6

Inverse Scattering Problems for Orthotropic Media

In this chapter we extend the results of Chap. 4 to the case of the inverse scattering problem for an inhomogeneous orthotropic medium. The inverse problem we shall consider in this chapter is to determine the *support* of the orthotropic inhomogeneity given the far-field pattern of the scattered field for many incident directions.

The investigation of the inverse problem is based on the analysis of a nonstandard boundary value problem called the *interior transmission problem.* This problem plays the same role for the inhomogeneous medium problem as the interior impedance problem plays in the solution of the inverse problem for an imperfect conductor, studied in Chap. 4. Having discussed the well-posedness of the interior transmission problem and the existence and countability of transmission eigenvalues, we proceed with a uniqueness result for the inverse problem. We will present here a proof due to Hähner [81] that is based on the use of a regularity result for the solution to the interior transmission problem. We then derive the linear sampling method for finding an approximation to the support of the inhomogeneity. Although the analysis of the justification of the linear sampling method refers to the scattering problem for an orthotropic medium, the implementation of the method does not rely on any a priori knowledge of the physical properties of the scattering object. In particular, we show that the far-field equation we used in Chap. 4 to determine the shape of an imperfect conductor can also be used in the present case where the corresponding far-field pattern is used for the kernel of this equation. Finally, since transmission eigenvalues carry qualitative information about the material properties of the inhomogeneous scattering object (cf. Sect. 6.2), we conclude this chapter by showing how transmission eigenvalues can be determined from the (noisy) far-field data.

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6.1 Formulation of Inverse Problem

Let D be the support and A and n the constitutive parameters of a bounded, orthotropic, inhomogeneous medium in $\mathbb{R}^2$, where D, A, and n satisfy the assumptions given in Sect. 5.1. The scattering of a time-harmonic incident plane wave $u^i := e^{ikx\cdot d}$ by the inhomogeneity D is described by the transmission problem (5.13)–(5.17) with $f := e^{ikx\cdot d}$ and $h := \partial e^{ikx\cdot d}/\partial\nu$, which we recall here for the reader's convenience:

$$\nabla\cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D, \tag{6.1}$$

$$\Delta u^s + k^2\, u^s = 0 \qquad \text{in} \quad \mathbb{R}^2\setminus\bar{D}, \tag{6.2}$$

$$v - u^s = e^{ikx\cdot d} \qquad \text{on} \quad \partial D, \tag{6.3}$$

$$\frac{\partial v}{\partial\nu_A} - \frac{\partial u^s}{\partial\nu} = \frac{\partial e^{ikx\cdot d}}{\partial\nu} \qquad \text{on} \quad \partial D, \tag{6.4}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{6.5}$$

where $k > 0$ is the (fixed) wave number, $d := (\cos\phi,\, \sin\phi)$ is the incident direction, $x = (x_1,\, x_2) \in \mathbb{R}^2$, and $r = |x|$. In particular, the interior field $v(\cdot) := v(\cdot,\phi)$ and scattered field $u^s(\cdot) := u^s(\cdot,\phi)$ depend on the incident angle ϕ. The radiating scattered field u^s again has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}}u_\infty(\theta,\phi) + O(r^{-3/2}), \qquad r\to\infty,$$

where the function $u_\infty(\cdot,\phi)$ defined on $[0,\, 2\pi]$ is the *far-field pattern* corresponding to the scattering problem (6.1)–(6.5) and the unit vector $\hat{x} := (\cos\theta,\, \sin\theta)$ is the observation direction. In the same way as in Theorem 4.2 it can be shown that the far-field pattern $u_\infty(\theta,\phi)$ corresponding to (6.1)–(6.5) satisfies the reciprocity relation $u_\infty(\theta,\phi) = u_\infty(\phi+\pi,\theta+\pi)$ and is given by

$$u_\infty(\theta,\phi) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}\int_{\partial B}\left(u^s(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial\nu} - e^{-ik\hat{x}\cdot y}\frac{\partial u^s(y)}{\partial\nu}\right)ds(y), \tag{6.6}$$

where ∂B is the boundary of a bounded domain containing D (it can also be ∂D).

The following result can be obtained as a consequence of Rellich's lemma (Theorem 4.1).

Theorem 6.1. *Suppose that the far-field pattern u_∞ corresponding to (6.1)–(6.5) satisfies $u_\infty = 0$ for a fixed angle ϕ and all θ in $[0,\, 2\pi]$. Then $u^s = 0$ in $\mathbb{R}^2\setminus\bar{D}$.*

Note that by the analyticity of the far-field pattern Theorem 6.1 holds if $u_\infty = 0$ only for a subinterval of $[0,\, 2\pi]$.

The *inverse scattering problem* we are concerned with is to determine D from a knowledge of the far-field pattern $u_\infty(\theta, \phi)$ for all incident angles $\phi \in [0,\, 2\pi]$ and all observation angles $\theta \in [0,\, 2\pi]$. We remark that for an orthotropic medium standard examples [77, 136] show that A and n are not in fact uniquely determined from the far-field pattern $u_\infty(\theta, \phi)$ for all $\phi \in [0,\, 2\pi]$ and $\theta \in [0,\, 2\pi]$, but rather what is possible to determine is the support of the inhomogeneity D.

We now consider the *far-field operator* $F : L^2[0,\, 2\pi] \to L^2[0,\, 2\pi]$ corresponding to (6.1)–(6.5) defined by

$$(Fg)(\theta) := \int_0^{2\pi} u_\infty(\theta, \phi) g(\phi) d\phi. \tag{6.7}$$

As the reader has already seen (Chap. 4), the far-field operator will play a central role in the solution of the inverse problem. The first problem to resolve is that of injectivity and the denseness of the range of the far-field operator. We recall that a Herglotz function with kernel $g \in L^2[0,\, 2\pi]$ is given by

$$v_g(x) := \int_0^{2\pi} e^{ikx\cdot d} g(\phi)\, d\phi, \tag{6.8}$$

where $d = (\cos\phi, \sin\phi)$. Note that by superposition, Fg is the far-field pattern of the solution to (6.1)–(6.5), with $e^{ikx\cdot d}$ replaced by v_g. For future reference we note that

$$\tilde{v}_g(x) := \int_0^{2\pi} e^{-ikx\cdot d} g(\phi)\, d\phi \tag{6.9}$$

is also a Herglotz wave function with kernel $g(\phi - \pi)$.

Theorem 6.2. *The far-field operator F corresponding to the scattering problem (6.1)–(6.5) is injective with dense range if and only if there does not exist a Herglotz wave function v_g such that the pair v, v_g is a solution to*

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \quad \text{and} \quad \Delta v_g + k^2\, v_g = 0 \qquad \text{in} \quad D, \tag{6.10}$$

$$v = v_g \quad \text{and} \quad \frac{\partial v}{\partial \nu_A} = \frac{\partial v_g}{\partial \nu} \qquad \text{on} \quad \partial D. \tag{6.11}$$

Proof. In exactly the same way as in Theorem 4.3, one can show that the far-field operator F is injective if and only if its adjoint operator F^* is injective. Since $\mathrm{N}(F^*)^\perp = \overline{F(L^2[0,\, 2\pi])}$, to prove the theorem we must only show that F is injective. But $Fg = 0$ with $g \neq 0$ is equivalent to the existence of a nonzero Herglotz wave function v_g with kernel g for which the far-field pattern u_∞ corresponding to (6.1)–(6.5) with $e^{ikx\cdot d}$ replaced by v_g vanishes. By Rellich's lemma we have that $u^s = 0$ in $\mathbb{R}^2 \setminus \bar{D}$, and hence the transmission conditions imply that

$$v = v_g \qquad \text{and} \qquad \frac{\partial v}{\partial \nu_A} = \frac{\partial v_g}{\partial \nu} \qquad \text{on} \quad \partial D.$$

Since v_g is a solution of the Helmholtz equation, we have that v and v_g satisfy (6.10) as well. This proves the theorem. □

Motivated by Theorem 6.2, we now define the *interior transmission problem* associated with the transmission problem (5.13)–(5.17).

Interior transmission problem. Given $f \in H^{\frac{1}{2}}(\partial D)$ *and* $h \in H^{-\frac{1}{2}}(\partial D)$, *find two functions* $v \in H^1(D)$ *and* $w \in H^1(D)$ *satisfying*

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad in \quad D, \tag{6.12}$$

$$\Delta w + k^2\, w = 0 \qquad in \quad D, \tag{6.13}$$

$$v - w = f \qquad on \quad \partial D, \tag{6.14}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} = h \qquad on \quad \partial D. \tag{6.15}$$

The boundary value problem (6.12)–(6.13) with $f = 0$ and $h = 0$ is called the *homogeneous interior transmission problem* or the *transmission eigenvalue problem.*

Definition 6.3. Values of k for which the homogeneous interior transmission problem has a nontrivial solution are called transmission eigenvalues.

In particular, Theorem 6.2 states that if k is not a transmission eigenvalue, then the range of the far-field operator is dense.

6.2 Interior Transmission Problem

As seen earlier, the interior transmission problem appears naturally in scattering problems for an inhomogeneous medium. Of particular concern to us in this section are the countability and the existence of real transmission eigenvalues, and the approach to studying the interior transmission problem depends on whether or not $n \equiv 1$. In our analysis of the interior transmission problem we exclude the case of $A = I$ and refer the reader to Chap. 8 in [54], which deals with (6.12)–(6.15) when $A = I$.

We begin by establishing the uniqueness of a solution to the interior transmission problem for complex-valued refractive indexes.

Theorem 6.4. *If either* $Im(n) > 0$ *or* $Im\left(\bar{\xi} \cdot A\,\xi\right) < 0$ *at a point* $x_0 \in D$, *then the interior transmission problem (6.12)–(6.15) has at most one solution.*

Proof. Let v and w be a solution of the homogeneous interior transmission problem (i.e., $f = h = 0$). Applying the divergence theorem to $\overline{v}$ and $A\nabla v$ (Corollary 5.8), using the boundary condition and applying Green's first identity to $\overline{w}$ and w (Remark 6.29) we obtain

$$\int_D \nabla\overline{v}\cdot A\nabla v\,dy - \int_D k^2 n|v|^2\,dy = \int_{\partial D}\overline{v}\cdot\frac{\partial v}{\partial \nu_A}\,dy = \int_D |\nabla w|^2\,dy - \int_D k^2|w|^2\,dy.$$

Hence

$$\operatorname{Im}\left(\int_D \nabla\overline{v}\cdot A\nabla v\,dy\right) = 0 \quad \text{and} \quad \operatorname{Im}\left(\int_D n|v|^2\,dy\right) = 0. \tag{6.16}$$

If $\operatorname{Im}(n) > 0$ at a point $x_0 \in D$, and hence by continuity in a small disk $\Omega_\epsilon(x_0)$, then the second equality of (6.16) and the unique continuation principle (Theorem 17.2.6 in [89]) imply that $v \equiv 0$ in D. In the case where $\operatorname{Im}\left(\bar{\xi}\cdot A\,\xi\right) < 0$ at a point $x_0 \in D$ for all $\xi \in \mathbb{C}^2$, and hence by continuity in a small ball $\Omega_\epsilon(x_0)$, from the first equality of (6.16) we obtain that $\nabla v \equiv 0$ in $\Omega_\epsilon(x_0)$ and from (6.12) $v \equiv 0$ in $\Omega_\epsilon(x_0)$, whence again from the unique continuation principle $v \equiv 0$ in D. From the boundary conditions (6.13) and (6.14), and the integral representation formula, w also vanishes in D. □

We now proceed to the solvability of the interior transmission problem following the approach in [20] and [34]. In the following analysis we assume without loss of generality that D is simply connected. We first study an intermediate problem called the *modified interior transmission problem*, which turns out to be a compact perturbation of our original transmission problem.

The modified interior transmission problem is as follows: given $f \in H^{\frac{1}{2}}(\partial D)$, $h \in H^{-\frac{1}{2}}(\partial D)$, a real-valued function $m \in C(\bar{D})$, and two functions $\rho_1 \in L^2(D)$ and $\rho_2 \in L^2(D)$, find $v \in H^1(D)$ and $w \in H^1(D)$ satisfying

$$\nabla\cdot A\nabla v - m\,v = \rho_1 \qquad \text{in} \quad D, \tag{6.17}$$

$$\Delta w - w = \rho_2 \qquad \text{in} \quad D, \tag{6.18}$$

$$v - w = f \qquad \text{on} \quad \partial D, \tag{6.19}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} = h \qquad \text{on} \quad \partial D. \tag{6.20}$$

We now reformulate (6.17)–(6.20) as an equivalent variational problem of the form (5.18). To this end, we define the Hilbert space

$$W(D) := \left\{\mathbf{w} \in \left(L^2(D)\right)^2 : \nabla\cdot\mathbf{w} \in L^2(D) \quad \text{and} \quad \nabla\times\mathbf{w} = 0\right\}$$

equipped with the inner product

$$(\mathbf{w}_1,\,\mathbf{w}_2)_W = (\mathbf{w}_1,\,\mathbf{w}_2)_{L^2(D)} + (\nabla\cdot\mathbf{w}_1,\,\nabla\cdot\mathbf{w}_2)_{L^2(D)}$$

and the norm

$$\|\mathbf{w}\|_W^2 = \|\mathbf{w}\|_{L^2(D)}^2 + \|\nabla\cdot\mathbf{w}\|_{L^2(D)}^2.$$

We denote by $\langle\cdot,\,\cdot\rangle$ the duality pairing between $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$. The duality pairing

$$\langle \varphi, \boldsymbol{\psi} \cdot \nu \rangle = \int_D \varphi \, \nabla \cdot \boldsymbol{\psi} \, dx + \int_D \nabla \varphi \cdot \boldsymbol{\psi} \, dx \tag{6.21}$$

for $(\varphi, \boldsymbol{\psi}) \in H^1(D) \times W(D)$ will be of particular interest in the sequel.

We next introduce the sesquilinear form $\mathcal{A}$ defined on $\{H^1(D) \times W(D)\}^2$ by

$$\mathcal{A}(U,V) = \int_D A\nabla v \cdot \nabla \bar{\varphi} \, dx + \int_D m \, v \, \bar{\varphi} \, dx + \int_D \nabla \cdot \mathbf{w} \, \nabla \cdot \bar{\boldsymbol{\psi}} \, dx + \int_D \mathbf{w} \cdot \bar{\boldsymbol{\psi}} \, dx \\ - \left\langle v, \, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle - \langle \bar{\varphi}, \, \mathbf{w} \cdot \nu \rangle \, , \tag{6.22}$$

where $U := (v, \mathbf{w})$ and $V := (\varphi, \boldsymbol{\psi})$ are in $H^1(D) \times W(D)$. We denote by $L : H^1(D) \times W(D) \to \mathbb{C}$ the bounded conjugate linear functional given by

$$L(V) = \int_D (\rho_1 \, \bar{\varphi} + \rho_2 \, \nabla \cdot \bar{\boldsymbol{\psi}}) \, dx + \langle \bar{\varphi}, \, h \rangle - \left\langle f, \, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle . \tag{6.23}$$

Then the variational formulation of the problem (6.17)–(6.20) is to find $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$ such that

$$\mathcal{A}(U,V) = L(V) \qquad \text{for all} \quad V \in H^1(D) \times W(D). \tag{6.24}$$

The following theorem proves the equivalence between problems (6.17)–(6.20) and (6.24).

Theorem 6.5. *The problem (6.17)–(6.20) has a unique solution* $(v, w) \in H^1(D) \times H^1(D)$ *if and only if the problem (6.24) has a unique solution* $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$*. Moreover if* (v, w) *is the unique solution to (6.17)–(6.20), then* $U = (v, \nabla w)$ *is the unique solution to (6.24). Conversely, if* $U = (v, \mathbf{w})$ *is the unique solution to (6.24), then the unique solution* (v, w) *to (6.17)–(6.20) is such that* $\mathbf{w} = \nabla w$*.*

Proof. We first prove the equivalence between the existence of a solution (v, w) to (6.17)–(6.20) and the existence of a solution $U = (v, \mathbf{w})$ to (6.24).

1. Assume that (v, w) is a solution to (6.17)–(6.20), and set $\mathbf{w} = \nabla w$. From (6.18) we see that, since $\nabla \mathbf{w} = w + \rho_2 \in L^2(D)$, then $\mathbf{w} \in W(D)$. Taking the L^2 scalar product of (6.18) with $\nabla \cdot \boldsymbol{\psi}$ for some $\boldsymbol{\psi} \in W(D)$ and using (6.21) we see that

$$\int_D \nabla \cdot \mathbf{w} \, \nabla \cdot \bar{\boldsymbol{\psi}} \, dx + \int_D \mathbf{w} \cdot \bar{\boldsymbol{\psi}} \, dx - \left\langle w, \, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle = \int_D \rho_2 \, \nabla \cdot \bar{\boldsymbol{\psi}} \, dx.$$

Hence, by (6.19),

$$\int_D \nabla \cdot \mathbf{w} \, \nabla \cdot \bar{\boldsymbol{\psi}} \, dx + \int_D \mathbf{w} \cdot \bar{\boldsymbol{\psi}} \, dx - \left\langle v, \, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle \\ = - \left\langle f, \, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle + \int_D \rho_2 \, \nabla \cdot \bar{\boldsymbol{\psi}} \, dx. \tag{6.25}$$

We now take the L^2 scalar product of (6.17) with φ in $H^1(D)$ and integrate by parts. Using the boundary condition (6.20) we see that

$$\int_D A\nabla v \cdot \nabla\bar{\varphi}\, dx + \int_D m\, v\, \bar{\varphi}\, dx - \langle \bar{\varphi},\, \mathbf{w}\cdot\nu\rangle = \langle \bar{\varphi},\, h\rangle + \int_D \rho_1\, \bar{\varphi}\, dx. \quad (6.26)$$

Finally, adding (6.25) and (6.26) we have that $U = (v, \nabla w)$ is a solution to (6.24).

2. Now assume that $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$ is a solution to (6.24). Since $\nabla \times \mathbf{w} = 0$ and D is simply connected, we deduce the existence of a function $w \in H^1(D)$ such that $\mathbf{w} = \nabla w$, where w is determined up to an additive constant. As we shall see later, this constant can be adjusted so that (v, w) is a solution to (6.17)–(6.20). Obviously, if U satisfies (6.24), then $(v, \mathbf{w})$ satisfies (6.25) and (6.26) for all $(\varphi, \boldsymbol{\psi}) \in H^1(D) \times W(D)$. One can easily see from (6.26) that the pair (v, w) satisfies

$$\nabla \cdot A\nabla v - m\, v = \rho_1 \qquad \text{in} \quad D, \quad (6.27)$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} = h \qquad \text{on} \quad \partial D. \quad (6.28)$$

On the other hand, substituting for $\mathbf{w}$ in (6.25) and using the duality identity (6.21) in the second integral we have that

$$\int_D (\Delta w - w)\, \nabla \cdot \bar{\boldsymbol{\psi}}\, dx + \left\langle w - v,\, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle \quad (6.29)$$

$$= - \left\langle f,\, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle + \int_D \rho_2\, \nabla \cdot \bar{\boldsymbol{\psi}}\, dx$$

for all $\boldsymbol{\psi}$ in $W(D)$.

Now consider a function $\phi \in L^2_0(D) = \left\{ \phi \in L^2(D) \;:\; \int_D \phi\, dx = 0 \right\}$, and let $\chi \in H^1(D)$ be a solution to

$$\begin{cases} \Delta\chi = \bar{\phi} \;\; \text{in } D, \\ \dfrac{\partial \chi}{\partial \nu} = 0 \;\; \text{on } \partial D. \end{cases} \quad (6.30)$$

The existence of a solution of the preceding Neumann boundary value problem can be established by the variational methods developed in Chap. 5 (Example 5.15). We leave it to the reader as an exercise [127]. Taking $\boldsymbol{\psi} = \nabla \chi$ in (6.29) [note that from (6.30) $\nabla \cdot \bar{\boldsymbol{\psi}} = \phi$ in D and $\bar{\boldsymbol{\psi}} \cdot \nu = 0$ on ∂D] we have that

$$\int_D (\Delta w \; - \; w - \rho_2)\, \phi\, dx = 0 \qquad \text{for all} \quad \phi \in L^2_0(D),$$

which implies the existence of a constant c_1 such that

$$\Delta w - w - \rho_2 = c_1 \quad \text{in } D. \tag{6.31}$$

We now take $\phi \in L^2_0(\partial D)$ and let $\sigma \in H^1(D)$ be a solution to

$$\begin{cases} \Delta\sigma = 0 & \text{in } D, \\ \dfrac{\partial \sigma}{\partial \nu} = \bar{\phi} & \text{on } \partial D. \end{cases} \tag{6.32}$$

Taking $\boldsymbol{\psi} = \nabla\sigma$ in (6.25) [note that (6.32) implies that $\nabla \cdot \bar{\boldsymbol{\psi}} = 0$ in D and $\bar{\boldsymbol{\psi}} \cdot \nu = \phi$ on ∂D] we have that

$$\int\limits_{\partial D} (w \,-\, v \,+\, f)\, \phi \, ds = 0 \qquad \text{for all} \quad \phi \in L^2_0(\partial D),$$

which implies the existence of a constant c_2 such that

$$w \,-\, v \,+\, f = c_2 \quad \text{on } \partial D. \tag{6.33}$$

Substituting (6.31) and (6.33) into (6.29) and using (6.21) we see that

$$(c_1 - c_2) \int\limits_D \nabla \cdot \bar{\boldsymbol{\psi}} \, dx = 0 \quad \forall\, \boldsymbol{\psi} \in W(D),$$

which implies $c_1 = c_2 = c$ [take, for instance, $\boldsymbol{\psi} = \nabla\varrho$, where $\varrho \in H^1_0(D)$ and $\Delta\varrho = 1$ in D]. Equations (6.27), (6.31), and (6.33) show that $(v, w - c)$ is a solution to (6.17)–(6.20).

We next consider the uniqueness equivalence between (6.17)–(6.20) and (6.24).

3. Assume that (6.17)–(6.20) has at most one solution. Let $U_1 = (v_1, \mathbf{w}_1)$ and $U_2 = (v_2, \mathbf{w}_2)$ be two solutions to (6.24). From step 2 earlier we deduce the existence of w_1 and w_2 in $H^1(D)$ such that $\mathbf{w}_1 = \nabla w_1$ and $\mathbf{w}_2 = \nabla w_2$ and (v_1, w_1) and (v_2, w_2) are solutions to (6.17)–(6.20), whence $(v_1, w_1) = (v_2, w_2)$ and $(v_1, \mathbf{w}_1) = (v_2, \mathbf{w}_2)$.
4. Finally, assume that (6.24) has at most one solution, and consider two solutions (v_1, w_1) and (v_2, w_2) to (6.17)–(6.20). We can deduce from step 1 earlier that $(v_1, \nabla w_1)$ and $(v_2, \nabla w_2)$ are two solutions to (6.24). Hence $v_1 = v_2$ and $w = w_1 - w_2$ is a function in $H^1(D)$ that satisfies

$$\begin{cases} \Delta w - w = 0 & \text{in } D, \\ w = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial D, \end{cases}$$

which implies $w = 0$.

□

We now investigate the modified interior transmission problem in the variational formulation (6.24).

Theorem 6.6. *Assume that there exists a constant $\gamma > 1$ such that, for $x \in D$,*

$$Re\left(\bar{\xi} \cdot A(x)\,\xi\right) \geq \gamma|\xi|^2 \quad for\ all \quad \xi \in \mathbb{C}^2 \quad and \quad m(x) \geq \gamma. \tag{6.34}$$

Then problem (6.24) has a unique solution $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$. This solution satisfies the a priori estimate

$$\begin{aligned} \|v\|_{H^1(D)} + \|\mathbf{w}\|_W \leq 2C\frac{\gamma+1}{\gamma-1}\Big(&\|\rho_1\|_{L^2(D)} + \|\rho_2\|_{L^2(D)} \\ &+ \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\Big), \end{aligned} \tag{6.35}$$

where the constant $C > 0$ is independent of ρ_1, ρ_2, f, h, and γ.

Proof. The trace theorems (Sect. 5.2) and Schwarz's inequality ensure the continuity of the conjugate linear functional L on $H^1(D) \times W(D)$ and the existence of a constant c independent of ρ_1, ρ_2, f, and h such that

$$\|L\| \leq C\left(\|\rho_1\|_{L^2} + \|\rho_2\|_{L^2} + \|f\|_{H^{\frac{1}{2}}} + \|h\|_{H^{-\frac{1}{2}}}\right). \tag{6.36}$$

On the other hand, if $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$, then, by assumption (6.34),

$$|\mathcal{A}(U,U)| \geq \gamma\,\|v\|^2_{H^1} + \|\mathbf{w}\|^2_W - 2\,\mathrm{Re}\left(\langle \bar{v},\, \mathbf{w}\rangle\right). \tag{6.37}$$

According to the duality identity (6.21), one has by Schwarz's inequality that

$$|\,\langle \bar{v},\, \mathbf{w}\rangle\,| \leq \|v\|_{H^1}\,\|\mathbf{w}\|_W\,,$$

and therefore

$$|\mathcal{A}(U,U)| \geq \gamma\,\|v\|^2_{H^1} + \|\mathbf{w}\|^2_W - 2\,\|v\|_{H^1}\,\|\mathbf{w}\|_W\,.$$

Using the identity $\gamma x^2 + y^2 - 2xy = \frac{\gamma+1}{2}\left(x - \frac{2}{\gamma+1}\,y\right)^2 + \frac{\gamma-1}{2}x^2 + \frac{\gamma-1}{\gamma+1}y^2$ we conclude that

$$|\mathcal{A}(U,U)| \geq \frac{\gamma-1}{\gamma+1}\left(\|\mathbf{w}\|^2_W + \|v\|^2_{H^1}\right),$$

whence $\mathcal{A}$ is coercive. The continuity of $\mathcal{A}$ follows easily from Schwarz's inequality, the trace theorem, and Theorem 5.7. Theorem 6.6 is now a direct consequence of the Lax–Milgram lemma applied to (6.24). □

Theorem 6.7. *Assume that there exists a constant* $\gamma > 1$ *such that, for* $x \in D$,

$$Re\left(\bar{\xi}\cdot A(x)\,\xi\right) \geq \gamma|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{C}^2 \quad \text{and} \quad m(x) \geq \gamma. \tag{6.38}$$

Then the modified interior transmission problem (6.17)–(6.20) has a unique solution (v, w) *that satisfies*

$$\begin{aligned}\|v\|_{H^1(D)} + \|w\|_{H^1(D)} \leq C\frac{\gamma+1}{\gamma-1}\Big(&\|\rho_1\|_{L^2(D)} + \|\rho_2\|_{L^2(D)} \\ &+ \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\Big),\end{aligned} \tag{6.39}$$

where the constant $C > 0$ *is independent of* ρ_1, ρ_2, f, h, *and* γ.

Proof. The existence and uniqueness of a solution follow from Theorems 6.5 and 6.6. The a priori estimate (6.39) can be obtained directly from (6.17)–(6.20), but it can also be deduced from (6.35) as follows. Theorem 6.5 tells us that $(v, \nabla w)$ is the unique solution to (6.24). Hence, according to (6.35),

$$\|v\|_{H^1} + \|\nabla w\|_{L^2} \leq C_1\frac{\gamma+1}{\gamma-1}\left(\|\rho_1\|_{L^2} + \|\rho_2\|_{L^2} + \|f\|_{H^{\frac{1}{2}}} + \|h\|_{H^{-\frac{1}{2}}}\right).$$

From Poincaré's inequality in Sect. 5.2 we can write

$$\|w\|_{H^1(D)} \leq C_2\left(\|\nabla w\|_{L^2(D)} + \|w\|_{L^2(\partial D)}\right).$$

Now, using the boundary condition (6.19) and the trace theorem we obtain that

$$\|w\|_{H^1(D)} \leq C_2\left(\|\nabla w\|_{L^2(D)} + \|v\|_{H^1(D)} + \|f\|_{L^2(\partial D)}\right)$$

for some positive constant C_2. The constants C_1 and C_2 can then be adjusted so that (6.39) holds. □

Now we are ready to show the existence of a solution to the interior transmission problem (6.12)–(6.15).

Theorem 6.8. *Assume that either* $Im(n) > 0$ *or* $Im\left(\bar{\xi}\cdot A\,\xi\right) < 0$ *at a point* $x_0 \in D$ *and that there exists a constant* $\gamma > 1$ *such that, for* $x \in D$,

$$Re\left(\bar{\xi}\cdot A(x)\,\xi\right) \geq \gamma|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{C}^2. \tag{6.40}$$

Then (6.12)–(6.15) has a unique solution $(v, w) \in H^1(D) \times H^1(D)$. *This solution satisfies the a priori estimate*

$$\|v\|_{H^1(D)} + \|w\|_{H^1(D)} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\right) \tag{6.41}$$

where the constant $C > 0$ *is independent of* f *and* h.

Proof. Set

$$\mathcal{X}(D) = \left\{(v,w) \in H^1(D) \times H^1(D) \ : \ \nabla \cdot A\nabla v \in L^2(D) \text{ and } \Delta w \in L^2(D)\right\}$$

and consider the operator $\mathcal{G}$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{G}(v,w) = \left(\nabla \cdot A\nabla v - mv, \Delta w - w, (v-w)_{|\partial D}, \left(\frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial \nu}\right)_{|\partial D}\right) \tag{6.42}$$

where $m \in C(\bar{D})$ and $m > 1$. Obviously $\mathcal{G}$ is continuous and from Theorem 6.7 we know that the inverse of $\mathcal{G}$ exists and is continuous. Now consider the operator $\mathcal{T}$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{T}(v,w) = \left((k^2 n + m)v,\ (k^2 + 1)w,\ 0,\ 0\right)$$

From the compact embedding of $H^1(D)$ into $L^2(D)$ (Sect. 5.2), the operator $\mathcal{T}$ is compact. Theorem 6.4 implies that $\mathcal{G}+\mathcal{T}$ is injective, and therefore, from Theorem 5.16 we can deduce the existence and the continuity of $(\mathcal{G}+\mathcal{T})^{-1}$, which means in particular the existence of a unique solution to the interior transmission problem (6.12)–(6.15) that satisfies the a priori estimate (6.43). □

The foregoing analysis of the interior transmission problem requires that the matrix A satisfy

$$\text{Re}\left(\bar{\xi} \cdot A(x)\,\xi\right) \geq \gamma|\xi|^2 \ \text{ for all } \ \xi \in \mathbb{C}^2, \ \ x \in D \ \text{ and some constant } \ \gamma > 1,$$

that is, $\|\text{Re}(A)\| > 1$. The case of $\text{Re}(A)$ positive definite such that $\|\text{Re}(A)\| < 1$ is considered in [34]. By modifying the variational approach of Theorems 6.5 and 6.6 one can prove the following result.

Theorem 6.9. *Assume that either $Im(n) > 0$ or $Im\left(\bar{\xi} \cdot A\,\xi\right) < 0$ at a point $x_0 \in D$ and that there exists a constant $\gamma > 1$ such that, for $x \in D$,*

$$Re\left(\bar{\xi} \cdot (A(x))^{-1}\,\xi\right) \geq \gamma|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{C}^2 \quad \text{and} \quad \gamma^{-1} \leq m < 1.$$

Then (6.12)–(6.15) has a unique solution $(v,w) \in H^1(D) \times H^1(D)$. This solution satisfies the a priori estimate

$$\|v\|_{H^1(D)} + \|w\|_{H^1(D)} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\right), \tag{6.43}$$

where the constant $C > 0$ is independent of f and h.

We remark that a solvability result under less restrictive assumptions on A is obtained later in this chapter (Remark 6.29).

In general we cannot conclude the solvability of the interior transmission problem if A and n do not satisfy the assumptions of the previous theorem. In particular, if $\mathrm{Im}(A) = 0$ and $\mathrm{Im}(n) = 0$ in D, then k may be a transmission eigenvalue (Definition 6.3). Do transmission eigenvalues exist and, if so, do they form a discrete set? The approach in [20] and [34] presented earlier is not suitable to handle these questions, and therefore we devote the next section of the book to address these issues. In particular, we will prove that under appropriate assumptions transmission eigenvalues exist and form a discrete set with infinity as the only accumulation point. As mentioned at the beginning of this section, the analysis of the transmission eigenvalue problem for cases where $n = 1$ and $n \neq 1$ are fundamentally different, and hence we consider each of these cases separately. For the study of the transmission eigenvalue problem if $A = I$ we refer the reader to [32] and to Chap. 10 in [54].

6.3 Transmission Eigenvalue Problem

We recall that the transmission eigenvalue problem is formulated as a problem of finding two nonzero functions $v \in H^1(D)$ and $w \in H^1(D)$ satisfying

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D, \tag{6.44}$$

$$\Delta w + k^2\, w = 0 \qquad \text{in} \quad D, \tag{6.45}$$

$$v = w \qquad \text{on} \quad \partial D, \tag{6.46}$$

$$\frac{\partial v}{\partial \nu_A} = \frac{\partial w}{\partial \nu} \qquad \text{on} \quad \partial D. \tag{6.47}$$

Since transmission eigenvalues do not exist for complex-valued A and n, henceforth we assume that both A and n are real-valued and define

$$\begin{aligned} a_{min} &:= \inf_{x\in D} \inf_{\xi\in\mathbb{R}^2, |\xi|=1} (\xi \cdot A(x)\xi) > 0, \\ a_{max} &:= \sup_{x\in D} \sup_{\xi\in\mathbb{R}^2, |\xi|=1} (\xi \cdot A(x)\xi) < \infty, \\ n_{min} &:= \inf_{x\in D} n(x) > 0 \quad \text{and} \quad n_{max} := \sup_{x\in D} n(x) < \infty. \end{aligned} \tag{6.48}$$

Example 6.10. In what follows, we will need to consider a particular case of the interior transmission problem where D is a ball B_R of radius R centered at the origin, $A := a_0 I$, and $n := n_0$, where a_0 and n_0 are positive constants not both equal to one. In this case the interior transmission eigenvalue problem reads as

$$\Delta v + k^2 \frac{n_0}{a_0} v = 0 \quad \text{in } B_R\,, \tag{6.49}$$

$$\Delta w + k^2 w = 0 \quad \text{in } B_R\,, \tag{6.50}$$

$$v = w \quad \text{on } \partial B_R\,, \tag{6.51}$$

$$a_0 \frac{\partial v}{\partial r} = \frac{\partial w}{\partial r} \quad \text{on } \partial B_R\,, \tag{6.52}$$

where $r = |x|$. To solve (6.49)–(6.52) in $\mathbb{R}^2$, we make the ansatz

$$w(r,\hat{x}) \;=\; a_\ell\, J_\ell(k\, r)\, e^{i\ell\theta}\,, \qquad v(r,\hat{x}) \;=\; b_\ell\, J_\ell\left(k\sqrt{\frac{n_0}{a_0}}\, r\right) e^{i\ell\theta},$$

where J_ℓ are Bessel functions of order ℓ introduced in Chap. 3. Then using separation of variables one sees that the transmission eigenvalues satisfy

$$W(k) \;=\; \det\begin{pmatrix} J_\ell(kR) & J_\ell\left(k\sqrt{\frac{n_0}{a_0}}\, R\right) \\ k\, J_\ell'(kR) & k\sqrt{n_0 a_0}\, J_\ell'\left(k\sqrt{\frac{n_0}{a_0}} R\right) \end{pmatrix} = 0. \tag{6.53}$$

6.3.1 The Case $n = 1$

The case where $n = 1$ corresponds to the electromagnetic scattering problem for an orthotropic medium when the magnetic permeability in the medium is constant and the same as the magnetic permeability in the background. The *transmission eigenvalue problem* reads: find two nonzero functions $v \in H^1(D)$ and $w \in H^1(D)$ satisfying

$$\nabla \cdot A \nabla v + k^2 v = 0 \qquad \text{in} \quad D, \tag{6.54}$$

$$\Delta w + k^2 w = 0 \qquad \text{in} \quad D, \tag{6.55}$$

$$v = w \qquad \text{on} \quad \partial D, \tag{6.56}$$

$$\frac{\partial v}{\partial \nu_A} = \frac{\partial w}{\partial \nu} \qquad \text{on} \quad \partial D. \tag{6.57}$$

Our approach follows the one introduced in [20] and developed further in [31], which generalizes the first proof of the existence of transmission eigenvalues given in [134].

The proof of the existence of transmission eigenvalues is based on the following abstract analysis. Let X be a separable Hilbert space with scalar product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$, and let $\mathbb{A}$ be a bounded, positive definite, and self-adjoint operator on X. Under these assumptions $\mathbb{A}^{\pm 1/2}$ are well defined (cf. [115]). In particular, $\mathbb{A}^{\pm 1/2}$ are also bounded, positive definite, and self-adjoint operators, $\mathbb{A}^{-1/2}\mathbb{A}^{1/2} = I$ and $\mathbb{A}^{1/2}\mathbb{A}^{1/2} = \mathbb{A}$. We shall consider the spectral decomposition of the operator $\mathbb{A}$ with respect to self-adjoint nonnegative compact operators. The next two theorems indicate the main properties of such a decomposition.

Definition 6.11. A bounded linear operator $\mathbb{A}$ on a Hilbert space X is said to be nonnegative if $(\mathbb{A}u, u) \geq 0$ for every $u \in X$. $\mathbb{A}$ is said to be strictly coercive if $(\mathbb{A}u, u) \geq \beta \|u\|^2$ for some positive constant β.

Theorem 6.12. *Let $\mathbb{A}$ be a bounded, self-adjoint, and strictly coercive operator on a Hilbert space, and let $\mathbb{B}$ be a nonnegative, self-adjoint, and compact linear operator with null space $N(\mathbb{B})$. Then there exists an increasing sequence of positive real numbers $(\lambda_j)_{j\geq 1}$ and a sequence $(u_j)_{j\geq 1}$ of elements of X satisfying*

$$\mathbb{A}u_j = \lambda_j \mathbb{B}u_j$$

and

$$(\mathbb{B}u_j, u_\ell) = \delta_{j\ell}$$

such that each $u \in [\mathbb{A}(N(\mathbb{B}))]^\perp$ can be expanded in a series

$$u = \sum_{j=1}^{\infty} \gamma_j u_j.$$

If $N(\mathbb{B})^\perp$ has infinite dimension, then $\lambda_j \to +\infty$ as $j \to \infty$.

Proof. This theorem is a direct consequence of the Hilbert–Schmidt theorem applied to the nonnegative self-adjoint compact operator $\tilde{\mathbb{B}} = \mathbb{A}^{-1/2}\mathbb{B}\mathbb{A}^{-1/2}$. Let $(\mu_j, v_j)_{j\geq 1}$ be the sequence of positive eigenvalues and corresponding eigenfunctions associated with $\tilde{\mathbb{B}}$ such that $\{v_j\}_{j\geq 1}$ forms an orthonormal basis for $N(\tilde{\mathbb{B}})^\perp$. Note that zero is the only possible accumulation point for the sequence μ_j. Straightforward calculations show that $\lambda_j = 1/\mu_j$ and $u_j = \sqrt{\lambda_k}\, \mathbb{A}^{-1/2} v_j$ satisfy $\mathbb{A}u_j = \lambda_j \mathbb{B}u_j$. Obviously, if $w \in \mathbb{A}N(\mathbb{B})$, then $w = \mathbb{A}z$ for some $z \in N(\mathbb{B})$, and hence $(u_j, w) = \lambda_j(\mathbb{A}^{-1}\mathbb{B}u_j, w) = \lambda_j(\mathbb{A}^{-1}\mathbb{B}u_j, \mathbb{A}z) = \lambda_j(\mathbb{B}u_j, z) = 0$, which means that $u_j \in [\mathbb{A}N(\mathbb{B})]^\perp$. Furthermore, any $u \in [\mathbb{A}N(\mathbb{B})]^\perp$ can be written as $u = \sum_j \gamma_j u_j = \sum_j \gamma_j \sqrt{\lambda_j}\mathbb{A}^{-1/2} v_j$ since $\mathbb{A}^{1/2}u \in \left[N(\mathbb{A}^{-1/2}\mathbb{B}\mathbb{A}^{-1/2})\right]^\perp$. This ends the proof of the theorem. □

Theorem 6.13. *Let $\mathbb{A}$, $\mathbb{B}$, and $(\lambda_j)_{j\geq 1}$ be as in Theorem 6.12, and define the Rayleigh quotient as*

$$R(u) = \frac{(\mathbb{A}u, u)}{(\mathbb{B}u, u)}$$

for $u \notin N(\mathbb{B})$, where $(\cdot\,,\,\cdot)$ is the inner product on X. Then the following min-max principle holds:

$$\lambda_j = \min_{W \in \mathcal{U}_j^{\mathbb{A}}} \left(\max_{u \in W \setminus \{0\}} R(u) \right) = \max_{W \in \mathcal{U}_{j-1}^{\mathbb{A}}} \left(\min_{u \in (\mathbb{A}(W + N(\mathbb{B})))^\perp \setminus \{0\}} R(u) \right),$$

where $\mathcal{U}_j^{\mathbb{A}}$ denotes the set of all j-dimensional subspaces of $[\mathbb{A}N(\mathbb{B})]^\perp$.

Proof. The proof follows the classical proof of the Courant min-max principle and is given here for the reader's convenience. It is based on the fact that if $u \in [\mathbb{A}N(B)]^{\perp}$, then from Theorem 6.12 we can write $u = \sum_j \gamma_j u_j$ for some coefficients γ_j, where the u_j are defined in Theorem 6.12 (note that the u_j are orthogonal with respect to the inner product induced by the self-adjoint invertible operator $\mathbb{A}$). Then using the facts that $(\mathbb{B}u_j, u_\ell) = \delta_{j\ell}$ and $\mathbb{A}u_j = \lambda_j \mathbb{B}u_j$ it is easy to see that

$$R(u) = \frac{1}{\sum_j |\gamma_j|^2} \sum_j \lambda_j |\gamma_j|^2.$$

Therefore, if $W_j \in \mathcal{U}_j^{\mathbb{A}}$ denotes the space generated by $\{u_1, \dots, u_j\}$, then we have that

$$\lambda_j = \max_{u \in W_j \setminus \{0\}} R(u) = \min_{u \in [\mathbb{A}(W_{j-1} + N(\mathbb{B}))]^{\perp} \setminus \{0\}} R(u).$$

Next, let W be any element of $\mathcal{U}_j^{\mathbb{A}}$. Since W has dimension j and $W \subset [\mathbb{A}N(\mathbb{B})]^{\perp}$, then $W \cap [\mathbb{A}W_{j-1} + \mathbb{A}N(\mathbb{B})]^{\perp} \neq \{0\}$. Therefore,

$$\begin{aligned} \max_{u \in W \setminus \{0\}} R(u) &\geq \min_{u \in W \cap [\mathbb{A}(W_{j-1} + N(\mathbb{B}))]^{\perp} \setminus \{0\}} R(u) \\ &\geq \min_{u \in [\mathbb{A}(W_{j-1} + N(\mathbb{B}))]^{\perp} \setminus \{0\}} R(u) = \lambda_j, \end{aligned}$$

which proves the first equality of the theorem. Similarly, if W has dimension $j-1$ and $W \subset [\mathbb{A}N(\mathbb{B})]^{\perp}$, then $W_j \cap (\mathbb{A}W)^{\perp} \neq \{0\}$. Therefore,

$$\min_{u \in [\mathbb{A}(W + N(\mathbb{B}))]^{\perp} \setminus \{0\}} R(u) \leq \max_{u \in W_j \cap (\mathbb{A}W)^{\perp} \setminus \{0\}} R(u) \leq \max_{u \in W_j \setminus \{0\}} R(u) = \lambda_j,$$

which proves the second equality of the theorem. □

The following corollary shows that it is possible to remove the dependence on $\mathbb{A}$ in the choice of the subspaces in the min-max principle for the eigenvalues λ_j.

Corollary 6.14. *Let $\mathbb{A}$, $\mathbb{B}$, $(\lambda_j)_{j \geq 1}$, and R be as in Theorem 6.13. Then*

$$\lambda_j = \min_{W \subset \mathcal{U}_j} \left(\max_{u \in W \setminus \{0\}} R(u) \right), \tag{6.58}$$

where $\mathcal{U}_j$ denotes the set of all j-dimensional subspaces W of X such that $W \cap N(\mathbb{B}) = \{0\}$.

Proof. From Theorem 6.13 and the fact that $\mathcal{U}_j^{\mathbb{A}} \subset \mathcal{U}_j$ it suffices to prove that

$$\lambda_j \leq \min_{W \subset \mathcal{U}_j} \left(\max_{u \in W \setminus \{0\}} R(u) \right).$$

Let $W \in \mathcal{U}_j$, and let $v_1, v_2, \ldots, v_k$ be a basis for W. Each vector v_j can be decomposed into a sum $v_j^0 + \tilde{v}_j$, where $\tilde{v}_j \in [\mathbb{A}N(\mathbb{B})]^\perp$ and $v_j^0 \in N(\mathbb{B})$ (which is the orthogonal decomposition with respect to the scalar product induced by $\mathbb{A}$). Since $W \cap N(B) = \{0\}$, the space $\tilde{W}$ generated by $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_j$ has dimension j. Moreover, $\tilde{W} \subset [\mathbb{A}N(\mathbb{B})]^\perp$. Now let $\tilde{u} \in \tilde{W}$. Obviously, $\tilde{u} = u - u^0$ for some $u \in W$ and $u^0 \in N(\mathbb{B})$. Since $\mathbb{B}u^0 = 0$ and $(\mathbb{A}u_0, \tilde{u}) = 0$, we have that

$$R(u) = \frac{(\mathbb{A}\tilde{u}, \tilde{u}) + (\mathbb{A}u^0, u^0)}{(\mathbb{B}\tilde{u}, \tilde{u})} = R(\tilde{u}) + \frac{(\mathbb{A}u^0, u^0)}{(\mathbb{B}\tilde{u}, \tilde{u})}.$$

Consequently, since $\mathbb{A}$ is positive definite and $\mathbb{B}$ is nonnegative, we obtain

$$R(\tilde{u}) \leq R(u) \leq \max_{u \in W \setminus \{0\}} R(u).$$

Finally, taking the maximum with respect to $\tilde{u} \in \tilde{W} \subset [\mathbb{A}N(\mathbb{B})]^\perp$ in the preceding inequality, we obtain from Theorem 6.13 that

$$\lambda_j \leq \max_{u \in W \setminus \{0\}} R(u),$$

which completes the proof after taking the minimum over all $W \subset \mathcal{U}_j$. □

The following theorem provides the theoretical basis of our analysis of the existence of transmission eigenvalues. This theorem is a simple consequence of Theorem 6.13 and Corollary 6.14.

Theorem 6.15. *Let $\tau \longmapsto \mathbb{A}_\tau$ be a continuous mapping from $]0, \infty[$ to the set of bounded, self-adjoint, and strictly coercive operators on the Hilbert space X, and let $\mathbb{B}$ be a self-adjoint and nonnegative, compact, bounded, linear operator on X. We assume that there exist two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$\mathbb{A}_{\tau_0} - \tau_0 \mathbb{B}$ is positive on X,*
2. *$\mathbb{A}_{\tau_1} - \tau_1 \mathbb{B}$ is nonpositive on a ℓ-dimensional subspace W_j of X.*

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \ldots, \ell$ has at least one solution in $[\tau_0, \tau_1]$, where $\lambda_j(\tau)$ is the jth eigenvalue (counting multiplicity) of $\mathbb{A}_\tau$ with respect to $\mathbb{B}$, i.e., $N(\mathbb{A}_\tau - \lambda_j(\tau)\mathbb{B}) \neq \{0\}$.

Proof. First we can deduce from (6.58) that for all $j \geq 1$, $\lambda_j(\tau)$ is a continuous function of τ. Assumption 1 shows that $\lambda_j(\tau_0) > \tau_0$ for all $j \geq 1$. Assumption 2 implies in particular that $W_j \cap N(\mathbb{B}) = \{0\}$. Hence, another application of (6.58) implies that $\lambda_j(\tau_1) \leq \tau_1$ for $1 \leq j \leq \ell$. The desired result is now obtained by applying the intermediate value theorem. □

The main idea in studying the eigenvalue problem (6.54)–(6.57) is to observe that by making an appropriate substitution one can rewrite it as an equivalent eigenvalue problem for a fourth-order differential equation. To this end, let $w \in H^1(D)$ and $v \in H^1(D)$ satisfy (6.54)–(6.57), and make the substitution

$$\mathbf{v} = A\nabla v \in L^2(D)^2, \quad \text{and} \quad \mathbf{w} = \nabla w \in L^2(D)^2.$$

Since from (6.48) A^{-1} exists and is bounded, we have that

$$\nabla v = A^{-1}\mathbf{v}.$$

Taking the gradient of (6.54) and (6.55), we obtain that $\mathbf{v}$ and $\mathbf{w}$ satisfy

$$\nabla(\nabla \cdot \mathbf{v}) + k^2 A^{-1}\mathbf{v} = 0 \tag{6.59}$$

and

$$\nabla(\nabla \cdot \mathbf{w}) + k^2\mathbf{w} = 0, \tag{6.60}$$

respectively, in D. Obviously, (6.57) implies that

$$\nu \cdot \mathbf{v} = \nu \cdot \mathbf{w} \qquad \text{on } \partial D. \tag{6.61}$$

Furthermore, from (6.54) and (6.55) we have that

$$-k^2 v = \nabla \cdot \mathbf{v} \qquad \text{and} \qquad -k^2 w = \nabla \cdot \mathbf{w},$$

and the transmission condition (6.56) yields

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{w} \qquad \text{on } \partial D. \tag{6.62}$$

We now formulate the interior transmission eigenvalue problem in terms of $\mathbf{w}$ and $\mathbf{v}$. In addition to the usual energy spaces

$$H^1(D) := \left\{u \in L^2(D) : \nabla u \in L^2(D)^2\right\},$$
$$H_0^1(D) := \left\{u \in H^1(D) : u = 0 \text{ on } \partial D\right\},$$

we introduce the Sobolev spaces

$$H(\mathrm{div}, D) := \left\{\mathbf{u} \in L^2(D)^2 : \nabla \cdot \mathbf{u} \in L^2(D)\right\},$$
$$H_0(\mathrm{div}, D) := \{\mathbf{u} \in H(\mathrm{div}, D) : \nu \cdot \mathbf{u} = 0 \text{ on } \partial D\}$$

and

$$\mathcal{H}(D) := \left\{\mathbf{u} \in H(\mathrm{div}, D) : \nabla \cdot \mathbf{u} \in H^1(D)\right\},$$
$$\mathcal{H}_0(D) := \left\{\mathbf{u} \in H_0(\mathrm{div}, D) : \nabla \cdot \mathbf{u} \in H_0^1(D)\right\},$$

equipped with the scalar product

$$(\mathbf{u},\mathbf{v})_{\mathcal{H}(D)} := (\mathbf{u},\mathbf{v})_{L^2(D)} + (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_{H^1(D)}\,.$$

Letting $N := A^{-1}$, in terms of new vector-valued functions $\mathbf{w}$ and $\mathbf{v}$, the transmission eigenvalue problem can be written as

$$\nabla(\nabla\cdot\mathbf{v}) + k^2 N\mathbf{v} = 0 \quad \text{in} \quad D, \tag{6.63}$$
$$\nabla(\nabla\cdot\mathbf{w}) + k^2 \mathbf{w} = 0 \quad \text{in} \quad D, \tag{6.64}$$
$$\nu\cdot\mathbf{w} = \nu\cdot\mathbf{v} \quad \text{on} \quad \partial D, \tag{6.65}$$
$$\nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{v} \quad \text{on} \quad \partial D. \tag{6.66}$$

Definition 6.16. Values of $k \in \mathbb{C}$ for which the homogeneous interior transmission problem (6.63)–(6.66) has nonzero solutions $\mathbf{w} \in (L^2(D))^2$, $\mathbf{v} \in (L^2(D))^2$ such that $\mathbf{w}-\mathbf{v} \in \mathcal{H}_0(D)$ are called transmission eigenvalues. If k is a transmission eigenvalue, then we call $\mathbf{u} := \mathbf{v}-\mathbf{w}$ the corresponding eigenfunction where $\mathbf{v}$ and $\mathbf{w}$ are a nonzero solution of (6.63)–(6.66).

It is possible to write (6.63)–(6.66) as an equivalent eigenvalue problem for $\mathbf{w}-\mathbf{v} \in \mathcal{H}_0(D)$ satisfying the fourth-order equation

$$\left(\nabla\nabla\cdot + k^2 N\right)(N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\right) = 0 \quad \text{in} \quad D. \tag{6.67}$$

Equation (6.67) can be written in the variational form

$$\int\limits_D (N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\right)\cdot\left(\nabla\nabla\cdot\overline{\mathbf{v}} + k^2 N\overline{\mathbf{v}}\right)\,dx = 0 \tag{6.68}$$

for all $\mathbf{v} \in \mathcal{H}_0(D)$. The variational equation (6.68) can in turn be written as an operator equation

$$\mathbb{A}_k\mathbf{u} - k^2\mathbb{B}\mathbf{u} = 0 \qquad \text{for} \quad \mathbf{u} \in \mathcal{H}_0(D), \tag{6.69}$$

where the bounded linear operators $\mathbb{A}_k : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ and $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ are defined by means of the Riesz representation theorem

$$(\mathbb{A}_k\mathbf{u},\mathbf{v})_{\mathcal{H}_0(D)} = \mathcal{A}_k u(\mathbf{u},\mathbf{v}) \qquad \text{and} \qquad (\mathbb{B}\mathbf{u},\mathbf{v})_{\mathcal{H}_0(D)} = \mathcal{B}(\mathbf{u},\mathbf{v}), \tag{6.70}$$

with the sesquilinear forms $\mathcal{A}_\tau$ and $\mathcal{B}$ given by

$$\mathcal{A}_k(\mathbf{u},\mathbf{v}) := \left((N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\right), \left(\nabla\nabla\cdot\mathbf{v} + k^2\mathbf{v}\right)\right)_D + k^4\,(\mathbf{u},\mathbf{v})_D$$

and

$$\mathcal{B}(\mathbf{u},\mathbf{v}) := (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_D\,,$$

respectively, where $(\cdot,\,\cdot)_D$ denotes the $L^2(D)$ inner product.

Lemma 6.17. $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ *is a compact operator.*

Proof. Let $\mathbf{u}_n$ be a bounded sequence in $\mathcal{H}_0(D)$. Then there exists a subsequence, denoted again by $\mathbf{u}_n$, that converges weakly to $\mathbf{u}$ in $\mathcal{H}_0(D)$. Since $\nabla \cdot \mathbf{u}_n$ is also bounded in $H^1(D)$, from the Rellich compactness theorem we have that $\nabla \cdot \mathbf{u}_n$ converges strongly to $\nabla \cdot \mathbf{u}_0$ in $L^2(D)$. But

$$\|\mathbb{B}(\mathbf{u}_n - \mathbf{u})\|_{\mathcal{H}_0(D)} \leq \|\nabla \cdot (\mathbf{u}_n - \mathbf{u})\|_{L^2(D)},$$

which proves that $\mathbb{B}\mathbf{u}_n$ converges strongly to $\mathbb{B}\mathbf{u}$. □

In our discussion we must distinguish between the two cases $a_{min} > 1$ and $a_{max} < 1$. To fix our ideas, we consider in detail only the case where $a_{max} < 1$ (similar results can be obtained for $a_{min} > 1$; cf. [21, 31, 33]). If $\lambda_1(x) \leq \lambda_2(x)$ are the eigenvalues of the matrix $A(x)$, then the condition $a_{max} < 1$ means that $\inf_{x\in D} \lambda_1(x) \leq \sup_{x \in D} \lambda_2(x) = a_{max} < 1$. In particular, we have $\sup_D \|A^{-1}\|_2 > 1/a_{max} > 1$, where $\|\cdot\|_2$ is the Euclidean norm of the matrix, and this implies that $\xi \cdot (N(x) - I)^{-1}\xi \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^2$, $x \in D$, and some constant $\alpha > 0$. More specifically,

$$\xi \cdot (A^{-1} - I)^{-1}\,\xi \geq \frac{1}{\|A^{-1}\|_2 - 1}|\xi|^2 \geq \frac{1}{\sup_D \|A^{-1}\|_2 - 1}|\xi|^2, \quad \xi \in \mathbb{R}^2, x \in D;$$

thus,

$$\alpha := \frac{1}{\sup_D \|A^{-1}\|_2 - 1}. \tag{6.71}$$

Theorem 6.18. *Assume that $a_{max} < 1$. The set of real transmission eigenvalues is discrete. If k is a real transmission eigenvalue, then*

$$k^2 \geq \frac{\lambda_0(D)}{\sup_D \|A^{-1}\|_2}, \tag{6.72}$$

where $\lambda_0(D)$ is the first eigenvalue of $-\Delta$ on D.

Proof. To prove the first part of the theorem, we consider the formulation (6.69). Since our assumption $a_{max} < 1$ implies $\xi \cdot (N(x) - I)^{-1}\xi \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^2$, and $x \in D$ with α given by (6.71), we have that

$$\mathcal{A}_k(\mathbf{u}, \mathbf{u}) \geq \alpha\|\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\|^2_{L^2(D)} + k^4\|\mathbf{u}\|^2_{L^2(D)}.$$

Setting $X = \|\nabla\nabla \cdot \mathbf{u}\|_{L^2(D)}$ and $Y = k^2\|\mathbf{u}\|_{L^2(D)}$ we have that

$$\|\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\|^2_{L^2(D)} \geq X^2 - 2XY + Y^2,$$

and therefore

$$\mathcal{A}_k(\mathbf{u}, \mathbf{u}) \geq \alpha X^2 - 2\alpha XY + (\alpha + 1)Y^2. \tag{6.73}$$

From the identity

$$\alpha X^2 - 2\alpha XY + (\alpha+1)Y^2 = \epsilon\left(Y - \frac{\alpha}{\epsilon}X\right)^2 + \left(\alpha - \frac{\alpha^2}{\epsilon}\right)X^2 + (1+\alpha-\epsilon)Y^2 \tag{6.74}$$

for $\alpha < \epsilon < \alpha + 1$, setting $\epsilon = \alpha + 1/2$ we now obtain that

$$\mathcal{B}_k(\mathbf{u}, \mathbf{u}) \geq \frac{\alpha}{1+2\alpha}(X^2 + Y^2). \tag{6.75}$$

From (6.21) we have

$$\|\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\|^2_{L^2(D)} = \|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)} - 2k^2\|\nabla\cdot\mathbf{u}\|^2_{L^2(D)} + k^4\|\mathbf{u}\|^2_{L^2(D)},$$

which implies that

$$2k^2\|\nabla\cdot\mathbf{u}\|^2_{L^2(D)} \leq X^2 + Y^2.$$

Finally, combining the preceding estimates yields the existence of a constant $c_k > 0$ (independent of $\mathbf{u}$ and α) such that

$$\mathcal{A}_k(\mathbf{u}, \mathbf{u}) \geq c_k\frac{\alpha}{1+2\alpha}\|\mathbf{u}\|^2_{\mathcal{H}(D)}. \tag{6.76}$$

Hence the sesquilinear form $\mathcal{A}_k(\cdot\,,\,\cdot)$ is coercive in $\mathcal{H}_0(D)\times\mathcal{H}_0(D)$, and consequently the operator $\mathbb{A}_k : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is a bijection for fixed k. Recall that from Lemma 6.17 the operator $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is compact. Hence, to prove that the set of real transmission eigenvalues is discrete, we apply the analytic Fredholm theorem (Theorem 1.24) to

$$\mathbb{A}_k - k^2\mathbb{B} \qquad \text{or} \qquad \mathbb{I} - k^2\mathbb{A}_k^{-1}\mathbb{B}. \tag{6.77}$$

To this end, we observe that the sesquilinear form $\mathcal{A}_k(\cdot,\cdot)$ is analytic in k, which means that the mapping $k \to \mathbb{A}_k$ is analytic (cf. Theorem 8.22 in [54]). By the Lax–Milgram theorem we can conclude that $\mathbb{A}_k^{-1}$ also exists in a neighborhood of the positive real axis and the mapping $k \to \mathbb{A}_k^{-1}$ is analytic. Consequently, the mapping $k \to k^2\mathbb{A}_k^{-1}\mathbb{B}$ is analytic in a neighborhood of the real axis and for each k the operator $k^2\mathbb{A}_k^{-1}\mathbb{B}$ is compact. Therefore, the analytic Fredholm theorem (Theorem 1.24) implies that the set of transmission eigenvalues is discrete provided that there exists a $k > 0$ that is not a transmission eigenvalue, i.e., $\left[\mathbb{I} - k^2\mathbb{A}_k^{-1}\mathbb{B}\right]^{-1}$ exists. In what follows, we will show that if $k > 0$ is sufficiently small, then k is not a transmission eigenvalue by showing that the operator $\mathbb{A}_k - \mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is an isomorphism for $k > 0$ small enough. To this end, for $\nabla\cdot u \in H^1_0(D)$, using the Poincaré inequality (Sect. 5.2), we have that

$$\|\nabla\cdot\mathbf{u}\|^2_{L^2(D)} \leq \frac{1}{\lambda_0(D)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)}, \tag{6.78}$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D. Hence, from (6.74) and (6.78) for $\alpha < \epsilon < \alpha + 1$ we have that

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{B}(\mathbf{u},\mathbf{u}) \geq \left(\alpha - \frac{\alpha^2}{\epsilon}\right)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)} + (1+\alpha-\epsilon)k^2\|\mathbf{u}\|^2_{L^2(D)} - k^2\frac{1}{\lambda_0(D)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)}.$$

Therefore, if $k^2 < \left(\alpha - \alpha^2/\epsilon\right)\lambda_0(D)$ for every $\alpha < \epsilon < \alpha+1$, then $\mathbb{A}_k - k^2\mathbb{B}$ is invertible. In particular, taking ϵ arbitrarily close to $\alpha+1$ we have that if $k^2 < \frac{\alpha}{1+\alpha}\lambda_0(D)$, then k is not a transmission eigenvalue. This completes the proof of discreteness of real transmission eigenvalues.

In the foregoing discussion, we showed that if $k > 0$ is a transmission eigenvalue, then it must satisfy $k^2 > \frac{\alpha}{1+\alpha}\lambda_0(D)$, and thus, from (6.71) we obtain that $k^2 \geq \frac{\lambda_0(D)}{\sup_D \|A^{-1}\|_2}$, which proves the theorem. □

In a similar way it is possible to prove a similar result if $a_{min} > 1$ (see [33] for details). In particular, the following theorem holds.

Theorem 6.19. *Assume that $a_{min} > 1$. The set of real transmission eigenvalues is discrete. If k is a real transmission eigenvalue, then*

$$k^2 \geq \lambda_0(D), \tag{6.79}$$

where $\lambda_0(D)$ is the first eigenvalue of $-\Delta$ on D.

Now we turn our attention to prove the existence of positive transmission eigenvalues. We again only consider in detail the case where $a_{max} < 1$.

Theorem 6.20. *Assume that $a_{max} < 1$. Then there exists an infinite number of positive transmission eigenvalues with $+\infty$ as the only accumulation point.*

Proof. As explained earlier, $k > 0$ is a transmission eigenvalue if and only if the kernel of the operator $\mathbb{A}_k - k^2\mathbb{B}$ or $\mathbb{I} - k^2\mathbb{A}_k^{-1}\mathbb{B}$ is not empty, where the bounded, self-adjoint, strictly positive definite operator $\mathbb{A}_k^{-1}$ and the bounded, self-adjoint, nonnegative, compact operator $\mathbb{B}$ are defined by (6.70). Note that

$$\mathrm{N}(\mathbb{B}) = \{\mathbf{u} \in \mathcal{H}_0(D) \quad \text{such that } \mathbf{u} := \operatorname{curl}\varphi,\ \varphi \in H(\operatorname{curl}, D)\}.$$

We first observe that the multiplicity of each transmission eigenvalue is finite since it coincides with the multiplicity of the eigenvalue 1 of the compact operator $k^2\mathbb{A}_k^{-1}\mathbb{B}$, which is finite. To analyze the kernel of this operator, we consider the auxiliary eigenvalue problems

$$\mathbb{A}_k\mathbf{u} - \lambda(k)\mathbb{B}\mathbf{u} = 0 \qquad \mathbf{u} \in \mathcal{H}_0(D). \tag{6.80}$$

Thus, a transmission eigenvalue $k > 0$ satisfies $\lambda(k) - k^2 = 0$, where $\lambda(k)$ is an eigenvalue corresponding to (6.80). To prove the existence of an infinite set of transmission eigenvalues, we now use Theorem 6.15 for $\mathbb{A}_k^{-1}$ and $\mathbb{B}$ with $X = \mathcal{H}_0(D)$. Theorem 6.18 states that as long as $0 < k_0^2 < \frac{\lambda_0(D)}{\sup_D \|A^{-1}\|_2}$, the operator $\mathbb{A}_{k_0} - k_0^2\mathbb{B}$ is positive in $\mathcal{H}_0(D)$, whence assumption 1 of Theorem 6.15 is satisfied for $\tau_0 := k_0^2$. Next, let $k_{1,a_{max}}$ be the first transmission eigenvalue for the disk B of radius $R = 1$ and constant index of refraction $n := a_{max}^{-1}$ [i.e., (6.63)–(6.66) for $D := B$ and $N(x) := nI$ or (6.49)–(6.52) with $R = 1$, $n_0 = 1$, and $a_0 = a_{max}$]. This transmission eigenvalue is the first zero of

$$W(k) \;=\; \det\begin{pmatrix} J_0(k) & J_0\left(k\sqrt{\frac{1}{a_{max}}}\right) \\ k\,J_0'(k) & k\sqrt{a_{max}}\,J_0'\left(k\sqrt{\frac{1}{a_{max}}}\right)\end{pmatrix} \tag{6.81}$$

[if the first zero of the preceding determinant is not the first transmission eigenvalue, then the latter will be a zero of (6.53) for $\ell \geq 1$]. By a scaling argument, it is obvious that $k_\epsilon := k_{1,a_{max}}/\epsilon$ is the first transmission eigenvalue corresponding to a disk of radius $\epsilon > 0$ with index of refraction a_{max}^{-1}. Now take $\epsilon > 0$ small enough such that D contains $m := m(\epsilon) \geq 1$ disjoint disks $B_\epsilon^1, B_\epsilon^2 \dots B_\epsilon^m$ of radius ϵ, i.e., $\overline{B_\epsilon^j} \subset D$, $j = 1 \dots m$, and $\overline{B_\epsilon^j} \cap \overline{B_\epsilon^i} = \emptyset$ for $j \neq i$. Then $k_\epsilon = k_{1,a_{max}}/\epsilon$ is the first transmission eigenvalue for each of these disks with index of refraction a_{max}^{-1}, and let $\mathbf{u}^j := \mathbf{u}^{B_\epsilon^j, a_{min}} \in \mathcal{H}_0(B_\epsilon^j)$, $j = 1 \dots m$, be the corresponding eigenfunctions. We have that $u^j \in \mathcal{H}_0(B_\epsilon^j)$ and

$$\int_{B_\epsilon^j} \frac{1}{n-1}(\nabla\nabla\cdot\mathbf{u}^j + k_\epsilon^2\mathbf{u}^j)\cdot(\nabla\nabla\cdot\overline{\mathbf{u}}^j + k_\epsilon^2 n\overline{\mathbf{u}}^j)\,dx = 0. \tag{6.82}$$

By definition, the vectors $\tilde{u}^j$ are not in the kernel of $\mathbb{B}$. The extension by zero $\tilde{\mathbf{u}}^j$ of $\mathbf{u}^j$ to the whole D is obviously in $\mathcal{H}_0(D)$ due to the boundary conditions on ∂B^j. Furthermore, the functions $\{\tilde{\mathbf{u}}^1, \tilde{\mathbf{u}}^2, \dots \tilde{\mathbf{u}}^m\}$ are linearly independent and orthogonal in $\mathcal{H}_0(D)$ since they have disjoint supports, and from (6.82) we have that

$$\begin{aligned} 0 &= \int_{B_\epsilon^j} \frac{1}{n-1}(\nabla\nabla\cdot\mathbf{u}^j + k_\epsilon^2\mathbf{u}^j)\cdot(\nabla\nabla\cdot\overline{\mathbf{u}}^j + k_\epsilon^2 n\overline{\mathbf{u}}^j)\,dx \\ &= \int_D \frac{1}{n-1}|\nabla\nabla\cdot\tilde{\mathbf{u}}^j + k_\epsilon^2\tilde{\mathbf{u}}^j|^2\,dx + k_\epsilon^4\int_D |\tilde{\mathbf{u}}^j|^2\,dx - k_\epsilon^2\int_D |\nabla\cdot\tilde{\mathbf{u}}|^2\,dx \end{aligned} \tag{6.83}$$

for $j = 1 \dots m$. Denote by W_m the m-dimensional subspace of $\mathcal{H}_0(D)$ spanned by $\{\tilde{\mathbf{u}}^1, \tilde{\mathbf{u}}^2, \dots \tilde{\mathbf{u}}^m\}$. Since each $\tilde{\mathbf{u}}^j$, $j = 1, \dots, m$, satisfies (6.83) and they have disjoint supports, we have that for k_ϵ and for every $\tilde{\mathbf{u}} \in W_m$

$$\begin{aligned}
&\left(\mathbb{A}_{k_\epsilon}\tilde{u} - k_\epsilon^2 \mathbb{B}\tilde{\mathbf{u}},\, \tilde{\mathbf{u}}\right)_{\mathcal{H}_0(D)} \\
&\quad = \int_D (N-I)^{-1} |\nabla\nabla\cdot\tilde{\mathbf{u}} + k_\epsilon^2 \tilde{\mathbf{u}}|^2\, dx + k_\epsilon^4 \int_D |\tilde{\mathbf{u}}|^2\, dx - k_\epsilon^2 \int_D |\nabla\cdot\tilde{\mathbf{u}}|^2\, dx \\
&\quad \le \int_D \frac{1}{n-1} |\nabla\nabla\cdot\tilde{\mathbf{u}} + k_\epsilon^2 \tilde{\mathbf{u}}|^2\, dx + k_\epsilon^4 \int_D |\tilde{\mathbf{u}}|^2\, dx - k_\epsilon^2 \int_D |\nabla\cdot\tilde{\mathbf{u}}|^2\, dx = 0.
\end{aligned} \tag{6.84}$$

This means that assumption 2 of Theorem 6.15 is also satisfied, and therefore we can conclude that there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\frac{\lambda_0(D)}{\sup_D \|A^{-1}\|_2}, \frac{k_{1,\, a_{max}}}{\epsilon}]$. Note that $m(\epsilon)$ and k_ϵ both go to $+\infty$ as $\epsilon \to 0$. Since the multiplicity of each eigenvalue is finite, we have shown, by letting $\epsilon \to 0$, that there exists an infinite countable set of transmission eigenvalues that accumulate at $+\infty$. □

In a similar way it is possible to prove an analogous result if $a_{min} > 1$ (see [33] for details). In particular, the following theorem holds.

Theorem 6.21. *Assume that $a_{min} > 1$. Then there exists an infinite number of positive transmission eigenvalues with $+\infty$ as the only accumulation point.*

The foregoing proof of the existence of transmission eigenvalues provides a framework in which to obtain lower and upper bounds for the first transmission eigenvalue. To this end, denote by $k_{0,A} > 0$ the first positive transmission eigenvalue corresponding to A and D (we omit the dependence on D in our notation since D is assumed to be known). Assume again that $a_{max} < 1$.

Theorem 6.22. *Assume that the index of refraction $A(x)$ satisfies $a_{max} < 1$, where a_{max} and a_{min} are given by (6.48). Then*

$$0 < k_{0,a_{min}} \le k_{0,A(x)} \le k_{0,a_{max}}. \tag{6.85}$$

Proof. From the proof of Theorem 6.20 we have that $k_{0,A}^2$ is the smallest zero of

$$\lambda(k, A) - k^2 = 0, \tag{6.86}$$

where

$$\lambda(k, A) = \inf_{\substack{\mathbf{u}\in\mathcal{H}_0(D)\\ \|\nabla\cdot\mathbf{u}\|_D = 1}} \int_D (A^{-1}-I)^{-1} |\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}|^2\, dx + k^4 \int_D |\mathbf{u}|^2\, dx \tag{6.87}$$

and $\mathbf{u}$ not in the kernel of $\mathbb{B}$. [Note that any zero of $\lambda(k, A) - k^2 = 0$ leads to a transmission eigenvalue.] Obviously, the mapping $k \to \lambda(k, A)$ is continuous on $(0,\, +\infty)$. We first note that (6.87) yields

$$\lambda(k, a_{min}) \le \lambda(k, A(x)) \le \lambda(k, a_{max}) \tag{6.88}$$

for all $k > 0$. In particular, for $k := k_{0,a_{min}}$ we have that

$$0 = \lambda(k_{0,a_{min}}, a_{min}) - k^2_{0,a_{min}} \leq \lambda(k_{0,a_{min}}, A(x)) - k^2_{0,a_{min}},$$

and for $k := k_{0,a_{max}}$ we have that

$$\lambda(k_{0,a_{min}}, A(x)) - k^2_{0,a_{min}} \leq \lambda(k_{0,a_{max}}, a_{max}) - k^2_{0,a_{max}} = 0.$$

By continuity of $k \to \lambda(\tau, A) - k^2$, we have that there is a zero $\tilde{k}$ of $\lambda(k, A) - k^2 = 0$ such that $k_{0,a_{min}} \leq \tilde{k} \leq k_{0,a_{max}}$. In particular, the smallest zero $k_{0,A(x)}$ of $\lambda(k, A) - k^2 = 0$ is such that $k_{0,A(x)} \leq \tilde{k} \leq k_{0,a_{max}}$. To end the proof, we need to show that $k_{0,a_{min}} \leq k_{0,A(x)}$, i.e., all the positive zeros of $\lambda(k, A) - k^2 = 0$ are greater than or equal to $k_{0,a_{min}}$. Assume by contradiction that $k_{0,A(x)} < k_{0,a_{min}}$. Then, from (6.88), on the one hand, we have

$$\lambda(k_{0,A(x)}, a_{min}) - k^2_{0,A(x)} \leq \lambda(k_{0,A(x)}, A(x)) - k^2_{0,A(x)} = 0.$$

On the other hand, from the proof of Theorem 6.18 we have that for a sufficiently small $k' > 0$, $\lambda(k', a_{min}) - k'^2 > 0$. Hence there exists a zero of $\lambda(k, a_{min}) - k^2 = 0$ between k' and $k_{0,A(x)}$ smaller than $k_{0,a_{min}}$, which contradicts the fact that $k_{0,a_{min}}$ is the smallest zero. Thus we have proven that $k_{0,a_{min}} \leq k_{0,A(x)} \leq k_{0,a_{max}}$, and this completes the proof. □

In a similar way [31, 33], one can prove the following theorem.

Theorem 6.23. *Assume that the index of refraction $A(x)$ satisfies $a_{min} > 1$, where a_{max} and a_{min} are given by (6.48). Then*

$$0 < k_{0,a_{max}} \leq k_{0,A(x)} \leq k_{0,a_{min}}. \tag{6.89}$$

Theorems 6.22 and 6.23 show in particular that for constant index of refraction $A = aI$ the first transmission eigenvalue $k_{0,a}$ is monotonically increasing if $0 < a < 1$ and is monotonically decreasing if $a > 1$. If fact we can show that this monotonicity is strict, which leads to the following uniqueness result for a constant index of refraction in terms of the first transmission eigenvalue.

Theorem 6.24. *The constant index of refraction $A := aI$ is uniquely determined from a knowledge of the corresponding smallest transmission eigenvalue $k_{0,a} > 0$, provided that it is known a priori that either $a > 1$ or $0 < a < 1$.*

Proof. We show the proof for the case $0 < a < 1$ (a similar proof works for the case $a > 1$). Consider two homogeneous media with constant indexes of refraction a_1 and a_2 such that $a_2 < a_1 < 1$, and let $\mathbf{u}_1 := \mathbf{w}_1 - \mathbf{v}_1$, where $\mathbf{w}_1, \mathbf{v}_1$ is the nonzero solution of (6.63)–(6.66), with $A(x) := a_1 I$ corresponding to the first transmission eigenvalue k_{0,a_1}. Now, setting $k_0 := k_{0,a_1}$ and after normalizing $\mathbf{u}_1$ such that $\nabla \cdot \mathbf{u}_1 = 1$, we have

$$\frac{1}{1/a_1 - 1} \|\nabla\nabla \cdot \mathbf{u}_1 + k_0^2 \mathbf{u}_1\|^2_{L^2(D)} + k_0^4 \|\mathbf{u}_1\|^2_{L^2(D)} = k_0^2 = \lambda(k_1, a_1).$$

Furthermore, we have

$$\frac{1}{1/a_2-1}\|\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\|^2_{L^2(D)}+k^4\|\mathbf{u}_1\|^2_{L^2(D)}$$
$$\le\frac{1}{1/a_1-1}\|\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}\|^2_{L^2(D)}+k^4\|\mathbf{u}_1\|^2_{L^2(D)}$$

for all $\mathbf{u}\in\mathcal{H}_0(D)$ such that $\|\nabla\cdot\mathbf{u}\|_D=1$, $\mathbf{u}$ not in the kernel of $\mathbb{B}$, and all $k>0$. In particular, for $\mathbf{u}=\mathbf{u}_1$ and $k=k_0$

$$\frac{1}{1/a_2-1}\|\nabla\nabla\cdot\mathbf{u}_1+k_0^2\mathbf{u}_1\|^2_{L^2(D)}+k_0^4\|\mathbf{u}_1\|^2_{L^2(D)}$$
$$<\frac{1}{1/a_2-1}\|\nabla\nabla\cdot\mathbf{u}_1+k_0^2\mathbf{u}_1\|^2_{L^2(D)}+k_0^4\|\mathbf{u}_1\|^2_{L^2(D)}=\lambda(k_0,a_1).$$

But

$$\lambda(k_0,a_2)\le\frac{1}{1/a_2-1}\|\nabla\nabla\cdot\mathbf{u}_1+k_0^2\mathbf{u}_1\|^2_{L^2(D)}+k_0^4\|\mathbf{u}_1\|^2_{L^2(D)}<\lambda(k_0,a_1),$$

and hence for this k_0 we have a strict inequality, i.e.,

$$\lambda(k_0,a_2)<\lambda(k_0,a_1). \tag{6.90}$$

Hence, (6.90) implies the first zero k_{0,a_2} of $\lambda(k,a_2)-k^2=0$ is such that $k_{0,a_2}<k_{0,a_1}$ for the first transmission eigenvalues k_{0,a_1} and k_{0,a_2} corresponding to a_1 and a_2, respectively. Hence we have shown that if $0<a_1<1$ and $0<a_2<1$ are such that $a_1\ne a_2$, then $k_{0,a_1}\ne k_{0,a_2}$, which proves the desired strict monotonicity. The uniqueness result now follows immediately from Theorem 6.22. □

From the proof of Theorems 6.22 and 6.23, one can see that the following more general monotonicity property of the first transmission eigenvalue with respect to the support of inhomogeneity and the refractive index holds true.

Corollary 6.25. *Let $D_1\subset D\subset D_2$ and $A_1<A<A_2$, where A_1,A,A_2 all satisfy the assumptions of either Theorem 6.22 or Theorem 6.23. If $k_{0,A,D}$ denotes the first transmission eigenvalue corresponding to D and A, then*

$$0<k_{0,A_2,D_2}\le k_{0,A_2,D}\le k_{0,A,D}\le k_{0,A_1,D}\le k_{0,A_1,D_1}$$

if the assumptions of Theorem 6.22 are satisfied and

$$0<k_{0,A_1,D_2}\le k_{0,A_1,D}\le k_{0,A,D}\le k_{0,A_2,D}\le k_{0,A_2,D_1}$$

if the assumptions of Theorem 6.22 are satisfied. Here $A_1<A$ means that the matrix $A-A_1$ is positive definite uniformly in D, with a similar definition for $A<A_2$.

Remark 6.26. The existence and discreteness of transmission eigenvalues for the problem (6.54)–(6.57) are also considered in [105] using a different approach. In particular, in [105] (see also [54] for the case where $A = I$ and $n \neq 1$) the transmission eigenvalue problem (6.54)–(6.57) is shown to be an eigenvalue problem for a quadratic pencil operator $I - k^2\mathbb{C} + k^4\mathbb{D}$, where $\mathbb{C}$ and $\mathbb{D}$ are self-adjoint compact operators and $\mathbb{D}$ is nonnegative. The latter becomes a linear eigenvalue problem for the non-self-adjoint, matrix-valued operator

$$\begin{pmatrix} \mathbb{C} & \mathbb{D}^{\frac{1}{2}} \\ -\mathbb{D}^{\frac{1}{2}} & 0 \end{pmatrix}.$$

We note that interesting analytical results for this type of non-self-adjoint eigenvalue problems were obtained in [36] and [145]. For more results on a transmission eigenvalue problem as an eigenvalue problem for a quadratic pencil operator see [84, 85, 86].

6.3.2 The Case $n \neq 1$

We now turn our attention to the general case where both $A \neq 1$ and $n \neq 1$. We recall that the *transmission eigenvalue problem* is the problem of finding two nonzero functions $v \in H^1(D)$ and $w \in H^1(D)$ satisfying

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D, \tag{6.91}$$

$$\Delta w + k^2\, w = 0 \qquad \text{in} \quad D, \tag{6.92}$$

$$v = w \qquad \text{on} \quad \partial D, \tag{6.93}$$

$$\frac{\partial v}{\partial \nu_A} = \frac{\partial w}{\partial \nu} \qquad \text{on} \quad \partial D. \tag{6.94}$$

We already discussed at the beginning of Chap. 6.2 the Fredholm property of the foregoing problem under the assumption that $A - I > 0$ or $I - A > 0$ in D. In fact, we can show that the interior transmission problem satisfies the Fredholm property if the preceding assumptions on the contrast are satisfied only in a neighborhood of the boundary, but in this case we need to impose the same assumptions on the contrast $n - 1$. In addition, the approach we are about to discuss also proves that the set of transmission eigenvalues is discrete. Note that for this general case the existence of transmission eigenvalues can be proven under much more restrictive assumptions using a different approach.

6.3.3 Discreteness of Transmission Eigenvalues

Let $\mathcal{N}$ be a δ-neighborhood of the boundary ∂D in D i.e.,

$$\mathcal{N} := \{x \in D : \ \operatorname{dist}(x, \partial D) < \delta\},$$

and introduce the following notations:

$$a_* := \inf_{x\in\mathcal{N}} \inf_{\xi\in\mathbb{R}^2, |\xi|=1} (\xi \cdot A(x)\xi) > 0,$$

$$a^* := \sup_{x\in\mathcal{N}} \sup_{\xi\in\mathbb{R}^2, |\xi|=1} (\xi \cdot A(x)\xi) < \infty, \tag{6.95}$$

$$n_* := \inf_{x\in\mathcal{N}} n(x) > 0 \quad \text{and} \quad n^* := \sup_{x\in\mathcal{N}} n(x) < \infty.$$

Note that in (6.95) the infimum and supremum are only taken over a neighborhood of the boundary ∂D as opposed to over the entire domain D as in (6.48).

We consider the Sobolev space

$$\mathcal{H}(D) := \left\{ (v, w) \in H^1(D) \times H^1(D) : \ v - w \in H_0^1(D) \right\}.$$

Our first observation is that $(v, w) \in H^1(D) \times H^1(D)$ is a solution to (6.91)–(6.94) if and only if

$$a_k((v, w), (v', w')) = 0 \quad \text{for all} \quad (v', w') \in \mathcal{H}(D), \tag{6.96}$$

where the sesquilinear form $a_k(\cdot, \cdot) : \mathcal{H}(D) \to \mathbb{C}$ is defined by

$$\begin{aligned} a_k((v, w), (v', w')) : = & \int_D A\nabla v \cdot \nabla \overline{v}' \, dx - \int_D \nabla w \cdot \nabla \overline{w}' \, dx \\ & - k^2 \int_D n v \, \overline{v}' \, dx + k^2 \int_D w \, \overline{w}' \, dx. \end{aligned}$$

Let $\mathbf{A}_k : \mathcal{H}(D) \to \mathcal{H}(D)$ be the bounded linear operator defined by means of the Riesz representation theorem

$$(\mathbf{A}_k(v, w), (v', w'))_{\mathcal{H}(D)} = a_k((v, w), (v', w')). \tag{6.97}$$

Obviously, $\mathbf{A}_k$ depends analytically on $k \in \mathbb{C}$, and furthermore, for any two k and k' the operator $\mathbf{A}_k - \mathbf{A}_{k'}$ is compact, which is a simple consequence of the compact embedding of $\mathcal{H}(D)$ into $L^2(D) \times L^2(D)$. Therefore, to prove the discreteness of transmission eigenvalues, it suffices to prove that $\mathbf{A}_{k'}$ is invertible for some $k' \in \mathbb{C}$ since then we can write $A_k = A_{k'} + (A_k - A_{k'})$ and appeal to the analytic Fredholm theorem (Theorem 1.24). The difficulty in obtaining this result is that the sesquilinear form $a_k((v, w), (v', w'))$ is not coercive for any $k \in \mathbb{C}$ due to the opposite signs in the terms containing the gradients. To show the invertibility of the $\mathbf{A}_k$, we follow the arguments in [9] and [39], which rely on proving that $a_k(\cdot, \cdot)$ is T-coercive (as it is called in [10]) for some k. More specifically, the idea behind T-coercivity is to consider an equivalent formulation of (6.96), where a_k is replaced by a_k^T defined by

$$a_k^T((v,w),(v',w')) := a_k((v,w),\mathbf{T}(v',w')) \tag{6.98}$$

for all $((v,w),(v',w')) \in \mathcal{H}(D)\times\mathcal{H}(D)$, with the operator $\mathbf{T}:\mathcal{H}(D)\to\mathcal{H}(D)$ being an isomorphism. Obviously, $(v,w)\in\mathcal{H}(D)$ satisfies

$$a_k((v,w),(v',w')) = 0 \qquad \text{for all} \qquad (v',w')\in\mathcal{H}(D)$$

if and only if it satisfies

$$a_k^T((v,w),(v',w')) = 0 \qquad \text{for all} \qquad (v',w')\in\mathcal{H}(D).$$

If we can choose $\mathbf{T}$ and k such that a_k^T is coercive, then using the Lax–Milgram theorem and the fact that $\mathbf{T}$ is an isomorphism we can deduce that $\mathbf{A}_k:\mathcal{H}(D)\to\mathcal{H}(D)$ defined by (6.97) is invertible.

Lemma 6.27. *Assume that either* $0<a^*<1$ *and* $0<n^*<1$*, or* $a_*>1$ *and* $n_*>1$*. Then there exists* $k=i\kappa$*, with* $\kappa\in\mathbb{R}$*, such that* $\mathbf{A}_{i\kappa}:\mathcal{H}(D)\to\mathcal{H}(D)$ *is invertible.*

Proof. Let us first consider the case where $0<a^*<1$ and $0<n^*<1$ and introduce $\chi\in\mathcal{C}^\infty(\overline{D})$, a cutoff function equal to 1 in a neighborhood of ∂D supported in $\mathcal{N}$ such that $0\le\chi\le 1$. We define the isomorphism $\mathbf{T}:\mathcal{H}(D)\to\mathcal{H}(D)$ by

$$\mathbf{T}:(v,w)\mapsto(v-2\chi w,-w).$$

(Note that $\mathbf{T}$ is an isomorphism since $\mathbf{T}^2=I$.) We then have that for all $(v,w)\in\mathcal{H}(D)$

$$\begin{aligned}\left|a_{i\kappa}^T((v,w),(v,w))\right| = &\,|(A\nabla v,\nabla v)_D+(\nabla w,\nabla w)_D-2(A\nabla v,\nabla(\chi w))_D\\ &+\kappa^2\left((nv,v)_D+(w,w)_D-2(nv,\chi w)_D\right)\right|,\end{aligned} \tag{6.99}$$

where $(\cdot,\cdot)_{\mathcal{O}}$ for a generic bounded region $\mathcal{O}\subset\mathbb{R}^2$ denotes the $L^2(\mathcal{O})$ inner product. Using Young's inequality

$$|ab|\le\epsilon a^2+\frac{1}{\epsilon}b^2, \qquad \epsilon>0,$$

we can write

$$\begin{aligned}2\left|(A\nabla v,\nabla(\chi w))_D\right| &\le 2\left|(\chi A\nabla v,\nabla w)_{\mathcal{N}}\right|+2\left|(A\nabla v,\nabla(\chi)w)_{\mathcal{N}}\right|\\ &\le \eta(A\nabla v,\nabla v)_{\mathcal{N}}+\eta^{-1}(A\nabla w,\nabla w)_{\mathcal{N}}\\ &+\alpha(A\nabla v,\nabla v)_{\mathcal{N}}+\alpha^{-1}(A\nabla(\chi)w,\nabla(\chi)w)_{\mathcal{N}}\end{aligned} \tag{6.100}$$

and

$$2\left|(nv,\chi w)_D\right|\le\beta(nv,v)_{\mathcal{N}}+\beta^{-1}(nw,w)_{\mathcal{N}} \tag{6.101}$$

for arbitrary constants $\alpha > 0$, $\beta > 0$, and $\eta > 0$. Substituting (6.100) and (6.101) into (6.99), we now obtain

$$\begin{aligned}\left|a^T_{i\kappa}((v,w),(v,w))\right| &\geq (A\nabla v,\nabla v)_{D\setminus\overline{\mathcal{N}}} + (\nabla w,\nabla w)_{D\setminus\overline{\mathcal{N}}}\\ &+\kappa^2\left((nv,v)_{D\setminus\overline{\mathcal{N}}} + (w,w)_{D\setminus\overline{\mathcal{N}}}\right)\\ &+((1-\eta-\alpha)A\nabla v,\nabla v)_{\mathcal{N}} + ((I-\eta^{-1}A)\nabla w,\nabla w)_{\mathcal{N}}\\ &+\kappa^2((1-\beta)nv,v)_{\mathcal{N}} + ((\kappa^2(1-\beta^{-1}n) - \sup_{\mathcal{N}}|\nabla\chi|^2 a^*\alpha^{-1})w,w)_{\mathcal{N}}.\end{aligned}$$

Taking η, α, and β such that $a^* < \eta < 1$, $n^* < \beta < 1$, and $0 < \alpha < 1-\eta$, we obtain the coercivity of $a^T_{i\kappa}$ for κ large enough, which proves the lemma. The case where $a_* > 1$ and $n_* > 1$ can be handled in a similar way using $\mathbf{T}(v,w) := (v, -w+2\chi v)$. □

Lemma 6.27, combined with the fact that $\mathbf{A}_k - \mathbf{A}_{i\kappa}$ is compact, and an application of the analytic Fredholm theorem (Theorem 1.24) implies the following theorem.

Theorem 6.28. *Assume that either* $0 < a^* < 1$ *and* $0 < n^* < 1$*, or* $a_* > 1$ *and* $n_* > 1$*. Then the set of transmission eigenvalues is discrete in* $\mathbb{C}$*.*

Remark 6.29. As a consequence of the proof of Lemma 6.27 we can conclude that the operator $\mathbf{A}_k$ is Fredholm with index zero (cf. [127]). This implies that under the assumptions that either $0 < a^* < 1$ and $0 < n^* < 1$, or $a_* > 1$ and $n_* > 1$, the interior transmission problem (6.91)–(6.94) with boundary data $f \in H^{\frac{1}{2}}(\partial D)$ and $h \in H^{-\frac{1}{2}}(\partial D)$ has a unique solution $(v,w) \in H^1(D) \times H^1(D)$, provided $k \in \mathbb{C}$ is not a transmission eigenvalue. Furthermore, the solution depends continuously on the data f, h.

We conclude this section by showing that if we require that the contrast keep the same sign in D, i.e., $0 < a_{max} < 1$ or $a_{min} > 1$, the T-coercivity approach allows us to prove the discreteness of transmission eigenvalues under more relaxed assumptions on $n-1$. To this end, taking $v' = w' = 1$ in (6.96) we first notice that the transmission eigenfunctions (v,w) [i.e., the solution to (6.91)–(6.91) corresponding to an eigenvalue k] satisfy $k^2\int_D(nv-w)dx = 0$. This suggests introducing the subspace of $\mathcal{H}(D)$

$$\mathcal{Y}(D) := \left\{(v,w) \in \mathcal{H}(D) \mid \int_D (nv-w)dx = 0\right\}.$$

Now, suppose $\int_D(n-1)dx \neq 0$. Arguing by contradiction, one can prove the existence of a constant $C_P > 0$ (which depends on D and on n) such that

$$\|v\|^2_D + \|w\|^2_D \leq C_P(\|\nabla v\|^2_D + \|\nabla w\|^2_D), \quad \forall (v,w) \in \mathcal{Y}(D). \tag{6.102}$$

Furthermore, we observe that $k \neq 0$ is a transmission eigenvalue if and only if there exists a nontrivial element $(v,w) \in \mathcal{Y}(D)$ such that

$$a_k((v,w),(v',w')) = 0 \text{ for all } (v',w') \in \mathcal{Y}(D).$$

Using the variational formulation in this new subspace and (6.102) we can now prove the following theorem, which completes the analysis of the solvability of the interior transmission problem discussed at the beginning of Sect. 6.2.

Theorem 6.30. *Assume that either* $0 < a_{max} < 1$ *or* $a_{min} > 1$*, and* $\int_D (n-1)dx \neq 0$*. Then the set of transmission eigenvalues is discrete in* $\mathbb{C}$*.*

Proof. For the sake of simplicity we only consider in detail the case where $0 < a_{max} < 1$. Letting $\lambda(w) := 2\int_D (n-1)w/\int_D (n-1)$ we consider the mapping $\mathbf{T} : \mathcal{Y}(D) \to \mathcal{Y}(D)$ defined by

$$\mathbf{T} : (v, w) \mapsto (v - 2w + \lambda(w), -w + \lambda(w)).$$

Note that $\lambda(\lambda(w)) = 2\lambda(w)$, which implies that $\mathbf{T}^2 = I$, and hence $\mathbf{T}$ is an isomorphism in $\mathcal{Y}(D)$. Then for all $(v, w) \in \mathcal{Y}(D)$ we have that

$$\begin{aligned}
&\left|a_k^T((v, w), (v, w))\right| \\
&\qquad = |(A\nabla v, \nabla v)_D + (\nabla w, \nabla w)_D - 2(A\nabla v, \nabla w)_D \\
&\qquad - k^2\left((nv, v)_D + (w, w)_D - 2(nv, w)_D\right)| \\
&\qquad \geq (A\nabla v, \nabla v)_D + (\nabla w, \nabla w)_D - 2\left|(A\nabla v, \nabla w)_D\right| \\
&\qquad - |k|^2\left((nv, v)_D + (w, w)_D + 2\left|(nv, w)_D\right|\right) \\
&\qquad \geq (1 - \sqrt{a_{max}})((A\nabla v, \nabla v)_D + (\nabla w, \nabla w)_D) \\
&\qquad - |k|^2(1 + \sqrt{n_{max}})((nv, v)_D + (w, w)_D).
\end{aligned}$$

Consequently, for $k \in \mathbb{C}$ such that

$$|k|^2 < (a_{min}(1 - \sqrt{a_{max}}))/(C_P \max(n_{max}, 1)\,(1 + \sqrt{n_{max}}))$$

a_k^T is coercive on $\mathcal{Y}(D)$. The claim of the theorem follows from the analytic Fredholm theorem. □

The case $a_{min} > 1$ can be handled in a similar way using the isomorphism $\mathbf{T} : \mathcal{Y}(D) \to \mathcal{Y}(D)$ defined by

$$\mathbf{T} : (v, w) \mapsto (v - \lambda(v), -w + 2v - \lambda(v)).$$

We refer the reader to [9] for estimates on transmission eigenvalues following from the foregoing analysis. For the discreteness of complex transmission eigenvalues in the case where $A = I$ see [154].

6.3.4 Existence of Transmission Eigenvalues for $n \neq 1$

We finally come to the discussion of the existence of positive transmission eigenvalues in the general case of anisotropic media with $n \neq 1$. Unfortunately, the existence of transmission eigenvalues for this case can only be shown under restrictive assumptions on $A - I$ and $n - 1$. The approach presented here follows the lines of [35], where, motivated by the case of $n = 1$, the transmission eigenvalue problem is formulated in terms of the difference $u := v - w$. However, due to the lack of symmetry, the problem for u is no longer a quadratic eigenvalue problem but takes the form of a more complicated nonlinear eigenvalue problem, as will become clear in what follows.

To simplify the expressions, we set $\tau := k^2$ in (6.91)–(6.94) and observe that, if (w, v) satisfies (6.91)–(6.94), then subtracting the equation for v from the equation for w we obtain

$$\begin{aligned} \nabla\cdot A\nabla u + \tau n u &= \nabla\cdot(A-I)\nabla w + \tau(n-1)\,w \quad \text{in } D\,, \\ \nu\cdot A\nabla u &= \nu\cdot(A-I)\nabla w \quad \text{on } \partial D, \end{aligned} \tag{6.103}$$

where $u := v - w$. In addition, we also have $u = 0$ on ∂D and

$$\Delta w + \tau w = 0 \quad \text{in } D. \tag{6.104}$$

It is easy to verify that (v, w) in $H^1(D)\times H^1(D)$ satisfies (6.91)–(6.94) if and only if (u, w) in $H^1_0(D)\times H^1(D)$ satisfies (6.103)–(6.104). The main idea of the proof of the existence of transmission eigenvalues consists in expressing w in terms of u, using (6.103), and substituting the resulting expression into (6.104) in order to formulate the eigenvalue problem only in terms of u. In the case where $A = I$, this substitution is simple and leads to an explicit expression for the equation satisfied by u (see [54], Sect. 10.5, and [105]). In the current case the substitution requires the inversion of the operator $\nabla\cdot\big[(A-I)\nabla\cdot\big]+\tau(n-1)$ with a Neumann boundary condition. It is then obvious that the case where $(A-I)$ and $(n-1)$ have the same sign is more problematic since in that case the operator may not be invertible for special values of τ. This is why we only consider in detail the simpler case where $(A-I)$ and $(n-1)$ have opposite signs almost everywhere in D.

Note that for given $u \in H^1_0(D)$, the problem (6.103) for $w \in H^1(D)$ is equivalent to the variational formulation

$$\int_D \big[(A-I)\nabla w\cdot\nabla\overline{\psi} - \tau\,(n-1)\,w\,\overline{\psi}\big]\,dx \;=\; \int_D \big[A\,\nabla u\cdot\nabla\overline{\psi} - \tau n\,u\,\overline{\psi}\big]\,dx \tag{6.105}$$

for all $\psi \in H^1(D)$. The following result concerning the invertibility of the operator associated with (6.105) can be proven in a standard way using the Lax–Milgram lemma. We skip the proof here and refer the reader to [35].

Lemma 6.31. *Assume that either $a_{min} > 1$ and $0 < n_{max} < 1$, or $0 < a_{max} < 1$ and $n_{min} > 1$. Then there exists $\delta > 0$ such that for every $u \in H^1_0(D)$ and $\tau \in \mathbb{C}$ with $Re(\tau) > -\delta$ there exists a unique solution $w := w_u \in H^1(D)$ of (6.105). The operator $A_\tau : H^1_0(D) \to H^1(D)$, defined by $u \mapsto w_u$, is bounded and depends analytically on $\tau \in \{z \in \mathbb{C} : Re(z) > -\delta\}$.*

We now set $w_u := A_\tau u$ and denote by $\mathbb{L}_\tau u \in H_0^1(D)$ the unique Riesz representation of the bounded conjugate-linear functional

$$\psi \mapsto \int\limits_D \left[\nabla w_u \cdot \nabla \overline{\psi} - \tau\, w_u\, \overline{\psi}\right] dx \quad \text{for } \psi \in H_0^1(D)\,,$$

i.e.,

$$(\mathbb{L}_\tau u, \psi)_{H^1(D)} \;=\; \int\limits_D \left[\nabla w_u \cdot \nabla \overline{\psi} - \tau\, w_u\, \overline{\psi}\right] dx \quad \text{for } \psi \in H_0^1(D)\,. \qquad (6.106)$$

Obviously, $\mathbb{L}_\tau$ also depends analytically on $\tau \in \{z \in \mathbb{C} : \mathrm{Re}(z) > -\delta\}$. Now we are able to connect a transmission eigenfunction, i.e., a nontrivial solution (v, w) of (6.91)–(6.94), to the kernel of the operator $\mathbb{L}_\tau$.

Theorem 6.32. *The following statements are true:*

1. *Let $(w, v) \in H^1(D) \times H^1(D)$ be a transmission eigenfunction corresponding to some eigenvalue $\tau > 0$. Then $u = w - v \in H_0^1(D)$ satisfies $\mathbb{L}_\tau u = 0$.*
2. *Let $u \in H_0^1(D)$ satisfy $\mathbb{L}_\tau u = 0$ for some $\tau > 0$. Furthermore, let $w := w_u = A_\tau u \in H^1(D)$ be as in Lemma 6.31, i.e., the solution of (6.105). Then τ is a transmission eigenvalue with $(v, w) \in H^1(D) \times H^1(D)$ the corresponding transmission eigenfunction where $v = w - u$.*

Proof. Formula (6.106) implies that $(\mathbb{L}_\tau u, \psi)_{H^1(D)}$ for all $\psi \in H_0^1(D)$, which means that $L_\lambda u = 0$.

The proof of the second part of the theorem is a simple consequence of the observation that (6.104) is equivalent to

$$\int\limits_D \left[\nabla w \cdot \nabla \overline{\psi} - \tau\, w\, \overline{\psi}\right] dx \;=\; 0 \quad \text{for all } \psi \in H_0^1(D). \qquad (6.107)$$

Hence $L_\lambda u = 0$ implies that w_u solves the Helmholtz equation in D. Since $v := w - u$, we have that the Cauchy data of w and v coincide. The equation for v follows from (6.105). □

The operator $\mathbb{L}_\tau$ plays a similar role as the operator $\mathbb{A}_k - k^2\mathbb{B}$ in (6.77) for the case of $n = 1$.

Theorem 6.33. *The bounded linear operator $\mathbb{L}_\tau : H_0^1(D) \to H_0^1(D)$ satisfies the following statements holds:*

1. *$\mathbb{L}_\tau$ is self-adjoint for all $\tau > 0$.*
2. *$(\sigma \mathbb{L}_0 u, u)_{H^1(D)} \geq c\,\|u\|^2_{H^1(D)}$ for all $u \in H_0^1(D)$ and $c > 0$ independent of u, where $\sigma = 1$ if $a_{min} > 1$ and $0 < n_{max} < 1$, and $\sigma = -1$ if $0 < a_{max} < 1$ and $n_{min} > 1$.*
3. *$\mathbb{L}_\tau - \mathbb{L}_0$ is compact.*

Proof. 1. Let $u_1, u_2 \in H_0^1(D)$ and $w_1 := w_{u_1}$ and $w_2 := w_{u_2}$ be the corresponding solution of (6.105). Then we have that

$$
\begin{aligned}
(\mathbb{L}_\tau u_1,\, u_2)_{H^1(D)} &= \int_D \big[\nabla w_1 \cdot \nabla \overline{u_2} - \tau\, w_1 \overline{u_2}\big]\, dx \\
&= \int_D \big[A\,\nabla w_1 \cdot \nabla \overline{u_2} - \tau n\, w_1\, \overline{u_2}\big]\, dx \\
&\quad - \int_D \big[(A-I)\nabla w_1 \cdot \nabla \overline{u_2} - \tau\,(n-1)\, w_1\, \overline{u_2}\big]\, dx\,.
\end{aligned}
$$

Using (6.105) twice, first for $u = u_2$ and the corresponding $w = w_2$ and $\psi = w_1$ and then for $u = u_1$ and the corresponding $w = w_1$ and $\psi = w_2$, yields

$$
\begin{aligned}
(\mathbb{L}_\tau u_1,\, u_2)_{H^1(D)} &= \int_D \big[(A-I)\nabla w_1 \cdot \nabla \overline{w_2} - \tau\,(n-1)\, w_1\, \overline{w_2}\big]\, dx \\
&\quad - \int_D \big[A\,\nabla u_1 \cdot \nabla \overline{u_2} - \tau n\, u_1\, \overline{u_2}\big]\, dx,
\end{aligned}
\tag{6.108}
$$

which shows that $\mathbb{L}_\tau$ is self-adjoint.

2. To show that $\sigma \mathbb{L}_0 : H_0^1(D) \to H_0^1(D)$ is a strictly coercive operator, we recall the definition (6.106) of $\mathbb{L}_0$ and use the fact that $w = w_u = u + v$ to obtain

$$
(\mathbb{L}_0 u,\, u)_{H^1(D)} \;=\; \int_D \nabla w \cdot \nabla \overline{u}\, dx \;=\; \int_D |\nabla u|^2\, dx + \int_D \nabla v \cdot \nabla \overline{u}\, dx\,. \tag{6.109}
$$

From (6.105) for $\tau = 0$ and $\psi = v$ we now have that

$$
\int_D \nabla v \cdot \nabla \overline{u}\, dx \;=\; \int_D (A-I)\nabla v \cdot \nabla \overline{v}\, dx\,. \tag{6.110}
$$

If $a_{min} > 0$, then we have $\int_D (A-I)\nabla w \cdot \nabla \overline{w}\, dx \ge (a_{min}-1)\|\nabla w\|^2_{L^2(D)} \ge 0$, and hence

$$
(\mathbb{L}_0 u,\, u)_{H^1(D)} \;\ge\; \int_D |\nabla u|^2\, dx\,.
$$

Since from Poincaré's inequality $\|\nabla u\|_{L^2(D)}$ is an equivalent norm in $H_0^1(D)$, this proves the strict coercivity of $\mathbb{L}_0$. Now if $0 < a_{max} < 1$, then from (6.108) with $u_1 = u_2 = u$ and $\tau = 0$ we have

$$-(\mathbb{L}_0 u,\, u)_{H^1(D)} = -\int_D (A-I)\nabla v\cdot\nabla\overline{v}\,dx \;+\; \int_D A\,\nabla u\cdot\nabla\overline{u}\,dx$$
$$\geq a_{min}\int_D |\nabla u|^2\,dx,$$

which proves the strict coercivity of $-\mathbb{L}_0$ since $a_{min} > 0$.

3. This follows from the compact embedding of $H_0^1(D)$ into $L^2(D)$. □

We are now in a position to establish the existence of infinitely many positive transmission eigenvalues, i.e., the existence of a sequence of $\tau_j > 0$, and corresponding $u_j \in H_0^1(D)$, such that $u_j \neq 0$ and $\mathbb{L}_{\tau_j} u_j = 0$. Obviously, these $\tau > 0$ are such that the kernel of $\mathbb{I} - \mathbb{T}_\tau$ is not trivial, which corresponds to 1 being an eigenvalue of the compact self-adjoint operator $\mathbb{T}_\tau$, where $\mathbb{T}_\lambda : H_0^1(D) \to H_0^1(D)$ is defined by

$$\mathbb{T}_\lambda := -(\sigma\mathbb{L}_0)^{-\frac{1}{2}}\left(\sigma(\mathbb{L}_\tau - \mathbb{L}_0)\right)(\sigma\mathbb{L}_0)^{-\frac{1}{2}}.$$

Thus, we can conclude that real transmission eigenvalues have finite multiplicity and are such that $\tau := k^2$ are solutions to $\mu_j(\tau) = 1$, where $\{\mu_j(\tau)\}_1^{+\infty}$ is the increasing sequence of the eigenvalues of $\mathbb{T}_\tau$. To prove the existence of positive transmission eigenvalues, we again apply Theorem 6.15 to the continuous operator-valued mapping $\tau \mapsto \mathbb{L}_\tau$, which in our case takes the following form.

Theorem 6.34. *Let $\sigma = 1$ if $a_{min} > 1$ and $0 < n_{max} < 1$, and $\sigma = -1$ if $0 < a_{max} < 1$ and $n_{min} > 1$, and make the following assumptions:*

1. There is a $\tau_0 \geq 0$ such that $\sigma\mathbb{L}_{\tau_0}$ is positive on $H_0^1(D)$.
2. There is a $\tau_1 > \tau_0$ such that $\sigma\mathbb{L}_{\tau_1}$ is nonpositive on some m-dimensional subspace W_m of $H_0^1(D)$.

Then there are m values of τ in $[\tau_0, \tau_1]$ counting their multiplicity for which $\mathbb{L}_\tau$ fails to be injective.

Using Theorem 6.34 we can now prove the main result of this section.

Theorem 6.35. *Assume that either $a_{min} > 1$ and $0 < n_{max} < 1$, or $0 < a_{max} < 1$ and $n_{min} > 1$. Then there exists an infinite sequence of positive transmission eigenvalues $k_j > 0$ ($\tau_j := k_j^2$) with $+\infty$ as the only accumulation point.*

Proof. We sketch the proof only for the case of $a_{min} > 1$ and $0 < n_{max} < 1$ (i.e., $\sigma = 1$ in Theorem 6.34). First, we recall that assumption 1 of Theorem 6.34 is satisfied with $\tau_0 = 0$ from Theorem 6.33 (*2.*). Next, from the definition of $\mathbb{L}_\tau$ and the fact that $w = v + u$, we have

$$(\mathbb{L}_\tau u,\, u)_{H^1(D)} \tag{6.111}$$
$$= \int_D \left[\nabla w\cdot\nabla\overline{u} - \tau\, w\,\overline{u}\right]dx = \int_D \left[\nabla v\cdot\nabla\overline{u} - \tau\, v\,\overline{u} + |\nabla u|^2 - \tau\,|u|^2\right]dx.$$

We also have that v satisfies

$$\int_D \left[(A-I)\nabla v \cdot \nabla\overline{\psi} - \tau(n-1)\, v\, \overline{\psi}\right] dx \;=\; \int_D \left[\nabla u \cdot \nabla\overline{\psi} - \tau\, u\, \overline{\psi}\right] dx \qquad (6.112)$$

for all $\psi \in H^1(D)$. Now taking $\psi = v$ in (6.112) and substituting the result into (6.111) yields

$$\begin{aligned}(\mathbb{L}_\tau u,\, u)_{H^1(D)} & \qquad (6.113)\\ = \int_D \left[(A-I)\nabla v \cdot \nabla\overline{v} - \tau\,(n-1)\,|v|^2 + |\nabla u|^2 - \tau\,|u|^2\right] dx\,. &\end{aligned}$$

Now let $\hat{\tau}$ be such that $\hat{\tau} := k_1^2$, where k_1 is the first transmission eigenvalue corresponding to (6.49)–(6.52) for the disk B_R with $a_0 := a_{min}$ and $n_0 := n_{max}$. We denote by $\hat{v}$, $\hat{w}$ the corresponding nonzero solutions and set $\hat{u} := \hat{v} - \hat{w} \in H_0^1(B_R)$. We denote the corresponding operator by $\hat{\mathbb{L}}_\tau$. Of course, by construction, we have that (6.113) still holds, i.e., since $\hat{\mathbb{L}}_{\hat{\tau}}\hat{u} = 0$,

$$\begin{aligned}0 = & \left(\hat{\mathbb{L}}_{\hat{\tau}}\hat{u}, \hat{u}\right)_{H^1(B_R)}, \qquad (6.114)\\ = & \int_{B_R} \left[(a_{min}-1)|\nabla\hat{v}|^2 - \hat{\tau}\,(n_{max}-1)|\hat{v}|^2 + |\nabla\hat{u}|^2 - \hat{\tau}\,|\hat{u}|^2\right] dx\,.\end{aligned}$$

Next we denote by $\tilde{u} \in H_0^1(D)$ the extension of $\hat{u} \in H_0^1(B_R)$ by zero to the whole of D and let $\tilde{w} := w_{\tilde{u}}$ be the corresponding solution to (6.105) and $\tilde{v} := \tilde{w} - \tilde{u}$. In particular, $\tilde{v} \in H^1(D)$ satisfies

$$\begin{aligned}&\int_D \left[(A-I)\nabla\tilde{v} \cdot \nabla\overline{\psi} - \hat{\tau}\, p\, \tilde{v}\, \overline{\psi}\right] dx \;=\; \int_D \left[\nabla\tilde{u} \cdot \nabla\overline{\psi} - \hat{\tau}\, \tilde{u}\, \overline{\psi}\right] dx\\ &= \int_{B_R} \left[\nabla\hat{u} \cdot \nabla\overline{\psi} - \hat{\tau}\, \hat{u}\, \overline{\psi}\right] dx \;=\; \int_{B_R} \left[(a_{min}-1)\nabla\hat{v} \cdot \nabla\overline{\psi} - \hat{\tau}\,(n_{max}-1)\, \hat{v}\, \overline{\psi}\right] dx\end{aligned}$$

for all $\psi \in H^1(D)$. Therefore, for $\psi = \tilde{v}$ we have

$$\begin{aligned}&\int_D (A-I)\nabla\tilde{v} \cdot \nabla\overline{\tilde{v}} - \hat{\tau}\,(n-1)\,|\tilde{v}|^2\, dx\\ &\qquad = \int_{B_R} (a_{min}-1)\,\nabla\hat{v} \cdot \nabla\overline{\tilde{v}} + \hat{\tau}\,|n_{max}-1|\,\hat{v}\,\overline{\tilde{v}}\, dx.\end{aligned}$$

Using the Cauchy–Schwarz inequality we obtain

$$\int_D (A-I)\nabla\tilde{v}\cdot\nabla\overline{\tilde{v}} - \hat{\tau}\,(n-1)\,|\tilde{v}|^2\,dx$$

$$\leq \left[\int_{B_R} (a_{min}-1)\,|\nabla\hat{v}|^2 + \hat{\tau}\,|n_{max}-1|\,|\hat{v}|^2\,dx\right]^{\frac{1}{2}}$$

$$\cdot \left[\int_{B_R} (a_{min}-1)\,|\nabla\tilde{v}|^2 + \hat{\tau}\,|n_{max}-1|\,|\tilde{v}|^2\,dx\right]^{\frac{1}{2}}$$

$$\leq \left[\int_{B_R} (a_{min}-1)\,|\nabla\hat{v}|^2 - \hat{\tau}\,(n_{max}-1)\,|\hat{v}|^2\,dx\right]^{\frac{1}{2}}$$

$$\cdot \left[\int_{D} (A-I)\nabla\tilde{v}\cdot\nabla\overline{\tilde{v}} - \hat{\tau}\,(n-1)\,|\tilde{v}|^2\,dx\right]^{\frac{1}{2}}$$

since $|n-1| = 1-n \geq 1-n_{max} = |n_{max}-1|$. Hence we have

$$\int_D \left[(A-I)\nabla\tilde{v}\cdot\nabla\overline{\tilde{v}} - \hat{\tau}\,(n-1)\,|\tilde{v}|^2\right]dx$$

$$\leq \int_{B_R} \left[(a_{min}-1)\,|\nabla\hat{v}|^2 - \hat{\tau}\,(n_{max}-1)\,|\hat{v}|^2\right]dx\,.$$

Substituting this into (6.113) for $\tau = \hat{\tau}$ and $u = \tilde{u}$ yields

$$\left(\mathbb{L}_{\hat{\tau}}\tilde{u},\tilde{u}\right)_{H^1(D)} = \int_D \left[(A-I)\nabla\tilde{v}\cdot\nabla\overline{\tilde{v}} - \hat{\tau}\,(n-1)\,|\tilde{v}|^2 + |\nabla\tilde{u}|^2 - \hat{\tau}\,|\tilde{u}|^2\right]dx$$

$$\leq \int_{B_R} \left[(a_{min}-1)|\nabla\hat{v}|^2 - \hat{\tau}\,(n_{max}-1)\,|\hat{v}|^2 + |\nabla\hat{u}|^2 - \hat{\tau}\,|\hat{u}|^2\right]dx \;=\; 0$$

by (6.114). Hence from Theorem 6.34 we have that there is a transmission eigenvalue $k>0$ such that in $k^2 \in (0,\hat{\tau}]$. Finally, repeating this argument for disks of arbitrarily small radius we can show the existence of infinitely many transmission eigenvalues exactly in the same way as in the proof of Theorem 6.18. In a similar way we can prove the same result for the case where $0 < a_{max} < 1$ and $n_{min} > 1$. □

From the preceding analysis it is possible to obtain bounds for the first transmission eigenvalue stated in the following theorem (here we omit the proof and refer the reader to [35]).

Theorem 6.36. *Let $B_R \subset D$ be the largest disk contained in D and $\lambda_0(D)$ the first Dirichlet eigenvalue of $-\Delta$ in D. Furthermore, let $k_0(A, n, D)$ be the first transmission eigenvalue corresponding to (6.91)–(6.94).*

1. If $a_{min} > 1$ and $0 < n_{max} < 1$ then

$$\lambda_0(D) \;\le\; k_0^2(A, n, D) \;\le\; k_0^2(a_{min}, n_{max}, B_R).$$

2. If $0 < a_{max} < 1$ and $n_{min} > 1$, then

$$\frac{a_{min}}{n_{max}}\,\lambda_0(D) \;\le\; k_0^2(A, n, D) \;\le\; k_0^2(a_{max}, n_{min}, B_R).$$

For other estimates of the same type we refer the reader to [9].

We end our discussion in this section with a few comments on the case where $(A-I)$ and $(n-1)$ have the same sign. As indicated earlier, if we follow a similar procedure, then we are faced with the problem that (6.105) is not solvable for all τ. For this reason it is only possible to prove the existence of a finite number of transmission eigenvalues under the restrictive assumption that $n_{max} - 1$ is small enough (for more details we refer the reader to [35]).

In a series of interesting papers [118, 119] and [120] Lakshtanov and Vainberg introduced an alternative approach to showing the discreteness and existence of transmission eigenvalues as well as initiating a studying of the counting function for transmission eigenvalues.

6.4 Uniqueness

The proof of uniqueness for the inverse medium scattering problem is more complicated than for the case of scattering by an imperfect conductor considered in Chap. 4. The idea of the uniqueness proof for the inverse medium scattering problem originates from [93,94], in which it is shown that the shape of a penetrable, inhomogeneous, isotropic medium is uniquely determined by its far-field pattern for all incident plane waves. The case of an orthotropic medium is due to Hähner [81] (see also [57]), the proof of which is based on the existence of a solution to the modified interior transmission problem. We begin with a simple lemma.

Lemma 6.37. *Assume that either $\bar{\xi} \cdot Re(A)\, \xi \ge \gamma |\xi|^2$ or $\bar{\xi} \cdot Re(A^{-1})\, \xi \ge \gamma |\xi|^2$ for some $\gamma > 1$. Let $\{v_n, w_n\} \in H^1(D) \times H^1(D)$, $n \in \mathbb{N}$, be a sequence of solutions to the interior transmission problem (6.12)–(6.15) with boundary data $f_n \in H^{\frac{1}{2}}(\partial D)$, $h_n \in H^{-\frac{1}{2}}(\partial D)$. If the sequences $\{f_n\}$ and $\{h_n\}$ converge in $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$ respectively, and if the sequences $\{v_n\}$ and $\{w_n\}$ are bounded in $H^1(D)$, then there exists a subsequence $\{w_{n_k}\}$ that converges in $H^1(D)$.*

Proof. Assume first that $\bar{\xi} \cdot \mathrm{Re}(A)\,\xi \geq \gamma|\xi|^2$, $\gamma > 1$, and let $\{v_n, w_n\}$ be as in the statement of the lemma. Due to the compact embedding of $H^1(D)$ into $L^2(D)$, we can select L^2-convergent subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$. Hence, $\{v_{n_k}\}$ and $\{w_{n_k}\}$ satisfy

$$\begin{aligned}
\nabla \cdot A\nabla v_{n_k} - \gamma v_{n_k} &= -(\gamma + k^2 n) v_{n_k} && \text{in} \quad D,\\
\Delta w_{n_k} - w_{n_k} &= -(1 + k^2) w_{n_k} && \text{in} \quad D,\\
v_{n_k} - w_{n_k} &= f_{n_k} && \text{on} \quad \partial D,\\
\frac{\partial v_{n_k}}{\partial \nu_A} - \frac{\partial w_{n_k}}{\partial \nu} &= h_{n_k} && \text{on} \quad \partial D.
\end{aligned}$$

Then the result of the lemma follows from the a priori estimate of Theorem 6.7. In the case where $\bar{\xi} \cdot \mathrm{Re}(A^{-1})\,\xi \geq \gamma|\xi|^2$, $\gamma > 1$, we use Theorem 6.9 and $1/\gamma$ instead of γ in the preceding equation for v_{n_k} to obtain the same result. □

Note that in the proof of Lemma 6.37 we use the a priori estimate for the modified interior transmission problem instead of the a priori estimate for the interior transmission problem. This allows us to obtain the result without assuming that k is not a transmission eigenvalue.

We can prove a result similar to that in Lemma 6.37 under different assumptions about the physical properties of the medium. In particular, assuming that $\mathrm{Im}(A) = 0$ and $\mathrm{Im}(n) = 0$ we recall definition (6.95) of a_*, a^*, n_*, and n^*.

Lemma 6.38. *Assume that either* $0 < a^* < 1$ *and* $0 < n^* < 1$*, or* $a_* > 1$ *and* $n_* > 1$*. Let* $\{v_n, w_n\} \in H^1(D) \times H^1(D)$*,* $n \in \mathbb{N}$*, be a sequence of solutions to the interior transmission problem (6.12)–(6.15) with boundary data* $f_n \in H^{\frac{1}{2}}(\partial D)$*,* $h_n \in H^{-\frac{1}{2}}(\partial D)$*. If the sequences* $\{f_n\}$ *and* $\{h_n\}$ *converge in* $H^{\frac{1}{2}}(\partial D)$ *and* $H^{-\frac{1}{2}}(\partial D)$*, respectively, and if the sequences* $\{v_n\}$ *and* $\{w_n\}$ *are bounded in* $H^1(D)$*, then there exists a subsequence* $\{w_{n_k}\}$ *that converges in* $H^1(D)$*.*

Proof. Similarly to the proof of Lemma 6.37, let $\{v_n, w_n\}$ be as in the statement of the lemma. Due to the compact embedding of $H^1(D)$ into $L^2(D)$, we can select L^2-convergent subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$. Hence, $\{v_{n_k}\}$ and $\{w_{n_k}\}$ satisfy

$$\begin{aligned}
\nabla \cdot A\nabla v_{n_k} - \kappa^2 n v_{n_k} &= (\kappa^2 - k^2) n v_{n_k} && \text{in} \quad D,\\
\Delta w_{n_k} - \kappa^2\, w_{n_k} &= (\kappa^2 - k^2) w_{n_k} && \text{in} \quad D,\\
v_{n_k} - w_{n_k} &= f_{n_k} && \text{on} \quad \partial D,\\
\frac{\partial v_{n_k}}{\partial \nu_A} - \frac{\partial w_{n_k}}{\partial \nu} &= h_{n_k} && \text{on} \quad \partial D,
\end{aligned}$$

where $\kappa > 0$ is chosen as in Lemma 6.27 (i.e., for $k := i\kappa$ the interior transmission problem is invertible). Then the result of the lemma follows from the boundedness of the inverse of the operator equivalent to the interior transmission problem for $k := i\kappa$ (Remark 6.29). □

We are now ready to prove the uniqueness theorem.

Theorem 6.39. *Let the domains D_1 and D_2, the matrix-valued functions A_1 and A_2, and the functions n_1 and n_2 satisfy the assumptions in Sect. 5.2 and the assumptions of either Lemma 6.37 or Lemma 6.38. If the far-field patterns $u^1_\infty(\theta, \phi)$ and $u^2_\infty(\theta, \phi)$ corresponding to D_1, A_1, n_1 and D_2, A_2, n_2, respectively, coincide for all $\theta \in [0, 2\pi]$ and $\phi \in [0, 2\pi]$, then $D_1 = D_2$.*

Proof. Denote by G the unbounded connected component of $\mathbb{R}^2 \setminus (\bar{D}_1 \cup \bar{D}_2)$, and define $D^e_1 := \mathbb{R}^2 \setminus \bar{D}_1$, $D^e_2 := \mathbb{R}^2 \setminus \bar{D}_2$. By Rellich's lemma, we conclude that the scattered fields u_1 and u_2, which are the radiating part of the solution to (5.13)–(5.17) with D_1, A_1, n_1 and D_2, A_2, n_2, respectively, and boundary data with $f := e^{ikx\cdot d}$ and $h := \partial e^{ikx\cdot d}/\partial\nu$, $d = (\cos\phi, \sin\phi)$, coincide in G. Let $\Phi(x, z)$ denote the fundamental solution to the Helmholtz equation given by (3.33).

We now show that the scattered solutions $u_1(\cdot, z)$ and $u_2(\cdot, z)$ also coincide for the incident waves $\Phi(\cdot, z)$ with $z \in G$, i.e., for $f := \Phi(\cdot, z)$ and $h := \partial\Phi(\cdot, z)/\partial\nu$. To this end, choose a large disk Ω_R such that $\bar{D}_1 \cup \bar{D}_2 \subset \Omega_R$ and k^2 is not a Dirichlet eigenvalue for Ω_R. Then, for $z \notin \bar{\Omega}_R$, by Lemma 4.4, there exists a sequence $\{u^i_n\}$ in $\text{span}\{e^{ikx\cdot d} : |d| = 1\}$ such that

$$\|u^i_n - \Phi(\cdot, z)\|_{H^{\frac{1}{2}}(\partial\Omega_R)} \to 0, \quad \text{as } n \to \infty.$$

The well-posedness of the Dirichlet problem for the Helmholtz equation in Ω_R (Example 5.15) implies that u^i_n approximates $\Phi(\cdot, z)$ in $H^1(\Omega_R)$. Then the continuous dependence on the data of the scattered field (5.41), together with the fact that the scattered fields corresponding to u^i_n coincide as linear combinations of scattered fields due to plane waves, implies that $u_1(\cdot, z)$ and $u_2(\cdot, z)$ also coincide for a fixed $z \notin \bar{\Omega}_R$. Since $\Phi(\cdot, z)$ and its derivatives are real-analytic in z, we can again conclude from the well-posedness of the transmission problem (5.13)–(5.17) that $u_1(\cdot, z)$ and $u_2(\cdot, z)$ are real-analytic in z and therefore must coincide for all $z \in G$.

Let us now assume that $\bar{D}_1$ is not included in $\bar{D}_2$. Since D^e_2 is connected, we can find a point $z \in \partial D_1$ and $\epsilon > 0$ with the following properties, where $\Omega_\delta(z)$ denotes the ball of radius δ centered at z:

1. $\Omega_{8\epsilon}(z) \cap \bar{D}_2 = \emptyset$;
2. The intersection $\bar{D}_1 \cap \Omega_{8\epsilon}(z)$ is contained in the connected component of $\bar{D}_1$ to which z belongs;
3. There are points from this connected component of $\bar{D}_1$ to which z belongs that are not contained in $\bar{D}_1 \cap \bar{\Omega}_{8\epsilon}(z)$;
4. The points $z_n := z + \dfrac{\epsilon}{n}\nu(z)$ lie in G for all $n \in \mathbb{N}$, where $\nu(z)$ is the unit normal to ∂D_1 at z.

Due to the singular behavior of $\Phi(\cdot, z_n)$ at the point z_n, it is easy to show that $\|\Phi(\cdot, z_n)\|_{H^1(D_1)} \to \infty$ as $n \to \infty$. We now define

$$w^n(x) := \frac{1}{\|\Phi(\cdot, z_n)\|_{H^1(D_1)}} \Phi(x, z_n), \qquad x \in \bar{D}_1 \cup \bar{D}_2$$

and let v_1^n, u_1^n and v_2^n, u_2^n be the solutions of the scattering problem (5.13)–(5.17) with boundary data $f := w^n$ and $h := \partial w^n / \partial \nu$ corresponding to D_1 and D_2, respectively. Note that for each n, w^n is a solution of the Helmholtz equation in D_1 and D_2. Our aim is to prove that if $\bar{D}_1 \not\subset \bar{D}_2$, then the equality $u_1(\cdot, z) = u_2(\cdot, z)$ for $z \in G$ allows the selection of a subsequence $\{w^{n_k}\}$ from $\{w^n\}$ that converges to zero with respect to $H^1(D_1)$. This certainly contradicts the definition of $\{w^n\}$ as a sequence of functions with $H^1(D_1)$ norm equal to one. Note that $u_1(\cdot, z) = u_2(\cdot, z)$ obviously implies that $u_1^n = u_2^n$ in G.

We begin by noting that, since the functions $\Phi(\cdot, z_n)$ together with their derivatives are uniformly bounded in every compact subset of $\mathbb{R}^2 \setminus \Omega_{2\epsilon}(z)$ and $\|\Phi(\cdot, z_n)\|_{H^1(D_1)} \to \infty$ as $n \to \infty$, then $\|w^n\|_{H^1(D_2)} \to 0$ as $n \to \infty$. Hence, if Ω_R is a large ball containing $\bar{D}_1 \cup \bar{D}_2$, then $\|u_2^n\|_{H^1(\Omega_R \cap G)} \to 0$ as $n \to \infty$ from the a priori estimate (5.41). Since $u_1^n = u_2^n$ in G, then $\|u_1^n\|_{H^1(\Omega_R \cap G)} \to 0$ as $n \to \infty$ as well. Now, with the help of a cutoff function $\chi \in C_0^\infty(\Omega_{8\epsilon}(z))$ satisfying $\chi(x) = 1$ in $\Omega_{7\epsilon}(z)$ (Theorem 5.6), we see that $\|u_1^n\|_{H^1(\Omega_R \cap G)} \to 0$ implies that

$$(\chi u_1^n) \to 0, \quad \frac{\partial(\chi u_1^n)}{\partial \nu} \to 0 \qquad \text{as } n \to \infty \tag{6.115}$$

with respect to the $H^{\frac{1}{2}}(\partial D_1)$ norm and $H^{-\frac{1}{2}}(\partial D_1)$ norm, respectively. Indeed, for the first convergence we simply apply the trace theorem, while for the convergence of $\partial(\chi u_1^n)/\partial \nu$ we first deduce the convergence of $\Delta(\chi u_1^n)$ in $L^2(\Omega_R \cap D_1^e)$, which follows from $\Delta(\chi u_1^n) = \chi \Delta u_1^n + 2\nabla \chi \cdot \nabla u_1^n + u_1^n \Delta \chi$, and then apply Theorem 5.7. Note here that we need conditions 2 and 4 on z to ensure $\Omega_{8\epsilon}(z) \cap D_1^e = \Omega_{8\epsilon}(z) \cap G$.

We next note that in the exterior of $\Omega_{2\epsilon}(z)$ the $H^2(\Omega_R \setminus \Omega_{2\epsilon}(z))$ norms of w^n remain uniformly bounded. Then the assertion about the boundary regularity of the solution to (5.13)–(5.17) stated in the second part of Theorem 5.28 implies that u_1^n is uniformly bounded with respect to the $H^2((\Omega_R \cap D_1^e) \setminus \Omega_{4\epsilon}(z))$ norm. Therefore, using the compact embedding of $H^2(\Omega_R \cap D_1^e)$ into $H^1(\Omega_R \cap D_1^e)$, we can select a $H^1(\Omega_R \cap D_1^e)$ convergent subsequence $\{(1-\chi)u_1^{n_k}\}$ from $\{(1-\chi)u_1^n\}$. Hence, $\{(1-\chi)u_1^{n_k}\}$ is a convergent sequence in $H^{\frac{1}{2}}(\partial D_1)$, and, similarly to the foregoing reasoning, we also have that $\{\partial((1-\chi)u_1^{n_k})/\partial \nu\}$ converges in $H^{-\frac{1}{2}}(\partial D_1)$. This, together with (6.115), implies that the sequences

$$\{u_1^{n_k}\} \qquad \text{and} \qquad \left\{ \frac{\partial u_1^{n_k}}{\partial \nu} \right\}$$

converge in $H^{\frac{1}{2}}(\partial D_1)$ and $H^{-\frac{1}{2}}(\partial D_1)$, respectively.

Finally, since the functions $v_1^{n_k}$ and w^{n_k} are solutions to the interior transmissionproblem (6.12)–(6.15) for the domain D_1 with boundary data $f = u_1^{n_k}$ and $h = \partial u_1^{n_k}/\partial\nu$, and since the $H^1(D_1)$ norms of $v_1^{n_k}$ and w^{n_k} remain uniformly bounded, then, according to Lemma 6.37, we can select a subsequence of $\{w^{n_k}\}$, denoted again by $\{w^{n_k}\}$, that converges in $H^1(D_1)$ to a function $w \in H^1(D_1)$. As a limit of weak solutions to the Helmholtz equation, $w \in H^1(D_1)$ is a weak solution to the Helmholtz equation. We also have that $w|_{D_1\setminus\Omega_{2\epsilon}(z)} = 0$ because the functions w^{n_k} converge uniformly to zero in the exterior of $\Omega_{2\epsilon}(z)$. Hence, w must be zero in all of D_1 [here we make use of condition 3, namely, the fact that the connected component of D_1 containing z has points that do not lie in the exterior of $\bar{\Omega}_{2\epsilon}(z)$]. This contradicts the fact that $\|w^{n_k}\|_{H^1(D_1)} = 1$. Hence the assumption $\bar{D}_1 \not\subset \bar{D}_2$ is false.

Since we can derive an analogous contradiction for the assumption $\bar{D}_2 \not\subset \bar{D}_1$, we have proved that $D_1 = D_2$. □

Remark 6.40. We remark that the proof of the uniqueness of the support of an anisotropic media presented in Theorem 6.39 is valid as long as the material properties A and n guaranty that the corresponding interior transmission problem is a compact perturbation of a well-posed problem.

6.5 Linear Sampling Method

Having shown that the support of an inhomogeneity can be uniquely determined from the far-field pattern, we now want to find an approximation to the support. To this end, we will use the linear sampling method previously introduced in Chap. 4 for the inverse scattering problem for an imperfect conductor. In particular, we shall show that, provided k is not a transmission eigenvalue, the boundary ∂D of the inhomogeneity D can be characterized by the solution of the far-field equation (4.33), where the kernel of the far-field operator is the far-field pattern corresponding to (6.1)–(6.5).

Given $(f,h) \in H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$, let $(v,u) \in H^1(D) \times H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ be the unique solution to the corresponding transmission problem (5.13)–(5.17). We recall that the radiating part u has the asymptotic behavior

$$u(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}) + O(r^{-3/2}), \qquad r \to \infty, \quad \hat{x} = x/|x|,$$

where u_∞ is the far-field pattern corresponding to (v,u).

Definition 6.41. The bounded linear operator $B : H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to L^2[0,\, 2\pi]$ maps $(f,h) \in H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ onto the far-field pattern $u_\infty \in L^2[0,\, 2\pi]$, where (v,u) is the solution of (5.13)–(5.17) with the boundary data (f,h).

Note that the fact that B is bounded follows directly from the well-posedness of (5.13)–(5.17).

As in the case of the scattering problem for an imperfect conductor, the operator B will play an important role in the solution of the inverse problem. To determine the range of the operator B, it is more convenient to consider its transpose instead of its adjoint. This is because operating with the duality relation between $H^{\frac{1}{2}}(\partial D)$, $H^{-\frac{1}{2}}(\partial D)$ is much simpler than using the corresponding inner products. In what follows we will define the transpose operator and derive some useful properties of this operator.

Let X and Y be two Hilbert spaces, and let X^* and Y^* be their dual spaces. For any linear mapping $A : X \to Y$, the *transpose* $A^\top : Y^* \to X^*$ is the linear mapping defined by

$$\left\langle A^\top v, u \right\rangle_{X,X^*} = \langle v, Au \rangle_{Y,Y^*}, \qquad \text{for all } u \in X \text{ and } v \in Y^*,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the denoted spaces.

It can be shown (see Lemma 2.9 in [127]) that the transpose $A^\top$ is bounded if and only if A is bounded. To describe the relation between the range and the kernel of A and $A^\top$, we use the following terminology. For any subset $W \subseteq X$, the *annihilator* W^a is the closed subspace of X^* defined by

$$W^a = \{g \in X^* : \ \langle g, u \rangle = 0 \text{ for all } u \in W\}.$$

Similarly, for $V \subseteq X^*$ the annihilator ${}^aV$ is the closed subspace of X defined by

$${}^aV = \{u \in X : \ \langle g, u \rangle = 0 \text{ for all } g \in V\}.$$

Lemma 6.42. *The null space and range of A and $A^\top$ satisfy*

$$\mathrm{N}(A^\top) = A(X)^a \quad \textit{and} \quad \mathrm{N}(A) = {}^aA^\top(Y^*).$$

Proof. Applying the various definitions we obtain

$$\begin{aligned} A(X)^a &= \{g \in Y^* : \ \langle g, v \rangle = 0 \ \text{ for all } \ v \in \operatorname{range} A\} \\ &= \{g \in Y^* : \ \langle g, Au \rangle = 0 \ \text{ for all } \ u \in X\} \\ &= \{g \in Y^* : \ \left\langle A^\top g, u \right\rangle = 0 \ \text{ for all } \ u \in X\} \\ &= \{g \in Y^* : \ A^\top g = 0\} = \mathrm{N}(A^\top). \end{aligned}$$

A similar argument shows that $\mathrm{N}(A) = {}^aA^\top(Y^*)$. □

It is an easy exercise using the Hahn–Banach theorem [115] to show that a subset $W \subseteq X$ is dense if and only if $W^a = \{0\}$. In particular, from Lemma 6.42 we have the following corollary.

Corollary 6.43. *The operator A has a dense range if and only if the transpose $A^\top$ is injective.*

With the help of the preceding lemma and corollary we can now prove the following result for the operator B.

Theorem 6.44. *The range of* $B : H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to L^2[0,\, 2\pi]$ *is dense in* $L^2[0,\, 2\pi]$.

Proof. We consider the dual operator $B^\top : L^2[0,\, 2\pi] \longrightarrow H^{-\frac{1}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$, which maps a function g into $(\tilde{f}, \tilde{h})$ such that

$$\langle B(f,h), g\rangle_{L^2\times L^2} = \left\langle f, \tilde{f}\right\rangle_{H^{\frac{1}{2}}\times H^{-\frac{1}{2}}} + \left\langle h, \tilde{h}\right\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}},$$

where $\langle \cdot,\, \cdot\rangle$ denotes the duality pairing between the denoted spaces. Now let $(\tilde{v}, \tilde{u})$ be the unique solution of (5.13)–(5.17) with $(f,h) := (\tilde{v}_g|_{\partial D}, \partial \tilde{v}_g/\partial\nu|_{\partial D})$, where $\tilde{v}_g$ is the Herglotz wave function defined by (6.9). Then from (6.6) we have

$$\langle B(f,h), g\rangle = \int_0^{2\pi} u_\infty(\theta) g(\theta)\, d\theta = \int_{\partial D} \left(u(y)\frac{\partial \tilde{v}_g(y)}{\partial\nu} - \tilde{v}_g(y)\frac{\partial u(y)}{\partial \nu}\right) ds(y).$$

Since u and $\tilde{u}$ are solutions of the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$ satisfying the Sommerfeld radiation condition, an application of Green's second identity implies that

$$\int_{\partial D} \left[u(y)\frac{\partial \tilde{u}(y)}{\partial\nu} - \tilde{u}(y)\frac{\partial u(y)}{\partial \nu}\right] ds(y) = 0.$$

Using the transmission conditions on the boundary for $\tilde{u}$ and $\tilde{v}$ we obtain

$$\begin{aligned}
&\langle B(f,h), g\rangle_{L^2\times L^2} = \\
&= \int_{\partial D} \left[u(y)\left(\frac{\partial \tilde{v}_g(y)}{\partial\nu} + \frac{\partial \tilde{u}(y)}{\partial\nu}\right) - (\tilde{v}_g(y) + \tilde{u}(y))\,\frac{\partial u(y)}{\partial \nu}\right] ds(y) \\
&= \int_{\partial D} \left(u(y)\frac{\partial \tilde{v}(y)}{\partial\nu_A} - \tilde{v}(y)\frac{\partial u(y)}{\partial \nu}\right) ds(y) \\
&= \int_{\partial D} \left[(v(y) - f(y))\,\frac{\partial \tilde{v}(y)}{\partial\nu_A} - \tilde{v}(y)\left(\frac{\partial v(y)}{\partial \nu_A} - h(y)\right)\right] ds(y).
\end{aligned}$$

Finally, applying Green's (generalized) second identity to v and $\tilde{v}$ we have that

$$\langle B(f,h), g\rangle_{L^2\times L^2} = \int_{\partial D} \left[f(y)\left(-\frac{\partial \tilde{v}(y)}{\partial\nu_A}\right) + \tilde{v}(y)h(y)\right] ds(y).$$

Hence the dual operator $B^\top$ can be characterized as

$$B^\top g = \left(-\left.\frac{\partial \tilde{v}}{\partial\nu_A}\right|_{\partial D}, \tilde{v}|_{\partial D}\right).$$

In what follows we want to show that the operator $B^\top$ is injective. To this end, let $B^\top g \equiv 0$, $g \in L^2[0, 2\pi]$. This implies that $\tilde{v} = 0$ and $\partial\tilde{v}/\partial\nu_A = 0$ on the boundary ∂D. Therefore, $\tilde{u}$ satisfies the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$, the Sommerfeld radiation condition, and, from the transmission conditions,

$$\tilde{u} = -\tilde{v}_g \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial \nu} = -\frac{\partial \tilde{v}_g}{\partial \nu} \qquad \text{on } \partial D.$$

Thus, setting $\tilde{u} \equiv -\tilde{v}_g$ in D we have that $\tilde{u}$ can be extended to an entire solution to the Helmholtz equation satisfying the radiation condition. This is only possible if $\tilde{u}$ vanishes, which implies that $\tilde{v}_g$ vanishes also and, thus, $g \equiv 0$, whence $B^\top$ is injective. Finally, from Corollary 6.43 we have that the range of B is dense in $L^2[0, 2\pi]$. □

From Lemma 6.42 we also have that

$$\mathrm{N}(B) = B^\top(L^2[0, 2\pi])^a := \left\{ (f_0, h_0) : \int_{\partial D} \left(-f_0 \frac{\partial \tilde{v}}{\partial \nu_A} + h_0 \tilde{v} \right) ds = 0 \right\},$$

where $\tilde{v}$ is as in the proof of Theorem 6.44. Hence, using the divergence theorem, we see that the pairs $(v|_{\partial D}, \partial v/\partial \nu_A|_{\partial D})$, where $v \in H^1(D)$ is a solution of $\nabla \cdot A\nabla v + k^2 n\, v = 0$ in D, are in the kernel of B. So B is not injective. We will restrict the operator B in such a way that the restriction is injective and still has a dense range.

To this end, let us denote by $\overline{H}$ the closure in $H^1(D)$ of all Herglotz wave functions with kernel $g \in L^2[0, 2\pi]$. Note that the space $\overline{H}$ coincides with the space of H^1 weak solutions to the Helmholtz equation. In other words, $\overline{H} = \overline{W(D)}$, where $\overline{W(D)}$ is the closure in $H^1(D)$ of $W(D)$ defined by

$$W(D) := \{u \in C^2(D) \cap C^1(\overline{D}) : \Delta u + k^2\, u = 0\}.$$

Indeed, if $u \in \overline{W(D)}$, then by seeing u as a weak solution of the interior impedance boundary value problem for the Helmholtz equation in D with $\lambda = 1$ we have from Theorem 8.4 in Chap. 8 (set $\partial D_D = \emptyset$) that there exists a positive constant C such that

$$\|u\|_{H^1(D)} \le C \left\| \frac{\partial u}{\partial \nu} + iu \right\|_{H^{-\frac{1}{2}}(\partial D)}.$$

Then the proof of Theorem 4.10 implies that for any $\epsilon > 0$ there exists a Herglotz wave function v_g such that $\|u - v_g\|_{H^1(D)} < \epsilon$, whence $\overline{H} = \overline{W(D)}$. For later use we state this result in the following lemma.

Lemma 6.45. *Any solution to the Helmhotz equation in a bounded domain $D \subset \mathbb{R}^2$ can be approximated in the $H^1(D)$ norm by a Herglotz wave function.*

Next, we define

$$H(\partial D) := \left\{ \left(u|_{\partial D},\, \frac{\partial u}{\partial \nu}\bigg|_{\partial D} \right) : u \in \overline{H} \right\}.$$

Lemma 6.46. *$H(\partial D)$ is a closed subset of $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$.*

Proof. Consider $(f, h) \in \overline{H(\partial D)}$. There exists a sequence $\{u_n, \partial u_n/\partial \nu\}$ converging to (f, h) in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$, where $u_n \in \overline{H}$. Since the sequence $\{u_n, \partial u_n/\partial \nu\}$ is bounded in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$, by considering u_n to be the solution of an impedance boundary value problem in D we can deduce that $\{u_n\}$ is bounded in $H^1(D)$. From this it follows that a subsequence (still denoted by $\{u_n\}$) converges weakly in $H^1(D)$ to a function u that is clearly in $\overline{H}$. From the continuity of the trace operators (Theorems 1.38 and 5.7) we deduce that $\{u_n, \partial u_n/\partial \nu\}$ converges weakly in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ to $(u, \partial u/\partial \nu)$ and by the uniqueness of the limit $(f, h) = (u, \partial u/\partial \nu)$. Hence $(f, h) \in H(\partial D)$, which completes the proof. □

From the preceding lemma, $H(\partial D)$ equipped with the induced norm from $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ is a Banach space.

Now, let B_0 denote the restriction of B to $H(\partial D)$.

Theorem 6.47. *Assume that k is not a transmission eigenvalue. Then the bounded linear operator $B_0 : H(\partial D) \longrightarrow L^2[0, 2\pi]$ is injective and has a dense range.*

Proof. Let $B_0(f, h) = 0$ for $(f, h) \in H(\partial D)$, and let (v, u) be the solution to (5.13)–(5.17) corresponding to these boundary data. Then the radiating solution to the Helmholtz equation in the exterior of D has a zero far-field pattern, whence $u = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$. This implies that v satisfies

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \quad \text{in} \quad D, \qquad v = f \quad \text{and} \quad \frac{\partial v}{\partial \nu} = h \quad \text{on} \quad \partial D.$$

From the definition of $H(\partial D)$, f and h are the traces on ∂D of a $H^1(D)$ solution w to the Helmholtz equation and its normal derivative, respectively. Therefore, (v, w) solves the homogeneous interior transmission problem (6.12)–(6.15), and since k is not a transmission eigenvalue, we have that $w \equiv 0$ and $v \equiv 0$ in D, whence $f = h = 0$.

It remains to show that the set $B_0(H(\partial D))$ is dense in $L^2[0, 2\pi]$. To this end, it is sufficient to show that the range of B is contained in the range of B_0 since from Theorem 6.44 the range of B is dense in $L^2[0, 2\pi]$. Let u_∞ be in the range of B, that is, u_∞ is the far-field pattern of the radiating part u of a solution (v, u) to (5.13)–(5.17). Let (v, w) be the unique solution to (6.12)–(6.15) with the boundary data $(u|_{\partial D}, \partial u/\partial \nu|_{\partial D})$. Hence (v, u) is the solution to (5.13)–(5.17) with boundary data $(w|_{\partial D}, \partial w/\partial \nu|_{\partial D}) \in H(\partial D)$ and has a far-field pattern coinciding with u_∞. This means that $B_0\,(w|_{\partial D}, \partial w/\partial \nu|_{\partial D}) = u_\infty$. □

Theorem 6.48. *The operator* $B_0 : H(\partial D) \longrightarrow L^2[0,\, 2\pi]$ *is compact.*

Proof. Given $w \in \overline{H}$, consider the solution (v, u) of (5.13)–(5.17) with boundary data $f := w|_{\partial D}$ and $h := \partial w/\partial \nu|_{\partial D}$. Let $\partial\Omega_R$ be the boundary of a disk Ω_R centered at the origin containing $\bar{D}$. The continuous dependence estimate (5.41) implies that the operator $G : H(\partial D) \to H^{\frac{1}{2}}(\partial\Omega_R) \times H^{-\frac{1}{2}}(\partial\Omega_R)$, which maps

$$\left(w|_{\partial D},\ \frac{\partial w}{\partial \nu}\bigg|_{\partial D} \right) \to \left(u|_{\partial\Omega_R},\ \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega_R} \right),$$

is bounded. Next we denote by $K : H^{\frac{1}{2}}(\partial\Omega_R) \times H^{-\frac{1}{2}}(\partial\Omega_R) \to L^2[0,\, 2\pi]$ the operator that takes $(u|_{\partial\Omega_R},\, \partial u/\partial\nu|_{\partial\Omega_R})$ to u_∞ given by

$$u_\infty(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int\limits_{\partial B} \left(u(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu} - e^{-ik\hat{x}\cdot y} \frac{\partial u(y)}{\partial \nu} \right) ds(y)$$

where $\hat{x} = x/|x|$. An argument similar to that in the proof of Theorem 4.8 shows that K is compact. Therefore, $B_0 = KG$ is compact since it is a composition of a bounded operator with a compact operator. □

For a Herglotz wave function v_g given by (6.8) with kernel $g \in L^2[0,\, 2\pi]$ we define $H : L^2[0,\, 2\pi] \to H(\partial D)$ by

$$Hg := \left(v_g|_{\partial D},\ \frac{\partial v_g}{\partial \nu}\bigg|_{\partial D} \right).$$

Corollary 6.49. *Assume that* $u_\infty \in L^2[0,\, 2\pi]$ *is in the range of* B_0. *Then for every* $\epsilon > 0$ *there exists a* $g_\epsilon \in L^2[0,\, 2\pi]$ *such that* Hg_ϵ *satisfies*

$$\|B_0(Hg_\epsilon) - u_\infty\|_{L^2[0,\, 2\pi]} \le \epsilon.$$

Proof. The proof is a straightforward application of the definition of the space $H(\partial D)$, the continuity of the trace operator, and the operator B_0, together with Lemma 6.45. □

Turning to our main goal of finding an approximation to the scattering obstacle D we consider the *far-field equation* corresponding to the scattering by an orthotropic medium given by

$$\int\limits_0^{2\pi} u_\infty(\theta, \phi) g(\phi) d\phi = \gamma e^{-ik\hat{x}\cdot z}, \qquad z \in \mathbb{R}^2, \tag{6.116}$$

where $u_\infty(\theta, \phi)$ is the far-field pattern of the radiating part of the solution to the forward problem (6.1)–(6.5) corresponding to the incident plane wave with incident direction $d = (\cos\phi,\, \sin\phi)$ and observation direction

$\hat{x} = (\cos\theta, \sin\theta)$. As in Chap. 4 the far-field equation can be written in the form

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z), \qquad z \in \mathbb{R}^2,$$

where Fg is the far-field operator corresponding to the transmission problem (6.1)–(6.5), and $\Phi_\infty(\hat{x}, z)$ is the far-field pattern of the fundamental solution $\Phi(x, z)$ to the Helmholtz equation in $\mathbb{R}^2$. We observe that the far-field operator Fg can be factored as

$$Fg = B_0(Hg).$$

Hence the far-field equation takes the form

$$\left(B_0(Hg)\right)(\hat{x}) = \Phi_\infty(\hat{x}, z), \qquad z \in \mathbb{R}^2. \tag{6.117}$$

As the reader has already encountered in the case of scattering by an imperfect conductor, the *linear sampling method* is based on the characterization of the domain D by the behavior of a solution to the far-field equation (6.117). By definition, $B_0(Hg)$ is the far-field pattern of the solution (v, u) to the transmission problem (5.13)–(5.17) with boundary data $(f, h) := Hg$. Therefore, for $z \in D$, from Rellich's lemma the far-field equation implies that this u coincides with $\Phi(\cdot, z)$ in $\mathbb{R}^2 \setminus \bar{D}$. In other words, for $z \in D$, $g \in L^2[0, 2\pi]$ is a solution to the far-field equation if and only if v and $w := v_g$ solve the interior transmission problem

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D, \tag{6.118}$$

$$\Delta w + k^2\, w = 0 \qquad \text{in} \quad D, \tag{6.119}$$

$$v - w = \Phi(\cdot, z) \qquad \text{on} \quad \partial D, \tag{6.120}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \qquad \text{on} \quad \partial D, \tag{6.121}$$

where v_g is the Herglotz wave function with kernel g. In general, this is not true. However, in what follows, we will show that one can construct an approximate solution to the far-field equation that behaves in a certain manner.

We first assume that $z \in D$ and that k is not a transmission eigenvalue. Then the interior transmission problem (6.118)–(6.121) has a unique solution (v, w). In this case $(v, \Phi(\cdot, z))$ solves the transmission problem (5.13)–(5.17) with transmission conditions $f := w|_{\partial D}, h := \partial w/\partial\nu|_{\partial D}$. Since the preceding solution has the far-field pattern $\Phi_\infty(\cdot, z)$, we can conclude that $\Phi_\infty(\cdot, z)$ is in the range of B_0. From Corollary 6.49 we can find a g_z^ϵ such that

$$\|B_0(Hg_z^\epsilon) - \Phi_\infty(\cdot, z)\|_{L^2[0,\, 2\pi]} < \epsilon \tag{6.122}$$

for an arbitrarily small ϵ. From the construction of B_0 and Corollary 6.49 we see that the corresponding Herglotz wave function $v_{g_z^\epsilon}$ approximates w in the $H^1(D)$ norm as $\epsilon \to 0$. Furthermore, for a fixed $\epsilon > 0$, the $H^1(D)$ norm $v_{g_z^\epsilon}$

blows up if z approaches the boundary from the interior of D, as does the $L^2[0,\,2\pi]$ norm of g_z^ϵ. To see this, we choose a sequence of points $\{z_j\}$, $z_j \in D$, such that

$$z_j = z^* - \frac{R}{j}\nu(z^*), \qquad j = 1, 2, \ldots,$$

with sufficiently small R, where $z^* \in \partial D$ and $\nu(z^*)$ is the unit outward normal at z^*. We denote by (v_j, w_j) the solution to (6.118)–(6.121) corresponding to $z = z_j$. As $j \to \infty$ the points z_j approach the boundary point z^* and, therefore, $\|\Phi(\cdot, z_j)\|_{H^{\frac{1}{2}}(\partial D)} \to \infty$. From the trace theorem and by using the boundary conditions we can write

$$\|v_j\|_{H^1(D)} + \|w_j\|_{H^1(D)} \geq \|v_j - w_j\|_{H^{\frac{1}{2}}(\partial D)} = \|\Phi(\cdot, z_j)\|_{H^{\frac{1}{2}}(\partial D)}. \quad (6.123)$$

In particular, we show that the relation (6.123) implies that

$$\lim_{j\to\infty} \|w_j\|_{H^1(D)} = \infty.$$

To this end, we assume, in contrast, that

$$\|w_j\|_{H^1(D)} \leq \bar{C}, \qquad j = 1, 2, \ldots,$$

for some positive constant $\bar{C}$. From the trace theorem we have

$$\|w_j\|_{H^{\frac{1}{2}}(\partial D)} \leq \bar{C} \qquad \text{and} \qquad \|\frac{\partial w_j}{\partial \nu}\|_{H^{\frac{1}{2}}(\partial D)} \leq \bar{C}, \qquad j = 1, 2, \ldots.$$

Recall that for every j the pair $(v_j, \Phi(\cdot, z_j))$ is the solution of (5.13)–(5.17) with $(f, g) := \left(w_j|_{\partial D}, \partial w_j/\partial\nu|_{\partial D}\right)$. The a priori estimate (5.41) implies that

$$\begin{aligned} \|v_j\|_{H^1(D)} &+ \|\Phi(\cdot, z_j)\|_{H^1(\Omega_R\setminus\bar{D})} \\ &\leq C\left(\|w_j\|_{H^{\frac{1}{2}}(\partial D)} + \|\frac{\partial w_j}{\partial \nu}\|_{H^{-\frac{1}{2}}(\partial D)}\right) \leq 2C\bar{C}, \end{aligned}$$

which contradicts the fact that $\|\Phi(\cdot, z_j)\|_{H^1(\Omega_R\setminus\bar{D})}$ does not remain bounded as $z_j \to z^* \in \partial D$. So we have that

$$\lim_{j\to\infty} \|w_j\|_{H^1(D)} = \infty.$$

Since for every $j = 1, 2, \ldots$ the corresponding Herglotz wave functions $v_{g_{z_j}^\epsilon}$ satisfying (6.122) approximate the solution w_j in the $H^1(D)$ norm, we conclude that

$$\lim_{j\to\infty} \|v_{g_{z_j}^\epsilon}\|_{H^1(D)} = \infty,$$

and hence

$$\lim_{j\to\infty} \|g_{z_j}^\epsilon\|_{L^2[0,\,2\pi]} = \infty.$$

Next we consider $z \in \mathbb{R}^2 \setminus \bar{D}$, and again we assume that k is not a transmission eigenvalue. We would like to show that if g_z^ϵ is such that

$$\|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2[0,\,2\pi]} < \epsilon$$

for a given arbitrary $\epsilon > 0$, then the $H^1(D)$ norm of the corresponding Herglotz wave functions $v_{g_z^\epsilon}$ is not bounded as $\epsilon \to 0$. Assume, to the contrary, that there exists a null sequence $\{\epsilon_n\}$ such that $\|v_n\|_{H^1(D)}$ remain bounded as $n \to \infty$, where $v_n := v_{g_z^{\epsilon_n}}$. From the trace theorem $\|Hg_n\|_{H(\partial D)}$ also remain bounded. Then without loss of generality we may assume weak convergence $Hg_n \rightharpoonup h$, where $h := \left(w|_{\partial D}, \left.\frac{\partial w}{\partial \nu}\right|_{\partial D} \right)$ for some $w \in \overline{H}$, i.e., that w is a $H^1(D)$ weak solution to the Helmholtz equation. Since $B_0 : H(\partial D) \to L^2[0, 2\pi]$ is bounded, we also have that $B_0 H g_n \rightharpoonup B_0 h$ in $L^2[0, 2\pi]$. But by construction, $B_0 H g_n \to \Phi_\infty(\cdot, z)$, which means that $B_0 h = \Phi_\infty(\cdot, z)$. This contradicts the fact that $\Phi_\infty(\cdot, z)$ does not belong to the range of the operator B_0 because this would mean that $\Phi(\cdot, z)$ solves the Helmholtz equation in the exterior of D.

We summarize the foregoing analysis in the following theorem. To this end, we state the following assumptions on the symmetric matrix-valued function $A = (a_{j,k})_{j,k=1,2}$, $a_{j,k} \in C^1(\bar{D})$ and $n \in C(\bar{D})$:

- *Assumption 1*: $\bar{\xi} \cdot \mathrm{Im}(A)\, \xi = 0$, $\mathcal{I}m\,(n) = 0$, and

  $$\text{either } 0 < a^* < 1 \text{ and } 0 < n^* < 1, \text{ or } a_* > 1 \text{ and } n_* > 1,$$

  where a_*, a^*, n_*, and n^* are defined by (6.95).
- *Assumption 2*: $\bar{\xi} \cdot \mathrm{Im}(A)\, \xi \le 0$, $\mathrm{Im}(n) \ge 0$, and

  $$\text{either } \bar{\xi} \cdot \mathrm{Re}(A)\, \xi \ge \gamma|\xi|^2 \text{ or } \bar{\xi} \cdot \mathrm{Re}(A^{-1})\, \xi \ge \gamma|\xi|^2$$

  for all $\xi \in \mathbb{C}^2$ and $x \in \overline{D}$ with a constant $\gamma > 1$.

Theorem 6.50. *Assume that D is a bounded domain having a C^2 boundary ∂D such that $\mathbb{R}^2 \setminus \bar{D}$ is connected, and A and n satisfy either Assumption 1 or Assumption 2. Furthermore, assume that k is not a transmission eigenvalue. Then if F is the far-field operator (6.7) corresponding to the transmission problem (6.1)–(6.5), we have that*

1. *For $z \in D$ and a given $\epsilon > 0$ there exists a function $g_z^\epsilon \in L^2[0, 2\pi]$ such that*

   $$\|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2[0,2\pi]} < \epsilon$$

   and the Herglotz wave function $v_{g_z^\epsilon}$ with kernel g_z^ϵ converges in $H^1(D)$ to w as $\epsilon \to 0$, where (v, w) is the unique solution of (6.118)–(6.121);
2. *For $z \notin D$ and a given $\epsilon > 0$ every function $g_z^\epsilon \in L^2[0, 2\pi]$ that satisfies*

   $$\|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2[0,2\pi]} < \epsilon$$

   is such that

   $$\lim_{\epsilon \to 0} \left\| v_{g_z^\epsilon} \right\|_{H^1(D)} = \infty.$$

The importance of Theorem 6.50 in solving the inverse scattering problem of determining the support D of an orthotropic inhomogeneity from the far-field pattern is clear from our discussion in Chap. 4. In particular, using regularization methods to solve the far-field equation $Fg = \Phi_\infty(\cdot, z)$ for z on an appropriate grid containing D, an approximation to g_z can be obtained, and hence ∂D can be determined by those points where $\|g_z\|_{L^2[0,\,2\pi]}$ becomes unbounded. More discussion on the numerical implementation is presented in Chap. 8.

6.6 Determination of Transmission Eigenvalues from Far-Field Data

In the previous section we showed how the linear sampling method could be used to determine the support of the inhomogeneous scattering object provided k is not a transmission eigenvalue. At the same time we showed that the transmission eigenvalues carried qualitative information about the material properties of the scatterer (cf. Theorems 6.22, 6.23, 6.25, and 6.36). To exploit the possibility of using this qualitative information, we are no longer interested in avoiding transmission eigenvalues as in the case of the linear sampling method but rather now want to be able to determine them from the (noisy) far-field data. This last section of our chapter is devoted to this problem.

At this point we assume that D (or a reconstruction of D using the linear sampling method) is known and fix an arbitrary point $z \in D$. In Theorem 6.50 it was shown that if k is not a transmission eigenvalue, then for a given $\epsilon > 0$ there exists a function $g_z^\epsilon \in L^2[0, 2\pi]$ such that

$$\|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2[0,2\pi]} < \epsilon \tag{6.124}$$

and the Herglotz wave function $v_{g_z^\epsilon}$ with kernel g_z^ϵ converges in $H^1(D)$ to w as $\epsilon \to 0$, where (v, w) is the unique solution of (6.118)–(6.121). We will now show that if k is a transmission eigenvalue, then the $H^1(D)$ norm of $v_{g_z^\epsilon}$ blows up as $\epsilon \to 0$. More specifically, we can prove the following theorem.

Theorem 6.51. *Assume that either* $0 < a^* < 1$ *and* $0 < n^* < 1$, *or* $a_* > 1$ *and* $n_* > 1$, *where* a_*, a^*, n_*, *and* n^* *are defined by (6.95). Let* k *be a transmission eigenvalue and* g_z^ϵ *satisfy (6.124). Then for every* $z \in D$, *except for a nowhere dense set,* $\|v_{g_z^\epsilon}\|_{H^1(D)}$ *cannot be bounded as* $\epsilon \to 0$.

Proof. Assume to the contrary that for a set of points $z \in D$ that has an accumulation point, there exists a sequence $\epsilon_n \to 0$ such that $\|v_n\|_{H^1(D)}$ remains bounded as $n \to \infty$, where $v_n := v_{g_z^{\epsilon_n}}$, with $g_z^{\epsilon_n}$ satisfying (6.124). Without loss of generality we may assume that v_n converges weakly to $w \in H^1(D)$. In a similar way as in the proof of Theorem 6.50 it is seen that

$w := w_z$, where v_z and w_z solve the interior transmission problem (6.118)–(6.121). But (6.118)–(6.121) is equivalent to the variational form (see the discreteness of transmission eigenvalues for the case $n \neq 1$ in Sect. 6.3)

$$a_k((v,w),(v',w')) = \ell(v',w') \quad \text{for all} \quad (v',w') \in \mathcal{H}(D), \tag{6.125}$$

where

$$\ell(v',w') = \int_{\partial D} \overline{v'}\,\frac{\partial \Phi(\cdot,z)}{\partial \nu}\,ds - \int_D (\nabla \phi_z \cdot \nabla \overline{w'} - k^2 \phi_z \overline{w'})\,dx,$$
$$\mathcal{H}(D) := \left\{(v,w) \in H^1(D) \times H^1(D) : \, v - w \in H_0^1(D)\right\},$$

and $\phi_z \in H^1(D)$ is a lifting function such that $\phi_z = \Phi(\cdot,z)$ on ∂D. As discussed in Remark 6.29, (6.125) satisfies the Fredholm alternative. Hence, noting that the operator determined by $a_k(\cdot,\cdot)$ via the Riesz representation theorem is self-adjoint, we have that $w := w_z$ and v_z solve (6.118)–(6.121) if and only if $\ell(v_k, w_k) = 0$, where (v_k, w_k) is a transmission eigenfunction corresponding to the transmission eigenvalue k. Using integration by parts and the facts that $\Delta w_k + k^2 w_k = 0$ in D and $v_k = w_k$ on ∂D we obtain that the solvability condition takes the form

$$\int_{\partial D} \left(\overline{w_k} \frac{\partial \Phi(\cdot,z)}{\partial \nu} - \frac{\partial \overline{w_k}}{\partial \nu} \Phi(\cdot,z) \right) ds = 0.$$

Now Green's representation formula and the analyticity of the solution to the Helmholtz equation imply that $w_k = 0$ in D and, consequently, $v_k = 0$ in D. This contradicts the fact that (v_k, w_k) is a transmission eigenfunction, which proves the theorem. □

Similarly, we can prove the following theorem, which we leave as an exercise for the reader.

Theorem 6.52. *Assume that $n = 1$ and either $a_{max} < 1$ or $a_{min} > 1$, where a_{max} and a_{min} are defined by (6.48). Let k be a transmission eigenvalue and g_z^ϵ satisfy (6.124). Then for every $z \in D$ except for a nowhere dense set, $\|v_{g_z^\epsilon}\|_{H^1(D)}$ cannot be bounded as $\epsilon \to 0$.*

Theorem 6.50, together with Theorems 6.51 and 6.52, suggests that for $z \in D$, $v_{g_z^\epsilon}$ exhibits different behavior if k is not a transmission eigenvalue and if k is a transmission eigenvalue. Hence the far-field equation can be used to determine the transmission eigenvalues in addition to determining the support of the inhomogeneity if the far-field data are available for a range of frequencies.

In practice, only the noisy far-field operator F_δ given by

$$F_\delta g = \int_0^{2\pi} u_\infty^\delta(\hat{x}, d) g(d)\,ds(d)$$

is available, where u_∞^δ is the noisy far-field data with noise level $\delta > 0$. Then we look for the Tikhonov regularized solution $g_z^{\alpha,\delta}$ of the far-field equation defined as the unique minimizer of the *Tikhonov functional*

$$\|F_\delta g - \Phi_\infty(\cdot, z)\|^2_{L^2[0,2\pi]} + \alpha \|g\|^2_{L^2[0,2\pi]},$$

where the positive number $\alpha > 0$ is known as the *Tikhonov regularization parameter* (cf. Sect. 2.1). This regularization parameter depends on the noise level and can be chosen such that $\alpha(\delta) \to 0$ as $\delta \to 0$. If $g_z^\delta := g_z^{\alpha(\delta),\delta}$, then it can be shown (see [22]) that

$$\lim_{\delta \to 0} \|F g_z^\delta - \Phi_\infty(\cdot, z)\|_{L^2[0,2\pi]} = 0.$$

Hence Theorems 6.51 and 6.52 hold true for the regularized solution g_z^δ, where ϵ is now replaced by δ.

The first part of Theorem 6.50 also holds true for the regularized solution g_z^δ of the far-field equation, but its justification involves the more elaborate argument developed for the Dirichlet obstacle scattering problem by Arens in [5]. This argument can be carried through for the case of inhomogeneous media with real-valued physical parameters, which is the case where transmission eigenvalues exist. It is essential to this generalization to show that $\Phi(\cdot, z)$ is in the range of $(F^*F)^{1/4}$ if and only if $z \in D$, which constitutes the so-called *factorization method*. More generally, the factorization method provides an analytical framework to justify the linear sampling method (i.e., Theorem 6.50) for the regularized solution of the far-field equation that is obtained in practice. The factorization method holds for a restrictive class of scattering problems and is the subject of the following chapter.

In conclusion, to determine the transmission eigenvalues from the far-field data, we choose a point $z \in D$ and the Tikhonov regularized solution g_z^δ to the far-field equation. The transmission eigenvalues will appear as sharp peaks in the plot of $\|v_{g_z^\delta}\|_{H^1(D)}$ or $\|g_z^\delta\|_{L^2[0,2\pi]}$ against the wave number k for a range of interrogating frequencies.

As an example of the use of transmission eigenvalues to determine information about the material properties of the scattering object from far-field data, we consider the scattering problem (5.8)–(5.12) with $n = 1$ and D the unit square $[-1/2, 1/2] \times [-1/2, 1/2]$. We consider four different possibilities for $A = A(x)$:

$$A_{iso} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/8 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1/6 & 0 \\ 0 & 1/8 \end{pmatrix}, \qquad A_{2r} = \begin{pmatrix} 0.1372 & 0.0189 \\ 0.0189 & 0.1545 \end{pmatrix},$$

noting that A_{2r} is obtained by rotating matrix A_2 by 1 radian. For each A the direct scattering problem is then solved using finite-element methods, and the far-field equation with noisy far-field data is then solved for 25 random source points z in the unit square (for details see [28]). It is assumed that D is known (for example, through the use of the linear sampling method). In Fig. 6.1 we show a plot of the average norm of the Herglotz kernel $\|g_z^\delta\|_{L^2[0,2\pi]}$ against

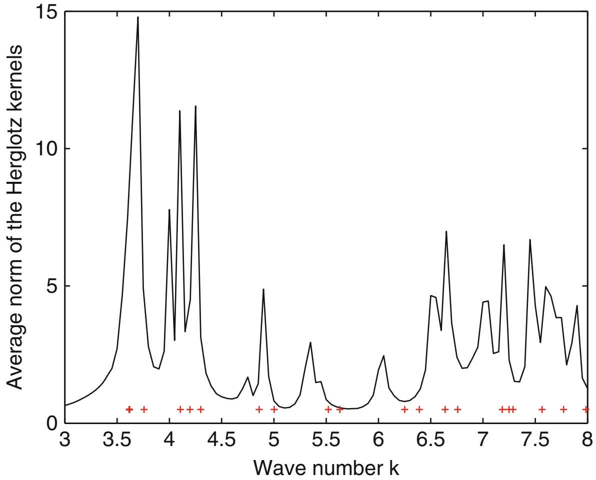

Fig. 6.1. Results for square using anisotropy A_{2r}. We show a plot of the average values of $\|g_z^\delta\|_{L^2(D)}$ against k. We also mark the computed eigenvalues from the finite-element code (shown as + along the bottom of the graph). Good agreement is seen with the lowest computed eigenvalue and the first peak of the norms of g_z^δ][1]

Table 6.1. Our theory implies that the scalar a reconstructed from the first nonzero real transmission eigenvalue should lie between the eigenvalues of matrix A. In the case of an isotropic A, the predicted a should reconstruct the diagonal of A. The table supports both these claims[1]

Domain	Matrix	Eigenvalues	Predicted $k_{1,D,A(x)}$	Predicted a
Square	A_{iso}	1/4,1/4	5.3	0.248
	A_1	1/2,1/8	4.1	0.172
	A_2	1/6,1/8	3.55	0.135
	A_{2r}	1/6,1/8	3.7	0.145

the wave number k corresponding to matrix A_{2r}. Given the first transmission eigenvalue $k_{1,D,A(x)}$ from Fig. 6.1, we can now compute a positive number a such that $k_{1,D,aI} = k_{1,D,A(x)}$. According to Theorem 6.22, a should lie between the smallest and largest eigenvalues of A. Table 6.1 below shows the results of this calculation for each of the preceding cases for A.

Additional numerical examples of the determination of transmission eigenvalues from far-field data and their use to obtain information on the refractive index of the inhomogeneity can be found in [21] and [28].

[1]Reprinted from F. Cakoni, D. Colton, P. Monk, and J. Sun, The inverse electromagnetic scattering problem for anisotropic media, Inverse Problems 26 (2010), 074004.