5

Scattering by Orthotropic Media

Until now the reader has been introduced only to the scattering of time -harmonic electromagnetic waves by an imperfect conductor. We will now consider the scattering of electromagnetic waves by a penetrable orthotropic inhomogeneity embedded in a homogeneous background. As in the previous chapter, we will confine ourselves to the scalar case that corresponds to the scattering of electromagnetic waves by an orthotropic infinite cylinder. The direct scattering problem is now modeled by a transmission problem for the Helmholtz equation outside the scatterer and an equation with nonconstant coefficients inside the scatterer. This chapter is devoted to the analysis of the solution to the direct problem.

After a brief discussion of the derivation of the equations that govern the scattering of electromagnetic waves by an orthotropic infinite cylinder, we proceed to the solution to the corresponding transmission problem. The integral equation method used by Piana [136] and Potthast [137] to solve the forward problem in this case is only valid under restrictive assumptions. Hence, following [81], we propose here a variational method and find a solution to the problem in a larger space than the space of twice continuously differentiable functions. To build the analytical frame work for this variational method, we first extend the discussion of Sobolev spaces and weak solutions initiated in Sects. 1.5 and 3.3. This is followed by a proof of the celebrated Lax–Milgram lemma and an investigation of the Dirichlet-to-Neumann map. Included are several simple examples of the use of variational methods for solving boundary value problems. We conclude our chapter with a solvability result for the direct problem.

5.1 Maxwell Equations for an Orthotropic Medium

We begin by considering electromagnetic waves propagating in an inhomogeneous anisotropic medium in $\mathbb{R}^3$ with electric permittivity $\epsilon = \epsilon(x)$, magnetic permeability $\mu = \mu(x)$, and electric conductivity $\sigma = \sigma(x)$. As the reader

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knows from Chap. 3, the electromagnetic wave is described by the electric field $\mathcal{E}$ and the magnetic field H satisfying the *Maxwell equations*

$$\operatorname{curl}\mathcal{E} + \mu\frac{\partial\mathcal{H}}{\partial t} = 0, \qquad \operatorname{curl}\mathcal{H} - \epsilon\frac{\partial\mathcal{E}}{\partial t} = \sigma\mathcal{E}.$$

For time-harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \tilde{E}(x)e^{-i\omega t}, \qquad \mathcal{H}(x,t) = \tilde{H}(x)e^{-i\omega t}$$

with frequency $\omega > 0$, we deduce that the complex-valued space-dependent parts $\tilde{E}$ and $\tilde{H}$ satisfy

$$\begin{aligned}&\operatorname{curl}\tilde{E} - i\omega\mu(x)\tilde{H} = 0,\\ &\operatorname{curl}\tilde{H} + (i\omega\epsilon(x) - \sigma(x))\tilde{E} = 0.\end{aligned}$$

Now let us suppose that the inhomogeneity occupies an infinitely long conducting cylinder. Let D be the cross section of this cylinder having a C^2 boundary ∂D, with ν being the unit outward normal to ∂D. We assume that the axis of the cylinder coincides with the z-axis. We further assume that the conductor is imbedded in a nonconducting homogeneous background, i.e., the electric permittivity $\epsilon_0 > 0$, and the magnetic permeability $\mu_0 > 0$ of the background medium is a positive constants, while the conductivity $\sigma_0 = 0$. Next we define

$$\tilde{E}^{int,ext} = \frac{1}{\sqrt{\epsilon_0}}E^{int,ext}, \qquad \tilde{H}^{int,ext} = \frac{1}{\sqrt{\mu_0}}H^{int,ext}, \qquad k^2 = \epsilon_0\mu_0\omega^2,$$
$$\mathcal{A}(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right), \qquad \mathcal{N}(x) = \frac{1}{\mu_0}\mu(x),$$

where $\tilde{E}^{ext}, \tilde{H}^{ext}$ and $\tilde{E}^{int}, \tilde{H}^{int}$ denote the electric and magnetic fields in the exterior medium and inside the conductor, respectively. For an orthotropic medium we have that the matrices $\mathcal{A}$ and $\mathcal{N}$ are independent of the z-coordinate and are of the form

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & a\end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} n_{11} & n_{12} & 0\\ n_{21} & n_{22} & 0\\ 0 & 0 & n\end{pmatrix}.$$

In particular, the field E^{int}, H^{int} inside the conductor satisfies

$$\operatorname{curl}E^{int} - ik\mathcal{N}H^{int} = 0, \qquad \operatorname{curl}H^{int} + ik\mathcal{A}E^{int} = 0, \tag{5.1}$$

and the field E^{ext}, H^{ext} outside the conductor satisfies

$$\operatorname{curl}E^{ext} - ikH^{ext} = 0, \qquad \operatorname{curl}H^{ext} + ikE^{ext} = 0. \tag{5.2}$$

Across the boundary of the conductor we have the continuity of the tangential component of both the electric and magnetic fields. Assuming that $\mathcal{A}$ is invertible, and using $ikE^{int} = \mathcal{A}^{-1}\text{curl}\, H^{int}$ and $ikE^{ext} = \text{curl}\, H^{ext}$, the Maxwell equations become

$$\text{curl}\, \mathcal{A}^{-1}\text{curl}\, H^{int} - k^2\mathcal{N}H^{int} = 0 \tag{5.3}$$

for the magnetic field inside the conductor and

$$\text{curl}\,\text{curl}\, H^{ext} - k^2 H^{ext} = 0 \tag{5.4}$$

for the magnetic field outside the conductor. If the scattering is due to a given time-harmonic incident field E^i, H^i, then we have that

$$E^{ext} = E^s + E^i, \qquad H^{ext} = H^s + H^i,$$

where E^s, H^s denotes the scattered field. In general the incident field E^i, H^i is an entire solution to (5.2). In particular, in the case of incident plane waves, E^i, H^i is given by (3.4). The scattered field E^s, H^s satisfies the Silver–Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0$$

uniformly in $\hat{x} = x/|x|$ and $r = |x|$.

Now let us assume that the incident wave propagates perpendicular to the axis of the cylinder and is polarized perpendicular to the axis of the cylinder such that

$$H^i(x) = (0,\, 0,\, u^i), \qquad H^s(x) = (0,\, 0,\, u^s), \qquad H^{int}(x) = (0,\, 0,\, v).$$

By elementary vector analysis, it can be seen that (5.3) is equivalent to

$$\nabla \cdot A\nabla v + k^2 nv = 0 \qquad \text{in } D, \tag{5.5}$$

where

$$A := \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

Analogously, (5.4) is equivalent to the Helmholtz equation

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in } \mathbb{R}^2 \setminus \overline{D}. \tag{5.6}$$

The transmission conditions $\nu \times (H^s + H^i) = \nu \times H^{int}$ and $\nu \times \text{curl}\,(H^s + H^i) = \nu \times \mathcal{A}^{-1}\text{curl}\, H^{int}$ on the boundary of the conductor become

$$v - u^s = u^i \quad \text{and} \quad \nu \cdot A\nabla v - \nu \cdot \nabla u^s = \nu \cdot \nabla u^i \qquad \text{on} \quad \partial D. \tag{5.7}$$

Finally, the $\mathbb{R}^2$ analog of the Silver–Müller radiation condition is the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0,$$

which holds uniformly in $\hat{x} = x/|x|$.

Summarizing the foregoing discussion we have that the scattering of incident time-harmonic electromagnetic waves by an orthotropic cylindrical conductor is modeled by the following transmission problem in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a nonempty, open, and bounded set having C^2 boundary ∂D such that the exterior domain $\mathbb{R}^2 \setminus \bar{D}$ is connected. The unit normal vector to ∂D, which is directed into the exterior of D, is denoted by ν. On $\bar{D}$ we have a matrix-valued function $A : \bar{D} \to \mathbb{C}^{2\times 2}$, $A = (a_{jk})_{j,k=1,2}$, with continuously differentiable functions $a_{jk} \in C^1(\bar{D})$. By $\mathrm{Re}(A)$ we mean the matrix-valued function having as entries the real parts $\mathrm{Re}(a_{jk})$, and we define $\mathrm{Im}(A)$ similarly. We suppose that $\mathrm{Re}(A(x))$ and $\mathrm{Im}(A(x))$, $x \in \bar{D}$, are symmetric matrices that satisfy $\bar{\xi} \cdot \mathrm{Im}(A)\,\xi \ \le 0$ and $\bar{\xi} \cdot \mathrm{Re}(A)\,\xi \ge \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$ and $x \in \overline{D}$, where γ is a positive constant. Note that due to the symmetry of A, $\mathrm{Im}\left(\bar{\xi} \cdot A\,\xi\right) = \bar{\xi} \cdot \mathrm{Im}(A)\,\xi$ and $\mathrm{Re}\left(\bar{\xi} \cdot A\,\xi\right) = \bar{\xi} \cdot \mathrm{Re}(A)\,\xi$. We further assume that $n \in C(\bar{D})$, with $\mathrm{Im}(n) \ge 0$.

For functions $u \in C^1(\mathbb{R}^2 \setminus D)$ and $v \in C^1(\bar{D})$ we define the normal and conormal derivative by

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h\to+0} \nu(x) \cdot \nabla u(x + h\nu(x)), \qquad x \in \partial D$$

and

$$\frac{\partial v}{\partial \nu_A}(x) = \lim_{h\to+0} \nu(x) \cdot A(x)\nabla v(x - h\nu(x)), \qquad x \in \partial D,$$

respectively. Then the scattering of a time-harmonic incident field u^i by an orthotropic inhomogeneity in $\mathbb{R}^2$ can be mathematically formulated as the problem of finding v, u such that

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \qquad \text{in} \quad D, \tag{5.8}$$

$$\Delta u^s + k^2\, u^s = 0 \qquad \text{in} \quad \mathbb{R}^2 \setminus \bar{D}, \tag{5.9}$$

$$v - u^s = u^i \qquad \text{on} \quad \partial D, \tag{5.10}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \qquad \text{on} \quad \partial D, \tag{5.11}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \tag{5.12}$$

The aim of this chapter is to establish the existence of a unique solution to the scattering problem (5.8)–(5.12). In most applications the material properties of the inhomogeneity do not change continuously to those of the background medium, and hence the integral equation methods used in [136] and [137] are not applicable. Therefore, we will introduce a variational method to solve our problem. Since variational methods are well suited to Hilbert spaces, in the next section we reformulate our scattering problem in appropriate Sobolev spaces. To this end, we need to extend the discussion on Sobolev spaces given in Sect. 1.5.

5.2 Mathematical Formulation of Direct Scattering Problem

In the context of variational methods, one naturally seeks a solution to a linear second-order elliptic boundary value problem in the space of functions that are square integrable and have square integrable first partial derivatives. Let D be an open, nonempty, bounded, simply connected subset of $\mathbb{R}^2$ with smooth boundary ∂D. In Sect. 1.5 we introduced the Sobolev spaces $H^1(D)$, $H^{\frac{1}{2}}(\partial D)$, and $H^{-\frac{1}{2}}(\partial D)$. The reader has already encountered the connection between $H^{\frac{1}{2}}(\partial D)$ and $H^1(D)$, that is, $H^{\frac{1}{2}}(\partial D)$ is the trace space of $H^1(D)$. More specifically, for functions defined in $\bar{D}$ the values on the boundary are defined and the restriction of the function to the boundary ∂D is called the *trace*. The operator mapping a function onto its trace is called the *trace operator*. Theorem 1.38 states that the trace operator can be extended as a continuous mapping $\gamma_0 : H^1(D) \to H^{\frac{1}{2}}(\partial D)$, and this extension has a continuous right inverse (see also Theorem 3.37 in [127]). The latter means that for any $f \in H^{\frac{1}{2}}(\partial D)$ there exists a $u \in H^1(D)$ such that $\gamma_0 u = f$ and $\|u\|_{H^1(D)} \le C\|f\|_{H^{\frac{1}{2}}(\partial D)}$, where C is a positive constant independent of f. (Map D in a one-to-one manner onto the unit disk, and use separation of variables to determine u as a solution to the Dirichlet problem for Laplace's equation. Then map back to D.)
For any integer $r \ge 0$ we let

$$C^r(D) := \{u : \partial^\alpha u \text{ exists and is continuous on } D \text{ for } |\alpha| \le r\},$$
$$C^r(\bar{D}) := \{u|_{\bar{D}} : u \in C^r(\mathbb{R}^2)\}$$

and put

$$C^\infty(D) = \bigcap_{r\ge 0} C^r(D) \qquad C^\infty(\bar{D}) = \bigcap_{r\ge 0} C^r(\bar{D}).$$

In Sect. 1.5, $H^1(D)$ is naturally defined as the completion of $C^1(\bar{D})$ with respect to the norm

$$\|u\|^2_{H^1(D)} := \|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)}.$$

Note that $H^1(D)$ is a Hilbert space with the inner product

$$(u, v)_{H^1(D)} := (u, v)_{L^2(D)} + (\nabla u, \nabla v)_{L^2(D)}.$$

It can be shown that $C^\infty(\bar{D})$ is dense in $H^1(D)$. The proof of this result can be found in [127].

Since $H^1(D)$ is a subspace of $L^2(D)$, we can consider the *embedding* map $\mathcal{I} : H^1(D) \to L^2(D)$ defined by $\mathcal{I}(u) = u \in L^2(D)$ for $u \in H^1(D)$. Obviously, $\mathcal{I}$ is a bounded linear operator. The following two lemmas are particular cases of the well-known *Rellich compactness theorem.*

Lemma 5.1. *The embedding* $\mathcal{I} : H^1(D) \to L^2(D)$ *is compact.*

In the sequel, we also need to consider the Sobolev space $H^2(D)$, which is the space of functions $u \in H^1(D)$ such that u_x and u_y are also in $H^1(D)$. Similarly, $H^2(D)$ can be defined as the completion of $C^2(\bar{D})$ [or $C^\infty(\bar{D})$] with respect to the norm

$$\|u\|^2_{H^2(D)} = \|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)} + \|u_{xx}\|^2_{L^2(D)} + \|u_{xy}\|^2_{L^2(D)} + \|u_{yy}\|^2_{L^2(D)}.$$

Lemma 5.2. *The embedding* $\mathcal{I} : H^2(D) \to H^1(D)$ *is a compact operator.*

The proof of the Rellich compactness theorem can be found, for instance, in [72] or [127]. For the special case of $H^p[0, 2\pi]$ this result is proved in Theorem 1.32.

We now define

$$C_0^\infty(D) := \{u : u \in C_K^\infty(D) \text{ for some compact subset } K \text{ of } D\},$$

where

$$C_K^\infty(D) := \{u \in C^\infty(D) : \operatorname{supp} u \subseteq K\}$$

and the support of u, denoted by supp u, is the closure in D of the set $\{x \in D : u(x) \neq 0\}$. The completion of $C_0^\infty(D)$ in $H^1(D)$ is denoted by $H_0^1(D)$ and can be characterized by

$$H_0^1(D) := \{u \in H^1(D) : u|_{\partial D} = 0\},$$

where $u|_{\partial D}$ is understood in the sense of the trace operator $\gamma_0 u$. This space equipped with the inner product of $H^1(D)$ is also a Hilbert space. The following inequality, known as *Poincaré's inequality*, holds for functions in $H_0^1(D)$.

Theorem 5.3 (Poincaré's Inequality). *There exists a positive constant* M *such that for every* $u \in H_0^1(D)$ *we have*

$$\int_D |u|^2\, dx \leq M \int_D \|\nabla u\|^2\, dx,$$

where M *is independent of* u *but depends on* D.

Proof. We first assume that $u \in C_0^1(D)$. Since D is bounded, it can be enclosed in a square $\varGamma := \{|x_i| \leq a, i = 1, 2\}$, and u will continue to be identically zero outside D. Then for any $x = (x_1, x_2) \in \varGamma$ we have, using the Cauchy–Schwarz inequality, that

$$\begin{aligned} |u(x)|^2 &= \left| \int_{-a}^{x_1} u_{x_1}(\xi_1, x_2)\, d\xi_1 \right|^2 \\ &\leq (x_1 + a) \int_{-a}^{x_1} |u_{x_1}|^2\, d\xi_1 \\ &\leq 2a \int_{-a}^{x_1} |u_{x_1}|^2\, d\xi_1, \end{aligned}$$

and hence

$$\int_{-a}^{a} |u(x)|^2 \, dx_1 \leq 4a^2 \int_{-a}^{a} |u_{x_1}|^2 \, d\xi_1.$$

Now integrate with respect to x_2 from $-a$ to a to obtain

$$\begin{aligned}\int_{\Gamma} |u(x)|^2 \, dx &\leq 4a^2 \int_{\Gamma} |u_{x_1}|^2 \, dx \\ &\leq 4a^2 \int_{\Gamma} |\nabla u|^2 \, dx.\end{aligned}$$

The theorem now follows from the fact that $C_0^1(D)$ is dense in $H_0^1(D)$. □

Remark 5.4. It can be shown that the optimal constant M in the preceding Poincaré's inequality is equal to $1/\lambda_0(D)$, where $\lambda_0(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D (cf. [95]).

Remark 5.5. Our presentation of Sobolev spaces is by no means complete. A systematic treatment of Sobolev spaces requires the use of the Fourier transform and distribution theory, and we refer the reader to Chap. 3 in [127] for this material.

For later use we recall the following classical result from real analysis.

Lemma 5.6. *Let G be a closed subset of $\mathbb{R}^2$. For each $\epsilon > 0$ there exists a $\chi_\epsilon \in C^\infty(\mathbb{R}^2)$ satisfying*

$$\begin{aligned} \chi_\epsilon(x) &= 1 && \textit{if} \quad x \in G, \\ 0 \leq \chi_\epsilon(x) &\leq 1 && \textit{if} \quad 0 < \textit{dist}(x, G) < \epsilon, \\ \chi_\epsilon(x) &= 0 && \textit{if} \quad \textit{dist}(x, G) > \epsilon, \end{aligned}$$

where $dist(x, G)$ denotes the distance of x from G.

The function $\chi_\epsilon(x)$ defined in the preceding lemma is called a *cutoff function* for G. It is used to smooth out the characteristic function of a set.

Keeping in mind the solution to the scattering problem in Sect. 5.1, we now extend the definition of the conormal derivative $\partial u/\partial \nu_A$ to functions $u \in H^1(D, \Delta_A)$, where

$$H^1(D, \Delta_A) := \{u \in H^1(D) : \; \nabla \cdot A \nabla u \in L^2(D)\},$$

equipped with the graph norm

$$\|u\|^2_{H^1(D,\Delta_A)} := \|u\|^2_{H^1(D)} + \|\nabla \cdot A \nabla u\|^2_{L^2(D)}.$$

In particular, we have the following *trace theorem.*

Theorem 5.7. *The mapping $\gamma_1 : u \to \partial u/\partial \nu_A := \nu \cdot A\nabla u$ defined in $C^\infty(\bar{D})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_1, from $H^1(D, \Delta_A)$ to $H^{-\frac{1}{2}}(\partial D)$.*

Proof. Let $\phi \in C^\infty(\bar{D})$ and $u \in C^\infty(\bar{D})$. The divergence theorem then becomes

$$\int_{\partial D} \phi\, \nu \cdot A\nabla u\, ds = \int_D \nabla\phi \cdot A\nabla u\, dx + \int_D \phi\, \nabla \cdot A\nabla u\, dx.$$

Because $C^\infty(\bar{D})$ is dense in $H^1(D)$, this equality is still valid for $\phi \in H^1(D)$ and $u \in C^\infty(\bar{D})$. Therefore,

$$\left| \int_{\partial D} \phi\, \nu \cdot A\nabla u\, ds \right| \le C\|u\|_{H^1(D,\Delta_A)}\|\phi\|_{H^1(D)} \quad \forall \phi \in H^1(D), \quad \forall u \in C^\infty(\bar{D}),$$

where C is a positive constant independent of ϕ and u but dependent on A and D. Now let f be an element of $H^{\frac{1}{2}}(\partial D)$. There exists a $\phi \in H^1(D)$ such that $\gamma_0\phi = f$, where γ_0 is the trace operator on ∂D. Then the preceding inequality implies that

$$\left| \int_{\partial D} f\, \nu \cdot A\nabla u\, ds \right| \le C\|u\|_{H^1(D,\Delta_A)}\|f\|_{H^{\frac{1}{2}}(\partial D)} \quad \forall f \in H^{\frac{1}{2}}(\partial D), \quad \forall u \in C^\infty(\bar{D}).$$

Therefore, the mapping

$$f \to \int_{\partial D} f\, \nu \cdot A\nabla u\, ds \qquad f \in H^{\frac{1}{2}}(\partial D)$$

defines a continuous linear functional and

$$\|\nu \cdot A\nabla u\|_{H^{-\frac{1}{2}}(\partial D)} \le C\|u\|_{H^1(D,\Delta_A)}.$$

Thus, the linear mapping $\gamma_1 : u \to \nu \cdot A\nabla u$ defined on $C^\infty(\bar{D})$ is continuous with respect to the norm of $H^1(D, \Delta_A)$. Since $C^\infty(\bar{D})$ is dense in $H^1(D, \Delta_A)$, γ_1 can be extended by continuity to a bounded linear mapping (still called γ_1) from $H^1(D, \Delta_A)$ to $H^{-\frac{1}{2}}(\partial D)$. □

As a consequence of the preceding theorem we can now extend the divergence theorem to a wider space of functions.

Corollary 5.8. *Let $u \in H^1(D)$ such that $\nabla \cdot A\nabla u \in L^2(D)$ and $v \in H^1(D)$. Then*

$$\int_D \nabla v \cdot A\nabla u\, dx + \int_D v\, \nabla \cdot A\nabla u\, dx = \int_{\partial D} v\, \nu \cdot A\nabla u\, ds.$$

Remark 5.9. With the help of a cutoff function for a neighborhood of ∂D we can, in a way similar to that in Theorem 5.7, define $\partial u/\partial \nu_A$ for $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ such that $\nabla \cdot A\nabla v \in L^2_{loc}(\mathbb{R}^2 \setminus \bar{D})$ (see Sect. 3.3 for the definition of H^1_{loc}-spaces).

Remark 5.10. Setting $A = I$ in Theorem 5.7 and Corollary 5.8 we have that $\partial u/\partial \nu$ is well defined in $H^{-\frac{1}{2}}(\partial D)$ for functions $u \in H^1(D, \Delta) := \{u \in H^1(D) : \Delta u \in L^2(D)\}$. Furthermore, the following Green's identity holds:

$$\int_D \nabla v \cdot \nabla u \, dx + \int_D v \, \Delta u \, dx = \int_{\partial D} v \frac{\partial u}{\partial \nu} \, ds \qquad u \in H^1(D, \Delta),\, v \in H^1(D).$$

In particular, Theorem 3.1 and Eq. (3.41) are valid for H^1-solutions to the Helmholtz equation.

We are now ready to formulate the direct scattering problem for an orthotropic medium in $\mathbb{R}^2$ in suitable Sobolev spaces. Assume that A, n, and D satisfy the assumptions of Sect. 5.1. Given $f \in H^{\frac{1}{2}}(\partial D)$ and $h \in H^{-\frac{1}{2}}(\partial D)$, find $u \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ and $v \in H^1(D)$ such that

$$\nabla \cdot A\nabla v + k^2 n \, v = 0 \qquad \text{in} \quad D, \tag{5.13}$$

$$\Delta u + k^2 \, u = 0 \qquad \text{in} \quad \mathbb{R}^2 \setminus \bar{D}, \tag{5.14}$$

$$v - u = f \qquad \text{on} \quad \partial D, \tag{5.15}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = h \qquad \text{on} \quad \partial D, \tag{5.16}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0. \tag{5.17}$$

The scattering problem (5.8)–(5.12) is a special case of (5.13)–(5.17). In particular, the scattered field u^s and the interior field v satisfy (5.13)–(5.17) with $u = u^s$, $f = u^i|_{\partial D}$, and $h := \left. \frac{\partial u^i}{\partial \nu} \right|_{\partial D}$, where the incident wave u^i is such that

$$\Delta u^i + k^2 u^i = 0 \qquad \text{in } \mathbb{R}^2.$$

Note that the boundary conditions (5.15) and (5.16) are assumed in the sense of the trace operator, as discussed previously, and u and v satisfy (5.13) and (5.14), respectively, in the weak sense. The reader already encountered in Sect. 3.3 the concept of a weak solution in the context of the impedance boundary value problem for the Helmholtz equation. In the next section we provide a more systematic discussion of weak solutions and variational methods for finding weak solutions of boundary value problems.

5.3 Variational Methods

We will start this section with an important result from functional analysis, namely, the *Lax–Milgram lemma.* Let X be a Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$.

Definition 5.11. A mapping $a(\cdot, \cdot) : X \times X \to \mathbb{C}$ is called a *sesquilinear form* if

$$\begin{aligned} a(\lambda_1 u_1 + \lambda_2 u_2, v) &= \lambda_1 a(u_1, v) + \lambda_2 a(u_2, v) \\ &\qquad \text{for all } \lambda_1, \lambda_2 \in \mathbb{C},\ u_1, u_2, v \in X, \\ a(u, \mu_1 v_1 + \mu_2 v_2) &= \bar{\mu}_1 a(u, v_1) + \bar{\mu}_2 a(u, v_2) \\ &\qquad \text{for all } \mu_1, \mu_2 \in \mathbb{C},\ u, v_1, v_2 \in X, \end{aligned}$$

with the bar denoting the complex conjugation.

Definition 5.12. A mapping $F : X \to \mathbb{C}$ is called a *conjugate linear functional* if

$$F(\mu_1 v_1 + \mu_2 v_2) = \bar{\mu}_1 F(v_1) + \bar{\mu}_2 F(v_2) \text{ for all } \mu_1, \mu_2 \in \mathbb{C},\ v_1, v_2 \in X.$$

As will be seen later, we will be interested in solving the following problem: *given a conjugate linear functional $F : X \to \mathbb{C}$ and a sesquilinear form $a(\cdot, \cdot)$ on $X \times X$, find $u \in X$ such that*

$$a(u, v) = F(v) \qquad \textit{for all} \quad v \in X. \tag{5.18}$$

The solution to this problem is provided by the following lemma.

Theorem 5.13 (Lax–Milgram Lemma). *Assume that $a : X \times X \to \mathbb{C}$ is a sesquilinear form (not necessarily symmetric) for which there exist constants $\alpha, \beta > 0$ such that*

$$|a(u, v)| \leq \alpha \|u\|\, \|v\| \qquad \textit{for all} \quad u \in X,\ v \in X \tag{5.19}$$

and

$$|a(u, u)| \geq \beta \|u\|^2 \qquad \textit{for all} \quad u \in X. \tag{5.20}$$

Then for every bounded conjugate linear functional $F : X \to \mathbb{C}$ there exists a unique element $u \in X$ such that

$$a(u, v) = F(v) \qquad \textit{for all} \quad v \in X. \tag{5.21}$$

Furthermore, $\|u\| \leq C\|F\|$, where $C > 0$ is a constant independent of F.

Proof. For each fixed element $u \in X$ the mapping $v \to a(u, v)$ is a bounded conjugate linear functional on X, and hence the Riesz representation theorem asserts the existence of a unique element $w \in X$ satisfying

$$a(u, v) = (w, v) \qquad \text{for all} \quad v \in X.$$

Thus we can define an operator $A : X \to X$ mapping u to w such that

$$a(u, v) = (Au, v) \qquad \text{for all} \quad u, v \in X.$$

1. We first claim that $A : X \to X$ is a bounded linear operator. Indeed, if $\lambda_1, \lambda_2 \in \mathbb{C}$ and $u_1, u_2 \in X$, then we see, using the properties of the inner product in a Hilbert space, that for each $v \in X$ we have

$$\begin{aligned}(A(\lambda_1 u_1 + \lambda_2 u_2), v) &= a((\lambda_1 u_1 + \lambda_2 u_2), v)\\ &= \lambda_1 a(u_1, v) + \lambda_2 a(u_2, v)\\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v)\\ &= (\lambda_1 Au_1 + \lambda_2 Au_2, v)\,.\end{aligned}$$

   Since this holds for arbitrary $u_1, u_2, v \in X$, and $\lambda_1, \lambda_2 \in \mathbb{C}$, we have established linearity. Furthermore,

$$\|Au\|^2 = (Au, Au) = a(u, Au) \le \alpha \|u\|\,\|Au\|.$$

   Consequently, $\|Au\| \le \alpha \|u\|$ for all $u \in X$, and so A is bounded.
2. Next we show that A is one-to-one and the range of A is equal to X. To prove this, we compute

$$\beta \|u\|^2 \le |a(u, u)| = |(Au, u)| \le \|Au\|\,\|u\|.$$

   Hence, $\beta\|u\| \le \|Au\|$. This inequality implies that A is one-to-one and the range of A is closed in X. Now let $w \in A(X)^\perp$, and observe that $\beta\|w\|^2 \le a(w, w) = (Aw, w) = 0$, which implies that $w = 0$. Since $A(X)$ is closed, we can now conclude that $A(X) = X$.
3. Next, once more from the Riesz representation theorem, there exists a unique $\tilde{w} \in X$ such that

$$F(v) = (\tilde{w}, v) \qquad \text{for all} \quad v \in X$$

   and $\|\tilde{w}\| = \|F\|$. We then use part 2 of this proof to find a $u \in X$ satisfying $Au = \tilde{w}$. Then

$$a(u, v) = (Au, v) = (\tilde{w}, v) = F(v) \qquad \text{for all} \quad v \in X,$$

   which proves the solvability of (5.21). Furthermore, we have that

$$\|u\| \le \frac{1}{\beta}\|Au\| = \frac{1}{\beta}\|\tilde{w}\| = \frac{1}{\beta}\|F\|.$$

4. Finally, we show that there is at most one element $u \in X$ satisfying (5.21). If there exist $u \in X$ and $\tilde{u} \in X$ such that

$$a(u, v) = F(v) \quad \text{and} \quad a(\tilde{u}, v) = F(v) \qquad \text{for all} \quad v \in X,$$

   then

$$a(u - \tilde{u}, v) = 0 \qquad \text{for all} \quad v \in X.$$

   Hence, setting $v = u - \tilde{u}$ we obtain

$$\beta\|u - \tilde{u}\|^2 \le a(u - \tilde{u}, u - \tilde{u}) = 0,$$

whence $u = \tilde{u}$.

□

Remark 5.14. If a sesquilinear form $a(\cdot,\cdot)$ satisfies (5.19), then it is said that $a(\cdot,\cdot)$ is *continuous*. A sesquilinear form $a(\cdot,\cdot)$ satisfying (5.20) is called *strictly coercive*.

Example 5.15. As an example of an application of the Lax–Milgram lemma we consider the existence of a unique weak solution to the Dirichlet problem for the Poisson equation: given $f \in H^{\frac{1}{2}}(\partial D)$ and $\rho \in L^2(D)$, find $u \in H^1(D)$ such that

$$\begin{cases} \Delta u = -\rho & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases} \tag{5.22}$$

To motivate the definition of a $H^1(D)$ weak solution to the preceding Dirichlet problem, let us consider first $u \in C^2(D) \cap C^1(\bar{D})$ satisfying $\Delta u = -\rho$. Multiplying $\Delta u = -\rho$ by $\bar{v} \in C_0^\infty(D)$ and using Green's first identity we obtain

$$\int_D \nabla u \cdot \nabla \bar{v}\, dx = \int_D \rho \bar{v}\, dx, \tag{5.23}$$

which makes sense for $u \in H^1(D)$ and $v \in H_0^1(D)$ as well. Note that the boundary terms disappear when we apply Green's identity due to the fact that $v = 0$ on ∂D. Now we will use (5.23) to define a weak solution. To this end, we set $X = H_0(D)$ and define

$$a(w,v) = (\nabla w,\, \nabla v)_{L^2(D)}\,, \qquad w, v \in X.$$

In particular, it is clear that

$$|a(w,v)| \le \|\nabla w\|_{L^2(D)} \|\nabla v\|_{L^2(D)} \le \|w\|_{H^1(D)} \|v\|_{H^1(D)}.$$

Furthermore, from Poincaré's inequality there exists a constant $C > 0$ depending only on D such that

$$a(w,w) = \|\nabla w\|^2_{L^2(D)} \ge C \|w\|^2_{H^1(D)},$$

whence $a(\cdot,\,\cdot)$ satisfies the assumptions of the Lax–Milgram lemma.

Now let $u_0 \in H^1(D)$ be such that $u_0 = f$ on ∂D and $\|u_0\|_{H^1(D)} \le C\|f\|_{H^{\frac{1}{2}}(\partial D)}$. If $u = f$ on ∂D, then $u - u_0 \in H_0^1(D)$. Next we examine the following problem.
Find $u \in H^1(D)$ such that

$$\begin{cases} u - u_0 \in H_0^1(D), \\ a(u - u_0, v) = -a(u_0, v) + (\rho, v)_{L^2(D)} & \text{for all } v \in H_0^1(D). \end{cases} \tag{5.24}$$

A solution to (5.24) is called a *weak solution* of the Dirichlet problem (5.22), and (5.24) is called the *variational form* of (5.22).

Since $a(\cdot,\,\cdot)$ is continuous, the mapping $F: v \to -a(u_0, v) + (\rho, v)_{L^2(D)}$ is a bounded conjugate linear functional on $H_0^1(D)$. Therefore, from the Lax–Milgram lemma, (5.24) has a unique solution $u \in H^1(D)$ that satisfies

$$\|u\|_{H^1(D)} \leq C(\|u_0\|_{H^1(D)} + \|\rho\|_{L^2(D)}) \leq \tilde{C}(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|\rho\|_{L^2(D)}),$$

where the constant $\tilde{C} > 0$ is independent of f and ρ.

Obviously, any $C^2(D) \cap C^1(\bar{D})$ solution to the Dirichlet problem is a weak solution. Conversely, if the weak solution u is smooth enough (which depends on the smoothness of ∂D, f, and ρ – see [127]), then the weak solution satisfies (5.22) pointwise. Indeed, taking a function $v \in C_0^\infty(D)$ in (5.24) we see that

$$\int_D (\Delta u + \rho)\, v\, dx = 0 \qquad \text{for all } v \in C_0^\infty(D),$$

and hence $\Delta u = -\rho$ almost everywhere in D. Furthermore, $u - u_0 \in H_0^1(D)$ if and only if $u = u_0$ on ∂D, whence $u = f$ on ∂D.

We now return to the abstract variational problem (5.18) and consider it in the following form: find $u \in X$ such that

$$a(u, v) + b(u, v) = F(v) \qquad \text{for all} \quad v \in X, \tag{5.25}$$

where X is a Hilbert space, $a, b : X \times X \to \mathbb{C}$ are two continuous sesquilinear forms, and F is a bounded conjugate linear functional on X. In addition:

1. Assume that the continuous sesquilinear form $a(\cdot,\,\cdot)$ is strictly coercive, i.e., $a_1(u, u) \geq \alpha\|u\|^2$ for some positive constant α. From the Lax–Milgram lemma we then have that there exists a bijective bounded linear operator $A : X \to X$ with bounded inverse satisfying

$$a(u, v) = (Au, v) \qquad \text{for all} \quad v \in X.$$

2. Let us denote by B the bounded linear operator from X to X defined by

$$b(u, v) = (Bu, v) \qquad \text{for all} \quad v \in X.$$

   The existence and the continuity of B are guaranteed by the Riesz representation theorem (see also the first part of the proof of the Lax–Milgram lemma). We further assume that the operator B is compact.
3. Finally, let $w \in X$ be such that

$$F(v) = (w, v) \qquad \text{for all} \quad v \in X,$$

   which is uniquely provided by the Riesz representation theorem.

Under assumptions 1–3, (5.25) equivalently reads as follows:

$$\text{Find } u \in X \text{ such that } Au + Bu = w. \tag{5.26}$$

Theorem 5.16. *Let X and Y be two Hilbert spaces, and let $A : X \to Y$ be a bijective bounded linear operator with bounded inverse $A^{-1} : Y \to X$, and $B : X \to Y$ a compact linear operator. Then $A + B$ is injective if and only if it is surjective. If $A + B$ is injective (and hence bijective), then the inverse $(A + B)^{-1} : Y \to X$ is bounded.*

Proof. Since A^{-1} exists, we have that $A+B = A(I-(-A^{-1})B)$. Furthermore, since A is a bijection, $(I-(-A^{-1})B)$ is injective and surjective if and only if $A+B$ is injective and surjective. Next we observe that $(-A^{-1})B$ is a compact operator since it is the product of a compact operator and a bounded operator. The result of the theorem now follows from Theorem 1.21 and the fact that $(A + B)^{-1} = (I - (-A^{-1})B)^{-1}A^{-1}$. □

Example 5.17. Consider now the Dirichlet problem for the Helmholtz equation in a bounded domain D: Given $f \in H^{\frac{1}{2}}(\partial D)$, find $u \in H^1(D)$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } D, \\ u = f & \text{on } \partial D, \end{cases} \tag{5.27}$$

where k is real. Following Example 5.15, we can write this problem in the following variational form: *find $u \in H^1(D)$ such that*

$$\begin{cases} u - u_0 \in H_0^1(D), & \\ a(u - u_0, v) = -a(u_0, v) & \text{for all } v \in H_0^1(D), \end{cases} \tag{5.28}$$

where u_0 is a function in $H^1(D)$ such that $u_0 = f$ on ∂D and $\|u_0\|_{H^1(D)} \leq C\|f\|_{H^{\frac{1}{2}}(\partial D)}$, and the sesquilinear form $a(\cdot,\, \cdot)$ is defined by

$$a(w, v) := \int_D \left(\nabla w \cdot \nabla \bar{v} - k^2 w\bar{v}\right) dx, \qquad w, v \in H_0^1(D).$$

Obviously, $a(\cdot,\, \cdot)$ is continuous but not strictly coercive. Defining

$$a_1(w, v) := \int_D \nabla w \cdot \nabla \bar{v}\, dx, \qquad w, v \in H_0^1(D)$$

and

$$a_2(w, v) := -k^2 \int_D w\bar{v}\, dx, \qquad w, v \in H_0^1(D)$$

we have that

$$a(w,v) = a_1(w,v) + a_2(w,v),$$

where now $a_1(\cdot,\,\cdot)$ is strictly coercive in $H_0^1(D) \times H_0^1(D)$ (Example 5.15). Let $A : H_0^1(D) \to H_0^1(D)$ and $B : H_0^1(D) \to H_0^1(D)$ be bounded linear operators defined by $(Au,v) = a_1(u,\,v)$ and

$$(Bu,v) = \int_D u\bar{v}\,dx \qquad \text{for all} \quad v \in H_0^1(D),$$

respectively. In particular, A is bounded and has a bounded inverse. We claim that $B : H_0^1(D) \to H_0^1(D)$ is compact. To see this, we first note that

$$\begin{aligned}\|Bu\|^2_{H^1(D)} = (Bu, Bu) = \int_D u\overline{Bu}\,dx &\le \|u\|_{L^2(D)}\|Bu\|_{L^2(D)} \\ &\le \|u\|_{L^2(D)}\|Bu\|_{H^1(D)},\end{aligned}$$

and hence $\|Bu\|_{H^1(D)} \le \|u\|_{L^2(D)}$. Now let $\{u_j\} \subset H_0^1(D)$ be such that $\|u_j\|_{H_0^1(D)} \le C$ for some positive constant C independent of j. Then, since by Rellich's theorem $H^1(D)$, and hence $H_0^1(D)$, is compactly embedded in $L^2(D)$, we have that there exists a subsequence, still denoted by $\{u_j\}$, such that $\{u_j\}$ is strongly convergent in $L^2(D)$, i.e., $\{u_j\}$ is a Cauchy sequence in $L^2(D)$. Since $\|Bu\|_{H^1(D)}$ is bounded by $\|u\|_{L^2(D)}$, we have that $\{Bu_j\}$ is a Cauchy sequence in $H_0^1(D)$, and hence $\{Bu_j\}$ is strongly convergent. This now implies that B is compact, as claimed.

We can now apply Theorem 5.16 to (5.28). In particular, the injectivity of $A - k^2B$ implies the existence of a unique solution to (5.28). The injectivity of $A - k^2B$ is equivalent to the fact that the only function $u \in H_0^1(D)$ that satisfies

$$a(u,v) = 0 \qquad \text{for all} \quad v \in H_0^1(D)$$

is $u \equiv 0$. This is the uniqueness question for a weak solution to the Dirichlet boundary value problem for the Helmholtz equation. The values of k^2 for which there exists a nonzero function $u \in H_0^1(D)$ satisfying

$$\Delta u + k^2 u = 0 \qquad \text{in} \;\; D$$

(in the weak sense) are called the *Dirichlet eigenvalues* of $-\Delta$ and the corresponding nonzero solutions are called the *eigensolutions* for $-\Delta$. Note that the zero boundary condition is incorporated in the space $H_0^1(D)$.

Summarizing the preceding analysis, we have shown that if k^2 is not a Dirichlet eigenvalue for $-\Delta$, then (5.27) has a unique solution in $H^1(D)$.

Theorem 5.18. *There exists an orthonormal basis u_j for $H_0^1(D)$ consisting of eigensolutions for $-\Delta$. The corresponding eigenvalues k^2 are all positive and accumulate only at $+\infty$.*

Proof. In Example 5.17 we showed that $u \in H_0^1(D)$ satisfies

$$\Delta u + k^2 u = 0 \qquad \text{in} \;\; D$$

if and only if u is a solution to the operator equation $Au - k^2 Bu = 0$, where $A : H_0^1(D) \to H_0^1(D)$ and $B : H_0^1(D) \to H_0^1(D)$ are the bijective operator and compact operator, respectively, constructed in Example 5.17. Since A is a positive definite operator, the equation $Au - k^2 Bu = 0$ can be written as (see [115] for the existence of the operator $A^{\frac{1}{2}}$)

$$\left(\frac{1}{k^2} I - A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) u = 0 \qquad u \in H_0^1(D).$$

It is easily verified that A (and hence $A^{-\frac{1}{2}}$) is self-adjoint. Since B is self-adjoint, we can conclude that $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ is self-adjoint. Now noting that $A^{-\frac{1}{2}} B A^{-\frac{1}{2}} : H_0^1(D) \to H_0^1(D)$ is compact since it is a product of a compact operator and bounded operators, the result follows from the Hilbert–Schmidt theorem. □

Remark 5.19. The results of Examples 5.15 and 5.17 are valid as well if D is not simply connected, i.e., $\mathbb{R}^2 \setminus \bar{D}$ is not connected.

The boundary value problems arising in scattering theory are formulated in unbounded domains. To solve such problems using variational techniques developed in this section, we need to write them as equivalent problems in a bounded domain. In particular, introducing a large open disk Ω_R centered at the origin that contains $\bar{D}$, where D is the support of the scatterer, we first solve the problem in $\Omega_R \setminus \bar{D}$ (or in Ω_R in the case of transmission problems) using variational methods. Having solved this problem, we then want to extend the solution outside Ω_R to a solution to the original problem. The main question here is what boundary condition should we impose on the artificial boundary $\partial\Omega_R$ to enable such an extension. To find the appropriate boundary conditions on $\partial\Omega_R$, we introduce the *Dirichlet-to-Neumann map*. We first formalize the definition of a *radiating solution* to the Helmholtz equation.

Definition 5.20. A solution u to the Helmholtz equation whose domain of definition contains the exterior of some disk is called radiating if it satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0,$$

where $r = |x|$ and the limit is assumed to hold uniformly in all directions $x/|x|$.

Definition 5.21. The Dirichlet-to-Neumann map T is defined by

$$T : w \to \frac{\partial w}{\partial \nu} \qquad \text{on} \quad \partial\Omega_R,$$

where w is a radiating solution to the Helmholtz equation $\Delta w + k^2 w = 0$, $\partial\Omega_R$ is the boundary of some disk of radius R, and ν is the outward unit normal to $\partial\Omega_R$.

Taking advantage of the fact that Ω_R is a disk, by separating variables as in Sect. 3.2 we can find a solution to the exterior Dirichlet problem outside Ω_R in the form of a series expansion involving Hankel functions. Making use of this expansion we can establish the following important properties of the Dirichlet-to-Neumann map.

Theorem 5.22. *The Dirichlet-to-Neumann map T is a bounded linear operator from $H^{\frac{1}{2}}(\partial\Omega_R)$ to $H^{-\frac{1}{2}}(\partial\Omega_R)$. Furthermore, there exists a bounded operator $T_0 : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ satisfying*

$$-\int\limits_{\partial\Omega_R} T_0 w\, \overline{w}\, ds \geq C\|w\|^2_{H^{\frac{1}{2}}(\partial\Omega_R)} \tag{5.29}$$

for some constant $C > 0$ such that $T - T_0 : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ is compact.

Proof. Let w be a radiating solution to the Helmholtz equation outside Ω_R, and let (r, θ) denote polar coordinates in $\mathbb{R}^2$. Then from Sect. 3.2 we have that

$$w(r,\theta) = \sum_{-\infty}^{\infty} \alpha_n H_n^{(1)}(kr) e^{in\theta}, \quad r \geq R \text{ and } 0 \leq \theta \leq 2\pi,$$

where $H_n^{(1)}(kr)$ are the Hankel functions of the first kind of order n. Hence T maps the Dirichlet data of $w|_{\partial\Omega_R}$ given by

$$w|_{\partial\Omega_R} = \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

with coefficients $a_n := \alpha_n H_n^{(1)}(kR)$ onto the corresponding Neumann data given by

$$Tw = \sum_{-\infty}^{\infty} a_n \gamma_n e^{in\theta},$$

where

$$\gamma_n := \frac{k H_n^{(1)'}(kR)}{H_n^{(1)}(kR)}, \qquad n = 0, \pm 1, \ldots .$$

The Hankel functions and their derivatives do not have real zeros since otherwise the Wronskian (3.22) would vanish. From this we observe that T is bijective. In view of the asymptotic formulas for the Hankel functions developed in Sect. 3.2 we see that

$$c_1|n| \le |\gamma_n| \le c_2|n|, \qquad n = \pm 1, \pm 2, \dots$$

and some constants $0 < c_1 < c_2$. From this the boundness of $T : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ is obvious since from Theorem 1.33 for $p \in \mathbb{R}$ the norm on $H^p(\partial\Omega_R)$ can be described in terms of the Fourier coefficients

$$\|w\|^2_{H^p(\partial\Omega_R)} = \sum_{-\infty}^{\infty}(1+n^2)^p|a_n|^2.$$

For the limiting operator $T_0 : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ given by

$$T_0 w = -\sum_{-\infty}^{\infty} \frac{|n|}{R} a_n e^{in\theta}$$

we clearly have

$$-\int_{\Omega_R} T_0 w\, \overline{w}\, ds = \sum_{-\infty}^{\infty} 2\pi|n||a_n|^2,$$

with the integral to be understood as the duality pairing between $H^{\frac{1}{2}}(\partial\Omega_R)$ and $H^{-\frac{1}{2}}(\partial\Omega_R)$. Hence

$$-\int_{\partial\Omega_R} T_0 w\, \overline{w}\, ds \ge C\|w\|^2_{H^{\frac{1}{2}}(\partial\Omega_R)}$$

for some constant $C > 0$. Finally, from the series expansions for the Bessel and Neumann functions (Sect. 3.2) for fixed k we derive

$$\gamma_n = -\frac{|n|}{R}\left\{1 + O\left(\frac{1}{|n|}\right)\right\}, \qquad n \to \pm\infty.$$

This implies that $T - T_0$ is compact from $H^{\frac{1}{2}}(\partial\Omega_R)$ into $H^{-\frac{1}{2}}(\partial\Omega_R)$ since it is bounded from $H^{\frac{1}{2}}(\partial\Omega_R)$ into $H^{\frac{1}{2}}(\partial\Omega_R)$ and the embedding from $H^{\frac{1}{2}}(\partial\Omega_R)$ into $H^{-\frac{1}{2}}(\partial\Omega_R)$ is compact by Rellich's Theorem 1.32. This proves the theorem. □

Example 5.23. We consider the problem of finding a weak solution to the exterior Dirichlet problem for the Helmholtz equation: given $f \in H^{\frac{1}{2}}(\partial D)$, find $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^2 \setminus \bar{D}, \\ u = f & \text{on } \partial D, \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u}{\partial r} - iku\right) = 0. \end{cases} \tag{5.30}$$

Instead of (5.30) we solve an equivalent problem in the bounded domain $\Omega_R \setminus \bar{D}$, that is, we find $u \in H^1(\Omega_R \setminus \bar{D})$ such that

$$
\begin{cases}
\Delta u + k^2 u = 0 & \text{in } \Omega_R \setminus \bar{D}, \\
u = f & \text{on } \partial D, \\
\dfrac{\partial u}{\partial \nu} = Tu & \text{on } \partial \Omega_R,
\end{cases}
\tag{5.31}
$$

where $f \in H^{\frac{1}{2}}(\partial D)$ is the given boundary data, T is the Dirichlet-to-Neumann map, and Ω_R is a large disk containing $\bar{D}$.

Lemma 5.24. *Problems (5.30) and (5.31) are equivalent.*

Proof. First let $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ be a solution to (5.30). Then the restriction of u to $\Omega_R \setminus \bar{D}$ is in $H^1(\Omega_R \setminus \bar{D})$ and is a solution to (5.31). Conversely, let $u \in H^1(\Omega_R \setminus \bar{D})$ be a solution to (5.31). To define u in all of $\mathbb{R}^2 \setminus \bar{D}$, we construct the radiating solution $\tilde{u}$ of the Helmholtz equation outside Ω_R such that $\tilde{u} = u$ on $\partial \Omega_R$. This solution can be constructed in the form of a series expansion in terms of Hankel functions in the same way as in the proof of Theorem 5.22. Hence we have that $Tu = \dfrac{\partial \tilde{u}}{\partial \nu}$. Using Green's second identity for the radiating solution $\tilde{u}$ and the fundamental solution $\Phi(x,y)$ (which is also a radiating solution) we obtain that

$$
\int_{\partial \Omega_R} \left[(Tu)(y) \Phi(x,y) - u(y) \frac{\partial \Phi(x,y)}{\partial \nu} \right] ds_y = 0, \qquad x \in \Omega_R.
$$

Consequently, the representation formula (3.41) (Remark 6.29) and the fact that $\dfrac{\partial u}{\partial \nu} = Tu$ imply

$$
\begin{aligned}
u(x) &= \int_{\partial D} \left[u(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial u}{\partial \nu} \Phi(x,y) \right] ds_y \\
&\quad - \int_{\partial \Omega_R} \left[u(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial u}{\partial \nu} \Phi(x,y) \right] ds_y \\
&= \int_{\partial D} \left[u(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial u}{\partial \nu} \Phi(x,y) \right] ds_y .
\end{aligned}
$$

Therefore, u coincides with the radiating solution to the Helmholtz equation in the exterior of $\bar{D}$. Hence a solution of (5.30) can be derived from a solution to (5.31). □

Next we formulate (5.31) as a variational problem. To this end, we define the Hilbert space

$$
X := \{ u \in H^1(\Omega_R \setminus \bar{D}) : \; u = 0 \quad \text{on } \partial D \}
$$

and the sesquilinear from $a(\cdot,\,\cdot)$ by

$$a(u,v) = \int\limits_{\Omega_R\setminus\bar{D}} \left(\nabla u\cdot\nabla\overline{v} - k^2 u\overline{v}\right)\,dx - \int\limits_{\partial\Omega_R} Tu\,\overline{v}\,ds,$$

which is obtained by multiplying the Helmholtz equation in (5.31) by a test function $v \in X$, integrating by parts, and using the boundary condition $\partial u/\partial\nu = Tu$ on $\partial\Omega_R$ and the zero boundary condition on ∂D. Now let $u_0 \in H^1(\Omega_R \setminus \bar{D})$ be such that $u_0 = f$ on ∂D. Then the variational formulation of (5.31) reads: *find* $u \in H^1(\Omega_R \setminus \bar{D})$ *such that*

$$\begin{cases} u - u_0 \in X, \\ a(u-u_0,v) = -a(u_0,v) \qquad \text{for all } v \in X. \end{cases} \tag{5.32}$$

To analyze (5.32) we define

$$a_1(w,v) = \int\limits_{\Omega_R\setminus\bar{D}} (\nabla w\cdot\nabla\overline{v} + w\overline{v})\,dx - \int\limits_{\partial\Omega_R} T_0 w\,\overline{v}\,ds$$

and

$$a_2(w,v) = -(k^2+1)\int\limits_{\Omega_R\setminus\bar{D}} w\overline{v}\,dx - \int\limits_{\partial\Omega_R} (T-T_0)w\,\overline{v}\,ds,$$

where T_0 is the operator defined in Theorem 5.22, and write the equation in (5.32) as

$$a_1(u-u_0,v) + a_2(u-u_0,v) = F(v), \qquad \text{for all } v \in X,$$

with $F(v) := a(u_0,v)$. Since T is a bounded operator from $H^{\frac{1}{2}}(\partial\Omega_R)$ to $H^{-\frac{1}{2}}(\partial\Omega_R)$, F is a bounded conjugate linear functional on X and both $a_1(\cdot,\,\cdot)$ and $a_2(\cdot,\,\cdot)$ are continuous on $X \times X$. In addition, using (5.29), we see that

$$a_1(w,\,w) \ge C\|w\|^2_{H^1(\Omega_R\setminus\bar{D})}.$$

Note that including a L^2-inner product term in $a_1(\cdot,\,\cdot)$ is important since the Poincaré inequality no longer holds in X. Furthermore, due to the compact embedding of $H^1(\Omega_R \setminus \bar{D})$ into $L^2(\Omega_R \setminus \bar{D})$ and the fact that $T - T_0 : H^{\frac{1}{2}}(\partial\Omega_R) \to H^{-\frac{1}{2}}(\partial\Omega_R)$ is compact, $a_2(\cdot,\,\cdot)$ gives rise to a compact operator $B : X \to X$ (Example 5.17). Hence from Theorem 5.16 we conclude that the uniqueness of a solution to (5.31) implies the existence of a solution to (5.31) and, consequently, from Lemma 5.24 the existence of a weak solution to (5.30). To prove the uniqueness of a solution to (5.31) we first observe that according to Lemma 5.24 a solution to the homogeneous problem (5.31) ($f = 0$) can be extended to a solution to the homogeneous problem (5.30). Now let u be a solution to the homogeneous problem (5.30). Then Green's first identity and the boundary condition imply

$$\int_{\partial\Omega_R} \frac{\partial u}{\partial \nu}\overline{u}\,ds = \int_{\partial D} \frac{\partial u}{\partial \nu}\overline{u}\,ds + \int_{\Omega_R\setminus\bar{D}} \left(|\nabla u|^2 - k^2|u|^2\right)\,dx \tag{5.33}$$

$$= \int_{\Omega_R\setminus\bar{D}} \left(|\nabla u|^2 - k^2|u|^2\right)\,dx, \tag{5.34}$$

whence

$$\mathrm{Im}\left(\int_{\partial\Omega_R} \frac{\partial u}{\partial \nu}\overline{u}\,ds\right) = 0.$$

From Theorem 3.6 we conclude that $u = 0$ in $\mathbb{R}^2 \setminus \bar{D}$, which proves the uniqueness and, therefore, the existence of a unique weak solution to the exterior Dirichlet problem for the Helmholtz equation. Note that in the preceding proof of uniqueness we have used the fact that off the boundary an $H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ solution to the Helmholtz equation is real-analytic. This can be seen from the Green representation formula as in Theorem 3.2, which is also valid for radiating solutions to the Helmholtz equation in $H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ (Remark 6.29).

In this section we have developed variational techniques for finding weak solutions to boundary value problems for partial differential equations. As the reader has already seen, in scattering problems the boundary conditions are typically the traces of real-analytic solutions, for example, plane waves. Hence, provided that the boundary of the scattering object is smooth, one would expect that the scattered field would not, in fact, be smooth. It can be shown that if the boundary, the boundary conditions, and the coefficients of the equations are smooth enough, then a weak solution is in fact C^2 inside the domain and C^1 up to the boundary. This general statement falls in the class of so-called regularity results for the solutions of boundary value problems for elliptic partial differential equations. Precise formulation of such results can be found in any classic book of partial differential equations (cf. [72] and [127]).

5.4 Solution of Direct Scattering Problem

We now turn our attention to the main goal of this chapter, the solution to the scattering problem (5.13)–(5.17). Following Hähner [81], we shall use the variational techniques developed in Sect. 5.3 to find a solution to this problem. To arrive at a variational formulation of (5.13)–(5.17), we introduce a large open disk Ω_R centered at the origin containing $\bar{D}$ and consider the following problem: given $f \in H^{\frac{1}{2}}(\partial D)$ and $h \in H^{-\frac{1}{2}}(\partial D)$, find $u \in H^1(\Omega_R \setminus \bar{D})$ and $v \in H^1(D)$ such that

$$\nabla \cdot A\nabla v + k^2 n\, v = 0 \quad \text{in} \quad D, \tag{5.35}$$

$$\Delta u + k^2\, u = 0 \quad \text{in} \quad \Omega_R \setminus \bar{D}, \tag{5.36}$$

$$v - u = f \quad \text{on} \quad \partial D, \tag{5.37}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = h \quad \text{on} \quad \partial D, \tag{5.38}$$

$$\frac{\partial u}{\partial \nu} = Tu \quad \text{on} \quad \partial \Omega_R, \tag{5.39}$$

where T is the Dirichlet-to-Neumann operator defined in Definition 5.21.

We note that exactly in the same way as in the proof of Lemma 5.24 one can show that a solution u, v to (5.35)–(5.39) can be extended to a solution to the scattering problem (5.13)–(5.17) and, conversely, a solution u, v to the scattering problem (5.13)–(5.17) is such that v and u restricted to $\Omega_R \setminus \bar{D}$ solve (5.35)–(5.39).

Next let $u_f \in H^1(\Omega_R \setminus \bar{D})$ be the unique solution to the following Dirichlet boundary value problem:

$$\Delta u_f + k^2 u_f = 0 \quad \text{in } \Omega_R \setminus \bar{D}, \qquad u_f = f \quad \text{on } \partial D, \qquad u_f = 0 \quad \text{on } \partial \Omega_R.$$

The existence of a unique solution to this problem is shown in Example 5.17 (see also Remark 5.19). Note that we can always choose Ω_R such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in $\Omega_R \setminus \bar{D}$. An equivalent variational formulation of (5.35)–(5.39) is as follows: find $w \in H^1(\Omega_R)$ such that

$$\int\limits_D \left(\nabla\overline{\phi} \cdot A\nabla w - k^2 n\overline{\phi} w\right) dx + \int\limits_{\Omega_R \setminus \bar{D}} \left(\nabla\overline{\phi} \cdot \nabla w - k^2\overline{\phi} w\right) dx \tag{5.40}$$
$$- \int\limits_{\partial\Omega_R} \overline{\phi}\, Tw\, ds = \int\limits_{\partial D} \overline{\phi} h\, ds - \int\limits_{\partial\Omega_R} \overline{\phi}\, Tu_f\, ds + \int\limits_{\Omega_R \setminus \bar{D}} \left(\nabla\overline{\phi} \cdot \nabla u_f - k^2\overline{\phi} u_f\right) dx$$

for all $\phi \in H^1(\Omega_R)$. With the help of Green's first identity (Corollary 5.8 and Remark 6.29) it is easy to see that $v := w|_D$ and $u := w|_{\Omega_R \setminus \bar{D}} - u_f$ satisfy (5.35)–(5.39). Conversely, multiplying the equations in (5.35)–(5.39) by a test function and using the transmission conditions one can show that $w = v$ in D and $w = u + u_f$ in $\Omega_R \setminus \bar{D}$ is such that $w \in H^1(\Omega_R)$ and satisfies (6.68), where v, u solve (5.35)–(5.39).

Next we define the following continuous sesquilinear forms on $H^1(\Omega_R) \times H^1(\Omega_R)$:

$$a_1(\psi, \phi) := \int\limits_D \left(\nabla\overline{\phi} \cdot A\nabla\psi + \overline{\phi}\,\psi\right) dx + \int\limits_{\Omega_R \setminus \bar{D}} \left(\nabla\overline{\phi} \cdot \nabla\psi + \overline{\phi}\,\psi\right) dx$$
$$- \int\limits_{\partial\Omega_R} \overline{\phi}\, T_0 \psi\, ds \qquad \phi, \psi \in H^1(\Omega_R)$$

and

$$a_2(\psi,\phi) := -\int_D (nk^2+1)\overline{\phi}\psi\,dx - \int_{\Omega_R\setminus\bar{D}} (k^2+1)\overline{\phi}\,\psi\,dx$$
$$-\int_{\partial\Omega_R} \overline{\phi}\,(T-T_0)\psi\,ds \qquad\qquad \phi,\psi\in H^1(\Omega_R),$$

where the operator T_0 is the operator defined in Theorem 5.22. Furthermore, we define the bounded conjugate linear functional F on $H^1(\Omega_R)$ by

$$F(\phi) := \int_{\partial D} \overline{\phi}h\,ds - \int_{\partial\Omega_R} \overline{\phi}\,Tu_f\,ds + \int_{\Omega_R\setminus\bar{D}} \left(\nabla\overline{\phi}\cdot\nabla u_f - k^2\overline{\phi}u_f\right)dx.$$

Then (6.68) can be written as the problem of finding $w\in H^1(\Omega_R)$ such that

$$a_1(w,\phi) + a_2(w,\phi) = F(\phi) \qquad \text{for all } \phi\in H^1(\Omega_R).$$

From the assumption $\bar{\xi}\cdot\mathrm{Re}(A)\,\xi \geq \gamma|\xi|^2$ for all $\xi\in\mathbb{C}^3$ and $x\in\overline{D}$ and (5.29) we can conclude that the sesquilinear form $a_1(\cdot,\,\cdot)$ is strictly coercive. Hence, as a consequence of the Lax–Milgram lemma, the operator $A: H^1(\Omega_R)\to H^1(\Omega_R)$ defined by $a_1(w,\phi) = (Aw,\phi)_{H^1(\Omega_R)}$ is invertible with bounded inverse. Furthermore, due to the compact embedding of $H^1(\Omega_R)$ into $L^2(\Omega_R)$ and the fact that $T-T_0 : H^{\frac{1}{2}}(\partial\Omega_R)\to H^{-\frac{1}{2}}(\partial\Omega_R)$ is compact (Theorem 5.22), we can show exactly in the same way as in Example 5.17 that the operator $B: H^1(\Omega_R)\to H^1(\Omega_R)$ defined by $a_2(w,\phi) = (Bw,\phi)_{H^1(\Omega_R)}$ is compact. Finally, by Theorem 5.16, the uniqueness of a solution to (5.35)–(5.39) implies that a solution exists.

Lemma 5.25. *The problems (5.35)–(5.39) and (5.13)–(5.17) have at most one solution.*

Proof. According to our previous remarks, a solution to the homogeneous problem (5.35)–(5.39) ($f=h=0$) can be extended to a solution $v\in H^1(D)$ and $u\in H^1_{loc}(\mathbb{R}^2\setminus\bar{D})$ to the homogeneous problem (5.13)–(5.17). Therefore, it suffices to prove uniqueness for (5.13)–(5.17). Green's first identity and the transmission conditions imply that

$$\int_{\partial\Omega_R} \overline{u}\frac{\partial u}{\partial\nu}\,ds = \int_{\partial D} \overline{u}\frac{\partial u}{\partial\nu}\,ds + \int_{\Omega_R\setminus\overline{D}} \left(|\nabla u|^2 - k^2|u|^2\right)^2\,dx$$
$$= \int_D \left(\nabla\overline{v}\cdot A\nabla v - k^2n|v|^2\right)^2\,dx + \int_{\Omega_R\setminus\overline{D}} \left(|\nabla u|^2 - k^2|u|^2\right)^2\,dx.$$

Now since $\bar{\xi}\cdot\mathrm{Im}(A)\,\xi \leq 0$ for all $\xi\in\mathbb{C}^2$ and $\mathrm{Im}(n) > 0$ for $x\in\overline{D}$, we conclude that

$$\operatorname{Im}\left(\int_{\partial\Omega_R} \overline{u}\frac{\partial u}{\partial \nu}\,ds\right) \leq 0,$$

which from Theorem 3.6 implies that $u = 0$ in $\mathbb{R}^2 \setminus \bar{D}$. From the transmission conditions we can now conclude that $v = 0$ and $\partial v/\partial \nu_A = 0$ on ∂D.

To conclude that $v = 0$ in D, we employ a unique continuation principle. To this end, we extend $\operatorname{Re}(A)$ to a real, symmetric, positive definite, and continuously differentiable matrix-valued function in $\overline{\Omega}_R$ and $\operatorname{Im}(A)$ to a real, symmetric, continuously differentiable, matrix-valued function that is compactly supported in Ω_R. We also choose a continuously differentiable extension of n into $\overline{\Omega}_R$ and define $v = 0$ in $\Omega_R \setminus \bar{D}$. Since $v = 0$ and $\partial v/\partial \nu_A = 0$ on ∂D, then $v \in H^1(\Omega_R)$ and satisfies $\nabla \cdot A\nabla v + k^2 nv = 0$ in Ω_R. Then, by the regularity result in the interior of Ω_R (Theorem 5.27), v is smooth enough to apply the unique continuation principle (Theorem 17.2.6 in [89]). In particular, since $v = 0$ in $\Omega_R \setminus \bar{D}$, then $v = 0$ in Ω_R. This proves the uniqueness. □

Summarizing the preceding analysis, we have proved the following theorem on the existence, uniqueness, and continuous dependence on the data of a solution to the direct scattering problem for an orthotropic medium in $\mathbb{R}^2$.

Theorem 5.26. *Assume that D, A, and n satisfy the assumptions in Sect. 5.1, and let $f \in H^{\frac{1}{2}}(\partial D)$ and $h \in H^{-\frac{1}{2}}(\partial D)$ be given. Then the transmission problem (5.13)–(5.17) has a unique solution $v \in H^1(D)$ and $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$, which satisfy*

$$\|v\|_{H^1(D)} + \|u\|_{H^1(\Omega_R\setminus\bar{D})} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\right), \tag{5.41}$$

with $C > 0$ a positive constant independent of f and h.

Note that the a priori estimate (5.41) is obtained using the fact that by a duality argument $\|F\|$ is bounded by $\|h\|_{H^{-\frac{1}{2}}(\partial D)}$ and $\|u_f\|_{H^1(\Omega_R\setminus\bar{D})}$, which in turn is bounded by $\|f\|_{H^{\frac{1}{2}}(\partial D)}$ (Example 5.17).

We end this section by stating two regularity results from the general theory of partial differential equations formulated for our transmission problem. The proofs of these results are rather technical and beyond the scope of this book.

Let D_1 and D_2 be bounded, open subsets of $\mathbb{R}^2$ such that $\bar{D}_1 \subset D_2$, and assume that A is a matrix-valued function with continuously differentiable entries $a_{jk} \in C^1(\bar{D}_2)$ and $n \in C^1(\bar{D}_2)$. Furthermore, suppose that A is symmetric and satisfies $\bar{\xi} \cdot \operatorname{Re}(A)\,\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$ and $x \in \overline{D}_2$ for some constant $\gamma > 0$.

Theorem 5.27. *If $u \in H^1(D_2)$ and $q \in L^2(D_2)$ satisfy*

$$\nabla \cdot A\nabla u + k^2 nu = q,$$

then $u \in H^2(D_1)$ *and*

$$\|u\|_{H^2(D_1)} \leq C \left(\|u\|_{H^1(D_2)} + \|q\|_{L^2(D_2)} \right),$$

where $C > 0$ *depends only on* γ, D_1 *and* D_2.

For a proof of this theorem in a more general formulation see Theorem 4.16 in [127] or Theorem 15.1 in [70]. Note also that a more general interior regularity theorem shows that if the entries of A and n are smoother than C^1 and q is smoother than L^2, then one can improve the regularity of u, and this eventually leads to a C^2 solution in the interior of D_2.

For later use, in the next theorem we state a local boundary regularity result for the solution to the transmission problem (5.13)–(5.17). By $\Omega_\epsilon(z)$ we denote an open ball centered at $z \in \mathbb{R}^2$ of radius ϵ.

Theorem 5.28. *Assume* $z \in \partial D$, *and let* $u^i \in H^1(D)$ *such that* $\Delta u^i \in L^2(D)$. *Define* $f := u^i$ *and* $h := \partial u^i / \partial \nu$ *on* ∂D.

1. *If for some* $\epsilon > 0$ *the incident wave* u^i *is also defined in* $\Omega_{2\epsilon}(z)$ *and the restriction of* u^i *to* $\Omega_{2\epsilon}(z)$ *is in* $H^2(\Omega_{2\epsilon}(z))$, *then the solution* u *to (5.13)–(5.17) satisfies* $u \in H^2((\mathbb{R}^2 \setminus \overline{D}) \cap \Omega_\epsilon(z))$ *and there is a positive constant* C *such that*

$$\|u\|_{H^2((\mathbb{R}^2 \setminus \overline{D}) \cap \Omega_\epsilon(z))} \leq C \left(\|u^i\|_{H^2(\Omega_{2\epsilon}(z))} + \|u^i\|_{H^1(D)} \right).$$

2. *If for some* $\epsilon > 0$ *the incident wave* u^i *is also defined in* $\Omega_R \setminus \Omega_\epsilon(z)$ *and the restriction of* u^i *to* $\Omega_R \setminus \Omega_\epsilon(z)$ *is in* $H^2(\Omega_R \setminus \Omega_\epsilon(z))$, *then the solution* u *to (5.13)–(5.17) satisfies* $u \in H^2(\mathbb{R}^2 \setminus (\overline{D} \cup \Omega_{2\epsilon}(z)))$ *and there is a positive constant* C *such that*

$$\|u\|_{H^2(\mathbb{R}^2 \setminus (\overline{D} \cup \Omega_{2\epsilon}(z)))} \leq C \left(\|u^i\|_{H^2(\Omega_R \setminus \Omega_\epsilon(z))} + \|u^i\|_{H^1(D)} \right).$$

This result is proved in Theorem 2 in [81]. The proof employs the interior regularity result stated in Theorem 5.27 and techniques from Theorem 8.8 in [72].