics of East Africa (SMEA) was started as Mathematics B (traditional) and Mathematics S (Modern) for the EAEC examinations. In 1970 the National Curriculum Development Centre (NCDC) was established in Uganda to control curricula in the country and the EAEC abolished mathematics alternative A. In 1980 the UNEB took over the functions of the EAEC. In 1981 a single syllabus, from the "mixed" syllabi B and S mathematics, was introduced and was expected to comprise the *good* parts of alternative B and the *good* parts of alternative S. UNEB first examined this mathematics syllabus in 1983. The recommended teaching approach was the discovery approach.

Additional Mathematics

The additional mathematics syllabus for O-level mathematically able students included Pure Mathematics, Vectors and Matrices, Mechanics, and Statistics (UNEB 2005).

Advanced Level

The Uganda Advanced Certificate of Education (UACE 2008) regulations offer two mathematics syllabi: (1) Principal Mathematics and (2) Subsidiary Mathematics. The Principal Mathematics course content includes Pure Mathematics and Applied Mathematics. The Subsidiary Mathematics course content includes Pure Mathematics, Vectors and Matrices, Mechanics, and Statistics to support subjects that use mathematical concepts.

Secondary school teachers need a 2-year Grade V (diploma) teaching certificate or 3-year degree graduate certificate, PGDE certificate, Masters, or doctoral degree certificate.

3.4.3 Tertiary (Mathematics) Education

In 1963 the University of East Africa was established. Today, there are over 30 universities and institutions of higher learning in Uganda offering Arts and Sciences courses. In the Sciences, some mathematics is offered. For example, in the Department of Mathematics at Makerere University, both undergraduate and postgraduate mathematics programs are offered. The undergraduate program lasting 3 years offers a major program in mathematics and a minor program in mathematics core and elective courses (Department of Mathematics 2008).

In the Department of Science, Technical and Vocational Education (DSTVE) in the School of Education, undergraduate, and postgraduate mathematics education programs are offered for the Bachelor of Education of Science with Education program, PGDE, Master of Science Education, and Doctor of Philosophy in Mathematics Education. The undergraduates follow tailored Mathematics Education courses (School of Education 2008). PGDE and Master's students follow prescribed mathematics education modules (School of Education 2007). Tertiary education-level teachers are required to be Master's or Doctoral degree holders.

3.4.4 Professional Development of Mathematics Teachers: Preservice and In-service

Attempts to improve mathematics and science education in Uganda have been run through externally initiated and funded projects. (1) The In-Service Secondary Teacher Education Project, which Britain started in 1994, was intended to improve the quality of teaching in mathematics, English, and science through In-service Teacher Training and to establish a national network of teacher resource centers but was not sustained. (2) The Secondary Science and Mathematics Teachers Program, which Japan started in 2005, aims to improve the teaching ability of Mathematics and Science teachers at the secondary school level; it is ongoing, but the outcomes have yet to be determined.

3.5 Conclusion

In Uganda, although mathematics education is still in its infancy, the country desires national and international acceptability and seeks to maintain international standards amid inevitable challenges. In this paper, I suggest that the long-term goal of mathematics education is to prepare students for the field of work and lifelong learning. Mathematics education is studied at all levels of education in the country. Over the years mathematics education has continued to grow in enrollment but not in quality, despite difficulties and shortages of materials. The mathematics curriculum has undergone adoption, adaptation, and minimum writing of new curricula (Wilson 1992). Due to space limitations, this paper could not cover issues of resources such as textbooks, ICT, the methodology in use, or assessment practices. There is a need to investigate the educational levels, professional careers, curricular practices, instructional approaches, and available resources that were not captured here. Studies are urgently needed on who teaches mathematics in our schools and institutions, the resources that are available to support mathematics teaching and learning, the current learning environment in schools and institutions, the methods teachers use for mathematics instruction, and the education and continuous professional development of teachers.

4 Former French Colonies

France governed a great number of colonies in Sub-Saharan Africa, mainly in West Africa. The schools established in these colonies were destined to form an elite among the colonized people, to train persons capable of assisting in administrating the colony, and thus to contribute to the economic exploitation of the country. Thus, Benin (also known as Dahomey) was considered an African "Latin Quarter" because it produced office workers and teachers to meet not only its own needs but also those of neighboring French colonies (Davis 1992).

While the focus of the schools was to transmit the cultural values and models of the metropolis, thus concentrating on literature studies, mathematics was marginal – only some rudiments being taught were indispensable for transmitting knowledge necessary to maintain functioning equipment and to train teachers charged to give this instruction. In addition to these schools, there were training centers to mold workers who were destined for auxiliary tasks assisting French technicians. No structures were in place to enable teaching at a scientific level, such as engineering schools, technological institutes or universities (Touré 2002, p. 175).

Thus, the end of colonialism and the beginning of independence confronted the former colonies with the enormous challenge of establishing a public education system and in particular of establishing a teaching of mathematics for all. The model of education often remained what it had been before for a very long time and continued to be based on the mathematics curriculum of the former colonizing state; its goals also resembled those set in the past, primarily to prepare administrative personnel. The role of the church likewise remained important: thus, in Benin in 1965, the Catholic Church ran 450 schools, attended by 45 % of the entire school population (Davis 1992).

After a few years of this practice, profound problems with the use of the French curricula were revealed:

– The then-dominating axiomatic approach of the French syllabus proved to be too abstract for the young Africans who were accustomed to audio-visual messages tied to their rural world.
– Teaching was conducted in French and thus conflicted with the vernacular languages.
– Pedagogical problems emerged explained by deep sociocultural differences within the country and their consequences for understanding mathematical concepts.
– The teachers were ignorant – of the mathematical knowledge existing in those societies to which their teaching was directed.
– Given the diversity of the students in the classes, teaching staff proved to be quantitatively and qualitatively insufficient to resolve ensuing teaching problems (Touré 2002, p. 175).

As an example of existing problems, one can point to the fact that even in Benin, which was relatively successful in the field of education, there was an extreme shortage of rural schools: the schools were in the cities; moreover, the school curriculum was entirely removed from the problems of village life. Attempts were made somehow to mitigate linguistic problems – for example, the first grade was devoted mainly to studying French – but this did not help very much (Davis 1992).

In all the former colonies, therefore, attempts were made to reform the curricula and adapt them better to their respective national situations. Thus, starting in 1975, the government of Benin began taking steps to devote more attention to the application of knowledge in practical life, including agriculture. During this process, a substantially greater number of children received access to education. From 1975 until 1983, the attendance rate for Benin's primary schools increased from 41 % to 62 %, while the number of students in secondary schools grew from 41,802 to 117,724 (Davis 1992).

An effective means of improving mathematics education proved to be workshops in which the francophone African countries discussed how to resolve these problems to establish a mathematics curriculum serving all. Such workshops were organized in 1983 (Abidjan/Ivory Coast), in 1985 (Cotonou/Benin), and in 1988 (Conakry/Guinea). Particularly important were the results of the workshop of 1992 in Abidjan. Since at this time New Math had also been abandoned in the metropolis, the syllabi now elaborated for all grades emphasized teaching a more concrete mathematics and becoming more attractive and better adapted to the respective sociocultural contexts, thus also reorienting teacher education (Touré 2002, p. 176).

It is important to emphasize, however, that the development of mathematics education was considerably hampered by existing social problems. Thus, in Benin, which we have already used as an example, it was discovered at a certain point that people with higher education could find no employment in the country; meanwhile, schools lacked resources, which on the one hand made the work of teachers very difficult, and on the other hand not infrequently caused serious delays in paying them their salaries. All of this undoubtedly fueled strong criticism of the educational reforms (Davis 1992).

5 Former Portuguese Colonies: Case Study of Mozambique

5.1 Independence and Its Influence

Mozambique became independent in 1975 as the People's Republic of Mozambique, after more than 10 years of a guerrilla campaign against the Portuguese rule led by the socialist Front for the Liberation of Mozambique (FRELIMO). This political movement, mainly supported by the Soviet Union, had given great attention to education. In the liberated areas, teaching was organized; mathematics textbooks were even elaborated.

> At Independence, the illiteracy rate in Mozambique was still 93%, one of the highest in the world.... School achievements in colonial time were very poor. For example, in 1970 only 39% of the children passed from one grade to the next. The mathematics curricula were the ones used in Portugal, during the 1960s 'enriched' by a formal introduction of the vocabulary of set theory, without, however, using it to understand mathematics. Mathematics seemed to be identified with written arithmetic. Very little attention was given to mental arithmetic and geometry. (Gerdes 1981, p. 457)

Since independence, creating a national educational system for all became one of the country's official goals.

> Private and missionary schools were abolished. Education was nationalised and made free of charge, resulting in an educational *explosion*. In 1973 there were 589,000 children in primary schools. In 1978 there were 1,419,000 out of a total population of about 11,000,000 at that time. There were 33,000 pupils in secondary schools in 1974. In 1978 there were 82,000. During the First National Literacy Campaign (1978), 130,000 were taught in the priority centers (sectors of collective production such as factories, state enterprises and agricultural cooperatives; the FRELIMO Party; the Mozambican Women's Organization, etc.).
>
> The nationalisation of formal education made it possible to standardise primary school education (from the first to the fourth year), and to allocate resources more equally among the different regions of the country and thus to greatly abolish the regional (town-countryside), racial, and social discrimination of the colonial education system. (p. 460)

Mathematics became one of the pillars of the new educational system. The practice of elaborating textbooks based in African culture – which had started even earlier in areas liberated from Portuguese rule – were continued and extended. The mathematics textbooks in Mozambique, *Eu vou à Escola: Matemática*, for primary grades, and *Matemática? Claro!*, for secondary grades,[1] turned out to be milestones for presenting mathematical concepts in their sociocultural contexts, based on the then-emerging research in ethnomathematics in Mozambique. Gerdes (1981) commented on the difficulties in elaborating the proper textbooks:

> The programmes were developed in a hurry in 1975, in an attempt to respond to the political and socio-economic changes in the country. The mathematics curriculum for primary school was simplified (e.g., the arithmetic programme was reduced so the pupils learned only to operate well with natural numbers and to handle linear measures and money), in order to cope with the difficulties that stem from the weak knowledge of the Portuguese

[1] The consolidated versions date from 1983 (primary) and 1988 (secondary).

> language, the medium of instruction and communication. Already in 1975 the first teachers' manuals had been produced. But the first national mathematics textbooks and exercise books for pupils of the first grade will leave the presses only in 1981 (who could write them before?). At the other end of the educational spectrum, professors of the Eduardo Mondlane University have now produced the mathematics textbooks for the tenth and eleventh grades. But much other educational material is still in short supply. (p. 461)

The realization of new policies in education was considerably slowed down and impeded by a devastating civil war (which ended only in 1992). During the war, schools and their personnel often were targeted; also at this time, Mozambique became one of the poorest nations on earth, which clearly had an effect on education.

5.2 *Teacher Training*

One of the most important causes of difficulties in the country's education was the shortage of qualified staff. Under colonial rule, two types of training schools prepared teachers for primary schools, reflecting the fact that there were two types of primary schools – public schools for the Portuguese and the *assimilados* and low-level Roman Catholic Church mission schools. The country had no institutions for the preparation of secondary school teachers.

After the country won its independence, almost all of the Portuguese (approximately one quarter of a million) left; this group included many teachers. Meanwhile, the growing number of schools called for an increased number of teachers. The manner in which this problem was solved is described by Gerdes (1981) as follows:

> pupils of the fourth year started to teach the second year, etc. In this way thousands of well-intentioned persons with no professional teacher training at all were appointed as teachers. By 1978, more than 12,000 of them had already attended 'refresher' courses. (p. 462)

Institutions for the preparation of secondary school teachers also began to appear in the country. Initially, these institutions offered a course of study that lasted only 10 months; to be admitted to these institutions, a person had to have completed only nine grades. Gradually, however, the requirements became more demanding – the institutions began to admit only those who had completed 12 grades and to offer a 5-year course of study – and this was partly facilitated by the fact that, although there was an objective shortage of teachers, it was not easy on a practical level for teachers to find work in a country that was in the grips of a war. In 1986, the *Instituto Superior Pedagógico,* later renamed *Universidade Pedagógica* (UP), was established in Maputo, the country's capital. Later, branches of this university were opened in other cities. Mathematics education is represented at this institution by two basic programs: one of them aimed at preparing teachers for secondary schools and the other preparing highly qualified experts in the mathematics of elementary schools who are capable of conducting programs of preparation and professional development for the teachers in these schools (Gerdes 1998).

It should be noted that significant assistance for the development of education in Mozambique, including mathematics education, came from other countries, particularly the USSR, East Germany, Sweden, and the Netherlands. Gradually, however, Mozambique developed its own professional corps of teachers who usually acquired some part of their higher education abroad. Ethnomathematics and building awareness of prospective teachers of mathematical roots in their own and other African cultures constituted a pillar for the entire course of study:

> One of the objectives of the required course "Mathematics in History" for the second year students enrolled in the *licenciatura* in the Teaching of Mathematics and Physics is to contribute to a broader historical, social and cultural perspective on and understanding of mathematics. The first theme, "Counting and Numeration Systems," gives a good start, because the students can begin to analyze and compare the various ways of counting and numeration they learned in their life. After they have discovered the rich variety at the national level, they then are brought into contact with systems both from other parts of Africa and the world, and from other historical periods. (Gerdes 1998, p. 41)

According to Gerdes (1998, pp. 36ff), much experience was gained by this course in making mathematics meaningful for the future teachers. Examples elaborated in this course juxtaposed mathematical practices from the countryside versus those from the city, manual work versus intellectual work, and models coming from abroad versus models with roots in the proper cultural environment. Thus, prospective teachers raised in the cities were surprised to learn from their colleagues, coming from the rural areas, their knowledge of practices to construct rectangles without the prior use of right angles (which is important, e.g., in building houses) (Gerdes 1998, pp. 38ff.).

6 South Africa

Khuzwayo (1997), drawing on interviews he conducted with a few mathematics educators, describes South African schools during the apartheid period (1948–1994). He emphasizes that education was based on rote memorization, which was sometimes completely senseless, and was suffused with the idea of inequality between boys and girls. These characteristics are probably not unique – in many countries in Africa as well as the rest of the world, neither characteristic was particularly unusual. What truly distinguished South African education of that time was its overt racism. Naturally, during earlier years and in other countries, the colonial system of education did not envision any equality between the Black and White populations but, on the contrary, prepared the former for the role of obedient servant to the latter (Abdi 2003). Nonetheless, the rhetoric of the apartheid period was far more explicit than anything previously established.

The Minister of Native Affairs Hendrik Verwoerd expressed his position in 1954 in the following way:

> When I have control over Native education I will reform it so that the natives will be taught from the childhood to realise that equality with Europeans is not for them.... People who believe in equality are not desirable teachers for Natives.... What is the use of teaching the Bantu mathematics when he cannot use it in practice? The idea is quite absurd. (cited in Khuzwayo 1997, p. 9)

At the same time, the Bantu Education Act, designed by Verwoerd, called for sufficiently varied educational services: not only primary and vocational education but also secondary education and teacher training. Abdi (2003) sees in this a recognition by the government for the need to some degree at least to pacify the indigenous population. Mathematics education, however, remained largely a "Whites only" zone – an overwhelming majority of qualified mathematics teachers were White, and no preparation of Black experts was considered. Khuzwayo (1997) also notes that while segregation and racism in the humanities were highly conspicuous and triggered protests, analogous processes in mathematics education attracted less attention.

7 Conclusion

The history of mathematics education in Africa requires further study. Reconstructing what took place during the pre-colonial period is difficult: little evidence has survived, particularly in written sources, and the researchers are forced to resort to analogies with what can be observed today. However, even information about later periods has not been sufficiently collected and systematized.

The influence exerted on the development of mathematics education by foreign states – even during the post-colonial period – is obvious. Peshkin (1965) noted the similarity between decisions made at educational conferences in Cambridge in 1952, still during the colonial period, and those in Addis-Ababa in 1961 at the dawn of independence:

> despite impressive quantitative gains, in their important qualitative features – the content of the textbooks and the examinations and the style of the teaching – the schools under consideration for reform in 1961 must have

> closely resembled those of 1952. As in the past, the consistently criticized, bookish, literary school with its urban and European focus still prevailed. Consequently, observers in 1952 and in 1961 viewed the same schools and the same problems in the light of generally identical expectations of the schools. And, not surprisingly, they made very much the same recommendations for reform. (p. 216)

These recommendations were reiterated subsequently as well, without necessarily being implemented. Perhaps the most famous international project was the Entebbe Mathematics Project (Williams 1971), during the course of which many new textbooks were prepared, although they hardly brought about radical improvements in teaching. Geoffrey Howson, who took part in projects offering international assistance to Africa, notes the optimism that existed during the 1960s, and comments that considerable financial resources proved to be spent in vain (Karp 2009). Attempts to transfer materials from abroad directly fell through. Purely methodological and mathematical innovations turned out to be ineffective within the context of economic, political, and social problems, which they all too often ignored.

International aid was provided under conditions of rivalry and conflict between the superpowers: Soviet experts in mathematics to a certain extent competed with their American counterparts; mathematics education and education in general became part of international politics; and consequently real education turned out to be less important than the achievement of one or another set of propagandistic and political goals.

Among the internal problems of African countries that made it difficult to build a high-quality system of education, along with poverty, lack of resources, and political instability, one must also mention the multilingual nature of student bodies: in many classes, students have different native tongues, which differ from the language in which they are taught (Setati and Adler 2000). It is evident that a growing number of children have access to formal education on the African continent; nonetheless, providing even elementary education to all children remains an unresolved problem.

The generalization of what took place during the colonial and post-colonial years in African mathematics education, and the identification of various trends, patterns, and differences and similarities between processes in different countries, are all important objectives of research going forward.

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Chapter 20
Mathematics Education in Islamic Countries in the Modern Time: Case Study of Tunisia

Mahdi Abdeljaouad

1 Introduction

Over the course of the last two centuries, the Islamic world went through significant transformations. Countries that had been colonies or semi-colonies of European states, or countries that had been provinces within the Ottoman Empire, which in its turn was already frequently seen as "sick" by the nineteenth century, acquired independence and subsequently considerable political influence as well. Education in general and mathematics education in particular developed in these countries in different ways. Detailed accounts of the specific characteristics of each country will be the work of the future: at present, a great detail remains unresearched – there are no studies devoted to mathematics education in many of the regions in question. This chapter, therefore, will take as its focus a single country in the Islamic world: Tunisia. A number of the features of mathematics education in this country are also characteristic of other countries in the Islamic world.

Correspondingly, the purpose of this chapter is to examine the teaching of mathematics in Tunisia first as a province of the Ottoman Empire in the nineteenth century, then as a French protectorate from 1881 to 1956, and finally as an independent country. Important political, economic, and societal reforms were proposed by the rulers of Tunis as early as 1840, but they were rejected, resulting in an important revolt in 1862. Comprehensive reforms initiated between 1873 and 1877 by Prime Minister Khayr al-Dīn included the reform of the educational system. During the French domination, various types of educational systems coexisted; they disrupted the old order and encouraged the formation of a new French educated elite profoundly connected with its Arab-Islamic roots. Eventually, these elite successfully led the nationalistic struggle against France. After 1956, Habib Bourguiba, the leader of the Tunisian independence, introduced numerous reforms particularly in the education system.

In presenting Tunisian educational institutions throughout these two centuries, the transformations occurring in the teaching of mathematics and their consequences on the Tunisian intellectual and political life will be demonstrated.

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2 The Political and Social Situation from 1830 to 1960

After two centuries of nominal allegiance to the Ottoman sultan and the establishment of the French protectorate in 1881, Tunisians started fighting for independence of their country and obtained it in 1956.

2.1 Toward the End of the Ottoman Rule in Tunisia

Tunisia and Egypt were the only provinces of the Ottoman Empire ruled by local Ottoman militaries that were almost free from effective control of the central authorities and only paying regular financial tribute to the sultan. The preservation of the autonomy of the Ḥusainlds (of Mameluk origin), reigning in Tunisia since the seventeenth century, was threatened in 1830 by the French occupation of Algeria. The pressures from the European consuls, the end of the Mameluk's dynasty reigning in neighboring Tripoli, and the start in 1839 of the *Tanzimat* reform era characterized by various attempts to modernize the Ottoman Empire resulted in important military reforms initiated by Ḥusayn Pasha (1824–1835) who expelled the remnants of the Janissary corps while still in barracks and replaced them with a *nizāmī* army trained by European advisors and composed for the first time of native Tunisians along with the Mameluks. These reforms were continued under his successors: Ahmed Bey (1835–1853), Muḥammad Bey (1855–1859), and Muḥammad al-Ṣādiq Bey (1859–1882). The establishing of *Ahd al-amān* (Security Covenant) guaranteed the equality of all citizens before the law and their freedom of movement and security, regardless of religion or origin. The proclamation in 1861 of the first Tunisian constitution led to the bloody riots of 1862–1864. The rulers' corruption, mismanagement, and lavish spending of public money led to bankruptcy and the creation in 1869 of a European-controlled debt liquidation commission headed by General Khayr al-Dīn.

Khayr al-Dīn (d. 1890) was a young boy from Circassia sold as a slave by Taḥsīn Bey, an Ottoman judge in Istanbul where he received a good education. He was offered in 1838 to the ruler of Tunis, Ahmed Bey. Then he was trained by French officers in military sciences and practices and joined the new *nizāmī* army as an aspiring officer. He became familiar with teachers and students of the new Bardo military school. Major general and minister of the navy in 1857, Khayr al-Dīn, led the movement of constitutional reforms and abandoned all public offices; he spent several years in France before being asked to head the *Commission Internationale Financière* in 1869, which was in charge of administering the revenues of the Tunisian state. "Once dismissed from the prime ministry," Khayr al-Dīn became Grand Vizier of the Ottoman Empire for a few months in 1878 and retired in Istanbul.

As the leader of the reformers, Khayr al-Dīn wanted

> to wake up the patriotism of the Ulema and of the Muslim statesmen, and urge them to cooperate together in choosing the most effective ways to improve the State of the Islamic nation, to expand the circle of knowledge and increase public wealth, by the development of agriculture, trade and industry. (Khayr al-Dīn al-Tūnusī 1876, p. 34)

As reported by Arnold H. Green (1978, p. 107), Khayr al-Dīn wanted also to forestall European imperialism by borrowing Western institutions which would strengthen Tunisia and at the same time eliminate such pretexts for direct European intervention as fiscal irresponsibility, administrative chaos, and the absence of law and order. He also needed to convince Tunisian political and intellectual leaders that reforms were necessary for survival as well as compatible with Islamic culture.

Promoted to the position of Prime Minister between 1873 and 1877, Khayr al-Dīn implemented as many Western-inspired reforms as possible without hurting the aristocratic and Ulema privileges. Some intellectuals led by Khayr al-Dīn such as Generals Ḥusayn and Larbi Zarrouk and the Ulema Maḥmūd Qābādū, Aḥmad Ibn Abī al-Ḍiyāf, Muḥammad Bayram V, and Sālim Būḥājib worked out the reforms which were introduced at the traditional *Zaytūna* mosque-university; they also organized and developed the curriculum of a new modern public school, the Ṣadiqi College, founded in 1875, from which many important reformers later graduated.

Khayr al-Dīn's successful reforms antagonized the local oligarchy and led to his quick dismissal. The French protectorate was established in Tunisia 4 years later in 1881.

2.2 The French Protectorate (1881–1956) and the Seeds of Tunisian Patriotism

The French army occupied Tunisia in 1881. Later, the Convention of La Marsa (June 8, 1883) was imposed. It gave the reigning Bey nominal sovereignty over domestic affairs while the French resident minister assumed whole responsibility of defense, security, foreign, and finance affairs. During this period (1890–1907), liberal French officials tried to cooperate with the Tunisian elite and favored the continuation of the reformers' policy. But after 1907, the French started a program of systematic colonization, confiscating public and private lands and offering them to the European settlers. A more overt segregation policy was then encouraged and all Tunisian demands were dropped. As a consequence of this policy, seeds of patriotism began to develop among the elite.

The Western-educated Ṣadiqi graduates shared their admiration for Khayr al-Dīn's reforms with a few nontraditionalist Ulema headed by Sālim Būḥājib, one of the most prominent *shaykh* at the Zaytūna mosque-university. They vigorously defended extending modern education, cultural enlightenment, and economic development to all Tunisians. In this position, they followed the ideas of Muhammad Abduh, the Islamic reformist and head of *Al-Azhar* mosque-university, as well as France's liberal values. In the educational field, they encouraged the development of bilingual Franco-Arab primary schools and the creation of new opportunities for non-French-speaking scholars to become acquainted with new ideas and techniques. In 1888, they launched the newspaper *al-Ḥāḍira*, in which the editors expressed their reformist ideals and proposals, and in 1896 they founded the *al-Khaldūniyya* association. In 1904, the *Association des Anciens du Collège Sadiki* was created, with the aim of reviving the school's alumni and promoting a modern bicultural society.

This historical development, however, was not very beneficial for the so-called *évolués*, the graduates of Khayr al-Dīn's Ṣadiqi College with their more Western education and support for educational reforms. In the decade 1900–1910, the *évolués* of the *Ṣādiqiyya* were increasingly disliked by the great majority of the ʻUlema and simultaneously repelled by the program of systematic colonization undertaken by the French after 1900 (Green 1978, p. 138).

In 1907, the Young Tunisian movement was launched and a new political newspaper *Le Tunisien* began publication on December 12. Five topics were considered as the most essential: education, justice, finance, agriculture, and administration. Ali Bach Hamba[1] declared in the first issue of *Le Tunisien:* "In front rank of our preoccupations we shall place the question of instruction." Faintly praising the administration's few reforms, he urged a construction program to provide facilities for free and universal primary education for all children in Tunisia, thereby rejecting the idea of segregation for the Muslim because it can result in their inferior education (Brown 1976, p. 79).

Protectorate authorities, however, were not concerned with these pleas and maintained their segregation policies, given that patriots started a long and sometimes dramatic fight against French domination. After World War I, the newly founded Tunisian nationalist movement rejected French rule in the name of modern values. It organized the masses and fought for independence. After World War II, French authorities tried to introduce some reforms in the country including reforms in education, but mass mobilization by the nationalist Neo-Destour party paved the way for independence in 1956.

[1] Ali Bach Hamba (1879–1918) entered Ṣadiqi College in 1891 and graduated with the *Diplôme supérieur d'arabe*. After he obtained *la licence de droit de l'Université d'Aix*, he worked as an interpreter in 1897 and became the administrator of Ṣadiqi College (1898–1907). Although he was a founding member of the *Jeunes Tunisiens association*, he first supported the policy of association with France, but soon became a spokesman for critics of French discriminatory measures. He was expelled from Tunisia in 1912 and died in Istanbul.

2.3 The First Years of Independent Tunisia (1956–1960)

The government established a modern Western type of secular state without neglecting the Arab and Islamic roots of the Tunisian people. The administration was reorganized and competent Tunisians were recruited to run it. The new rulers' priorities were to reform all economic and social sectors, including modernizing laws, emancipating women, and establishing a coherent, unified, and modern educational system, centrally run and extended to all cities and villages.

3 The Traditional System of Education Run by the Ulema

In nineteenth-century Tunisia, several educational systems coexisted. The first and oldest system consisted of the traditional trilogy – *kuttāb, madrasa*, mosque-university – managed for centuries by the Ulema. This system was the provider of most clerks, judges, teachers, and clerics. The other existing systems were far less popular: they aimed at providing first education to Jewish children or to pupils from small foreign minorities.

After 1881, colonial rulers did not want to oppose the Ulema; therefore, they did not introduce any significant changes in the organization of the traditional Muslim system and in its curricula. On the other hand, they encouraged all non-Muslim students to study in French schools. The abolition of the traditional system run by the Ulema started only after independence and left room for only one traditional institution: the faculty of theology.

Throughout the nineteenth century and into the beginning of the twentieth, the Ulema ran primary courses in *kuttābs* in cities and villages and some intermediary-level courses in some urban mosques and *madrasa* in Tunis. Higher-level courses were given only at the *Zaytūna* University. No real changes were introduced into this system during the nineteenth century, although Ahmed Bey (1837–1855) attempted to reorganize the studies and duties of the teachers and General Khayr al-Dīn tried in 1875 to introduce an in-depth reform of the *Zaytūna* curriculum. During the protectorate period, the French did not want to intervene and left to the conservative Ulema the power to oppose all proposed reforms. Only after World War II did significant changes occur in the traditional system.

3.1 The Kuttābs

At this time, the *kuttāb* was still the main institution of education open to children ages 6–12, but it no longer met the basic training requirements of the communities.

Researchers have estimated the numbers of students in *kuttābs* before the Protectorate. Moore (1965, p. 22) wrote that 13,000 pupils were educated in 1,250 *kuttābs*; Machuel (1889, p. 62) wrote that 979 *muaddibs* (teachers) taught 17,361 pupils in 971 *kuttābs*. In describing the process of education in a *kuttāb* in 1907, the leader of the Young Tunisian movement, Khairallah Ben Mustapha,[2] was critical:

> In summer as in winter, children arrive at the sunrise…. They drop out their tablets hanged on nails against the wall and sit in their respective places. After a first collective prayer, they start the individual dictation which consists in continuation of the last verse written the day before by each student. Once the dictation is completed,

[2] The son of Hassouna Ben Mustapha (d. 1912), the first Tunisian engineer graduating from the Military school of Bardo, Khairallah Ben Mustapha (1867–1964) entered Ṣadiqi College in 1875 and obtained the *Diplôme supérieur d'arabe*. He taught at *al-Madrasah al-ᶜalawiyah* and at Ṣadiqi College. He created and headed the *Association des anciens élèves du Collège Sadiki* from 1906 to 1914. He worked as an interpreter and later became head of the Protocol for Nāṣir Bey and finally the director of the Ḥ*abūs* (religious and welfare foundations) administration.

> the master takes the tablets one after the other to fix errors. Woe to the student who made mistakes! The hand of the *muaddib* is quick to slap his head and pull his ears (…). Then comes the recitation, still in a very loud voice, then washing tablets (…). The most advanced students rewrite with red ink letters or words already shaped with a non-cut reed pen on a tablet coated with clay. Then they learn them by repeating each letter or each word after the monitors, imitating his movements made by swinging the top of the body, until they learn them well (…). Later all the students recite very loudly the first *Qur'an surat* (*al-fatiha*) and a prayer. (Ben Mustapha, Khairallah 1909, p. 553)

Considered by Khairallah as an absolute priority, the reform of the *kuttābs* was to be implemented through the initial training of the *muaddibs*, the teachers of these schools. He convinced the French authorities to authorize the creation in 1894 of *al-madrasa al-*ᶜ*usfūriyya*, a training school for *muaddibs*. Along with a perfect knowledge of the Qur'an and of religious matters, students in this school were trained for 5 years in the Arabic language and had to learn enough French to study in this language some elements of arithmetic, geometry, surveying, and geography. This undertaking was partially unsuccessful because most of the 75 graduates did not teach in *kuttābs* but in public primary schools. In 1908, these schools for *muaddibs* were closed.

All throughout the French protectorate and after independence, the creation of *kuttābs* was left to poor local communities or to some private citizens. Boys and girls ages 4 to 6 learned the Qur'an in the traditional way. Their teacher received some religious education but no formal pedagogical training. Some modern kindergartens offered alternative schooling for these children. At the age of 6, all children entered public primary schools.

3.2 Education in Mosques and at the Zaytūna Mosque-University

In the nineteenth century, formal education usually started in one of the most prestigious mosques of provincial cities and could be pursued at the second and higher levels at the *Zaytūna* mosque-university in Tunis.

3.2.1 Education in Provincial Mosques and at the Lower Level of the Zaytūna

At the lower level, students were supposed to learn and recite the Qur'an, as well as learn reading and calligraphy necessary to write Qur'an verses on tablets. At the end of the elementary cycle, students had to learn by heart and be evaluated on parts of some *urjūzas*. An *urjūza* is a didactic poem or rhymed prose introducing different terms used in a given academic subject. Memorizing it was necessary for attending higher-level courses, usually consisting of commentaries on parts of the *urjūza*. In arithmetic and inheritance, pupils had to learn the poem: *ad-Durra al-bayḍā' fī aḥsan al-funūn wa'l-ashyā'* [White Pearl on the Better of Arts and Things] by ʿAbd al-Raḥmān al-Akhḍarī (1510–1575). Teachers were graduates from the Tunisian *Zaytūna* mosque-university and sometimes from *Al-Azhar* mosque-university in Cairo.

All second-level courses were focused on religious (*Qur'an*, *ḥadith*, *tafsīr*) or juridical (*sharī*ᶜ*a*) subjects and only very few concerned Arabic language and grammar. Students studied the elements of arithmetic (numbers and fractions) necessary for the sciences of inheritance, as well as certain elements of geometry and astronomy used to determine the *Qibla* (Mecca direction) and the hours of prayers and precise dates of religious holidays; however, these subjects were taught the old way based on rote learning and recitation.

In nineteenth-century Tunisia, the *madrasa* which were affiliated with some prestigious mosques became residences for poor students and for those coming from distant cities and villages.

Higher education was the monopoly of the *Zaytūna* in Tunis. Here is a description of a typical procedure of teaching.

> Each Professor backs one of the columns of the *Zaytūna* mosque, his listeners grouped in a semicircle around him, cross-legged on mats covering the floor. Whatever the subject of the lesson, it starts first by a summary of the *metn* (text), then it is followed by a classical commentary, in which the Professor strives to identify the author's idea expressed in archaic terms. Practical exercises are reduced to the reading of the text by one of the students who applies and discusses the rules set. Listeners do not take notes, but all have a copy of the book being studied in the lesson. (Lasram 1909a, p. 156)

Among the 217 courses taught in 1871, only five were reserved for the commentaries of *farā'iḍ* and mathematics rhymed poems. Unchanged for centuries, the educational system generated elites incapable of meeting the imminent threat of French colonization.

In 1875, Khayr al-Dīn decided to reorganize the pedagogy and content of courses at the Zaytūna and to structure the new curricula composed of 28 disciplines, with textbooks recommended for each. For mathematics, 10 works were prescribed:

Lower cycle:

- *Al-Nukhba al-ḥisābiyya* (Selections from arithmetic) of Ḍiyā' al-Dīn al-Khazrāj
- *Sharḥ al-durra al-bayḍā' fī aḥsan al-funūn wa'l-ashyā'*. A commentary on al-Akhḍarī's *urjūza* on arithmetic and inheritance

Intermediate cycle:

- *al-Murshida fi'l-ḥisāb*. A guidebook in arithmetic by Ibn al-Hā'im (d. 1412) (Rosenfeld and Ihsanoșğlu 2003, n° 783, M5)
- *Sharḥ talkhīs aᶜmāl al-ḥisāb lī Ibn al-Bannā*. A commentary by al-QalaṢādī (d. 1486) on Ibn al-Bannā's *Concise book on works in arithmetic* (Rosenfeld and Ihsanoğlu 2003, n° 696, M1 and n° 865, M7)
- *Sharḥ ashkāl at-ta'sīs lī al-Samarqandī*. A commentary by Qāżī-zāde al-Rūmī (ca. 1440) on Shams al-Dīn al-Samarqandī's textbook on geometry *Substantional Propositions* (Rosenfeld and Ihsanoğlu 2003, n° 655, M1 and n° 808, M2)
- *Sharḥ al-Chagmīnī*. A commentary by Qāżī-zāde al-Rūmī on al-Chagmīnī's (d. 1221) textbook *Compendium of Astronomy* (Rosenfeld and Ihsanoğlu 2003, n° 547, A2 and n° 808, A1)

Superior cycle:

- *Sharḥ munyat al-ḥussāb*. A commentary by Ibn Ghāzī (d. 1513) on his own *urjūza* on arithmetic and algebra (Rosenfeld and Ihsanoğlu 2003, n° 606, M1)
- *Sharḥ talkhīs aᶜmāl al-ḥisāb*. A commentary on Ibn al-Banna's *Concise book on works in arithmetic* written by 'Abd al-'Azīz al-Hawārī al-Misrātī (1345) (Rosenfeld and Ihsanoğlu 2003, n° 747, M1)
- *Sharḥ al-taḍkira*. Commentary on Naṣīr al-Dīn al-Ṭūsī's work on astronomy written by al-Sayyīd. (For al-Ṭūsī's book, refer to Rosenfeld and Ihsanoğlu (2003), n° 606, A10, but no reference was found for al-Sayyīd)
- *Taḥrīr al-Ṭūsī li kitāb Uqlīdis*. Presentation of Euclid's *Elements* by NaṢīr al-Dīn al-Ṭūsī (Rosenfeld and Ihsanoğlu 2003, n° 606, M1)

This new curriculum was in fact intended to merely preserve traditional knowledge and methods (Abdeljaouad 2006), even though the *Zaytūna*'s professors severely criticized the new regulations and hindered their effective implementation. Mathematics instruction continued to be based on rote learning of *urjūzas* and on commentaries of the linguistic and stylistic aspects of their wording, preventing any effective practice of arithmetic or geometry (Abdeljaouad 1986).

3.2.2 The Testimony of a Successful Student

Some efforts were made to diversify the recruitment of students at the *Zaytūna* around 1912. In his autobiography, the Tunisian poet Abū l-Qāsim al-Shābbī (1909–1934) described his own studies. Let us look at what mathematics he received:

At the primary level: Three first lessons from *Durūs* Shafīq Manṣūr[3]

- 1st lesson: Oral and written numeration of integers, addition, and subtraction
- 2nd lesson: Multiplication and division of integers
- 3rd lesson: Divisibility criteria; decimal fractions and numbers; measuring lengths, time, surfaces, and volumes using traditional and metric units; mixed numbers

Second level

- *Al-Risāla al-sibṭiyya* (The Epistle of Ṣibṭ al-Māridīnī)
- *Al-Nukhba al-ḥisābiyya* (Selections from arithmetic) of Ḍiyā' al-Dīn al-Khazrājī
- The 4th lesson of *Durūs* Shafīq Manṣūr. Squares and cubes of whole and fractional numbers; rule of three; percentages; proportions; rules of exchange; rules of society
- *Durūs fi al-handasa,* Shafīq Manṣūr's textbook in elementary geometry
- Some chapters from *Sharḥ al-taḍkira* by al-Sayyīd

This description shows that, at the primary level, al-Shābbī studied a modern type of mathematics utilizing the work of a contemporary Arab author. The courses were certainly given at one of the newly founded modern Qur'anic schools, and the professor must have been a graduate of *al-Madrasah al-ᶜalawiyah* or *al-Khaldūniyya*. (Details on these institutions will be given later in Sect. 3.4.). However, second-level students continued to follow traditional courses from the Ulema in the old way, sitting in a semicircle around their master at the *Zaytūna* mosque-university.

3.2.3 Example of Traditional Teaching of Arithmetic in the 1930s at Zaytūna

No serious changes occurred either in the administration or in the curriculum of *Zaytūna*. French authorities supported the role of the influential conservative shaykhs. The example below shows the old-fashioned teaching of arithmetic by Shaykh al-Shaṭṭī. To follow it, one should be aware that the particular importance of the Islamic laws of inheritance made them a compulsory subject for all *Zaytūna* students. The elaborate system of rules for the devolution of property required the knowledge of some elements of arithmetic. Throughout the nineteenth and early twentieth centuries, courses consisted of learning didactic poems of the Middle Ages and compendiums and commentaries including problem-solving techniques using operations with fractions.

In 1934, Shaykh Muḥammad al-Ṣādiq al-Shaṭṭī published *Lubāb al-farā'iḍ* (The Heart of Inheritance Laws) which was accepted as an official textbook. This book included three parts:

- The bases of the Islamic law of succession, illustrated with case studies (pages 3–43)
- The arithmetic needed for inheritance problems (pages 44–80)
- The last and longest chapter devoted to examples of solving inheritance problems using arithmetic (pages 80–221)

[3] Shafīq Manṣūr (1856–1890) was an important mediator for European mathematics in Egypt because he authored numerous school textbooks in mathematics, a treatise in differential and integral calculus, and scientific papers in the review *al-Muqtataf* (Crozet 2008: 478). *Al-Durūs al-ḥisābiyya lil-madāris al-ibtidā'iyya* (Courses in arithmetic for primary schools) is one of the textbooks he published in 1886 and used in Egyptian governmental primary schools.

Table 20.1 al-Shaṭṭī's notations for fractions (1934)

Designation	al-Shaṭṭī's notation	Traditionally	Modern notation	Value
Kasr munfarid (simple fraction)	$\frac{5}{11}$	Numerator: 5 Denominator:11	$\frac{5}{11}$	$\frac{5}{11}$
Kasr muntasib (continuous fraction)	$\frac{1\,2\,3}{3,4,5}$	Numerator: 43 Denominator:60	$\frac{3}{5}+\frac{2}{4\times5}+\frac{1}{3\times4\times5}$	$\frac{2}{3}$
Kasr mubaaᶜḍ (partitioned fraction)	$\frac{1\mid2\mid3}{3\mid4\mid5}$	Numerator: 6 Denominator:60	$\frac{1\times2\times3}{3\times4\times5}$	$\frac{1}{10}$
Kasr mukhtalif (distinct fraction)	$\frac{2\mid1}{3\mid4},\frac{2\mid3}{3\mid5}$	Numerator: 34 Denominator:60	$\frac{2\times1}{3\times4}+\frac{2\times3}{3\times5}$	$\frac{17}{30}$

In introducing the arithmetic chapter, al-Shaṭṭī indicated that knowledge of the arithmetic of whole numbers was a prerequisite to this course, but he added that fractions should be presented in a way that differed from what is usually taught in primary schools.

Al-Shaṭṭī presented fractions with the typology and notations created in Andalusia and the Maghreb sometime in the twelfth century. They were exactly the same as in arithmetic textbooks written by Ibn al-Hā'im (d. 1412) and al-Qalaṣādī (d. 1486). All modern operating signs (+, −, ×, :) are absent from al-Shaṭṭī's textbook and are replaced, على ، في ، ا respectively, by ( ). In this type of) arithmetic, one-learned to compute common denominators of two or more fractions, to decompose the denominator of a fraction into factors (not necessarily primes), and then to identify its numerator. There was also a short section on divisibility by 9, 8, and 7 and a short section on proportions. The only major difference with the old books is that the author adds to each section a set of unsolved revision exercises to recapitulate previously covered mathematics concepts.

The (Table 20.1) shows each type of fraction provided with its particular notation.

3.3 *Implementing Reforms at the Zaytūna Mosque-University*

In a book written in 1907, Shaykh Muḥammad al-Ṭāhir Ibn ʻĀshūr[4] identified all the reforms that should be introduced in the *Zaytūna* curriculum and organization. As Arnold Green (1978, p. 214) wrote:

> For each of the deficiencies which he found in the *Zaytūna* educational system Ibn ʻĀshūr proposed a specific remedy. In general these proposals included more stringent regulations and closer supervision, broadening the scope of the curriculum and making "modern sciences" required courses, and introducing pedagogical methods which do not rely on rote memorization but seek to develop the students' critical powers.

Ibn ʻĀshūr's proposals and arguments were refused by the majority of the Ulema.

The following 20 years were marked by some serious students' unrest and strikes in 1910–1912 and in 1928–1930 in support of the reforms. However, changes occurred only under the supervision of Ibn ʻĀshūr who was appointed a rector of the *Zaytūna* in 1932.

According to the Decree of 3 March 1933, *Zaytūna* became the only institution offering secondary and higher Muslim education (article 22). Other mosques in Tunis and in provincial cities became branches offering only lower-level courses. Students had to "open their minds by being trained in those parts of mathematics that are not opposed to the *sharīᶜa* and do not disturb learning the basic

[4]Muḥammad al-Ṭāhir Ibn ʻĀshūr (1879–1972) was one of the most prominent reformers and nationalistic Tunisian shaykh. After graduating from *Zaytūna*, he successively became judge, *muftī*, *shaykh al-Islām*, and rector of *Zaytūna* University. He remained in this position after independence.

Table 20.2 Degrees offered by *Zaytūna*

	Number of years	Degree
Lower level	4	*al-Ahliya*
Second level	4	*al-Taḥṣīl*
Higher level	3	*al-ᶜĀlimiya*

religious sciences" (article 20). However, Ibn ʻĀshūr was able to implement these reforms only after World War II: from 1949 to 1951, 25 provincial branches of *Zaytūna* were created in premises outside of the mosques, with two of the branches reserved for girls.

Among compulsory matters taught outside the mosques at all levels were arithmetic, algebra, geometry, surveying, cosmography, and timekeeping. Examinations for three degrees were created, as shown in Table 20.2.

In 1950, 8,179 boys and 80 girls passed the *al-Ahliya* examination. A second-level section for girls was then created to help them prepare for the *al-Taṣṣīl* examination. In 1956, 10,933 boys and 408 girls were registered in the *Zaytūna* second-level schools.

Some of the *al-Taṣṣīl* graduates went to Middle East universities to continue their studies.

Following Tunisian independence, *Zaytūna* lower and second levels were integrated into the newly created public system of education, and the *Zaytūna* higher level grew into a faculty of theology.

3.4 The Khaldūniyya Association

In 1896, Ṣadiqi's graduates Béchir Sfar and Mohamed Lasram founded the *Khaldūniyya* association, the aim of which was to offer to non-French-speaking Tunisians courses and conferences in modern science and the humanities (philosophy, political sciences, judiciary matters, history, economics, and geography) all taught in Arabic. In 1898, regular daily courses in mathematics (arithmetic, algebra, metric system, and geometry), history, and geography were taught, as well as a three-level course in the French language and grammar. All these courses were helpful for passing the examination of the newly created *Diplôme de connaissances pratiques* which allowed access to administrative positions. By 1907, 176 students from *Zaytūna* and civil servants earned this diploma.

Among the 1,228 Arabic books and magazines of the *Khaldūniyya* library, 635 were on modern scientific subjects and most were Egyptian and Syrian translations of French textbooks.

Throughout the protectorate period, *al-Khaldūniyya* continued to be considered an annex to the *Zaytūna*, and the leaders of the association encouraged the emergence of the teaching of mathematics and sciences in Arabic at the lower and intermediary levels of the *Zaytūna* system. In 1936, they started to offer a 3-year secondary-level course for those students who obtained the *Diplôme de connaissances pratiques*, and in 1947, the association organized *Le Baccalauréat arabe*, a private examination required for entry into one of the Middle Eastern universities (Egypt, Syria or Iraq).

From 1947 to 1950, Mohamed Souissi[5] published textbooks covering the program of mathematics required at the modern *taṣṣīl* examination: one volume of *Khulāṣat al-ḥisāb* (Essence of arithmetic) which contained basics in arithmetic, measurement, and surveying and three volumes of *UṢūl al-jabr* (Elements of algebra) the contents of which included:

> Algebraic expressions. First degree equations with one unknown. Systems of two or more linear equations. Algebraic functions. Linear functions. Second degree equations. Arithmetic and geometric sequences. Logarithms.

[5] After graduating from Ṣadiqi College, Mohamed Souissi (1915–2007) prepared *la licence de mathématiques* at the University of Paris and returned to Tunis as a professor of mathematics at Ṣadiqi College. At the same time, he taught mathematics in Arabic at *al-Khaldūniyya* association and at the secondary level of *Zaytūna*. He later became a professor at the University of Tunis, specializing in the history of Arab mathematics.

In 1951, a group of teachers headed by Mohamed Mili[6] published two textbooks on elementary geometry which were used in the modern branches of the *Zaytūna*.

4 New Types of Education in Precolonial Tunisia

Yet in the first third of the nineteenth century, the success of the reforms undertaken by Mehmet ali in Egypt served as models for the Tunisian rulers (Abdeljaouad 2010). In 1820, Maḥmūd Pasha (1814–1824) recruited a French engineer from the Polytechnic School of Paris and assigned him to reorganize the army and organize the factories for military purposes. In 1838, Ahmed Bey (1837–1855) strengthened the new *nizāmī* army and founded a military school at the Bardo Palace Citadel headed by the Italian expatriate officer Louis Calligaris.

Born in 1808 in the Piemonte region, Louis Calligaris had learned Arabic in Aleppo in his childhood. As an independent Italian officer, he participated in the first war of the Ottomans against Mehmet ali's army in Syria and spent some time teaching geometry in the Polytechnic School of Istanbul. Then, in 1833–1834, he moved to Tunis and trained some cadets before creating and heading the Bardo military school from 1838 to 1850. After he left Tunis, he became a professor of Arabic at the University of Turin. In 1856, he published a *History of Napoleon I*; in 1863, a theoretical course on the Arabic language; and in 1864–1870, a multilingual dictionary, *the Rafīq fī kullī Ṭarīq* (The Companion in all roads). He died in Italy in 1870.

4.1 The Bardo Military School

It was in fact Hussein Bey (1824–1835) who invited the Italian officer Louis Calligaris to set up a school for cadets on the model of similar European schools. In a letter dated June 8, 1834, Calligaris wrote:

> I learned that Hussein Bey agreed to adopt a plan for a military school on the model of Ükseraï of Constantinople, where I worked as an interim professor of practical geometry and trigonometry. Having been the sponsor of this institution, it is quite natural that I should be its Director. To make it cheaper at its beginning, I started by teaching two classes myself, one for mathematics and one for the art of war. (Monchicourt 1929, pp. 322–323)

But the actual renovation of the Army was the work of Ahmed Bey who also gave new impetus to the Bardo military school. A nineteenth-century Tunisian historian, Aḥmad Ibn Abī al-Ḍiyāf, appointed in 1837 as the chief secretary of Ahmed Bey and the first non-Mameluk to occupy that position, was one of the most prominent advocates of reforms. According to Ibn Abī al-Ḍiyāf,

> On the first of *Muḥarram* 1256 (March 5, 1840), Ahmed Bey set up a military school at the Bardo and installed it in his palace which he left for a new one. <The school> was intended for teaching all the sciences that *nizāmī* soldiers needed to know, such as fortification, geometry, arithmetic, and the French language since most reference books were written in this language. Its Director was the competent and educated colonel Calligaris, who recruited a *madrasa* professor for teaching the Koran and religious subjects. (Ibn Abī al-Ḍiyāf 1989, p. 41)

[6] Mohamed Ezzedine Mili was a graduate of the *Ecole normale de Tunis* (1938) and the *Ecole normale de Saint-Cloud* (1944). He then completed a *licence de mathématiques* and graduated as engineer from the *Ecole des Télécommunications de Paris* (1946). On his return to Tunis, he was recruited only as an assistant engineer. At the same time, he taught mathematics in Arabic at *al-Khaldūniyya association* and at the secondary level of *Zaytūna*. After independence, he was elected secretary general of the International Communication Union.

Between 1837 and 1852, the school was run by Calligaris. French, mathematics, and the art of war were taught in Turkish and Arabic. From 1841 to 1868, a prominent madrasa professor Shaykh Maḥmūd Qābādū taught Arabic and literature at the Bardo military school and also at the *Zaytūna* mosque-university. He is considered to be a mentor of nineteenth-century Tunisian reformers. He was appointed *muftī* in 1868 and died in 1871.

Most students were Mameluks living in the Tunisian palaces, but for the first time, cadets of Tunisian origin were also trained. The best students were asked to translate important works on military science into Arabic. For example, in 1844, a student, the Mameluk Ḥusayn, translated Baron Henri de Jomini's *Précis de l'art de la guerre* into Arabic and his translation *Ṣina^{c}at al-muḥāraba* was revised by Qābādū and published in 1869. In the preface of this book, Qābādū wrote a fervent plea for the translation of European scientific treatises into Arabic:

> Muslims had fallen backward in spite of the possibility of progress which Islam had given them. It would be necessary, therefore, to search for the reasons of this regression not in religion but elsewhere. The comparison between the Muslim world and Europe shows that the determining factor is the lack of *al-^{c}ulūm al-kawnīyya* (universal sciences). After Europeans acquired the sciences from Muslims, Europe dominated the Muslim world, while the Muslim in the meantime abandoned the sciences. It is hardly possible that the Muslims will regain their happiness as long as they do not restore the sciences which they abandoned. As the Europeans granted <sciences> the position they deserved, improved and enriched them, it remains now for the Muslims to reclaim them, either by restoring them, or by learning them. (Qābādū 1869, vol. 2, p. 31)

Other students, such as Rāshid, Muḥammad Ben Ḥajj ʿUmar, and Aḥmad al-*Mourali*, translated their courses into Arabic and had them corrected by Qābādū. It was in this school that many innovations were introduced in Tunisia. They included the following:

- Teaching traditional subjects (arithmetic, geometry, measurement, trigonometry, algebra) the European way, that is, with simplified, easy-to-retain algorithms
- Learning foreign languages (French and Italian) and their effective use
- Learning classical Arabic, grammar, and theology with new pedagogical methods not only based on memory

For the first time in Tunisia, the Bardo military school gathered native young Tunisians with young Mameluks at a prestigious teaching institution. They were to play an important role in the high military and possibly administrative and political hierarchy.

Some of the school's graduates became leaders in the process of Tunisia's modernization.[7] The Bardo military school was closed in 1855.

4.2 The School of War (1855–1869)

When Muḥammad al-Ṣādiq Bey (1859–1881) came to power, he replaced the Bardo military school by a *Maktab al-ḥarb* (School of War) which trained only noncommissioned officers. According to Marty (1935, p. 321), its new director, the French Capitaine de Taverne (1819–1865), required that cadets (14–16 years old) be versed in the reading and writing of Arabic. One hundred twenty students,

[7] Ḥusayn (d. 1887) became general, president of the municipality of Tunis (1858–1865), and minister of instruction and civil works (1874–1887). Rustum (d. 1886) became general and minister of the war (1870–1878). Of Circassian origin as was Khayr al-Dīn, both men promoted reforms and once in power tried to implement them. All three died in exile. Hassouna Ben Mustapha (1815–1901) became an engineer who built numerous public works and taught some courses at the School of War after 1855. French authorities promoted him to the function of *qāid* (governor) and head of an important agriculture department. The student Iskander became a lieutenant in the Tunisian army and later in 1875, assisted Larbi Zarrouk, the director of Ṣadiqi College.

distributed into six separate classes, had to follow the curriculum of similar military schools in France, with the teaching of mathematics and military sciences conducted in French. The Arabic language was used for religious and literary studies only. The course of the classes was as follows:

1st year: Qur'an, French language
2nd year: Qur'an, Arabic syntax, French language
3rd year: Arabic syntax, French language, Arithmetic
4th year: Arabic syntax, French language, Arithmetic, Geometry, Algebra
5th and 6th years: Military sciences and their applications

French officers Etienne Soullier (recruited in 1856) and Zephyrin Eymond (recruited in 1858) taught mathematics at the school. Other French officers taught military techniques and were aided by officers who graduated from the Bardo military school, such as the engineer Hassouna Ben Mustapha.

According to François Arnoulet (1954, p. 167), during the third and fourth years, students were supposed to receive 7 h per week of theoretical courses and 8 h of sessions in exercises and applications; 18 months were assigned for arithmetic and 6 for algebra. Mathematics teachers could not take any personal initiative, but they had to follow the pedagogical instructions scrupulously. Students' examination sheets consisted of a standard list of exercises. The school library contained 200 French books donated in 1875 by al-Ṣādiq Bey to the library of Ṣadiqi College. One of them was Capitaine Hudelot's *Traité de géométrie descriptive à l'usage des sous-officiers de toutes armes*, sent to the School of War in 1852 from Marseille by the author himself.

After 6 years of studies, graduates would gain a position in the army. More than 100 noncommissioned officers were prepared between 1855 and 1869, but most of them received very poor training. The only exception were two graduates (Mohamed Mourali and Ali Qadri), who were admitted to the French military school at Saint-Cyr from which they graduated in 1864, while some others became important civil administrators even after the French occupation of Tunisia in 1881.[8] Students were often mocked by older officers. For example, Mohamed El Karoui, one of the best students of the School who graduated in 1869, related the following distressing story: After fortuitously attending a trigonometry lesson, a general of the Garde asked for explanations. Two pupils tried to give them successively. Then the general walked out, ceremoniously shouting: "And say that Our Highness spends his money for such tall stories!" (Marty 1935, p. 331).

With the death of Ernest de Taverne who was replaced by Jean-Baptiste Campenon between 1862 and 1864, and given the rulers' lack of concern about serious financial difficulties, the school was gradually neglected and finally closed in 1869.

In terms of their original missions, the Military School of Bardo and the School of War did not succeed in changing the Tunisian administration; however, some of their graduates became new champions of the development of the Western-style education system and staunchly supported General Khayr al-Dīn's reforms.

4.3 The Foundation of Ṣadiqi College by Khayr al-Dīn

The most important innovation introduced by General Khayr al-Dīn in the Tunisian educational system was the foundation in 1875 of Ṣadiqi College, following the model of French secondary schools. Ṣadiqi College aimed to disseminate new sciences along with the practice of Arabic language and literature and provide a solid knowledge of Islam.

[8] Mohamed El Karoui became first interpreter and then head of the Archives office; Ṣāliḥ ʻAbd al-Wahhāb was administrator in Mahdia; Laroussi Ben Ayed headed Ṣadiqi College; and Amor ben Barquète became first an assistant to the minister Khayr al-Dīn, then headed Ṣadiqi College, and ended his career in 1888 as Chief of the administration of the *Ḥabūs* (religious and welfare foundations).

4.3.1 Organization of Ṣadiqi College

The school enrolled 150 pupils and offered three levels of courses:

The first level was the *Kuttāb*. This traditional institution aimed to teach some elements of writing and reading Arabic and a significant portion of Qur'an. Traditional *muaddibs* were teachers at this level.

The next 6 years of studies covered the elementary level (4 years) and the early secondary level (2 years). Strong emphasis was placed on foreign languages, mainly French and Italian, although Arabic was not neglected nor were Islamic matters. Mathematics, geography, and the sciences were taught in French or Italian. It is important to note that not only were the teaching methods new and favored understanding and comprehension, but each class had its own classroom; moreover, the teacher had his own desk placed on a platform and could use a blackboard on the wall behind him in full view of the desks, each of which was shared by two pupils. Also, printed reading tables, geographical maps, metric system tables, and synoptic tables featuring images of plants or animals were hung on the walls. The library of the school contained hundreds of French and Arabic books. Examinations were obligatory for moving from one class to the next.

The head of the Tunis municipal council, General Larbi Zarrouk, ran the school and was assisted by Colonels Iskander and Amor Ben Barquète.

Khayr al-Dīn used to visit the college many times a year and attend the weekly evaluation sessions in foreign languages and mathematics. He was especially interested in brilliant students such as Béchir Sfar and encouraged them by inviting them to have meals with him at his palace. He also organized important ceremonies at the end of the school year with the presence of all the aristocracy, including the Bey.

Khayr al-Dīn had intended Ṣadiqi College to prepare Tunisian modern elites (teachers, engineers, architects, doctors, lawyers, and civil servants) to be highly trained in all domains and capable of transforming the country into a modern state, preserved from being devoured by other countries. This project was completed too late, however, and could not stop the occupation of Tunisia by the French army. Nevertheless, Khayr al-Dīn's idea "succeeded beyond his dreams, for the Franco-Arabic education provided the beginnings of cultural synthesis for the new nation" (Moore 1965, p. 23).

4.3.2 The Testimony of a French Visitor

On April 1878, after a visit to Ṣadiqi College, the head of the French *Lycée d'Alger*, Daniel Grasset (1878, p. 191), described the functioning of the school. Grasset reported that among the 150 students registered in the school 3 years after its founding, 75 were studying sciences in French, 40 in Italian, and 35 in Turkish. They included elements of arithmetic from first year to fourth year (at this time, students had to learn the Rule of Three). Learning geometry and drawing figures started in the fourth year and the metric system in the fifth. Squares, cubes, progressions, and logarithms were introduced in the fifth year and algebra and trigonometry in the sixth.

Grasset met also the teaching staff: Tahar Ben Salah, a graduate of the Franco-Arab College of Constantine who was a teacher of French and arithmetic at the elementary level; Mr. Clement, teacher of Italian and arithmetic at the same level; and Mr. Eymond, who was professor of mathematics, physics, and cosmography. Many *Zaytūna* shaykhs taught Arabic and Islamic subjects. Grasset expressed his admiration for the high level attained by the students and his great esteem for the pedagogical work of the professors.

4.3.3 One of the First Ṣadiqi Graduates: Béchir Sfar

The son of a general, Béchir Sfar (1865–1917) learned the Qur'an and some elements of Arabic reading and writing at his neighborhood *kuttāb* in Tunis, and he entered the newly founded Ṣadiqi College in 1875. Among the brightest students of the school, he headed a group of six Ṣadiqi graduates from the class of 1880 sent to study in Paris. He was admitted to the *Lycée Saint-Louis* where he studied highly specialized mathematics preparing him to enter one of the prestigious schools of engineering. However, in July 1882, the new French authorities in charge of education in Tunis forced him and his fellow Tunisians to stop their studies and return to Tunis.

Béchir Sfar was appointed as head of the accounting department in 1884, then as director of the public administration of the *Ḥabūs* in 1892, and finally as governor of the region of Sousse in 1908.

In addition to his important administrative positions, Béchir Sfar became the leader of the young, Western-educated intellectuals. He was fond of French liberal ideas and wanted to extend modern education, cultural enlightenment, and economic development to all Tunisians. He shared his defense of reforms with Shaykh Sālim Būḥājib, one of the most prominent professors at the *Zaytūna* and a companion of Khayr al-Dīn.

Béchir Sfar launched the newspaper *al-Ḥāḍira* in 1888, in which the editors expressed their reformist ideas. In 1896, he was one of the leading founders of the *Khaldūniyya* association and its president in 1897. He also taught courses in history emphasizing the importance of the role played by Arabs and Muslims from 1897 to 1908. Finally, in 1904, he was among the founders of the *Association des Anciens du Collège Sadiki*.

4.3.4 The First Tunisian Modern Textbook in Geometry

In 1880, Amor Ben Barquète,[9] the second aide of the director of the newly founded Ṣadiqi College, completed the translation into Arabic of a French geometry textbook (Guilmin 1864).[10] In the preface of the translation, Ben Barquète explained that he accomplished this task as "an answer to a request made by a person he cannot dare contradict, but must satisfy." Abrougui (2010) has assumed that the important person alluded to in this preface must have been Larbi Zarrouk, then head of Ṣadiqi College and a known reformist and admirer of Khayr al-Dīn. He added, "I desired to help Muslims and offer it to teachers and students." Two manuscripts of the translation can actually be found at *Bibliothèque nationale de Tunis*: the first one (no 8099) was written in February 1880 – it is a complete translation of Guilmin's textbook. The second (no 16579) was an incomplete copy offered by Wanīs Ibn ʻĪsā in 1922 to the Library of the *Khaldūniyya* association.

The topics of Ben Barquète's book and those of Guilmin's textbook are similar, starting with some introductory definitions (volume, surface, line, and so on), followed by two parts.

Part I concerns plane geometry and is comprised of four chapters: (1) on angles and polygons (essentially triangles); (2) on circumferences of circles; (3) on proportional lines, similar polygons, regular polygons, similar triangles, metric relations in a triangle, measure of the circumference of a circle, and computing the number pi; and (4) on surfaces of polygons and circles.

Part II concerns space geometry and is comprised of four chapters: (5) on planes and lines in the space, (6) on polyhedrons, (7) on the three round corps, and (8) on spherical triangles and polygons.

Guilmin ended his book with some elements of applied mathematics. Ben Barquète presented them as *iḍāfat mukhtalifa* (different useful remarks).

[9] A graduate of the Bardo military school and the best student of the second promotion, Amor Ben Barquète had served upon his graduation as an aide de camp to General Khayr al-Dīn. In 1875, he was appointed second aide of Larbi Zarrouk, the director of newly founded Ṣadiqi College. He was appointed by the French rulers head of the college in 1882 and head of the *Ḥabūs* organization in 1884 but died in 1888.

[10] Guilmin, Ch. M. A. (1812–1884) was a graduate of the *Ecole normale supérieure de Paris* and a professor of mathematics. He wrote numerous textbooks in mathematics for the primary and secondary levels, as for example, *Cours d'arithmétique*, 1853; *Cours d'algèbre élémentaire*; and *Cours de géométrie élémentaire*, 1854 (26 editions).

While Ben Barquète's translation is mostly close to the original text, there are some significant formal differences in the presentation of the topics. These differences show that the translator had substantial experience in the pedagogy of mathematics and followed his own approach for teaching it. This is illustrated by the following:

- Ben Barquète modified more than a third of the proofs given by Guilmin in Chap. 1 (Abrougui 2010, p. 61).
- Specifically, in many instances he subdivided a complex theorem into two parts, each becoming a theorem with its proof on its own.
- In each of the chapters, Ben Barquète added to Guilmin's problems with solutions translated into Arabic, exercises "to be proved by the students," not found in Guilmin's textbook. Some of these problems are corollaries of theorems, some are construction problems, and some are numerical applications. For example, 29 exercises are added to Chap. 1 and 47 to Chap. 3.
- To Guilmin's geometrical figures, Ben Barquète added eight new figures, either in some proofs or in some of his own exercises, to provide a better understanding of the text.

Abrougui (2010, p. 64) believes that a detailed study of the supplementary exercises proposed by Ben Barquète needed to be done to determine their source (i.e., may be some other French textbook) or prove their originality. This study of all the exercises might also show some of their incompatibility with the topic studied just before. For example, exercises 49, 50, and 51, page 15, concern the bisector even though that concept had not yet been defined.

Ben Barquète's work was never published; in fact, no other Tunisian textbook in mathematics was to be written or published until 1946.

In sum, one may conclude that the nineteenth century can be regarded as the starting period for modern teaching in Tunisia. As discussed, for the first time in this Ottoman province, Mameluks and non-Mameluks graduating from the military school were allowed to become officers in the army; they constituted a homogeneous elite, open to the outside world, well-rooted in Arab and Tunisian cultures. Some of them became new champions of the development of a Western-style educational system and staunch supporters of Khayr al-Dīn's reforms. However, the dynamics of the progressive training of a new Tunisian elite prepared to build a modern country broke up when the French troops occupied Tunisia in 1881.

5 The Educational System Under French Rule

When the French protectorate was instituted in Tunisia in 1881, the most prominent reformers, such as the President of the municipal council of Tunis and the Director of Ṣadiqi College, Larbi Zarrouk, were exiled abroad. The French authorities required the first graduates of Ṣadiqi College to occupy positions as intermediaries between them and the population. The young mathematical talent, Béchir Sfar, whose ambition was to continue his graduate studies at the prestigious *Lycée Saint-Louis de Paris* and embrace a career as a professor of mathematics or an engineer, along with his fellows from the Tunisian student mission in France, was summoned to return to Tunisia and hold junior positions as interpreters in the administration.

On May 6, 1883, Louis Machuel (1848–1921), a French professor of Arabic in Algeria, was appointed as head of the *Direction de l'Instruction Publique*. From the start, Machuel planned to control the whole education system, including the traditional one, but was careful not to be antagonistic to the Ulema and the administrative elite. He started by downgrading the College Ṣadiqi and encouraging European settlers and Tunisian Jews to obtain a French education. His strategy consisted first of setting up parallel to the existing schools (Muslim, Jewish, Christian, and Italian) a new system of schools where teaching was not religious and was given essentially in French (primary and vocational schools, some colleges, and one lycée). Top priority was offering jobs to graduates from these new French schools.

5.1 Ṣadiqi College After 1881

After 1881, once Ṣadiqi College was definitively established, the French decided to maintain it as a Muslim special school preparing civil servants. However, they downgraded Khayr al-Dīn's ambitious project of "preparing young Tunisians for high-level liberal and public careers" and imposed a curriculum which objective was only to train middle-level civil servants who were well grounded in the Muslim culture, fluent in Arabic and French, and capable of becoming good interpreters and intermediaries between the French authorities on the one hand and the Tunisian administration and people on the other.

5.1.1 The Early College Ṣadiqi Graduates

As previously discussed, Béchir Sfar had to stop his scientific career and become a civil servant. Another example of an interrupted scientific career is the life of Khairallah Ben Mustafa, who became an interpreter and a teacher at the newly created normal school. As a brilliant defender of the rights of Tunisian children to a modern education, he founded the first modern Qur'an school where a renovated teaching of mathematics and science was available in Arabic. There is also the sad example of Salah Ben Salah, who was able to enter the *Ecole supérieure des mines de Saint-Etienne* but was denied any work as an engineer. He was forced into exile abroad where he died at the age of 39. According to Ahmed Abdessalem (1975, p. 35), the French thought that

> what was needed first in Tunisia were primary schools and professional training centers in the fields of commerce, agriculture and industry. We have to train workers, cultivators and shopkeepers, and before all take care that no alienated persons were brought up.

5.1.2 Changes in Ṣadiqi College

From 1883 to 1892, Louis Machuel, the French *directeur de l'instruction publique*, supervised the functioning of the school through his designation of several successive Ṣadiqi directors[11] and watched over all financial, organizational, and pedagogical matters concerning the college. Beginning in 1892 and for the first time, a French director, Marius Delmas, was appointed as director of Ṣadiqi. While applying French laws and regulations, Delmas introduced important changes in the organization of the school; in 1897, Ṣadiqi was moved to new and modern premises where the primary school was completely separated from the secondary school. Students entered at the age of 6 and studied for 5 or 6 years. A vast room was used as a traditional *kuttāb* where pupils collectively learned the Qur'an by heart and obtained some writing and reading skills in Arabic using clay tablets. Half of the study time during the first and second years, one fourth of the time in the third and fourth years, and 1 h per week in the fifth year were spent at the *kuttāb*. The remaining time was devoted to the same curriculum offered in French primary schools, with Arabic language and grammar courses added and some training in translation between French and Arabic. In these courses, pupils used French textbooks for most subjects and a few strictly controlled Arabic books not published in Egypt.

[11] After Ṣadiqi's first director, Larbi Zarrouk left to go into exile in Turkey because of a protest against the French occupation of Tunisia in 1881, more than ten directors headed the college for short periods, including Hassouna Mettali (1881–1883), Amor Ben Barquète (1883–1885), Mohamed El Karoui (1885–1886), Laroussi Ben Ayed (1886–1888), and Tahar Ben Salah (1888–1892).

5.1.3 Teaching Mathematics at Ṣadiqi College

In mathematics, Ṣadiqi College students had to learn arithmetic and the metric system.

The Tunisian *Bulletin officiel de la Direction de l'Instruction Publique* (January 1887) specified what knowledge of mathematics was required for entering Ṣadiqi College:

- For the first year (maximum age, 12 years): reading and writing numbers, addition and subtraction of numbers with three figures
- For the second year: the same with the table of multiplication
- For the third year: addition, subtraction, and multiplication of numbers
- For the fourth year: all operations on numbers
- For the fifth year: all operations on numbers and decimal numbers. Elements of the metric system

In 1890, school textbooks published in France by the *Inspecteur général de l'éducation* Pierre Leysenne were used. The author recommended problems taken from an environment familiar to the students. The teacher was asked to define the principles necessary for practicing the four operations, to train pupils to solve everyday problems especially those concerning the decimal systems of weights and measures, and to use an intuitive approach to geometry. Pupils were prepared to take the *Certificat d'études primaires* examination.

The following example of an arithmetic problem proposed at the 1886 Tunisian *Brevet élémentaire* shows that in reality no students' religious beliefs and familiar setting were taken into account: Muslim students were asked to solve problems about wine.

> An accountant wants to supply a provision of wine for a year; he anticipated that there will be 84 people to feed, consuming each 64 centiliters per day. He deducted 48 days of leave during which five-sixths of the fed people will be out, and 58 days of vacation during which there will be only 6 people. What is the quantity of wine that he should buy and how much shall he pay to the merchant if the hectoliter costs 36 francs?

5.1.4 The 1906 Reform of Ṣadiqi College

Delmas supported the creation of a *Conseil de perfectionnement* for Ṣadiqi College. Its Tunisian members were chosen from among the brightest Ṣadiqi graduates and the most progressive *Zaytūna* Shaykhs (Shaykh Muḥammad al-Nakhlī, a famous reformer whose ideas were largely inspired by Egyptian Muhammad Abduh; Béchir Sfar, the president of the *Ḥabūs* council; Abdeljalil Zaouche, an industrial member of the Municipal council; Ali Bach Hamba, administrator of Ṣadiqi College; Mohamed Lasram, president of the *Khaldūniyya* Association; Khairallah Ben Mustafa). The Tunisians insisted on upgrading the level of studies of the college and encouraging the graduates to take the *Baccalauréat* examination. This would allow the best of them to attend courses in French universities and thus get access to high administrative positions in Tunisia. With the help of Delmas, they greatly improved the level of studies by adding a new section to the one preparing students for the *Diplôme supérieur d'arabe et de traduction*, later called *Diplôme de fin d'études du Collège Sadiki*. Graduates from the new section were permitted to prepare the *Baccalauréat* in the neighboring French *Lycée Saint-Louis*.

In 1907, new regulations were adopted: only students receiving the *Certificat d'études primaires* could study at Ṣadiqi College. All subjects were taught in French (22.5 h per week) plus 6 h for Arabic and 1 h for calligraphy. The curriculum for the first 4 years of study was exactly the same as in France, except that courses in drawing were deleted. Most teachers were French. At the end of the fourth year, students had to take the *Brevet élémentaire d'arabe* examination which was exactly the same as the *Brevet élémentaire* in France, Arabic being considered a foreign language. During the fifth and last year of study, students were trained in French, Arabic, Islamic history, translation exercises, Islamic laws, Tunisian organizations and regulations, and commercial and administrative accounting. They were supposed to get the *Diplôme supérieur d'arabe et de traduction* and eventually be recruited as interpreters or administrative agents who were required to master the four operations on numbers and fractions, the rule of three, the rule of society and proportions in arithmetic, the metric system, measuring surfaces and volumes, elementary practical geometry, and surveying.

5.1.5 Testimony of a Tunisian Intellectual

Abed Mzali (1906–1997) was the sixth Tunisian who passed the *agrégation d'arabe* in France. He became a professor and was author of several publications. Fifty years after graduating from Ṣadiqi College in 1923, he recalled his studies there:

> I recall a professor whose method of teaching was particularly safe and precise, and in the language of which no word could be used instead of another, no turn of phrase could unreasonably be substituted by another. His conclusions were attractively clear and its reasoning marked by the common sense and the most rigorous logic. All features essential for molding methodical and precise minds and learning the requirements of both solid and subtle reasoning, and avoiding omissions or duplications were there. This man was Mr. Lelu our Professor of mathematics who strongly instilled in me his method so that I obtained in the graduating class the first prize in mathematics, as well as that of physics. Then I dreamed to make it my specialty, but the barriers on my road were such that I had to abandon this idea. Half a century later, I can have forgotten some problems in trigonometry or the demonstration of some theorems in geometry, but the spirit of the method is still present in me and had made an indelible mark in me. (Abdessalem 1975, pp. 153–154)

5.1.6 The 1927 Reform of Ṣadiqi College

A decisive upgrading of the status of the college took place in 1927. It became not only a training school for interpreters and low-ranking civil servants, but also a secondary school with special attention to the sciences and mathematics.

A new "modern" section was added (designated 2e and 1e) for those students who had passed the *Brevet d'arabe* and wished to pursue scientific studies similar to those offered in the modern section of the French *lycées*, while continuing to receive a serious education in Arabic language and literature and Muslim civilization. The final examination was *le Diplôme de fin d'études sadikiennes.* Graduates could prepare for the *Baccalauréat* either in the class of philosophy or the class of mathematics at the *Lycée Carnot*.

After World War II, "Tunisian" sections were created in the provincial *lycées* (Bizerte, Sousse and Sfax). Their program was modeled on that of Ṣadiqi College. After 1956, this model became the only one used in the now-unified system of secondary schools.

5.2 The System of Primary Schools During the French Protectorate

In addition to French schools attended mostly by non-Muslim pupils, Franco-Arabic primary schools were founded in most of the cities and run by the *Direction de l'Instruction Publique*. Also, a new kind of *kuttābs* was created by private individuals and associations: the modern Qur'anic schools.

5.2.1 French Primary Schools and Franco-Arabic Primary Schools

French primary schools were intended specifically for French and European children, some for boys only, others for girls only, and some for both. Curriculum and teaching in these schools were identical to those in France. They covered ages 5–15 in six levels of study, one for each year: *Cours enfantin* (1 year), *Cours élémentaire* (2 years), *Cours moyen* (2 years), and, the last one, *Cours supérieur*

Table 20.3 Information from *Bulletin Officiel de l'enseignement public* (Tunis, 1926/01)

Type of primary schools	Tunisian boys	Tunisian girls	Total students
French primary schools for boys	704	3	9,257
French primary schools for girls	32	279	9,469
French primary schools for boys and girls	324	27	2,193
Franco-Arabic primary schools for boys	17,658	80	18,511
Franco-Arabic primary schools for girls		1,778	1,779
Total	18,718	2,167	41,109

Table 20.4 Information from *Bulletin Officiel de l'enseignement public* (Tunis, 1926/01)

Type of *Ecoles primaires supérieures*	Tunisian boys	Tunisian girls	Total students
Alaoui College	116		375
E.P. Supérieures (Sfax and Sousse)	31		213
E.P. Sup. for girls (Bizerte-Sousse-Sfax)		1	276
Total	147	1	864

(1 year) also called "Classe de septième, première division." This last course led to the examination of the *Certificat d'études primaires*. All subjects, taught in French, included arithmetic, the metric system, elements of geometry, and bookkeeping.

Franco-Arabic primary schools were schools with the same curricula as the French primary schools, but Arabic language courses were added for European pupils. Tunisian boys had Qur'an, religious lessons, and Classical Arabic language and grammar in addition. The primary schools for Muslim girls had two branches, one similar to the Franco-Arabic schools leading to the *Certificat d'études primaires* and one significantly more oriented to household education. Arab pupils were encouraged to attend these schools so they could become good workers in French-owned farms and factories (Table 20.3).

L'école primaire supérieure was an extension of the primary schools, offering courses in geometry, linear drawing and surveying, elements of physical and natural sciences with a practical purpose, and an introduction to history and geography, especially of France. Pupils prepared during 1 or 2 years for the examinations of the *Brevet élémentaire* and *le Brevet supérieur*. This second examination was considered the last one for most boys and for all girls and only after 1924 were young girls admitted to *Lycées* and could prepare for the *Baccalauréat*. *Ecoles primaires supérieures* were ultimately transformed into *lycées* and received primarily European children. Examples of this type of school were the Alaoui College (for boys) which ultimately became a *lycée* and *l'Ecole Jules Ferry* (for girls) which became in 1915 the *Lycée Armand Fallières* (the secondary school for girls) (Table 20.4).

5.2.2 The Modern Qur'an Schools

Khairallah and his companions considered reforming the *kuttābs* an absolute priority. In 1907, he launched the first private modern Qur'an school in Tunis. In this school, the curricula and the pedagogy were to be the same as those used in the Franco-Arab primary schools, but all subjects were given in Arabic. Compulsory courses in the French language were also offered to pupils. Studies lasted 5 years.

In the fifth year of education, children ages 12–14 used textbooks from Egypt and Syria in the teaching of Arabic language and mathematics. In mathematics, these pupils had to master the following in arithmetic: prime numbers, divisibility, decomposition of a number into prime factors, GCD, problems of interest, discount, sharing, and means. In addition to arithmetic, they studied accountability, the metric system of measuring volumes and weights, and plane geometry (Khairallah 1909, p. 580).

Table 20.5 Subjects taught at the Lycée Carnot and the distribution of time

Subjects	6e (h)	5e (h)	4e (h)	3e (h)
French	9	9	5	5
Arabic	3	3	3	3
Italian	2	2	2	2
Mathematics	2	2	3	4
Other subjects	6	6	10½	10½
Total	22	22	23½	24½2

Parents striving to educate their children and those opposed to Franco-Arabic schools on nationalistic or religious grounds ultimately founded similar schools in cities and villages where public schools were missing. These schools prepared students to study in modern secondary schools.

Souad Bakalti (1996, p. 144) showed that modern Qur'an schools for girls were opened after World War II. In 1955, the number of girls attending these schools was 5,474 out of 38,662 pupils. After independence, they were transformed into regular primary schools within the unified system of education.

5.3 The French Secondary Schools in Tunisia

A few years before 1881, a Catholic school was created offering a secondary-level education mostly to European pupils. With the Protectorate, this school became a *lycée* run by the French according to the curricula existing in France. As a centrally located school in Tunis, the *Lycée Saint-Charles* was composed of a primary school and a college. In 1893, the school was given a new name, *Lycée Carnot,* and became the most prestigious secondary school in Tunisia. The first secondary school for girls was the *Lycée Armand Fallières* which replaced the secondary section of *L'école Jules Ferry* in 1915. More colleges were created later.

5.3.1 The Lycée Carnot

Following exactly the same curricula and methods of teaching as those in similar secondary schools in France, the *Lycée Carnot* offered two types of education: a "classical" one based on the study of Latin and a "modern" one emphasizing mathematics and foreign languages. Pupils had to choose their type of studies upon leaving primary school.

Each section was subdivided in two stages: le *premier cycle* which was a 4-year course and *le second cycle* which was a 3-year course. Pupils were then prepared to take one of the four *Baccalauréat* examinations (A, B, C, or D). C and D included the science sections which constituted separate classes in the *second cycle*. Classical secondary sections were strictly identical to those in France, while modern sections were more adapted to the needs of colonial Tunisia. After the premier cycle, pupils could choose either the modern second cycle which was exactly like the one in France, or they chose the commercial second cycle which was a 3-year course leading to a diploma in business studies.

In 1895, the *Bulletin Officiel de l'enseignement public* published a decree defining the official curriculum of the modern premier cycle at the *Lycée Carnot* (Table 20.5).

The content of the mathematics courses is briefly described below.

Sixth grade (2 hours/week)

Operations on integers. Ordinary fractions. Decimal numbers.

Earth sphere. Vertical. Horizon. Simple notions on the Poles, meridians, parallels and the Earth equator. Cardinal points. Longitude and geographical latitude. (The teachers were supposed to avoid any theoretical discussion.)

Fifth grade (2 hours/week)

Prime numbers. Metric system. Measures of surfaces and volumes. Extraction of a square root of a number. Rule of three. Simple interest. Commercial discount. Pension. Problems related to mixtures and alloys. (Teachers were supposed to insist on using the metric system and devoted much attention to solving problems.)

Fourth grade (3 hours/week)

Ratios and proportions. Proportional sizes. Application to commercial arithmetic: interest, discount, commissions, bank accounts; different methods of calculation. Rules of society and mixtures. Calculation of annuities. Metric system. Comparison with the major systems of weights, measures and currencies. Cost and parities in goods.

Segment, line and plane. Angles. Triangles. Perpendicular lines. Right triangles. Equal triangles. Geometric loci. Parallel lines. Sum of the angles of a triangle, of a convex polygon. Properties of parallelograms. Symmetrical figures. Use of the ruler and the square. Circles. Measures of angles. The use of the ruler and the compass. Elementary problems of geometric loci.

Third grade (3 hours/week)

Theoretical aspects of numeration. Operations on integers. Divisibility of integers. Remainder of the division. The greatest common divisor. Prime numbers. The least common multiple. Operations on fractions. Decimal fractions. Square and square roots.

Using letters. Geometric formulas. First degree equations. Negative numbers. Number line. Operations on negative numbers. Algebraic fractions. Monomial and polynomial expressions. Equations with several unknowns. First degree inequalities. Applications to commercial arithmetic.

Proportional lines. The side splitter theorem and its reciprocal theorem. Angle bisectors of a triangle. Geometrical locus of all points the distances of which from the segment's endpoints are in a given ratio. Similar triangles. Similar polygons. Lines intersecting a circle. Metric relations in a triangle. Proportions. Regular polygons. Inscribing a polygon in a circle. Inscribing a triangle. Right triangle. Pythagorean theorem. Length of an arc in a circle. Length of the circumference. Ratio of the circumference of a circle to the diameter of the circle. Computing π. Areas of polygons and circles. Ratio of the areas of two polygons and of two circles.

Concept of surveying. Use of the chain and the square of surveyor. Concepts of plane leveling. (Applications in the classroom and in the field.)

It is difficult to determine the impact on Tunisia of the famous reform of 1902, which for the first time in France had "established a decisive improvement of the status of mathematics as a teaching subject in the secondary modern sections" (Schubring 2002, p. 52). Since programs of Tunis *lycées* were an exact duplication of those in France, this reform did "reinforce the experimental nature of the new teaching of mathematics" in the four first years of studies. Belhoste (1990, p. 395) quoted the official instructions:

> The teaching of geometry must be essentially concrete. (…) Any purely verbal definition being excluded, one should speak of a new element by giving its concrete representation and indicating its construction. (…) Professor will have to appeal to the experience and resolutely accept as experimental truth all what seems obvious to children.

Belhoste (1990, p. 395) enumerated the principal innovations introduced in the program of the modern section of the second cycle:

> In 2e and 1e: Subjects are more developed than in preceding years. The study of variation and graphic representation and derivation of algebraic expressions such as: ax+b; ax+b/a'x+b', and ax^2+bx+c. In geometry, the study of space geometry and trigonometry.
>
> Descriptive geometry; introducing vectors in courses of mechanic and elements of analytic geometry.
>
> Also in 1e: Usual curves—ellipse, parabola, hyperbola—are discussed in detail. Principles and methods of analytic geometry, of descriptive geometry and of vectors are gradually introduced through applications.
>
> In mathematics class, all the preceding subjects are studied thoroughly. In algebra, the concepts of function and derivative are introduced and applied to the circular functions. In geometry, symmetry, translation, rotation (in the plane and about an axis in space) and homothety are studied.

5.3.2 Further Developments of Secondary Schools

Education and schooling were imposed by the constant increase of French settlers and were also used for a better assimilation of Europeans and Tunisian Jews during the Protectorate period. Over time,

Table 20.6 Schooling for Tunisian children at the primary level

Year	Number of students	Rate
1953–1954	214,484	100
1956–1957	231,312	108
1959–1960	390,150	182

primary schools were first transformed into colleges and some of these into *lycées* preparing students to all types of *Baccalauréat*.

In 1944, a new section was introduced in these schools, called "the Tunisian section." This was an extension of the courses taught in Ṣadiqi College and prepared for the diploma of this college. Eventually the Franco-Tunisian *Baccalauréat* was created in 1950.

After reporting that 14,497 students were registered in secondary schools, Sraieb (1968, p. 66) pointed out that among them, only 6,682 were Tunisians (section classique, 289 Tunisians out of 3,189 students; section moderne, 2005 Tunisians out of 6,918 students; section tunisienne, 4,382 Tunisians out of 4,390 students). These numbers were definitely insignificant when considering the 157,000 pupils registered in primary schools. In fact, one out of every six French pupils could go to a secondary school, while the ratio was 1 out of 25 for Tunisians.

The leaders of the French colony were opposed to any extension of the secondary school system and refused to open higher education to Tunisians. One example of the policy of segregation which the colonial rulers systematically followed gives the following fact. Tunisian graduates from French engineering schools were not allowed to enter any position of authority and control in the technical and industrial areas. French authorities maintained a strict monopoly of Europeans in these domains. During the 70 years of colonization, less than 30 engineers could thus be trained (Ben Salem 1994).

In 1945 *l'Institut des Hautes Etudes Supérieures* (IHES) was founded as an annex in Tunis of the *Faculté de Paris*. Preparatory courses were offered in law, sociology and history, Arabic language and literature, and the sciences, medicine, and mathematics. Moore (1965, p. 176, note 39) reported that in 1952–1953, the number of students at the university level was 212 at the IHES, 600 at the *Zaytūna*, and 300 in France. In 1956, 107 Tunisians (107 men and 5 women) were among the 531 students registered in the Sciences section of the IHES. After independence, in 1960–1961, there were 254 at the *Zaytūna*, 1746 at the IHES, and between 1,500 and 2,000 at France universities.

5.4 Teaching Mathematics in Independent Tunisia

Tunisian education quickly developed after 1956. All boys and girls of ages 6–16 began receiving a basic preparation. All three levels of schooling were unified based on the model of Ṣadiqi College: primary schools (6 grades), intermediate level (3 grades), and secondary schools (3 grades). The Table 20.6 shows the rate of development of schooling for Tunisian children at the primary level after independence in 1956.

Secondary education ended with one of the four new Tunisian *Baccalauréats* (humanities, mathematics, sciences, and economics). The best students graduating from the professional and vocational training schools were accepted in higher education. The University of Tunis was established by incorporating the existing higher schools and institutes, including the faculty of theology.

Teaching mathematics was reinforced at all levels of the educational system. It was taught in Arabic during the 6 years of primary school and in French at the intermediate and secondary levels. Developing a highly qualified cadre of scientists, engineers, entrepreneurs, and other professionals became a major objective. To accomplish this goal, the preparation of high-achieving mathematics students was encouraged. A great number of pupils passing the mathematics *Baccalauréat* were sent to French schools of engineering, while others pursued learning mathematics at the *Faculté des sciences de Tunis*.

6 Conclusion

At the beginning of the nineteenth century, three North African countries were under Ottoman rule: Algeria, Tripoli, and Tunisia. The first was invaded by the French army in 1830 and became a French colony where the colonial policy of settlement and assimilation aimed essentially to establish French as the only official language while trying to downgrade the position of the Arabic language. The educational system was then structured so that the curriculum was delivered entirely in French and allowed no place for Arabic studies. Tripoli had its ruling dynasty dismissed and was directly ruled by an Ottoman governor until 1911–1912 when Italy invaded Libya.[12] Tunisia's response to European pressures and threats was to follow Egypt's path by reforming its army and somehow modernizing its institutions (Abdeljaouad 2010). The Prime Minister Khayr al-Dīn (1873–1877) tried to change the traditional system of education run by the *Zaytūna* shaykhs and founded a new school, Ṣadiqi College, modeled after the French *lycées*. The seeds of modernity that he sowed by creating this college flourished in the following century and gave rise to the Tunisian modern nation.

After the establishment in 1881 of the French protectorate over Tunisia, reforms were stopped and direct colonial administrations replaced the old-fashioned institutions. The *Direction de l'Instruction Publique* founded French primary schools and colleges essentially for settlers' children and other primary schools for Tunisians; however, these pupils were generally not encouraged to go beyond some kind of vocational training, and barriers were set up that deprived them from secondary and higher education.

Ṣadiqi College graduates played an important role in developing multiple actions through political parties, educational and cultural associations, and the publication of newspapers and magazines in order to remove these barriers. They created modern Qur'anic primary schools and the cultural association *al-Khaldūniyya*, where modern sciences and mathematics were taught; they insisted on enhancing the level of studies at Ṣadiqi College. They finally secured that graduates from the college be permitted to prepare for the French *Baccalauréat* and thereafter continue to study in universities.

European mathematics had a natural place in the curriculum of the military schools founded by the nineteenth-century reformers and in the classes of Ṣadiqi College. During the Protectorate period, the mathematics taught in schools and colleges in France was part of the curriculum of similar schools in Tunis. It is only after independence in 1956 that a unified system of education was progressively installed and ultimately generalized to all Tunisian pupils and expanded to secondary and higher schools.

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[12] On the extreme West, the French Protectorate over Morocco was established in 1912. The French authorities did not duplicate either the policy of assimilation of the Arabs as in Algeria or the policy of encouraging bilingual primary schools as in Tunisia. They accepted that the Moroccans were trying to develop teaching in the Muslim schools and colleges, but prevented the establishment of any gateways to French higher education. However, as early as 1930, the Moroccan bourgeoisie imposed the teaching of French in Muslim colleges and the creation in these schools of a section preparing for the *Baccalauréat*, so that as early as 1936, eleven Moroccan students obtained the French *Baccalauréat* and pursued university studies.

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Part V
History of Teaching Mathematical Subjects in School

Chapter 21
History of Teaching Arithmetic

Kristín Bjarnadóttir

1 Introduction

This chapter on the history of teaching arithmetic concerns the period from the thirteenth century to the third quarter of the twentieth century or nearly seven centuries. The starting date is marked by the origin of reckoning schools, *scuole d'abbaco*, in Italy, while the closing date is determined by end of the worldwide reform movement, frequently termed "New Math".

Various forms of textbooks have been written since the origin of writing. Printing centuries later greatly facilitated their distribution in book form. Written and printed sources in the form of arithmetic textbooks therefore abound, while sources on the personal experiences of arithmetic teaching are scarcer. Furthermore, organised schools were few until the nineteenth century. Originally, schools were established to educate certain professions. Cathedral and monastery schools of the Christian Church were obliged to ensure the clergy's arithmetic knowledge of the calendar. Apprentices to trades such as masons, merchants, and moneylenders could expect to learn such practical mathematics as was relevant to their profession. Schools for educating merchants are of special interest in this respect for the need for arithmetic in monetary transactions and for conversion of monetary and measuring units.

Arithmetic textbooks fell early on into a certain pattern. However, authors had some freedom on the pedagogical ordering of the material, to present mathematics in context and to develop natural curiosity. The early printed textbooks were not written for use in institutionalised school systems in which students were working towards examination qualifications. The relation of the text to the reader differed from that of a present-day school text. There were usually not sets of exercises; the works were not presented in a readily identifiable "lessons" and the presence of a teacher was not assumed (Howson 1995). This gradually changed as schools became more widespread.

Textbooks not only have a role for individuals or within individual classrooms but also within a system as a whole. In some circumstances, a textbook may attempt to establish new content or pedagogical norms. Normally they may be considered in the contemporary school system to flesh out a centrally prescribed curriculum. However, they may attempt to update pedagogy within a centrally prescribed curriculum or even help to define a new curriculum (Howson 1995).

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In the following, the history of teaching arithmetic will be traced through sources such as arithmetic textbooks, scholarly papers, and some personal experiences of learning arithmetic, keeping in mind that history of mathematics of the past is no longer regarded simply as a precursor to the mathematics of the present but as an integral part of its own contemporary culture (Stedall 2012). For simplicity, this history will mainly be confined to Western countries, Europe and North America, with the awareness that European arithmetic is wholly based on Chinese, Indian, and Arabic heritage. Because of its diversity, the development of this history in other parts of the world will only be referred to in general terms.

2 Origins in the Middle Ages

2.1 Theoretical Tracts and Practica

In late mediaeval times and early modern times, arithmetic literature can be divided into *practica* and *theoretical tracts*. The *theoretical tracts* were generally written in Latin and reflected at first classical Greek mathematical concerns. They gradually developed into advanced mathematics. The theoretical works which provided a foundation for scholastic mathematical contemplation were modelled after the writings of Boethius (c. 480–524), whose work was based on translations of the works by the neo-Pythagorean mathematician Nicomachus of Gerasa (c. 60–c.120) and Euclid (c. 300 B.C.). From them a perennial idea is traced, appearing in literature until the late nineteenth century, that the unit was not a number but the origin of number. The text *Dixit Algorizmi* by the Islamic scholar al-Khwarizmi says "… this is what is said in another book on arithmetic that the unit is the origin of all numbers and is outside numbers" (Allard 1992, p. 1), believed to be referring to either Euclid's *Elements*, book VII, or *Arithmetica* by Nicomachus. Boethian arithmetics were the primary source of all arithmetic taught in the Latin schools and universities, at least to the end of the twelfth century (Swetz 1987).

Algorisms evolved from descendant translations of al-Khwarizmi's text. An algorism whose problem situations dealt with affairs of business and commerce was also called *practica*. Commercial arithmetics were not devoted to philosophical speculations on the nature of number as the theoretical tracts often were; rather, they were handbooks on readily usable mathematics, preferably written in the vernacular.

2.2 Origin of Reckoning Schools

In late mediaeval times, occupational opportunities in Europe broadened from agricultural and pastoral pursuits to include participation in activities of manufacture and commerce (Swetz 1987). This development originated alongside with the Renaissance in northern Italian cities, such as Florence, Venice, Milan, and Bologna.

The rise of the Florentine woollen cloth industry and of banking provided a basis of capital. Just before the middle of the fourteenth century, Florence had become a metropolis of about 90,000 people, making it one of the great cities of Europe. From the fourteenth-century Florence, the rising commercial centre of Western Europe, emerged the *scuole d'abbaco*, reckoning schools, which provided training in calculating techniques necessary for the trading commerce (Schubring 2012).

A widespread knowledge of the Arabic numeral system, which is first noted among Italian merchants of the thirteenth century, was undoubtedly the result of their close contacts with the Arab world and with new demands created by their increasingly complex system of business organisation (van Egmond 1980). The new numeral system, however, was met with suspicion which would last for a long time to come. In 1299, its use was forbidden in Florence by the corporation of money changers, so merchants began only gradually to utilise it. One objection against it was that the zero, 0, could be misinterpreted as 6. Even in 1494, reckoning masters in the city of Frankfurt am Main in Germany were forbidden to make use of the Indian figures in the arithmetic books of the city. Abacus counter boards were used for centuries alongside written algorithms (Lüneburg 2008).

If a student wished to learn commercial arithmetic, he or she usually did not go to university where arithmetic was taught as one of the subjects of the *quadrivium* under the influence of scholasticism, but he or she sought out a reckoning master, a man skilled in the arts of commercial computation, with whom to study. In Italy these masters were called *maestri d'abbaco*, and in the German territories, *Rechenmeister*. Many of them accepted students for private tuition or conducted formal group classes in their art which gave rise to reckoning schools, whose numbers rapidly increased in the commercial cities and along the trade routes of Europe (Swetz 1987).

The earliest attested abbacus master taught in Bologna in 1265, probably in a private arrangement. Within the next four decades, abbacus masters turned up in numerous other towns from Umbria and Tuscany in the south to Genoa, Lombardy, and Venice in the north. Masters paid by city communes surfaced in sources from 1280s onwards – mainly in smaller communes. Venice and Florence appear to have felt no need for a public undertaking. The Florentine schools were soon considered to be the best, and many Florentines went to teach in other places (Høyrup 2007). There were six abbacus schools in Florence in 1343 and an average of three or four such schools operating continuously in Florence from the earliest decades of the fourteenth century right through the sixteenth century and probably beyond (van Egmond 1980). In 1613, Nuremberg, Germany, alone had 48 such institutions (Swetz 1987).

The normal entrance age in the abbacus schools was 10–11 years, and the normal duration of the training was around 2 years. At first, students were taught to write numbers in the Arabic number system, followed by the multiplication tables and their applications (Høyrup 2007). They were taught how to deal with fractions and how to solve basic mathematical problems. Sections of the course were devoted to understanding the complex Florentine monetary system. The school day followed a familiar routine of lessons, exercises, and recitations. Nearly all educated men of the Renaissance gained their basic understanding in schools such as these. When grouped with earlier schools of reading and writing, higher schools of Latin grammar, and the educational apex of the university, it is apparent that the abbacus schools were an integral part of a well-designed educational system (van Egmond 1980).

2.3 *Libri d'abbaco*

Since the *abbacus* books (spelled with two b's in modern Italian), *libri d'abbaco,* deal with the elementary operations of counting and reckoning that were performed on the *abacus* board and are contemporary with it, it was perhaps natural to assume that these books described or taught how to use this device. In reality, however, there is no connection whatsoever between the Italian *abbacus* books and the counter *abacus* other than etymological. The abbacus books are instead exclusively concerned with teaching and using the forms and methods of the new Hindu-Arabic numeral system and not a single mention of the counter abacus has been found in any abbacus book (van Egmond 1980).

The first use of the word *abbacus* in the sense of reckoning, independent of the counter called *abacus* that was widely used for calculations, probably appeared in the title of Leonardo Pisano's *Liber abbaci* (Sigler 2003) of 1202. Leonardo Pisano (1170/1180–1240/1250) was a son of a Pisan merchant who had been an official in the Arab port of Bugia in North Africa where Leonardo grew up. Leonardo, who is now better known by his pen name Fibonacci, studied under Arab teachers and travelled widely in the Arab world where he learned about Arab commercial and mathematical systems. His book, a massive compendium of mathematical practices conducted according to the Arabic systems, was written in Latin, but towards the end of the thirteenth century, many shorter versions and extracts based on it and written in Italian began to appear (van Egmond 1980). The book showed the practical importance of the new numeral system and its calculation methods. It contained practical topics such as calculations of profits, currency conversions, and measurements, supplemented by the now standard topic of current algebra texts such as mixture problems and the like (Katz 1993).

During the twelfth through fifteenth centuries, algorithms, based on al-Khwarizmi's arithmetic, appeared in great numbers and in a diversity of languages other than Latin (Swetz 1987). Numerous Italian *libri d'abbaco* were written by teachers who spoke the ordinary language of the marketplace. The average abbacus book contains around 400 individual problems, each one different from the other. A characteristic feature of the abbaci is that the solution and detailed working out of the problem are always given immediately after the enunciation. The modern habit of printing only the problem and leaving the solution to the reader, or at most giving only the answers at the back of the book, is never found in the abbaci. The books provide a full description of the syllabi of the abbacus schools. They contained some or the entire list below:

1. Preliminary material
   - Numeration – a description of the ten Hindu-Arabic numerals and explanation of the principle of place value
   - The four arithmetic operations – addition, subtraction, multiplication, and division – usually applied successively to whole numbers, fractions, and the compound quantities of monies, weights, and measures
   - Tables – multiplication tables for numbers and monetary units, tables of squares, and lists of the parts of monetary units
2. Business problems
   - Prices and products – finding the price or amount of a good by means of the Rule of Three
   - Currency exchange and conversion of units of weights and measures
   - Problems relevant to bartering
   - Partnership – dividing the profits between members of partnership
   - Interest and discount – distinguished between simple interest and compound interest
   - Equation of payments – a series of loans made over a period of several months or years combined for repayment on a single date
   - Alligation – silver and gold of varying purities combined in a mixture of a desired purity
3. Recreational problems – including algebraic questions and series and progressions
4. Geometrical problems
   - Introduction – definition of geometrical objects and figures
   - Geometrical problems – dealing with measurements of abstract geometrical figures
   - Measure problems – dealing with the measurement of real objects by geometrical principles
5. Methodological sections
   - Rule of Three – the "Golden Rule" or rule of proportion
   - Rule of the false or simple position
   - Rule of double false position
   - Algebra – solution by unknowns and equations
6. Miscellaneous material
   - Number theory – classical Boethian arithmetic
   - Tariffs – compilations of mercantile information for various cities in the European trading sphere
   - Astronomy and astrology

- Calendars – tables and charts for finding the initial days of the month or ecclesiastical holidays
- Medicine – recipes and home remedies for various illnesses
- Literature – including poems, romances, and chronologies (van Egmond 1980)

The *libri d'abbaco* were not schoolbooks in the modern sense. The character of school instruction in this period, the time-consuming nature of and expense of manuscript production, and the rather advanced nature of the books themselves all make it improbable that every 10-year-old boy who attended an abbacus school could have had an abbacus book for his personal use. It is more likely that some of the manuscripts served as a collection of problems or reference books which teachers could draw upon for their lessons (van Egmond 1980).

3 Early Modern Times

3.1 The First Printed Arithmetic Textbooks

It was in the climate of the Venetian trade that the *Treviso Arithmetic* appeared, a textbook by an anonymous author that has been named after its place of origin, Treviso, a town in the vicinity of Venice (Swetz 1987). *Treviso Arithmetic* provides a glimpse of the mathematical transition taking place. Printed in 1478, it is the earliest known, dated, printed arithmetic book written for popular use. The *Treviso Arithmetic* is a *practica*, intended for self-study and relevant to the commercial and reckoning needs of the Venetian trade. It was written in a vernacular, the common Venetian dialect of its period, and intended for all who wished to learn the art of computation, not just for the privileged few as had been the case previously. Vernacular texts eliminated a monopoly of knowledge and gave great impetus to the rise of a middle class (Swetz 1987).

It has been estimated that between the origin of European printing and the end of the fifteenth century, 30 practical arithmetics were printed, of which more than one-half were written in Latin, seven in Italian, four in German, and one in French. During the same period, about 26 theoretical Boethian-style Latin arithmetics were also produced. Mathematics was moving from the realm of scholastic speculations to the applications of manufacturing and the marketplace (Swetz 1987). Practical books led the demand for mathematical literature. A number of Italian, for example, *Il Libro de Abacho de Arithmetica e De Arte Mathematiche* (1484) by Piero Borgi (1424–1484); Portuguese, for example, *Tradado da pratica de arismetica* (1519) by Gaspar Nicolas; German; and English textbooks were produced (Swetz 1992).

In Germany, Nuremberg was the focus of mathematics education. Boys from families of merchants and skilled workers came to study at the Nuremberg *Rechenschulen* and brought the knowledge of the mercantile art all around Germany and across borders. The mathematician Johannes Müller (1436–1476), better known by his Latinised name Regiomontanus, from Königsberg in Unterfranken, studied there. A great number of arithmetic books were published in the German-speaking towns: Leipzig, Frankfurt am Main, Vienna, Erfurt, Nuremberg, etc. In 1483, the first printed German arithmetic textbook was published by Heinrich Petzensteiner, *Rechnung in mancherley weys* (Jänicke 1888). The author of the book, also called *Das Bamberger Rechenbuch*, was Ulrich Wagner (?–1490), one of the famous *Rechenmeister* in Nuremberg in the fifteenth century (Schröder 1988).

Among the best known German reckoning masters is Adam Ries (1492–1559), who in 1522 was *Rechenmeister zu Erfurt*. Ries, who later lived in Annaberg, wrote a number of works, among them *Rechenung auff der linihen und federn im zal, mass und gewicht*, first published in 1522 and republished many times since. *Rechenung auff der linihen* (translated as "calculating on the lines") refers to calculations on the abacus counter, to which doubling and halving belonged alongside the four basic operations. Use of the counter board is visually simple in the case of addition and subtraction, but more difficult in handling multiplication and division. Therefore, *federn* (feathers) were used to write numerals, where the first nine, 1…9, were "bedeutlich" or meaning something, while the tenth, 0, meant nothing. This book by Adam Ries mirrored the battle of the abacists, who favoured the use of the abacus counter, against the algorists, who favoured the performance of arithmetic operations through the positional system of numbers.

Care was taken to explain the position system and how to read large numbers by grouping three numerals together, beginning from the right side. Ries and other similar writers of arithmetic textbooks then carefully explained how to perform the arithmetic operations by examples. Simple number tricks were used to ease mental multiplication, while regular written multiplication fell into a pattern still being practised. Written division was less comprehensible for a modern person as only the remainders and not the intermediate products were written down. In addition to the four basic operations, Ries taught progressions, the Rule of Three, fractions, currency exchange, and other topics relevant to trade (Jänicke 1888).

Division was considered the most difficult operation for the teacher to teach and for the student to understand. Knowledge of division implies a proficiency in the three other basic operations. Perhaps the most impressive algorithm of basic computation used through the seventeenth century was the galley method of division, which reminded early writers of the sails of a ship. This method was also referred to as "scratch division", given the necessity of crossing or scratching out various digits throughout the process. The method can be traced to Eastern societies; Hindus employed it on their sand tables. The galley method was actually efficient and demanded less paper than the standard long-division method, which was important because paper was still an expensive commodity (Swetz 1987).

A simple example from *Arithmetica Historica* by Suevus (1593), of dividing 1593 by 4 in order to check if the year 1593 was a leap year, illustrates the algorithm, which was modified by writing the intermediate products below the divisor, immediately below the digit of the dividend to be divided. The intermediate products, such as 36 which is the result of 4 times 9(0), are not necessarily written in a line. Here, the digit 3 is written in the lowest line, in the hundredths column, and 6 in the second lowest line, in the tenths column. The following product, 28 (4 times 7), is similarly written. Remainders, which are three in all cases, are written above the dividend.

  ~~3 3~~ 3
~~1 5 9 3~~ (3 9 7
  ~~4 4 4~~
~~1 2 6~~ 8
  ~~3~~ 2

Dura cosa è la partita (division is a difficult thing), said the Italians (Swetz 1987).

Mathematics textbooks were written in other European countries also. In 1540, the Dutch mathematician Gemma Frisius (1508–1555) published his *Arithmetica Practicae* which appeared in 59 editions in the sixteenth century and several centuries thereafter (Swetz 1992). In *The Grounde of Artes* by the Welshman Robert Recorde (ca. 1512–1558), published in 1543, the author used the Socratic method of dialogue to deal with matters of definition and understanding (Kilpatrick 1992).

During the sixteenth and seventeenth centuries, education was gradually releasing itself from scholastic characteristics. Particularly for arithmetic, the claim that the unit was not a number was questioned. This was, however, a recurring theme in arithmetic textbooks, even up to the nineteenth century.

3.2 Stevin's Number Concept and Decimal Fractions

Simon Stevin (1548–1620), who lived in the Netherlands, contributed to a substantial change in mathematical thinking. His contribution was the creation of a notation for decimal fractions, and he strongly advocated its use. He also played a fundamental role in changing the basic concept of number and erasing the Aristotelian distinction between number and magnitude. These contributions are set forth in his works *De Thiende* and *l'Arithmétique*, both published in 1585. Decimal fractions were not used in Europe in the late Middle Ages or in the Renaissance. If fractions were needed, they were written as common fractions or, in many trigonometric works, as sexagesimal fractions (Katz 1993).

Stevin began his *l'Arithmétique* with two definitions: (i) arithmetic is the science of number, and (ii) number is that which explains the quantity in each thing; that is, Stevin made the point that a number represents any type of quantity at all. A number is no longer to be a collection of units as defined by Euclid and Nicomachus, and therefore not a basis for a distinction between the discrete and the continuous. Stevin also defined irrational numbers, such as the square root of 8, as numbers, and the decimal number system of *De Thiende* enabled him to represent the square root of 8 to any accuracy desired. Not until the nineteenth century, however, was the work of imbedding "discrete arithmetic" into "continuous magnitude" completed. Nevertheless, Stevin stood at a watershed of mathematical thinking (Katz 1993).

3.3 Reform Movements and Textbooks

In the seventeenth century, two great movements determined the educational structures found on the continent of Western Europe until the French Revolution: the *Protestant Reform* and the *Catholic Counter-Reform* (Schubring 2012).

The Protestant Reform in the early 1500s, according to its general approach of assuming the population was literate, set out to issue school ordinances in subsequent decades to establish *Gymnasia* in larger towns that would prepare its students for university studies. Martin Luther's (1483–1546) collaborator, Philipp Melanchthon (1497–1560), was particularly keen on nurturing mathematics education. For Melanchthon, knowledge existed primarily for the service of moral and religious education. He praised mathematics for its ethical role (Grosse 1901, p. 13). The arithmetic, however, was mainly left to the *Hochschulen*, the vocational schools. In the Latin schools, it had only little space in the sixteenth century (Jänicke 1888).

Melanchthon served as professor at the University in Wittenberg where Sigismund Suevus (1526–1596) attended his lectures. Suevus, who was thus well acquainted with the Protestant Reform, wrote his *Arithmetica Historica – Die Löbliche Rechenkunst* of 455 pages, published in 1593. Suevus did not discuss the number concept at all in his *Arithmetica Historica*, but went directly to numeration, explaining the decimal place value system and the four arithmetic operations using examples from the Bible and historical literature about antiquity. The author combined the educational goals of arithmetic and its ethical aims through the careful choice of examples. The content of *Arithmetica Historica* comprised numeration, the four arithmetic operations, progressions, and the Rule of Three, in addition to its variations; the reversed rule, the double rule, the virgin rule, the false rule – *Regula falsi* – etc.; square and cubic numbers and their roots; and finally the area of a circle using 3 1/7 as pi, but no fractions. In his foreword, Suevus dedicated the book to the mercantile class, but it was also intended for the Latin and German schools and even for the young and old in all social classes (Grosse 1901).

Most textbooks of this time were written with self-instruction in mind, while the mercantile community was the main target group. Therefore, the main emphasis was on nice solution methods, and an emphasis on inner reasons was considered unnecessary and inappropriate. The ideology of arithmetic teaching of the times was as follows: (a) arithmetic was also intended to serve ethic education, and (b) practical arithmetic had less importance. Poetic rhymes were used to support the ethical role of arithmetic instruction, even up to the nineteenth century. The literature is rich; about 300 arithmetic texts were published in Germany in the seventeenth century (Grosse 1901).

One of the first arithmetic textbooks in the English language was *Cocker's Arithmetick* by Edward Cocker (1631–1676). The book ran to more than 100 editions over a period of about 100 years. It was first published in 1677 or 1678, after Cocker's death, and edited by John Hawkins. *Cocker's Decimal Arithmetick* was also published posthumously by John Hawkins in 1684. *Decimal Arithmetick* along with its companion volume, *Cocker's Arithmetick*, was used in schools in the United Kingdom for more than 150 years. The concept of decimal fractions and the advantages of using them in calculations were well known, but a wide variety of different notations were in use. After surveying various notations, *Decimal Arithmetick* recommended the decimal point notation introduced by John Napier (1550–1617). *Decimal Arithmetick* gives instructions for calculations involving decimals, methods of extracting roots, and an overview of the concept of logarithms. There are many worked examples, some of which involve solid geometry and the calculation of interest. Another author whose books became widespread and translated into other languages is Edward Hatton (b. 1664?), whose titles include *Tradesman's Treasury* (1695) and *An Intire System of Arithmetic* (1721).

3.4 Teaching Methods and Learning

During the seventeenth century, progress in teaching methods was underway. The teacher was expected not to exclusively turn his attention to students' memorisation skills. His duty was to explain and bring forward students' understanding and not only explain the examples in the existing book but add to them in order to enlighten understanding. The multiplication table was not to be learned only rotely without understanding. Mathematicians were not much concerned with regular arithmetic; this was left to reckoning masters, teachers at school institutes, the clerical class, technicians, and other interested parties (Grosse 1901).

Jan Amos Comenius (1592–1670) was a Czech-speaking teacher, educator, and writer who became the prime educational leader of the seventeenth century. He emphasised in his *Didactica Magna* (*Great Didactic*) of 1628–1632 that arithmetic and geometry must be taught partly for the various needs for life and partly for the scholarly topics that awakened and sharpened the mind. In his *schola universalis sapientiae* (school of universal knowledge), mathematics was to be taught in all classes. The arithmetic topics to be learned were as follows: 1st class, understanding the numbers; 2nd class, addition and subtraction; 3rd class, multiplication and division; 4th class, proportions and the Rule of Three; 5th class, varieties of proportions; 6th class, logistic; and 7th class, the holy and mystic numbers of the Bible (Grosse 1901). In formulating the general theory of education, Comenius is the forerunner of Pestalozzi.

In other European countries, young people were learning arithmetic as well, and books were published for this target group. Around 1630, John Wallis (1616–1703), later professor of geometry at Oxford, had not learned arithmetic at school or at Cambridge University where he studied, but from his younger brother studying to go into trade. Some years later, the intelligent and literate Samuel Pepys (1633–1703), also educated at Cambridge, struggled to learn his multiplication tables (Stedall 2012).

4 The Eighteenth Century

Enlightenment thinking in France emphasised rationalism as the dominant epistemology and elevated mathematics to the leading discipline, capable of promoting social and scientific progress. While rationalism at first remained restricted to Enlightenment in France, Enlightenment occasioned initiatives of the state for education which transcended for the first time the hitherto dominant focus on professional finalities in several European states, both Protestant and Catholic (Schubring 2012).

4.1 Education and Schooling

Primary or elementary schools were founded in the German territories at the turn of the seventeenth century, for example, in Prussia, where such an initiative was introduced in 1717. The intention was to have compulsory schooling for the entire population. Teaching subjects should be reading, writing, basic reckoning, and religion. There were, however, many practical obstacles for realising these noble aims of education of compulsory schooling for the entire population, and its effectiveness was limited.

Having initiated schools with a quantitative measure of their effectiveness, efforts would eventually be made to focus on improving the quality of instruction in these schools through better teachers, better material, better school equipment, and access for all, independent of children doing agrarian work for the family. In fact, by the middle of the eighteenth century, the so-called normal schools were founded in various states – for instance, in Austria – to train teachers for the elementary schools.

In the eighteenth and nineteenth centuries, economic development, no longer exclusively based on agrarian modes, led to an enormous increase in urban populations. Basic numeracy skills, such as the ability to tell the time, count money, and carry out simple arithmetic, became essential in this new urban lifestyle. Within the new public education systems, mathematics eventually became a central part of the curriculum from an early age. The *Gymnasia* in the Protestant regions of Northern Europe provided at first some basic arithmetic. What used to be professional, vocational training eventually transformed in some towns into kinds of elementary schools. During the eighteenth century, a German form of Enlightenment philosophy called for instruction in useful knowledge: Philanthropismus, which challenged the monopoly of classical learning in the *Gymnasia* and featured mathematics and the sciences. The first *Realschule* implementing these goals was founded in Berlin in 1747 (Schubring 2012).

The French Revolution in 1789 triggered the decisive step towards realising education for all, including arithmetic for all. The ultimate rationale of a civic society became the equality of all citizens. A key to such a system became the educational system, organised to provide equality via education for all citizens. Beyond primary schooling, states instituted and organised secondary schools as general education. Within these school systems, mathematics became a major teaching subject for all children who acquired access to these schools. In terms of social reality, equality was far from realised, particularly because of social obstacles to access to education (Schubring 2012).

4.2 The Number Concept and Didactical Concerns

Although Stevin transformed the number concept to include the unity as a number, many textbooks claimed the opposite in the eighteenth and early nineteenth centuries. The *Demonstrative Rechenkunst* by Christlieb von Clausberg (1689–1751), which contained a total of 1,544 pages, was published in Leipzig in three editions, in 1732, 1748, and 1762. The number is defined in a way that is typical of ancient understanding: “the unit or one for itself is only the nadir or the root of the numbers; that is an accepted magnitude from which the numbers grow and are observed…” (Clausberg 1732, p. 15).

The content of *Demonstrative Rechenkunst* is traditional, while it also contains outstanding didactical examples on mental arithmetic. It emphasises that calculation procedures are traditions that may be bended, not mathematical laws, for example, multiplying a number by units, tens, and hundreds, may be done in any order. In one example, the author explains how multiplying by 215 is related to multiplying by 152 and 251 (Clausberg 1732, p. 96). The work is divided into four volumes and the whole second volume is dedicated to what the author calls *die Vorteile*, the advantages, of each of the operations. For example, multiplying by 75 can be done more easily by multiplying by 100 and subtracting one fourth (p. 505).

Multiplying and dividing by repeated doubling and halving seems to have been a useful practice in mental arithmetic. The "scratch division" was difficult for laymen to comprehend. It was on the wane in the eighteenth century and Clausberg did not use it. He, however, recommended passing the problem of division when possible by halving, repeatedly if necessary. Dividing by 12 seems to have been considered more difficult than dividing successively by 2, 2, and 3. The work is in general a useful didactical handbook for those concerned with frequent mental calculations.

Leonhard Euler (1707–1783), the famous Swiss mathematician, wrote an elementary textbook in Arithmetic, *Einleitung zur Rechenkunst zum Gebrauch des Gymnasii bey der Kayserlichen Academie der Wissenschafften in St. Petersburg/Introduction to the Art of Reckoning, for use in the Gymnasium of the Imperial Academy of Sciences in St. Petersburg*, published in 1738. In this book, Euler used many of Suevus's historical problems (Bjarnadóttir 2011), but avoided discussion about the number concept (Euler 1738). In his influential *Vollständige Einleitung zur Algebra*, published in German in 1770 and translated into many languages, English among them, Euler defined a number as follows:

> Whatever is capable of increase or diminution, is called *magnitude*, or *quantity*…fix at pleasure upon any known magnitude of the same species with that which is to be determined, and consider that as the *measure* or *unit*…a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit. From this it appears that all magnitudes may be expressed by numbers. (Euler 1822, pp. 1–2)

Euler thus introduced a definition of a number that could include the unit and irrational numbers. Not only were the unit and number concept difficult to cope with but also the zero. In his algebra textbook, Euler introduced the zero as nothing and negative numbers as less than nothing, referring to debts, and illustrated by examples (Euler 1822, pp. 9–10) the necessity that the product of a positive and a negative number must be negative and the product of two negative numbers therefore must be positive.

Euler's contemporary, the German Professor A. G. Kästner (1719–1800), wrote in his *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspectiv,* a work of great importance, first published in 1758:

> This term 'less than nothing' presupposes…a meaning of the word Nothing which relates to a certain manner to regard the 'something' (*nihilum relativum*) and which discerns it from the Nothing, regarded without any relation (*nihilum absolutum*).... If one does not regard the term 'less than nothing' in this meaning it becomes wrong… and, in fact, mathematical experts have been seduced to erroneous conceptions of negative quantities…. (Kästner 1792, pp. 73–74)

This quotation relates to differentiation between the philosophical/metaphysical nothing and the mathematical relative zero, which d'Alembert had rejected but which the Germans have propagated (Schubring 2005). "Less than nothing" was a necessary term when introducing negative numbers which were rapidly entering arithmetic in the eighteenth century. The works of an Icelandic arithmetic textbook author, who studied Kästner's work but lived in mathematical solitude for most of the nineteenth century, reflect the difficulties (Gunnlaugsson 1865). He had doubts about the expression "less than nothing", which he used initially to refer to negative numbers, but he contradicted himself by writing that "0 must be less than the negative numbers" when he introduced division. His trouble was related to the discrepancy between introducing the zero within a set of numbers in the sequence … −3, −2, −1, 0, +1, +2, +3, …, on one hand, and regarding the zero as no quantity when it came to division by zero, on the other hand. His conclusion was that the zero, 0, lies "on the inner limit of quantity", while infinity, ∞, is the "outer limit of quantity". This dilemma was first solved in the 1880s when the construction of natural numbers was realised in the works of Gottlob Frege (1846–1925) and Giuseppe Peano (1858–1932).

Decimal fractions did not occur in either of Euler's two elementary textbooks, the *Rechenkunst* and the influential *Einleitung zur Algebra*. However, decimal fractions were treated in *Cocker's Decimal Arithmetick* (Cocker 1703), first published in 1684. There is evidence that decimal fractions were taught by a professor named Geuss to novices at the University of Copenhagen in the early 1780s

(Lbs. 408, 8vo). From there, decimal fractions appeared in a textbook on arithmetic and algebra for use in the Icelandic Latin schools by 1785; it was written by Stefánsson (1785), the main champion of the Enlightenment in Iceland.

4.3 The Rule of Three

The Rule of Three, *Regula Trium*, was a simple proportion, involving three known quantities from which a fourth must be found. Today, such a problem would be considered trivial, but before the advent of mathematical symbolism when problems were presented rhetorically, the solving of such problems posed considerable conceptual difficulties. The Rule of Three as a mathematical technique can be traced back to problems in *Rhind papyrus* (c. 1650 B.C.) and the Chinese mathematical classic *Jiŭzhāng Suànshù, The Nine Chapters* (around the first century CE) (Tropfke 1980).

The rule was usually presented without a mathematical rationale. Theoretical justifications were of less interest than the quickness and accuracy of the results obtained, so it was merely a procedure to be memorised and used. Authors of commercial arithmetic books took extensive care to establish the manner in which the proportion was set up and solved (Swetz 1987).

In standard textbooks up through the nineteenth century, the three known terms were labelled. In *Cocker's Arithmetick*, the explanation of the Rule of Three goes as follows:

> Again, observe, that of the three given numbers, those two that are of the same kind, one of them must be the first, and the other the third, and that which is of the same kind with the number sought, must be the second number in the rule of three; and that you may know which of the said numbers to make your first, and which your third, know this, that to one of those two numbers there is always affixed a demand, and that number upon which the demand lieth must always be reckoned the third number. (Cocker 1703, p. 103)

In the eighteenth- and nineteenth-century textbooks, the procedure was more organised. The three known terms were, for example, labelled front term, middle term, and rear term, and the procedure could run as follows:

(i) Write the three terms in horizontal order with bars between them.
(ii) Front term and rear term must be of the same kind.
(iii) The rear term must contain a question.
(iv) Multiply the middle term by the rear term and divide by the front term to bring the fourth term.
(v) Front term and rear term may be multiplied or divided by the same number; this may also be done to front term and middle term.

The rules were differently posed in the various textbooks. For example, Rule v, on simplifying the computations by cancelling out common factors, was found in Briem (1880) but not in either Clausberg's (1732) *Demonstrative Rechenkunst* or in Stefánsson's book (1785).

Naturally, the correct order of terms was crucial. The main risk was related to which of tasks for which the rule was fit. For trade, the typical tasks were to convert between the various currencies and to compute proportional prices against quantity and vice versa, or labour and provisions against time. Another type of the Rule of Three was the inverted rule, *Regula inversum*, which was to be used for an amount of labour, provisions, etc., suitable for a certain number of persons for some length of time, to be adapted to a different number. Those two types could be easily confused. Another risk was that the method would be applied to nonproportional phenomena.

Some textbooks of the twentieth century, for example in Iceland, would emphasise that the two first terms made a conditional sentence, and then a unit sentence was inserted where the students were to decide whether to divide or multiply, that is, use the direct or inverted Rule of Three. The procedure continued by a question sentence with the third term. Hatami (2007) has written a comprehensive overview of the different types of solving problems involving the technique of the Rule of Three.

The Rule of Three was also termed *Regula Aurum*, the Golden Rule, for its superiority, much like gold is superior to other metals; some would term it *Regula Philosophorum & Mercatorum*, the rule of philosophers and merchants (Stefánsson 1785). These terms reflect its high reputation. The rule survived well into the twentieth century, but was practically eradicated by the school mathematics reforms of the 1960s. Its target group had shifted from merchants and farmers and their servants and clients to school youth preparing tasks in life that were significantly different from those of youngsters in earlier societies that were devoid of schools for the general public.

5 The Nineteenth Century

The "common school movement" in the United States refers to the establishment of state elementary school systems in the first half of the nineteenth century. The term *common* meant that these state-supported public elementary schools, exalted as the school that "educated the children of all the people", were open to children of all socioeconomic classes and ethnic and racial groups. Nevertheless, many children, particularly enslaved African Americans, did not attend. Not a selective academic institution, the common school sought to develop the literacy and numeracy needed in everyday life and work. Its basic curriculum stressed reading, writing, spelling, arithmetic, history, and geography. It was regarded as the educational agency that would assimilate and Americanise the children of immigrants.

The common school movement in the United States paralleled some trends taking place in Western Europe in the first half of the nineteenth century. In the 1830s, the British parliament, though not creating a state-school system, began to provide grants to educational societies for primary schooling. In France, under Minister Guizot (1787–1874), a primary school system was also established during the regime of Louis Philippe I. These transnational trends, found in Europe and America, indicated that governments were beginning to take responsibility for providing some kind of elementary schooling. Unlike in France, which had started to create a highly centralised national educational system, US public schools were decentralised. Educational decision-making was reserved to each state (*Educational Encyclopedia,* Elementary Education n.d.a).

German primary schools were already established in the eighteenth century and in the following period, with educators discussing didactical questions there. Early nineteenth-century German-speaking educators, the successors of Comenius, such as Johann Pestalozzi, proposed teaching methods based on concrete experience. Progress in implementing education for all, including arithmetic for all, and the eventual propagation of the influences of these educators will be exemplified in Central Europe (Germany, France, England), Northern Europe (Denmark, Iceland, Estonia), and North America.

5.1 Germany and Pestalozzi

Johann Heinrich Pestalozzi (1746–1827) was a Swiss pedagogue who made a profound impact on educational theories in the continent of Europe as well as in the United States. As a young man, he was influenced by the philosopher Rousseau (1712–1788) and his story *Emile*. In his earlier age, Pestalozzi wrote books on his ideas on society, political philosophy, and education. He had already passed the age of 50 when he began to run schools for the poor, where he developed his pedagogical method of educating children, in which all understanding can be achieved though a pedagogically ordered sequence. He had to move his institute from one location to another to be able to realise his aims. His writings and reports of his work attracted to his institute a great number of visitors who introduced the Pestalozzian method far and wide. The visitor came, for example, from Bremen and Prussia, such as Berlin and provinces of Poland. The government of Denmark was the first to officially send students to Pestalozzi's institute to be trained as teachers. Finally, Pestalozzi could establish his institute at Yverdon in Switzerland during 1804–1825 (Silber 1976).

In Yverdon, mathematics gained particular importance. In number teaching, Pestalozzi set out to find ways of helping children to understand number and not merely to develop speed and accuracy in the mechanical working of examples. Children were to discover for themselves the mathematical rules through activities based on sense-impressions. They were encouraged to work in groups and instruct each other. Pestalozzi's educational aims were based on his idea of *Anschauung*, which concerns the mind's reception of a sense-impression, observation, perception, or intuition – the development of mental faculties. The children were not given the products of learning but were guided to find them for themselves. They were taught to use their own eyes, hands, and minds. Exercises in arithmetic were in the first instance related to circumstances in their environment and objects, such as beans and pebbles or whatever was at hand. The principle was the use of units, at first solid and later as marks on paper (Silber 1976).

The counting of real objects, the grouping, the adding, and the subtracting were the essential basic activities of early number work, so that the primitive constitution of numbers should be impressed upon the mind without being complicated and confused by written symbols. Pupils would gain an intuitive knowledge of the real properties and proportions of numbers, a knowing in the mind, independent of sense-impression or reasoning.

To facilitate progress to division, multiplication, and understanding fractions, Pestalozzi devised his Table of Units in which the unit adopted was the square, a figure which lends itself to simple visual subdivision and partition. Through activities with these divisible squares, pupils were to gain an intuitive knowledge of the proportions of the different fractions and could proceed to their symbolic representation with clear ideas of their true significance. Pestalozzi said that "if my life had any value, it consists in the fact that I raised the square to the fundament of an *Anschauung*, which people had never done"[1] (Jänicke 1888, p. 68).

Pestalozzi himself did not teach mathematics, but his ideas generated theories in mathematics education which were put forward by other educators. The mathematics teaching at Yverdon was in the hands of Joseph Schmid, Pestalozzi's former pupil, who had a genius for mathematics. Pestalozzi's opponents did not agree with his methods, but his followers who included distinguished educators propagated his ideas. Each writer composed his own version of the pedagogical theories which continued to develop among educators during the nineteenth century and beyond. Among the writers who developed Pestalozzi's theories were Ernst Tillich (1780–1867) in his *Allgemeines Lehrbuch der Arithmetik oder Anleitung zur Rechenkunst für Jedermann* of 1806 and Joseph Schmid who published *Die Elemente der Zahl als Fundament der Algebra nach Pestalozzis Grundsätzen* in 1810 (Jänicke 1888; Silber 1976).

By expanding the frame of Pestalozzi's reformatory *Anschauungslehre,* Tillich managed both to make this model more complete on both pedagogical and theoretical levels and to adapt it to already existing mathematics education. Although Tillich based the most coherent system on Pestalozzi, he was not the only one to improve upon it. Schmid independently came up with the same improvements but with other names attached to them. After Tillich's and Schmid's improvements on Pestalozzi's method, the full combination of their contributions, were absorbed into the more general and state-organised reforms of elementary schools from 1820 onwards. Wilhelm von Türk (1774–1846) in Potsdam and Adolph Diesterweg (1790–1866) in Bonn and Berlin developed their specific brand of Pestalozzian reckoning. In their versions of school reckoning, the *Rechenbücher* (reckoning books) returned, after a vigorous introduction into mental arithmetic along the lines of Pestalozzi (Bullynck 2008).

[1] Wenn mein Leben einen Wert hat, so besteht er darin, dass ich das Quadrat zum Fundamente einer Anschauung erhob, die das Volk nie hatte.

After 1800, reckoning as the art of manipulating ciphers in certain ways – as it was learned in the old *Rechenbücher* – was a synonym for turning the pupils into parrots or lifeless calculating machines. The abstractions of Pestalozzi and Tillich had turned "reckoning" into "arithmetic", a more abstract thing, useful as a general, if not propaedeutic, discipline in education. Tillich went the farthest in this by identifying counting with thinking, but textbooks after 1820 also stressed that arithmetic had a general use for learning to think. All introductions stressed this propaedeutic role of arithmetic over its use in everyday life (Bullynck 2008).

Although pedagogical reforms around 1800 fell under the title of *Anschauung* and understanding, external factors to replace mechanical and memory-based education, such as social demands for reckoning in schools and the problem of mass education, largely determined the form and content of educational methods. Pestalozzi was the first educator to develop methods suitable for mass education, which involved an oralisation of the content. As a consequence, the "mechanical" aspect of arithmetic, or writing ciphers, disappeared and was replaced by mental arithmetic. Also, the Rule of Three and fractions, belonging to the more advanced core content of *Rechenbücher*, were rephrased completely through relationships between numbers, with the solution methods to these problems recurring over the unit of one (Bullynck 2008).

5.2 France

The idea of national government support for popular education and teacher training first became an important social and political issue during the French Revolution of 1789. Important plans were proposed during the turmoil that followed. Condorcet (1743–1794) played a decisive role in transforming the new conceptions into the educational structures after the Revolution (Quartararo 1995; Schubring 2012).

During Napoleon's Consulate and Empire, education was not a priority and the church's control over popular education that had been characteristic of the prerevolutionary era was gradually regained. Under the July Monarchy, however, the 1833 law of Guizot focused on curricula and the duties of local officials in primary instruction. The basic course of study in the public primary schools would include moral and religious instruction, reading, writing, and arithmetic. Each commune was required to support a public elementary school for boys. In order to staff these schools, each department would establish and run its own normal school. Guizot was particularly interested in teacher-training schools for primary instruction. Their syllabi included the same topics as the primary schools. There was no comparative law for girls' public primary schooling, and the church provided their popular education for a while (Quartararo 1995). Renaud d'Enfert (2003) has written an extensive account of mathematics education in primary schools in France during 1791–1914.

5.3 England

In England, there was only little state interference in education throughout the nineteenth century, and mathematics was considered a marginal teaching subject in secondary schools (Schubring 2012). From the 1830s onwards, elementary education increased, owing to partial financial support of the government. The parliament passed the 1870 Elementary Education Act (The Forster Education Act), implementing compulsory elementary education for all children in Britain. Elementary schools provided a curriculum that emphasised reading, writing, and arithmetic (Griffith and Howson 1974).

Along with the popular schools, private schools flourished, sometimes as family enterprises. Jacqueline Stedall recounts learning at a school, named Greenrow Academy, founded in 1780. Records from 1809 reveal that the pupils could be from below 10 years old up to age 23, although most were about 14 or 15. The curriculum stressed mathematical studies. Not only does the curriculum of the school exist but pupils' copybooks have also been preserved. Pupils carefully inscribed copied examples of standard problems, thus creating for themselves a collection of worked examples. Many of them were taken from popular textbooks of the time, in particular from *The Tutor's Assistant* by Francis Walkingame (1723–1783), first published in 1751 (Stedall 2012). They provide insight into the pupils' tasks. The Rule of Three was taught by rote; a nineteenth-century English schoolboy was not expected to start working on his own initiative. In 1832, a particular pupil had to learn the Rule of Three Direct, the Rule of Three Inverse, and the Double Rule of Three. These topics were followed by, among others, barter, interest, the Rule of Fellowship (the sharing of costs and profits), vulgar fractions, decimal fractions, arithmetic and geometric progressions, and some examples on permutations. Many examples were taken from Walkingame. The pupil's two arithmetic books contained almost 900 pages combined. Girls and women were more commonly taught at home by their fathers, husbands, or brothers. Also their copybooks reveal Walkingame's examples on compound addition, the Rule of Three, reduction, and fractions of monetary and measuring units, up to compound interests. Girls' copybooks do suggest that mathematical education for girls had strong practical emphasis and further, by modern standards, the pace was sometimes slow and repetitive (Stedall 2012).

5.4 Denmark, Iceland, and Estonia in Northern Europe

In Denmark, arithmetic teaching to children began with a 1739 ordinance by which around one-tenth of the population received instruction. A primary school ordinance was issued in 1814 whereby arithmetic became a compulsory subject for all children. The number of lessons per week was not defined. The arithmetic teaching was based on Pestalozzi's theories on concrete experience, while a couple of school memoirs reveal doubts that the reform ideas and concrete materials ever reached the children (Hansen 2002).

Iceland, a colony of Denmark until the twentieth century, only saw legislation on education in 1880, prescribing the four operations in whole numbers and decimal fractions as the responsibility of the families in home education under the supervision of parish ministers. The first two lower secondary schools for the general public were established in the early 1880s. Decimal fractions were emphasised as decimal monetary units were being implemented concurrently (Bjarnadóttir 2006). Textbooks in the vernacular that were published from the mid-nineteenth century contained traditional topics about whole numbers and fractions in addition to decimal fractions, percentages, interests, and the Rule of Three. Numerous examples offered advice on the cautious allocation of income and properties. The textbooks were primarily aimed at self-instructing youth, prospective farmers, and craftsmen who may not have had free access to paper; thus, emphasis on memorising prevailed. In his foreword to his *Arithmetic* intended for young adults and use in upcoming schools, the author wrote:

> …for the chapter about algebra, equations and logarithmic calculations I have, however, expected that people had some instruction; in that chapter I have, as elsewhere, avoided supporting the rules prescribed by reasoning; when I have made exceptions in several places, it is because the reasoning could as well be an exercise or it was so clear that it could be used to support the memorizing of the rule. (Briem 1880, p. iii)

The few students who attended Latin school studied Danish textbooks. In his memoirs, lawyer Ari Arnalds (1872–1957) described his quest for education in the last decades of the nineteenth century in Icelandic rural society (1949). His parents, a farmer and a midwife, could only afford a private teacher, a Latin school graduate, during one winter term for their nine children. The children read and wrote in all their spare time from their farm tasks. While they were knitting or weaving in winter or watching sheep in summer, they recited verses. Arnalds went through Briem's (1880) *Arithmetic* without any external instruction. Because the parish minister knew Arnalds had been working hard on arithmetic, his problem for confirmation was to compute the sum to be expected if a penny had been carrying 4 % interest from the year when Christ was born. Arnalds only knew the method to find the length of the period needed to double the sum, which in this case were 17 years and several months. From this he calculated up to the year 1886, arriving at a figure with 30 digits – this took him more than an hour.

Around 1800, as a reflection of the Enlightenment movement, schools began to be organised in Estonia. In the first decades of the century, folk education remained underdeveloped and home schooling dominated, while it was demanded that all peasants' children, who were not taught at home, attend school. Children had to study two or three winters; they were taught reading, writing, arithmetic, and choral singing. They had to memorise multiplication tables, master the four basic arithmetic operations, and learn measurements and weights. During the 1860s, a general 2- or 3-year compulsory education for the common people in elementary schools was adopted. No instructional literature in Estonian existed until 1850, while a number of textbooks written in the Estonian language were issued during the second half of the nineteenth century. The most influential arithmetic book of this period, *Mõistlik rehkendaja/A Sensible Reckoner*, was compiled by Rudolf Gottfried Kallas (1851–1913) in 1874, written in Estonian mainly for primary school teachers. The first part of the book is devoted to general problems of arithmetic teaching, referring to the works of Comenius and Pestalozzi. With his textbook, Kallas altered traditional understandings of arithmetic teaching and set new standards for many years to come (Andresen 1985; Kruze et al. 2009; Prinits 1992).

5.5 The United States

In the early 1800s, many school-age children in the United States rarely attended school. Of those who did attend, many, especially boys in the New England colonies or states, attended during the winter months only and did not study any mathematics beyond elementary arithmetic (Clements and Ellerton 2012). The mathematical textbooks of the eighteenth century were published for several reasons but not for the pupils. Pupils rarely had a textbook. They copied in their own notebooks rules for mathematics and exercises with which to practise the rules. The textbook was mainly for the teachers or for self-taught individuals. The mathematical topics of arithmetic texts in the first decades of the nineteenth century were generally concerned with proficiency in whole number and fractional operations. Most early texts had little or no coverage of decimals. This may seem surprising since US currency has always been based on the decimal system. However, until almost mid-century, business in the United States was conducted in three primary ways: barter, foreign coinage, and local currencies (Michalowicz and Howard 2003).

Arithmetic was regarded as a vocational subject in the early 1800s, a skill whose chief application was for the world of commerce. The appropriate pupil for such study was the 12–14-year-old boy, judged to be mature enough to absorb the techniques of computation as well as sufficiently competent in writing a permanent copybook. Topics beyond arithmetic operations were in line with what van Egmond (1980) described above for the abbacus schools: Rule of Three, fellowship, tare and tret, stock and brokerage, insurance, and compound interest (Cohen 2003).

In 1821, Warren Colburn (1793–1833) published an innovative arithmetic textbook, *First Lessons in Arithmetic on the Plan of Pestalozzi*, based on Pestalozzi's pedagogy, that gave students the opportunity to discover rules by induction from examples (Kilpatrick 1992). Earlier, arithmetic teaching had been rule and memory based, based on the British *Cocker's Arithmetic*, first published in 1677, and a large collection of textbooks which tried to simplify arithmetic and adapt it to the monetary units of current times. Two related but distinct pedagogical techniques were embedded in Colburn's method. First, Colburn wanted children to learn arithmetic as a mental exercise, done in the mind without pencil and paper, before they learned abstract symbols for numbers and operations. The problems were illustrated by plates showing concrete objects, allowing the children to count out the right answer. The second pedagogical technique involved inductive reasoning. Students were to discover the basic rules of arithmetic for themselves by working out carefully chosen examples. The Rule of Three was entirely omitted from Colburn's books. Colburn wanted to end children's slavish reliance on rules and rote learning (Cohen 2003).

Colburn's texts were an instant sensation among educators in the 1820s. Inevitably, a backlash set in against his inductive method. Merchants complained that students came to work with them with only a chaotic jumble of ideas about numbers and no practical knowledge. However, Colburn's methods built on a seedbed of dissatisfaction with the old methods and electrified educators with the startling notion that children could learn arithmetic basics even before they could read and write. The vast diffusion of numerical skills in the United States from the 1820s to 1900 owed much to his influence. Numeracy spread and flourished under the democratic political revolution and developing capitalism. Successive generations of educators carried on the struggle to find the right mix of approaches to mathematics education, and the debate which entered into the early nineteenth century continues to resonate in educational theory and practice in the twenty-first century (Cohen 2003).

Although the rule method of the eighteenth century continued to have its champions throughout the nineteenth century, the inductive and analytic approaches promoted by Rousseau and Pestalozzi on the European continent and Colburn in North America were clearly popular with teachers and found wide use in their classrooms. Using these methods, nineteenth-century teachers promoted what in the late twentieth century was referred to as mental mathematics, logical reasoning, and number sense (Michalowicz and Howard 2003).

5.6 Rational Arithmetic and the Number Concept

In the nineteenth century, efforts were made to put arithmetic on a firm foundation. In Italy, rational arithmetic was introduced for 14–15-year-old pupils in upper secondary classical instruction in 1867. Rational arithmetic refers to the part of algebra that deals mainly with the properties of integers and rational numbers exposed with theorems derived from axioms and definitions, parallel with the rational teaching of geometry. The aim was to show that all of mathematics, not only geometry, is a deductive science (Menghini 2012).

In his book *Die Grundlagen der Arithmetik*/*The Foundations of Arithmetic*, first published in 1884, Gottlob Frege (1848–1925) wrote that some writers defined "number" as a set or multitude or plurality, but these views suffered from the drawback that the concept did not cover 0 and 1 (Frege 1953, p. 28). The Italian Giuseppe Peano (1858–1932) had similar aim to give the number concept a sound basis. Peano chose three primitive concepts: zero, number (i.e. non-negative whole number), and the relationship "is the successor of", satisfying five postulates:

1. Zero is a number.
2. If *a* is a number, the successor of *a* is a number.
3. Zero is not the successor of a number.
4. Two numbers of which the successors are equal are themselves equal.
5. If a set S of numbers contains zero and also the successor of every number in S, then every number is in S.

In the Peano axioms, the postulational method attained a new height of precision, with no ambiguity of meaning and no concealed assumptions (Boyer and Merzbach 1991, pp. 597–598).

6 The Twentieth Century

6.1 The Period Until the 1950s

The early twentieth century was characterised by differing opinions on arithmetic instruction, now that primary school had become the norm in most countries in Europe and North America. At the early secondary school level, a dual justification for the teaching of mathematics prevailed in the United States. On one hand, the goal was to develop the intellect in general and the reasoning faculty in particular. On the other hand, there were utilitarian and business-oriented reasons (Kliebard and Franklin 2003).

This was also the case elsewhere. In Denmark, a kind of wave model in shifts of emphasis in arithmetic teaching existed between mechanical skills and understanding. This continued from the 1880s when skills were in focus, to new school legislation in 1903 which shifted the focus to understanding, to the mid-1920s with its return to a Back-to-Basics movement, to 1958 when school legislation returned again to less formal training and more functional development. The Back-to-Basics movement was a reaction to the emphasis on understanding based on experimental psychology that the transfer of training – which had been the guideline at the beginning of the century – did not seem built on solid evidence, and testing showed that the situation was not satisfactory in reckoning classes (Hansen 2002).

The topic of transfer of training occupies a unique position in the relationship between psychologists and mathematics educators. It has played a vital role in arguments for the place of mathematics in the curriculum. In the United States, Edward L. Thorndike (1874–1949) termed his psychology connectionism. It was one of the forerunners of the behaviourism that dominated American psychology until 1930. Thorndike said that transfer of training cannot be assumed to occur, that it is rarely automatic, and that direct teaching is usually more efficient and economical (Kilpatrick 1992).

The father of progressive education, John Dewey (1859–1952), claimed that children learned more and at quicker rates when teachers encouraged their natural curiosity instead of subjecting them to the rigid discipline and corporal punishment of traditional nineteenth-century classrooms. Dewey and his fellow educational progressives drew from the works of Pestalozzi and others. Dewey used games and various forms of play as vehicles for teaching.

Introduced in 1919 in the Chicago suburb of that name, the Winnetka Plan emerged as a result of John Dewey's work, inspiring teachers to attempt innovative pedagogies in their classrooms. The Winnetka Plan experimented in individualised ungraded learning in order for students to progress at their own rate of learning. The curriculum was set up in two components: "common essentials" and "creative group activities". The first component concentrated on common knowledge and mastery skills, such as spelling, reading, writing, and counting. Quality, rather than time, was emphasised. According to this plan, a child must master material at 100 % to progress to the next level. No student ever "failed" or "skipped a grade" in the common essentials (Schugurensky n.d.).

The children learned in informal settings. Progressive schools enlisted the spontaneous interests of the pupils and adapted the curriculum to each child's interests and needs. By the 1940s, progressive ideology and rhetoric (but not necessarily progressive practices) had become conventional in American classrooms. In the cold war atmosphere of the 1950s, however, educational progressivism came under serious attack. Progressive curricula were held responsible for the lag in preparing for scientific and technological careers, culminating in the Sputnik crisis of 1957 (*Educational Encyclopedia,* Progressive education n.d.b).

The basic task of public education in the United States in the late 1950s and early 1960s shifted from providing basic education to all children to creating a technocratic elite to make the United States competitive with the Soviet Union. Hard thinking and learning to deal with abstractions became skills that were considered critical for the survival of society (Becker and Perl 2003). Through the efforts of the Organisation for European Economic Co-operation (OEEC), later the Organisation for Economic Co-operation and Development (OECD), this became a characteristic attitude to mathematics education in the Western world for the following period.

6.2 CIEAEM and Modern Mathematics

In the 1950s, questions arose in many countries about mathematics teaching. An international reform movement in mathematics education had several origins, both in the United States and Europe. New arenas opened with the aim of finding new approaches to mathematics education suitable to the changed mathematical and social context. An outstanding arena was the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM). Among its founding members was the Swiss psychologist Jean Piaget (1896–1980) as well as mathematicians Caleb Gattegno (1911–1988) from Italy, Gustave Choquet (1915–2006) and Jean Dieudonné (1906–1992) from France, and Hans Freudenthal (1905–1990) from the Netherlands. Furthermore, there were secondary school teachers Emma Castelnuovo (1913–) from Italy, Lucienne Félix (1901–1994) from France, and Willy Servais (1913–1979) from Belgium. The main concern of the CIEAEM was a growing attention to students and the process of teaching.

The Bourbaki group consisted of French mathematicians, led by Dieudonné, who worked at a mathematical encyclopaedia, wherein the borders between different mathematical topics were abolished. The Bourbaki group's central concept was "structure". The mathematical structures were called *Mathématique Moderne* (Modern Mathematics). When describing these structures, the importance lay in the elements' relationships, determined by axioms. The enthusiasm and activity of the CIEAEM generated new issues that had a great impact, particularly in Europe (Furinghetti et al. 2008; Furinghetti 2008):

- The relevance of psychology in mathematics education
- The attention to teaching methodology
- The key role of concrete materials and active pedagogy
- The relevance of studies carried out in practice, even with the teacher's participation
- The need to take all school levels (from primary to university) into consideration
- Empirical research
- The relation between mental and mathematical structures and Modern Mathematics
- The emergence of the figure of the researcher in mathematics education
- The democratisation of mathematics

Important books focusing on these new issues were produced. In the first one, published in 1955 and written by Piaget, Dieudonné, Gattegno, Choquet, Beth, and Lichnerowitz, *L'enseignement des mathématiques*, all authors agreed on the opportunities that Modern Mathematics offered to mathematics teaching. Piaget's contribution dealt with the foundations of mathematical theories, and he looked for links between mathematical structures, as introduced by the Bourbaki group, and the structures of the mind. Dieudonné claimed that the essence of mathematics was reasoning on abstract notions. Even if an intellectual effort was needed to reach the abstraction of algebraic structures, Dieudonné did not consider that the solution to the problems of mathematics teaching was to delay abstraction: it was necessary to reveal to adolescents the very aspect of mathematics following the formation of their mental structures. Gattegno explicitly mentioned concrete teaching materials, such as Cuisinaire rods, geo-planes, and films. The books represent two important innovations supported by the CIEAEM: Modern Mathematics and concrete materials. While the first innovation was mainly confined to a period of less than 20 years, at least in its radical interpretations, the second topic is still current and comprises educational ideas that remain under discussion (Furinghetti et al. 2008).

Already in 1960, Freudenthal expressed doubts about Piaget's research, first because Piaget's mathematical background was rather weak, but mainly because Piaget's approach hardly reflected the teaching situation in the classroom; it better fits the rather unusual laboratory situation of the psychologist (Furinghetti et al. 2008).

6.3 *Emergence of New Math*

The actions of the CIEAEM, containing important germs of didactic research, were paralleled by the New Math movement in the United States. Induction testing for the US armed forces presented evidence that many young people were incompetent in mathematics. Furthermore, World War II had focused national attention on the growing need for trained personnel to serve an emerging technological society (Osborne and Crosswhite 1970). These circumstances drew attention to school mathematics. During the 1950s, several important reform projects were launched. At the time of the Sputnik Shock in 1957, nearly fully developed reform programmes already existed to respond to the national call for improvement in mathematics and physics education (Gjone 1983, vol. 1). Edward Begle (1914–1978) at Yale University directed the most extensive project, School Mathematics Study Group (SMSG), which created and implemented a primary and secondary school curriculum between 1958 and 1977. Initially it was aimed at college-bound students but was later adapted for all students. It had been translated into 15 languages before its production was terminated (Briscoe Center for American History n.d.; Gjone 1983, vol. I).

An important conference was held in Woods Hole, Massachusetts, in September 1959, where university professors in mathematics and natural sciences met professors of psychology and pedagogy for the first time to discuss the basis for further development of reform projects in mathematics and natural sciences (Bruner 1960). This was also the first time that the reform movement in the United States came into contact with European reform movements. The European contact was Bärbel Inhelder (1913–1997), a close collaborator of Jean Piaget in Geneva (Gjone 1983, vol. II). Among the mathematicians present were Marshall Stone (1903–1989), the president of the International Commission on Mathematics Instruction (ICMI), and the leaders of the largest projects, such as Begle, in addition to psychologist Jerome Bruner (1915–) at Harvard University, who was the leader of the conference. Bruner wrote a report of the conference, *The Process of Education*, where he presented his well-known ideas on a spiral curriculum and the hypothesis: "Any subject can be taught efficiently in some intellectually honest form to any child at any stage of development" (Bruner 1960, p. 33). The CIEAEM's activity and the New Math movement in the United States shared common roots with the Bourbaki group: set theory, functions, relations, and logic should find their places in the new curricula, supported by the methodology of discovery.

Interestingly, some of the reforms incorporated elements of progressivism. For example, the work of Bruner was based on the developmental psychology of Jean Piaget and incorporated many of Dewey's ideas of experiential education. Dewey and Bruner both viewed experience as essential to the learning process. Bruner's psychological approach has deep connections to the work of Dewey and led to a resurgence of his ideas in the second half of the century.

6.4 *The Royaumont Seminar*

From 1959 the reform started to expand – as psychologists and pedagogues became more interested in mathematics and natural science teaching – to new pupil groups and new grades. OEEC experts found that reform was necessary within the member countries to meet the demands and new techniques from industry. An important seminar on mathematics teaching was held in November 1959 in Royaumont in France, arranged by the OEEC. Each member country and the United States, Canada, and Yugoslavia were invited to send three delegates: an outstanding mathematician, a mathematics educator or a person in charge of mathematics in the Ministry of Education, and an outstanding secondary school teacher of mathematics. The seminar was attended by all the invited countries, except Portugal, Spain, and Iceland (OEEC 1961). Rarely noticed, this famous meeting at Royaumont took place almost at the same time as the Woods Hole Conference.

The Royaumont meeting can be seen as the beginning of a common reform movement to modernise school mathematics in the world. Marshall Stone was the president of the meeting. Among its thirteen plenary speakers were Dieudonné, Choquet, Servais, and Felix from the CIEAEM; Begle, the director of the SMSG project in the United States; and Svend Bundgaard (1912–1984) from Denmark. Dieudonné's contribution makes clear that the conference was run under the influence of the Bourbaki group (Furinghetti et al. 2008), and contributions by Stone and Begle demonstrate the link to the Woods Hole Conference.

In the seminar's conclusions, it was said that arithmetic – or rather computations – was traditionally considered a tool needed in life and business affairs (OEEC 1961). Therefore, most of the teaching of this subject had been a mechanical rote learning of facts and algorithms. The psychological implications of learning procedures used in primary schools, and the shift of school aims to developing concepts and modes of thinking (as well as skills), were conceived to necessitate a corresponding change in arithmetic instruction. The learning should be the result of understandings arising from guided experimentation and discovery, with the use of physical objects of one sort or another. In this way, the child must be led to the abstraction of the *quality of a set* called its *number*. In getting to this abstraction, it was considered necessary to use the ideas – but not necessarily the language – of sets, subsets, correspondence, and order. The concepts must be correctly developed right from the start.

The understanding and use of a decimal place system of numeration were considered necessary components of early instruction. With this place system – and the intuitive use of the laws of commutativity, associativity, and distributivity – all operations on whole numbers, common fractions, and fractions in decimal notation could be developed reasonably rather than seem like a set of magical tricks. In this development, the use of models at the outset was considered essential, while care was needed not to replace arithmetic with the model.

Children were to learn to calculate with reasonable speed and accuracy, as was demanded in everyday adult life. Beginning in the fifth school year and for the next few years, brighter children could be introduced to the study of number relations, involving odd and even numbers, primes, factorisation, greatest common factor, least common multiple, and place numeration systems to bases other than 10. Such study certainly needed to be under the supervision of a teacher who understood all of the implied mathematical relations and the relation of the material to the subsequent study of mathematics. Generalisations of arithmetical relations through the use of literal symbols could serve as an informal introduction to algebra.

There were areas of disagreements, such as introducing negative numbers at an early year as an adjunct to whole numbers and fractions and using symbols such as $8+1$ and $7+2$ as another name for 9 rather than as an operation. There was also agreement that mathematics should be taught in a way that exhibited its structure, stressing the use of the usual laws and placing more stress on the role of the numbers 0 and 1.

In a summary of the report of the meeting, a *change in purpose* was emphasised – namely, the building of mathematical concepts, structure, and understanding as paramount to manipulative skills; *change in the use of ideas*; and a *different organisation and treatment of the several branches* (based on psychological knowledge of mental growth) in order to bolster understanding and provide a more common basis for the continued study and application of mathematics.

In a justification for the planned reform, it was argued that changes in the cultural, industrial, and economic patterns of many nations called for a basic change in educational patterns. More people must be better trained in scientific knowledge. Even laymen must come to understand science; today, knowing mathematics is basic to understanding science.

Each country could reform its mathematics teaching according to its own needs, but it was recommended to establish as much cooperation as possible. Each country would have its own unique way of making the reform – of introducing new material, of organising the sequential study, and of experimenting with possible programmes. Channels should be provided for communicating the results of these programmes and experiments between all countries in order to draw on the best thinking available in these countries that could stimulate new ideas (OEEC 1961).

6.5 *Nordic Cooperation*

The Nordic participants at Royaumont agreed on organising Nordic cooperation for the reform of mathematics teaching (Gjone 1983, vol. II). Primary teacher Agnete Bundgaard was the leader of the primary school project. Bundgaard and her collaborator wrote a series of textbooks for age 7–13. Jens Høyrup (1979) deemed that material as a most orthodox adjustment to the mathematicians' demands.

The content of the series was highly theoretical. Numbers were introduced as the quality of sets. The commutative, associative, and distributive laws; even and odd numbers; the zero in multiplication; pairing numbers by a given function and finding a function; Roman numerals; and place value notation to the base five were all introduced before the close of the third grade. The same applied to prime numbers, permutation of three digits, and the transverse sum and its relation to the nine times table. When preparing to borrow in subtraction, the idea of an easy plus-name was introduced: for example, $10+3$ was an easy plus-name for $6+7$. Multiplication names were also introduced, for example, $2 \cdot 5+3$ was a multi-plus-name for 13, in preparing for division. In the fourth year, notation for set theory was introduced, with pairing, subsets, intersection, and union, in addition to various bases to the place value system and prime factoring. Multiplication modulo nine was introduced in grade 5. Decimal fractions were presented before common fractions, which were delayed to grade 6. Subtraction was presented as a search for a missing addend and division as a search for a missing factor (Bundgaard 1969–1972).

The introduction of this material may be considered an example of what Howson (1995) termed "attempting to update pedagogy within a centrally prescribed curriculum", at least in Iceland where it reached the majority of children in several cohorts around 1970 without revision of the national curriculum document of 1960.

6.6 *Backlashes*

The enthusiasm for the New Math declined in the early 1970s. A longitudinal study was made in the United States on more than 110,000 students in more than 1,500 schools, starting in 1961. The study showed that students who studied the New Math demonstrated fewer computing skills than those who had received traditional instruction (Gjone 1983, vol. V).

In reality, the changes were more often in content than in pedagogy. Parents' reactions to the New Math were similar in many countries: they did not understand the procedures and algorithms their children applied and many saw only confusion. In numerous cases, the actual implementation caused disappointments, negative reactions, criticism, and various changes in conditions, such as beliefs concerning the economic efficacy of education, which promoted further redefinition. This process, however, led to a permanent redefinition of school mathematics (Cooper 1985). Despite these difficulties and disappointments, Fey (1978) noted that even at the time his article was written, it was hard to imagine that so many ingredients of the New Math proposal were completely foreign to most mathematics programmes and teachers in 1960.

7 Activities Around the World

This account of the history of arithmetic teaching has focused on the development in Europe and North America, tracing its roots to the Middle and Far East. The account has focused on mainstream currents, originating in Italy, Germany, and France, but also on its development in more remote areas, for example, Northern Europe. While there were channels between the civilisations on the Eurasian continent, other civilisations, such as in the Western Hemisphere, developed their own arithmetic education which was not realised by others until much later. The Inca civilisation flourished in what is now Peru and surrounding areas from about 1,400 to 1,560. They possessed a logical numbering system of recording knots and chords of what is called *quipus* (Katz 1993). As this and other civilisations were eventually destroyed, they did not make an impact on the global picture of the history of arithmetic teaching.

Before opening to the West in 1867, schools in the feudal Edo period in Japan were run by each Samurai clan and reserved for the children of their proper clan. With the Meiji Era, the former private feudal schools were abolished, and a system of public elementary schools was created all over the country, with arithmetic as one of the major teaching subjects. In China, a state-school system was established by the end of the nineteenth century (Schubring 2012).

Activities towards basic arithmetic training have been driven in many corners of the world, either domestic or ethnic, or influenced by foreign currents, often from former colonial powers. Colonialism ruled countries in Latin America, Africa, and Asia for long periods. Educational traditions in those countries have in part taken shape through the adaptation of borrowings from abroad. Foreign assistance, however, has not always taken these traditions into account. The study of ethnomathematics arose in Latin America, for example, as a counterweight to this: it stressed the importance of the cultural context in mathematics education and quickly won recognition around the world (Karp 2013).

Since the 1980s, ethnomathematics has become a topic of growing research interest, unravelling mathematical practices, in particular as revealed by the arts and crafts traditions of indigenous groups, which have been neglected by historiography. Ethnomathematics exists and has existed in one form or another in all societies. Its subjects are the topics of everyday life: they concern nourishment, space, and time in preparing seasonal phenomena – knowing where (space) and when (time) to plant, harvest, and store. Practical mathematics is practised at markets, dealing with money, making change, and offering discounts. In different environments, ethnomathematics differs in terms of counting time, measurements of land and distances, systems of taxation, and arithmetic dealing with the economy. In many places, for example, trade was practised in the form of barter, by which trading goods were allocated monetary value that varied from place to place (d'Ambrosio 2006).

A topic that may be ascribed to ethnomathematics is multiplication turned into addition. This technique exists in several versions and may be traced back to the Rhind papyrus, dated ca. 1650 B.C. It has been termed Egyptian multiplication, but a so-called Russian peasant multiplication is closely related to it. In short, the multiplier is doubled until it reaches the next result below itself. Then those factors are picked out that add up to the multiplier and used to multiply the multiplicand. An example is the multiple 23×25:

$$23 = 1 + 2 + 4 + 16, \text{ so}$$

$$23 \cdot 25 = \left(1 + 2 + 4 + 16\right) \cdot 25$$

$$= 25 + 50 + 100 + 400 = 575$$

The technique is basically a use of the binary system (Seppala-Holtzman 2007) and is built on the fact that doubling a number, that is, adding it to itself, is easier than multiplication, using a table of many multipliers. Division is based on similar principles.

The former colonial powers have continued to exert cultural influences. Consequently, many countries became involved, even if often with some lag time, in broad international movements. The different countries, although constrained by their own specific characteristics, have nonetheless taken part in worldwide movements to renew and reform school curricula, such as introducing the New Math (Karp 2013).

8 Summary and Conclusions

The content of arithmetic gradually expanded from the days of the early *scuole d'abbaco* in Italian towns to the reforms of the 1960s, and computation methods have become smoother. Counting, numeration, and the decimal placement system, however, still constitute the basic curriculum of arithmetic. The four arithmetic operations in whole numbers are still practised, expanded to common fractions. Only in the nineteenth century did decimal fractions become widely presented and percentages increasingly seen in textbooks. The number concept was controversial until the late nineteenth century. The great content change arose in the reforms of the 1960s when introductory statistics and probabilities, elements of number theory, and abstractions and generalisations were included in the compulsory education curriculum to a much greater extent than before.

Learners of arithmetic have become younger and more numerous; indeed, every child in developing as well as industrialised countries is expected to study arithmetic as a part of his or her basic education. There have been shifts in the perceived goals of learning arithmetic, and teaching methods have gained increasing attention. The greatest change in the pursuit of arithmetic learning and teaching – the introduction of electronic calculators – is beyond the scope of this chapter, but it goes without saying that they have made an immense impact on arithmetic teaching.

The aim of arithmetic education was originally utilitarian. The preferred target group of arithmetic instruction was boys who were preparing for some definite profession as merchants, civil servants, owners of properties to be run economically, or clergy who needed to compute dates of ecclesiastical feasts. In larger societies, reckoning schools were established and textbooks became the teachers' tools, while in other more rural and peripheral societies, home education and self-education were the norm and textbooks were tools for the pupils themselves.

Arithmetic textbooks have increasingly become directed at children, taking their interests, social environments, and circumstances into consideration. Thinkers and educators, such as Comenius, Pestalozzi, Colburn, and Dewey, and later Piaget and Bruner, have exerted great influences on later generations in the direction of child centeredness. The methodical emphasis on elaboration of the number concept with children has changed radically.

The aims of arithmetic education have oscillated from utilitarian aims to mental exercises and back again. New instruction methods have been developed. Reforms have, however, often had their backlashes. The anticipated progress was less than hoped for, parents and teachers may not have been informed well enough, or the planned reform was not implementable on a large scale. Nevertheless, writers, who were followers of the prominent thinkers and educators, usually suggested some improvements which gradually worked their way into the schools. Experience shows that no advocate of a single method has been able to impress that method on any considerable number of followers. The best teacher has been the one who, being interested in the subject, conveys that interest to the pupils. Good teachers are seldom limited to any one set of objects or to any particular device; they have made arithmetic modern in its applications and followed the best curricula of the day.

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Chapter 22
Notes for a History of the Teaching of Algebra

João Pedro da Ponte and Henrique Manuel Guimarães

1 Introduction

Abundant literature is available on the history of algebra. However, the history of the teaching of algebra is largely unwritten, and as such, this chapter essentially constitutes some notes that are intended to be useful for future research on this subject. As well as the scarcity of the works published on the topic, there is the added difficulty of drawing the line between the teaching of algebra and the teaching of arithmetic – two branches of knowledge whose borders have varied over time (today one can consider the arithmetic with the four operations and their algorithms and properties taught in schools as nothing more than a small chapter of algebra). As such, we will be very brief in talking about the more distant epochs, from which we have some mathematics documents but little information on how they were used in teaching. We aim to be more explicit as we travel forwards into the different epochs until modern times. We finish, naturally, with some reflections on the present-day and future situations regarding the teaching of algebra.

2 From Antiquity to the Renaissance: The Oral Tradition in Algebra Teaching

In the period from Antiquity to the Renaissance, algebra was transmitted essentially through oral methods. The written records, such as clay tablets and papyri, were used to support oral teaching but duplication was expensive and time-consuming. These artefacts, however, were important for the continuity of memory in the long term.

Mathematics began to be developed through the facets of geometry and arithmetic. However, we also find primitive manifestations of mathematics that we can link to algebra in several epochs and dispersed regions around the world. In Mesopotamia, earlier than 3000 BC, records exist of relatively abstract problems, often presented as practical problems. Some Mesopotamian tablets may even be regarded as small textbooks, such as one that lists 21 problems. Whereas traditional accounts (Boyer and Merzbach 1980) assume that these problems already have an algebraic nature, more recent views suggest that they involved simply calculating with quantities (Høyrup 2002). As far back as then, there

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is evidence of institutionalised teaching and the mathematical texts that included exercises and problems for the student to solve and items for the teacher to use (Schubring 2003).

Diophantus of Alexandria (second half of the third century AD) authored one of his most important works, *Arithmetic* (split into 13 books, of which only 10 are known). It presents a sequence of problems that can be written in the form of equations and are solved using numerical methods. For each problem the author seeks an integer solution, without attempting to study all the possible solutions or drawing up a general theory on the matter. Diophantus used a sophisticated notation with different abbreviations for the square of an unknown number, for the cube and so on successively (Boyer and Merzbach 1980; Hawking 2010).

Al-Khwarizmi (ninth century AD) wrote what can be considered the founding text of algebra (*Al-Kitab al mukhtasar hisab al-jabr wa'l-muqabala, Abrégé du calcul par la restauration et la comparaison*), dedicated to the solution of second-degree equations. The book starts with a very small presentation of the positional principle of the number system and then states that "the numbers which are required in calculating by *al-jabr* and *al-muqābāla* are of three kinds, namely, roots, treasures and simple numbers relative neither root nor treasure". The author goes on to solve all six possible combinations of these kinds of numbers – which yield six kinds of equations, presenting the solution methods through the use of examples. In the second part, he presents geometric demonstrations for the different procedures presented (Boyer and Merzbach 1980). During this epoch in the Islamic Empire, numerous scriptures were handwritten for educational use. According to Schubring (2003), the Islamic civilisation was the first to create an institution destined for large-scale education at a high level: the madrasa. In this school, the teaching was oral; the teacher read the texts aloud, dictating them and giving some explanations. The students had to memorise exactly what had been taught and show that they had written a faithful transcription.

The Renaissance led to the emergence of several different algebras, such as *Die Coss* in Germany, published in 1524 by Adam Riese (1492–1559), and *O Libro de Algebra en Arithmetica Y Geometria* in Portugal, published in 1567 by Pedro Nunes (1502–1578). But the most significant event of this period was the solution to third-degree equations by Niccolò Tartaglia (ca. 1500–1557) and fourth-degree equations by Lodovico Ferrari (1522–1565), in a complicated story that also involved Scipione del Ferro (1465–1526). Girolamo Cardano (1501–1576) published these results in his *Ars Magna* (in 1545), a book that follows the style of Al-Khwarizmi's examples. For around three centuries, the solution for polynomial equations became the essential problem of mathematics (Boyer and Merzbach 1980).

The first European universities or "general studies" arose in the twelfth century under the aegis of the Catholic Church, with a view to educate jurists and doctors. The preparative studies, administered by the Faculty of Arts, included *Trivium* (grammar, rhetoric and logic) and *Quadrivium* (arithmetic, geometry, music and astronomy). Mathematics was part of the *Quadrivium* but it sometimes was not taught; when it was taught, however, there was no significant presence of algebra. Also in this case, the education was oral. The teacher dictated the texts of the chosen book aloud and that was considered the definitive explanation of the subject. The students had to memorise exactly what was taught, asking for clarification from the masters and showing their registers to be absolutely correct. The teaching therefore comprised of reading (*lectio*) and discussion (*disputatio*) (Carvalho 1986; Schubring 2003).

In all these civilisations, the teaching was essentially of an oral nature. Written notes played an important role for the preservation of memory, but did not constitute a tool for teaching and learning. Over time, the teaching of the subject through problems gave way to new forms of presentation, through explanations in common language as well as – from the time of François Viète de Fontenay (1540–1603) (*In Artem Analyticen Isagoge*, 1591) and René Descartes (1596–1650) (*La Géométrie*) – symbolic language.

Algebra began to constitute a generalisation of the methods of arithmetic, enabling broader classes of problems to be solved, and it also generated its own problems, transforming itself into a theory of solving polynomial equations. Although many geometric problems and sharing legacies had led to its development, its applications to the arithmetic of administration and commerce or astronomy did not give rise to interesting new problems. Therefore, algebra was cultivated above all because of its intrinsic value, as pure mathematics.

3 Seventeenth to Eighteenth Centuries: The Emergence of the Algebra Textbooks

The invention of print (1445) marks an important turning point: commercial arithmetic textbooks soon appeared. They were followed by geometry textbooks and then algebra textbooks, which were used to support education in the universities, colleges and other educational institutions that were being created, some of them for the purpose of professional training.

In Catholic countries where the Counter-Reformation prevailed, the Society of Jesus, founded in 1534, took on a leading role in education. The Jesuits created colleges ("collège" in France), where there was less emphasis on mathematics, which was only taught in the last year of the college; they adopted parts of *Euclid's Elements* as textbook in 1552 (Schubring 2005). Hence, algebra was not part of the Jesuit curriculum. Also in Portugal, at the time of D. João III (end of the sixteenth century), in the Jesuit colleges (and at Coimbra University, which was also run by the Jesuits), arithmetic, geometry and astronomy were taught, but algebra was not (Teixeira 1934).

Meanwhile, treatises specifically dedicated to algebra – which was understood as the study of polynomial equations – began to appear. Antoine Arnauld's book (1612–1694), *Nouveaux Éléments de Géometrie* (1st edition in 1667, 2nd edition in 1683), which was used in the "petites écoles" of Port Royal, began with the study of operations with quantities ("la quantité ou grandeur en general"); in other words, it developed algebra first and only in a later step applied these notions to geometry. This work (published anonymously) represents a revolution in style, using Descartes' algebraic notation. The book was aimed at a broad, non-specific target public. Afterwards, Jean Prestet (1642–1691) published his *Éléments des Mathématiques* (1st edition in 1675, 2nd edition in 1689), which was totally dedicated to algebra and was used in the colleges of the Oratory Order, whose curriculum was independent of that of the Jesuits. The author considered algebra as the most general field of mathematics and regarded geometry as a simple applied branch. Both books were followed by other textbooks aimed at the university public (Schubring 2005). Alexis Claude Clairaut (1713–1765) published a textbook in 1746, *Éléments d'Álgébre*, elaborated for private teaching. It included the solution of fourth-degree equations. The author intended to follow an allegedly "genetic approach" dealing with the matters in the order of invention, which in practice meant following a problem-solving approach. Boyer and Merzbach (1980) indicate that Clairaut was very successful in showing that the introduction of algebraic notation was necessary and inevitable. Meanwhile, the textbooks that followed this approach can be criticised for avoiding the main conceptual difficulties of the matters dealt with (Schubring 2005).

In Germany, Christian Wolff (1679–1754) wrote a multivolume textbook, *Elementa Matheseos*, in Latin (between 1710 and 1713) and in German, following Arnauld's style. The first volume had a chapter on algebra. The work was praised by d'Alembert as the best mathematics book for teaching of the time (Schubring 2003). In England, in the eighteenth century, there were several elementary algebra textbooks. The *Elements of Algebra* by Nicholas Sauderson (1682–1739) had five editions (between 1740 and 1792). Thomas Simpson (1710–1761) wrote a *Treatise of Algebra* that had 18 editions (1st in 1745, last in 1809). Maclaurin (1698–1746) also wrote a *Treatise of Algebra* that had about 12 editions (1st in 1748, last in 1796). These books were essentially composed of rules and examples. According to Boyer and Merzbach (1980), English algebra textbooks of this century illustrated a tendency towards increasing algorithmic emphasis, but left considerable uncertainty about the logical foundations. Euler wrote a popular algebra textbook that was published in German and Russian (1770–1772), in French (1774) and in other languages including English. These *Elements of Algebra* are acknowledged as being of excellent didactic quality (Dunham 2000).

In Portugal, the Marquis of Pombal reformed Coimbra University in 1772, which was endowed with a Mathematics Faculty with four consecrated subjects: algebra and infinitesimal calculus, geometry, mechanics and astronomy. For the algebra course, a translation (by Monteiro da Rocha) of *Elements d'Analyse Mathematique* by Etienne Bézout (1739–1783) (1774, 1793, 1801 and 1818 editions) was used (Teixeira 1934).

As in the universities, mathematics also began to be taught in the military schools that arose in France during the eighteenth century to educate the officers of the most powerful army of the epoch. The so-called Artillery and Fortification Schools, created in 1720, had strong mathematics education. As such, the Frenchman Bernard Forest de Belidor (1697/1698–1761), a teacher in these schools, published the *Nouveau Cours de Mathematique* (a book containing 656 pages with many diagrams, which had two editions, in 1725 and 1757). It was basically a geometry book applied to artillery and fortifications, but included elements of algebra when necessary. In Portugal, Belidor's book, translated by Manuel de Sousa, was adopted for more than 20 years (1764–1786) in military courses, up until these courses were reassigned to Coimbra University and to the Royal Navy Academy in Lisbon (Valente 1999). The first chapter of Belidor's book contains an introduction to geometry and the second chapter deals with ratios, proportions, sequences, logarithms and first- and second-degree equations. Chapter 9 deals with conical sections (see Valente 1999, pp. 70–71; algebra does not play an important role in the other chapters). Meanwhile another Frenchman, Etienne Bézout (1739–1783), published his *Cours de Mathématiques* from 1770 in two series: one with five volumes for the students to become navy officers and one with four volumes for future artillery officers. In the navy series, the third volume was dedicated to algebra and its application to geometry, while in the artillery series it was the second volume.

According to Valente (1999), Belidor and Bézout's books were complete elementary mathematics courses, where the authors sought to compile updated knowledge of this science with potential interest for the military preparation courses they were destined for. Bézout's textbooks, however, continued to be used after the French Revolution in general education. In Portugal in 1796, at the Lisbon Royal Academy of Coast Guards (aimed at 14–18-year-olds and later to children from 12 years upwards), the second year was dedicated to algebra, following Bézout's book (Valente 1999, p. 91). At Coimbra University, algebra was taught in the second year, dealing with topics such as literal calculus, analysis, sequences, conical sections, infinitesimal algebra and differential and integral calculus (Carvalho 1986). Bézout's books were used until the end of the nineteenth century in secondary schools and colleges all around Europe, the USA and Brazil.

The sixteenth, seventeenth and eighteenth centuries witnessed significant changes in education, with printed textbooks playing an increasingly important role in pushing the traditional practice of reading texts aloud into a secondary position. Gradually, in the teaching, the printed material weakened the predominance of the oral transmission. From that point onwards, the learning method was no longer simply listening and writing down notes; the students could now manage their own studies on their own initiative.

As part of Enlightenment thinking, which prepared a new type of textbooks for public school systems, d'Alembert published in the *Encyclopédie* his important reflection on "elementarising" science. The advantages and disadvantages of the synthetic approach were also discussed (in line with Euclid's paradigm) as was the analytical approach, which dominated shortly after the Revolution. D'Alembert also theorised on the learning process, criticising the prevalence of oral methods. He stated that one only masters what one learns for oneself, and he did not believe that the teacher carried out the fundamental work in the teaching process; rather students did because they were required to reflect and work hard ("méditation et travail"). Furthermore, D'Alembert criticised Clairaut's approaches as being unsuited for textbooks (Schubring 2003).

4 Nineteenth Century: The Slow Rise of Algebra in Secondary Education

The nineteenth century witnessed the constitution of algebra as an important subject in the secondary education curriculum. During this period, its borders with arithmetic and geometry were delimited. The end of the century witnessed an important transformation in the didactics of this topic, with an increased emphasis on exercises.

4.1 Algebra in Secondary Education in Different Countries

The 1789 French Revolution ended the feudal system and led to State-organised public education throughout the nation. As a result, the *écoles centrales* were established as the first form of secondary schools in 1795, paving the way for the creation of the *École Polytechnique* in Paris in 1794.

In the first public secondary education mathematics programmes after the Revolution in 1802, algebra was given an important place (2ème and 1ère classes) using Lacroix's books to solve first- and second-degree equations, equation systems, operations involving algebraic expressions, proportions, sequences, logarithms and the numerical resolution of equations. The textbook procedure organised in the Napoleonic period established the principle of a compulsory book for the entire country (Schubring 2005, pp. 72–80).

However, the importance of mathematics in secondary education began to be reduced from 1809 onwards and drastically since the Restoration in 1815. By the mid-nineteenth century in France, it occupied an extremely weak position in public education. Mathematics had a significant role in supplementary education only, especially for the students who wanted to apply for enrolment into the polytechnic school (Schubring 2005, pp. 16–17). For these students, Louis Pierre Marie Bourdon (1779–1854) wrote his *Eléments d'Álgèbre*, published in 1817, which was broadly used up until the end of the nineteenth century. The author himself accepted the influence of Lacroix's book and his concern to cater for the programme in place in the *École Polytechnique* in Paris.

In contrast, in Prussia, after a slow growth in the eighteenth century, algebra took on a noticeably strong presence in the 1810 programme. Of the 9 years for the Gymnasium curriculum, algebra had to begin in the third year with the theory of first- and second-degree equations and continue in the fourth year with quadratic equations, calculations with powers, powers of a binomial, logarithms and elements of analytical geometry; in the fifth year with algebraic equations and numerical solutions, analytical geometry in 2 and 3 dimensions and conical sections; and in the seventh year with third- and fourth-degree equations (sixth, eighth and ninth year were devoted to other topics). Although this programme served as a guideline and was not officially prescribed, it documented the analytic approach and the strong role given to mathematics education, thereby triggering the birth of the profession of the secondary education mathematics teacher (Schubring 2005, p. 17). Prussia, from the moment of Napoleon's defeat (1814), adopted a policy that emphasised the role of the teacher, and consequently oral methods, along with the acceptance of the dual teaching/researching function (Schubring 2003, p. 129). A new kind of schoolbook was produced by Johann Andreas Matthias (1761–1837), with successive reprints from 1813 onwards: a concise guide manual (*Leitfaden*) for students, a methodological complement for teachers and a collection of exercises (*Aufgabensammlung*). Meanwhile, at the same time as the pool of well-trained professional teachers was increasing, methodological aids for the teacher disappeared. Working documents began to be formed by the *Leitfaden-Aufgabensammlung* pair, making the collections of exercises extremely popular (Schubring 2003, pp. 137–140).

In Brazil, the *Real Colégio Militar* in Rio de Janeiro, which began giving lessons in 1811 (for youths aged 15 and over), included a complete mathematics course with algebra. The academy looked to the *École Polytechnique* in Paris as its reference point, using the new textbooks by Lacroix and Legendre written to replace Bézout's textbooks (in this country, the algebras of Euler and Lacroix were published in 1811 and 1813, but only the second received larger use). According to Valente (1999), the first year of this course covered several topics, "sometimes being a mathematics course which would be given in secondary education that did not exist at the time in Brazil" (p. 94). Years later, in 1837, the *Colégio Pedro II* was created to serve as the secondary school model throughout Brazil. In its study plan, algebra was taught in the sixth year, after 3 years of arithmetic and two of geometry. In secondary schooling, which prepared students to enter higher education (for engineering, law and medicine degrees), mathematics assumed a new role. It no longer represented a technical knowledge that was specific to the military schools, but it was integrated into the general school syllabus for those who wanted to attend higher education (Valente 1999, pp. 105–119).

In Portugal, the *Real Colégio Militar*, specifically geared towards the children of officers, had been in operation since 1803. This course, which was 6 years, first makes a reference to algebra as a subject in the third year, alongside arithmetic, geometry and trigonometry. Upon the creation of the secondary schools by Passos Manuel in 1836, the subjects of arithmetic, algebra, geometry and trigonometry were offered. It is pointed out that the depth at which algebra is treated in the following periods depended on successive reforms. In higher education, the new polytechnic schools created in Lisbon and Porto in 1837 included subjects like elementary algebra and transcendent algebra on their syllabi, which were prerequisites for the differential and integral calculus and mechanics courses (Carvalho 1986). In Spain, in the secondary education syllabus of 1857, the fourth year was dedicated to arithmetic and algebra and the fifth year to geometry and trigonometry (Sierra et al. 2007).

4.2 Reorganisation of the Topics

A major change in the policy of admission into higher education (especially engineering courses) during the eighteenth century increased the importance of algebra at the secondary school level. At the time, algebra was a fundamental prerequisite for the study of infinitesimal calculus and ultimately occupied a central position in the final years of secondary school education. On the other hand, the introduction of algebra in secondary education was accompanied by a reorganisation of the topics of elementary mathematics. At the start, the arithmetic-geometry-algebra sequence was followed, aligning with the organisation of Bézout's course (1764–1769) and as shown in the syllabus of initial studies of the *Colégio Pedro II* (in 1837). However, Lacroix's books (1797–1805) introduced a new arithmetic-algebra-geometry sequence, based on the idea that algebra is nothing more than universal arithmetic. This became the sequence adopted by the *Colégio Pedro II* from 1841 onwards. An identical phenomenon occurred in other countries, leading to an algebraic processing of certain aspects of geometry. Therefore, as an example, Pedro de Alcântara Bellegarde's compendium (1807–1864), published in Brazil for use in the *Escola de Arquitetos Medidores* (1838), began with elements of arithmetic (operation with integers, fractions, decimals and the rule of three), before moving on to algebra, with such topics as the extraction of roots and logarithms, which were previously treated in the arithmetic course only. *Apostillas de Algebra*, by Luiz Pedro Drago, published in 1868 and used in the *Colégio Pedro II*, included algebraic operations and first- and second-degree equations in the first part and ratios and proportions, sequences and logarithms in the second part (Valente 1999, pp. 119–127).

Hence, an evolution took place for what was treated within the scope of each of two subjects (arithmetic and algebra) and with a progressive study of topics such as logarithms and sequences in arithmetic and algebra. But an even more important evolution happened with the way certain topics of arithmetic or geometry were gradually tackled using algebra. *Aritmética* by Louis Pierre Marie Bourdon (1779–1854), published in France (1st edition in 1820, 20th edition in 1872) and organised into two parts, exemplifies this important landmark. The first part was purely arithmetic and the second part used algebraic language dealing with general properties of numbers, powers and roots, ratios and proportions, sequences and logarithms. This approach was introduced in Brazil by Cristiano Benedito Ottoni (1811–1896) in books that were used from 1855 onwards (Valente 1999, pp. 150–168). However, this adjustment between arithmetic and algebra occurred at a slow pace. As such, João António Coqueiro's (1837–1910) arithmetic book, published in 1860, included algebraic elements (namely, algebraic expressions, first-degree equations and system of equations). In contrast, António Trajano's *Arithmetic* (1843–1921), which was extraordinarily popular (1st edition in 1879, 136th edition in 1958), did not make any use of algebra (Valente 1999, pp. 159–165).

4.3 New Didactics

At the end of the nineteenth century, a new didactics for algebra emerged. We mentioned earlier the appearance of collections of exercises in Prussia from the start of the century. In France, Joseph Louis François Bertrand's books (1822–1900) contained a summary and a set of exercises at the end of each section. In Brazil, Coqueiro's arithmetic book, published in 1860, was the first to include exercises with and without solutions at the end of the chapter. The *Apostillas de Algebra* by Luiz Pedro Drago, published in 1868, was organised into sections (which included problems, definitions or rules to learn), with exercises at the end of each chapter (Valente 1999, p. 167). In Portugal, the books published by José Adelino Serrasqueiro (1835–?) from 1869 onwards were heavily influenced by Joseph Louis François Bertrand (1822–1900); like the French author, they included exercises at the end of the different sections.

Two didactics began to coexist. On the one hand was the "didactics of the lesson" (the "lecture"), which would be used in the military schools and which became typical in the preparatory colleges for higher education. As Valente points out (1991), in these didactics "saying prevails over doing" (p. 173). The course was conducted in a purely oral – often dictated – form, whereby the students took notes. The education, both oral and written, was based on long explanations about how to carry out certain procedures. On the other hand, the "didactics of the exercise" began to take shape, starting in the school practice. Here the students were given a different role: facing directly up to the difficulties posed by the exercises, whose complexity would lead us today to call them "problems" (Valente 1999; Ponte et al. 2007). The use of exercise solving was made much easier by the introduction of a new tool – the blackboard – introduced by Frères de Lessale. This made it possible to correct exercises in direct interaction with the students, giving rise to the practice of "calling a student to the blackboard". The didactics of the exercise progressively broadened its sphere of influence, also entering higher education.[1] The two didactical approaches – the lesson-lecture and the exercise – are still being used today, usually in combined form (Valente 1999, pp. 173–176).

5 Twentieth Century: Apogee and Decline of the Teaching of Algebra

The teaching of algebra in the twentieth century, at the secondary school level, is split into three large periods: the first, where algebra was viewed above all as the theory of polynomial equations and as preparation for the subsequent study of infinitesimal calculus, playing a very important role in the

[1] In Portugal in the twentieth century, the algebra classes of a university mathematics course continued to be split into "theoretical classes" (with explanations and examples) and "practical classes" (with exercises). Similar practices are followed in many secondary schools, especially in the more advanced years.

syllabi; the second, which was marked by the modern mathematics movement, algebraic structures and algebra, inspiring the approach to studying numbers and geometry; and the third, which was marked by great uncertainty about algebra's place in the curriculum. In fact, throughout the 1950s, a number of wide-ranging initiatives were carried out with the common aim of modifying the mathematics teaching syllabi in order to update the mathematical topics taught, as well as to introduce new curriculum reorganisations and new teaching methods (Moon 1986; NACOME 1975).

5.1 Algebra in Secondary Education in Different Countries

In the USA in 1894, the Committee of Ten established elementary algebra as an important part of the secondary school curriculum (Myers 1979). A statement on college entrance requirements presented to a meeting of mathematicians in 1899 summarises what was expected for the candidates' mastery of algebra:

(i) The arithmetical side of algebra, including problems with literal quantities
(ii) The "equational" side of algebra, including word problems
(iii) Algebraic translation, including skills attained in manipulating long and complex algebraic expressions
(iv) Topics such as exponents, surds, quadratic equations, solving equations by factoring and making algebraic statements from problems given in words

Most algebra textbooks began with a presentation of "literal" numbers, that is, algebraic expressions. Then the student worked on solving equations, usually presented through world problems. According to Hirstein, Weinzweig, Fey and Travers (1980), from the start of the century until the 1950s, the introductory algebra course served two purposes:

(i) To prepare the student to use algebra to solve "practical problems"
(ii) To furnish them with the manipulation skills needed for higher studies

In Brazil, Serrasqueiro's algebra was used in the *Colégio Pedro II* from 1891 until at least 1923. This book introduced into algebra teaching the elementary theory of the determinants and their application in solving first-degree equations (Valente 1999, p. 168). The importance of the *Elementos de Álgebra* by FIC (1st edition at the end of the nineteenth century and the virtually identical 10th edition in 1950) is evident. This book included algebraic calculus, first- and second-degree equations, sequences, logarithms, interest, likelihood and (in the appendix) a graphical representation of first- and second-degree equations, Newton's binomial, exponential equations, notions of series and so on (Valente 1999, pp. 187–188). Euclides Roxo (1890–1950) introduced a new curriculum at the *Colégio Pedro II* in 1929 that drastically changed secondary school mathematics education in Brazil. With this reform, mathematics was taught at every grade and the separation between arithmetic, algebra and geometry disappeared. These subjects were taught in an integrated way and the function concept was introduced very early in the curriculum. However, in a later reform in 1942, some separation between the subjects recurred and the study of functions was postponed to a later stage of the curriculum (Carvalho 2006).

It is interesting to note that in Spain, the 1934 Reform (which only lasted until the end of the civil war in 1939) introduced the "cyclical method". As such, the first course presented the initial notions of arithmetic and intuitive geometry, and the recommendation was for more emphasis on an intuitive and practical approach. In the third and fourth years, the same concepts were tackled, but this time through a "rational presentation". In subsequent years, real numbers, limits and continuity of functions and topics such as logarithms, arithmetic and geometric sequences and complex numbers were studied (Sierra et al. 2007). Some topics began to be taught earlier, such as the second-degree equation that moved from the fifth to the fourth year.

5.2 *Mathematics as a Unified Subject?*

Finally, another question that marked the twentieth century, in terms of the secondary school mathematics curriculum, was the issue of whether it must be organised into separate sub-subjects or if mathematics must be presented as an integrated single subject. As early as 1903 in the USA, Eliakim Hastings Moore (1862–1932) called for a curriculum where the different branches of mathematics would be unified (Kilpatrick 1992); later, the Bourbakist movement made the issue of the unity of mathematics one of its central themes.

An important moment in this discussion was the CIEM (Commission internationale de l'enseignement mathématique) meeting in 1911 in Milan, which intended to "pursue the study of current trends in mathematics education" with special attention on "the issue of merging the different branches of mathematics" (CIEM 1911a, pp. 122–123). The account of the discussion (CIEM 1911b, pp. 468–471) makes a distinction between "purist" and "fusionist" trends. The former separates the various areas of mathematics in a strict way, while the latter does not make this separation but rather creates some kind of "integration" between different branches of mathematics. In the presentation of a given branch of mathematics, the "fusionists" use examples or representations of another branch: for example, "graphical representations in arithmetic or algebra", "trigonometry formulas in geometry" and geometrical examples in the theory of proportions (CIEM 1911b, pp. 468–469). The account of this meeting indicates that "the fusion of subjects can be made in very different levels of depth" and notes that there was a large variation among different countries to this respect among syllabi and methods (CIEM 1911b, p. 469). The discussion mainly addressed aspects related to the teaching of geometry, but noted that "there is fusion in most countries" between algebra, or arithmetic, and geometry (one exception was the Italian lyceums). The most important highlight of this "fusion" was the very frequent use of graphical representations and the teaching of "one year" of analytic geometry in countries such as Austria and Switzerland (CIEM 1911b, p. 469).

Despite this discussion, Carvalho (2006) indicates that in Brazil, in the 1920s, the subjects of secondary school mathematics (arithmetic, algebra, geometry and trigonometry) were still taught in a strictly compartmentalised way, separated by school years. However, an increasing number of mathematics study plans integrating the different topics, especially algebra and geometry, began to appear in many countries. For example, in Spain, as a result of the 1934 Reform, mathematics was presented in a unified manner, eliminating the separation into disconnected subjects. Curiously, one of the few countries in the world where this integration did not yet occur was the USA, where most of the schools continued to have an algebra course in the 9th grade followed by a geometry course in the 10th grade.

5.3 *Approach Centred on Equations or on Functions?*

According to Artigue, Assude, Grugeon and Lenfant (2001), in France in the twentieth century, until modern mathematics, algebra was an essential topic on the curriculum, alongside arithmetic and geometry, whereby its central concept was the equation. This trend also occurred in most countries, although in some cases through an extremely abstract approach which was essentially backed up by work with algebraic expressions (with one or various variables) and in other cases with a significant emphasis on word problems.

In the early twentieth century in Germany, Felix Klein (1849–1925) suggested that the concept of function should occupy a central position throughout the entire secondary education curriculum, and thus the teaching of algebra should be changed. Klein's proposal became one of the most keenly debated issues in the initial phase of the ICMI. The role of the concept of function and its importance and relationship with the concepts of traditional algebra were discussed. This perspective pleased those who valued the applications, given that the concept of function is an important modelling tool.

It also pleased many who were concerned about the learning of differential and integral calculus, where the concept of function plays a central role. In contrast, it did not please those who considered the learning of algebraic manipulation (expressions and equations) as the fundamental basis for the subsequent studies of mathematics. The fact is that this idea generated an important international movement of curriculum development, leading Kilpatrick (1992) to say that “the banner for the reform of secondary school mathematics would be the concept of function” (p. 7). By the mid-twentieth century, Klein’s ideas had won a half victory – the concept of function occupied a central position in the syllabus of the final phase of this teaching level (grades 10–11), but it had a very small role in the initial phase (grades 7–9).

5.4 *Evolution of the Treatment of Algebraic Topics*

Along with the evolution of the role of the concept of function, we witnessed throughout the twentieth century a significant evolution in the treatment of algebraic topics. Some progressively lost importance and even disappeared from syllabus (such as solving third-degree equations, solving systems of three or more equations, determinants, solving equations involving algebraic fractions and irrational expressions, series and Newton’s binomial). Furthermore, the approach to central concepts also changed. For example, in Portugal between the end of the nineteenth century and the end of the twentieth century, the teaching of the first-degree equation was carried out earlier and in a more simplified manner. Initially, algebraic expressions were studied in a single year (including algebraic fractions), along with numerical and literal equations, and finally systems of equations. Later, separate stages of this area of study were differentiated, with each topic studied in different years. Algebraic expressions were no longer treated as before, as a prerequisite for equations; instead, they were introduced based on the actual study of equations in an attempt to motivate the study of this topic, which was acknowledged as being arduous and uninteresting (Ponte 2004).

The role of applications in mathematics teaching is a subject of debate during the entire twentieth century. Both equations and functions are concepts that have many applications both outside and inside mathematics. In England, John Perry developed a new curriculum on practical mathematics and argued for a more intuitive and laboratory-based approach in a 1901 address. Klein, as we have seen, was also in favour of an emphasis on the applications of mathematics. Despite this, the approach that prevailed in most countries placed applications in a secondary position (Carvalho 2006).

5.5 *The Modern Mathematics Movement*

At the end of the nineteenth century, algebra changed its focus as a field of research. It no longer focused foremost on the study of polynomial functions but centred on the study of structures, starting with group structure. The influence of the *Modern Algebra* textbook written by Bartelt van der Waerden (1903–1996), published in 1930–1931, represented a revolution not only in algebra teaching but also in algebra research. The Bourbakist movement became the visible face of a programme that placed algebra as the integrating part of mathematics and which would have a large influence on the curriculum movement of modern mathematics.

In the USA, initially the changes were cautious, as we can see in the 1959 recommendations report by the Commission on Mathematics of the CEEB (College Entrance Examination Board):

> The recommendations of the CM do not envisage changes in the mechanics of formal manipulations of algebra. They will be the same as hitherto taught, and the subject matter will be largely the same. The differences will be in the concept, in terminology, in the symbolism, in graphs on the line, and in the inclusion of a rather new segment

> of new work dealing with inequalities. There will be a shift in emphasis from mechanical manipulations to the development of concepts, which is equally important. Of course, the development of adequate skills continues to be an important objective of the high school algebra course. (Hirstein et al. 1980, p. 377)

In 1959, the Organisation for European Economic Cooperation (OEEC) decided to hold a working session aimed at promoting a deep-rooted reform in the teaching of this subject. This work session took place at the *Cercle Culturel de Royaumont* in France, lasting 2 weeks and gathering around 50 delegates from 18 countries (OEEC 1961a). It came up with general guidelines for a proposal to reform mathematics education, heavily influenced by the dominant structural ideas of the time, particularly with regard to mathematics and psychology; the most ardent advocate was Jean Dieudonné. This proposal was specified a year later, in Dubrovnik, with the writing of the book *Synopses for Modern Secondary School Mathematics* (OEEC 1961b).

Two main guidelines stand out in Royaumont's proposal (OEEC 1961a) with regard to the content and curriculum organisation for a new mathematics programme: on the one hand, it gave emphasis to the unity of mathematics; on the other, it introduced new topics and approaches of mathematics, labelled modern. These recommendations above all empowered algebra and vector geometry and placed a correspondingly lesser value to Euclid's geometry, in the axiomatic orientation given to the study of mathematics whereby the mathematical language and symbolism are emphasised.

Algebra appears in the programmatic recommendations of Royaumont as a unifying item par excellence. An example of this is Choquet's proposal (OEEC 1961a) for the teaching of arithmetic; he argued for the merging of arithmetic and algebra, which he believed was possible based on the study of mathematical structures. Other examples are those suggested by an algebraic approach to geometry, which is in line with the proposal to begin its study with geometric transformations "starting with translations, rotations and reflections and proceeding step by step to a more generalised application of groups of transformations" (OEEC 1961a, p. 77), as well as recommendations for teaching vector geometry as early as possible at school, in order to synthesise algebra and geometry.

Also relative to the emphasis on algebra in education, Servais (OEEC 1961a) argued that presenting its study as a simple generalised extension of arithmetic gives a distorted vision of that domain of mathematics. Therefore, he stated, algebra should not be restricted to the numerical field immediately at the inception of its study in the secondary schools. Hence, he made his proposal for the early introduction of algebra with sets, justified by the fact that with the use of sets "from the start the field of algebra is not confined to the algebra of numerical operations" (OECE 1961a, p. 69). Servais proposed the study of set theory and argued that sets should become familiar to students as early as possible. He suggested that the notion of function should be introduced based on this theory and directly from the beginning of school. According to Servais, with the preliminary study of the notions of set, function, Cartesian product, relations and operations, "algebra would be given the role in mathematics which it is increasingly acknowledged, it deserves" (OEEC 1961a, p. 71).

In tandem with the prominence of algebra, vector geometry was adopted which led to an accordingly lower emphasis on Euclid's geometry. Dieudonné (OEEC 1961a) played a leading role in this shift. In his famous statement, which he himself accepted as encapsulating his ideas, he argued: "If the whole programme I have in mind had to be summarised in one slogan it would be: *Euclid must go*!" (p. 35). In truth, Dieudonné was extremely critical about the teaching of geometry that was practised at the time, proposing that it should be completely replaced by transformational geometry and vector geometry. As such, algebra also penetrated geometry, as was stated in the final conclusions of the seminar: "a more thorough and effective treatment of logical aspects of these subjects is imperative (…) and this adaptation further demands the earliest possible exposition of the connections between geometry and algebra—particularly linear and vector algebra" (OEEC 1961a, p. 122).

The programme drawn up by Dubrovnik (OEEC 1961b) incorporated many of Royaumont's guidelines and proposals. The two main topics were precisely algebra and geometry, spread over two cycles – 11–15 years and 15–18 years – whereby, in this last topic, the programme proposed an algebraic approach with the systematic study of geometric transformations, first introducing the notions of vector, angle and symmetry (and reflection), then studying other transformations (OEEC 1961b). For the first cycle (11–15 years), algebra started with set theory, from elementary notions of the theory to the study of properties, relations and operations with sets, including notions of binary relation, Cartesian product of two sets and function and mapping of sets. Despite the disagreements expressed in Royaumont, Dubrovnik's programme included the study of these structures right at the outset of the first 1st cycle, considering that "[t]here is no intention of introducing, at this level, a systematic or formal study of rings and fields" (OEEC 1961b, p. 81). In the second cycle (15–18 years), algebra deepened set theory and prolonged until linear spaces, linear applications and matrices.

Hence, regarding the modern mathematics period, we can talk about a structural approach (emphasis on the structures) as well as a functional approach (emphasis on the concept of function). In addition to introducing specific topics (such as elements of set theory and the study of abstract algebraic structures such as group, field and linear space), it began to inspire all the work in arithmetic and geometry. While, in preceding periods, the study of the first-degree equations was based on the principles of equivalence, presented formally (as theorems) or as practical rules, in this period another practice was put into place, initially seeking to solve equations based on the definition of numerical operations and their properties. Only later were guidelines presented and practical rules formulated for solving equations (Ponte 2004).

5.6 The Post-Reform Period

Modern mathematics represents the overwhelming triumph of algebra, but it was a short-lived triumph. In the following period, there was a sizable reduction in topics of an algebraic nature in the curriculum and a substantial simplification of symbolism. Functions were dealt with separately and, at the elementary level, at times in association with statistics. In several countries such as France and Portugal, although many algebra topics continued to be covered at various points in the curricula, they disappeared as separate topics. In France, the emphasis was put on modelling; in Portugal, algebra was regarded only as computation (operations with expressions, solving equations, inequalities and systems), and functions were taught as a different topic (Artigue et al. 2001; Ponte 2006). Artigue et al. (2001) believe that today several trends live alongside one another, which they label empirical (emphasising modelling), structural (emphasising structures) and centred on equations (the "classic" approach).

In the 1990s in the USA, a new trend emerged that gave algebra a prominent role in the school syllabus; this was supported by the NCTM (2000), making algebra a curriculum *strand* from the start of primary school. This movement, which has had visible impact on the curriculum in several countries, was underpinned by the central idea of "development of algebraic thinking" which focused on the notions of generalisation and symbolisation, valuing the work with sequences, a functional approach to proportion and the structural properties of the number sets.

6 Conclusion: What Is the Outlook for Algebra in Education?

The history of algebra education set forth here in general terms gives rise to several reflections, with which we end the chapter. First, we need to consider the technologies available in each epoch, which clearly had a decisive role in the way algebra was taught over time. The difficulty in carrying out and reproducing written records led to the lengthy predominance of oral education. The emergence of print enabled the intensive use of written material, which then was considered and presented as a superior alternative to oral methods. Later, the oral/written dichotomy became a combination of oral and written teaching, with variations to a greater or lesser degree depending on local traditions.

Today, we are in a period of rapid development of new digital technologies (Internet, Web 2.0, interactive boards), which will certainly heavily influence mathematics teaching, in particular the teaching of algebra, although its direction is not yet known (Moreno-Armella et al. 2008).

A second reflection is linked to approaches to algebra education and especially to the tasks used. Verbal problems played an important role in algebra education until well into the twentieth century. From a certain time, algebra began to constitute, in itself, a wealth of problems, giving rise to two large didactic movements – one following a purer mathematics line using problems formulated solely in mathematical terms and the other profusely using word problems within the scope of extra-mathematical contexts. Upon the consolidation of the concept of function as the central idea of algebra teaching, a third kind of task has progressively gained predominance – tasks involving modelling of real situations. One can argue, like Puig (2004), that students' difficulties in word problems are linked to the lack of teaching this topic, meaning that they cannot understand the logic underlying the Cartesian method. But in truth, it is very difficult to give a central role to word problems without making them artificial, especially when moving into a more advanced study of algebra. On the other hand, it is possible that the issue of tasks is not restricted to finding the key task that can serve as the basis of all education, but rather the combination of different kinds of tasks that provide the experiences needed to bring about desired learning.

Other issues pertaining to the approach to follow underwent successive inflections throughout history, such as the role of explanations versus the role of problems, the role of the teacher and the role of the student, the synthetical approach versus the analytical approach, the relationship of algebra with other fields outside and inside mathematics and the role given to symbolism and the problem of abstraction, which began as notable progress until it became an obstacle to learning for most students. The fact is students who study algebra change from epoch to epoch, much like their motivations and the conditions in which they study also change. A successful approach in a given epoch with a given group of students may prove disastrous in another epoch and with other students.

Finally, we see how, over time, the presence of algebra in secondary school syllabi has evolved greatly. As such, algebra has moved through progressive affirmation in the mathematics curriculum, reaching dominance in the first half of the twentieth century. Afterwards, with modern mathematics, algebra was given an absolutely hegemonic position, virtually inspiring the whole curriculum at this level. Not only did algebra occupy a prominent role, but the approach to numbers and geometry was undertaken from an algebraic perspective, in line with its respective structures. However, in the subsequent period, the role of algebra was questioned drastically, losing visibility as an independent topic and becoming a simple residue, with algebraic computation as a prerequisite for the study of infinitesimal calculus and a modicum of algebraic structures serving as glue for different topics. It will be interesting to see to what extent current-day conceptualisations about algebraic thinking, early algebraic thinking and new modelling-based approaches are able to guarantee a stable and lasting place for algebra on the school curriculum and to what degree they contribute to the success of students' learning in mathematics.

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Chapter 23
History of Teaching Geometry

Evelyne Barbin and Marta Menghini

1 Introduction

From the beginning of the transmission of geometric knowledge, two different aspects of geometry have been present: the abstract "speculative," represented by Plato and by the Elements of Euclid, and the practical, represented by the work of Heron and by applications of geometry. The teaching of geometry inherited both of these two aspects, and the tension between a deductive/rational approach and a pseudo-practical/intuitive one manifested itself many times throughout history. The emphasis on the second aspect in the nineteenth and twentieth century, in a period during which great attention was paid to the mind of the child, led to experimental geometry and to methods that allowed the shifting of the subject to the lower grades.

After the work of Descartes, another tension developed in the teaching of geometry and specifically in approaches to problem solving between the pure methods of reasoning and the methods making use of arithmetic, algebra, or analysis. Attempts to find a new language for school geometry reached the apex in the substitution of geometry by linear algebra in the 1960s. This chapter attempts to give a brief description of the major changes and developments in the teaching of geometry.

2 The Beginnings of Teaching Geometry in Secondary Schools

2.1 The Heritage of the Antiquity

Plato assigned an important place to the teaching of geometry, which for him meant going far beyond technical calculations and following the spirit of abstract mathematics as presented by Theaetetus or Eudoxus. The most famous book from this point of view is unquestionably *The Elements* of Euclid, which dates back to the Hellenistic period. In the Euclidean text, the order of the propositions is essential: This order suggests going from the most simple to the increasingly challenging. Figures represented in Book I go from the most regular triangles to parallelograms. The first theorems in Book I are deduced

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directly from the axioms, while the following theorems are deduced from both the axioms and previous theorems: There is a hierarchy of theorems. The search for an "order of simplicity," which suits both the content and the learner, will recur in the means and purposes of the teaching of geometry throughout its history (Barbin 2007), and the Euclidean order will become a major theme of discussion.

Antiquity also presents a different perspective from the Euclidean kind of geometry. The books of Heron of Alexandria in the first century attest to a teaching of practical geometry, including various tools for measuring in *Metrics* or solving problems in *Geometry* and *Stereometry*. Heron collected earlier geometrical rules and formulas, which were helpful in teaching prospective artisans and engineers.

The Arabic practical geometries inherited both Greek and Indian knowledge. Many Arabic textbooks were devoted to practical geometry, such as *A Book on Those Geometric Constructions Which Are Necessary for a Craftsman* by Abu al-Wafa in the tenth century or *Sufficient on Calculation* by al-Karaji in the following century (Djebbar 2001).

From the thirteenth century (thanks to the important movement of translation associated with the Andalusians in the twelfth century), Latin Europe received part of the geometrical knowledge available in the Arabic language, originating in Greece or developing in Arabic mathematics (Moyon 2008). In 1220, Leonardo Pisano, named Fibonacci, published his *Practica Geometriae* in Latin, which contains both proofs in the manner of Euclid and solutions to problems for craftsmen.

Geometry was part of the *quadrivium*, which also included *arithmetic*, *astronomy,* and *music*; together with the subjects of the *trivium,* it formed the teaching program in the universities of the Middle Ages. For a long time in the Middle Ages, the Cathedral schools (see Chap. 6) used textbooks derived from later Roman writers of the fifth–seventh centuries, such as Boethius, Capella, and Isidore of Seville. Textbooks attributed to Boethius were used in English universities until the sixteenth century (Howson 1982, p. 1). This course of geometry begins with axioms and postulates; however, its contents are those of Books I to IV of Euclid, with very few proofs and some rules for the measure of simple geometric figures. After the time of Gerbert (eleventh century), who served as a head of the Cathedral school in Reims for some time and later became Roman Pope Sylvester II, the teaching of geometry became even more practical. The textbook of practical geometry *De arte mensurandi* by Jean de Murs, who was a master of the Sorbonne in Paris in the fourteenth century, was greatly devoted to problems of measuring or division of figures.

Several authors of the sixteenth century, interested not only in the practical but also in the deductive aspect of geometry, criticized the demonstrative discourse of Euclid. So, in *Scholarum mathematicarum* of 1569, Pierre de la Ramée (Petrus Ramus) criticized Euclidean logic, principles, and order of propositions.

Christoph Clavius introduced mathematics to the Jesuit Colleges, considering it necessary for the general culture (Dainville 1954). In his Latin *Elements* of 1574, Clavius started from the Greek version, translated by Commandinus some years before, but he doubled the number of propositions, modified the proofs, developed the constructions, and completed them with several annotations. He used a mixed approach with both quantities and numbers to explain the geometrical propositions more clearly.

In 1550, Johann Scheubel, in his six books printed in Latin in Bale, used the symbolism of German algebraists. His *Elements*, printed until 1612, was adapted into Jesuit teaching for numerous years (Romano 2006). Jesuits also wrote textbooks devoted to geometry's practical uses and in vernacular languages. For instance, in France, Pardies edited his *Elements of Geometry* "where by a short and easy method we can learn all it is well to know from Euclid, Archimedes, Apollonius and the best inventions of Ancients and Modern Geometers"[1] (1671, preface); Milliet Dechalles wrote his *Elements of Euclid,* in which he gave many applications to gnomonic, optics, or astronomy (1682). At the beginning of the eighteenth century, some Jesuits were teachers of hydrographics in schools devoted to the marine (Lamy 2006).

[1] This and subsequent translations are by the authors unless indicated otherwise.

2.2 Geometry Teaching from Universities to Secondary Schools

It is not easy to define the level at which geometry was taught in schools in the Middle Ages and Renaissance. The practical geometry of Fibonacci, for instance, was taught – along with his arithmetic – in the last grades of the Italian *Scuole d'abaco*, which were parish-schools for pupils who wanted to learn a trade, as well as in the universities. The *Sphaera* of Sacrobosco – a treatise on the circle and the sphere linked to the teaching of celestial bodies – was taught from the end of the thirteenth century in English universities, the access also possible to students of 14 years of age, while in grammar schools there was practically no geometry teaching (Howson 1982, p. 3). Also in continental Europe after the thirteenth century, schools interested mainly in educating employees or clergymen reduced geometry to the sort of popular astronomy represented by the "Sphere" (Günther 1900).

In the sixteenth century, the first five or six books of Euclid were regularly used in European universities. Some elementary teaching was then transferred to the secondary level. The transference of Euclid from the universities to the secondary schools lasted about 200 years.

The Jesuit colleges represented the first form of extended secondary schooling in European Catholic regions, with an organized program of mathematics, as defined in the *Ratio Studiorum* (1599) and including the reading of Euclid's Elements and a part of the *Sphere*. The scholarly version of Clavius' Elements was substituted by the *Elementa geometriæ* of André Tacquet (1654), which concerned plane and solid geometry and also provided some of Archimedes' theorems. This text remained for over a century as the most important reference for teaching both in the last grades of secondary schools (not only those run by Jesuits) and in universities.

Notwithstanding the contents of Tacquet's text, little time was devoted to geometry in the *Ratio Studiorum* and thus in the Jesuit colleges. At first, we find in secondary schools (mainly in the Catholic regions) only propositions of the first book of Euclid supplemented by practical exercises. In Germany, texts mixing logical and practical geometry were used at both the university and secondary level, as those by Sturm (1699) and later by Kaestner (1710). In the second half of the seventeenth century, the teaching of practical geometry was done in some Protestant schools, such as the Gymnasium in Erfurt, which used a text by Schroeter containing elements of surveying. Euclid's *Elements* were introduced in successive years but only "with ocular demonstrations" (reported by Walker Stamper 1909, p. 64). Tacquet's *Elements* were explained together with field practice in the Pädagogium of Halle in 1702. Generally, the Protestant countries in Northern Europe, such as Holland, emphasized the practical side of teaching geometry, and the passage from the university to the school level occurred there some decades later than in Germany. In most European countries, the teaching of geometry as a general course in secondary schools appeared only during the nineteenth century. This development was supported by the influence of military schools (and in some cases by the expulsion of the Jesuits) or forced by the requirements for university admission (see Sect. 4).

3 The Influence of Cartesian Geometry

René Descartes, a former student of the College of Jesuits in la Flèche, published *La Géométrie* in 1637. This text was not a new version of the *Elements* but aimed to furnish a universal method to "solve all the problems of geometry" (Descartes 1637, p. 2). Descartes's approach differed from other methods of presenting geometrical knowledge proposed in the sixteenth century, like that of Petrus Ramus. For his purpose, Descartes considered that all figures could be seen as composed of segments so that simple arithmetical operations (addition, subtraction, multiplication, division, and extracting roots) can be applied. It is an arithmetization of geometry that permitted Descartes introducing the

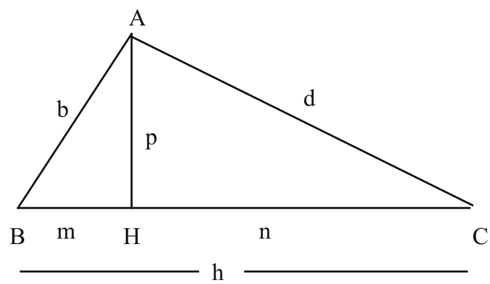

Fig. 23.1 The theorem of Pythagoras by Arnauld

equations of geometrical curves (as he called it). This systematical arithmetization of geometry, which requires the introduction of a unity segment, greatly influenced geometrical teaching.

3.1 The Cartesian Influence on the "New Elements of Geometry" in the Seventeenth Century

Les Nouveaux elements de Géométrie of d'Antoine Arnauld was intended for Jansenist teaching at the convent of Port Royal. Arnauld wrote that "Euclid's *Elements* are so confusing and muddled that are far from giving to the mind the idea and the image of the true order; on the contrary they accustom it to the mess and the confusion" (Arnauld 1667, preface). The main novelty consisted in following "the natural order" promoted by Descartes in his *Règles pour la direction de l'esprit* which went from simple figures to compound figures. After treating magnitudes, Arnauld examined straight and circular lines, angles, proportional segments, figures made of lines, and then surfaces.

The "natural order" required new proofs, particularly for the results which Euclid obtained by using areas. To prove properties of segments without using angles or delimited areas, Arnauld introduced a "theory of perpendicular and oblique lines," which was taken up again in the schoolbooks of Lacroix (1799) and in French schoolbooks until the 1950s (Barbin 2009a). To establish his theory, Arnauld called upon "obviousness," which is very far from the Euclidean spirit but is in accordance with the Cartesian one. For instance, he asserted that, by only considering "the nature of the straight line" (Arnauld 1667, p. 87), if two points A and B have equal distances from two given points, then all the points of the straight line defined by A and B have this property.

Arnauld's proofs of Pythagorean theorem and of the "intercept theorem" (about the ratios of line segments that are created if two intersecting lines are intercepted by a pair of parallels) replaced those of Euclid in many books until the twentieth century, for example, in Amiot (1857) and Clairaut (1741), but also in such schoolbooks from the years 1950–1960, as Lebossé and Hemery (1962). Arnauld proved the Pythagorean theorem by using the arithmetization of geometry (Arnauld 1667, p. 294). He considered a right triangle ABC with height AH (Fig. 23.1) letting AB = b, AC = d, AH = p, BH = m, HC = n, and BC = h. There are three similar triangles in the figure; proportions obtained from their similarity give the following equalities: pp = mn, dd = hn, and bb = hm. He concluded that bb + dd = h m + hn = h(m + n) = hh.

In this proof, the visual obviousness is replaced by the calculation on magnitudes utilizing algebraic symbolization (Barbin 2010). Bernard Lamy followed this method in his *Éléments de géométrie ou de la mesure des corps*, seven editions of which were published from 1685 to 1758. He referred to Arnauld's book but moved away from the strict order followed by his predecessor. This Oratorian priest pursued the algebraization of geometry, proposed by Descartes, and offered geometry without any figure. He devoted a book to the Cartesian method of problem solving, praised as a "method of invention" opposed to a "method of doctrine (Lamy 1685)."

4 The Teaching of the "Elements of Geometry" in the Eighteenth and Nineteenth Centuries

After the criticisms of the seventeenth century concerning the proofs and order of Euclid's *Elements* and after the novelties introduced by Descartes, several geometers of the following two centuries wrote new "Elements of Geometry." Directly after the French Revolution of 1789, two important French textbooks appeared. Their new spirit was perceived throughout Europe, as shown by their numerous editions and translations. They marked the end of the monopoly of religion and Church in teaching but also the beginning of the new industrial era (Schubring 2007).

4.1 The "Elements of Geometry" of Legendre

In 1794, Adrien-Marie Legendre wrote a book similar to Euclid's *Elements,* but one in which he sought "to simplify and to repair" the Greek text (Legendre 1794, preface). All the later editions contained an extensive part on plane and spherical trigonometry.

Legendre's definitions of the words "theorems" and "problems" are typical: "a theorem is a truth which becomes obvious by means of a reasoning called proof. A problem is a proposed question that demands a solution" (Legendre 1794, p. 4). The call for obviousness is Cartesian, but here it concerns reasoning and not a method. The term "problem" took a scholarly meaning. Like Arnauld, Legendre arithmetized the geometrical magnitudes and avoided the difficulties of proportions.

The beginning of the book differs from that of Euclid. Legendre gives only five axioms instead of the various postulates and common notions. The first proposition is a Euclidean axiom, which states that "all right angles are equal"; it is proved by using the notion of distance. Legendre's work consists of eight books, the first four of which follow the Euclidean order: triangles and quadrilaterals, circles and measure of angles, proportions of figures, and regular polygons. Book VIII is devoted to "the three round solids" (cylinder, cone, and sphere).

The changes in the various editions essentially concerned the proofs. From the first edition (1794) to the twelfth edition (1823), Legendre tried to prove the "axiom of parallels" and gave four proofs, which were unsatisfactory (Bkouche 2007). In his notes, Legendre emphasized that the terminology was intended to "give more exactness and precision to the geometrical language." This aspect probably charmed the teachers, since the *Elements of Geometry* was a great publishing success.

An important novelty was the introduction of the idea of "symmetry," which appeared in the first version of 1794. Here, Legendre introduced solid objects which were equal (congruent, in modern terminology) but could not be superposed one onto another (Hon and Goldstein 2005). Legendre also defined symmetric polyhedra and listed their properties.

From 1845, the *Elements* of Legendre were followed by the *Éléments de Géométrie de A.M. Legendre avec Additions et Modifications par M.A. Blanchet*, which was the object of several editions over 40 years, although the book was strongly criticized (Schubring 2007).

4.2 The Elements of Geometry of Lacroix

At the end of the eighteenth century, Silvestre François Lacroix edited the *Éléments de Géometrie* intended for the École Centrale of Paris (1799), a school created just after the Revolution. His book was written in a climate of rivalry with Legendre's book (Schubring 1987). The success of Lacroix's book, as well as the success of some other French books of the earlier nineteenth century, is due to the influence and export of the model of the *Grandes Écoles* (specifically, the *École Polytechnique*).

In the "Preliminary discourse" of his book, Lacroix discusses the order to be followed in the *Elements of Geometry*, the approach to write them, and the method to be used in mathematics. He presented himself as an heir of Arnauld and of Alexis Clairaut; at the same time, he adopted a sensualist conception of geometrical knowledge: He quoted from *Traité des sensations* of Condillac and also *Emile* of Jean-Jacques Rousseau. According to Lacroix, geometrical ideas are not extracted from the imagination but come from nature.

Lacroix's opinion about proofs was very eclectic and he proposed to provide different kinds of proofs to students. Again, in the "Preliminary discourse," he gave many explanations for the distinction between synthesis and analysis. He defined them not in the sense of the Ancients but (like Descartes and Arnauld) as ways to move from the simple to the compound and vice versa. According to Lacroix, it is important to use both ways. For him, the choice of a proof is a question of style, because what is important are the "connections between ideas" as well as the art of reasoning (Barbin 1991). He was opposed to the excess of theorization and abstraction and encouraged students to practice the physical sciences, which offer numerous and useful discoveries.

All of Lacroix's proofs of the three theorems on the equalities (congruence) of two triangles use the method of superposition. Then, he introduces Arnauld's theory of perpendiculars and obliques. The proof of the theorem about the sum of the three angles of a triangle entirely follows the Euclidean proof. This eclecticism is witness to a certain lack of interest in the rigor of axiomatic deduction inherited from Euclid. It is one key difference from Legendre, with his numerous attempts to prove "the axiom of parallels."

4.3 The International Diffusion of Legendre's and Lacroix's Books

Legendre's book was translated into Italian in 1802 and into English in 1825; it was published in Germany from 1822 to 1858. The translation adapted by Charles Davies for American teaching in 1828 contained an additional chapter on the theory of proportions and paid particular attention to the treatment of irrational numbers; it was used in West Point (Rickey and Shell-Gellash 2004). The Portuguese translation was published in Rio de Janeiro in 1809, 1812, and 1815.

Legendre's *Elements of Geometry* influenced the teaching of geometry in many countries (Schubring 2007), interestingly including Greece (Kastanis and Kastanis 2006) and Egypt. This seems paradoxical since Euclid's *Elements* had been written in Greek and the Arabic edition by Nasir al-Din at-Tusi was printed before 1801. However, Legendre's book was the object of two Arabic translations, and several editions of it were published in Egypt between 1839 and 1872. This success facilitated the adoption of descriptive geometry in Egypt (Crozet 2009). The criticisms of and objections to Legendre's *Elements* were numerous in the nineteenth century, but the book always remained a standard reference (Thibault 1844).

Lacroix's textbook became known in England and the USA fairly soon after its publication. For instance, it is mentioned in *Elements of Geometry* by John Radford Young (1827) and James Hayward (1829). But it did not meet with the same success as Lacroix's earlier book *Treatise of Trigonometry* (1798), which was translated into English in 1820. The influence of Lacroix's practical spirit can be found in the textbook by Ralf Pomeroy, entitled *The Engineer's Practical Elements,* edited in Philadelphia (1836).

Lacroix's *Elements of Geometry* was translated into Spanish by Jose Rovello y Morales and published in Madrid (first edition in 1807). It was also published in Buenos Aires in 1821 by the "Impresa de la independencia"; the complete title noted that the book was written for the use of "our youth." Finally, *Elements of* Geometry was translated into Italian in Florence (first edition in 1813).

5 Developments in School Geometry in the Nineteenth Century

5.1 "Advanced" Contents for School Geometry in the Technical Instruction

The development that had occurred in revolutionary France in descriptive geometry influenced the emergence of new mathematical subjects and their teaching. Gaspard Monge, the author of *Géométrie descriptive* (1799), was strongly involved in developing the initial conception of the *Ecole Polytechnique* and played a considerable role in its creation (Grattan-Guinness 2005). The military and civilian needs of well-prepared engineers and administrators which required the creation of the *Ecole Polytechnique* were common to many countries. The *Ecole* influenced the development of studies of a technical nature in nineteenth-century Europe and North America; for example, *Technische Hochschulen* in Germany (Schubring 1989, p. 180), *Politecnici* in Italy, and military schools in the USA (Simons 1931) were all created to address the needs of their students in technical studies.

This development contributed to the transfer of the theoretical and practical aspects of new mathematical subjects (such as the same descriptive geometry) to "modern" or technical secondary schools. In 1859, for instance, in the "Realschulen" of Prussia, we find descriptive geometry, analytic geometry, and studies of conic sections along with static and mechanics (Schimmack 1911, p. 4). It is interesting to note that subjects that eventually would have important developments within mathematics first entered secondary instruction via technical schools which had a relatively minor rank in the social hierarchy of educational institutions.

The introduction at the secondary level of *projective geometry*, the so-called new geometry, was particularly relevant in Italy, where new syllabi for mathematics and physics were issued in 1871 for the technical institutes. The programs – in keeping with the synthetic approach developed in Italy – included involution, the duality principle, and the projective properties of conics and spheres. The parallel teaching of descriptive geometry included central projection, the projective properties of figures, and collinearities with attention to homology, all the way up to the construction of intersections of surfaces of the second degree.

The textbook following this syllabus, written by Luigi Cremona (1873), was translated in many countries (Germany, France, England), but only in Italy was it used at the level of secondary school (Menghini 2006).

The Italian curricula of projective and descriptive geometry were reduced at the end of the nineteenth century and became more aligned with those of secondary technical instruction throughout the rest of Europe. Now projective geometry consisted of the study of projective ranges and pencils, of harmonic ratios, and of projective relationships in a circle. Descriptive geometry was restricted to orthogonal and central projections, which were taught together with equalities, similarities, affinities, and perspectivities.

Teaching in technical schools also influenced teaching in other institutions. At the turn of the nineteenth century, elements of projective geometry could be found in many European gymnasia or lycées, like the treatment of harmonic and anharmonic groups, polarity, and the projective properties of conic sections (Henrici and Treutlein 1881; Sannia and d'Ovidio 1869).

In higher technical instruction, *conic sections* were also initially treated within *pure* projective geometry, but soon this treatment became analytic and spread to all the universities. It rapidly passed into secondary instruction, mainly in its synthetic treatment. By contrast, *analytic geometry* (in the sense of the geometry of coordinates) entered schools very late and for a long time primarily in technical instruction (see Sect. 4.2). The teaching of analytic geometry began in polytechnics and military schools and was introduced for three-dimensional space (Pepe 2004). The use of this analytic technique in space was in fact the real motivation for its introduction as a method of study. Analytic geometry was used for the study of curves and surfaces, as in *Essai de géométrie analytique* (1805) by Jean-Baptiste Biot.

Spherical trigonometry and its applications also passed from technical higher institutions to technical secondary instruction; some elements of spherical trigonometry could be found in certain non-technical institutions as well as gymnasia and lycée because of its applications to geography. According to Simon (1906, p. 2), the treatment of trigonometry at the turn of the century had been widened to an extent "that would have been sufficient for an astronomer in earlier periods."

5.2 The Analytic Approach to Geometry: The Case of Conic Sections

The first entrance of "modern" topics in secondary instruction, such as those mentioned in Sect. 5.1, occurred in Germany in the 1800s and not only in technical schools. Debates over their introduction in the gymnasia would last the whole century and were interwoven with frequent rearrangements of the syllabi in the different German states (lands). One can understand how early German teachers began to introduce these topics, considering that in 1834 the inclusion of spherical trigonometry and conic sections in the school curriculum, which had been tolerated until then, was explicitly forbidden by the Ministry of Prussia (Schimmack 1911, p. 4). The Prussian syllabi sufficiently influenced teaching in other German states so that in the following years we find the teaching of conics in its synthetic treatment only in some gymnasia. Alongside the official syllabi, textbooks circulated during that period for both gymnasia and Realschulen, which presented conic sections as well as analytic geometry (Bretschneider 1844; Schellbach 1843). Bretschneider's text even presents equations of translations and rotations.

The debates slowly resulted in the complete acceptance of conic sections in the German gymnasia, which became a standard subject with a mainly synthetic treatment. Conic sections also appeared in schoolbooks in other countries (Amiot 1857; Cremona 1873; Borel 1905). The definition was mostly given as loci; the relations between the lengths of the segments were described with letters, but the treatment was not analytic. For instance, tangents to the conic sections were constructed geometrically.

Though not intended as a schoolbook, the *Treatise on Conic Sections* written by George Salmon in Great Britain in 1848 (enlarged in 1855 and translated into French, Italian, and German) introduced "junior readers" to the first elements of analytic geometry (Salmon 1855).

There was, however, a certain resistance to analytic geometry, which was still sometimes considered a topic for higher studies. In the 1880s, those German states that had previously accepted analytic geometry – Sachsen already in 1847; Hessen in 1877 – removed it from the syllabi for the gymnasia. This had the likely aim of eradicating the increasing use of analytic methods in teaching conics. We note, however, that in Austria, analytic geometry was introduced in 1849 and reapproved with some methodological suggestions in 1884. In Italian technical schools at the end of the nineteenth century, the use of analytic geometry was recommended as a tool, even though it was not listed among the contents.

In the final examinations of the gymnasia in Baden (Treutlein 1907; Schönbeck 1994), exercises required the construction of the conic sections given – for example, a focus, the length of the greater axis, and two points. The same examination reserved analytic geometry mainly for the *Oberrealschulen*.

The issue with the introduction of analytic geometry related to the fact that it was mostly seen from the point of view of equations defining a locus rather than from the functional point of view of dependent variables and graphical representation. It was considered an essential tool for the introduction of calculus (Laisant 1907) but less important for the initial study of geometry.

In France, there were three types of secondary mathematical teaching in the second part of nineteenth century: the "classical" teaching (for schools where Latin and Greek were core subjects), the scientific classes of the lycées (which prepared for the examination of the Grandes Écoles), and "special teaching" (later on, "scientific teaching"), which contained more mathematics than "classical" and was intended for future middle-level technicians (Belhoste 1995). In the 1870s, Jules Dufailly edited three schoolbooks of geometry, which corresponded to each program: The first one was in the style of the Euclidean *Elements* and was intended for lycée scientific classes (Dufailly 1874); the two other books contained a section on analytic geometry. In the second part of the century, schoolbooks appeared in which analytic geometry was treated autonomously (one example is the *Lessons of Analytical Geometry with Two or Three Dimensions* of Roguet 1854). They were intended for classes in secondary schools as preparation for the baccalaureate and for examinations in polytechnic or normal schools (Barbin 2012).

5.3 The Search for New Methodologies in Geometry Teaching: Teaching via Problems

From the beginning of the nineteenth century, the space assigned to problems in geometry textbooks increased. Additionally, schoolbooks appeared devoted only to problems and their solutions. In France, this can be explained by the existence of entrance examinations for the *Grandes Écoles*, which consisted of problems. Two other reasons triggered the interest in problems in teaching, however: First, textbook authors believed that problem solving was a good way for students to gain understanding. Second, the problems were found to be a good tool for teaching the relatively new geometrical methods introduced in the nineteenth century.

The first reason had already manifested itself in 1741 in *Éléments de géométrie* by Clairaut, which was reprinted in eight editions until 1920 and influenced many authors, including Lacroix. In his preface, Clairaut criticized Euclid's *Elements* because they begin with a great number of preliminary principles, "which seem to promise nothing but something dried." He wrote that he preferred to engage his readers in solving problems so that "the beginners will easier acquire the spirit of invention" and will remain interested (Clairaut 1741, preface).

The book did not follow a logical order but an order of proposed geometrical inventions: The notions were introduced when they become necessary to solve a problem (Barbin 1991). Thus, Clairaut defined a perpendicular line as an answer to a problem of finding the distance between two sides of a river. Then, he introduced rectangles and squares because the construction of civil works, like ramparts, canals, or streets, required drawing parallel straight lines, defined as lines the distance between which was measured everywhere by perpendiculars of the same length. Therefore, the notion of perpendicular was linked to the idea of distance. The triangle came later on when an area had to be measured. The notion of angle was introduced when an obstacle (a hill, a wood or a pond) prevented measuring the side of a triangle. But, as Clairaut wrote, his textbook did not concern practical geometry.

Clairaut's book received a large audience, mainly in the nineteenth century. A Latin translation was published in 1749, followed by Italian from 1751 to 1885, German from 1753 to 1789, Dutch in 1760 and 1792, Polish in 1772 and 1856, and five editions in English between 1836 and 1851. The book was included in the French official curricula in 1852 but was removed 10 years later. In Italy, the text was largely adopted by technical schools until the end of the nineteenth century (Castelnuovo 1946).

The second reason mentioned above represents the intent to teach the pure (or synthetic) geometrical methods of the nineteenth century, which had been invented to solve problems in a more elegant and easy way than the analytic methods. This point of view was adopted in *Examen des différentes méthodes pour résoudre les problèmes de géométrie* written in 1818 by Gabriel Lamé (Lamé 1818), a young student of *École Polytechnique*, and devoted to entrance examinations for the *Grandes Écoles* (Barbin 2009b). Throughout the century, many schoolbooks used problems to teach these methods and theories, such as that written by Desboves (1875). The second part of the textbook was entirely devoted to "the methods to solve problems," such as theories of transversals or homothetic figures.

Transformations were also essential in the geometry developed in the nineteenth century as a tool to extend the theorems to new situations. In 1866, Julius Petersen from Denmark wrote a textbook in which he solved problems by means of translations, symmetries, and rotations. This book achieved rapid success in the Scandinavian countries and was translated into English, French, German, Italian, and Russian. In the preface of the 1879 edition, Petersen wrote that the book had to be useful not only for elementary geometry but also as preparation for the study of modern geometry (Petersen 1879).

Some schoolbooks were composed as sequences of problems, such as those by Frère Gabriel-Marie: The sixth edition of 1920 began with an outline of geometrical methods and then continued with 2,000 solved questions (Frère Gabriel-Marie 1920). In this edition, one part was devoted to "the geometry of the triangle." This geometry started in the 1870s, when different authors in France, Great Britain, and the USA (Casey 1885; Rouché and de Comberousse 1883; Coolidge 1916) introduced new objects like the Lemoine point, the Brocard circle, or the older Simson line. The level was that of an upper class in a secondary school or college level; the aim was not so much that of teaching new theorems but of showing new methods – such as the geometry of correspondences or the trilinear coordinates – but limited to simple figures (Romera-Lebret 2009).

5.4 Discussions About the "Elements"

In the last decades of the nineteenth century, manuals that rivaled Euclid's appeared in Great Britain (Moktefi 2011). The "Association for the Improvement of Geometrical Teaching," whose first president was Thomas A. Hirst, was even created in 1871 to contrast the age-long tradition of having the *Elements* as the principal textbook for teaching geometry in British universities, colleges, and, later on, schools.

In the same period, in 1867, Euclid's *Elements* were introduced as the only geometry text in Italian gymnasia and lycées. This decision aimed to free the newly unified country from foreign textbooks, such as Legendre's or Mocnik's (1846), which were considered by Italian mathematicians to lack adequate rigor. The decision caused a debate over whether to favor or oppose Euclid that became interwoven with the British one (Menghini 1996).

These debates, which were not limited to Great Britain and Italy, mainly concerned the question of the purity of Euclidean geometry. The teaching of synthetic geometry was considered gymnastics for the mind that could only be fostered by a system of propositions as those set by Euclid. Objections to Euclid concerned exactly this closed system, which did not permit the use of algebra to work with measures of segments, nor did it even allow one to say that a straight line marks the shortest distance between two points. Purists affirmed that the use of algebraic means would hinder pupils in using their intuition and finding a simple solution to a problem. To the contrary, their opponents considered it wrong to prevent pupils from using all available instruments to solve a problem (Wilson 1868). Notwithstanding the fear of abandoning a long and well-established tradition, the protests against Euclid had a greater effect in Britain (see Sect. 6) than in Italy, where attention to the foundations of geometry brought about only slight changes in textbooks which always had to follow the "Euclidean method (Barbin Menghini Moktefi 2013)."

As in Great Britain, the study of Euclid's Elements in the USA was lowered from universities to secondary schools, as forced by the requirements for university admission. This occurred only in the second half of the nineteenth century. The requirements were not as rigid as the British ones, for example, Harvard required two books of Playfair's *Euclid* in 1855 (1795) and later all plane geometry, but without explicit reference to Euclid (Walker Stamper 1909, p. 97).

6 The Twentieth Century: The International Reform Movements

At the beginning of the twentieth century, most countries recognized the need for new methodologies in geometry teaching. The dominant thought was a fuller consideration of the child's mind and the recognition that school curricula should be built upon pedagogical principles (Young 1920, pp. 1–8; Howson et al. 1981, p. 29). An introductory (propaedeutic) course of geometry based on intuitive or experimental considerations and on drawing before starting the treatment of demonstrative geometry was considered essential. Such an intuitive approach allowed geometry teaching to begin in what we now call middle school (Treutlein 1911, pp. V-VI; Howson 1982, p. 148). These ideas had a prelude in the nineteenth century but now received strong international interest as a result of mathematicians becoming involved in elementary teaching.

6.1 "Experimental Geometry"

In 1901 John Perry delivered a speech at the British Association for the Advancement of Science, emphasizing the educational value of experimental procedures in the first approach to Euclidean geometry. He proposed that a larger part of elementary geometry be assumed as axiomatic and that the subject matter be taught with reference to its utility and to the interest of the child (Perry 1901). In the USA, Eliakim H. Moore (1903) appreciated Perry's emphasis on the practical side of mathematics. He stressed the need for mechanical drawing and graphical methods, calling for a laboratory method that develops the spirit of research in the students.

In 1905, with the influential support of Felix Klein, the *Meraner Lehrpläne* (reprinted in Klein and Schimmack 1907) proposed that geometrical teaching would start at age 11 with the observation of objects of everyday life and continue till age 13 with the study of the basic concepts of figures and with the use of rulers and compasses to make constructions and measurements; the theoretical study of the properties of the figures would cover ages 13–17. Peter Treutlein had also constructed concrete models for geometry teaching, even if mostly for the post-secondary level (Schönbeck 1994, p. 61).

Experimental geometry, however, remained linked to Perry's name. Perry's movement became known throughout the entire world, from Japan (Siu Man Keung 2009) to the USA, and gave rise to many curricular changes. Also, the French reforms of 1905, based on the approach of the textbook written by Méray (1874), shared much in common with this movement.

Nevertheless, Perry's method was accused of giving insufficient emphasis to "reasoning" (Howson 1982, pp. 149–151) and so was considered suitable only for the lower grades (ages 10–13). In any event, there were significant exceptions in many countries, namely, the book by Godfrey and Siddons (1903) in England, the text by Borel (1905) in France, and further editions of the book by Henrici and Treutlein of 1881 in Germany. In the USA, the teaching of deductive geometry began later than in most of the European countries, anyway (Smith 1913, p. 250).

6.2 Geometrical Constructions

In the *Ecole Polytechnique*, Monge assigned great importance to the union of geometry with drawing, which he considered an important means of *communication* for engineers. Also the entrance examination demanded for the construction, with ruler and compass, of algebraic entities (Paul 1980, p. 113). In some countries (e.g., Italy), *geometric drawing* became an independent topic derived from descriptive geometry, with rules for constructions mostly given without corresponding theorems.

With experimental geometry, drawing acquired a didactical role, helping the pupil to become familiar with a geometric figure and its main properties (Smith 1913, p. 240). Borel proposed the creation of laboratories to ensure the connection between geometric drawing and geometry teaching (Bkouche 2003).

In Germany, the curricular reforms of the beginning of the century assigned a role to geometrical constructions and not only in the lower grades. Often in German books, theorems were interpreted from a constructive point of view: For instance, the theorem according to which two triangles are equal if they have three equal sides can be reformulated in the sense that we can construct exactly one triangle if three of its sides are given.

6.3 The Geometry of Transformations

We have seen that geometric transformations were included in descriptive geometry in the nineteenth century. From the middle of the 1800s, geometry of motions was a school matter in different countries (Bretschneider 1844; Méray 1874; Henrici and Treutlein 1881; Frattini 1901; see also Houel 1867). Isometries (translations, rotations, and reflections) were used to introduce congruence and to discover the properties of geometric figures.

The credit given to Klein's Erlanger Program of 1872 for fostering the teaching of geometry of transformations in school does not find unanimous support among scholars (Bender 1982; Struve 1984). Isometries were not considered in the Erlanger Program; moreover, groups were seen as a formal means of classifying geometries and not as a "working tool *within* a geometry" (Freudenthal 1973, p. 108). When reconsidering the Erlanger Program in his *Elementary Mathematics from an Advanced Standpoint*, Klein (1909, *Zweiter Hauptteil*) defined *affine geometry* (which studies the invariants of an affine transformation) and *metric geometry* (which studies the invariants of the so-called principal group, i.e., the group of similarities) and underscored that the invariants of a group correspond to those concepts that are meaningful in the related geometry. These ideas did not enter schools before 1930 and often not until the 1960s, when they merged with the concepts of structural geometry.

6.4 The Issues of the International Commission on Mathematical Instruction (ICMI)

As president of the ICMI from its foundation in 1908, Klein succeeded in extending to an international level the reform movement started in Germany. Under the banner of "functional thinking," the reform aimed to include in the secondary school curriculum such topics as analytic geometry and calculus (Schubring 1987). ICMI's enquiries also investigated other questions referring to geometry.

Rigor and Intuition. At the ICMI Congress in Milan in 1911, Guido Castelnuovo presented a report on the teaching of geometry in the various countries which made use of the work of Walther Lietzmann, specifically the classification that Lietzmann had proposed in order to introduce a preliminary discussion (Fehr 1911, p. 461). According to Castelnuovo's description, one can broadly affirm that Great Britain, France, and Germany were shifting from a Euclidean method – in which some of the axioms were given empirically and the subsequent development was logical – towards a method that alternated intuitive and deductive reasoning. Italy, on the other hand, was shifting towards a more rigorous method in which all necessary axioms were stated, even if on an empirical basis. At the ICMI Congress of Cambridge, David Eugene Smith (1912) stressed the role of intuition and experience in the first stages of child education and emphasized that the most important question was "how much of the teaching of geometry in secondary school should be inductive?"

The Fusion of Plane and Solid Geometry. The idea of fusion originated from projective and descriptive geometry, which worked with projections in space and sections. Different textbooks (starting from Bretschneider in 1844 to Méray in 1874/1903; de Paolis in 1884; Lazzeri and Bassani in 1891, also translated into German by Treutlein in 1911) adopted this idea, mixing plane and solid considerations. For instance, the chapter on the properties of incidence also referred to the mutual position of a plane and a straight line, while homothety was defined in space and then on the plane. Pupils were supposed to have a better intuition of spatial relations when passing from space to plane and to reason by analogy. Moreover, proofs could be presented of plane theorems using projections in space of simple known configurations.

This important methodological question was also considered at the ICMI Congress of 1911 – within the more general theme of the fusion of different branches of mathematics – by giving examples of successful textbooks.

7 The Heritage of Axiomatic and Structural Mathematics

7.1 New Axioms for Euclidean Geometry

At the end of the nineteenth century, research on the foundations of geometry, particularly David Hilbert's *Grundlagen der Geometrie* of 1899, brought some changes to the axioms listed in textbooks. Some mathematicians tried to improve Euclid by using new axioms of superposition to prove the equality (congruence) of geometric figures, that is, by creating new axioms to characterize movements (Sannia and D'Ovidio 1869; De Paolis 1884). Other authors assumed equality between certain objects (generally segments and angles) using the concept of one-to-one correspondence (Veronese 1897; Halsted 1904, also translated into French). But the recognition that many axioms were in fact tacitly used and that it is impossible to be completely rigorous in school resulted mainly in the maintenance of geometry in Euclid's way, although also very different approaches were developed (Young 1920, p. 198). Importantly, discussions about non-Euclidean geometries which happened at this time resulted only in bringing somewhat more attention to the introduction of the theory of parallels.

In the 1930s, American mathematicians, including David Birkhoff, proposed new systems of axioms for Euclidean geometry that avoided not only Euclid's superposition but also Hilbert's "betweenness." Congruence was established by means of measure, that is, of a correspondence between segments and real numbers. The first didactic presentation of these ideas was made by Birkhoff and Beatley, in their high school text *Basic Geometry* (1941). The authors postulate that points on a line and rays through a point can be put in a one-to-one correspondence with real numbers so that the difference between two of these numbers indicates the measure of a segment or of an angle. By adding an axiom stating that two points determine a line and an axiom stating the conditions of the similarity of two triangles, all the basic theorems of Euclidean geometry can be deduced. The pupil's work was supposed to be aided by the practical use of a scale and a protractor. Birkhoff's approach shows that synthetic proofs can also be performed using an intuitive notion of real numbers, which guarantee the possibility of comparing segments and angles. These ideas slowly found their way into American teaching and inspired many other textbooks on geometry, such as those produced by the School Mathematics Study Group.

The structuralist and formalist vision of geometry, starting with the work of Hilbert, culminated, as far as school geometry is concerned, in reforms that originated in the work of the group of mathematicians created in 1932 and presented under the name Bourbaki.

7.2 The Effects of Bourbakism

The first influences of Bourbakism in mathematics education manifested themselves in 1950, when Caleb Gattegno founded the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement Mathématique*. This group consisted of individuals from different backgrounds (Furinghetti et al. 2008), among whom were the mathematicians Jean Dieudonné and Gustave Choquet. Members of CIEAEM played an important role in 1959 at the Conference of Royaumont (Paris), which gave rise to reforms in many countries. This conference, chaired by the ICMI president Marshall Stone, was organized by OECE (now OECD), an international body of 18 European countries. The USA, Canada, and Yugoslavia were also invited. In Western countries, the so-called Sputnik shock had turned the attention of politicians to educational systems, and therefore, the conference had the concrete aim of producing curricular reform that was not limited to Europe. The various contributions of the conference show that it was strongly influenced by the French mathematical school. Under his famous slogan "A bas Euclide," Dieudonné proposed that the treatment of geometry should start from the assumed as given knowledge of real numbers, followed by the definition of rules for operations on a set of non-defined objects (basically, defining a vector space). The introduction of a scalar product would allow for the introduction of such metric relations as perpendicularity (OECE 1961). In this way, elementary geometry effectively coincides with linear algebra.

The speech of Willy Servais from Belgium, then secretary of CIEAEM, concentrated on algebraic structures and vector spaces; geometric transformations were considered an example of group structure. After Servais' speech, the German Otto Botsch objected to substituting traditional methods with an algebraic approach, which was still deductive. Botsch declared that more than half of the German secondary schools were treating geometrical transformations using a synthetic method. The treatment was inspired by Klein's Erlanger Program, even if the order was from the simplest transformations (symmetries) to the more complex. The discussion that followed, however, was centered on axiomatic structures and reduced to how the method suggested by Botsch could ensure an understanding of axioms, definitions, and theorems. Klein's Program was therefore relegated to the "back burner" (OECE 1961, p. 80).

One year after Royaumont, OECE gathered a group of experts in Dubrovnik (Yugoslavia) to produce "a modern program for mathematics teaching in secondary schools." Affine geometry had to precede metric geometry, as it was considered simpler from a mathematical point of view (Gispert 2009). In line with Choquet's proposals, the axiomatic approach did not refer explicitly to the axioms of vector spaces (OECE 1962; Choquet 1964) and instead suggested a consideration of axioms of incidence and order, axioms of affine structure, and axioms of metric structure, along with the teaching of transformations (it was recommended that all teachers be introduced to all of them; in practice, students were often taught them as well). For ages 11–15, a more intuitive approach to geometry was recommended, in line with proposals made by Paul Libois. Although the program was presented as a synthesis of proposals from various countries, it did not represent what actually happened in succeeding years. In particular, France and Belgium followed the proposal of Dieudonné.

Led by the Bourbakist André Lichnerowicz (a former president of ICMI) and run under the influence of a "pédagogie active," the 1969 reforms in France imposed the teaching of the axiomatic theory of finite-dimensional vector spaces from the first year of the lycée (age 15) (Dorier 2000), together with set theory, logic, and structures.

In Belgium, Georges Papy led a reform in which geometrical contents were abstract even at the 11–15 age level. At age 12–13, affine geometry was introduced axiomatically or by means of vectors and parallel projections. Then, real numbers and the two-dimensional vector space were introduced. Metric geometry was introduced utilizing symmetry, in the way proposed by Choquet in the appendix of his book; then, starting from 15 years, linear algebra was offered following Dieudonné's proposal. In some Italian books of the 1970s, the axioms of symmetry and distance according to Choquet (1964) were used to start from metric geometry instead of from affine geometry.

In the 1960s, commissions and pedagogical centers from all over the world were created to discuss new programs, such as the New Math movement in the USA and the School Mathematics Project

(SMP) in England. Even in countries that did not adopt a Bourbakist view, new topics appeared such as vectors, transformations, matrixes, and set theory. The SMP proposed topics that became characteristic of alternative geometry teaching in the 1960s: non-Euclidean geometries, finite geometries, topology, graph theory, "artisan" type of work on surveying, perspective and technical drawing of three-dimensional objects, and geometrical transformations and their composition (Howson 2003, see also Castelnuovo 1963). In the USA as well, linear algebra did not become the main content of the *New Math* reform, which was based instead on set theory, functions, and diagrams and included modular arithmetic, symbolic logic, and abstract algebra. There were other different projects; only in some cases, the focus was on axiomatic set theory, but often the presentation was formal and abstract and at a very early stage. In Spain, Italy, and Brazil, courses based on new topics such as set theory, correspondences, algebraic structures, logic, and vector spaces were delivered to teachers, but only in Spain did this lead to a new curriculum (Ausejo 2010). New Math in the USA lasted till the 1970s. The modern mathematics movement in Europe lasted till the 1980s.

8 The Teaching of Geometry in Elementary Schools: The Influence of Educators

At the beginning of the twentieth century, many mathematicians and educators, such as Charles Godfrey, Francisco Ferrer, and Charles-Ange Laisant, were strong supporters of the opportunity to teach geometry in the lower grades. The proposed contents were mostly confined to geometrical drawing and mensuration (see Smith 1913, p. 239; Howson 1982, p. 148) and were thought appropriate for not-too-young students. Only a few authors (e.g., Young 1920, p. 180) proposed activities expressly intended for students in elementary schools, such as "constructive and inventional geometry." Laisant led an important movement at the beginning of the twentieth century for the teaching of young children, particularly those from the lower class of society, based on visual geometry ("Éducation nouvelle"); this movement acquired an international importance (Auvinet 2013).

8.1 The Inquiry of OEEC in 1959

Annexed to the proceedings of the Royaumont Conference was an inquiry concerning the teaching of mathematics in the 18 member countries and in the USA and Canada (OECE 1961, p. 172). In questions concerning elementary schools, teachers were referred to as "Professeurs d'arithméthique," while in the general overview about the programs, there was no mention of geometry. Among the contents of the examination for admission to secondary schooling, only Italy and Portugal included geometry. Thus, we can infer that, until 1960, little attention was paid to geometry in the elementary schools of many countries: It was either not taught or the teaching was limited to the names of the principal geometric figures and the rules for calculating simple measures. However, there were *significant exceptions* which paved the way to a new pedagogy of the subject.

8.2 The Educators: The Schools of Pestalozzi, Dewey, Montessori, and Decroly

One of the first contributions of the educational sciences to schools was the introduction of Pestalozzi's *Formenlehre* in Dutch primary schools, which, at the beginning of the nineteenth century, marked the start of education in geometry at this level (de Moor 1995). *Formenlehre* was based on the drawing and verbal description of simple geometric forms. Pestalozzi's follower, Friedrich Fröbel, used concrete materials (*gifts*), in particular concrete geometric forms, to enable the pupils of the first German *Kindergartens* in the 1840s to discover their properties through play.

Interest in early childhood education grew, particularly in the USA; a leading figure in this movement was John Dewey, with his Laboratory School at the University of Chicago created in 1896, a progressive institution comprised of nursery school through twelfth grade. Followers of Dewey's idea of *active learning* were Maria Montessori, who created a school for children (*Casa dei bambini*) in Rome in 1907, and Ovide Decroly, who in the same year created the École de l'Ermitage in Bruxelles, where he used the pedagogical methods developed since 1901 with children who had minor disabilities.

These schools, apart from being places of didactical research and often devoted to children with particular needs, notably influenced future teaching in the lower grades. The main mathematical content of such experimental teaching was arithmetic, but the simple fact of handling concrete materials, of observing space and forms, and of drawing allowed an approach to geometry. In Montessori's opinion, with the study of geometry, the child makes the most autonomous discoveries (Montessori 1934): Manipulatives allow children to discover relations between the parts of a figure. In addition to the well-known pink tower (cubes with different sides to be ordered to form a tower), there were geometric forms that had to fit in the holes of a wooden tablet and iron geometric forms that could be used as a help in drawing.

8.3 The Psychology of Piaget: From the Diagnostic Level to the Educational One

Materials and experiments devoted to geometry can be found in the work of the psychologist Jean Piaget. While studying the behavior of the child in a clinical manner, Piaget developed methods that broadened the range of mathematical topics in primary school, including geometry (Sime 1973). Examples of experiences in geometry, for instance, were the well-known comparison of a quantity of water contained in cups with different heights and bases and the comparison of the areas of grass that a cow can eat in a field on which barns are added (Piaget and Inhelder 1956). These methods became widespread in elementary schools when they joined the reform movements of the 1960s.

8.4 The Projects

A direct impact on schools came from national initiatives such as the "Nuffield Project," which – inspired by the work of Edith Biggs in 1957 – encouraged the use of concrete materials in British primary schools and of laboratory techniques. As for geometry, the project started from solid geometric forms and moved on to plane figures and their transformations: symmetry, rotation, similarity, and so on (Nuffield Project 1967).

In the Netherlands, the "Realistic Math" movement led by Hans Freudenthal, with the project "Wiskobas," brought geometry back into Dutch primary schools in the 1960s, thanks to the use of rich contextualized problems. Freudenthal also strongly supported the work of Pierre van Hiele, who gave a mathematical and didactical format to the stages (already mentioned by Montessori and Piaget) performed by a pupil when passing from the perception of geometrical figures to their description and definition and on to higher levels of Euclidean deduction. Van Hiele's *levels of thought* describe the passage from the *symbol* (a figure with the range of its properties) to the *significant signal* (properties which recall the entire symbol) and from the *signal* to the *definition* (the property sufficient to distinguish a figure) (van Hiele and van Hiele-Geldof 1958). These ideas influenced curricular designs also outside the Netherlands, starting from the Soviet Union and eventually reaching the USA. The practical work of the van Hieles was at school level 11–14, but their ideas have been used for late primary education as well as for secondary education, thus transporting the contributions of pedagogy of geometry to an upper level.

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Chapter 24
History of Teaching Calculus

Luciana Zuccheri and Verena Zudini

1 Introduction

The history of teaching calculus is obviously more recent than, say, of teaching geometry. Still, it is noteworthy that shortly after the creation of calculus by Newton and Leibniz, it was taught at the university level and soon thereafter at the secondary level as well. It is not possible to present the entire detailed history of calculus teaching here, both because the size of the chapter is limited and the many sides of this history have not been until now sufficiently researched. This chapter will mainly focus on the European history (while also recommending more research and publications on the history of teaching calculus in other parts of the world). We will attempt to document some major approaches, suggestions, debates, and changes in this history since the eighteenth century to the relatively recent past, emphasizing when possible key international movements and studies and simultaneously making use of these studies. Important sources for future research including the most influential calculus textbooks will also be listed.

2 On the Major Mathematical Concepts in Calculus

The modern calculus that is taught at secondary school (traditionally divided into *integral calculus* and *differential calculus*) represents a part of real analysis (called in the past *infinitesimal calculus*), which mainly focuses on real functions of one real variable. Starting with the concepts of *real number*, *function*, and *limit* of a function, calculus is concerned with problems of *integration* and *differentiation* of a function and other topics such as the *infinite series*. Integral calculus arose from the problem of finding the area of plane figures and the volume of solid bodies. The development of differential calculus began when the tangent lines to curves and the speed of motion were considered (in both cases, a rate of change is discussed).

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These problems originated in ancient times and, with the contribution of different cultures,[1] evolved over time. In the seventeenth century, the focus was set on the close connection of these problems by establishing that integration and differentiation are the inverse of each other (see the so-called Fundamental Theorem of Calculus); in the nineteenth century, integration was not exclusively considered as the inverse operation of derivation and became used to provide a rigorous definition of area and volume (then the concept and definition of *definite integral* appeared).

The concept of *real number* is deeply connected with the concept of *continuity*, which had already been present in one sense in ancient mathematics; both were formalized only in the second half of the nineteenth century. To quantify continuous *magnitudes*,[2] the Greek mathematicians developed (utilizing a geometrical context) the *proportion theory* reported in Euclid's *Elements*, which constituted the basis for any rigorous treatment of this subject almost until the seventeenth century. The formalization of the concept of real number led to a complete arithmetization of analysis.

The concept of *limit* occurs while considering some *infinite processes*: The concept of *infinite* had been present since antiquity in mathematical and philosophical thought, together with the ideas of *infinitely large* and *infinitely small.* In the history of calculus, the distinction between *actual infinite* and *potential infinite*, already stressed by Aristotle, was crucial, especially regarding the concept of *infinitely small*. Some idea of *actual infinitesimal* occurred many times, in periods of great mathematical creativity, producing not only new results and insights but also doubt, confusion, and controversy, as, for instance, with the *method of indivisible*s. The idea of *infinitesimal* increment was also present in the different approaches taken by Newton and Leibniz, the founders of calculus. In particular, Leibniz (in his *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus*, 1684) defined the *differential* saying that dx represents an arbitrary increment of the abscissa x and dy is the corresponding increment of the ordinate y of the corresponding point on the tangent line of a curve but in performing calculations tended to consider dx infinitely small and to confuse dy with the difference of the ordinates corresponding to x and $x + dx$ on the curve itself.[3]

In systematizing the theory, mathematicians generally preferred to formalize the concepts avoiding the concept of actual infinity and actual infinitesimal. An example is the so-called exhaustion method used by Greek mathematicians (e.g., in works of Euclid and Archimedes) for proving theorems involving infinite processes and considered the model of mathematical rigor until the eighteenth century. Another example is the definitions of limit given by Cauchy and Weierstrass in the nineteenth century: Cauchy's definition of limit[4] is more dynamic than the *(ε,δ)-definition* of Weierstrass, but both highlight the infinite in a potential rather than an actual sense.

The concept of *function* is the most evident feature that differentiates modern mathematics from that of antiquity. This is a key concept in modern calculus and its use represents a sort of Copernican revolution in mathematics. The development of this concept followed another mathematical revolution produced by defining curves by algebraic equations rather than by a particular geometrical property. This approach became increasingly used from the seventeenth century after the publication of Descartes' treatise *Géométrie*.

[1] For instance, see Russo (2004) and Katz (1995). For a careful study of these subjects from historical and epistemological points of view, see Schubring (2005) who also includes extensive bibliographical information.

[2] *Magnitude* is a basic concept: We can briefly say that the magnitude of an object is "what we want to measure of it," that is, the length (of a line), the extension (of a surface), the volume (of a solid body), and the duration (of an interval of time).

[3] Leibniz considered curves defined by variables (i.e., y) depending on the abscissa (i.e., x). The slope of the tangent line was given by the *differential ratio*, which Leibniz denoted as dy/dx. This notation was called "Leibniz's notation" and was also used for the limit of the incremental ratio of a function.

[4] "*When the values successively attributed to a particular variable indefinitely approach a fixed value in such a way as to end up differing from it by as little as we wish, this fixed value is called the limit of all the other values*" (in Bradley and Sandifer 2009, p. 6).

The cinematic approach to calculus stressed the importance of the concept of *variable*. Newton (in *Tractatus de quadratura curvarum,* 1704) considered quantities increasing or decreasing during time (*fluentes*) with given speeds (*fluxiones*). Initially, the focus was set only on the relationship between all the variables involved in the same (algebraic) equation; later attention was turned to the dependence of one variable on another. A crucial point in this development (and in the teaching of this concept) was introduced by Leonhard Euler (*Introductio in analysin infinitorum,* 1748). According to Euler, function y of x was every "analytical expression in x" – that is, every expression containing powers, logarithms, trigonometric functions, and so on in the variable x. Later, the definition of a function (and its continuity) was generalized even further.

Over the last three centuries, the development of mathematical analysis has contributed not only to the radical modification of the character of mathematics itself but also to its role in the sciences. Increasingly, mathematicians became not so much interested in static single mathematical objects but rather in their variation. The concept of function created the basis for a unified treatment of different problems and produced numerous results. Analysis became a fundamental instrument for application in a variety of fields, contributing to the development of a new conception of science, particularly in physics, and to the birth of new research sectors in other fields, including economics and psychology. All these changes reinforced and further developed the respect of the roles that mathematics and science played in the society and impacted a greater understanding of their educational and cultural value.

Analysis has been taught since the eighteenth century in scientific and technical courses of higher education as well as in military schools. However, the problem of preparing students for future high-level scientific and technical studies gradually arose. This required changes in the traditional mathematics curriculum at the secondary school level, previously based primarily on Euclid's *Elements* and offering a static view of mathematical objects. A radically different approach was needed which would make central the discussion of the dynamic concept of function (see Schubring 2007).

3 The Beginning of Teaching Calculus

The first textbook on the "new calculus" was written by the Marquis de l'Hôpital (*Analyse des infiniment petits pour l'intelligence des lignes courbes,* 1696) and served to instruct the scientists in this new branch. Shortly after its publication, other textbooks appeared for higher education and, in some countries, even for secondary schools. In these countries until the eighteenth century, calculus was included in courses of scientific and technical higher education. This resulted in the preparation of several textbooks both for self-instruction and for higher education[5] which, in turn, contributed to the identification of fundamental principles of the discipline.

In France, before the revolution began, courses of differential and integral calculus had been offered in some *Collèges* (although only elementary mathematics was taught in the *Collèges* run by religious orders). Importantly, calculus courses were also offered in *Écoles militaires* which had developed since the 1750s.

The text *Leçons élémentaires des mathématiques* (published in 1741, republished and translated many times) by the Abbé Nicolas Louis de La Caille, professor at the *Collège Mazarin* of Paris University, contained a section on differential and integral calculus from the fourth edition of 1756 and onwards. Starting with a definition of "differential" as an infinitely small step, both differentiation and integration were discussed (most of the attention, however, was paid to calculations). Integration was defined formally as the inverse operation of differentiation.

[5] For information on teaching calculus at the higher level in the eighteenth and nineteenth centuries and on relevant textbooks, see Gispert (2009), Grattan-Guinnes (2009), Pepe (2009), and Schubring (2005, 2009).

This text was continued, extended, and reorganized by the Abbé Joseph-François Marie after 1768 (with many reprints and translations[6] before and after the revolution). Regarding the principles of calculus, this textbook reported the positions taken by Newton, Euler, and Leibniz, as well as d'Alembert's view about the concept of *limite* (limit).

The textbook *Institutions Mathématiques* by the Abbé Jean Sauri (published in 1770 and in press until 1834) contained a substantial section on differential and integral calculus, although this was considered by presenting the concept of *quantité infiniment petite* (infinitely small quantity) without any foundations and was devoted almost only to calculations. The textbook *Éléments de Mathématiques* (1781) written for a larger public by Roger Martin is also worth mentioning. Here, infinitesimal calculus was based on the concept of limit; the *infiniment petits* were introduced with the help of this concept, but not in a rigorous way.

The first textbook for French military schools, written in 1752 by Charles Étienne Louis Camus, did not include a discussion of calculus; unlike this one, the textbook written by Étienne Bézout contained sections on both differential and integral calculus. At first it was published in six volumes, starting from 1764, for navy schools (*Cours de Mathématiques à l'usage des Gardes du Pavillon et de la Marine*), followed by a version for artillery schools (1770–1772). This text was based on the concept of *infiniment petits* and introduced integral calculus as the inverse of differentiation (Schubring 2005, pp. 122, 213 ff., 338).

In 1797 the treatise *Théorie des fonctions analytiques* of Josef Louis Lagrange appeared, which based calculus on an algebraic approach that was disseminated from France to several European countries (Schubring 2009). Lagrange was the first to introduce the term *dérivées* (*de la fonction primitive*), that is, derivatives, which he used for the coefficients of the terms $(x-x_o)^n$ of Taylor's series of a function f at point x_o. This definition made no use of the ratio between infinitesimal quantities, but was defective in being based on the supposed convergence of the considered series.

In this context, then, when a complete public school system appeared during the French Revolution, the teaching of calculus was introduced in secondary schools. In 1802 Sylvestre François Lacroix published the *Traité élémentaire du calcul différentiel et intégral* (with several subsequent editions), a textbook based on his more extensive text *Traité du calcul différentiel et du calcul integral.* In this text, which was popular in many countries, the derivative of a function is called the *differential coefficient* and is defined using an intuitive limit notion (as the limit of the ratio of concurrent increments of the function and its variable). This text addressed not only higher education students but also classes in the *Lycées* (*classe de mathématiques transcendantes,* later *classe de mathématiques spéciales*) which prepared for admission to the *École polytechnique* (see Caramalho Domingues 2008, pp. 283 ff.). The *Lycées* had been established by Napoleon Bonaparte in 1802 and were reformed during the French Restoration; these classes were abolished in 1814 and later reestablished.

Simultaneously in Germany, each of the country's many independent states had its own educational system. In 1810 in Prussia, Wilhelm von Humboldt conducted a reform that established a public school system and reorganized secondary education (which was modelled on 1808 reforms in Bavaria). The new curriculum included a course in mathematics which led from arithmetic up to differential and integral calculus; however, after 1830 it was not possible to teach calculus in the Prussian *Gymnasium* (see Schubring 1985, 2009; Schubring et al. 2008). In the second half of the nineteenth century in Hamburg, in the *Realgymnasium*, differential calculus was obligatory and integral calculus was optional (see Beke 1914).

From 1810 to 1830 in Germany, a considerable number of calculus textbooks were published (including those for secondary schools), which applied the approach of either Lagrange or Carl Friedrich Hindenburg, professor at Leipzig University. Hindenburg founded the so-called combinatorial school, inspired by Lagrange's program of algebraic representation of functions through a

[6] For instance, an Italian translation by Stanislao Canovai and Gaetano del Ricco was used at the end of the 1700s for teaching calculus at the University of Pavia (see Pepe 2009).

polynomial series. He assumed it to be applicable to all types of functions, with an entirely formal approach to analysis and without consideration of continuity, limits, and convergence. An example of this approach is Christian Gottlieb Zimmermann's textbook *Anfangsgründe der Differential- und Integral-Rechnung: Aus der Theorie der Funktionen hergeleitet und entwickelt* (1816).[7]

Cauchy offered a radical critique of the theory of series used in analysis as a main tool (see Pepe 2009). In his textbooks *Cours d'analyse de l'École royale polytechnique* (1821) and *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal* (1823), the basic concepts of calculus were defined formally and more rigorously. He based this treatment on the concept of limit of a function and defined the *fonction dérivée* (derivative function, denoted by y' or by the so-called Lagrange's notation $f'(x)$) as the limit of the incremental ratio of the function.

It is worth noting, finally, that in the second half of the nineteenth century, the teaching of calculus in some countries was completely abolished in the secondary school for a variety of reasons. In Prussia, for example, the government removed calculus in 1882 from the *Realschulen* and in 1892 from the *Realgymnasien,* because, according to the critique, it was treated without critical spirit and rigor and was based on texts which presented the notions in a metaphysical context (an approach utilizing infinitesimal magnitudes which were presented as truly existing, like "the spirit of an ephemeral magnitude") (see Beke 1914; Klein[8] 1924, pp. 233–234).

In Italy, in some regions that were not yet part of the Kingdom of Italy, proclaimed in 1861, the first elements of calculus were taught in secondary schools (Vita 1986, p. 4, reports the example of the *Liceo* in the Kingdom of Naples). However, this ended with the Coppino Reform of 1867, which reintroduced into the *Ginnasio-Liceo* the study of Euclid's *Elements* according to the classical model. This reform was carried out for reasons of rigor as well as for ideology in order to replace the German and French textbooks used up to then and to encourage the writing of good Italian school textbooks (see Schubring 1996; Giacardi 2006). Notions of calculus were not included even for the program of the *Sezione fisico-matematica* of the *Istituto tecnico*, thus leaving the task of teaching differential and integral calculus to the universities (see Pepe n.d.).

4 The Demand for National Reforms in Several European Countries

At the end of the nineteenth century, the strong need to reform the teaching of mathematics was realized in several countries. This reform movement arose within the wider context of a renewal of the teaching of the sciences, which was motivated by both social and economic reasons. In particular, the emphasis on traditional Euclidean geometry, which was dominant in European countries, or at least in classically oriented secondary schools, needed to be changed. It seemed important to introduce the notions of variable and function and to establish the inclusion of calculus as the main item on the reform agenda along with the modernization of the teaching of mathematics.

4.1 The Situation in France

In France, secondary education, which assigned mathematics and the sciences in general a lower position in relation to the humanities, was deemed insufficient given the progress being made in science

[7] See Schubring (2009), who also includes a comparative evaluation of textbook production for analysis in Europe from the 1680s up to 1830.

[8] The text for self-instruction by H. B. Lübsen (1855), *Einleitung in die Infinitesimal-Rechnung zum Selbstunterricht* is cited by Klein as an example of this conception. The text was based on the theory of series – at that time already outdated – which Lübsen considered more suitable for beginners.

and technology and new demands to develop education accordingly. Traditional teaching, especially in mathematics, was bitterly criticized for its limitations. This critique paved the way for the reform of 1902 (see *L'Enseignement Mathématique* 1905, vol. 7, pp. 491–497; see also Fehr 1905). The reform – with positivist overtones – not only placed humanistic and scientific teaching on an equal footing but also resulted in radical changes in curricula, programs, and teaching instructions. The committee which was set up to reform scientific programs was divided into three subcommittees for mathematics, physics, and natural sciences and consisted of secondary education teachers and experts. The overall committee was chaired by the mathematician Jean-Gaston Darboux and its members included Jules Tannery, the historian and philosopher of mathematics.

Before 1902, the notion of derivative was reserved for higher education courses and, in secondary education, apart from a few exceptions, only for preparatory classes for higher education in the scientific field (*Mathématiques spéciales*; Bioche 1914). By contrast, the new programs of 1902 provided the introduction of calculus in secondary schools. There were some differences between the humanity and scientific *Sections*. In the scientific *Sections*, the notion of function and graphic representations were already introduced to 13–14-year-old students at the end of the first cycle in the so-called *Section B*; in the second cycle, graphic representations and derivatives were introduced to 14–15-year-old students in the *Classes de seconde*, *Sections C* and *D*. As for the humanity *Sections*, 16–17-year-old students dealt with notions of function, derivative, and the first elements of differential and integral calculus in the *Classe de Philosophie*.

Generally, the programs included calculation of areas and volumes using infinitesimal methods. Topics pertaining to calculus were included in those of algebra; that is, this was an "algebrized" and "algorithmized" analysis, based on the calculation of derivatives and of primitives. An example of this type of treatment is already outlined in the textbook *Leçons d'Algèbre élémentaire* (1896) by Carlo Bourlet (also intended for those preparing for admission to École de Saint-Cyr, a military academy founded by Napoleon Bonaparte in 1802). Bourlet explained in his preface that he had accepted Darboux's suggestion to use the method of derivatives to study the variation of functions. In fact, instead of the ad hoc methods used so far, he preferred to focus on methods that could be more extended. Work on a classical concept of "calculation" dominated the French secondary school teaching of analysis (see Artigue 1996).

Textbooks were written for different school courses to follow the new programs. Among these should be mentioned the manuals written during the years 1903–1905 by Emil Borel, Carlo Bourlet, and Auguste Grévy, as well as Jules Tannery's textbook, *Notions de Mathématiques* (1903), for the *Classe de Philosophie*, which also contained a historical part written by his brother Paul (see Fehr 1905). In particular, Jules Tannery's text developed the concepts of differential and integral calculus, starting from the basics of classical geometry and then systematically exploring analytic geometry. Introducing the derivation of a function (which was supposed to be continuous) stemmed from the calculation of tangents and velocities, using the concept of limit according to Cauchy and the so-called Lagrange's notation $f'(x)$. Teaching the integral started from the intuitive concept of area.

In line with these approaches were some observations about differential and integral calculus made by the French mathematician Henri Poincaré in a conference at the Paris *Musée pédagogique* (see Gispert 2007). In particular, Poincaré (1904) stated that it was better to begin with the derivative "in the way of Lagrange," while Leibniz's differential notation was "dangerous" because it could induce pupils to hold an erroneous idea of a ratio, not to mention create difficulties in understanding composition of functions and partial derivatives. Poincaré considered the introduction of the notion of *integral* as a way of rigorously defining the concept of the area of a surface (as done at a higher level) to be pedagogically wrong at the secondary school level. He believed that, because pupils would have already acquired this concept intuitively, presenting it this way would not have inspired interest and would have even seemed an incomprehensible nuance to them. To the contrary, he suggested presenting the definite integral of a function as only the area bounded by its graph and the x-axis.

4.2 The Case of Germany

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In Germany, every state (*Land*) was autonomous in its organization of schools and teaching practice. At this time, Germany enjoyed a booming industry with significant economic developments. From 1890 on, a strong movement had developed in favor of understanding more deeply the applications of mathematics in all the branches of natural sciences (particularly in the technical sciences) and its importance for all aspects of life (see Klein 1925, pp. 253 ff.). Although a special association of teachers of mathematics and natural sciences (*Deutscher Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts*) had been founded, curricular and methodological reform initiatives did not come from the teachers (see Schubring 2007). Rather, the broader reform movement throughout the 1890s came "from above." The most influential in this movement was mathematician Felix Klein, professor at the University of Göttingen, who was involved with the reform of mathematics education from universities to secondary schools. Of all the initiatives attributed to him, special mention should be made of the suggestion to integrate the *Technische Hochschulen* into the universities and to establish a single type of secondary school which would unify the humanity schools and technical-scientific schools – that is, *Humanistische Gymnasien* or simply *Gymnasien,* and *Realschulen* and *Realgymnasien* (this suggestion never materialized). In the 1890s, Klein realized the challenges of teacher training. He specifically identified a damaging "double discontinuity" encountered by a student moving from school to university and then back to school as a teacher, given that the distance between the mathematics taught at secondary school and the one taught at university was too great (see Klein 1924, Introduction to the 1st Edition of 1908, pp. 1ff.).

An important step in the development of the reforms was a report for the *Conference on School* (*Schulkonferenz*) in 1900, which Klein was commissioned to prepare by the Prussian Ministry of Culture. This conference focused, on the one hand, on the further development of teaching mathematics and natural sciences in general and, on the other hand, on identifying the best type of preparation at the secondary schools for university studies, especially for technical higher education. Klein's report of May 1900 contained his plan for the new structure of the relations between secondary and university education. (Importantly, in the report, he made use of some critique and suggestions which emerged from an "anti-mathematical movement" of engineers in the 1890s.) According to Klein, all types of secondary schools offered sufficient opportunity for a successful university study of mathematics; however, mathematics in technical universities was facing a deep crisis. Originally, following the *École Polytechnique*, the most famous model of technical higher education, mathematics was represented there as the main element of studies common to all students and as the basis for the specialized study of technical disciplines. This "polytechnic" function of mathematics, however, was long obsolete. Mathematical research and teaching had increasingly developed in the direction of studies of pure mathematical foundations. However, the technical disciplines had become very specialized and demonstrated little interest in problems of higher mathematics. In his report, Klein suggested a radical solution to this situation. Since the mathematics classes at university technical schools consisted of a general part and an advanced or higher part, he proposed that the general part be taught in secondary schools and only the advanced part be taught in university courses.

The basic subjects that Klein believed should be taught at the secondary level were analytic geometry and differential and integral calculus. In the years 1900–1901, Klein attempted to convince the Prussian Ministry of Culture to organize the reform of education according to the new plan. The ministry, however, refused to prescribe it "from above" and, in turn, proposed to Klein to organize the introduction of curricular changes "from below," thereby gaining the consent of properly trained mathematics teachers who would act to promote the reforms in the schools by themselves. To make the spirit of his reform proposals more understandable to teachers, Klein worked on the concept of "functional thinking" (see

Schubring 2007). He underscored the importance of teaching applications to understand the mathematical concepts better. He also identified function as a key concept which had played a fundamental role in mathematics over the last two centuries and was in its instruction the cornerstone of all innovations carried forward in the reform. Klein stated that such a concept should be taught early, beginning with the permanent use of the graphic method to represent functional dependencies in the Cartesian system, which is customary in all practical applications. Also, Klein considered it useful to introduce the first notions of infinitesimal calculus, which are used in applications to the natural sciences and to problems in the field of insurance, thereby establishing more connections with real-world practice.

Klein's vision of the method of teaching mathematics (see Klein 1924, pp. 82ff.) was influenced by his understanding of the development of mathematics. He identified two major tendencies in the historical development of mathematics. One (A) tended towards a fossilized, logical, and enclosed format of every individual mathematical branch; the other (B) tended towards an organic systematization of mathematical knowledge as a whole. (According to Klein, a bit less important third tendency (C) emerged from algorithmic thinking.) Klein included in (A) all rigorous works of systematization in the disciplines, such as Euclid's *Elements* or the foundational works of infinitesimal analysis of the nineteenth century. Tendency (B), which was followed, above all, by the founders of the infinitesimal analysis in the seventeenth century but was not unknown in Ancient Greece (note *The Method* by Archimedes), was, according to Klein, most productive for the development of mathematics. Klein also championed it as a model for teaching the discipline, in contrast to what had been done until then.

Above all, Klein proposed to present to pupils at an early stage the concept of function with the fusion of space-number representation. According to Klein (1924, in particular, pp. 221, 231, 238ff.), the concept of function should be a kind of "yeast" rising throughout all mathematics teaching at the secondary school level. However, it should be taught as defined by Euler while by all means avoiding formal definitions based on Cantor's theory of sets. Numerous elementary examples should be offered during the process of teaching, beginning in the lower secondary school with simple graphical representations on squared paper of functions such as $y = ax + b$ and $y = x^2$. In this way, knowledge born out of concrete examples at a lower level would naturally carry pupils forward towards the acquisition of the fundamentals of the differential and integral calculus at a higher level. Thus, pupils would understand the concepts of infinitesimal calculus without any "mystical misconceptions."

The idea of the reform movement was synthesized thus:

> We want the concepts, expressed via the symbols $y=f(x)$, dy/dx, $\int y dx$, to become familiar to the students with these characters, i.e. not as a new abstract discipline, but in an organic construction, in the context of the whole teaching, starting from the simplest examples and going up, step by step. (Klein 1924, p. 240)

Finally, Klein (1924, p. 255) highlighted four fundamental points to keep in mind when dealing with the subject:

- to illustrate abstract considerations with figures (obtained by approximations with the Fourier and Taylor series);
- to underscore the link with similar fields (for example, calculus of differences and also philosophical research);
- to underscore the historical process;
- to utilize some examples taken from "popular literature" in order to point out the gap between the conception of the general public, influenced by this literature, and that of professional mathematicians.

Klein sought the collaboration of associations (see Seyfarth 1924 and Schubring 2007). The support came from the *Verein deutscher Ingenieure*, the *Deutscher Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts*, the *Gesellschaft deutscher Naturforscher und Ärzte*, and the *Hochschulkreisen* (these last two were under the headmastership of Klein himself). The general idea was to give a new order to the teaching of mathematics which would reflect its applications and the great progress achieved in the eighteenth and nineteenth centuries and which would elevate its cultural significance. Following Klein's exhortations, mathematicians and biologists joined forces and in 1904 set up a committee, the *Breslauer Kommission*, charged with the task of creating a reform project to teach mathematics and natural sciences in all types of secondary schools. In 1905, in Meran, this committee contributed to the formulation of a proposal for new teaching programs, which were later widened in Stuttgart in 1906 and in Dresden in 1907. They were called *Meraner Reform* or *Meraner Lehrpläne* (teaching programs).

These programs were highly innovative. They contained recommendations for teaching and strongly corresponded with Klein's thought, in particular, the emphasis on the importance of the concept of function and of applications of mathematics. In the lower cycle (10/12–15-year-old pupils), they included a preliminary discussion of the concept of variable and of functional dependency. In the upper cycle (16–17/18-year-old pupils), graphical representations were recommended to provide connections among arithmetic, algebra, and geometry. Later, the course was intended to include analytic geometry and the study of functions, up to differential and integral calculus. The study of differential and integral calculus, however, was proposed only for the *Realschule* and the *Realgymnasium,* while, for the *Gymnasium*, it was left to the decision of individual teachers by giving only the vague instruction that teaching "should push forward to the threshold of infinitesimal calculus." This did not correspond with Klein's program, which provided a flexible transfer from secondary school to all higher courses and so required that curricular reforms should concern all types of schools.

Texts were written that reflected the content of the new programs, such as the textbooks of Otto Behrendsen and Eduard Götting for the two levels of secondary school (*Lehrbuch der Mathematik nach modernen Grundsätzen*, for the *Unterstufe* 1908, 1st ed., and for the *Oberstufe* 1912, 1st ed.). For the lower level, a modernized teaching was proposed both in arithmetic and geometry, emphasizing where possible their functional aspects. For arithmetic, this was achieved, for example, by showing how the results of arithmetic operations vary when terms vary; also, the coordinates were later introduced and examples of functions were studied graphically. In geometry, geometric transformations were introduced. For the higher level, the study of functions was continued with the discussion of trigonometric functions. The derivative of a function (called *Differentialquozient* and later *abgeleitete Funktion*, for which the symbols dy/dx or $df(x)/dx$ and later the symbol $f'(x)$ were used) was then introduced in the process of studying tangent lines. The derivative was defined as the limit of the incremental ratio (*Differenzenquozient*), using the notation *lim,* but without the formalism of the "(ε,δ)-definition." The integral was introduced starting from the concept of area (bounded by the graph of a function), and the inverse character of derivation and integration was discussed. Later, the definite integral was presented as the area bounded by the graph of a function; limits of sums like $\sum_{i=0}^{n} f(x)\Delta x_i$ were considered. All functions were continuous in the considered intervals; however, the problem of the existence of limit was not mentioned.

4.3 The Situation in England

In England at the end of the nineteenth century, the teaching of mathematics at the secondary school level had become fossilized with classical aspects of the so-called pure mathematics. As such, mathematics came to be considered as merely mental gymnastics; this understanding hardly encouraged students' interest and adversely affected the preparation of the majority of students. This approach also resulted in a less than adequate preparation for future mathematicians, drawing criticism from mathematicians (Perry 1902, p. 1). The call for reform came from the world of engineering, which at that time appreciated mathematics for its applications.

Of note was the work of John Perry, professor of mechanics at the Royal College of London who was very active within the *British Association for the Advancement of Science* (Godfrey 1909; Beke 1914). Perry addressed a meeting of the *British Association* held in Glasgow in 1901[9] and, based on his 30-plus

[9] It is of interest to mention that some female mathematicians actively took part in this conference including Charlotte Angas Scott. Also present was the American mathematician David Eugene Smith.

years of experience, proposed a "syllabus" for the reform of the teaching of mathematics. In this syllabus, he assigned a place to introduce the first elements of differential and integral calculus at the secondary school level. Perry observed that a "thoughtful teacher" could impart the first elements of infinitesimal calculus to his young pupils, making use of squared paper and simple algebra as well as laboratory experiments; such a teacher could also freely use the symbolism of calculus when teaching elementary mechanics even before beginning more advanced algebra, trigonometry, and analytical geometry (Perry 1902, p. 106). The method which Perry proposed was well illustrated in a textbook which he had written to prepare students who wanted to begin studying engineering (Perry 1897). He began with plotting graphs, then gradually introduced differentiation and integration, and moved from polynomial functions to exponential and logarithmic functions as well as trigonometric functions, supplementing the course with several practical examples of the application to mechanics. Perry used the concept of limit in a dynamic and intuitive form and determined the derivative of a function (called *differential coefficient*, for which the notation dy/dx was used) as the limit of the incremental ratio. The problem of the existence of the limit in these considered cases was never mentioned or discussed.

After the Glasgow meeting of 1901, committees were formed to discuss the problems facing mathematics education and to overcome them in accordance with the practical needs of the industrial society. This caused universities and other bodies involved in school examinations to modify examination papers so that they would conform to the new points of view (Godfrey 1909). Perry's suggestions, which argued for the utility of mathematics rather than its rigor and for laboratory-based experience rather than abstraction, had a great influence, especially in the Anglo-Saxon world (Howson 1984).

As an example, we can consider the textbook written by Charles Godfrey and A. W. Siddons (*Elementary Algebra* 1913, 1st ed.) in line with official documents of the years 1909–1911 (including the *Algebra Syllabus for the School Certificates of the Oxford and Cambridge Joint Board*). The textbook covers the mathematical subject to be learned by a "pupil of average ability" during a full school course. In it we find particular care in treating certain educational aspects. Each new notation was introduced gradually, with a problem showing its usefulness and exercises to practice in it; the pupils were expected to understand and use the "words" before a formal definition was offered. Much attention was paid to functional aspects, and the idea of functional dependency was emphasized (e.g., in solving equations, the graphical representation was systematically used). The differentiation of a function was also treated gradually, utilizing an intuitive concept of limit and introducing the *derived function* with a nonstandard notation; only in the last chapters was the notation dy/dx used (a note was made that it did not represent a ratio). Integration was explained as the way to solve a differential equation (without using the integration sign).

5 International Reform Initiatives Before the First World War

In the meantime, the field of mathematical teaching became more open; various European and non-European countries opened communication and conducted international conferences on mathematics education with the aim of setting up a common language and culture. An important step was the foundation of the journal *L'Enseignement Mathématique* in 1899 (Schubring 1996).

The question of teaching infinitesimal calculus in secondary school was discussed for the first time at a large audience at the *International Conference of Mathematicians* held in Heidelberg in 1904; this question was raised in the session dedicated to the teaching of mathematics. Klein put forward his ideas, eliciting consensus but also provoking opposition (even among the Germans, particularly Robert Fricke). On this occasion, the Frenchman Jules Frédéric Charles Andrade observed that, in his opinion, the greatest difficulty to be met in the new mathematics teaching was not the assimilation of the concepts relative to the infinitely small or the method of limits, but the "terror which the algebraic transformations instilled in the pupils" (*L'Enseignement Mathématique*, vol. 6, pp. 392ff.).

An important occasion for analyzing and discussing the status of mathematics teaching in various countries was the fourth international congress of mathematicians, held in Rome from April 6 to 11, 1908.

Reports on teaching mathematics in secondary schools in some European countries were presented in "Section IV – Philosophical, historical and didactic questions" (Castelnuovo 1909, pp. 441ff.). Regarding the introduction of calculus into secondary school teaching, the movements that were already in existence in some countries were discussed. The French were cited more than once as a model for the rapidity with which the reform of 1902 was put into practice and for the production of suitable textbooks. The state of the reform movement in Germany was considered, specifically stressing Klein's ideas on the centrality of the concept of function and stating that it was not so much a revolution as an evolution. In fact, what was required was a lightening up of the school programs, a weeding-out of subjects that no longer had significance (e.g., exercises in formal computation and complicated geometrical constructions) in order to introduce new topics that were useful to practical life and applicable to the sciences while also keeping the pupils' mental development in mind. It was noted that the work of the German reform committee had also been inspired by the ideas of the Austrian physicist and philosopher Ernst Mach, particularly by his conception of science as "economy of thought." In England, there was talk of a strong movement for the early introduction of differential and integral calculus in schools, following the Glasgow meeting of 1902 and J. Perry's contribution. Further, it was observed that in Switzerland, already for more than half a century, the programs in most *Gymnases scientifiques* contained the first elements of differential calculus and, in certain types of schools, also integral calculus (although there was a great divergence between the study programs and the organization of schools in the various cantons and the teaching staff was afforded a great amount of liberty).

In other countries, the need to reform the teaching of mathematics and to introduce calculus into secondary schools began to be felt. In Hungary, for example, the influence of the German reform movement promoted by Klein had already led to the setting up of a national committee, under the presidency of Emanuel Beke, one of Klein's former students. The first steps in this direction had been taken by introducing such concepts into some schools, even a school for girls (*Mädchengymnasium*); further, a maximum problem, to be solved with differentiation, had been included in an examination for the final school diploma. In Italy, Giovanni Vailati had drafted the mathematical part of a project to reform the Italian secondary school, keeping in mind the European reform movements and supporting the introduction of fundamental concepts of infinitesimal calculus. The reform project was presented in February 1908 but was judged to be too radical and never implemented (see Giacardi 2009). In Austrian secondary schools, program renewal was deemed necessary to keep up with modern progress in science. At the same time, it was considered important to take into account the students' rate of development and make use of their intuition in introducing advanced concepts, which only later would be treated more rigorously. Reform in Austrian school curricula came into effect in the 1909/1910 school year; it introduced the first elements of calculus into secondary school teaching in the *Gymnasium*, *Realgymnasium*, and *Realschule* (see Zuccheri and Zudini 2007, 2008).

At the Congress of Rome, the *Commission Internationale de l'Enseignement Mathématique* (CIEM) or *Internationale Mathematische Unterrichtskommission* (IMUK) was established, with Felix Klein as its first president (Howson 1984). The Commission, together with the journal *L'Enseignement Mathématique*, played a very important role as *trait d'union* among scholars, allowing for their communication and cooperation at an international level (Schubring 2003, 2008; Furinghetti 2003; Furinghetti et al. 2008). The work of CIEM-IMUK was fundamental in launching the first international reform movement in mathematics education; the Commission decided to write international comparative reports on topics which were considered major reform concerns, among them the introduction of calculus at the secondary school level (Schubring 2003).

Concerning calculus, stock of the situation was taken at the *Conference internationale de l'enseignement mathématique* held in Paris in April of 1914. On this occasion, Emanuel Beke, on behalf of the presidency of the CIEM-IMUK, presented a detailed report on the results obtained from the introduction of differential and integral calculus in the upper grades of the secondary schools, based on a questionnaire answered by representatives of the committee for various countries (see Beke 1914; see also Kahane 2003). Regarding the European countries, those cited in the report were Germany, Austria, Belgium, Denmark, France, Holland, Hungary, the United Kingdom, Italy, Norway, Romania, Russia, Serbia, Sweden, and Switzerland. Another report was presented by C. Bioche (1914; Artigue 1996) on the situation with the *Lycée* in France, a country which boasted a greater degree of experience thanks to the 1902 reform, which was partially modified in 1912.

Beke's extensive report began by underscoring how much the educational value attributed to the sciences had changed, thanks to the "positive sciences." He emphasized that this called for a renewal of teaching mathematics in order to bring it into line with new ideas on the development of minds. In all the countries where new study programs for secondary schools had been in use since 1902, there was a relatively important place reserved for the concept of function and (apart from a few exceptions) the first elements of differential and integral calculus. The changes had been rapid and widespread, even if the concept of function had not yet been positioned as the central focus of all secondary school teaching, as Klein had hoped. Despite differences in the rules and laws of the various countries, it can be said that calculus appeared in the official programs of Bavaria, Baden, Württemberg, and Hamburg (for Germany), Austria, Denmark, France, the United Kingdom, Romania, Russia, Sweden, and Switzerland; in some countries, it did not appear in the official programs but was taught in many schools (Prussia and Saxony for Germany; Hungary) or only in exceptional cases (Italy). Some countries (Holland, Norway, Belgium, Serbia, and Italy) had announced a probable inclusion of calculus in the programs within a short time. The schools where differential and possibly integral calculus had been introduced were, above all, of a scientific nature (such as the *Realgymnasien* or *Realschulen*) or schools that prepared the students for further scientific studies as well as some military schools and a few girls' schools (e.g., in Russia). Beke praised the textbooks written on this subject by the French and published in Darboux's collection, recalling that Klein had encouraged the translation of the textbook by Jules Tannery (see Sect. 4.1) into German. Further, Beke cited some innovative textbooks which appeared in Germany (e.g., Behrendsen and Götting; see Sect. 4.2) and the English textbook written by Godfrey and Siddons (see Sect. 4.3).

Regarding the content of teaching calculus, Beke observed the following:

- Nearly everywhere, only functions of one variable were dealt with.
- The differentiation of polynomial and rational functions was considered everywhere, while in some places it was supplemented by the differentiation of exponential and trigonometric functions with their inverse functions.
- Generally, Lagrange's notation was preferred to that of Leibniz.
- In most countries, the notion of integral was also introduced, with a clear preference for beginning with the indefinite integral and then dealing with the definite integral.

As for the applications of calculus, it was noted that:

- Taylor's series figured in few programs and were discussed only in schools where the infinite series had been treated longer.
- All countries dealt with the search for maxima and minima of functions.
- In physics, calculus was used to define concepts of velocity and acceleration and at times found a wider application in the study of the center of gravity, the moment of inertia, and potential.
- While calculus was used in geometry to determine areas and volumes, old methods and Cavalieri's Principle were still also applied.

The rigor with which the subject was treated drew some of the greatest controversy. On the one hand, some maintained that teaching without adequate scientific rigor was more harmful than beneficial, while, on the other hand, teachers made clear that the intellectual level of secondary school pupils did not allow for the use of rigorous methods and that it was necessary to start from intuitive methods with geometrical and mechanical considerations and then arrive at abstract concepts. With regard to this, Beke had noted that:

- Practically everywhere, irrational numbers were introduced with the extraction of roots and only in exceptional cases was general theory considered (using Dedekind's method, as was done, e.g., in certain Italian schools and in the majority of Austrian schools).

- In all countries, the notion of limit was introduced, using relatively elementary theorems without providing any explanation for them.
- "Nowhere derivable" continuous functions were not considered, but, in some schools, functions not derivable in one point were considered.
- In most schools, the concept of differential was not treated and, in general, it was a cause of misunderstanding.

Where there had been reform, the necessary fusion of calculus with other subjects was kept in mind; in this way, attempts were made to avoid overloading the pupils by reducing the program elsewhere. Beke, in his overall optimistic report, described a shared reform movement, in the context of a spirit of great collaboration between countries. Unfortunately, very soon after the conference, the beginning of the First World War put a brake on such enthusiasm.

6 Between the First and Second World Wars

A new recognition of the state of the teaching of mathematics at an international level was forming at the end of the 1920s and in the 1930s. Thanks to these international reports, more sources than ever before were now available for a greater number of countries. From these reports published in the journal *L'Enseignement Mathématique*, it is clear that during the period between the First and Second World Wars, the school systems of many European countries had been subjected to several reforms, some of which followed the direction outlined by Klein's movement and placed the concept of function at the core of the teaching of mathematics. Calculus, in particular differential calculus and the study of functions, was included almost everywhere in the programs of secondary scientific schools. However, especially around the 1930s, the teaching of mathematics in secondary schools was reduced in many countries.

The French secondary school (*Lycées* and *Collèges*) was subjected to a reform in 1923–1925 (see *L'Enseignement Mathématique* 1937) that established that all students of the first six classes follow the same programs for mathematics, physics, and chemistry; derivatives were still introduced into the seventh year of the course, within the context of the algebra program.

The secondary school in Germany was initially under a reform in 1925 (*Richterreform*), the plans of which were aligned with Klein's ideas (see *L'Enseignement Mathématique* 1929, where several textbooks and volumes on teaching methodology for teachers were reported). Among the most important aims of mathematical teaching, there was the development of functional thinking, which was supposed to be provided in all secondary schools and includes the teaching of differential and integral calculus in the *Prima* class (the upper secondary school class); the historical and cultural aspects of mathematics also needed to be emphasized. With the advent of National Socialism (Rüping 1954), these last aspects of mathematics became a particularly controversial issue, while the utilitarian aspects became primary. In 1938, the new minister promulgated a reform of the organization of the higher secondary school, which was in force for the whole *Reich.* The number of hours for mathematical teaching was reduced and some topics (such as the infinite series) were removed from the programs. Practical applications in mathematical teaching were particularly emphasized after the outbreak of the Second World War.

In England, where the public secondary school had been subjected to wide changes, the graphical representation of functions was included in the minimal standard program of algebra (given for all students' final obligatory examination). More and more schools had introduced into their programs at least the differentiation and integration of polynomial functions. Calculus was considered the natural future development of the algebra taught at the secondary school level (see *L'Enseignement Mathématique* 1929 and 1937).

In Switzerland (see *L'Enseignement Mathématique* 1929), a decree of the Federal Council in 1925 established a broader uniformity in the teaching programs of the cantons; in the programs of the federal examination for the final school diploma of mathematics, the concept of function and the graphical representation of functions were included. The program for C-type (mathematical-scientific) education also included differential calculus.

In Italy (Vita 1986; Marchi and Menghini 2011), at the beginning of the Fascist period, the Gentile Reform of 1923 had triggered the setting up of the *Liceo scientifico*; its study plans included the elementary concepts of infinitesimal analysis, although weekly mathematical hours were reduced, compared to those of the *Sezione fisico-matematica* of the *Istituto tecnico*, which was replaced by this *Liceo*. A textbook of reference was *Nozioni di Matematiche* by Federigo Enriques and Ugo Amaldi (see *L'Enseignement Mathématique* 1929). With subsequent modifications in the programs, which continued to be made until the end of the Second World War, one recommendation was to connect the notions of infinitesimal calculus with their geometrical and physical meaning, while real numbers had to be considered in their significance of ratios of magnitudes.

In Holland, a reform had modified the teaching programs in 1919. In the *Gymnases,* the graphical representation of functions was introduced; in higher scientific-type classes (*Section B*), supplementary hours were also added to treat, among other things, the elements of infinitesimal calculus. In a type of school called "HBS" (*Hoogere Burgerscholen*), where Ancient Greek and Latin were not offered but by contrast scientific and linguistic courses were taught, the programs were modified in 1920, although only the graphical representation of functions was added (see *L'Enseignement Mathématique* 1929).

In Austria, a reform for secondary schools had started in 1918 (and ended with the rules of 1927 and 1928); it set up a lower 4-year course for all students, while the higher course (also 4 years) was subdivided into a classical class, a scientific class, a modern language class, and a so-called *Deutsche Oberschule* with only one foreign language. In mathematics programs, the concept of function was still treated, but even though infinitesimal calculus continued to be treated in the scientific *Gymnasium*, this subject was reduced in the other secondary schools (see *L'Enseignement Mathématique* 1930). In 1935, the school system was modified again by widely reducing the number of hours for mathematics and the sciences in all secondary schools, apart from the *Realschule*. The concept of functional dependence, graphical representations, and practical applications were highlighted (see *L'Enseignement Mathématique* 1937, where also textbooks were cited).

In Sweden, the organization and programs of the secondary school were reformed by decrees of 1928 and 1933, which aimed to make the school more democratic by unifying the lower courses. However, hours of mathematical teaching were reduced in all secondary scientific schools and, drastically, in the classical schools. Although in the first years of the lower course the preparatory activities for introducing the notion of function, including the plotting of simple graphs, were still maintained, the teaching of differential and integral calculus in the *Gymnase réal* was soon removed from obligatory courses for all students. Thereafter, because these subjects were necessary to students who intended to continue studying at the *École polytechnique*, an optional course was created for them (see *L'Enseignement Mathématique* 1938).

In Greece (see *L'Enseignement Mathématique* 1937), the secondary school was reformed in 1929. In 1935, the first elements of differential and integral calculus, including the notion of derivative and primitive function, were introduced into the programs of the last year of secondary school.

In Hungary, the school system was modified in 1926, and the programs included the first elements of differential and integral calculus for the penultimate year of the *Gymnasium* and *Realgymnasium* (see *L'Enseignement Mathématique* 1937).

Russia (and then the Soviet Union) underwent many changes over the period discussed here. Until 1917, the country had many different kinds of secondary educational institutions. The teaching of calculus in these schools had given rise to strong polemics; nonetheless, to one degree or another, calculus was taught in most cases. The situation changed after the revolution of 1917, when the public school system was radically transformed. The aim of mathematical instruction was officially changed: the official reason for teaching mathematics became its usefulness in practical life only (see *L'Enseignement Mathématique* 1933) rather than the development of thinking and reasoning. The elements of calculus were still represented in syllabi at this time but in reality were only very rarely taught. Beginning in

1932, after another series of reforms, calculus disappeared from the curricula altogether – the goal was now to give students a firm grasp of elementary mathematics. It is noteworthy that the elements of calculus began to reappear in the mathematics curricula only in the 1970s.

In Norway, following official programs according to a law of 1910, teachers were able to replace parts of the study plan with an introduction to differential calculus and its applications (see *L'Enseignement Mathématique* 1930). However, in the 1930s, an anti-mathematics movement had been formed. While strengthening the teaching of other subjects, such as foreign languages, physics, and biology, was requested, some educators wanted to make mathematics an optional subject; others, however, highlighted the practical and applied aspects of mathematics (see *L'Enseignement Mathématique* 1937).

In Poland, the school system had been modified by the reform of 1932. The program of mathematics in the *Gymnasium* provided the development of the concept of function and the extension of the concept of number towards the introduction of the real numbers (using Dedekind's sections or Cauchy's sequences); by contrast, with regard to geometry, the program was inspired by the Italian treatise of Enriques and Amaldi, which concerned geometrical magnitudes. In the physical–mathematical classes of the *Lyzeum*, infinitesimal calculus was introduced (see *L'Enseignement Mathématique* 1937).

In Romania, a reform of 1928 had radically reduced mathematical teaching. In 1935, after noting the low scientific preparation of the students who went on to higher studies, a new scientific class was introduced into the higher course of the *Lycée*, where the study of the derivatives of elementary functions was provided (see *L'Enseignement Mathématique* 1937).

A special case is that of Portugal, where the concept of derivative had been introduced in 1905 in the seventh and last year of the secondary school, in scientific classes. In 1918, a reform also introduced the theory of limits and the notion of integral into the sixth year of these classes. In 1919, the notions of derivative, differential, and integral were introduced in literary classes. However, after the coup d'état of 1926 and several subsequent reforms, teaching hours were reduced until, in 1936, the mathematics hours in the last 2 years of the secondary school were reduced to only two a week and the teaching of calculus was cancelled. It was again reintroduced in 1947 (Florêncio Aires 2006).

7 Further Developments After the Second World War

Regarding the situation in West Germany (Federal Republic of Germany) in the 1950s, Rau (1954) wrote:

> In 1907 Weinmeister could still write: 'A problem has never so involved the world of the German mathematics teachers as that still unsolved of the introduction of calculus into the secondary school. Very much has been said in favor and very much against; expert and inexpert people have spoken of it; almost everyone has spoken with conviction, so that one has the reasonable doubt about a concordant answer, without any controversy.' Today this involvement has widely passed and almost nobody has doubts about treating the subject. (p. 47)

Rau (1954), however, noticed:

> Generally, in all types of schools, an extreme formalism is deplored. This danger was already known about 50 years ago when infinitesimal calculus was introduced into schools.
>
> In the mentioned paper of Weinmeister it is said: 'As it is well known, infinitesimal calculus contains many formulas which have to penetrate into the flesh and the blood of pupils if we want teaching to achieve its aim. Here there is the impending danger that a pupil thinks that the essence of the teaching lies in these formulas and that it is sufficient to know and apply them for his mathematical training.'
>
> The didactic plans of mathematics of Lower Saxony refuse the ways of observation which are not accessible by intuition (for example, the pure 'Epsilontik' [epsilon technique]) as something which overcomes the capacities of comprehension of the student. (p. 48)

As in Germany, the introduction of calculus into the secondary schools of the majority of European countries had already been carried out, at least for the scientific courses. Klein's conception of teaching calculus in each type of secondary school had not asserted itself, however. Today, the widespread practice remains to differentiate teaching calculus according to different curricula. Furthermore, the demand for rigor and clarity, sometimes in the absence of specific instructions in official school plans, had often promoted the adoption of teaching methods which were based on the formal definition of limit by means of ε–δ.

Characteristic examples of the development of the teaching of calculus in secondary schools after the Second World War are illustrated by two countries: France and the United Kingdom.

In France since the end of 1950s, the wide distance between the teaching of mathematics at the secondary level and at university level reemerged. The first reform of 1960 modified the teaching programs concerning calculus by introducing more rigor and demanding that its treatment be based on a more formal introduction of the concept of limit. The request for abstraction became so widespread with the reform of 1970, as inspired by the ideas of Gustave Choquet and Jean Dieudonné, that a reaction came about in 1972–1973 from the *Association of Mathematics Teachers* (APM, later APMEP) and a counterreform was conducted at the beginning of 1980s. The programs of 1970s made analysis independent from algebra and provided the introduction of the derivative of a function through the more general notion of a "linear function tangent to a given function at a point" (Artigue 1996).

In the United Kingdom, the distance between the mathematics at the school level and the mathematics at the university level became the issue of concern at the end of the 1950s. Neither school mathematics nor university mathematics was found to meet the expectations of the world of industry and trade. Also, much stimuli had pervaded the United Kingdom from the international congresses where, following Dieudonné's ideas, a revision of school mathematics was requested to take into account the more recent developments of mathematics and especially of pure mathematics. The great impulse for a renewal came in 1958 after the launch of the Soviet Sputnik, which was followed by grants offered by the US National Science Foundation (NSF) and various conferences on mathematics teaching. After the Southampton Congress of 1961, the activity of the *School Mathematics Project* (SMP), one of the most important education reform projects of the 1960s in the United Kingdom, began. SMP prepared a new curriculum and new textbooks (with guidelines and recommendations for teachers). This highly innovative project was developed under the direction of Bryan Thwaites as a research program of the University of Southampton in cooperation with schools. Regarding calculus, SMP tried (meeting with fierce opposition!) to change the English tradition of presenting differential calculus together with the concept of motion. In particular, other definitions of derivative were introduced using the notation of Lagrange $f'(x)$ instead of that of Leibniz dy/dx (Howson 2009).

8 Teaching Calculus in Other Countries

More attention is given in this chapter to European countries. It is not possible to give a detailed discussion of the development of the teaching of calculus at secondary school in all countries here; however, we will provide below a brief case study of two non-European countries, confining ourselves to what happened in the United States and Brazil.

8.1 The United States in the Twentieth Century

Alexander Karp

Answering questions about the Paris Report on calculus in secondary schools, David Eugene Smith wrote:

> Calculus does not figure in the secondary curriculum in the United States; it cannot even be made elective, since the pupils of the upper classes are very much absorbed in the preparation for college entrance examinations. (The Paris report 1914, p. 326)

Naturally, this did not mean that calculus was not taught in any school in the country. Given the diversity of curricula and given that the line between secondary and higher education remained vague until a certain time, calculus could be found in the curricula of certain schools (Edmonds 1903). Yet, on the whole, the picture painted by Smith was accurate.

Smith's student and follower Noah Rozenberg (1921) argued in his dissertation for the necessity and feasibility of teaching calculus in high school as an elective. The College Entrance Examination Board and NCTM also supported the introduction of calculus in secondary schools (Nordgaard 1928; Swenson 1934). Nonetheless, Swenson (1934) noted that all of these recommendations had been carried out by comparatively few schools. Furthermore, calculus was taught differently in different schools, and in all cases educators aimed to teach it differently from the way it was taught in colleges (e.g., usually only the simplest functions – polynomials – were studied).

After the Second World War, the situation changed somewhat. Moise (1962) wrote:

> Many schools teach calculus in the twelfth grade; and the experience of the last few years shows that it is quite workable, if qualified teachers are available. But the *if* is important. High-school calculus courses have become status symbols, pursued, in some cases, without proper regards to staff resources. (p. 86)

In 1972–1973, four times as many students were studying calculus in high school as had done so in 1960 (Rash 1977). However, the effectiveness and usefulness of teaching calculus in school were the subjects of numerous debates.

Rash (1977) and Sorge and Wheatley (1977), for example, showed that students who had taken the prestigious calculus course in high school had no grasp of seemingly more elementary sections of the mathematics curriculum (algebra or trigonometry) and therefore experienced difficulty in college. Indeed, similar skepticism had already been voiced much earlier. Neelley (1961) wrote straightforwardly that "high school calculus is largely a waste of time" (p. 1005), but this remark did not lead to any reduction in the number of students taking calculus in high school, as noted above.

Skeptical comments were accompanied by positive ones – educators pointed out the effectiveness of teaching calculus in classes for the gifted (Mezynski and Stanley 1980) and also in those cases when the calculus course was extended over a long period of time (Austin 1979).

8.2 The Situation in Brazil

João Bosco Pitombeira de Carvalho

The first experience of teaching calculus in Brazil dates back to 1889. The introduction of calculus came as part of the educational reform which occurred after the institution of the republic at the suggestion of the head of the newly created Ministry for Instruction, Benjamin Constant, a follower of Auguste Comte's positivism. Following this reform, differential and integral calculus were included in the mathematics curriculum for *Colégio Pedro II*, the model secondary school institution in Brazil, created in 1837 and maintained by the central government.

Eugênio Raja Gabaglia, the chair of mathematics at *Colégio Pedro II* and representative of Brazil at IMUK, wrote in 1914 that the results of teaching calculus were satisfactory:

> Differential and integral calculus were taught together with analytic geometry, from 1891 to 1901; then, despite satisfying results, it became suppressed, so that at present it is not included in the secondary school syllabus; only the definition of the derivative is given in some schools. (*L'Enseignement Mathématique* 1914, p. 251)

The textbook by Sonnet (1869) was used. However, the inclusion of these parts of "higher mathematics" met strong opposition, mostly because of the lack of teacher knowledge of the subject.

The presence of calculus in the secondary school curriculum was short-lived this time (Carvalho 1996; Silva 1996). In 1900, it no longer figured in the programs which, later on (e.g., in 1915) did not even include the concept of function. This remained unchanged until 1929, when the program for the sixth year – the complementary course for students planning to pass the entrance examination for the *Escola Politécnica* – included a first course in differential calculus.

With the sweeping educational reforms of the 1930s and 1940s, calculus was reintroduced as a mandatory subject in secondary school mathematics, though with less ambitious coverage than in the 1890s. Then until 1951, this part of the program was continuously diminished. After 1961, national curricula were no longer mandatory and the teaching of calculus ceased in almost all secondary schools.

9 Concluding Remarks

Since the 1960s, internationally, although with variations from country to country, a new era for mathematics education began. Psycho-pedagogical studies became very influential and the specific research field of mathematics education emerged, influencing secondary school education, particularly from the end of 1970s. Many studies were conducted on teaching calculus, specifically highlighting difficulties in learning fundamental notions such as limit and function. Some authors have linked them with the epistemological obstacles (Artigue 1998).

In the meantime, the rapid development of computing technological tools and specific software and their introduction into school teaching have made possible the emphasis on algorithmic approaches in teaching the basic notions of calculus (recall the tendencies in the development of mathematics highlighted by Klein and discussed in Sect. 4.2 – they manifest themselves in mathematics education as well).

Furthermore, enrollment in upper secondary schools has increased substantially, compared to previous centuries, and the learning procedures of young people are changing. Some of these changes may be attributed to the rapid spreading of digital technologies among young people, especially in the communication via the web and social network communities.

The necessary search for new forms of teaching calculus will take advantage from the study of past methodologies. However, as A. G. Howson (1984) writes, we have to remember that

> Changing social and mathematical contexts ensure that any 'solution' can only be a temporary one. (p. 77)

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Chapter 25
History of Teaching Vocational Mathematics

Rudolf Sträßer

1 Introduction: Impossibility of Writing a History of Teaching Vocational Mathematics

Writing a history of teaching vocational mathematics is like trying to cope with a paradox: it is simply impossible to write such a history in the same way that one can with a history of teaching Arithmetic, Algebra, Geometry, Calculus, and Statistics/Stochastics. There are two major reasons for this paradox. First, in most educational settings, vocational mathematics is not a special subject. In most countries, 'vocational mathematics' does not exist as a taught subject or even as part of a taught subject. Unless we understand any general education as vocational education[1] and attempt to keep some specific features of general education as vocational education, we will find that some nations do not have vocational education at all, even if we consider the idea of vocational education in the largest sense possible.

Second, and even worse, there is virtually no written history of vocational education on a global level because 'vocational education' is defined differently throughout the world – if it even exists in some form in certain countries. Consequently, the main section of this chapter presents case studies on the development of teaching vocational mathematics in two nations. In the fourth section, the author makes some suggestions in the form of speculation about how the teaching of vocational mathematics developed throughout history and how it may be understood in broader, nonlocal, nonnational trends. Given the history of education as it is known at present, I do not see a way to describe teaching vocational mathematics in a more uniform way.

In order to approach this paradox, I begin by trying to give a 'definition' of vocational education in general terms and then to locate mathematics and its teaching inside the phenomena related to vocational education. My starting point is an identification of a borderline between general and vocational education. As the distinguishing feature of vocational education, I suggest taking the relation to 'industry', or more precisely the relation of mathematics, to 'any activity of economic or social value, including the service industry, regardless of whether it is in the public or private sector' (the definition of 'industry' as offered in OECD 2008, p. 4). This understanding of industry implies a rather broad understanding of vocational education, but nevertheless allows us to distinguish it from general education by pointing to the 'economic or social value' of the subject to be taught. In this respect, following the Discussion Document of the ICMI Study No. 20 on 'Educational Interfaces between

[1] This is not a joke, but a serious consideration.

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Mathematics and Industry (EIMI)', I also use the description of the term 'Mathematics' in this document: For this chapter, Mathematics 'comprises any activity in the mathematical sciences, including mathematical statistics' (OECD 2008, p. 4). 'Workers at all levels utilise mathematical ideas and techniques, consciously or unconsciously, in the process of achieving the desired workplace outcome' (EIMI Discussion Document, published as Damlamian et al. 2009, p. 526). In using this approach from the EIMI Discussion Document, we have at least a fuzzy borderline between general and vocational education by looking at the teaching of Mathematics as it is somehow related to industry in a broad sense, but we also have the option of looking at the teaching of concepts and procedures, which use elementary or advanced Mathematics, for the sake of using them in industry but not immediately taken as Mathematics.

I will also limit the text to vocational education for the adolescent group, aged 15–19 years. This saves me from commenting on the complex issue of tertiary education, which will not be analysed here. This decision also eliminates consideration of perhaps the most interesting roles of academic and pre-academic institutions in the preparation for jobs and professions, even if – at least at the tertiary/university level – most nations have special academic institutions to train future engineers and qualified business and administration people. Some of these institutions nowadays may have the same academic standing as traditional universities (like the former 'Technische Hochschulen' in Germany) and some are even considered among the most prestigious academic institutions in the nation (e.g. the 'École polytechnique' in Paris, France). The technical colleges in the United Kingdom and the 'Ingenieurschulen' in Germany (today often figuring as 'universities of applied sciences') demonstrate different types of organisational solutions to recruiting and training higher-level specialists in various professional activities. In some nations, one cannot even determine from an institution's name or its private or public way of functioning the level of vocational hierarchy for which they are training. Moreover, certain well-known academic institutions, which started as vocational training institutions at the highest level, have now developed into 'general' academic institutions of high standard (e.g. the 'Technische Hochschulen' in Germany are now offering nonvocational programmes or MIT in the USA). Focusing on the 15–19 age group and looking specifically at courses, institutions, and students related to industry in the broadest sense provides an opportunity to arrive at a well-defined subject for this chapter.

Given this very broad perspective on teaching vocational mathematics and taking into account an international perspective on this type of teaching, we can conclude that different countries have highly varied approaches to vocational mathematics education. If we look at any type of teaching vocational mathematics, we find this as a sometimes major part of general education, with types of secondary education explicitly preparing for the workplace. Some countries at least differentiate their upper secondary schools, offering specific types of institutions that prepare for entire vocational fields (like some types of upper secondary institutions in Austria and France). Other countries even offer special arrangements for the majority of its youth who intend to enter the workplace at the end of obligatory schooling (typically around the age of 16). The so-called dual system in Germany may be the most prominent example of vocational education, which still caters to more than a third of young adults in the 16–20 age range.

2 Two National Case Studies in Mathematics in Vocational Education

I have chosen Australia and Germany as cases in the history of vocational mathematics for two main reasons. The first is the obvious reason of access and availability. Being German, I have always been interested in the history of German vocational education in general – and I am happy to have some Australian colleagues to help me with their history of vocational mathematics. The second reason is that the history of vocational mathematics in the two nations, even though quite different, shows some prototypic features which are worth exploring and may help to understand historical developments in other nations. I have to add that it is impossible to describe these national developments completely within the space given for this text, so shortcuts, omissions, and generalisations over specific cases cannot be avoided.

To situate the two cases in a reasonable way, one should mention that vocational education, at least in Europe, is not a brand-new, modern development. As far back as the Middle Ages and 'all over Europe', there was training for specific competencies in 'industry' growing out of the training for future artisans, which in some places was institutionalised in specific organisations, especially guilds. The Arithmetic of the tradesmen (bookkeeping) and the birth of perspective drawing are examples of Mathematics which historically grew out of these institutions. What was left of vocational education in England for some time after the Second World War had clearly descended from this tradition, and specific mathematical topics were part of this training. The development of vocational training and schools in 'Germany' at the beginning of the eighteenth century also grew out of this source, both developments being a reaction to the needs of industry (in the usual, narrow sense) for qualified personnel.

2.1 Germany

Writing about the history of vocational mathematics in Germany is easier than for my second case, Australia, because there are publications on the German history of technical and vocational education in general (e.g. see Blankertz 1982, pp. 171ff) and at least one paper on the history of vocational mathematics in schools for adolescents aged 15–19, the vocational training schools ('Berufsschulen', see Grüner 1985). For Germany, only the history of the last 25 years has yet to be written from scratch. Nevertheless and before going into details, one should mention an important feature of German vocational education and training, which somewhat complicates the (hi)story: With the exceptions of Nazi Germany (1933–1945) and after the unification of the German Democratic Republic (DDR) with the Federal Republic of Germany (BRD) in 1989, there was no uniform political and administrative unit called Germany that could be reported on. In terms of the educational system, Germany was (and still is!) a conglomerate of very different educational systems, with especially large differences in the vocational parts of these systems. What follows, then, should be read as an identification of major trends in larger parts of what is now known as Germany and not a description of what occurred in all regions of Germany.

Germany is well known for one feature of vocational education and training, namely, the apprenticeship type of training with government and enterprises sharing vocational education and training. Even if the final examination of this system is usually run by professional bodies, the standard 3 years of training are conducted in part in state-run vocational training schools ('Berufsschulen', often 1–2 days per week or corresponding whole weeks during the 3 years) and work-based training in enterprises for the larger portion of these years of apprenticeship. Official sources state that the system takes more than 60 % of the young adolescents leaving the school system. As approximately 35 % do not leave schools around the age of 16 (most of them going to 'Gymnasium' to earn a university entrance certificate), we can assume that at present, more than one third of 16+ youth attends vocational schools as part of the apprenticeship system (for the actual figures, see, e.g. Bundesministerium für Bildung und Wissenschaft [BMBF] 2011, pp. 37ff).

The history of this system is telling: It came into being as late as the beginning of the twentieth century, but grew out of two different school systems. One forerunner catered to the majority of youth who had left general education (at that time usually around the age of 14) and was meant as a continuation and repetition of general education – in the beginning, normally held on Sundays – with the political purpose of securing loyalty to the church and the government ('Altar und Thron'). For the then mostly nationalist and conservative governments, this was a stronghold against rising Marxism and social democracy. Consequently, the mathematics 'taught' in this type of schools repeated arithmetic, the rule of three with applications to everyday life and political issues, and not specific vocational topics or areas of work. In some regions and especially for some ‘advanced’ professional areas like metalwork and later electricity, from the end of the eighteenth century, a sort of parallel but clearly vocational Sunday school developed. Since the middle of the nineteenth century, this led to industrial training schools (‘Gewerbliche Fortbildungsschule’), which specifically trained future masters for handicraft and small trade enterprises and consequently introduced mathematics related to specific professional areas like metalwork, electricity, and accounting. Only at the beginning of the twentieth century were these two sources collapsed into a vocational type of school (‘beruflich gegliederte Fortbildungsschule’), with specific classes for special vocations or vocational areas (e.g. turners in metalwork, hairdressers or retail salespersons). In Prussia, this process came to an end as late as 1911, when the Ministry of Trade and Commerce decreed curricula which (with the exception of the Nazi era) were valid until the 1950s, especially in the northern part of the then newly founded Federal Republic of Germany. Mathematics in these schools had to serve vocational subjects in these schools (‘Berufs-/Fachkunde’), but also looked into the mathematical issues of citizens and private households (‘Bürgerkunde’). Borrowing from the previous general repetition, mathematics normally started with elementary arithmetic and basic quantities (especially money, weight, and geometrical measures like area and volume) and percentages and then looked into the mathematics of the vocational area for which the school was training (like surfaces of curved bodies for metalwork, calculation of resistance for electricity or calculation of interest rates, discounts, and accounting for future tradespersons). Even in the early 1970s, I personally saw this structure when analysing textbooks for this type of schools (I hesitate to call them colleges) in my dissertation (Sträßer 1974, also see Sträßer 1978 and 1983).

Already in the late eighteenth century, these part-time schools were paralleled by full-time professional colleges, especially for handicraft people, which according to Grüner (1985, pp. 16ff) can be characterised as heavily relying on algebraic formulae. Initially meant as colleges for future masters of handicraft, the formulae needed for special vocations were taught and had to be learned by heart without deriving them or explaining them in detail: a full load of formulae in a short time with mechanical applications to professional ‘practice’. From the early twentieth century onwards, and because of teachers from these colleges were starting to teach in the vocational schools mentioned above, mathematics in vocational schools (especially in metalwork and construction) turned away from everyday arithmetic (‘bürgerliches Rechnen’) to calculations from mechanics (forces, velocity, gear transmission ratio, physical work, and power), even if most of the students, who came from schools without any preparation for algebra, had severe difficulties with this type of formula-bound vocational mathematics (for a discussion of related problems, see Sträßer 1981).

After the Second World War and the creation of specific classes for specific vocations or vocational areas (‘Fachklassen’), mathematics teaching in part-time vocational schools gradually leaned toward tasks from professional practice. For the mid-1980s, Grüner (1985, p. 19) distinguishes between two groups of vocational areas. The first group (in descending order of the importance of mathematics) is made up of electricity, chemistry, physics, biology, metalwork, building, wood, and printing technology. These areas can be characterised by a heavy use of algebra for formulae, diagrams, and the like. Calculations in these areas mostly came from the physical sciences in a broad sense. The second group contains health care, nutrition, domestic economics, the textile industry, painting, interior design, and agriculture and can be characterised by an absence of nonelementary mathematics. The vocations of business and administration are special groups involving many specific algorithms to calculate interest rates and other elementary algorithms from accounting and bookkeeping. A recent development partly influenced by competency-based ideas from politics and vocational education in general seems to be an integration of mathematics into vocational knowledge, implying the disappearance of a separate subject of mathematics in vocational schools. Some federal curricula (e.g. for metalwork and electricity) were organised along the structure of their respective vocational knowledge and practice, with mathematical knowledge integrated into modules on turning or resistance (to give two examples). With examination still manned and structured locally from professional bodies, this integration of mathematics into vocational knowledge seemed at least contested in some places and vocational fields.

What about German adolescents who did not move toward university entrance (i.e. going to 'Gymnasium') or apprenticeship? They represent about 20 % of those leaving general education (see BMBW 2011, p. 42) and, in most cases, enter full-time vocational colleges to prepare for specific vocations or vocational areas. It is not possible to describe the full variety of types and organisational solutions that exist for this clientele because the individual regions ('Länder') fervently play out their autonomy in this educational sector. Consequently, one cannot even sketchily describe the variety of mathematics taught in these colleges – and some institutions do not have any mathematics at all. However, one can identify a trend which had worked centuries ago (see the description of trade and agricultural schools by Schubring 1989, pp. 178ff). Starting at a very low level of vocational school, these institutions often developed into institutions of secondary but still vocational education and then into parts of general education preparing for university, if not eventually becoming a university type of institution. This move from vocational education to general education also occurred in the 'Fachoberschulen' in twentieth-century Germany: They started as full-time colleges to prepare for vocations like business or metalwork, but – for the larger portion of their students – gradually turned into institutions preparing for university entrance while the aim of preparation for vocational practice gradually faded. Especially in 'Fachoberschulen', this had consequences for the mathematics taught in these institutions: In most 'Fachoberschulen', mathematics became a poor image of mathematics taught in 'Gymnasium'. At present, their now-obligatory teaching of calculus heavily (and even more than in 'Gymnasium') relies on the inculcation of algorithms without aiming at conceptual understanding, with very little attention to applications specific to the vocations they are nominated to train for. This development is detailed in Blum and Sträßer (1992, pp. 243f).

2.2 Australia

Given the lack of a 'standard' text on the history of vocational education in Australia,[2] I follow a short description from Griffith University, Brisbane. Vocational education in Australia started in the late 1880s with the foundation of 'mechanics institutes' and 'schools of arts' – somehow imitating the system of vocational education in the United Kingdom, the home country of the majority of immigrants and persons in power. 'Technical education remained the sole responsibility of the States and Territories after Federation in 1901. Technical education evolved gradually in each State and Territory, generally remaining under-recognised and undervalued but with increases in funding and support occurring mainly in times of crises such as during the Depression years of the 1930s and World War II'. Financial assistance and responsibility for vocational education remained with the Australian states until the 1970s. Again because of a crisis (this time possibly one in the supply of a skilled workforce in 'new industries like communications, finance, and other service industries'), federal action was taken, resulting in the creation of 'Technical and Further Education (TAFE)' institutions following the Kangan report (1974). The next major step seems to be the 'establishment of the Australian National Training Authority (ANTA) in 1992, [which] led to the development of the current national system of vocational education and training (VET) working through the co-operation of all States and Territories' (for this very brief sketch and all quotations, see the description at http://www.griffith.edu.au/vc/ate/content_vet_hist.html). As we will see below, the establishment of the TAFE system and the ANTA, with government-influenced research into vocational education often done within the 'National Center for Vocational Education Research (NCVER)' (URL: http://www.ncver.edu.au/), led to important changes within vocational education in Australia generally and within vocational mathematics specifically on this continent.

[2] The 'reference list' offered by the official NCVER website on the 'History of VET in Australia' is quite revealing, see http://www.ncver.edu.au/resources/timeline/overview.html#Overview

According to descriptions from my Australian colleagues (here I basically follow Fitzsimons 1997, 2002 and Javed 2008) and till the 1970s, if not the 1980s, vocational mathematics in Australia echoed similar neglect of vocational education that pervaded Australia in general. For the past (pre-1990s), Fitzsimons 1997 (p. 299) describes adult and further education courses in mathematics as 'unregulated', while courses in vocational and technical education 'have generally been taught from standard traditional epistemological and pedagogical positions'. She writes: 'The emphasis (in vocational mathematics courses, insert RS) was on basic skills and pragmatism. In the higher level certificate classes for technicians, areas with a heavy commitment to mathematics had strictly enforced syllabi and external examinations' (Fitzsimons 1997, p. 301).

This situation seems to have changed in the early 1990s with an orientation to competency-based training ('CBT') supported by companies and industry and enforced by law. With the introduction of a 'nationally consistent hierarchical curriculum' 'across all vocational areas', 'fractions and decimals … reappeared', even if 'these mandated arithmetical processes do not reflect the actual mathematical needs of industrial workers.… In fact such workers are likely to be using and understanding much more sophisticated mathematics and statistics, simply because they are contextually significant, but may be unable to satisfy competency requirements on a basic skill test without the use of a calculator' (Fitzsimons 1997, p. 302). From the analysis of Fitzsimons and a study of more recent government documents, one may conclude that the rise of CBT in vocational mathematics education first led to a revival of basic elementary mathematics in vocational education, implying difficulties with the workplace context and its more specialised, sometimes more demanding nature. Ongoing developments are difficult to describe because the negligence of vocational mathematics and vocational education in general has led to sparse research in the area. Only two pertinent dissertations could be found (Fitzsimons 2001; Javed 2008), with Javed's thesis suggesting that interest in the introduction of new technology drained any last bit of academic attention on teaching and learning vocational mathematics.

On a formal curriculum level, we can find a number of courses on 'mathematics' in the TAFE system. In November 2011, a search for the TAFE system under http://coursesnow.com.au/search/?query=mathematics produced a list of more than 30 courses, with a large number offered by TAFE NSW under the headline 'apprenticeships' and two with two levels named 'certificate III' and 'certificate IV'. TAFE NSW offers an explanation for these courses: 'Apprenticeships and traineeships combine work and structured training. They allow you to learn a trade or workplace skill and receive a nationally accredited qualification. Apprenticeships usually last 3 or 4 years and cover 'traditional' trades such as carpentry, electrical, hairdressing and plumbing'. The explanation given for the 'Certificate II in General and Vocational Education (CGVE)' is 'This course is for participants who have not previously achieved a Year 10 qualification. This qualification develops general education, employability and industry specific knowledge and skills at a Year 10 equivalent standard. It aims to develop literacy, numeracy, introductory science skills, computing, vocational & sociology skills to a recognised level of education' (quote from the course description on the website). Based on this explanation, I assume that certificate III and IV courses can truly be considered vocational. A look at one of these courses ('Certificate III in Engineering – Fabrication Trade (Light Fabrication)', supplied by TAFE NSW) shows a description of what is regarded here as vocational mathematics (I quote names of 'modules/units'): 'Perform engineering measurements', 'Perform computations' and 'Interact with computing technology' are part of the course's mandatory units; the units 'Perform geometric development', 'Perform advanced geometric development cylindrical/rectangular', 'Perform advanced geometric development conical' and 'Perform advanced geometric development transitions' comprise an elective group: 'Set computer controlled machines/processes', 'Set and edit computer controlled machines/processes' and 'Write basic NC/CNC programs' form another group of electives; and with the highest 'weight' in the modules, 'Prepare basic engineering drawing' and 'Perform basic engineering detail drafting' is a fourth elective group. A module with less weight 'Apply mathematical techniques in a manufacturing engineering or related environment' completes this list of elective 'mathematical' modules in the certificate III course. One sees that the mandatory courses are very general and could also be part of general mathematics education, while the elective courses attempt to identify workplace-related content involving mathematics. In particular, the geometry and drawing content seems to be or can be taught in close relation to a person's workplace in light fabrication. In addition, the explicit role of new technology should be acknowledged and is confirmed (together with the important role of drawing and geometry) if one looks at the modules of the course 'Certificate IV Engineering (Fabrication)'.

I need to temper this positive statement by mentioning that the level III-certificate of 'Food Processing (Retail Baking – Bread)' only had 'Use basic mathematical concepts' and 'Create and use simple spreadsheets' as modules obviously related to mathematics. To say the least, vocational mathematics seems to differ in nature and scope according to different economic sectors. As a complement, however, it should be mentioned that a search in the present 'Australian Qualifications Framework' (2011) or the respective 'Handbook' from 2007 shows no mention of the word 'mathematics' (or 'calculation' or the like; see AQF advisory Board 2007). Mathematics does not seem to be explicit in the general and official framework for (workplace) qualifications. As a consequence, I assume that its role in the TAFE certificates is due to the influence of industry (in a broad sense), which requires a competent workforce trained in the TAFE system, despite what the government is prepared to acknowledge as workplace qualifications. In stating this, I do not start from the assumption that industry is in the best position to identify vocational mathematics (see above for the German situation).

Recently published statistics also offer information on the relevance of these activities in Australian society (see NCVER 2011). In Australia, '78.8% of 15- to 19-year-olds participated in education and training as of August 2010', with '52.7% at school, (1.1% participated in school-based apprenticeships and traineeships, 13.1% participated in other VET in Schools programs; 38.5% did not participate in VET in Schools programs). Moreover, 13.8% were enrolled in higher education, 4.7% participated in trade apprenticeships or traineeships, 1.8% participated in non-trade apprenticeships or traineeships, 5.8% were enrolled in publicly-funded VET programs' (quote from the 'highlights' of NCVER 2011, first page). This indicates that slightly more than a quarter of those between the ages of 15 and 19 somehow studied 'in relation' to the vocational sector, while half of them received part of their general education in schools. Using the 'course characteristics' analysed later in this document, 32.2 % and 37.9 % of VET students (if I understand correctly: from outside school!) took Certificate II and Certificate III courses (classified according to the above-mentioned Australian Qualifications Framework; see p. 18 of the document). This implies that only a small minority of adolescents were somehow exposed to some sort of vocational mathematics.[3]

3 Additional Remarks

People reflecting on vocational mathematics may be tempted to simply ask a worker what mathematics he or she needs. An alternative but comparable approach could be to ask management what mathematics it thinks is necessary to do the job. This approach is often suggested and actually tried in the early 1980s, especially in the United Kingdom with the Cockcroft report. No. 66 of the report comments on the two 'research studies' preparing the report:

> Both studies draw attention to the diversity of types of employment which exists, to the variety of mathematical demands within each and to the considerable differences which were found to exist even within occupations which might be assumed from their titles to be similar. It is therefore not possible to produce definitive lists of the mathematical topics of which a knowledge will be needed in order to carry out jobs with a particular title. (See Cockcroft 1982, section no. 66 of the report)

[3] Unfortunately, the term 'VET-student' is not clearly defined in the document.

Consequently, the easiest way to determine vocational mathematics – by asking those who work in the area or manage the work – seems to be blocked. The Cockcroft report continues:

> It is, however, possible to describe in general terms the types and levels of mathematics which are likely to be encountered by certain broad categories of employees.

Thus, there is some legitimacy in the broad descriptions of vocational mathematics given in the case study on Germany in Sect. 2.1, even if a one-to-one relation between mathematics and specific jobs or occupations does not seem feasible. Sections 2.1 and 2.2 attempted to cope with the impossibility of identifying vocational mathematics for specific vocations by citing examples from very specific courses in Australia and mentioning broad areas of mathematics as typical for larger vocational fields for Germany.

Another and more recent development in the area of vocational mathematics complicates the issue of identifying vocational mathematics. In some way a consequence of the integration of mathematical knowledge into vocational and because of the growing use of information technology linked to mathematics in industry, mathematics educationists have further analysed the intimate link between specific mathematical and professional knowledge. Starting with a study on the use of mathematics in the banking sector (see Noss and Hoyles 1996), English and Dutch scholars introduced the theoretical concept of 'techno-mathematical literacies' to name 'the mathematical knowledge that is required to be effective in an ICT-rich context.… It is akin to literacy in that it involves interpretation as well as the ability to appreciate and communicate with others about mathematical information; and it is mediated by technology – *techno* – in that the information is expressed through symbolic artefacts generated by automated systems' (from Hoyles et al. 2010 and 2013). Techno-mathematical literacy ('TmL', as Hoyles calls it) may be an appropriate way of conceptualising the intimate relationship between mathematical and vocational knowledge in modern, industrialised societies. The recent curricular integration of mathematics into vocational courses and contexts described for Australia and Germany may illustrate this theoretical development in the didactics of vocational mathematics.

From the case studies above and a plethora of similar cases in other countries and circumstances, one can learn about an important development of vocational education institutions, which has severe consequences for vocational mathematics. More often than not, an institution starts as a school or college to cater to the need of training for a specific economic development, i.e. as a clearly vocational school or college. In most cases, this institution will have a distinct curriculum answering the mathematical (and other!) needs of this economic demand. Such institutions often prove to be so successful that they gradually change into institutions of secondary, if not tertiary, education, totally losing their specific vocational flavour, thereby perhaps ending up as very well-respected academic institutions (like the École polytechnique in France, some 'Technische Hochschulen' in Germany or MIT in the USA). Viewed from the inside – and as can be seen in the Australian TAFE institutions and German Fachoberschulen, vocational colleges integrate curricular modules preparing for university entrance and – by this internal process – weaken the vocational character of the institution. Such developments also make it difficult to write a history of vocational mathematics.

4 Conclusion: Speculating on a History of Vocational Mathematics Education

I would like to end this brief text on the history of vocational mathematics with a general remark about how this issue will develop in the future. From what can be seen from the different nations with their different ways of handling the role of mathematics in professional knowledge and practice, one can predict a rather sad development. As long as vocational education and the role of mathematics in it are neglected areas of research, development, and practice, it will be difficult to write a history of vocational education. Moreover, as long as there is no account of the history of this most important part of applied mathematics (if this concept makes sense), 're-inventing the wheel' will be a standard but wasteful procedure in vocational mathematics and vocational training and education in general.

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Chapter 26
Mathematics Teaching Practices

Amy Ackerberg-Hastings

While instructional methods are interwoven with the stories of nations, subjects, and time periods told elsewhere in this volume, this chapter concentrates specifically on the historical existence and significance of mathematics teaching practices. In particular, this chapter highlights two fundamental characteristics of the history of teaching practices: the diversity of practices that existed in the past and the meandering, nonprogressive paths along which practices evolved. Addressing the diversity of practices is especially important, if only because all too often there appear in the mathematics education literature expressions such as "traditional teaching," which imply that formerly teaching was more or less uniform. In reality, formerly teaching was quite varied. Many different traditions not only existed in different parts of the world but coexisted in the same regions. This diversity reveals apparent similarities between the past and present in aims and techniques, as well as differences shaped by changes in context over time. For instance, available technologies, cultural values, and pedagogical preferences all influenced whether teachers required students to present their work orally or in writing. Elsewhere, some forms of practices have appeared repeatedly, through rediscovery or through the evolution of societal goals and, in turn, educational fashions.

Thus, like all of history, the history of teaching practices is messy and complex. It is not possible to list every teaching practice ever encountered by students. After all, formal teaching and learning of mathematics began at least as early as 2600 BCE, when Sumerian scribal students copied and memorized arithmetical and multiplication lists on cuneiform tablets (Robson 2008). The overlapping events and complexity of the subject inhibit attempts to construct overarching chronological narratives. Indeed, a comprehensive monograph on teaching practices has yet to be written. We have not attempted to describe every important practice nor even to provide detailed tracings of transformations in any one practice or analysis of the causes of these transformations. Instead, this chapter presents a selection of snapshots from the teaching and learning of mathematics, which happen to be largely confined to Europe and North America since the eighteenth century, with a particular emphasis on the United States during the nineteenth century. This approach allows us to suggest techniques that may be applied to researching and writing other aspects of the history of teaching practices.

This chapter thus additionally introduces the processes of identifying examples of practices and of historically evaluating the examples found. For instance, documents containing information on teaching practices are not typically already organized and synthesized but rather must be unearthed and mined for data. Textbooks are an obvious primary source for the history of teaching practices, yet this form of evidence is inherently limited and must be analyzed carefully. As a genre of literature,

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textbooks did not exist before the eighteenth century (Karam 2000–2001). Whether they were treatises utilized in instruction before the invention of textbooks, such as Euclid's *Elements of Geometry*, or volumes written by mathematicians or educators for students and marketed by publishers to schools and colleges, such as Leonhard Euler's *Elements of Algebra*, textbooks are notably unreliable as narrators of classroom practices. These materials are prepared in advance, sometimes at a great distance of space and time from the people who used the textbooks, and even annotations added by teachers and students can only approximate how a class period unfolded. Even though textbooks may provide the author's ideal conception for the content of a course and can even suggest a recommended approach to teaching through the book's format and style, these sources are not sufficient sources of information for how practices were employed. To tell a complete story that approaches the truth of what really happened in classrooms, historians of mathematics education must consult and compare a wide variety of materials beyond textbooks: syllabi and lesson plans, student notes, diaries and memoirs by teachers or students, obituaries of beloved instructors, institutional histories, administrative records, copies of examinations, and documents prepared to further campaigns for reform. For early eras, archaeological and anthropological evidence may be necessary.

Readers often must also engage in inference and comparison when they consult secondary sources for information on teaching practices. As in Cajori (1890), books and articles tend to mention the actual conduct of classes only in passing while concentrating on other aspects of the history of mathematics education. Data may be reported but not interpreted, under a false assumption that facts speak for themselves. Anecdotes may be presented outside of their historical context. Even worse, mathematics education literature can fall prey to celebrating "heritage" and neglect "history" (Grattan-Guinness 2004). This is another reason why this chapter limited itself to a few of the countless possible examples of teaching practices, so that we may model thoughtful and thorough historical analysis of primary and secondary sources in a manageable number of cases.

Despite the challenges that arise in documenting a subject based on transitory, everyday events and experiences, recorded in hard to find and yet overwhelming quantities of source material, the history of teaching practices offers fruitful training grounds for students of the history of mathematics education. (Primers on historical research and writing include Tosh 2010 and Storey 2009.) Historians of science and historians of education have demonstrated how methods developed in the professional discipline of history in the late twentieth century, including the "new" history of the book, new social history, and the "cultural turn," permit scholars to uncover the meanings embedded in educational sources. For example, Susan Lindee (1991) showed how teachers and administrators at American female academies took a stand on gender by consciously choosing Jane Marcet's *Conversations on Chemistry*, because they wanted young women to understand chemistry as a theoretical subject more than as an aid for domesticity, while Jennifer Monaghan (2005) asked questions about instructional purpose and method in order to identify and explain a transformation in American children's literacy around 1750.

As a first step toward unpacking the rich possibilities promised by reading primary sources "against the grain," like historians utilizing the cultural turn, and to provide a tool for conceptualizing similarities and differences among characteristically diverse teaching practices, we suggest three categories of practices that correspond to three stages in the learning process: practices of acquiring knowledge, practices of rehearsing and reinforcing knowledge, and practices of assessing knowledge. These categories serve as an organizing principle for this chapter, although we recognize that any attempt to impose order on the broad variety of historical examples is necessarily an arbitrary schematic.

1 Practices of Acquiring Knowledge

Until recently (with works such as Ellerton and Clements 2012), historians of mathematics education have typically been more concerned with the content of the curriculum than with how teaching was conducted. In part, at least according to our reading of eighteenth- and nineteenth-century British and

American literature, this is because educators were usually silent on the issue before the twentieth century. For instance, in a meeting of teachers held in Ypsilanti, Michigan, in 1852, mathematics textbook compiler Charles Davies argued that effective school instruction was most dependent on the moral character of the teacher, making no mention of specific classroom methods or structures (Davies 1852). Since there are no obvious historical lists of types of practices, we have divided examples of practices into groups according to a broad conceptualization of how students are usually expected to learn. First, they are presented with new information and concepts so that they may acquire knowledge from the presentations. Second, they repeatedly practice the material until they or their teachers believe that they are ready for formal assessment. Finally, they are examined on their mastery. In other words, we emphasize processes rather than styles of teaching and learning, unlike educational theorists such as Fischer and Fischer (1979).

When they think of educational presentations, readers trained in Western twentieth-century mathematics classrooms may initially think of teachers lecturing at the front of a classroom. The technique of lecturing dates at least to medieval Europe. By the fourteenth century, European university professors read from a manuscript copy of the text and students wrote down what they heard (Novikoff 2012, pp. 351–355). Both before and after the advent of the printed book, this tradition continued to be a favored technique in higher education, especially wherever textbooks were expensive or otherwise scarce. By the seventeenth and eighteenth centuries, lecturers might also write out their own presentations, perhaps in preparation for publishing textbooks in the subjects they were teaching. Students copied what they heard, in as verbatim a form as possible.

Since there is little surviving evidence for lecturing in school mathematics, particularly for periods before the twentieth century, we provide examples from colleges and universities. John Playfair of the University of Edinburgh, who was Joint Professor of Mathematics from 1785 to 1805 and then Professor of Natural Philosophy from 1805 to 1819, was well known for reading directly from his script, albeit in an erudite and thorough manner. While no lecture notes by the professor or by students are known to have survived from Playfair's algebra, geometry, or fluxions courses,[1] five sets of student notes survive for natural philosophy lectures Playfair delivered between 1806 and 1812. (The subject of natural philosophy was roughly equivalent to what a modern student would learn in a non-calculus general physics class.) These notes indicate that students recorded what was read to varying levels of completeness and thus remind us that the care taken by individual students with their own educations also varies significantly. (On assessing student notes, see also Leme da Silva and Valente (2009) and Blair (2004).) Even so, since the order of the topics is similar in each set of notes, these manuscripts help researchers trace the evolution of Playfair's course material before he published the two-volume textbook, *Outlines of Natural Philosophy*, in 1812 and 1814 (Jeffrey and Stewart 1822, pp. xxii, lxiv–lxv; Ackerberg-Hastings 2009).

Rather than read lectures verbatim, other professors and teachers have adopted more extemporaneous or even theatrical styles. John Farrar of Harvard College (1779–1853) taught algebra, geometry, natural philosophy, and astronomy, and he apparently occasionally offered individual instruction in calculus. One of his students, Andrew Peabody, wrote:

> His were the only exercises at which there was no need for a roll-call. No student was willingly absent. The professor had no notes, and commenced his lecture in a conversational tone and manner, very much as if he were explaining his subject to a single learner…. His face glowed with the inspiration of his theme. His voice, which was unmanageable as he grew warm, broke into a shrill falsetto; and with the first high treble notes the class began to listen with breathless stillness, so that a pin-fall could, I doubt not, have been heard through the room. (Peabody 1888, pp. 70–71)

Another student in the mid-1820s, Charles Francis Adams, did not offer quite as positive a review of Farrar's classroom presentations, noting that, even though he lectured from memory, he in fact closely followed the textbook, William Enfield's *Institutes of Natural Philosophy* (1783). Additionally,

[1] A list of topics Playfair covered in a course of "practical mathematics" is found in Playfair 1793.

Adams reported that the natural philosophy experiments demonstrated by Farrar often failed – Playfair and Farrar, like many natural philosophy professors in the eighteenth and nineteenth centuries, supplemented their lectures with in-class demonstrations performed for the students. Adams also noted that a copy of Farrar's astronomy lectures was in circulation (Donald and Donald 1964, pp. 113–133, 358–359, 381). This enabled students to focus on the classroom experience without worrying about note-taking, since they could fill in anything they missed from the content later.

Perhaps efforts to blend entertainment with information delivery, such as Farrar's, foreshadowed attempts in the twentieth century to communicate material over audiovisual media. For example, soon after the invention of television in 1939, some educators asked whether technology might replace teachers. Mention should also be made of the use of films in class (one example is found in the works of Jean Louis Nicolet from the 1950s; see Gattegno 2007). By the 1960s, there were numerous initiatives developing mathematics programming to be shown in classrooms or broadcast over the air. Institutions (including both governmental and commercial media providers such as the BBC in the UK or special educational ones such as Telescola in Portugal) delivered mathematical content through television. Some programs were formal in style and shown for teacher training or in outreach to remote areas. Others, perhaps because they targeted children, used dramatized situations to set up mathematical applications and parodied popular commercial programs. The 52 episodes of "Infinity Factory," conceived by emeritus MIT physics professor Jerrold Zacharias in 1974 and aired by the US Public Broadcasting System (PBS) in 1977, employed rock music, sketches based on "Laugh In," and settings familiar to impoverished African-American and Hispanic children to catch the attention of 8–11-year-olds ("By the Numbers" 1976). Topics covered included mapping and scaling, graphs, negative numbers, and weights and measures. Possibly because desires to make its experiences interactive went unrealized, the medium remained a relatively small part of mathematics teaching practices – in one typical statistic, television was utilized in only 10 % of American university mathematics classrooms in 1975 (Fey et al. 1976, p. 56).

The textbook also can be an example of material presented to students without opportunity to interact directly with the presenter, particularly if a learner read the textbook independently. This practice began to develop when textbooks became accessible to many students in the eighteenth century and persists in the twenty-first century, although it was apparently never as popular as formal forms of instruction provided in classrooms. In the late nineteenth and early twentieth centuries, authors and publishers introduced features to enhance readability, including drawings and photographs as well as varied typography. In 1888, George Wentworth argued that attention to the appearance of the page was vital to catching the interest of young students:

> Great pains have been taken to make the page attractive…. This arrangement presents obvious advantages. The pupil perceives at once what is given and what is required, readily refers to the figure at every step, becomes perfectly familiar with the language of Geometry, acquires facility in simple and accurate expression, rapidly *learns to reason*, and lays a foundation for the complete establishing of the science. (Wentworth 1888, pp. iii–iv, emphasis in source)

Boldface and italic typefaces signaled distinctions between proofs and examples, highlighted key principles, and connected ideas students encountered earlier in the textbook with new concepts. In doing so, books such as Wentworth's and *Plane and Solid Geometry* (1915), by Webster Wells and Walter W. Hart, also changed the ways in which subjects were organized and taught.

Once blackboards gained wide introduction into mathematics classrooms at all levels of education in the early nineteenth century, teachers could complement lectures by drawing diagrams in advance or in real time, by solving illustrative problems for students to imitate, and by pausing to discuss their processes. For instance, the Russian poet Yakov Polonsky praised one of his teachers in the 1830s for explaining and writing out formulas at the blackboard, so that students could memorize them (Karp 2007, pp. 110–111). There is not much concrete evidence that teachers regularly did these things before the twentieth century, when the advent of overhead projectors further fostered interaction by permitting the teacher to face the class (Kidwell et al. 2008, pp. 21–34, 53–68; Wylie 2012).

As we will see in the next section, it is certain that blackboards became vital necessities for practices of rehearsing and reinforcing knowledge.

Even with tools such as blackboards or audiovisual media, students have often failed to engage with teacher-centered, one-sided practices of acquiring knowledge. Many recollections have survived about how dull and uninteractive mathematics lessons were. One nineteenth-century Russian memoirist, for example, recalled his teacher in the following way (cited in Karp 2007):

> Walking into the classroom with his eyes lowered to the ground, he would come up to the blackboard, stand with his back to the class, and begin his lecture in a quiet, monotonous drone or almost in a whisper, reading from his notes and making various calculations and sketching mathematical figures on the blackboard…. For the most part, the cadets slept through his classes.

In part to combat this student apathy, the ideas of what today would be called "active learning" appeared centuries ago. Without attempting to discover the historical origins of these practices, let us note that the pedagogical views of Rousseau had an influence on the teaching of mathematics. Teaching by observing nature and engaging in practical tasks became increasingly popular; as a consequence, there appeared more and more classrooms in which emphasis was placed on principles that students discovered for themselves rather than through what teachers told them.[2] Textbook writers of the French Enlightenment and Revolution, for instance, explicitly advocated this type of approach. Their emphasis on practical training stemmed in part from the clear ordering of ideas espoused by the seventeenth-century Port-Royal school of logic. Eighteenth-century textbooks such as Alexis Clairaut's *Éléments de géométrie* (1741) required readers to conduct measurements with instruments before they were introduced to abstract geometrical concepts. The emphasis on self-guided, practical work continued throughout the text. Sylvestre François Lacroix's *Traité élémentaire d'arithmétique* (1797) and *Elémens de géométrie* (1799) argued that students learned best by following the order of a subject's historical development (Dhombres 1985; Barnard 1969; Boyer and Merzbach 1989).

Widely influential on the nineteenth-century attempts to teach through student discovery of concepts was the Swiss educational reformer Johann Pestalozzi. His plans for schools urged that education focus on the child by providing active and sensory experiences of the natural world that develop the mind, the body, and the moral nature. He advocated consideration of what is now called educational and developmental psychology. Initially, his ideas were propagated in other countries through textbooks, such as American Warren Colburn's *First Lessons in Intellectual Arithmetic* (1821), *Sequel to the First Lessons* (1822), and *An Introduction to Algebra upon the Inductive Method of Instruction* (1825). Colburn started by asking students to solve simple practical examples, from which they articulated general rules. He concentrated on one principle at a time, and he led students from acquiring knowledge to repeating and rehearsing what they had learned through a practice called recitation, which is discussed in the next section (Monroe 1912). Meanwhile, in the emerging subject of "informal geometry," German instructors applied Pestalozzian ideas by using drawing and arithmetic to encourage students to discover geometrical truths on their own. In the United States in the second half of the nineteenth century, there was a spate of informal geometry textbooks aimed at children and bearing buzzwords such as "inductive," "concrete," "observational," and "useful" in their titles (Coleman 1942).

At the turn of the twentieth century, mathematics educators around the world tried to organize mathematics classrooms as if they were science laboratories (Turner 2012). For example, in Italy between 1902 and 1909, Giovanni Vailati proposed replacing crowded classrooms and mind-numbing presentations of content with an arena where students figured out how to solve problems and discovered mathematical truths. Drawing, experimental geometry, and connections between algebra and geometry were to be emphasized. Although some of his ideas were adopted by the Royal Commission

[2] It is impossible here to go into a discussion of how teaching practices are related to the number of students in a class and to the accessibility of education in general; this problem, however, must be mentioned. In particular, many practices arose in the context of individual instruction.

charged with school reform and his plan received international attention, Italian teachers ultimately rejected the program as too radical (Giacardi 2009). In 1902 in the United States, E. H. Moore proposed that the University of Chicago take the lead in training prospective secondary school and college teachers to have their students make drawings and build models. Although he failed to secure funding to construct a laboratory building at Chicago, Moore continued to work on the concept and corresponded with John Perry of England about his version of the laboratory method, which also included the use of graph paper (Roberts 1997, pp. 247–253). American educational leaders remained interested in teaching through experiment for the next few decades, with Raleigh Schorling of the 1923 American National Committee on Mathematical Requirements explaining that laboratories in secondary schools needed to be equipped with compasses, protractors, yard stick, and balances, all sized for the blackboard. The blackboard was to be marked in cross sections or squares. Individual students were each to be supplied with compasses, protractors, rulers, and squared (graph) paper (NCMR 1923, pp. 277–278).

Tools and objects periodically resurged in popularity, often while different practices were employed sequentially in the second half of the twentieth century. American educators in the 1950s and 1960s who employed the "new math" used drawing instruments to introduce geometrical constructions. Then, they moved on to present abstract principles and to create diagrams formally, although they suggested that elementary students check their work with protractors (School Mathematics Study Group 1963, pp. 301–302, 675–741, 887–905). Manipulatives such as Cuisenaire rods and Dienes's blocks were also employed through combinations of teaching practices. New technologies, such as graphing calculators, provided teachers with additional means for encouraging students to discover mathematical ideas through experiment – as well as for interweaving laboratory class sessions with presentations of content by teachers. Instead of facilitating student work on experiments, other teachers had students encounter new material through problem solving. This instructional structure was used especially in classes for the mathematically gifted,[3] as in certain Moscow schools from the 1960s on (Vogeli 1997). The notion of this kind of instruction was employed even prior to this in colleges – it is often associated with R. L. Moore and called the "Texas method" or the "Moore method" (Parker 2005).

Many practices fell between the extremes of content delivered exclusively by teachers versus students actively constructing all of their own knowledge. At the turn of the nineteenth century, Andrew Bell and Joseph Lancaster found a particularly interesting way of blending these aims. While serving as a chaplain in Madras, India, Bell observed how students in Tamil schools worked together to memorize essential principles under the watchful eye of the community. In England, Lancaster adapted the system into a factory of learning, with a student monitor assigning work to a class of students slightly younger and less knowledgeable than the monitor and then checking the results, written on inexpensive and reusable slates. The monitors followed detailed daily schedules, which were carefully planned by the teacher. The slate work taught children to keep busy at practical tasks. The system spread around the world – and in fact was reintroduced by British colonial officials in India (Babu 2012; Hall 2003). American Mary Lyon described the use of the monitor system at her school as follows:

> We have been attending to Adams's Arithmetic on the monitorial plan for a long time, with usual, or rather increasing success. We shall very soon lay it aside for the season. Between fifty and sixty have attended this exercise together, comprising all the regular classes except the senior class; and, indeed, most of this class have been engaged all the time as monitors. We have adopted the plan of having a regular monitor for every section, consisting of from five to twelve, according to the capacity of the monitors and of the students.

[3] This touches on another issue, which cannot be discussed in detail here, but which we cannot omit to mention nevertheless: the differences among, and the specific characteristics of, teaching practices in classes with students with different levels of abilities.

Somewhat later she remarked: “This exercise is very pleasant (Lansing 1937).”

For at least 100 years, another teaching practice by which students acquire new knowledge has involved introducing new material in the course of a discussion with the whole class. Young (1925, p. 65) called this practice the “genetic mode” and characterized it as follows:

> In the genetic mode, the subject matter is developed by the class guided by the teacher. All work and think together, the pupils expressing their views as permitted or requested by the teacher, who acts as chairman or leader, assists by questions, hints, and suggestions, sees to it that the discussion reaches the desired result in a reasonable length of time, but allows it all the latitude consistent herewith.

In general, many twentieth-century teaching practices involved breaking up classes into groups or even requiring students to study new materials individually (as was done in medieval and Renaissance schools, where students studying completely different things could be found in the same room simultaneously). With the appearance of computer technology, the individualization of education in public schools began to rely, for example, on the use of the internet. Throughout the twentieth century, students were involved in learning through work on practical tasks. This was a chief aim of the Progressive movement in the United States, but such teaching practices perhaps reached their greatest scope, in terms of the number of students affected by them, in Soviet Russia after 1917. One such practice, for example, consisted of mathematical field trips. One Soviet teacher describes the beginning of such a field trip in the following way (cited in Karp 2012): “The first thing that we did, en route to the state farm, was to determine distances by eye and then immediately verify them by using a tape measure, counting steps.” Groups of students worked on mathematical projects (in effect, following the recommendations of William Heard Kilpatrick), such as drawing up plans for various constructions.

2 Practices of Rehearsing and Reinforcing Knowledge

We saw in the previous section that boundaries between practices are permeable within any given stage of the learning process. The practices of acquiring knowledge included those in which the class worked as a whole, those in which the class broke up into groups, those in which students worked on their own, and those in which one class sometimes worked as a whole and at other times in groups or individually. Similarly, boundaries are easily crossed between these categories of teaching practices. One can ask specifically where the dividing line is drawn between informal and formal means of assessment. For our purposes, although there is also some overlap with presentations of information and concepts, practices of rehearsing and reinforcing knowledge include daily activities completed by students in class or as homework. Tests and examinations, as more periodic events with higher stakes, are discussed in the next section of this chapter. The examples in this section are drawn almost exclusively from the history of the United States, but, as with the examples in the other sections, cases from other countries of rehearsing and reinforcing knowledge could have been given.

One form of written work with a lengthy history and that functioned both as a practice of acquiring knowledge and as a practice of rehearsing and reinforcing knowledge was the cyphering book. Nerida Ellerton and Ken Clements (2012) argue that cyphering books descended from medieval Italian manuscripts of commercial arithmetic, accounting, and geometry that boys prepared over 2 years of study in an *abacus* school. In Great Britain and North America in the eighteenth and nineteenth centuries, boys and girls worked on their penmanship as well as on the mathematics merchants needed for daily life by preparing books they kept as references for the rest of their lives (Doar 2006). The students worked under the supervision of a tutor or private school teacher. Some copied down dictation, and some copied directly from a book owned by the instructor or from problems worked out by the instructor.

Perhaps the oldest and certainly the easiest to implement practice of daily student work was rote memorization. Students learned rules from the teacher or, beginning in the late eighteenth century, from a textbook such as Thomas Dilworth's *The Schoolmaster's Assistant* (1743), in part by repeating them as a class. Then, when they were presented with a problem in the classroom or out in the world, they were expected to know how to call forth and apply the appropriate rule (Wren and McDonough 1934). Although in the past and in the present rote, drill has appeared to be the default teaching practice in historical schools, students and teachers alike found it boring, lacking utility and the ability to develop thorough understandings of concepts. Educators recommended alternatives. Official Russian documents going as far back as 1810 emphasized that "teachers must be required to know not any mechanical methodology, but one that can facilitate a genuine enrichment of the mind with useful and necessary truths" (Karp this volume). In order to spur students consciously to assimilate what they had learned, and not to memorize it by rote, students were asked, for example, to carry out proofs of theorems they had studied using diagrams with different letters from those in the textbook (Karp 2007).

In the United States, educators and administrators attempted to avoid rote memorization by highly developing a form of oral daily work called recitation. Like many mathematics teaching practices, recitation was created at institutions of higher learning – in this case, most notably at the US Military Academy at West Point – and adopted wholesale into high schools from the time of the invention of these institutions in the middle third of the nineteenth century. To an extent, methods and styles from colleges also entered into common schools, academies, and primary schools throughout the nineteenth century (Kidwell et al. 2008; Jones and Coxford 1970). Thus, we describe the West Point origins of the practice (Albree et al. 2000). Blackboards were mounted on all four walls of a mathematics classroom, where no more than 15 cadets met for 3 h each day, 6 days a week. Upon entering the classroom, 5 or 6 cadets proceeded to the blackboard, where the instructor assigned each a proposition to prepare. Once the first cadet was ready, he presented his work to the instructor, who followed up the presentation by quizzing the cadet on the proposition, its diagram, and any aspect of the material recently studied in class. As the instructor moved on to the next cadet, another young man would replace the first cadet and begin drawing his proposition. The instructor assigned daily grades on a three-point scale; the class sections were regularly reorganized according to these grades so that students of like abilities would be in the same section (Rickey and Shell-Gellasch n.d.). While their practices were perhaps not as regimented as those at West Point, colleges and common schools in Massachusetts and Connecticut simultaneously adopted recitation at the blackboard in the early decades of the nineteenth century (Cline Cohen 1982; Tolley 2003). With its emphasis on reproducing diagrams, the teaching practice was perhaps especially well suited for subjects such as geometry and trigonometry.

One of the ways Pestalozzian principles were adopted into classrooms was through efforts to make repetition and recitation more engaging and more rigorous. In the United States, many educators were inspired by Warren Colburn's *First Lessons in Intellectual Arithmetic* (1821). Colburn advocated the use of techniques that were referred to by a variety of terms, including "oral instruction," "mental arithmetic," and the "inductive method of instruction." The teacher was to introduce arithmetic at an earlier age than was typical in the early nineteenth century, around 8 or 9 instead of 12 or 13. Students were presented with problems using small integers and relating to real-life situations; they were encouraged to puzzle out a solution to the problem by thinking out loud on their own and by computing with their fingers. As they devised general rules for solving problems, they were guided into problems of increasing complexity. Teachers were to concentrate on one concept at a time, and they were encouraged to employ their own personality and style during the lesson (Colburn 1831). Similar attempts to implement this "good kind of drill" were made throughout Europe, both independently and as a result of the influence of Colburn's textbooks (Monroe 1912). The verbal exchange of questions and answers in some ways foreshadowed the twentieth-century Socratic method identified with R. L. Moore, although with Colburn's form of instruction by interview, the textbook remained the source of information and concepts. In contrast, the teacher who followed the Moore method cast aside the textbook in favor of developing the material himself or herself (Parker 2005).

The practice of solving problems orally in class was undoubtedly used before Colburn, too, and was subsequently at least sometimes extended to other mathematical subjects (e.g., Young (1925) recommended using oral problems not only in classes in arithmetic but also in algebra classes).

These practices were reflected in textbooks as well. In the mid-eighteenth century, Thomas Dilworth and other authors of arithmetic textbooks began to include "questions" or "exercises" that allowed readers to practice applying the rules that otherwise filled the textbooks. As with the "practical" and "promiscuous" (miscellaneous) story problems that Robert Adrain added to the American edition of Charles Hutton's two-volume *Course of Mathematics* (1812), the final answers to the questions were provided in the text – not in a separate answer key or at the back of the book. The point of these questions was for readers to think through the solving process mentally; perhaps some teachers and students discussed them in class. These students were not writing up work at home and bringing it to class. Even with the advent of long lists of exercises and explanations of general solving processes in "mental arithmetic" textbooks, which were accompanied by volumes of instructors' keys, the intent was that the questions would be answered orally during class time (Doar 2006, pp. 9–13; Dilworth 1810, pp. 173–184; Colburn 1827).

Unlike rote drill and recitation, which fit within centuries-old traditions of oral daily work, the practice of written individual problem and exercise solving in class (by students in their own notebooks) did not become widespread until a relatively late date. Technological reasons for this may be cited. Although the slates favored by Lancaster were a long-existing technology, paper was expensive and scarce in Europe and North America until late in the eighteenth century and blackboards were not available until the turn of the nineteenth century. Old textbooks often contain no written exercises at all or offer them in quantities that are clearly insufficient by today's standards. This in itself suggests that in classes, too, not much attention was devoted to written problem solving.

Textbooks containing exercises explicitly designed for pen-and-paper work gradually began to appear in the United States in the second half of the nineteenth century. The expansion of education to larger and younger student populations – which in turn raised concerns about classroom management – and the increasing availability of inexpensive paper and artificial light, which permitted evening work at home, were contributing factors. Increasing class sizes made it difficult to call every student to the board every day, and those in charge of schools sometimes questioned whether teachers were sufficiently prepared to direct recitation sessions. (This is also a gender issue, as the profession of teaching was being transformed into a job for women.) Written work prepared out of class offered an alternative means of reviewing the progress of an entire class. For example, while Thomas Hill's *First Lessons in Geometry* (1855) contained study questions in the footnotes and "practical questions and problems" that directed readers through thought experiments, his exercises in *Second Book of Geometry* (1863) instructed students to "draw," "measure," and "prove" (Hill 1855, pp. 137–144; Hill 1863, pp. 71–72, 74, 85, 100–105). Similarly, in 1875 Charles Davies added "graded" exercises, meaning they increased in difficulty from beginning to end, to his *Elements of Geometry*, which had been in print for nearly 50 years. After Davies's death, J. Howard Van Amringe contributed even more exercises. The solutions to these questions and proofs of these propositions were to be written out by students and brought into class for correction (Van Amringe 1882, pp. 49*[4]–49**, 92*–92**, 135*–135**, 177*–177**, 209*–209**, 234*–234**, 260*–260**, 261–275). Turning to European publications, we note that there were even more extensive lists of propositions for students to prove in Eugène Rouché and Charles de Comberousse *Traité de géométrie* (2 vol. (Paris: Gauthier Villars, 1879), i: pp. 307–362, ii: pp. 485–540).

Yet, earlier practices persisted alongside this new emphasis on homework. For instance, the review exercises in Henry B. Maglathlin's *New Practical Arithmetic* (1869) appeared alongside their final answers, while the main text emphasized rules to memorize. Meanwhile, Daniel W. Fish's 1874 *Complete Arithmetic* noted in its title that it combined oral and mental work with written assignments, which was also a feature of J. W. A. Young and Lambert L. Jackson's 1912 *The Appleton Arithmetics, Third Book*.

[4] The pages are asterisked because they were interpolated between pages of the previous, 1875 edition of Davies's geometry textbook.

As we noted in the previous section, this was also the period during which the appearance of textbooks was transformed. As publishers increased their influence over the textbook market and new educational goals were incorporated into schoolbooks, changes were introduced into content as well as into style. This in turn led to changes in the forms in which students were directed to prepare homework. The influence of new textbook formats on the daily conduct of classes may perhaps be observed most vividly in the emergence of the two-column proof in secondary school geometry. As typographical changes were introduced into textbooks, authors gradually moved from proofs written out in paragraphs, modeled on those in Euclid's *Elements of Geometry*, to proofs that listed each step on a separate line. For instance, in *Plane Geometry* (1906, p. 20), Edward Rutledge Robbins divided the sentences within the proof of each theorem into "Given," "To Prove," and "Proof." By 1911, authors such as Clara Avis Hart and Daniel D. Feldman (1911) separated the proof into a table with two columns, containing the "argument" in one column and the "reasons" in the other. Educators used this format to teach geometrical proof because their primary concern was requiring students to demonstrate the actual proving process, both during blackboard exercises in class and during students' preparations for the next day's class; teachers were no longer asking pupils to memorize a completed proof. This change in teaching practice was recommended by the 1893 Committee of Ten as well as by other late nineteenth-century reformers (NEA 1894, pp. 112–116). As Patricio Herbst (1999) points out, though, the emergence of this teaching practice meant that topics in geometry that were not well suited for the format were de-emphasized, such as solid geometry and proof by contradiction.

3 Practices of Assessing Knowledge

As noted above, the practices of assessment and reinforcement were interconnected – by calling on a student in class to solve a problem, the teacher also assessed the student. In addition, however, there existed special assessment events. While written examinations were administered in the ancient and medieval eras, most notably in order to award civil service positions in China, they did not become commonplace in Europe and North America until the modern period, when mass-produced paper became available and class sizes increased. In the West, oral tests preceded written forms. In describing examination day at school in *The Adventures of Tom Sawyer* (1876), Mark Twain wrote ironically about presentations of "compositions by young ladies": "The themes were the same that had been illuminated upon similar occasions by their mothers before them, their grandmothers, and doubtless all their ancestors in the female line clear back to the Crusades." This was an exaggeration, of course, but the overall character of the event was indeed based on a long tradition of disputations and public presentations while at the same time resembling formats that are attracting attention in the twenty-first century, such as portfolio presentations.

Evidence documenting formal, periodic examinations is typically more available and voluminous for higher education than for schools until the twentieth century. At Harvard, the tradition of annual oral examinations persisted from the college's first commencement in 1642 to the graduation ceremonies held in 1839. By the late eighteenth century, students prepared separate mathematical theses that included an out-of-class written component. As a year-end activity in junior and senior classes, they hired scribes to illustrate eclipses, architectural elevations, or mathematical problems on large (approximately 20″×30″) pieces of paper. They may have displayed the papers, and they explained the geometry, algebra, or calculus involved in their theses during commencement celebrations (Badger 1888; Harvard Mathematical Theses and Commencement Theses). In other words, the practice of oral examination at Harvard was interwoven with other practices and cultural events: culminating writing projects, not unlike the master's theses and doctoral dissertations prepared by today's graduate students, as well as public displays of learning and achievement. At West Point, in January and June of each year from the institution's founding in 1802, the Academic Board, composed of the superintendent and faculty, examined the cadets, section by section. Each cadet was randomly assigned a topic from the mathematics course and then asked questions about that topic. For instance, in 1825, the highest level of students may have been required to explain methods of integration or the differentiation of logarithms. They faced two tables of inquisitors and gave their explanations at one of two blackboards on easels, while three other cadets prepared their answers. Each cadet's examination lasted up to 5 h (Rickey and Shell-Gellasch n.d.). As with recitation, West Point's form of oral examination was adopted by many colleges founded in the nineteenth century – which often hired West Point graduates as mathematics professors – and trickled down to secondary institutions, including academies and high schools.

After it evolved into written form around 1790, the Cambridge Mathematical Tripos became perhaps the archetype of the high-stakes examination. By the late 1820s, over 4 days each January, students spent 16 h regurgitating proofs of theorems they had memorized from textbooks and another 7 h solving problems, all from printed examination papers (Craik 2008, pp. 91–92; Enros 1979). To prepare for the examinations, students found little use in attending professorial lectures and instead spent most of their time cramming for the examination questions with their college tutors, who taught directly to the test (Gascoigne 1984; Warwick 2003). Mathematics education at Oxford also was centered around tutors and written examinations (Slee 1988).

American colleges also first turned to written tests, around the 1830s, in order to administer graduation examinations. For convenience and uniformity, college presidents gradually replaced oral entrance examinations with written instruments as well. For instance, Harvard added admission requirements in arithmetic (up to the Rule of Three) in 1803, in algebra in 1820, and in geometry in 1844. By 1851, these examinations were consistently administered in written form. The textbooks from which the questions were taken were publicized, so that young men could study the texts on their own or engage tutors before attempting the examinations (Smallwood 1935, pp. 8–15). Where high schools were available and offered college preparation, the mathematics instructors might teach from the textbooks used by the nearby college. In 1869, the 21 mathematical questions asked of prospective students at the Massachusetts Institute of Technology included the following:

- Multiply 73 thousandths by 19 hundredths.
- Multiply $3a^2 + ab - b^2$ by $a^2 - 2ab - 3b^2$, and divide the product by $a + b$.
- Prove that the area of a trapezoid is equal to the half sum of its parallel bases multiplied by its altitude.

Students also answered 15 questions in grammar, composition, geography, history, and literature – administered as the "English" examination (MIT n.d.).

Entrance examinations tied to particular institutions and textbooks rather than to general mathematical subjects and concepts disappeared with the establishment of the College Board, which administered the first Scholastic Aptitude Test in 1926 on behalf of a national group of colleges. Meanwhile, written semester tests became a standard way of assessing performance in individual courses (Thelin 2004, pp. 302–303). Likewise, standardized mathematical testing in schools has a history that goes back more than 100 years in the United States (Madaus et al. 2003). Some states, such as New York, used examinations to certify that all students in a state had met minimum educational standards. In secondary and higher education, the semester calendar and final exams were thus interwoven with expectations for course content and teaching practices for acquiring and rehearsing knowledge. Nonetheless, particularly in schools, attitudes toward tests conducted "by an outside authority" have often been ambivalent. Young (1925, p. 149) contrasts such exams with those that constitute a "culminating class exercise" and explains that the former "may be regarded as necessary evils and their influence upon instruction as bad." And he adds at once: "Fortunately, this extreme form of the examination is by no means predominant in the United States."

This could not have been said of European countries in the nineteenth century. The content of final examinations for students in secondary schools shaped the teaching practices used in their courses, particularly for the last grade. The form of an exam varied by nation, depending in part on the extent to which an educational system was centralized or retained the local autonomy of the mathematics teacher. There were also differences with respect to whether the examination constituted a part of secondary education or whether it was already an element of higher education.

Italy provides an example of a unified examination, as all teaching subjects, including mathematics, were organized centrally by the ministry. Thus, all questions were identical for all the secondary schools in the entire country. It was difficult to prepare the students for questions that could not be predicted. From the 1870s on, the results of the national examinations in mathematics were so bad that mathematics was excluded from the subjects of the final exam in 1882. This reduced mathematics to the status of a minor subject in the curriculum (see the chapter on Italy in this volume). The other extreme is represented by Germany. There, due to the neo-humanist conception of autonomy for the teacher with respect to his teaching method, it was the task for each Gymnasium to define the questions for the final exam, called *Abitur*. The examinations were supervised by a school inspector from the regional authority. The mathematics teachers of each school chose the questions and thus the topics studied by the students. The only exception among the German states was Bavaria, where the government introduced in the 1820s the *Abitur* as a centrally organized examination – due to mistrust against the teachers, supposing them susceptible to pressures by the local public when writing examinations. France lies somewhat in the middle between these two extremes; the questions for the *baccalauréat* – in France simply called the "bac" – were not established by the local teacher staffs. This became organized semi-centrally; the questions were defined independently in each *Académie*, the regional school authority.

At the same time, the final examination and the syllabus were influenced by differences in the institutional structures of national educational systems. In France, the *bac* was not an element of secondary education, but of higher education: there, the *Facultés des Sciences* were responsible for administering the examinations, as a part of giving access to higher education. This was a vestige of how Jesuit colleges functioned in premodern times – the Jesuits had transferred all teaching of the arts faculties to their colleges, but these faculties retained the only right to award the degree for entering the professional faculties. The *Facultés des Sciences* were the successors of these arts faculties. In Germany and Italy, however, the *Abitur* clearly functioned as an element of secondary education; thus, the schools were the institutions that awarded the right to access higher education. The *Abitur* was the final step of secondary education. The syllabus was hence under the exclusive control of secondary education and not influenced by other levels of education.

Without elaborating on this topic here, we point out in conclusion that the general label "written exams" in fact encompasses the most diverse practices. Exams have varied in terms of the manner in which they were conducted, in terms of the problems that they contained, and in terms of the way in which they were put together and graded. Both written and oral exams have at times (e.g., in Russia after the revolution of 1917) been declared harmful and destructive to students' health and at the same time contributing nothing to the formation of an accurate assessment of students' knowledge and intellectual development. Moreover, both in explaining and in reinforcing the material, and in assessing students, individual and group formats alike have been employed. Assessments of the knowledge of a group as a whole (e.g., based on the response of one of its representatives) are far from the practices that are current today, but were nonetheless commonly employed at a former time (Karp 2012).

4 Conclusion

This chapter has introduced the variety of teaching practices developed by instructors over time, across space, and at all levels of education, through three useful, if somewhat artificially distinguished, stages in the learning process: the initial exposure to information and concepts, techniques and exercises that repeat the concepts until they are mastered, and assessments that test the level of mastery. An introduction can only whet the appetite for further study. Indeed, even by limiting the scope of most examples to North America and Europe since 1800, innumerable forms of teaching practices have been omitted. For instance, we have said almost nothing of the practice of independent student reading, notably engaged in at the turn of the nineteenth century by Nathaniel Bowditch, among others. Nor have we thoroughly treated the work of private tutors, such as those who guided the study of the most prominent women of eighteenth-century mathematics, such as Maria Agnesi and Emilie du Châtelet (Mazzotti 2007; Arianrhod 2012). Even though a book chapter cannot be comprehensive, we can suggest the diversity encompassed by its topic.

The second fundamental characteristic of the history of teaching practices addressed in this chapter is the significance of change over time. For most Westerners who came of age in the twentieth century, a mathematics class was a teacher at the front of a room describing principles and explaining problems, perhaps with the aid of a blackboard or its successor technologies. This chapter disabuses readers of the notion that the way things are now is the way they have always been. Even a historical classroom that is similar in appearance to its contemporary counterpart may have operated very differently; eighteenth- and nineteenth-century classrooms were often decentralized in organization, with groups of students working on different problems and subjects (Ellerton and Clements 2012). While we have briefly discussed how and why teaching practices changed with respect to the particular cases included in this chapter, there was no space for a broader discussion of causation. Key themes have been raised: the influence of technology on changes in teaching practices; relations between teaching practices and the accessibility of education, especially the rise of mass education; relations between teaching practices and students' abilities; mutual influences of practices and philosophies of mathematics education; and practices that first appeared in colleges and then penetrated into schools. Numerous other relationships have been left for exploration in other studies: mutual influences between classroom activities and teacher education; mutual influences between practices associated with different levels of schooling (elementary, middle, and high schools); cultural influences on the types of practices chosen by teachers and administrators; comparisons between teaching practices in mathematics and other subjects; and whether and how political changes influence teaching practices in mathematics, to name only a few possibilities. Many opportunities for investigating the history of teaching practices are available to historians of mathematics education.

This chapter thus offers samples of the history of teaching practices and of the skills and techniques employed in writing the history of teaching practices. Readers are encouraged to use their ingenuity to locate published books and articles, manuscript documents, objects, and other forms of relevant and reliable primary sources that were created by teachers and students. Scholars should not confine themselves to merely describing the contents of these sources but rather must look to the evidence for answers to questions about the process, context, meaning, and significance of teaching practices. The organizational structure of this chapter suggests one framework for historically evaluating the conduct of daily lessons and the methods of assessment that measure the progress and accumulated knowledge of students. Just as there are additional modes of practice deserving of attention, other methodological lenses that analyze as well as narrate are certainly possible and promising. There is much room for further research into particular pedagogical approaches and methods as well as for the collaborative preparation of synthetic and comparative accounts. Even though teaching practices are often mentioned within general histories of mathematics education, specialized and comprehensive monographs on the topic remain the work of the future.

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Part VI
Issues and Processes Across Borders

Chapter 27
History of International Cooperation in Mathematics Education

Fulvia Furinghetti

1 Introduction

International cooperation originated as a structured network of activities aimed at mutually helping people after the Second World War. Since the beginning, education has been one of the main themes of action for political international bodies such as UNESCO (*United Nations Educational, Scientific and Cultural Organization*) established in 1945 and OECD (*Organisation for Economic Co-operation and Development*) born as OEEC (*Organisation for European Economic Cooperation*) in 1948. As illustrated by Jacobsen (1996), UNESCO's primary emphasis on mathematics education has been to promote the exchange of information, to work nationally, and to cooperate with regional and international groups, in particular with the governments of its member states. The main idea underlying this action is that a convenient level in primary and secondary education is a necessary condition for developing countries to improve their economic and social standards.

The movement of cooperation is the point of arrival on a path that started in the nineteenth century, when an international[1] perspective entered the world view. This perspective was fostered by new and more efficient forms of communication offered by technology and industrial production. In that period, most countries had acquired the state of a modern nation, and systems of education were constructed or reshaped according to the needs of these new societies. Communication and internationalization touched different aspects of societal life, among them mathematical research first and mathematics education later. Solidarity followed the same path, starting in the nineteenth century as a movement among people of a country, and afterwards extending to relationships among countries.

[1] A discussion about the word "internationalization" and its related neologisms is in Parshall and Rice (2002a). The issue of internationalization of mathematics is treated in Parshall and Rice (2002b).

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In this chapter, cooperation is treated by following the path that goes from communication in mathematics to communication in mathematics education and from communication within countries to communication among countries. The attention paid to the International Commission on Mathematical Instruction (ICMI)[2] with its official organ *L'Enseignement Mathématique* is justified by the role it plays as a catalyst. ICMI is a privileged interlocutor between mathematicians and mathematics teachers/educators as well as between these two communities and the bodies accredited for international cooperation in education. Despite the many constraints in which its action develops, ICMI contributes to making international cooperation action oriented so that verbal declarations on international understanding or vague intercultural exchanges become concrete, scientific, technical, cultural, and economic projects and reinforce the capacity for self-development in countries.

2 Communication in Mathematics

2.1 Communication Within Countries

Progress in mathematics, as in the other sciences, has always been accompanied by the impetus of communicating them to colleagues, pupils, and laymen. Mesopotamian, Egyptian, and Greek civilizations left tablets, papyri, and books evidencing this behavior. In the classical period, there are a few traces of private communication in mathematics: this is the case of Archimedes's message to Eratosthenes explaining his method. The tradition of private correspondence in mathematics continued for a long time and, in certain cases, promoted important achievements in mathematics. A famous example is the exchange of letters on probability between Blaise Pascal and Pierre de Fermat in seventeenth century. The spread of ideas received a fundamental boost by the invention of the printing press at the end of the fifteenth century. The first half of the seventeenth century saw the emergence of new forms of communication, such as the birth of groups whose members gathered to promote discussion and disseminate new ideas from different fields. In Rome the *Accademia dei Lincei* was founded in 1603; in the 1620s, the *Académie Parisiensis* (a.k.a. "Mersenne's Circle") was organized in Paris; circle precursors of the *Royal Society*, officially born in 1662, originated in the 1640s; in 1657 the *Accademia del Cimento* was established in Florence; in 1666 the *Académie des Sciences* in Paris was founded. Academies and cultural circles fostered more structured exchanges and debates on different disciplines and the diffusion of results through their proceedings. Later, the need for a more specialized milieu for discussing mathematical ideas led to the foundation of journals specifically dedicated to mathematics. After some ephemeral attempts carried out at the end of the eighteenth century, two important journals, still existing nowadays, appeared in the first half of the nineteenth century: *Journal*

[2] Though in this chapter the acronym ICMI is used to refer to all periods, it should be noted that the name of the Commission underwent many changes. Earlier, the Commission was known by the French acronym CIEM (*Commission Internationale de l'Enseignement Mathématique*) or by the German acronym IMUK (*Internationale Mathematische Unterrichtskommission*). In English the denomination was "International Commission on the Teaching of Mathematics" or other slightly different expressions; no acronyms were used. In Italian acronyms were not used for the C*ommissione Internazionale per l'insegnamento Matematico*. When the International Mathematics Union had reconstituted the Commission (6–8 March 1952), the name "International Mathematical Instruction Commission" (IMIC) was adopted. This denomination was ephemeral: since 1954, ICMI (*International Commission on the Teaching of Mathematics*) became the denomination internationally used. Furinghetti and Giacardi (2008) present information and documents on the first hundred years of ICMI and links with the official organ *L'Enseignement Mathématique* and *ICMI Bulletin*, which are the main sources of the information reported in this chapter.

für die Reine und angewandte Mathematik founded by August Leopold Crelle (1826, Berlin) and *Journal de Mathématiques Pures et Appliquées* founded by Joseph Liouville (1836, Paris). The number of journals dedicated to mathematics grew so impressively that 182 mathematical periodicals were listed in *Catalogue of current mathematical journals* (1913).

In the second half of the nineteenth century, mathematical societies were established: one in Moscow in 1864, the *Société Mathématique de France* in 1872, the Italian *Circolo Matematico di Palermo* in 1884, the *New York Mathematical Society* in 1888 becoming in 1894 the *American Mathematical Society*, the *Deutsche Mathematiker-Vereinigung* in 1890, and the *London Mathematical Society* in 1895. As noted by Parshall (1995), the organization of national mathematical societies evidences that in the last decades of the nineteenth century, mathematicians recognized the importance of communication both in person and in print and acquired an awareness of their professional identity as researchers in mathematics.

2.2 Communication Among Countries

These events not only allowed communication to extend beyond personal relationships, but also played a substantial role in fostering the process of internationalizing mathematical research. Dhombres and Otero (1993) have studied *Annales de Mathématiques Pures et Appliquées*, an important journal that preceded the two mentioned before, founded by Joseph Diez Gergonne and Joseph Esprit Thomas-Lavernède in France and appearing in the period 1810–1832. They report that at least 22.8 % of authors, corresponding to 12.6 % of articles, were foreign. This trend of publishing foreign contributions continued in the journals of Crelle and Liouville (see Lützen 2002) and in other periodicals that appeared successively in the UK, Italy, and Sweden, among other countries. Communication and internationalization were made easier when early joint works in the field of bibliography were published: *Jahrbuch über die Fortschritte der Mathematik* (first issued in 1871) and *Répertoire bibliographique des sciences mathématiques* (first issued in 1894).

The Chicago Congress of Mathematicians in 1893 was a milestone in the process of making mathematics unbound: it was a kind of rehearsal for the tradition of International Congresses of Mathematicians (ICMs), which started in 1897. Since the Congress of Paris in 1900, ICMs are held every 4 years (with breaks due to the two World Wars). This regular event contributed remarkably to shaping the identity of an international community of researcher mathematicians. Communication forced the adoption of standards for this profession so that "[b]y the end of nineteenth century, to be a mathematician meant the same thing internationally" (Parshall 1995, p. 1588). Only in 1920 was the International Mathematical Union (IMU) founded. After its dissolution in 1932, there were efforts to reestablish it; the rebirth happened formally in 1951 and the first General Assembly of the new IMU was held in 1952.

In the section entitled "Ideas of international mathematical cooperation awaken," Lehto (1998) carefully describes the steps that led to the creation of the first ICM. These steps may be summarized as follows. Georg Cantor was a pioneer in conceiving the idea of international mathematical cooperation beyond the bibliographical initiatives created in the second half of the nineteenth century. Already in 1888, he proposed a meeting of German and French mathematicians to be held at a neutral site. In 1890, Walther von Dyck wrote to Felix Klein (see Schubring 2008a) about Cantor's project of an international congress of mathematicians. In a letter to Alexander Vassiljewitsch Vassiliev, a mathematician at Kasan, Cantor wrote that he had had the idea of an international congress in mind for 5 years. He corresponded with several mathematicians (Charles Hermite, Camille Jordan, Henri Poincaré, Charles-Ange Laisant, Émile Lemoine, Klein, von Dyck), some of whom also played an important role in internationalizing mathematics education. Klein too was convinced of the importance of international cooperation: in his opening address at the Chicago congress, he advocated the need of a mathematical union. Several concrete attempts in this direction were made by Cantor until March 1896. The idea of international congresses of mathematicians was explicitly launched by Laisant and Lemoine (1894). In the preface of the debut issue of their journal, they developed reflections similar to those Klein had presented in Chicago and put forward indications for future congresses of mathematicians that were largely followed at the first ICM in Zurich (1897).

3 Communication in Mathematics Education

3.1 Communication Within Countries

The development of mathematics education is obviously linked to the development of mathematics, but more than mathematics is affected by social, economical, and political situations, as well as national laws concerning instruction, teacher education, and recruitment. Moreover, the status and career of mathematics teachers have always been very different from those of mathematicians; in particular, for them, the occasions and places for discussing educational issues were rather local. For a long time, these constraints made mathematical education mainly a regional business.

The constitution of the modern states raised the need to share information among the teachers of a country. A network of communication was created by the journals dedicated to mathematics teaching or to spreading mathematical knowledge, which appeared around the half of the nineteenth century. They had different purposes and audiences. Some journals were addressed to teachers and pupils of classes which prepared admission to special schools, while others contained in their titles words such as "elementary mathematics" and others published mathematical questions addressed to amateurs. The publication of journals devoted to mathematics teaching is linked to the creation of national associations of mathematics teachers. In some cases, the periodicals were the terrain for promoting the idea of founding such professional associations, as shown by the following examples. In the USA, the MAA (*Mathematical Association of America*), founded in 1915, stemmed from the *American Mathematical Monthly* founded in 1894. The Association of Teachers of Mathematics of the Middle States and Maryland began publishing a quarterly journal, *The Mathematics Teacher,* in September 1908, which eventually was adopted as the official journal of NCTM (*National Council of Teachers of Mathematics*) upon its founding in 1920. In Italy, the journal *Periodico di Matematica* was founded in 1886 and became the official organ of the Italian national association of mathematics teachers *Mathesis* when it was established in 1895. In other cases, the foundation of teacher associations stressed the need to have journals spread information and ideas. In Germany, the *Deutscher Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts* was founded in 1891 and the journal *Unterrichtsblätter für Mathematik und Naturwissenschaften* followed in 1895. In the UK, the *Association for the Improvement of Geometrical Teaching* (AIGT), founded in 1871, evolved into the *Mathematical Association* in 1897. The Association continued to publish *The Mathematical Gazette,* which had first appeared in 1894. In France, the *Association des Professeurs de Mathématiques de l'Enseignement Public* (APMEP) began (with the name *Association des Professeurs de Mathématiques de l'Enseignement Secondaire Public*) its activities and the publication of its *Bulletin* in 1910.

The national journals and teacher associations became an important means for transmitting ideas and information among teachers inside a country and proved of crucial importance in shaping the professionalization of mathematics teachers. The themes treated were related to the national systems of education and the readership was composed of teachers of the country of publication; most contributors were national and the action of teacher associations was mainly focused on dealing with national problems.

3.2 Communication Among Countries

For a long time in mathematics education, exchanges among countries only concerned textbooks. A famous case was that of Adrien-Marie Legendre's *Éléments de géométrie*, which replaced Euclid's *Elements* as a textbook in some European countries and, in succeeding translations, in the USA. This habit of adopting foreign manuals contributed to sharing mathematical knowledge, but it was slowly abandoned when the construction of instructional systems in various countries tried to avoid cultural colonialism and encouraged the production of local manuals. In this concern, a telling example is the case of Italy, where very good manuals for teaching mathematics in secondary schools were published in the last decades of the nineteenth century, when the new unified state had been established. This outstanding production enjoyed the background provided by the brilliant development of Italian mathematical research during that period.

Except for the case of textbooks, school teachers at the end of the nineteenth century who had international contacts and read foreign mathematical journals were rare. Professional mathematicians, having more international contacts, had opportunities to know what was happening abroad in the field of mathematics education. When the internationalization of mathematics took place, the need for an analogous phenomenon also emerged in mathematics education. In the first ICM in Zurich (1897), no section was dedicated to mathematics education (see Furinghetti 2007), but 3 years later, such a section existed in the second ICM in Paris. What happened in the meantime and how the ideas of internationalization, communication, and solidarity entered the world of mathematics education through the foundation of the first international journal in the field of mathematics education is the topic of the next section. The events of those years were the very premises of international cooperation.

4 The Cradle of International Cooperation in Mathematics Education: The Journal *L'Enseignement Mathématique*

The first step towards actual international cooperation in mathematics education may be identified in the foundation of the journal *L'Enseignement Mathématique* (Mathematical Teaching, hereafter *EM*) in 1899. In the presentation of its debut issue, the editors Laisant and Fehr explained the mission and vision of the journal (see Les directeurs 1899). They observed that despite the international congresses of mathematicians inaugurated in Zurich in 1897 and accepted as a principle for the future, the world of education had not up to that point been able to join this great movement of scientific solidarity as fully as would have been desirable. Through the publication of the journal, Laisant and Fehr wished to overcome the obstacles to reciprocal communication. Their first concern was to provide *EM* with a clear and open international character. In the presentation of volume six, the editors claimed that mathematics teaching for them concerned both students and teachers (see Laisant and Fehr 1904). They were interested in the social role of science and its links with progress in various forms (industry, technology, etc.). The key words of the journal's program were *internationalism*, *information*, *communication* (all linked with solidarity) and, of course, *mathematics teaching*. Professional mathematicians were invited to collaborate actively to the project of the journal in order to keep the teaching of mathematics in touch with advances made in the discipline.

The mission and vision of *EM* mirror the spirit of the times of the foundation. In the second half of the nineteenth century, progress in transportations and other technological realizations was changing societal life. The tradition of universal exhibitions inaugurated in London (1851) epitomizes these achievements and the ideas of internationalization and communication that were pervading the world. Also in those years, organizations and voluntary associations formed for the mutual assistance of workers were flourishing. This meant that solidarity was an idea felt in society, as evidenced also by the birth of political parties which addressed the needs of less affluent people.

The two founders of *EM* were the incarnation of this Zeitgeist and the right persons to deal with the enterprise. Charles-Ange Laisant (see Ortiz 2008) was born in Basse-Indre (France) on 1 November 1841 and died in Paris (1920). Buhl (1920) described him as man of science, educator, philosopher, and politician, whose spirit was logical, idealistic, and revolutionary, with links to anarchism and radicalism. Ortiz (2008) claims that "In his time, contrary to a 'Darwinian' world-view based on fierce competition, Laisant and his friends attempted to construct a community in which cooperation and association was the rule." According to Ortiz (2001, p. 83), Laisant showed clear consistency between his political and scientific behaviors; in both fields he was seriously concerned with the introduction of new ideas, of opening up new possibilities, and looking for unity in things that appear different. Lamandé (2011) shows how Laisant's life as an educator allows us to understand the inextricable links between politics, science, and teaching.

Laisant's thesis in the *Faculté des Sciences* of Paris concerned quaternions and their applications to mechanics. He battled for the reception of equipollences and quaternions in France. The relevance of his role in the French mathematical community relies on the effort he made to support the transmission of ideas, to give the international mathematical community a more structured organization, and to promote personal contacts through regular international meetings of mathematicians. He taught at the École Polytechnique. He founded *L'Intermédiaire des Mathématiciens* (1894 with Lemoine) and edited the *Nouvelles Annales de Mathématiques* and the *Annuaire des Mathématiciens*[3] published in Paris by C. Naud in 1902. Laisant was, with Poincaré, a member of the Commission charged with the production of the *Répertoire Bibliographique des Sciences Mathématiques* (a precursor of the present journals of mathematical reviews). In 1888 he was president of the *Société Mathématique de France* and in 1904 president of the *Association Française pour l'Avancement des Sciences* and the delegate of France in ICMI. He also directed the newspaper of the French Third Republic, *Le Petit Parisien*.

Henri Fehr was born in Zurich on 2 February 1870 and died in Geneva in 1954 (see Anonymous 1955; Schubring 2008b). His doctoral dissertation concerned Grassmann's vectorial analysis. In 1900, he became professor of algebra and higher geometry at the *Faculté des Sciences* of the University of Geneva. He was regarded as an exceptional teacher. A prominent characteristic of Fehr's personality was his interest in the social and organizational aspects of the mathematical community and of academic life. He applied his skill as an organizer in founding the *Swiss Mathematical Society* (of which he was president), the *Foundation for the Advancement of Mathematical Sciences*, and the journal *Commentarii Mathematici Helvetici*. He was also involved in social commitments such as the committee on the fund of pensions of his colleagues. Fehr received national and international honors and appointments. He was the vice-president of IMU (1924–1932), secretary-general of ICMI since its foundation, and honorary president in 1952–1954. He attended all of the first eleven ICMs, as his country's delegate in the ICMs of 1924, 1928, 1932, and 1936 and also as a vice-president of the Congress in the ICMs of 1924, 1928, and 1932.

Until the Second World War, articles in *EM* were mainly written in French, although the editorial address in the volume of 1913 stated that papers written in the official languages of the ICMs (English, French, German, and Italian) and Esperanto were accepted. The wide use of French (the language of diplomacy at the time of the foundation) made the circulation and the range of authors limited. In the scientific world, the problem of the variety of languages was emerging at the very moment when internationalization was pursued. Actions to create an international language for science were underway. During the ICM in Paris (1900), there was a lively discussion about old and new international languages (see Gray 2002). In the end, the proposal by Vassilief was carried out: he asked that academies and learned societies study the proper means of facing the problem of the different languages used in scientific literature (see Gray 2002).

[3] The *Annuaire des Mathématiciens* was a publication that contained the names and addresses of living mathematicians, scientific societies, and scientific periodicals.

As discussed in Furinghetti (2009), the journal in the 1960s changed its mission and scope and became a mathematical journal with articles only related to mathematical research. For this reason, when emphasizing the role of the journal in the process of internationalizing mathematics education, the years before the First World War need to be considered. About the topics dealt in *EM,* the reader has to remember that the journal was edited by university mathematicians with an interest in what was happening in schools. Then, as Howson (2001, p. 182) put it:

> it is the writings of such authors that are best represented in the pages of the journal, rather than those of school-teachers, teacher trainers, or what came to be known as mathematics educators. Indeed, there have been periods when the periodical's papers were much more concerned with mathematics than with its teaching. As a result any survey based on papers to be found in *L'Enseignement Mathématique*, is likely to reflect not what was actually happening in schools but what some influential mathematicians thought might with advantage happen in them.

Moreover, although the words mathematical teaching should refer to any school level, *EM* was largely concerned with tertiary teaching. In the eyes of a modern reader, the content of the journal is far from the scope of present research in mathematics education, but *EM* was important for at least three reasons. Firstly, it presented some traditional and new mathematical contents in an accurate form. Secondly, it drew attention to issues rather neglected then, such as the nature of mathematical invention and the influence of psychological aspects on it, as exemplified by Poincaré (1908) for the former issue and by Binet (1899) for the latter. Alfred Binet, director of the laboratory of physiological psychology in Sorbonne, illustrated the achievements of experimental pedagogy and psychology in France and other countries (England, Germany, Italy, USA) and supported the need for making pedagogical studies scientific by introducing experimentation. In Binet's article are the seeds of some developments of mathematics education studies. The psychologists of the University of Geneva, Édouard Claparède and Théodore Flournoy, collaborated to the journal.

The third reason for the importance of *EM* is its informative character. It published surveys on national systems of instruction and on mathematical programs as well as news on university courses, mathematical publications, and other academic and scientific events. The goals of the journal were connecting people who teach mathematics and informing on pedagogical and instructional themes. Laisant and Fehr wished to introduce the exchanges of ideas and experiences that mathematicians were establishing through their international congresses into the world of mathematics teaching. They were convinced that the future of civilizations would mainly depend on the kind of scientific education offered to the young and that mathematics would play a central role in this education. The importance of this journal in the history of mathematics education may be appreciated by considering that only in 1968, another truly international journal devoted to mathematics education, though very different from *EM*, was founded.

5 A Concrete Realization of International Cooperation in Mathematics Education: The *International Commission on the Teaching of Mathematics*

5.1 The Foundation in 1908 and the Initial Agenda

The milieu of *EM* contributed remarkably to raising awareness of the need to share information about systems of education in the world. The internationalization of and communication about mathematics were seen as models to be applied also in education. Already in the debut issue an article on mathematics teaching in Spain appeared, and afterwards other articles of this kind were published. In volume 7, the editors reported on the indications given at the ICM held in Heidelberg (1904) about reforms to carry out in universities and they asked for reactions (see La Rédaction 1905). The theme of reforms in mathematics teaching was hot in that period not only at the university level (see Nabonnand 2007): at the beginning of the century, Perry had launched his proposals in the UK, in France the big reform had been launched in 1902, and the Meran meeting on reforms in German schools was held in 1905. In the same volume, the journal published reactions by Gino Loria[4] (pp. 383–386), Émile Borel (pp. 386–387), Jules Andrade (pp. 462–469), David Eugene Smith[5] (pp. 469–471), and Francisque Marotte (pp. 471–472). In his note, Smith (1905, p. 469) wrote that the best way to reinforce the organization of the teaching of pure mathematics would be the establishment of a Commission appointed by an international congress which would study the problem in its entirety.

As observed by Schubring (2003), *EM*, which had promoted international communication, encouraged "taking the next step, i.e. establishing *cooperation*" (p. 54). The Commission advocated by Smith was founded in Rome during the fourth ICM. The endeavor was not easy. Schubring (2008d) describes the political and academic difficulties faced during the ICM in Rome to achieve the goal. On 11 April 1908, the afternoon general assembly approved with vigorous applause the following agenda, adopted in the morning by the fourth section devoted to philosophical, historical, and didactic questions:

> The Congress, having recognized the importance of a careful examination of the syllabi and of the methods of teaching mathematics in the secondary schools of the various nations, entrusts the professors Klein, Greenhill and Fehr with the task to constitute an international Committee, which should study the question and report about it at the next Congress. (Castelnuovo 1909, p. 33, author's translation)[6]

Klein was elected president, Alfred George Greenhill (see Rice 2008) vice-president, and Fehr secretary-general. Smith became member (as a vice-president) of the Commission only in 1912, but, as evidenced in Schubring (2008d), already in the initial phase he played an important role, by bringing the three together, "essentially serving as the midwife for IMUK" (p. 6). The presence of Greenhill in the committee is quite surprising (Rice 2008). When appointed as a member of the Committee, he was already retired; moreover, his professional biography does not evidence a significant involvement in education, apart from his personal teaching where he succeeded in being a well-appreciated professor in applied mathematics. The explanation reported by Schubring (2008d) is plausible, namely, that it was taken into consideration that the next ICM would be held in Cambridge (UK).

The official names given to the Committee were CIEM (*Commission Internationale de l'Enseignement Mathématique*) in French and IMUK (*Internationale Mathematische Unterrichtskommission*) in German. The mission of the Commission was not to make uniform methods and an organization of systems of instruction, but to produce an overall study of reforms already realized or on the agenda.

Giving a structure to the Commission, a body that had never existed before, was a hard task. Firstly, an international net has to be created. After bilateral correspondence and meetings managed by Smith (see Schubring 2008d), Fehr, Greenhill, and Klein met in Cologne (23–24 September 1908) without Smith, but with Klein's assistant Walther Lietzmann (see Schubring 2008e). At this meeting, the main lines of the future activities and organization of the Commission were set up (see Klein et al. 1908). The Commission was formed by delegates representing the countries that had participated in at least two ICMs with an average of two members. These countries, belonging to Group 1, had one delegate. Among these countries, those having sent 10 or more members had two or three delegates and constituted Group 2 (see Table 27.1 taken from Klein et al. 1908, p. 447).

In order to have a complete representation of countries and continents, the countries not meeting the requirements, but having institutions suitable to contribute to the work of the Commission, formed the group of *associated countries*, namely, Argentina, Australia, Brazil, Bulgaria, Canada, Cape Colonies, Chile, China, Egypt, British India, Mexico, Peru, Serbia, and Turkey. Their delegates were permitted to follow the activities of the Commission, without having the right to vote.

[4] See Mercanti (2008).

[5] See Schubring (2008c).

[6] Il Congresso, avendo riconosciuto la importanza di un esame accurato dei programmi e dei metodi d'insegnamento delle matematiche nelle scuole secondarie delle varie nazioni, confida ai Professori KLEIN, GREENHILL e FEHR l'incarico di costituire un Comitato internazionale che studii la questione e ne riferisca al prossimo Congresso.

Table 27.1 List of countries invited to participate in the works of the Commission

Austria, 2 or 3 delegates	Japan[a], 1 delegate
Belgium, 1 delegate	Norway, 1 delegate
British Isles, 2 or 3 delegates	Portugal, 1 delegate
Denmark, 1 delegate	Romania, 1 delegate
France, 2 or 3 delegates	Russia, 2 or 3 delegates
Germany, 2 or 3 delegates	Spain, 1 delegate
Greece, 1 delegate	Sweden, 1 delegate
Holland, 1 delegate	Switzerland, 2 or 3 delegates
Hungary, 2 or 3 delegates	USA, 2 or 3 delegates
Italy, 2 or 3 delegates	

[a]In Klein et al. (1908), Japan was listed by mistake among the associated members. In Fehr (1911, p. 133) this country was declared to have the right to full membership

The delegates from a given country were invited to constitute a *national subcommission* encompassing people of all school levels and all types of schools (general, technical, professional). They would help the delegates in preparing the national reports.

The Commission was directed by the three members appointed in Rome (1908), who constituted the *Central Committee*. They were responsible for organizing the Commission's work and publishing its reports. *EM* was the official organ of the Commission. The official languages of the publications and the meetings were those admitted at the ICMs: German, English, French, and Italian.

The Commission claimed that its general aim was to make an inquiry and publish a general report on the current trends in mathematics teaching in the different countries (Klein et al. 1908, p. 450). It was expected that the work of the Commission would identify general principles that must inspire teachers rather than uniformity in details or in programs for all countries. The delegates, with the help of national subcommissions, were asked to prepare reports on the situation of mathematical instruction in their countries. This work was facultative for the associated countries. The resolutions taken in Rome, which confined the Commission's scope to secondary schools, were modified so that all kinds and levels of schooling had to be considered (Klein et al. 1908, p. 452). Among the recommendations given by the Central Committee was the encouragement to discuss the preparation of reports among teachers, associations, and other parties. This shows how the action of the Commission in principle could promote not only international cooperation, but also cooperation within countries. Moreover, it was recommended to pay particular attention to applied mathematics: this fact was linked to the widespread interest in this theme by the increasingly industrialized world and, perhaps, also by Greenhill's expertise in the field.

The general plan to be followed by the reports was (Klein et al. 1908, pp. 452–458):

Part 1. Current situation of the organization and of the methods of mathematical instruction:
The various kinds of schools
Aim of mathematics instruction and branches of mathematics taught in the various kinds of schools
Exams
Teaching methods
The training of prospective teachers
Part 2. Modern trends in the teaching of mathematics relative to the points listed above.

Successive meetings of the Central Committee refined the organization and the agenda of the Commission. The most relevant issues are reported in Circular n. 1 signed by Klein and Fehr (1909) for the meeting in Karlsruhe (5–6 April 1909) and Circular n. 2 signed by Klein and Fehr (1910) for the meeting in Basel (28 December 1909).[7]

[7]Fehr (1910, p. 371) mentions a meeting in Göttingen (April 1910). Howson (1984) mentions a meeting in the Harz Mountains (Germany).

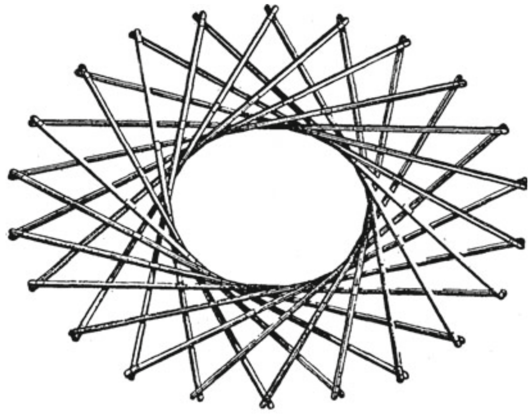

Fig. 27.1 A geometric model from Treutlein and Wiener (1912, p. 14)

5.2 International Meetings on Mathematical Education and International Inquiries Before the First World War

A turning point in the activities of the Commission was the meeting held in Brussels (9–10 August 1910) on the occasion of the Universal Exhibition. This meeting was followed by talks on scientific teaching in Germany and technical teaching in French secondary schools and by the international congress on secondary teaching (15–16 August 1910) (see Fehr 1910). Since the afternoon of August 10, the activities were open to a large audience. Various member states of the Commission presented their reports. In his opening address, Klein identified three typologies of reports: reports based on the method of systematic exposition, those based on the statistic method, and monographs.

Carlo Bourlet delivered a talk on the intertwining of pure and applied mathematics in secondary teaching. Peter Treutlein presented his method for teaching geometry aimed at developing pupils' geometric intuition, also with the aid of geometric models produced by the firm Teubner in Leipzig (Fig. 27.1). Klein illustrated the use of the Brill and Schilling models in teaching advanced mathematics.

With the Brussels event, the tradition of having real international conferences specifically dedicated to mathematics teaching began. The most remarkable of these meetings were held in Milan (18–21 September 1911) and Paris (1–4 April 1914).

The activities of the Commission were also presented in the subsection entitled *Didactics* of the fourth section at the fifth ICM in Cambridge (21–28 August 1912). The themes discussed were (see Furinghetti 2003; Schubring 2003):

1. Systematic exposition of mathematics (axioms, rigor, etc.) in secondary schools (Milan 1911)
2. The fusion of different branches of mathematics in secondary schools (Milan 1911)
3. The teaching of mathematics to university students of physics and of the natural sciences (Milan 1911)
4. Intuition and experimental evidence in secondary schools (Cambridge 1912)
5. Mathematics for university students of physics (Cambridge 1912)
6. Results of the introduction of calculus in secondary schools (Paris 1914)
7. Mathematics teaching for the technical professions in higher educational institutions (Paris 1914)
8. Theoretical and practical training of secondary mathematics teachers in different countries

The Commission planned to reconvene in Munich in August 1915 to discuss theme 8, but the First World War impeded the meeting and the report was presented at the ICM of Zurich in 1932.

The discussion of the themes was initiated by preparatory questionnaires and the reports were published in *EM*. These themes were pivotal to the reform movement of those years, but they maintain until today a paramount importance in mathematics education. Themes 1, 2, and 4 concerned secondary teaching, while the others focused on problems in tertiary teaching. This is not surprising if one considers the links the Commission had with mathematicians since its foundation.

5.3 Significance of the Action of the International Commission on the Teaching of Mathematics in Its Early Years

The First World War halted the impetus of the activities carried out by the Commission, but the achievements in its first decade of life were nonetheless impressive:

- A network of information and communication was established
- An organizational structure with precise rules was created
- Reference points for the discussion were offered: a journal as an official organ, and international meetings
- International inquiries were launched that served as catalysts for discussions
- The production of reports on mathematics teaching in various countries was promoted

The table of publications of the Commission, reported in Fehr (1920–1921, p. 339), lists 187 volumes, 310[8] reports, and 13,565 pages. The value of these reports is manifold. Firstly, they were an actual evidence of the feasibility of a project of international cooperation. Secondly, they promoted cooperation inside the countries involved in the preparation of the reports. Eventually, for the present reader, they are an invaluable picture of mathematical teaching during those years (Fig. 27.2).

Some secondary teachers (e.g., in Italy) authored national reports and participated in meetings open to a large audience, but, as a general impression, the dominant point of view seems to have been that of the mathematicians. This is linked to the establishment of the Commission inside the community of mathematicians and its dependence on this community through the mandate given every 4 years during the ICMs. The great majority of people working in the Commission in that period were mathematicians and they tackled educational problems from the mathematician's point of view. This had some consequences; for example, little attention was paid to primary and lower secondary levels.

6 Broadening International Cooperation in Mathematics Education

6.1 International Cooperation in Mathematics Education as a Joint Endeavor of Political Bodies and Mathematics Educators

In the period between the two World Wars, the Commission suffered a crisis affecting all scientific associations. From an educational point of view, it was no longer a period of reforms in mathematics curricula; from a political point of view, the dramatic wound caused by the war in the relationships

[8] The sum of the reports in the table reported by Fehr (1920–1921) is 319 and not 300, as Fehr writes.

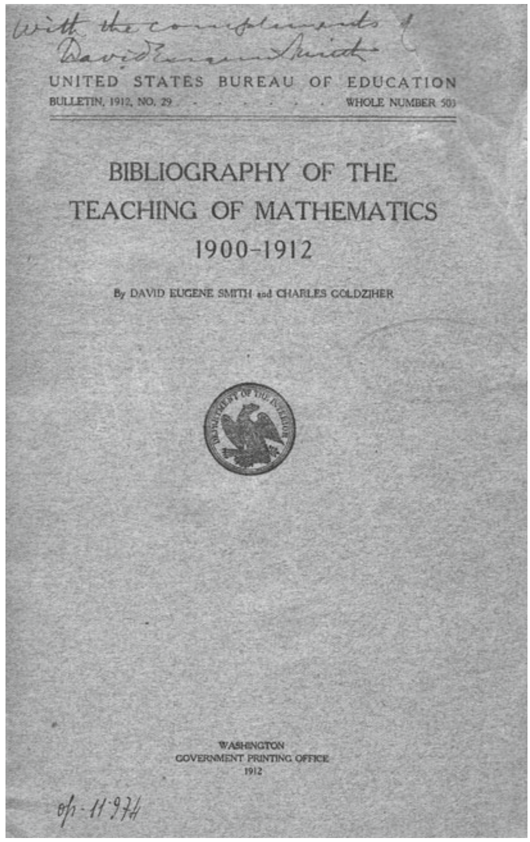

Fig. 27.2 Front cover (with Smith's signature) of the booklet (Smith and Goldziher 1912) prepared for the ICM 1912. It lists 1849 works dedicated to mathematics teaching from different countries

among countries had weakened the stimulus for international cooperation. As a consequence of the new ideology based on the restrictions on internationalism, the Commission was formally dissolved in 1920. The actual renewal of its activities at an international level (after the meeting held in Paris in 1914) occurred at the ICM of 1928, but, as evidenced by Schubring (2008d), international cooperation between the two World Wars suffered as a result of the difficult relationships among countries.

The situation was totally different after the Second World War. In 1952, when the Commission was transformed into a permanent subcommission of IMU, the political changes also modified the view of cooperation in mathematics education. Newborn international bodies such UNESCO and OECD focused attention on cooperation and offered a concrete and official support to initiatives.

ICMI stressed its mission through particular attention on developing countries and broadened its geographical scope. Ram Behari from India was elected member at large of the Executive Committee for the period 1955–1958 (see Furinghetti, 2008a): he was the first officer from outside Europe and North America. When the *Conference on Mathematical Education in South Asia* was organized by the mathematician Komaravolu S. Chandrasekharan at the Tata Institute of Fundamental Research in Bombay (22–28 February 1956), the vice-president of ICMI Marshall H. Stone (see Kilpatrick 2008) delivered an invited talk, together with the mathematicians Gustave Choquet, Hans Freudenthal (see Schubring 2008f), Alexander Danilovich Alexandrov[9] (see Siu 2008), Enrico Bompiani, and others. Some plenary talks treated mathematical instruction in China, Malaysia, and Singapore (see Report of

[9] Alternative transliterations are Aleksandr Aleksandrov and Aleksandr Alexandrov.

a Conference … 1956). The conference, organized in cooperation by IMU, UNESCO, the Ministry of Natural Resources and Scientific Research of the Government of India, the Sir Dorabji Tata Trust, and the Tata Institute of Fundamental Research, "was the first of this kind and it [was] hoped that [it] may serve as a model for similar events in the future" (Report of the Executive … 1956, p. 9).

The 1950s were again years of mathematical reforms, as it had been at the beginning of the twentieth century, and cooperation was needed: now the innovation in question was Modern/New Math. At the ICM in Edinburgh (1958), participants from the USA representing various groups who worked on renewing mathematics teaching delivered contributions related to the problem of reforms (see Todd 1960, pp. xli–xlii). In Europe, the new Commission CIEAEM (*Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques*), whose name refers explicitly to the study and improvement of mathematics teaching, had come into being in the early 1950s. Among its members were such people as the French Jean Dieudonné and Choquet, chief characters in the European movement of Modern Mathematics. Secondary teachers were also active in this Commission.

Members of the ICMI and CIEAEM officially met at the Conference of Royaumont (23 November to 4 December 1959) organized by OEEC, where European countries, the USA, and Canada were represented and to which schoolteachers were invited (see OEEC 1961). The composition of the Royaumont Seminar evidences an actual international cooperation that reflected an atmosphere of collaboration pursued during those years in a sociopolitical environment. Its concrete aim was to produce a curricular reform that was not limited solely to Europe. Social and political conditions were fostering reform movements all around the world. In particular, in Western countries, "Sputnik shock" turned politicians' attention to educational systems.

After Royaumont and until the mid-1970s, meetings in Europe and North America were organized to discuss new curricula and ways to implement them. National and international projects were launched; centers dedicated to mathematics education were created. Also primary school was touched by this movement of reform.

Since the second half of the 1960s, particularly in 1969–1974, the role of UNESCO in mathematics education was crucial. In 1968, the creation of the Center for Educational Research and Innovation (CERI), a major division of the OECD Directorate for Education, which was set up as an independently funded program by member countries and other organizations, provided the means for examining the course of reform. Behind this international cooperation and the movement of Modern/New Math, some political factors of that period stand out – the space race for one.

International cooperation was strengthened by the UNESCO-supported publication of a series of volumes entitled *New trends in mathematics teaching*. Anna Zofia Krygowska was appointed as the supervisor of the publication (see Report of the Executive … 1966, p. 3). The first volume appeared in 1967 (UNESCO 1967).

6.2 Turning Points in ICMI Life and International Cooperation

In the 1960s, two main points for giving ICMI a status suitable for action in international cooperation were realized. Heinrich Behnke, first as a secretary and after as president of ICMI (1955–1958), had addressed the problem of clarifying the relationship with IMU (see Furinghetti and Giacardi 2010). Afterwards, the action of presidents Stone (1959–1962) and André Lichnérowicz (1963–1966, see Gispert 2008) promoted attention to the peripheries. Nevertheless, the movements of reform and changes in mathematics and education stressed that international cooperation was not really progressing in how educational problems were being faced. While mathematical communication was increasing through journals and conferences, the only international journal in mathematics education was *EM*, which, indeed, had slowly become a mathematics journal (see Furinghetti 2009). The only regular occasions for discussions on mathematics education issues at an international level were the sections devoted to didactics in the ICM programs. The old agenda of ICMI was not suited to new times. In his introductory talk at Royaumont, Stone (1961) outlined a program of research in mathematics education (study and experimentation) encompassing the creation of research institutes in mathematics education and the inclusion of this subject into university research projects. He pointed out that teaching has to meet the needs of applications and devise new methodologies; he further stressed the enormous development of mathematics and the need to not widen the gap between school and university mathematics (see Furinghetti et al. 2008). Some years later, the combination of social forces and cultural changes during the 1960s pushed Freudenthal[10] to give new impulse to the old ICMI agenda by taking (independently from the mathematical community) two important decisions: founding the journal *Educational Studies in Mathematics* (1968) and establishing the tradition of having an international conference with regular dates (see Furinghetti 2008b). Such a conference was called ICME (*International Congress on Mathematical Education*): the first was held in 1969 in Lyon; since the second in 1972, ICMEs have been regularly organized around the world every 4 years. In this process of innovation, ICMI relied heavily on support from UNESCO. In 1976, during ICME-3, the first study groups affiliated with ICMI, HPM (*the International Group on the relations between the History and Pedagogy of Mathematics*) and PME (*the International Group for the Psychology of Mathematics Education*) were established. With these groups, which focused on specific themes in mathematics education, a new period began: their regular meetings and proceedings contributed to marking the evolution of research and reaching a large number of scholars.

Directly after the birth of *Educational Studies in Mathematics,* other international journals were founded: *Zentralblatt für Didaktik der Mathematik* (now *ZDM. The International Journal on Mathematics Education*) in 1969 (first editors Emmanuel Röhrl and Hans-Georg Steiner – see Schubring 2008g), *Journal for Research in Mathematics Education* in 1970 (first editor David C. Johnson), and *Journal of Mathematical Behavior* in 1971 (first editor Robert B. Davis). These initiatives provided international cooperation with a background based on scientific patterns that had fallouts in mathematics education (both in research and in the field). The dissemination of information created the terrain that fostered the maturation of local competences and critical thinking in mathematics education. As a consequence, the peripheries could attempt to avoid a passive acceptance of foreign patterns and develop their own way.

After the ICM-2006 in Madrid, important changes were achieved. According to the Terms of Reference of 2007,[11] the Executive Committee of ICMI is now elected by the General Assembly of ICMI itself, while in the past it was elected by the General Assembly of IMU. With the revision of the Terms of Reference in 2009, ICMI has introduced the notion of the affiliation of multinational societies active in mathematics education with ICMI. This has resolved the long-felt need to formalize links with ICMI by various bodies sharing its aims. CIAEM (Comité Interamericano de Educación Matemática or IACME, Inter-American Committee on Mathematics Education) was affiliated in 2009, CIEAEM in 2010, ERME (European Society for Research in Mathematics Education) in 2010, and MERGA (Mathematics Education Research Group of Australasia) in 2011. These changes constitute a further step in providing mathematics education with a cooperative character. They stress, on one hand, the autonomy of ICMI as an international body devoted to the care of mathematical instruction; on the other hand, they support the existence of an international network in which regional bodies interact at an international level.

[10] Already at the meeting of the Executive Committee in Geneva (2 July 1955), Freudenthal criticized the style of the reports and advocated for scientific research carried out by the Commission (see Desforges 1955).

[11] See http://www.mathunion.org/icmi/about-icmi/icmi-as-an-organisation/terms-of-reference/.

The duration of the Executive Committees in the period 2007–2012 is 3 years (not 4 as it was in the past). After this period of transition, the mandates will revert to a duration of 4 years as before. Further information about the new procedure may be found in Bass and Hodgson (2004).

The new procedure was adopted by IMU General Assembly (Santiago de Compostela, 19–20 August 2006). Further information may be found in Hodgson (2009) and on the IMU website, on the page dedicated to election procedures for ICMI http://www.mathunion.org/organization/ec/procedures-for-election/#ICMI.

6.3 Peripheries

Despite the purpose of internationalism that inspired the action of the bodies concerned with mathematics education, the countries driving the discussion and reforms were mainly North American and major European nations, although since its beginning, other continents were also represented in the ICMI (Japan in Group 1, Argentina, Australia, Brazil, Cape Colonies, Chile, China, Egypt, British India, Peru as associated countries). Thanks to Stone, president of IMU in 1952–1954 and of ICMI in 1959–1962, and André Lichnérowicz, president of ICMI in 1963–1966, collaborations – both scientific and organizational – were established with associations such as OEEC and UNESCO. This led to greater internationalism and to thematic congresses in various parts of the world. As outlined below, this action had different effects, depending on many factors such as proximity with some countries, languages, economic situations, external financial support, political issues, culture, beliefs, and values.

Science education had become an important area of cooperation with newly independent and developing countries, many of which established their own agencies for curriculum development. The long-standing collaboration between UNESCO and the International Council of Scientific Unions (ICSU) was strengthened by ICSU's establishment in 1961 of CIES (*Commission Interunions de l'Enseignement des Sciences*), the forerunner of its Committee on the Teaching of Science (CTS). CIES was a mechanism which served to coordinate the educational activities of the various scientific unions. Simultaneously, UNESCO established a Division of Science Teaching with the aim of strengthening UNESCO's programs for improving the teaching of science at the preuniversity level in developing countries. Pilot projects to incorporate modern approaches, methods, and materials were started – physics for Latin America, chemistry for Asia, biology for Africa, and mathematics for the Arab world. Various projects were carried out under the framework of EFA (Education for All).[12] In 1993, the participants in the Project 2000+ Forum (Paris, 5–10 July 1993) adopted a Declaration on scientific and technological literacy for all (see Project 2000 + Declaration 1993; Jacobsen 1996, p. 1253). The World Conference on Science organized by UNESCO and ICSU in Budapest (26 June to 1 July 1999) discussed the following issues: achievements, shortcomings, and challenges in science; science and society; and new commitments. The final Declaration of Budapest stressed the importance of scientific education for all. The book later published by UNESCO (UNESCO 2011) refreshed this idea with a particular focus on mathematics. At the *World Education Forum* (Dakar, Senegal, 26–28 April 2000),[13] 164 governments pledged to achieve EFA and identified six goals to be met by 2015 (see Peppler Barry 2000). In this global momentum, scientific literacy is one of the main challenges.

6.3.1 Asia

The Bombay conference mentioned earlier was an important step in establishing contacts outside Europe and North America. After Behari from India, Yasuo Akizuki from Japan was appointed as a member at large of ICMI Executive Committee (1963–1966, see Isoda 2008). In the ICMI meeting in Paris (7–8 December 1959), he emphasized the need for attention to all cultures in dealing with mathematics:

> Oriental philosophies and religions are of a very different kind from those of the West. I can therefore imagine that there might also exist different modes of thinking even in mathematics. Thus I think we should not limit ourselves to applying directly the methods which are currently considered in Europe and America to be the best,

[12] EFA movement is a global commitment to provide quality basic education for all children, youth, and adults; see http://www.unesco.org/new/en/education/themes/leading-the-international-agenda/education-for-all/, accessed July 2011.

[13] http://www.unesco.org/education/efa/wef_2000/, accessed July 2011.

> but should study mathematical instruction in Asia properly. Such a study might prove to be of interest and value for the West as well for the East. (Stone and Walusinski 1959, Annexe I, p. 289)

The effect of the actions of mathematics educators varied in the different countries of Asia. Nebres (2008) and Suat Khoh (2008) describe some regional activities (many of them linked to ICMI) and the benefits for the nations in the region. Southeast Asia's connections with ICMI began with the founding of the *Southeast Asian Mathematical Society* (SEAMS) and the national mathematical societies in 1972. In 1976, Yukiyoshi Kawada (ICMI secretary in 1975–1978, see Iitaka 2008) promoted the organization of a regional conference on mathematics education cosponsored by ICMI. The first *Southeast Asia Conference on Mathematical Education* (SEACME) took place in Manila, Philippines (29 May to 3 June 1978). Since 1978, SEAMS has organized SEACMEs regularly. As Suat Khoh (2008, p. 248) put it:

> The basic philosophy of the SEACME series … hinged on the principle that each conference was a national conference with regional and perhaps some international participation. The objective was to primarily benefit the host country and this engendered a sense of ownership and shared interest in the issues discussed and therefore better focus and relevance. Each conference was organized by the hosting nation which chose its own theme, invited speakers and encouraged attendance by mathematics teachers. The host country benefited, not only through mutual learning on relevant issues within the local mathematics education community, but also through providing their teachers and other participants the opportunity to learn from the regional and international speakers and participants.

In 1986, when China joined IMU and ICMI,[14] a formal collaboration among the Northeast Asian countries was established. The first ICMI-East Asian mathematics education conference was the *ICMI-China Regional Conference on Mathematics Education* held in Beijing (1991), followed by the second in Shanghai (1994). To be more inclusive, a third conference was organized in South Korea (1998) and had a more international character, with five keynote speakers from Australia, the UK, and the USA and six from East and Southeast Asia. The series became the *ICMI-East Asian Regional Conference on Mathematics Education* (ICMI-EARCOME). In 2002, EARCOME-2 and SEACME-9 were held together in Singapore. After that, the SEACME series was subsumed into the EARCOME series, and the resulting new EARCOME series began in Shanghai (2005). Subsequently, international cooperation was further strengthened. In 2000, Japan hosted ICME-9.

6.3.2 Australasia

In the middle of 1976, proposals for the creation of a national group interested in mathematics education research were put forward, and in 1977, the first MERGA (Mathematics Education Research Group of Australia) annual conference was organized. MERGA publishes two journals. Later on, the "A" in the acronym came to represent Australasia instead of Australia to stress the inclusion in the association of a vast area of countries (Pacific Islands, New Zealand, South Asia) participating in MERGA conferences. In 1984, ICME-5 was held in Adelaide.

6.3.3 Latin America

In Latin America, international cooperation was first/initially stimulated by the New Math that had arrived from the USA through the textbooks of the *School Mathematics Study Group* (SMSG) (see Barrantes and Ruiz 1998). A decisive impulse came from the *First Interamerican Conference on Mathematics Education*, held in Bogotá (4–9 December 1961). This conference, sponsored by ICMI

[14] The section entitled "China joins the IMU" (Lehto 1998, pp. 242–250) outlines the story of this episode. In *Bulletin* (1986, p. 16), two new members were welcomed to ICMI: the Chinese Mathematical Society representing the People's Republic of China and Kuwait. In the list of national representatives, the Chinese Mathematical Society and Mathematical Society located in Taipei are mentioned.

and by the Organization of American States (OAS), received large financial aid from the US National Science Foundation (NSF). Mathematicians and mathematics teachers, as representatives or guests, from 23 American countries and a few European special guests attended the conference. The talks delivered by eminent scholars (Choquet, Stone, Laurent Schwartz, Edward Griffith Begle,[15] Howard F. Fehr) reported on what was going on in Europe and the USA and on the means to introduce innovations into Latin America: suitable textbooks, curriculum changes, teacher training, and so on. On this occasion, the first Executive Committee of the *Inter-American Committee on Mathematics Education* (IACME) or *Comité Interamericano de Educación Matemática* (CIAEM) was established. After this event, the conferences of CIAEM were regularly held in different locations throughout Latin America. Later international cooperation was strengthened by CIBEM (*Congreso Iberoamericano de Educación Matemática*), proposed in 1987 during the eighth CIAEM in Dominican Republic. These congresses were organized by the *Associação de Professores de Matemática* (Portugal), the *Federación Española de Educación Matemática,* and CIAEM. At least two journals published in Latin America have reached an international audience: *Bolema (Boletim de Educação Matemática* first issued in 1985, published by the Departamento de Matemática IGCE–UNESP, Brazil) and *Relime* (*REvista Latinoamericana de Investigación en Matemática Educativa*, first issued in 1997, published by Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Colonia San Pedro Zacatenco, México, DF). *Relime* is the organ of CLAME (*Comité Latinoamericano de Matemática Educativa*).

6.3.4 Africa and Arabian Countries

In Africa, international cooperation in mathematics education was promoted through different initiatives by ICMI, UNESCO, and later IMU-CDE (*Commission on Development and Exchange*, established in 1979). In the summer 2006, I interviewed Emma Castelnuovo (Furinghetti 2008c) who reported on her experience in Niger, where UNESCO had sent her (four times from 1978 to 1982) to teach in local schools. She was very proud to remember her students enjoying the new method she applied in the classroom.

CIES organized in Dakar (14–22 January 1965) a congress on science teaching and its role in economic progress; this was attended by 84 members (specialists in sciences) from nine African countries, four American countries, five Asian countries, and nine European countries. ICMI was represented by its president Lichnérowicz and by secretary André Delessert. On this occasion, ICMI in collaboration with the National Senegalese Commission of the teaching of mathematics organized a congress (13–16 January) on mathematics teaching in conjunction with the teaching of other sciences (see Commission … 1966). During this same period, *L'Enseignement Mathématique* published the first paper authored by an African (Niang 1966), in which Africans' difficulties in mathematics were related to the absence of a written mother language and to the inadequate preparation of teachers. About teacher training education, Niang reports that UNESCO had created *Higher pedagogical centers* (Centres pédagogiques supérieurs) in francophone Africa (principally in Senegal and Ivory Coast), but they were transformed into "Higher normal schools" ("Écoles normales supérieures"). He adds that their plan was old, very classic, and inadequate for the milieu for which it was planned.

At the end of the 1960s, UNESCO and ALECSO (Arab League Educational, Cultural and Scientific Organization) launched a project aimed at introducing New Math into Egypt and other Arab countries. Textbooks were published for this purpose: the first appeared in 1970 for secondary school and in 1974 for primary school (see Malaty 1999).

Adler (2008) reports on mathematics and mathematics education activity in Africa. AMU (African Mathematical Union) has commissions linked with education: AMUCHMA (African Mathematical

[15] See Kilpatrick (2008)

Union Commission on the History of Mathematics in Africa), AMUCME (African Mathematical Union Commission on Mathematics Education in Africa), and AMUCMO (African Mathematical Union Commission on Mathematics Olympiad in Africa). Regional bodies, such as SAMSA (Southern Africa Mathematical Sciences Association) and SAARMSTE (Southern African Association for Research in Mathematics, Science and Technology Education) have regular conferences. The professional association of mathematics educators in South Africa AMESA (Association for Mathematics Education of South Africa) organizes annual meetings. The Maghrebian Colloquium on the History of Arab Mathematics also encompasses contributions related to education. During 22–25 June 2005, the first Africa regional ICMI congress, named AFRICME, was organized in Johannesburg. Other AFRICME congresses have been held since. Journals on mathematics education are published in Africa. Researchers in mathematics education from Africa are active at the international level.

7 Conclusions

The history outlined here evidences the synergies of such bodies as OEEC/OECD, UNESCO, ICMI, ICSU, and IMU, which aim to spread the outcomes of work in mathematics education around the world. In the peripheries and in many developing countries, these synergies have produced a background suitable to the growth of mathematics education communities aware of these problems and competent to face them. Unfortunately, problems still remain because, beyond the aspects of solidarity, international cooperation in mathematics education has political dimensions which are outside the control of the scientific bodies. Greater attention paid to mathematics programs, which are based upon the needs and cultures of the ethnic mixes found in most countries, attempted to fill the gap between rich and poor countries, but, as discussed by Jacobsen (1996), the curtailing of funds from these international agencies makes it "more difficult to look for governments for improved international cooperation in mathematics education" (p. 1253). The program launched by the president of ICMI Miguel de Guzmán (see Jaime Carvalho e Silva 2008), contemplates an ICME Solidarity Fund, developed from a percentage of the registration fees gathered at each ICME and aimed at supporting the participation of delegates from nonaffluent countries to ICME congresses, see (de Guzmán 1992). Since ICME-8 in Sevilla (1996), this tradition has ensured a more balanced representation from different areas of the world. Nevertheless, during the celebrations for the centenary of ICMI, Setati (2008), reporting on Africa, has pointed out that providing universal primary education by the target date of 2015 – one of the goals stated in the Millennium Declaration of Dakar Summit – requires further efforts, notably financial, if it is to be reached.

Reinforcing international cooperation is an exacting task, but it is worthwhile. History has shown that the benefits offered by international cooperation may affect not only the life of developing countries, but also mathematics education (both in research and in the field) in other countries. On one hand, international cooperation has put at the disposal of the peripheries the initiatives and results of studies and activities in the domain of mathematics education. However, on the other hand, thanks to the contact with contexts which were previously neglected, new concerns and issues have entered the mathematics education discourse. Among them, Nebres (2008, p. 152) lists:

- Ethnomathematics, especially from Latin America and, in particular, the work of Ubiratan d'Ambrosio
- Mathematics for All, coming in great part from UNESCO
- The impact of society, economics, and culture on mathematics education

Past experience has shown that the most effective results, both in research and in the field, did not involve only exporting foreign patterns to peripheries, but also raising local initiatives and contacts. The movement of international cooperation has made mathematics education and, more generally, culture an issue which concerns all people and has changed the perception of cultural differences. Stone (1956), in his plenary talk at the pioneer conference in Bombay, referred to the progressivism of Western culture and to cultures "of which we have anything like an adequate knowledge [that] appear to have been strongly conservative and essentially inimical to rapid, radical, or extensive change" (p. 32). Is this statement still sound in our days?

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Chapter 28
History of Tools and Technologies in Mathematics Education

David Lindsay Roberts

1 Introduction

Since the advent of the electronic calculator, it has become customary for discussion of "technology" in mathematics education to refer almost exclusively to use of electronic devices. For example, in a daily newspaper in 2011, we find the following:

> Ever since the first elementary school teacher rolled the first television set into the first classroom to air the first course offerings from "educational television," there's been the hope and the promise that technology would revolutionize the way teaching and learning would be done. (Pearlstein 2011)

The implication here is that there was no such thing as technology in education before electronic technology and that there were no great hopes for revolutionizing teaching and learning prior to such technology. However, this narrow view is highly misleading. The employment of tools to assist teaching and learning of mathematics in fact has a history long predating electronic technology, and some of them have been proclaimed as revolutionary. In this chapter we will endeavor to look at the history of educational technology in a more integrated fashion, giving no special preference to electronic technology. Indeed, such an approach provides a useful perspective from which to view the debates surrounding the electronic tools of today.

In order not to go to the other extreme, with the concept of technology encompassing an unmanageably large range of human activities, perhaps including mathematical notation and language in general, we will limit ourselves to material devices. Thus, for example, we will not count logarithms as a technology, while the slide rule, a physical device based on logarithms, will be within our purview.

It must also be acknowledged that even within these bounds, this chapter fails to cover the history of technology in mathematics education uniformly across the globe. Space limitations, combined with the special interests of the present writer, have resulted in a treatment that often focuses on developments in the United States, occasionally provides brief discussions of education in Europe, and regrettably offers very little direct commentary on other parts of the world. The reader may also detect a presumption that there has been significant homogenization of educational technology worldwide in recent decades, with little effort to present any supporting evidence. It is hoped, nevertheless, that this chapter will be usefully provocative even for those with interests different from the writer and will suggest fruitful avenues for further research.

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We organize our discussion in relation to a technology's scope of use, classifying technology into two primary groups: general-purpose tools and specialized technologies. By general-purpose tools, we refer to those of wide importance in many walks of life outside classrooms but put to special use in an educational setting. The specialized technologies, in contrast, are most likely to be encountered in technical work such as science or engineering. Some of these have been explicitly developed for teaching mathematics and have been largely confined there.

Educational use of technology has been subject to overarching educational philosophies prevailing at any given time and place. We will comment on the influence of some of these philosophies (sometimes disparaged as fads or fashions) where appropriate.

2 General-Purpose Technologies Used in Mathematics Education

2.1 The Textbook

At times technology has been invented specifically to serve mathematical purposes. At other times technology has entered mathematics, and specifically mathematics education, from the larger world outside, notably from commerce and from science. Probably the most ubiquitous of such tools, retaining a powerful presence in worldwide mathematics education to the present day, is the book. As an educational tool, the book serves as a medium for storing and displaying information to be conveyed to students. The book has a history almost as old as civilization itself, from clay tablets to the papyrus scroll, to the handwritten codex, to the printed book, and on to the modern e-book (Hobart and Schiffman 1998; Schubring 1999, 2003.). The manifold contributions of this technology to civilization are well known and need not be recounted. But the history of the mathematics textbook is much shorter, especially if we neglect advanced monographs in favor of books actually used in schools. Certainly for many centuries, individuals have learned mathematics independently from books, and likewise tutors have used books to teach mathematics to individuals and small groups, but a new era begins with the advent of mass schooling and the mass-produced textbook. These interconnected phenomena did not become prominent until the nineteenth century in Europe and the Americas and were materially aided by both political and economic developments. On the political side, there was rising support for providing education for a larger proportion of children. On the economic side, there were increasing efficiencies in the production of the physical book, and increasing facilities for transporting them over long distances, resulting in the ability to manufacture and distribute large numbers of books relatively cheaply (Kidwell et al. 2008).

When books were scarce, if a school class had a book at all, it would frequently be the exclusive possession of the teacher. If the class was of any appreciable size, this encouraged the recitation method of teaching, which frequently entailed the teacher simply reading aloud from the book and the pupils attempting, through writing or brute memorization, to retain what was read and then to recite it back to the teacher. Notable attempts to scale this system up were made in England and its colonies in the late eighteenth and early nineteenth centuries with the so-called monitorial system, in which the teacher would first teach a group of more advanced students, who would in turn teach less advanced students. In mathematics in particular, the recitation method and the monitorial system primarily supported a curriculum centered on the rote learning of the rudiments of arithmetic (Butts 1966).

Prior to the emergence of both the textbook and the blackboard, it was also common practice in many schools in Europe and North America for each student to produce a "copybook" or "cyphering book." Beginning with a collection of blank pages (paper and binding quality could vary widely, depending on economic circumstances), the student would copy out the material spoken aloud by the teacher. In the case of a teacher reading from a printed book, this could often mean that the student was almost literally producing a handwritten copy of the book or the problems from the book. Here again the use of copybooks primarily supported arithmetic instruction, but in some cases this could be fairly elaborate, including square and cube roots and complicated problems from commerce and business. The teacher could periodically inspect the copybooks, so that they could have functioned as what more recent educators would term a "portfolio." But how rigorously eighteenth- and nineteenth-century copybooks were evaluated for mathematical correctness is unclear, and some may have been assessed more on aesthetic grounds, such as penmanship (Cohen 1982; Clements and Ellerton 2010).

But with cheaper books came the possibility (though still often not the reality) that students as well as teachers could have individual access to a textbook. A student with a book could now be asked to read that book both during and outside of class and to work problems assigned from the book. It was now easier than previously to provide more sophisticated mathematics instruction for a classroom of pupils. Thus, the rising presence of algebra and geometry in addition to arithmetic in the curriculum of nineteenth-century schools surely owes a good deal to the proliferation of textbooks. It is also likely that the use of textbooks served to hide problems with inadequate teacher preparation. This was certainly the case in the nineteenth-century United States (Tyack 1974).

Moreover, the system of textbook usage amplified itself: a greater supply of books produced a greater demand for books, which in turn produced yet more books, and so on. In mathematics this resulted not merely in the creation of individual textbooks but entire series of textbooks covering the whole range of the curriculum from the lowest grades to the colleges: basic arithmetic to the differential and integral calculus. Conditions in the United States, especially the free-market economy and the separation from Britain, seem to have been especially favorable for establishing a vibrant textbook industry in the nineteenth century. In the United States notable nineteenth-century authors of mathematics textbooks include Charles Davies, Joseph Ray, and George Wentworth (Kidwell et al. 2008). In contrast, Australia relied for far longer on British textbooks and was thus slower to establish its own textbook industry. The educational influence of Europe on colonized regions is complex and is the subject of recent scholarly attention (Ellerton and Clements 2008).

One notable effect of textbooks has been to standardize and codify curriculum. Educators have often found it difficult to dislodge curriculum topics once they are printed in widely distributed textbooks. This is especially striking in the United States, which despite a long tradition of local control of schools, and avoidance of an official national curriculum, rapidly converged on a de facto standard curriculum in mathematics, as a relatively small number of textbooks began to dominate the market. Genuinely innovative mathematics textbooks have never fared well in the US market. Even during the 1950s and 1960s, supposedly a time of major upheaval, we can observe important textbooks exhibiting substantial continuity from earlier decades. The largest American program for curriculum reform during that era, the School Mathematics Study Group (SMSG), produced a variety of text materials, which were published in an inexpensive format by the Yale University Press. The hope was that these texts, some highly innovative, would serve as models for commercial textbooks. But this hope was realized in only rare cases, the most successful of which was the Houghton Mifflin algebra textbook series with Mary Dolciani as the lead author. If one examines the Dolciani textbooks, it is clear that although there is a sprinkling of new material, they owe a great deal to Houghton Mifflin texts from the days prior to SMSG (Freilich et al. 1952; Dolciani et al. 1965; Wooton 1965; Roberts 2009).

2.2 The Blackboard

The blackboard or chalkboard and its offshoots are today widely used outside education, especially in business and government, but unlike the book this technology seems to have found its first extensive use in the classroom and only then moved outward. Educational use of this tool is tightly bound to the rise of mass education, which brought a pressing need for multiple individuals to view the same information simultaneously. Prior to the wall-mounted blackboard, there had been a slow evolution of handheld writing surfaces, culminating in the slate, which could be written on with chalk. In Europe and North America, this was often a facet of the recitation method of instruction. The teacher could read a problem from the book, and the students could copy and display their solutions on their slates (Cajori 1890; Burton 1850).

The erasable blackboard, written on with chalk, spread quietly into schools in the early 1800s and was well established by the end of that century (Kidwell et al. 2008). It allowed the teacher to display complicated verbal or pictorial details with far more exactitude than merely reading aloud from a book. Moreover, it allowed students to work out problems on the board themselves, displaying their efforts for both the teacher and for other students to see and comment on, thus changing the personal dynamics of the classroom. In mathematics the blackboard worked in conjunction with the textbook to promote the rise of both algebra and geometry in the curriculum.

Blackboards have continued in use in mathematics classrooms to the present time. In many cases, the chalkboard has been replaced by the "dry-erase" or "whiteboard," but with no essential change in functionality. The interactive whiteboard, developed in the late twentieth century, represents a major innovation, allowing the material displayed on the board to be connected directly to a computer. Opinions vary widely on the value of this technology in the classroom (Smith et al. 2005; Wood and Ashfield 2008). Tablet personal computers offer similar functionality, including handwriting recognition, whereby the computer is able to interpret handwriting drawn on the screen, not merely type entered via a keyboard (Anderson 2011).

2.3 *The Overhead Projector*

A more recent classroom display technology is the overhead projector. Its earliest manifestations seem to have been related to education but not in school classrooms: public nineteenth-century science lecturers seeking added visual flair. Such use began to enter schools in the early twentieth century, as part of a wider movement for "visual education" that included photographic slides and filmstrips. About the same time, this technology also received a boost from a noneducational venue, the bowling alley, where it was used as a convenient way to project scores for bowlers to view. Overhead projectors then received substantial use by the US military during World War II for training purposes, probably contributing to a major expansion of school use in the postwar years (Kidwell et al. 2008).

Much more than the blackboard, this technology as used in schools has remained the exclusive domain of the teacher. It has two primary attractions. First, it allows the teacher to continue to face the students while displaying materials to them. Second, it allows the teacher to display elaborate transparencies created before class. For example, a teacher of solid geometry can prepare or purchase complicated diagrams of an exactitude that could never be hoped for in hand-drawn diagrams quickly improvised while watched by the students. There is however a drawback, in that reliance on prepared slides can encourage a too rapid succession of material that can overload the students' ability to assimilate the information presented.

Overhead projectors have continued in use to the present but in many cases have been superseded by new technologies allowing greater ease of use and a greater range of display functionality. Computer projection systems permit the display of any image, static or moving, available to the host computer and in particular allow slide shows formerly done via transparencies on an overhead projector to be accomplished via software such as PowerPoint. Another enhancement of the overhead projector is the document camera (also known as an image presenter or visualizer), which permits any document, or even a three-dimensional object, to be displayed on the overhead screen without any prior preparation of the document or object (Ash 2009).

Many classrooms in the twenty-first century provide not only a computer and projector for the teacher but also a computer for each student, networked with the teacher's computer. In some ways this is a return of the handheld slate, with a vast increase in functionality. Its potential for mathematics instruction is just being tapped.

2.4 The Computer

Like the book, this tool's wider societal uses are enormous. It has now established itself in mathematics education throughout the world, although its ultimate role is perhaps not yet clear. It can be argued that much educational use of computers is trivial compared to the full capabilities of the technology. For example, many students today can read textbooks on a computer screen, but this is surely not a profound capability. Probably the most common use of computers in elementary instruction is to provide instant feedback to students working on problems. This is undoubtedly an increase in convenience that might amaze earlier generations of students and teachers, but in principle it is no different from looking up the answer in the back of the book.

Unlike the book, the advent of computers in education is not lost in the mist of time and indeed is still within living memory. The original "main-frame" computers, developed during and just after World War II, were too expensive, too bulky, and required too much maintenance to have much attraction for educators. It was only in the 1960s, with time-sharing systems and with so-called minicomputers that there began to be any appreciable use of computers in education. It was now possible for several students to simultaneously interact with the same computer. A pioneering instance was the University of Illinois's Project PLATO (Programmed Logic for Automatic Teaching Operations) (Bitzer et al. 1961). This was built upon earlier, nonelectronic, "programmed" learning efforts which had become popular beginning in the 1950s. Programmed learning experiments, much of which were inspired by the work of B. F. Skinner and other psychologists, featured ordered sets of problems which the student was asked to work through (Vargas and Vargas 1996). The student's passage through the problems depended on whether the student gave correct or incorrect answers at each step; a student might be asked to cycle back through some material or else move on briskly to new topics. This could be accomplished merely with a book, by covering up the answers. Computers allowed this to be done more easily and with more flexibility. As already noted, this basic functionality continues to be one of the most widely used applications of computers in mathematics education.

A different tactic for computer use in education was explored at Dartmouth College, again beginning in the 1960s. Here the aim was to have undergraduates program the computer themselves, thus learning the fundamental logical principles behind the machines. They succeeded in making computer programming a feature not only of mathematics classes but of other classes where mathematics was applied, including business and the social sciences. A key piece in achieving this was the development by Dartmouth professors John Kemeny and Thomas Kurtz of the BASIC computer language, which subsequently spread worldwide among novice and expert computer users alike. Ultimately one major effect of the Dartmouth work, and other similarly oriented projects throughout the world, was that computer science branched off from mathematics as a separate academic discipline at the college and university level (Kemeny and Kurtz 1985).

The emergence of the microcomputer, or personal computer, in the 1970s and 1980s, gave further impetus to educational use of computers, especially below the college level. For the first time computers became a home appliance, which made school use much more comfortable for both students and teachers. Computer games, some with an educational component, such as *Lemonade*, for the Apple II personal computer, began to proliferate (Apple 1982). And now that the crude teletype terminals of earlier days were being replaced by video display screens, it was possible to generate much more elaborate graphics, with obvious application in geometry instruction. *Geometer's Sketchpad* and

Cabri-géomètre are two examples of computer programs taking advantage of these capabilities (DeTurck 1993). Statistical software such as *Minitab* and algebra software such as *Derive* also came on the market in the 1980s (Ryan and Joiner 1973; Grinberg 1989). Large software packages incorporating a full range of algebraic capabilities, together with sophisticated graphics, included *Maple* and *Mathematica* (Chonacky and Winch 2005). Such software has raised as yet unanswered questions about the content and methods of mathematics instruction. Even general-purpose software such as Microsoft *Excel* offers extensive mathematical capability which potentially could totally reshape the mathematics curriculum.

It must be noted, however, that computer use in mathematics classrooms varies greatly worldwide. The cost of purchasing and maintaining computers, together with training instructors to use them effectively, remains a significant obstacle in many places, especially compared with the older technology, the book.

3 Specialized Technologies Used in Mathematics Education

In addition to general-purpose tools, mathematics education has made use of specialized tools. Some of these have originated outside education, especially in commerce, science, and engineering. Others have originated within education and then moved outside. A few are essentially unique to mathematics education. We classify them here into three broad categories: tools for calculation, tools for drawing and display, and tools for physical manipulation.

3.1 Calculating Tools

Calculation is an activity that many in the general public consider synonymous with mathematics, to the distress of many mathematicians and mathematics educators. Of course there can be little doubt that the historical roots of much mathematics are found in the practical need for calculation, and consequently calculation has been a central justification for mathematics education since antiquity. In general, physical tools for calculation have first received extensive use outside the classroom, in realms where speed and efficiency are more pressing issues, before becoming an accepted part of standard school instruction. The slide rule, for example, was a tool of practicing engineers for decades before it was seriously taught in schools. Possibly the abacus, as used in Asia, is an exception to this trajectory. No physical calculating device has been a part of mathematics instruction in the West in the manner, or for the long duration, that the abacus has been part of such instruction in Asia.

3.1.1 The Abacus

The abacus depicts numbers by means of beads on wires. It apparently evolved from marks in sand or counters on a board. The device seems to have developed somewhere in the eastern Mediterranean world in antiquity, moved east to Asia, then moved back west via Russia into Europe and thence to the Americas. The transmission to Asia is conjectural, and it is possible that it originated there independently. What is clear is that whereas the abacus became a widely used tool of calculation in China and Japan, without serious competitor until very recent times, it never attained the same level of popularity in this role in Europe and North America. Instead, in the latter regions, it was primarily confined to use as a demonstration tool for teaching elementary arithmetic to young children.

The Chinese abacus (*suanpan*) appears to have been in substantial use by 1200 and probably much earlier. Transmission to Japan seems to have occurred via Korea. The Japanese modification of this instrument (called the *soroban*) was in use by 1600 (Smith 1958). The abacus has been part of education in both nations for centuries, and the device has continued to be part of mathematics instruction in many East Asian nations to the present day, although not without some controversy and competition from newer technology. In Malaysia, for example, although abacus use in schools declined for a time after handheld calculators became widely available, the abacus (*sempoa* in Malay) has more recently experienced an educational resurgence in connection with an increased emphasis on mental arithmetic (*China Daily* 2010; Shibata 1994; Siang 2007).

While in East Asia the beads move on vertical wires, the version of the abacus that became common in Russia featured horizontal wires. This would prove advantageous for using it as a display device for young children, since the teacher could hold up the abacus in front of the class and the beads would remain in place. It was used in Russia for early education until recent decades. The French mathematician Jean Victor Poncelet encountered the abacus while imprisoned in Russia following Napoleon's invasion of 1812 and introduced it to France on his return. It spread widely across France as a teaching tool in the nineteenth century (Gouzévitch and Gouzévitch 1998; Régnier 2003).

A similar teaching device began to appear in the United States in the 1820s, likely inspired at least in part by the French version. Here it meshed well with the Pestalozzian object-teaching philosophy that was gaining in popularity, and by the 1830s, it was being sold under various names, including "numeral frame" by companies catering to the growing educational market. These teaching abaci were not without detractors, however, some of whom felt they might even stifle the imagination of the child. They remained as a tool for only the youngest learners of arithmetic (Kidwell et al. 2008). In more recent years, apparently reacting to the perceived success of Asian students in mathematics, some educators have advocated more use of the Asian abacus in Western schools (Ameis 2003).

3.1.2 The Slide Rule

The slide rule incorporates in physical form the theory of logarithms pioneered by Scottish mathematician John Napier and English mathematician Henry Briggs in the early 1600s. By marking two straightedges with logarithmic scales and sliding one with respect to the other, it was possible to quickly calculate approximate answers to multiplication problems. Even more complicated problems could be handled with sufficient ingenuity, although the fact that the slide rule was an analog instrument meant that it always provided only approximate answers and thus was not appropriate for accounting or other commercial applications. Variations involving circular rules were also possible, and both possibilities had been explored by the middle of the seventeenth century in England. These slide rules were slowly improved over the next century and became a tool used by British engineers such as James Watt. By the early 1800s, they had spread to the European continent and to the United States (von Jezierski 2000).

It was not until the late nineteenth century that the slide rule became an educational tool, beginning first with colleges featuring an engineering curriculum. In Britain, the engineer John Perry included the slide rule among the practical tools that he advocated for reforming the training of engineers, scientists, and mathematicians, which he promoted first at Finsbury Technical College and later at Imperial College, London (Perry 1913; Gooday 2004). In the United States, institutions such as Rensselaer Polytechnic, the United States Military Academy, and the Massachusetts Institute of Technology were leaders in educational use of this technology. In the early twentieth century, the slide rule began to filter down into the secondary schools, helped by the movement, in both Europe and the United States, to establish mathematical "laboratories" which emphasized the mathematics of measurement and applications to the physical sciences. Instrument makers were selling slide rules to the high school market by the 1920s and some were also selling oversized models that could be displayed in front of a classroom for all students to see. The slide rule remained a recognized feature, although in most cases not a central one, of many mathematics and science classrooms until the advent of cheap electronic calculators in the 1970s (Kidwell et al. 2008).

3.1.3 The Calculator

Unlike the slide rule, the calculator is fundamentally a digital instrument, which seems to have given it a decided advantage in achieving a place in mathematics instruction. Its place in the classroom is still in an experimental stage. European development of mechanical calculators dates from the seventeenth century, with such notable mathematicians as Pascal and Leibniz prominently involved (Goldstine 1972). But it was not until the middle of the nineteenth century that industrial processes were sufficiently advanced to allow construction of calculating devices on a commercial basis, both in Europe and the United States. By the 1920s they had become a standard feature of many office settings. But it appears that it was not until after World War II that they received much consideration as educational assistants. In the 1950s there was some minor experimentation in classrooms with mechanical calculators, or mechanical calculators with electrical assistance, but the size and cost of these machines made them inconvenient as personal devices (Kidwell et al. 2008).

The major breakthrough occurred in the 1970s, with the arrival of inexpensive, fully electronic calculators. Initially these calculators were still relatively bulky and were able to perform little beyond the familiar four operations of arithmetic. But by the 1980s calculators had become readily portable and were able to compute trigonometric and other transcendental functions and to display graphs, thus far surpassing the functionality of mechanical calculators and slide rules. Classroom use became practical and although very uneven, soon became widespread enough to create disputes between enthusiasts and detractors. Calculators greatly increased the range of feasible problems that could be given to students, but concern was expressed about the effect on basic arithmetic skills, and doubts were raised about the readiness of teachers to use calculators effectively (Kelly 2003; Waits and Demana 2000). By the mid-1990s computer algebra systems (CAS) were available on handheld devices, leading to further debate. Now, in the twenty-first century, although the generic name persists, high-end devices referred to as “calculators” in fact provide a huge range of information storage, information display, and demonstration capabilities, in addition to pure calculation (Aldon 2010; Trouche 2005). Some controversy has persisted, but in recent years the use of calculators has been increasing around the world in secondary and elementary schools and at the college level as well.

3.2 Tools for Drawing and Display

Among such tools we include both devices for making marks and the media upon which the marks are made. (That the slate and blackboard could be considered in this latter category shows that our classification scheme is far from clear cut.) Most of these tools have found abundant use outside of education, most especially by surveyors and engineers. The most ubiquitous of such tools, the straightedge and the compass, are so old that their origin in education or anywhere else is highly obscure. It is of course well known that the ancient Greeks sought to investigate which figures could be constructed with straightedge and compass alone and that this led to classrooms throughout the West featuring these instruments in geometry instruction. We can add very little to this general outline here. Instead we will focus on some more recently invented devices, used less universally, but whose histories are nevertheless revealing.

3.2.1 The Protractor

The protractor, a semicircular device with markings to measure degrees of angles, emerged in Europe in the sixteenth century, out of the confluence of tasks generated by surveyors, navigators, and mapmakers. Early protractors were generally made from brass or horn. Horn, though less rugged than brass, and subject to wrinkling or curling, offered the advantage of being semitransparent, so that a draftsman could see an existing drawing underneath the protractor. By the eighteenth century, discussion of protractors began to appear not only in manuals for instrument makers but also in geometry textbooks, especially in France (Kidwell et al. 2008).

Geometry instruction remained more formal in the English-speaking world well into the nineteenth century. In those texts where straightedge and compass constructions were emphasized, protractors were not considered appropriate. A few "practical geometry" textbooks began to appear in the early nineteenth century, but it was not until the middle of the century that there was substantial movement away from formal Euclidean treatments of geometry. In the United States this approach, much later dubbed "informal geometry," was driven in part by educational philosophies emphasizing greater emphasis on using sense data, especially visual, to convey the abstract concepts of mathematics. The Swiss educator Johann Pestalozzi and his follower Friedrich Froebel were influential in this regard. American reform educators also observed the contemporary German efforts to develop geometry instruction for those not intending to attend university (Coleman 1942).

American geometry instruction through the remainder of the nineteenth century featured an eclectic mixture of formal and informal and of varying focus on the practical utility of geometry versus the merits of the subject for training the mind. Harvard president Thomas Hill's geometry textbook of 1863 explicitly directed protractor use for solving many of its problems and even described how students could create their own instruments (Hill 1863). But other books, such as those of Charles Davies and George Wentworth appearing later, continued to make little concession to practical matters, treating geometry as a purely abstract subject in which the protractor had no place (Davies 1885; Wentworth 1877). A rapprochement began to be effected in the 1890s as part of the general effort to standardize the entire secondary school curriculum. The formula adopted, first enunciated by the mathematics subcommittee of the Committee on Secondary School Reform (better known as the "Committee of Ten"), was to urge initial geometry instruction to be "concrete," while older students would be taught in a rigorously "demonstrative" manner. The protractor was an important tool for the former but was laid aside for the latter. Soon after this period, instrument manufacturers began to market much cheaper protractors (of cardboard or celluloid) for the growing demand. The protractor has continued to hold a similar place in geometry instruction to the present (Kidwell et al. 2008).

3.2.2 Linkages

Although the straightedge was long considered an unproblematic instrument, there was a brief period in the nineteenth century where there was agitation to change this. The impetus originated in engineering, specifically with the work of James Watt. In the course of refining his steam engine in 1784, he devised a system of rods and pins to convert rotary motion to approximately straight-line motion. This later caught the attention of Russian mathematician P. L. Chebyshev, who asked a question that seemingly had never before been asked explicitly: is it possible to produce an exact straight line by mechanical means? Whereas the compass provides a means of producing an exact circle with the simplest of means, it is not at all obvious how to produce a straight line in a similar fashion, without simply tracing along an already-existing straight line, which is how a straightedge is conventionally used (Kidwell et al. 2008).

The problem was solved in 1860s and 1870s by use of inversive geometry. A point on a system of rods or bars, connected by hinges or pivots, could be made to trace an exact straight line as another point on the device was made to traverse a circle. The discovery of these devices produced a brief flurry of intense interest among some mathematicians. English mathematician J. J. Sylvester invented the term “linkage” to describe all such systems of rods and pins (Hilsenrath 1937). It was shown how to produce other curves and to perform such feats as trisecting angles (Yates 1945). There were even calls to refashion geometry education. In 1895, the American mathematician G. B. Halsted unsuccessfully proposed the following:

> Henceforth Peaucellier’s Cell and Hart’s Contraparallelogram [two linkages producing exact straight lines] will take their place in our text-books of geometry, and straight lines can be drawn without begging the question by assuming first a straight edge or ruler as does Euclid. (Halsted 1895)

These devices have never become more than an enrichment topic in the classroom (Kidwell et al. 2008), but they have continued to create enthusiasm among mathematics teachers and teacher educators to the present time (Bartolini Bussi and Maschietto 2008).

3.2.3 Graph Paper

Graph paper, a now familiar medium for depicting geometric figures, has a shorter history than is often realized. Although today’s student of “Cartesian” geometry is often requested to “graph the following equation,” such problems are foreign to Descartes’ own seventeenth-century work. Indeed, it was only in the nineteenth century that this procedure became a standard part of the mathematical repertoire and not until the twentieth century that it became entrenched in school instruction.

Special ruled paper, designed to facilitate the depiction of relationships between two varying quantities, is essentially a nineteenth-century innovation of civil engineers, although some intimations can be found in eighteenth-century astronomy and chemistry. Builders of roads, canals, and especially railroads found it increasingly important to compare the vertical change on a route in relation to its horizontal progress. At first, individual users created their own paper to accomplish such tasks, but by the 1870s commercially produced paper was available. The cost of such paper rapidly declined in the last decades of the nineteenth century, making possible its use as an educational tool in the twentieth (Kidwell et al. 2008).

But engineering use and cheap production costs would not in themselves have created demand for graph paper in mathematics instruction, were it not for changes in the philosophy of mathematics education. The case is especially clear in Britain and the United States, both of which countries experienced reform movements related to emphasizing the value of visualizing abstract concepts. In Britain, John Perry, already mentioned, promoted a more concrete and visual approach to mathematics education, helping to break the unquestioned dominance of formal Euclidean geometry in British education. His influence extended to both Japan (where he worked for a time in the 1870s) and the United States. One of his specific proposals was for the substantial use of “squared paper” to facilitate mathematics instruction at all levels (Brock 1975; Brock and Price 1980). In the United States, Perry’s most prominent disciple was pure mathematician E. H. Moore of the University of Chicago, who championed a “laboratory method” of teaching mathematics at both the secondary and college levels. This involved strong emphasis on developing intuition in the student through physical models, weighing and measuring, and drawing on squared paper. Moore hoped to help students aiming to be scientists and engineers while also supporting future teachers of mathematics and research mathematicians. Moore’s long-term influence on the American curriculum was slight, one major exception being an increased use of graphs in algebra instruction, something rarely found in the nineteenth century (Roberts 2001).

Between 1910 and 1930, graph paper became established as a regular feature of much mathematics instruction in the United States. The picture-free algebra books of the nineteenth century were replaced by a new generation of textbooks containing pictures of graphs on grids of perpendicular lines and featuring problems inviting students to graph equations and other mathematical objects. Graphing has continued to have a regular place in mathematics instruction to the present time, although the role of graph paper itself is often replaced by the graphing calculator (Kidwell et al. 2008).

3.3 Tools for Physical Manipulation

In this category we consider any device that primarily serves its purpose by being physically handled and examined in three dimensions. In Europe and North America, there has been a discernable increased use of such tools from the beginning of the nineteenth century, although even within this period, the history is often strikingly erratic (Bartolini Bussi et al. 2010). Pestalozzi and Froebel, already mentioned, were especially influential in bringing material objects into the classroom to be seen or touched by the students. These included objects associated with mathematics, such as geometric solids. Froebel, teaching in Swiss and German towns in the 1830s and 1840s, recommended organized play with blocks to introduce the child to geometric shapes and to arithmetic ideas up to simple fractions. Froebel's ideas spread across Europe and to the United States in the late nineteenth century (Allen 1988).

An example of a tool of this kind that has come and gone with little trace is the cube root block. It is based on a method of extracting cube roots based on the binomial expansion of $(a+b)^3$, which can be illustrated with a cube of side $a+b$. (There is a corresponding method for extracting square roots, more well known, which can be illustrated with a diagram of a square of side $a+b$.) Illustrations of this cube can be found in English arithmetic texts from the seventeenth century (Recorde 1632), but it was not until the middle of the nineteenth century that it became an actual classroom device. With the aim of helping students understand the aforementioned cube root algorithm, scientific instrument companies in the United States began to produce and market wooden cube root blocks that could be dissected into constituent parts. These blocks, for advanced arithmetic students, were often advertized in conjunction with other classroom objects, such as cones for displaying conic sections and Froebel's blocks for kindergarten children. Diagrams based on the blocks were a staple of school arithmetic textbooks for many years, but the topic had detractors. The cube root block algorithm never gained any favor with engineers and other users of mathematics for practical purposes, since the efficiency of the algorithm is low compared to other methods, such as logarithms or Newton's method. Moreover, how often did mathematical practitioners even need to compute cube roots? By the 1890s many mathematics educators in the United States were campaigning against cube root extraction, but it persisted in the curriculum well into the twentieth century. Cube root blocks were still being sold in the 1920s. No studies of the effectiveness of the cube root block as a teaching technique are known. The block must be judged a demonstration tool of unclear benefit to support an algorithm of dubious value, but nevertheless for a time, it was a standard topic in the schools (Kidwell et al. 2008).

The cube root block can also be considered as part of a wider movement in Europe and North America to use geometric models in classrooms. This is built on a tradition originating in France in the early nineteenth century, especially with mathematician Gaspard Monge. Models made of plaster, string, wood, metal, and paper were developed in France and Germany. These went beyond the simple solids of Pestalozzi and Froebel to include hyperboloids and other more advanced structures, all the way to objects at the forefront of mathematical research, such as Riemann surfaces. Some of the string models could be manipulated to change shape. In Germany in the 1880s, at the instigation of the prominent mathematician Felix Klein, models, mainly of plaster, were manufactured and sold worldwide. Colleges and universities in the United States were among the buyers, but there is little evidence to support extensive classroom use of these models; more likely they were treated more as museum pieces. There were also isolated enthusiasts in the United States in the twentieth century, who built models or helped students build models. Their influence is very hard to gauge (Kidwell et al. 2008). In France, in this period, mathematicians Émile Borel and Jules Tannery encouraged construction of models by both teachers and students as part of their "laboratoire d'enseignement mathématique" (Châtelet 1909).

Meanwhile in Italy, Maria Montessori inherited Froebel's emphasis on teaching young children through tactile experience, buttressing her theories by appealing to more recent developments in psychology and anthropology. She advised that beginning students be given the opportunity to continually handle objects of various shapes, such as cylinders of varying heights and diameters. Colored cubes and rods were a central feature of her approach to arithmetic. Montessori schools were opened in Italy and Switzerland. After an initially rapid growth of interest in her work in the United States in the 1910s, her influence declined, in part due to criticism from American educational theorists such as William Heard Kilpatrick of Columbia University (Whitescarver and Cossentino 2008).

The United States experienced a Montessori revival beginning in the 1950s, and this closely coincided with, and perhaps helped to support, renewed interest in both the United States and Europe in using physical objects specifically in teaching mathematics. Other sources of support were found in the work of educational psychologists whose influence extended well beyond mathematics, such as the Swiss, Jean Piaget, and the Russian, L. S. Vygotsky. Among those in the 1960s who helped popularize what came to be called "manipulatives" in mathematics instruction were the Belgian educator Emile-Georges Cuisenaire, the Egyptian-born British educator Caleb Gattegno, and the Hungarian-born educator Zoltan Dienes, who worked in Britain, Australia, Canada, and elsewhere (Jeronnez 1976; Seymour and Davidson 2003). This period also saw a ferment of curriculum reform, notably in France, the U.S.S.R., and the United States, but present to varying degrees in many other nations. Some would see manipulatives such as Cuisenaire rods as incongruous with the emphasis on axiomatics and abstraction characteristic of many of the "New Math" programs (to use the designation popular in the United States), although Dienes, for one, saw no contradiction (Dienes 1971). It does appear that the popularity of certain manipulatives to some extent rose and fell with public perceptions of the New Math as a whole. Nevertheless, while New Math programs often experienced severe backlash, the use of manipulatives never went into total eclipse.

The presence of manipulatives in classrooms in the last 50 years is reflected in the large quantity of empirical research on the topic from the 1960s to the present. This research paints a mixed picture of the effectiveness of these tools. While some studies have detected very positive effects, others find these effects negated by poor teaching techniques (Karshmer and Farsi 2008; Moyer 2001; Sowell 1989). Some research even suggests that manipulatives can harm students by burdening them with the problem of "dual representation."

> That is, a given manipulative needs to be represented not only as an object in its own right but also as a symbol of a mathematical concept or procedure (McNeil and Jarvin 2007).

The computer, especially as connected to the Internet, makes readily available to students and teachers all of the objects mentioned above, and many more, in virtual form. Is this comprehensive technology platform something fundamentally new for mathematics education, or does it merely provide the means for delivering the services of the older technologies more quickly and efficiently? It remains to be seen.

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Chapter 29
History of Mathematics Teacher Education

Harm Jan Smid

1 Introduction

The learning of mathematics almost always involves the teaching of mathematics. When young scribes or priests in Egypt and Mesopotamia more than 4,000 years ago learned mathematics, they needed elder and more experienced scribes and priests to teach it to them. In this sense, not only the learning but also the teaching of mathematics has a long history.

It would be misleading, however, to say that the profession of being a mathematics teacher is as old as that. Most people, who taught mathematics at this time, would have seen themselves in the first place as scribes or priests, or, in other cultures like the Greek, even as scholars, not as teachers. Much later, from the time of the Renaissance, a number of people in Europe earned a living in teaching mathematics, mainly arithmetic for future tradesmen. Sometimes they were specialized in mathematics, like the so-called *maestri d'abbaco* or *Rechenmeister* ("masters of arithmetic"), but most of them, whether they were appointed as schoolmasters or were independent entrepreneurs, taught other subjects besides mathematics. They considered themselves schoolmasters in general, not mathematics teachers specifically.

Perhaps in Egypt or Mesopotamia, the state attempted to ensure the mathematical training of its scribes, but in modern history, such measures were not taken until the beginning of the nineteenth century. In fact, although secondary schools became established in Western Europe since the sixteenth century, following the model of the Gymnasium in Strasbourg established in the 1530s, neither the Protestant nor the Catholic educational systems organized any teacher training. While the various religious orders in the Catholic regions prepared novices within their own houses, teachers at Protestant Gymnasia were graduates of theological faculties that had not obtained a parish. The first initiatives by governments for teacher education were undertaken for elementary schools, establishing normal schools or seminaries from about the 1760s, in various European states.

The French Revolution ended the dominance of the churches in educational affairs. Later, states accepted responsibility for what we call now secondary education (particularly in France and Prussia). Mathematics became an important part of that education. As a consequence, the education of future mathematics teachers became an object of concern for those states, which started to issue decrees and laws to ensure the quality of the mathematics teachers at the new, state-regulated schools for secondary education. Being a mathematics teacher became a profession one could enter only by following a prescribed programme and by acquiring the necessary diplomas. By doing so, one could become a

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professional mathematics teacher, having a well-defined and respected place in society. The education of such a professional mathematics teacher is the focus of this chapter.

Certain important representative stages and distinctive features of the development of mathematics teacher education systems in different countries will be examined below. The discussion will be necessarily limited – only a few European countries and the United States will be mentioned – while the prolonged and occasionally dramatic history of mathematics teacher education in other countries remains beyond the bounds of this chapter. However, an investigation and comparison of what took place in the countries of this comparatively small (geographically speaking) region that will be examined also appear necessary and substantive.

2 Beginnings

Throughout the nineteenth century, the idea that the state had a responsibility for secondary education, and that mathematics should play an important role in that education, was gradually accepted in most European countries; as a consequence, these countries also began to organize some form of mathematics teacher education. Of course, there were differences in the way mathematics teacher education was organized in all these countries, but there were also similarities. Secondary education was considered necessary only for the future *elite,* not for the majority. Therefore, there was a clear distinction between schoolmasters, engaged in primary education and teachers engaged in secondary education. Mathematics teachers were educated at universities, schoolmasters at special training institutes. For teachers, the emphasis was on the subject they taught; for schoolmasters, pedagogy and didactics were at least equally important.

The motivating factor behind the teaching of mathematics for the future elite was the supposed *formative value* of mathematics: learning mathematics was intended to contribute to the development of one's mental capacities. Practical value or applicability of mathematics was typically much less important. Correspondingly, in the education of mathematics teachers, the emphasis was on pure mathematics.

There were clear differences between the European countries in the way professional teacher education was handled. In Prussia, already early in the nineteenth century, there was some attention to teacher education, and it gradually gained more importance. This was followed by other countries, as those in other German states, Austria or the Scandinavian countries. In France, on the other hand, teacher education was seen as superfluous; the best and only needed preparation for becoming a mathematics teacher was becoming a good mathematician. Most countries followed the Prussian or French model, but not all. In England, for instance, the basic condition – that the state had a responsibility for secondary education – was hardly met during the nineteenth century. That lack of responsibility had its consequences not only for mathematics teacher education but also for the position of mathematics teachers themselves. The analogous abstinence of governmental policies on mathematics teacher education is evidenced in the United States.

2.1 Prussia

After the Napoleonic wars, the government of Prussia concluded that a thorough reform of the state was necessary. Prussia was a state without colonies; natural resources there were not abundant. The country decided to focus the reform on education (the German phrase for it, *Bildung*, is almost untranslatable). The central means of achieving the goals of reform was the creation of the *gymnasium*, the type of school that needed to replace the old Latin school. Mathematics had to become one of the main subjects in the gymnasium, and mathematics was required to be taught for 5 or 6 h a week in all nine grades of the gymnasia. Prussia had developed an impressive curriculum for mathematics.

From the beginning of this reform, it was clear to the government that the availability of a corps of competent mathematics teachers was necessary for the successful implementation of the plans. The reform, therefore, included the upgrading of the faculties of philosophy of the universities so that the teachers for the new gymnasia could be educated there. To become a teacher, one had to study at least six (in later years 8) semesters in this new faculty, and then pass an examination before a state commission. A demo lesson was part of this examination. The fact that only a doctor's degree from a Prussian university gave exemption from this state exam (but not from the demo lesson) illustrates the high level of expertise that was required.

Teachers were not expected to become pure specialists in their field of teaching only. On the contrary, future teachers were examined in all the main subjects taught at the gymnasia, complemented by a special examination on the specific subject they wanted to teach. The ideal teacher was a scholar who was not only able to pursue research in his own field of interest but could also understand the principles of the other school disciplines. The underlying motivation for the idea was the concept of the unity of science, which had to lead the education of the gymnasium students. It is not surprising, then, that in the following decades the dominating position of mathematics in the gymnasia came under attack, and these attacks were successful to some extent. Mathematics, however, remained an important subject in the Prussian gymnasia. In teacher education, the equivalence of all teaching subjects slowly diminished. To become a mathematics teacher in a *Realschule*[1] (the type of school without Latin and Greek but with greater emphasis on mathematics and science), the state examination was required for the prospective gymnasium teacher, but Latin and Greek were not part of it.

The level of knowledge required for teaching in the middle and lower grades only also was discussed. In the long run, that level was lowered, and a sort of two-level system for mathematics teachers was introduced. The mathematics teachers who prepared for the upper grades, however, were allowed to teach the lower and middle grades, and standards for their preparation were kept rigid or even raised. A mathematics teacher increasingly became a specialist in his own field.

The scientific education of Prussian mathematics teachers was at a high level, but their professional education was not of the same standard. Pedagogy was a part of the state examination, but the education primarily provided theoretical knowledge which was not of much use in the classroom. The demo lessons were judged by the same university professors who strongly objected to the increased emphasis on professional education. Nevertheless, this does not mean that the demo lesson was of no importance at all.

In 1814, A.L. Crelle had to give his demo lesson (Schubring 1983, p. 292). The given task was a lesson of 1 h on the general properties of the conic sections, and part of the task required Crelle to select those properties and theorems that could be treated within 1 h and fashion a coherent lecture from that material. The examiner, J.G. Tralles, was aware that Crelle was a very competent mathematician and thought he did him a favour with this task, but Crelle's lesson was a disaster. Crelle thought he had to treat all the properties of the conic sections within 1 h, and, as Tralles ironically added, he also apparently thought he had to demonstrate his own knowledge of mathematics instead of displaying his ability to lecture. Tralles ended his report with the following devastating remarks:

> Mr Crelle started right from the beginning with such general enquiries and circumstantial calculations, which to be sure proved his own insight in the totality of his task, that he was induced soon to end with free and oral lecturing and to take refuge to his manuscript, and to write down the formulas from it, which he could not gradually

[1] In the course of the nineteenth century, besides the *gymnasia,* also *Realschule* and *Realgymnasia* (a mixed form) came into existence. Teacher education for these schools was the same as for the gymnasia, based on the ideal of pure mathematics. At the end of the nineteenth century, this emphasis on pure mathematics would cause problems at the technical schools and was one of the reasons for Felix Kleins proposals for reform.

> develop on the blackboard to the listeners, because of the many different mathematical signs and the necessary lemma's. Such a lecture cannot but miss the necessary liveliness and the attention of the pupils cannot be hold in the course of such enquiries, which the teacher himself on some moments did not grasp in all details.

Crelle did not enter teaching but became a successful engineer, advisor to the government, and founding editor of his famous journal – which, no doubt, was a wise decision.

In the following example, the demo lesson was a success, but the report provides details about the mathematics that the future teacher had to master. It concerns Franz Ley, 23 years old, whose examination took place in 1820 (Schubring 1983, p. 297).

> The examination in mathematics and physics with this candidate, who had studied these subjects with special predilection and with very good results, could be extended to all parts. So topics from geometrical analysis, analytical geometry, algebra, newer analysis, differential and integral calculus, mathematical geography, mechanics and the theory of electricity were discussed. On all fields he showed thorough knowledge and he expressed himself on all topics with preciseness, certainty and cleverness, also with sharp description of the concepts. In his paper, written in the German language, he responded with insight, in a clear presentation, with pure and simple language, the question that was put before him: How can we treat the theory of the conic sections in the classroom?

Franz Ley was also examined in Latin and Greek and had to write a paper in Latin on a didactical subject. Maybe his outstanding performance was an exemption, but it is remarkable that he not only showed thorough knowledge and gave a good demo lesson but that the writing of two papers on didactical topics was also part of the examination.[2]

In 1826, a probation year was introduced. From then on, a prospective teacher was required to attend lessons given by experienced teachers and give lessons himself under the supervision of such teachers. Only when the director of the school at the end of this year gave a positive judgement could one be appointed as a qualified teacher. No doubt this was an important step forward, although in practice it was not easy to provide competent supervisors for prospective teachers.

In most but not all universities, pedagogical seminars offered opportunities for future teachers to receive professional education; sometimes their courses lasted several years. Topics such as organizing content matter in satisfying scientific and didactical ways and different teaching methods (e.g. lecturing, heuristic teaching, Socratic dialogue) were featured in these seminars. However, they were never obligatory; the candidates could decide whether or not they wanted to attend them. Seminars' position remained marginal for several more decades. Only from 1890, the scientific state exam was followed by a second, professional state exam.

One can conclude that the education of prospective mathematics teachers in Prussia was at a very high scientific level, while professional elements were certainly not unimportant. The result was the creation of a corps of competent mathematics teachers, who held high prestige in the society – just as the government had in mind when it initiated the reform in 1810 (Noble 1927).

2.2 France

Prussia was not the only or even the first state to modernize its educational system. During the years of the Revolution, the French government replaced the old *Collèges* with the *Ecoles Centrales*, schools where mathematics played an important role. These schools, however, did not give a systematic and coherent education; rather, they offered lectures of all kind, some theoretical, some vocational, often attended by adults. Many of the mathematics teachers came from the former *Collèges,* and some had

[2] Another example is the paper that Karl Weierstrass wrote in 1840 for his final exam at the seminar in Münster: *About the Socratic dialogue and its applicability in school teaching.*

attended lectures at the *Ecole Normale.*[3] Napoleon replaced these schools with the *Lycées*, and after the Restoration, those were replaced with the *Collèges Royaux.* In these schools, classical languages, and literature dominated, while mathematics was only taught in the final grades (Schubring 1985b).

The most striking difference with Prussia, however, was its totally different approach to the position of the teachers. By contrast, the revolutionary French government did not want to create a new elite, based on formal education and diplomas. To become a mathematics teacher, one had to participate in the *concours*, a competitive examination theoretically open to everybody for all available slots. It was up to the candidates how to prepare themselves for the *concours.* Throughout the nineteenth century, a growing number of mathematics teachers became *licenciés* or *agrégés*; that is, they received university degrees that provided a sound or even excellent mathematical background. While in Prussia, the mathematics teachers were free to choose the textbooks they wanted – and many wrote textbooks of their own – the French government strictly prescribed which textbooks to use.[4] Textbooks were also selected by means of a *concours*, and there was a strong belief that by using the right textbook, the best possible results can be achieved regardless of the teacher's qualities or preferences. As a result, mathematics teaching was uniform throughout the country. Mathematics teachers, even if they were excellent mathematicians, had little professional freedom (Schubring 1985a, pp. 362–375).

2.3 Italy

After its unification in 1859/1861, Italy, like nearly all other European countries, trained its prospective mathematics teachers at the universities. They were entitled to teach when they had obtained their degrees. Italian mathematicians were interested in teaching and teacher training, and some of them wrote excellent textbooks on school mathematics, but the universities offered no professional training. For this purpose, the Italian government, in 1875, instituted the *Scuole di Magistero*, to be attended by future teachers upon their graduation from university. But these institutes could never fulfil their task properly. In many universities, such an institute was never established, their teaching staffs were incompetent for this task, and their supporting structures inadequate. Despite a series of legislative measures and heated discussions between Italian mathematicians, professional teacher education remained problematic (Giacardi 2010, 2011).

2.4 Russia

The organization of a systematic mathematics education in Russia began with Peter the Great. Consequently, the issue of mathematics teacher preparation arose at the same time. Actually, it would be accurate to say that during the entire eighteenth century and even beyond, the duties of a mathematics teacher could be performed by any graduate of a military school, which usually provided a decent education in mathematics, but no pedagogical education of any kind. Special teacher education was established only at the end of the eighteenth century, when Fyodor Yankovich de Mirijevo, a Serb who came to Russia from Austria, first put together a uniform system of education including a teachers' seminary (which later became the foundation for the Pedagogical Institute and still later for St. Petersburg University). It was expected that this seminary would prepare teachers first and foremost for public schools, but subsequently its graduates taught even at the Pedagogical Institute itself

[3] The so-called École Normale de l'an III (1795) existed only for a few months.

[4] In the *Ecoles Centrales,* the teachers were still free to choose the textbook, but the great majority used the same book, that of E. Bezout. In the *Lycées* and *Collèges,* the textbooks of Lacroix were dominating.

(Karp 2012). While certain works in methods of teaching already existed by this time (including those written by Yankovich de Mirijevo himself), the students were offered no courses specifically devoted to methodology (although it may be supposed that methodological issues were discussed in one way or another in the context of courses in general mathematics). The seminary's graduates went on to teach the most varied subjects, including not only mathematical ones but also, for example, Russian literature, civil architecture, Latin, and double-entry bookkeeping (Karp 2012).

Over time and into the nineteenth century, the preparation of mathematics teachers for gymnasia began taking place at universities, where students received a systematic education in mathematics, but studied practically no pedagogy or school-level mathematics (nonetheless, during certain periods, students were required to teach demo lessons). Teachers for less prestigious educational institutions were prepared in various kinds of seminaries, where students did study school-level mathematics and consequently the methodology of its teaching, but devoted far less attention to more advanced mathematics (Stefanova 2010).

2.5 The Netherlands

The Netherlands form an interesting example of a smaller country that underwent strong influences from its greater neighbours. In the old Republic, the central government was (much like England, see below) almost not involved in secondary education. During the eighteenth century, French ideas on education were widely discussed, and the country was even a part of France during 1810–1813. On the other hand, the new king, Willem I, had experienced strong Prussian influences; both his mother and his wife were Prussian princesses, and during the French occupation of the Netherlands, he had spent most of his time on his Prussian estates.

In 1816, the Latin schools experienced a modest reform; one novelty was that "the principles of mathematics" had to be taught at these schools. In 1826, a more explicit programme was prescribed. The intention of the government was that mathematics (like all other subjects) had to be taught by the regular teacher of the class, whose main task was the teaching of Latin and Greek. Although during their studies at the university, these teachers had to pass a fairly basic examination in mathematics, their mathematical competence was often poor. For that reason, many teachers did not want to teach mathematics. To replace them, primary schoolteachers were hired to do the job. Eventually, this led to a still-existing alternative road to become a mathematics teacher: a primary schoolteacher can, by passing special exams, become a mathematics teacher in a secondary school.

In 1828, the government decided that modifications to the professional education of prospective mathematics teachers were necessary. It rejected explicitly the idea of a centralized institute for teacher training and sent the inspector of the Latin schools to Prussia to study teacher education there. He wrote an enthusiastic report, especially advising a probation year. But the government did not want to go that far. It ordered that special courses for prospective teachers should be given at the universities, including pedagogy, methodology, and practical exercises on Latin schools (Smid 1997).

In Leiden, Jacob de Gelder, a mathematics professor, textbook author, and strong propagator of mathematics instruction, was in charge of teacher education and wrote a lengthy report to the Board of the university in 1838 about his lectures for future mathematics teachers. His report makes clear that the prescribed lectures on general pedagogy were not given at all, and De Gelder admitted that it had been impossible to organize practice at the Latin schools.[5] Therefore, De Gelder's lectures, 4 h

[5] At the University of Groningen, the students instead visited primary schools in the countryside where they attended lessons, but sometimes they caused scandals by ridiculing the schoolmaster. Letter from the board of governors to the minister of the interior, 1838, State Archives, The Hague.

each week during the year, were the only preparation for future mathematics teachers. De Gelder started his lectures with an overview of school mathematics and continued as follows:

> Most lessons are devoted to the so called analytic way of teaching (…) For that purpose I start each lecture with assigning some theorem or problem, with an indication of the way it could be proved or solved, with the invitation to try to find the solution by themselves, for which purpose I give a certain amount of time. When this time has passed, I treat the question myself, not by giving an explanation in the form of a monologue, but by asking questions and having discussion. (De Gelder 1838)

In this way, De Gelder wrote, he could train his students to organize their lessons at school in the same way. He realized it was far from perfect, but as he said, in the given circumstances it was the best possible option. De Gelder, who also wrote a *methodus docendi* about didactical and methodological questions that he used in his lectures, was serious about his task, but when he retired in 1840, his successor did not continue his lectures for mathematics teacher education.

Other professors tried to make the best of teacher education courses, but on the whole, they were unsuccessful. Universities were unwilling to cooperate, and when it became clear that these courses were not obligatory, the students' interest waned. After a decade or so, these courses disappeared without a trace. The laws on secondary and higher education (1863 and 1876) secured the scientific education of prospective teachers, but only in 1952 was some professional education for prospective mathematics teachers organized again at the universities (Bunt 1962). Surprisingly, it consisted of the same three elements as in 1828, but now this education was obligatory (Smid 2008).

2.6 England

In England, the government did not want to play a role of any importance in (secondary) education. In fact, it saw no reason for reform: England had been very successful in the eighteenth century and was indisputably the strongest power in Europe. Moreover, different religious groups rejected the idea that the state should have any involvement in their educational activities. As a consequence, the government did nothing for the education of mathematics teachers. Some smaller teacher training institutes, founded by churches or private organizations, educated primary schoolteachers. In 1878 and 1885, teacher training for secondary education for girls was established, but attempts to establish training departments for men in Oxford and Cambridge initially failed. Later, some teacher training colleges were established, and in the first decade of the twentieth century, the consecutive model of first obtaining a degree, followed by a year of professional training, became dominant.

At grammar or public schools, however, professional or even scientific competence was considered less important than having the right *character,* and no doubt, the ruling classes, themselves educated in these schools, shared this view. Having a degree from Oxford or Cambridge was considered of value per se, while special knowledge of the subject matter was often considered something accidental. In 1902, the central government assumed some responsibility for some form of secondary education. But unlike France or Prussia, for instance, the schools for the *elite*, the public schools remained untouched by these reforms. Although courses and opportunities for professional education on teacher training colleges and universities became more widespread during the twentieth century, around 1960 still about half of the mathematics graduates who entered teaching had not received any professional training (Howson 2010).

2.7 The United States

The history of mathematics teacher education in the United States forms a striking contrast with continental Europe. First of all, teacher education was not a matter for the central government, as education in general was not. During a large part of the nineteenth century, teaching licences were given by local or county officials; only slowly did the different states centralize this competence. But teacher education was to remain a responsibility for the individual states, not for the federal government. Formal mathematics teacher education started around 1890 in Michigan State Normal School ("a college-level institution that could prepare teachers for all public schools of the state, including the high schools" Donoghue 2003, p. 161), soon followed by institutions in Chicago and New York. In the first decade of the twentieth century, at least 25 institutions offered courses for future mathematics teachers. At first, these courses were often offered at special teacher colleges (normal schools), later on increasingly by university departments. Often primary schoolteachers were also educated at another department of the same college or university. The sharp distinction between schoolmasters and teachers that existed in Europe did not exist the same way in the United States.

Importantly, in the nineteenth and first part of the twentieth centuries, the American *High School* served a different goal than the German *Gymnasium* or the French *Lycée.* While the latter were designed to educate (and select) the future *elite*, the American High School, the "people's college", was intended (at least in theory) to furnish education for a broader audience. The level of mathematics taught in American high school at that time cannot be compared with the level of instruction at elitist European secondary schools. As a consequence, the competence in mathematics, required of an average American high school mathematics teacher at the end of the nineteenth and the early decades of the twentieth centuries, did not reach the standard of his or her European colleagues. Professional education, however, was much more prominent in the American system. In the first half of the twentieth century, special courses on subject-specific pedagogy such as mathematics became standard in most of the teacher training institutes. Graduate programmes and PhDs in mathematics education had already been developed at the beginning of the twentieth century, when such programmes in Europe were still non-existent (Donoghue 2003).

3 Mathematics Teacher Education Between the World Wars: A Few Examples

In 1931–1933, the *Commission Internationale de l'Enseignement Mathématique (CIEM,* later called ICMI) published a large-scale study on the education of mathematics teachers in its member countries (Fehr 1931).[6] Gino Loria wrote the *Rapport Général* (Loria 1932), followed by reports on teacher education in Austria (Wirkinger 1933), Belgium (Mineur and Sterkens 1933), Czechoslovakia (Vetter 1933), Denmark (Mollerup 1933), England (Neville 1933), France (Iliovici and Desforge 1933), Germany (Lietzmann 1933), Hungary (Beke et al. 1933), Italy (Perna 1933), Norway (Heegard 1933), Poland (CIEM 1933), Switzerland (Amberg 1933), the United States (Hedrick 1933), and Yugoslavia (Karamata 1933). This study demonstrated that in most countries, the state had accepted responsibility for mathematics teacher education. Typically, to become a mathematics teacher, one had to study mathematics at the university level. Great differences remained, however, regarding the pedagogical education of teachers (see, e.g. Turner 1939).

3.1 Competition for the Best: The French Way

With a total of 32 pages, the CIEM report on France is the longest of all national reports for a few reasons. The organization of French teacher education was complicated and not easy to understand for

[6] Raymond Clare Archibald published already in 1918 an extensive study on the education of teachers of mathematics in the countries represented in the CIEM, based on material then already available. His study covers partly other countries than the 1932 report and gives also detailed information on secondary and higher education.

an outsider. It was the result of a development of more than a century of both secondary and higher education, in which the role of the *concours* and the relations between the *grandes écoles*[7] and the universities was essential.

The *École Normale Superieure (ENS)*, established by Napoleon in 1808 as a training institute for prospective teachers, evolved in the nineteenth century into an institute for higher studies and research. The entrance as well as the final graduation took the form of a *concours* [competition], with a fixed number of available places where only the best were admitted. Part of the final *concours* was an oral presentation, which (at least in theory) was used to judge if the candidate was eligible to become a teacher. In the course of the nineteenth century, several attempts were made to introduce some professional education like lectures on pedagogy or teaching practices. Given the emphasis on scientific education, the resistance of the teaching staff, and the enormous pressure on the students created by the *concours*, these attempts remained futile.

In 1902, a great reform of French secondary education was conducted. Secondary education was organized in two consecutive cycles of 4 and 3 years, correspondingly; the dominance of classical languages was diminished and the position of mathematics improved (Gispert and Schubring 2011). In the process of the reform, a mass survey was conducted: in their responses to its questions, most of the teachers still rejected the idea of a better professional preparation, such as compulsory teaching practices. The majority of them, and no doubt also the great majority of the staff of the *ENS,* still held the opinion that a thorough scientific preparation was the best and only education needed to become a teacher. But politicians were strongly in favour of a better professional preparation, and from 1905 on, some form of teaching practices became compulsory for passing the *agrégation,* the highest degree for teaching. Professional education, however, remained in a marginal position. The French CIEM report quotes a lengthy statement of one of the former directors of the *ENS*, in effect saying that the best professional education is a thorough scientific one, like that given on the *ENS*, implying that a good scientist could become a good teacher even without any professional preparation. No doubt this feeling was much shared.

The CIEM report describes the situation around 1930. To become a mathematics teacher, one has to obtain the grade of *licencié.* This university degree was awarded when a candidate obtained three "certificates of higher studies" – in mathematics, those were analysis, rational mechanics, and general physics. This award was not connected with a *concours.* With this degree (but without any professional education), a candidate was eligible for teaching at the *collèges* (municipal secondary schools). To be eligible for teaching at the *lycées* (secondary schools run by the state, with the same programme as the *collèges* but more prestigious), a candidate has to be an experienced teacher of a *collège* or be admitted to the oral part of the *aggregation*.[8] That would suggest that one had passed the written part of that examination. The *agrégées* – those who additionally passed the oral part, that is, the *concours* which only a previously established number of candidates could pass – were appointed to one of the *lycées.* The *agrégées* received at least some professional education.

Strictly speaking, it was not necessary to enter the *ENS* to pass these requirements. A candidate could also study at other departments of science. But the additional intensive high-level lessons at the *ENS* gave greater preparation for passing these requirements, and of course a *normalien* later had better chances to be appointed at one of the prestigious *lycées*, especially in one with special preparatory mathematics classes for the *grandes écoles.*

There was another interesting way to become a mathematics teacher: to be educated as a prospective female teacher at the *École Normale Superieure des Jeunes Filles.* As in many countries, secondary education in France before World War II was strictly separated for boys and girls. In the 1880s,

[7] Institutes for higher education such as the *École Normale Superieure* or the *École Polytechnique.* These institutions were more prestigious than the universities.

[8] These rules contained some more refinements, which are omitted here.

the French government began to set up a system of secondary schools for girls. These schools were not on the same level as the schools for boys, but mathematics nonetheless played an important role in them, and slowly the level of these schools rose. A special school for prospective female teachers, the *École Normale Superieure des Jeunes Filles*, was established. From the beginning, professional education, including teaching practices, played a somewhat important role at this school, and in 1920 even a special *collège* for girls was attached to the school to provide better opportunities for such teaching practices. In many ways, the school resembled its masculine equivalent, including the use of a *concours* for admission and, after a while, also for the final exams. At the outset, its level was distinctively lower than that of the *ENS*, but slowly but steadily the level increased. During the first half of the twentieth century (especially when in 1924 the *concours* for the *agrégration* was opened for both sexes), the level of teaching at the *École Normale Superieure des Jeunes Filles* became increasingly comparable with the level of the *ENS.* While in many countries female mathematics teachers were still highly exceptional, in France talented girls could acquire a thorough scientific education combined with at least some form of professional training and become mathematics teachers.

3.2 Scandinavian Examples

R.C. Archibald (1918) wrote that few countries could compare with France in the high standards maintained in mathematics teacher education. He meant the high scientific level, but France was also exceptional in another aspect: the low level or even absence of professional education. In some other countries like Spain, Portugal, and the Netherlands, there was no or very little professional training. In other countries, like England and Switzerland, there were some opportunities for professional training, but only on a voluntary basis; in these countries, the central government had not even laid out rules for the scientific education of prospective teachers.

But in the majority of the European countries, both scientific and professional education was compulsory. In those countries (including the other German states), the Prussian example was in some way or another followed. A prospective mathematics teacher had to obtain a university degree – usually in a combination of mathematics with such subjects as physics or chemistry – and then to follow a trajectory of professional education. Such a trajectory, often facilitated by a pedagogical institute, endured no less than ½ year all the way up to 2 years and consisted of a mix of theoretical lessons and practical exercises.

This model was, for instance, introduced in the Scandinavian countries of Denmark, Norway, and Sweden.[9] Although the programmes in these three countries have distinct differences, they also share the common feature that the first part contains, apart from mathematics, several other subjects (e.g. physics, chemistry, or astronomy) as well as lectures on logic, philosophy, psychology, or pedagogy. The second part of the programme was a specialization in some areas of mathematics. After obtaining the degree, which usually took 4–5 years, the candidates entered a pedagogical seminar for a half-year (Norway) or a year (Denmark and Sweden). In this seminar, they had to pass examinations on such subjects as history of education, pedagogy, and didactics of specific school subjects. The most important part no doubt was formed by the practical experience of the secondary school to which they were assigned. The overall pattern was that they started with observing lessons of experienced teachers, then gave lessons (or parts of lessons) supervised by these teachers, and concluded by giving lessons

[9] For Norway, see *L'enseignement mathématique*, 32, 360–364, 1933; for Denmark, *L'enseignement mathématique*, 32, 202–204, 1933; for Sweden, R.C. Archibald, *The training of teachers of mathematics for the secondary schools of the countries represented in the international commission on the teaching of mathematics.* Washington 1918.

on their own. These lessons were discussed with other students and their supervisors. In Denmark and Sweden, the prospective teacher was required upon graduating from the pedagogical seminar, to serve as an assistant teacher for 1 or 2 years before he could become a teacher with full responsibilities.

3.3 Deviating Philosophies: The Cases of Italy and the USSR

It is interesting to consider how political theories and practices influence mathematics education as a whole and mathematics teacher education in particular. In Italy in 1923, the new Fascist government carried out a broad reform of the secondary school system. The greatest emphasis was placed on the classical-humanistic branch, designed for the ruling classes, while the technical-scientific branch was of a markedly lower status. The previously existing *Scuole di Magistero* had been suddenly abolished yet by 1920. In the new prevailing philosophy, "knowing" was believed to be equivalent to "knowing how to teach". Knowledge of mathematics was no longer considered important for the future elite. This attitude towards mathematics would endure until around 1980, when new initiatives in mathematics teacher education emerged (Furinghetti 1998; Bernardi and Arzarello 1996).

Altogether different was the fate of mathematics education and mathematics teacher education in the USSR. A brief post-Revolutionary period was followed by the onset of Stalin's industrialization (Karp 2010). Mathematics became an indispensable subject for future engineers, and the task of preparing them was assigned to the new schools. Many millions of schoolchildren began to be taught serious and substantial mathematics; consequently, the need for mathematics teachers exploded. Mathematics teachers began to be prepared not only in universities but also, most importantly, in special pedagogical institutes (in addition, special 2-year "Teachers' Institutes" were designed to accelerate the education of the teachers needed for schools). Pedagogical institutes provided a fundamental mathematics education, which was similar to what students obtained at universities, but at the same time offered extensive courses in pedagogical, psychological, and methodological disciplines, as well as prolonged teaching practice in schools (Stefanova 2010). Certain types of lessons and approaches to lesson construction, which presupposed a high level of mathematical knowledge and methodological competence, were popularized and even made mandatory in the country (Karp and Zvavich 2011). Future teachers were systematically taught to give such lessons.

3.4 Mathematics Teacher Education in the United States

As during other periods, the United States during the period under examination presents an example of a system in which the central government played a very limited role in the establishment of standards and procedures in teacher education, while the influence of various public organizations and committees proved to be significant, even though it did not rule out local developments that opposed their decisions. The period following the First World War was, on the one hand, a time of consolidation for the mathematics education community, when already existing organizations of mathematics educators developed, new ones appeared, new texts were published, and important texts were reissued many times (such as the work of Young 1925, which was published outside the United States as well). On the other hand, it was a time when progressive educators enjoyed great influences and frequently spoke out in favour of abandoning the traditional approach to teaching mathematics. A report prepared under the supervision of William Heard Kilpatrick decisively denied that mathematics played any role in the development of thinking in general (and rejected the idea that reasoning ability might be transferred from the study of one subject to another); he also questioned the usefulness of mass education in many areas of mathematics. Consequently, it was proposed that mathematics education be structured not in accordance with the inner logic of the subject but, on the one hand, as part of a general "laboratory" investigation of surrounding reality and, on the other hand, more "psychologically" and in keeping with the actual ideas and interests of the students (Donoghue 2003). These views, although immediately met with objections, undoubtedly exerted an influence on the practice and rhetoric of mathematics teacher education both in the United States and beyond its borders.

4 Mathematics Education in a Changing World

After World War II, the system of teacher education underwent radical change. Already during the nineteenth and first half of the twentieth century, the modernizing society gradually asked for more education for the lower classes. For a long time in Europe, however, vocational education or extended primary education was seen as sufficient. Sometimes mathematics was taught in these schools and the teachers there had to acquire some knowledge of mathematics but were not considered mathematics teachers and essentially belonged to a lower social class and regarded as schoolmasters. Only after World War II was this pattern breached under the pressure of changing economic and social circumstances. The idea gained ground that the art of teaching had some essential aspects in common for all age groups, and teaching mathematics implied being a mathematics teacher, regardless of the level of mathematics involved. Simultaneously, the content of mathematics that a future mathematics teacher had to study became more application-oriented, and professional education gained more weight.

4.1 Developments in Europe: The Breakdown of the Old System

In the nineteenth century, primary schools were mostly attended by the lower social strata, while the children of the elite received private education or went to special schools before moving into secondary schools. In the twentieth century, most children went to primary schools, but for the lower classes, this often was all the education they received. After the Second World War, some form of secondary education became open for all children. Yet, there was a great variety among secondary schools: preparing for university, general secondary education, extended primary education, and all kinds of vocational schools. The choice for a certain type of school (if any) was often influenced by the social class to which the parents of the child belonged and determined the child's future career to a large extent.

This system became the object of severe criticism. It was supposed to contribute to social stratification and have an anti-democratic character. There were also economic arguments; the talents of children of lower classes, badly needed in modern society, were neglected and lost by the forced choice of a school type at the age of 12. There was a strong movement to postpone this choice: to view education more as a continuous process, with less diversification, and to have better and longer secondary, and even higher, education for all children after primary education.

As a consequence, a mathematics teacher was no longer seen as a mere specialist in his field, working in an isolated branch of the educational system and having little to do with his colleagues, let alone with teachers or schoolmasters from other school types. A mathematics teacher became viewed as a member of a team, first of all within his own school but also more broadly within the whole of the educational system. As a mathematics teacher, this teacher had his or her own specialization, of course, but first of all this teacher was a teacher, an educator, who needed professional knowledge and skills – as was basically the same for all involved in the teaching profession. Professional education, and especially general didactics, became more prominent in mathematics teacher education.

In modern society, mathematics education is now seen as indispensable for every citizen, varying (of course!) in scope, content, and depth for different groups of students. The ideas of the *formative* *value* of mathematics lost its domination in mathematics education. Thus, mathematics content was not only modernized under the influence of the *new mathematics* movement but also new branches of useful mathematics such as stochastic and statistics, operational research, and discrete mathematics were added to school curricula, while the importance of traditional subjects as geometry diminished. As a result, the mathematical content of mathematics teacher education also changed.

Finally, in many countries, the shortage of mathematics teachers became acute. Not only did many more students attend secondary and even higher education but in the modern society, becoming a mathematics teacher was no longer the only option one had after studying mathematics. A growing proportion of mathematics graduates entered the industry, and it became more difficult to interest young mathematicians in the career of teaching (Kellaway 1949).

All these elements contributed to the constant pressure to modernize mathematics teacher education. Post-war developments in France provide a good example of what, with all of its possible variations, took place in many European countries (e.g. King 1971; Shuard 1976; Aydin 1990).

4.2 New French Ways

The French system of secondary education traditionally consisted of three branches: *collèges* and *lycées* for the elite, extended primary education for the middle classes, and vocational training for the working class. The teaching staff on the extended primary schools and vocational schools were schoolmasters, trained for the primary schools on the *Écoles Normales*, while the *licenciés* and *agrégées* taught on the *collèges* and *lycées.*

In the post-war period, this system was slowly transformed through a series of reforms. For the age group of 12–16 years old, new 4-year *collèges secondaire* were created, with some differentiation within the schools. These schools were followed by a system of various *lycées*: classical, scientific, vocational, and so on. This reform, as well as the increased number of pupils going to secondary education, made the traditional build-up of the teaching staff untenable. Already in 1950, a new type of teaching licence for secondary education was created, somewhere between the level of schoolmaster, whose scientific education was often too low to teach at the secondary level, and the scientifically educated teachers of the old *collèges*, whose professional education was usually poor. These new teachers received their professional training at the *Centres Pédagogiques Régionaux,* regional pedagogical centres.

In 1957, new institutions for teacher training were created. They could be attended by those who had passed the propedeutical examination; students were offered financial support during their further studies; if they were willing to sign up to be a teacher for at least 10 years. Nevertheless, many vacancies available for mathematics teachers could not be filled by adequately qualified teachers. Retired teachers and/or engineers wanting to be retrained as mathematics teachers were hired on annual contracts to fill the gaps.

During the 1980s, the Regional Pedagogical Centres became the object of serious debate. This debate was ongoing for some time about the *Ecoles Normales* and the diverse institutions for the education of teachers at vocational schools. In 1989, a final and decisive step was taken: all teacher training institutes were abolished and replaced by the *Institutes Universitaires de Formation des Maîtres (IUFM)*, university institutes for the education of all teachers from primary schools through *lycées*.

To be admitted to the first year of studies at *IUFM*, one must possess a bachelor's degree, usually after 3 years of study at a university. After the first year, there is a state-organized *concours,* a selective examination by which only the best (a predetermined number) are admitted. One can also participate in the *concours* without attending the first year of the *IUFM,* and those who passed it are admitted to the second and last year of the *IUFM* which is wholly devoted to professional education. Students who have obtained the *agrégation* have also to right to enter the second year of the *IUFM.* If one has completed this second year successfully, the government then is obligated to offer this person a job (Condette 2007).

4.3 Variations on a Theme

The French system described above is an example of the “consecutive” model: starting with some years of scientific preparation, it is followed by 1 or 2 years of professional education. Usually both the scientific and professional education take place in a university; however, sometimes the professional part is offered not at the university itself but at an institute that is somehow linked to a university. Even years earlier, this model was in use for mathematics teachers in secondary schools. In the meantime, for primary school teachers, probably the concurrent model had been much more widespread: scientific or subject-oriented education was combined with professional education at special teacher training colleges.

The shift towards the consecutive model did not occur only in France. In England, for instance, special teacher training colleges existed not only for primary school teachers but were also offered as courses for mathematics teaching certificates (Williams 1963). However, in the 1970s and 1980s, the number of these colleges of education declined rapidly. Nowadays, one has to acquire a “qualified teacher status”, obtained by completing an initial teacher training course. To be admitted to such a course, one needs a university degree. To become a mathematics teacher, the academic degree should be in mathematics (Howson 1982, pp. 199–202).

In Germany, the consecutive model is now the standard for all types of teacher education (Keitel-Kreidt 1997). While in France the tradition of the *concours* adds a specific aspect to the model, in Germany the tradition of the two state examinations – the first concerning academic subject matter, the second professional education – is still an essential element in the system. Preparation for the first state examination takes place at the universities and for the second examination in didactical seminars.

The consecutive model presently seems dominant, but the concurrent model has not entirely disappeared. In the Netherlands, for instance, mathematics teachers for the upper grades of secondary schools follow the academic consecutive path, but primary school teachers and mathematics teachers for the lower school grades take the concurrent path by studying at special colleges of education which do not belong to the university system. Once one is a primary schoolteacher or a mathematics teacher for the lower grades, one can acquire a teaching certificate for the highest grades by succeeding on special examinations. The courses preparing for these examinations are given in the evenings by the colleges of education, not by the universities.

5 Final Remarks

Mathematics teacher education remains at the centre of debate. It is noteworthy that not infrequently in contemporary discussions, models for it are sought in the experience of Asian countries (Ma 1999), which were beyond the scope of the discussion in this relatively brief chapter. This chapter is also limited because it does not address professional development or in-service training (see, e.g. Freudenthal 1969; Revuz 1969; Genzwein 1970 and Karp 2004), which constitute an important part of teacher preparation and continuing development.

Nonetheless, what has been discussed in this chapter is thought-provoking. Over the past 200 years, a conclusion was reached that now seems obvious: mathematics teachers must be specially prepared and this preparation must include a variety of components – mathematical, pedagogical, and methodological. The proportions of these components in any programme of study are evidently determined by multiple factors, including the prevailing conception of the nature and aims of education, and therefore by political factors as well. One of the most important among these political factors is the government’s role in education, including the degree of centralization in education – and in this respect, history knows different models.

Today's attempts to use foreign experience in mathematics teacher education are merely the continuation of a long history – the experience of Prussia, France, or Austria was decisive for many institutions far beyond the borders of these countries, and many similar examples of such influences could be adduced.

The history of mathematics teacher education is far from complete. Today, one may observe the appearance of nontraditional, so-called alternative methods of obtaining a teaching licence, in which applicants complete their education during the course of their own teaching, which bears a certain resemblance to the manner in which people became teachers hundreds of years ago. On the other hand, increasing attention is being paid to pedagogical content knowledge (Shulman 1986) as indispensable for teachers, that is, to a knowledge that varies for teachers of different subjects, yet is not identical to the knowledge of subjects themselves. Increasing attention is being paid to assessing teachers and, consequently, to assessing programmes for mathematics teacher education.

If history teaches us anything, it is at the very least that views about the aims, nature, and assessment of mathematics teacher education change over time. In this way, history again helps us to see and examine the complexity of the concepts and processes discussed above.

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