3

Scattering by Imperfect Conductors

In this chapter we consider a very simple scattering problem corresponding to the scattering of a time-harmonic plane wave by an imperfect conductor. Although the problem is simple compared to most problems in scattering theory, its mathematical resolution took many years to accomplish and was the focus of energy of some of the outstanding mathematicians of the twentieth century, in particular Kupradze, Rellich, Vekua, Müller, and Weyl. Indeed, the solution of the full three-dimensional problem was not fully realized until 1981 (cf. Sect. 9.5 of [54]). Here we will content ourselves with the two-dimensional scalar problem and its solution by the method of integral equations. As will be seen, the main difficulty of this approach is the presence of eigenvalues of the interior Dirichlet problem for the Helmholtz equation, and we will overcome this difficulty using the ideas of Jones [96], Ursell [156], and Kleinman and Roach [109].

The plan of this chapter is as follows. We begin by considering Maxwell's equations and then derive the scalar impedance boundary value problem corresponding to the scattering of a time-harmonic plane wave by an imperfectly conducting infinite cylinder. After a brief detour to discuss the relevant properties of Bessel and Hankel functions that will be needed in the sequel, we proceed to show that our scattering problem is well posed by deriving Rellich's lemma and using the method of modified single layer potentials. We will conclude this chapter by giving a brief discussion on weak solutions of the Helmholtz equation. (This theme will be revisited in greater detail in Chap. 5).

3.1 Maxwell's Equations

Consider electromagnetic wave propagation in a homogeneous, isotropic, nonconducting medium in $\mathbb{R}^3$ with electric permittivity ϵ and magnetic permeability μ. A time-harmonic electromagnetic wave with frequency $\omega > 0$ is described by the electric and magnetic fields

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$$\begin{aligned} \mathcal{E}(x,t) &= \epsilon^{-1/2} E(x) e^{-i\omega t}\,, \\ \mathcal{H}(x,t) &= \mu^{-1/2} H(x) e^{-i\omega t}\,, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^3$ and $\mathcal{E}$ and $\mathcal{H}$ satisfy *Maxwell's equations*

$$\begin{aligned} \operatorname{curl} \mathcal{E} + \mu \frac{\partial \mathcal{H}}{\partial t} &= 0\,, \\ \operatorname{curl} \mathcal{H} - \epsilon \frac{\partial \mathcal{E}}{\partial t} &= 0. \end{aligned} \tag{3.2}$$

In particular, from (3.1) and (3.2) we see that E and H must satisfy

$$\begin{aligned} \operatorname{curl} E - ikH &= 0\,, \\ \operatorname{curl} H + ikE &= 0\,, \end{aligned} \tag{3.3}$$

where the *wave number* k is defined by $k = \omega\sqrt{\epsilon\mu}$.

Now assume that a time -monic electromagnetic plane wave (factoring out $e^{-i\omega t}$)

$$\begin{aligned} E^i(x) = E^i(x;d,p) &= \frac{1}{k^2} \operatorname{curl}\operatorname{curl} p e^{ikx\cdot d}\,, \\ H^i(x) = H^i(x;d,p) &= \frac{1}{ik} \operatorname{curl} p e^{ikx\cdot d}\,, \end{aligned} \tag{3.4}$$

where d is a constant unit vector and p is the (constant) polarization vector, is an incident field that is scattered by an obstacle D that is an *imperfect conductor*, i.e., the electromagnetic field penetrates D by only a small amount. Let the total fields E and H be given by

$$\begin{aligned} E &= E^i + E^s\,, \\ H &= H^i + H^s\,, \end{aligned} \tag{3.5}$$

where $E^s(x) = E^s(x;d,p)$ and $H^s(x) = H^s(x;d,p)$ are the scattered fields that arise due to the presence of the obstacle D. Then E^s, H^s must be an "outgoing" wave that satisfies the *Silver–Müller radiation condition*

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0\,, \tag{3.6}$$

where $r = |x|$. Since D is an imperfect conductor, on the boundary ∂D the field E must satisfy the boundary condition

$$\nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu = 0\,, \tag{3.7}$$

where $\lambda = \lambda(x) > 0$ is the surface impedance defined on ∂D. Then the mathematical problem associated with the scattering of time-harmonic plane waves by an imperfect conductor is to find a solution E, H of Maxwell's equations (3.3) in the exterior of D such that (3.4)–(3.7) are satisfied. In particular, (3.3)–(3.7) define a *scattering problem* for Maxwell's equations.

Now consider the scattering due to an infinite cylinder with cross section D and axis on the x_3-coordinate axis where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Assume $E = (0, 0, E_3)$, $p = (0, 0, 1)$, and $d = (d_1, d_2, 0)$, i.e.,

$$E^i(x) = e^{ikx\cdot d}\hat{e}_3\,,$$

where $\hat{e}_3$ is the unit vector in the positive x_3 direction. Then E and H will be independent of x_3, and from Maxwell's equations we have that $H = (H_1, H_2, 0)$, where E_3, H_1, and H_2 satisfy

$$\begin{aligned} \frac{\partial E_3}{\partial x_2} &= ikH_1\,, \\ \frac{\partial E_3}{\partial x_1} &= -ikH_2\,, \\ \frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} &= -ikE_3. \end{aligned}$$

In particular,

$$\Delta E_3 + k^2 E_3 = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}. \tag{3.8}$$

In order for E_3^s to be "outgoing," we require that E_3^s satisfy the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial E_3^s}{\partial r} - ikE_3^s\right) = 0. \tag{3.9}$$

Finally, we need to determine the boundary condition satisfied by

$$E_3(x) = e^{ikx\cdot d} + E_3^s(x)\,, \tag{3.10}$$

where now $x \in \mathbb{R}^2$. To this end, we compute for $E = (0, 0, E_3)$ and $\nu = (\nu_1, \nu_2, 0)$ that $\nu \times \operatorname{curl} E = (0, 0, -\partial E_3/\partial\nu)$ and $(\nu \times E) \times \nu = E$. This then implies that (3.7) becomes

$$\frac{\partial E_3}{\partial \nu} + i\lambda E_3 = 0. \tag{3.11}$$

Equations (3.8)–(3.11) provide the mathematical formulation of the scattering of a time-harmonic electromagnetic plane wave by an imperfectly conducting infinite cylinder, and it is this problem that will concern us for the rest of this chapter.

3.2 Bessel Functions

We begin our study of the scattering problem (3.8)–(3.11) by examining special solutions of the *Helmholtz equation* (3.8). In particular, if we look for solutions to (3.8) in the form

$$E_3(x) = y(kr)e^{in\theta} \quad , n = 0, \pm 1, \pm 2, \cdots ,$$

where (r, θ) are cylindrical coordinates, we find that $y(r)$ is a solution of *Bessel's equation*

$$y'' + \frac{1}{r}y' + \left(1 - \frac{\nu^2}{r^2}\right) y = 0 \tag{3.12}$$

for $\nu = n$. For arbitrary real ν we see by direct calculation and the ratio test that

$$J_\nu(r) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{r}{2}\right)^{\nu+2k} , \tag{3.13}$$

where Γ denotes the gamma function, is a solution of Bessel's equation for $0 \le r < \infty$. J_ν is called a *Bessel function* of order ν. For $\nu = -n, n = 1, 2, \cdots$, the first n terms of (3.13) vanish, and hence

$$\begin{aligned} J_{-n}(r) &= \sum_{k=n}^{\infty} \frac{(-1)^k}{k!(k-n)!} \left(\frac{r}{2}\right)^{-n+2k} \\ &= \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(n+s)!s!} \left(\frac{r}{2}\right)^{n+2s} \\ &= (-1)^n J_n(r) , \end{aligned}$$

which shows that J_n and J_{-n} are linearly dependent. However, if $\nu \neq n$, then it is easily seen that J_ν and $J_{-\nu}$ are linearly independent solutions of Bessel's equation.

Unfortunately, we are interested precisely in the case where $\nu = n$, and hence we must find a second linearly independent solution of Bessel's equation. This is easily done using Frobenius' method, and for $n = 0, 1, 2, \cdots$ we obtain the desired second solution to be given by

$$\begin{aligned} Y_n(r) := \frac{2}{\pi} J_n(r) \log \frac{r}{2} - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{r}{2}\right)^{2k-n} \\ - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{r}{2}\right)^{n+2k}}{k!\,(n+k)!} \left[\psi(k+1) + \psi(k+n+1)\right] , \end{aligned} \tag{3.14}$$

where $\psi(1) = -\gamma$, $\psi(m+1) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ for $m = 1, 2, \cdots$, $\gamma = 0.57721566\cdots$ is Euler's constant, and the finite sum is set equal to zero if $n = 0$. From (3.13) and (3.14) we see that

$$J_n(r) = \frac{1}{n!} \left(\frac{r}{2}\right)^n \left[1 + O(r^2)\right] \quad , \quad r \to 0 , \tag{3.15}$$

and, for $n \ge 1$,

$$Y_n(r) = -\frac{(n-1)!}{\pi}\left(\frac{r}{2}\right)^{-n}\begin{cases}1+O(r^2\log r), & n=1,\\ 1+O(r^2), & n>1,\end{cases} \qquad r\to 0\,, \tag{3.16}$$

whereas for $n = 0$ we have that

$$Y_0(r) = \frac{2}{\pi}\log r + O(1) \quad , \quad r \to 0. \tag{3.17}$$

Note that in (3.15) and (3.16) the constant implicit in the order term is independent of n for $n > 1$. Finally, for n a positive integer we define Y_{-n} by

$$Y_{-n}(r) = (-1)^n Y_n(r)\,,$$

which implies that J_n and Y_n are linearly independent for all integers $n = 0, \pm 1, \pm 2, \cdots$. The function Y_n is called the *Neumann function* of order n.

Of considerable importance to us in the sequel are the *Hankel functions* $H_n^{(1)}$ and $H_n^{(2)}$ of the first and second kind of order n, respectively, which are defined by

$$\begin{aligned} H_n^{(1)}(r) &:= J_n(r) + iY_n(r)\,,\\ H_n^{(2)}(r) &:= J_n(r) - iY_n(r) \end{aligned} \tag{3.18}$$

for $n = 0, \pm 1, \pm 2, \cdots$, $0 < r < \infty$. $H_n^{(1)}$ and $H_n^{(2)}$ clearly define a second pair of linearly independent solutions to Bessel's equation.

Now let y_1 and y_2 be any two solutions of Bessel's equation

$$(ry_1')' + \left(r - \frac{\nu^2}{r}\right) y_1 = 0\,, \tag{3.19}$$

$$(ry_2')' + \left(r - \frac{\nu^2}{r}\right) y_2 = 0\,, \tag{3.20}$$

and define the *Wronskian* by

$$W(y_1, y_2) := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Then multiplying (3.19) by y_2 and subtracting it from (3.20) multiplied by y_1 we see that

$$\frac{d}{dr}(rW) = 0\,,$$

and hence

$$W(y_1, y_2) = \frac{C}{r}\,,$$

where C is a constant. The constant C can be computed by

$$C = \lim_{r\to 0} rW(y_1, y_2).$$

In particular, making use of (3.15)–(3.18) we find that

$$W(J_n, H_n^{(1)}) = \frac{2i}{\pi r}\,, \tag{3.21}$$

$$W(H_n^{(1)}, H_n^{(2)}) = -\frac{4i}{\pi r}. \tag{3.22}$$

We now note that for $0 \le r < \infty$, $0 < |t| < \infty$, we have that

$$e^{rt/2} e^{-r/2t} = \sum_{j=0}^{\infty} \frac{r^j t^j}{2^j j!} \sum_{k=0}^{\infty} \frac{(-1)^k r^k}{2^k t^k k!}\,,$$

and, setting $j - k = n$, we have that

$$\begin{aligned} e^{r/2(t-1/t)} &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(-1)^k r^{n+2k}}{2^{n+2k}(n+k)!k!} \right] t^n \\ &= \sum_{-\infty}^{\infty} J_n(r) t^n . \end{aligned} \tag{3.23}$$

Setting $t = ie^{i\theta}$ in (3.23) gives the *Jacobi–Anger expansion*

$$e^{ir\cos\theta} = \sum_{-\infty}^{\infty} i^n J_n(r) e^{in\theta}. \tag{3.24}$$

In the remaining chapters of this book we will often be interested in entire solutions of the Helmholtz equation of the form

$$v_g(x) := \int_0^{2\pi} e^{ikr\cos(\theta-\phi)} g(\phi)\, d\phi\,, \tag{3.25}$$

where $g \in L^2[0, 2\pi]$. The function v_g is called a *Herglotz wave function* with kernel g. These functions were first introduced by Herglotz in a lecture in 1945 in Göttingen and were subsequently studied by Magnus [125], Müller [131], and Hartman and Wilcox [83]. From (3.25) and the Jacobi–Anger expansion, we see that since g has the Fourier expansion

$$g(\phi) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} a_n (-i)^n e^{in\phi}\,,$$

where

$$\sum_{-\infty}^{\infty} |a_n|^2 < \infty\,, \tag{3.26}$$

v_g is a Herglotz wave function if and only if v_g has an expansion of the form

$$v_g(x) = \sum_{-\infty}^{\infty} a_n J_n(kr) e^{in\theta}$$

such that (3.26) is valid. Note that v_g is identically zero if and only if $g = 0$.

Finally, we note the asymptotic relations [121]

$$\begin{aligned} J_n(r) &= \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(r^{-3/2}), \quad r \to \infty\,, \\ H_n^{(1)}(r) &= \sqrt{\frac{2}{\pi r}} \exp i\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(r^{-3/2}), \quad r \to \infty\,, \end{aligned} \tag{3.27}$$

and the *addition formula* [121]

$$H_0^{(1)}(k\,|x-y|) = \sum_{-\infty}^{\infty} H_n^{(1)}(k\,|x|) J_n(k\,|y|) e^{in\theta}\,, \tag{3.28}$$

which is uniformly convergent together with its first derivatives on compact subsets of $|x| > |y|$, and θ denotes the angle between x and y.

3.3 Direct Scattering Problem

We will now show that the scattering problem for an imperfect conductor in $\mathbb{R}^2$ is well posed. We will always assume that $D \subset \mathbb{R}^2$ is a bounded domain containing the origin with connected complement such that ∂D is in class C^2. Our aim is to show the existence of a unique solution $u \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C(\mathbb{R}^2 \setminus D)$ of the exterior *impedance boundary value problem*

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}\,, \tag{3.29}$$

$$u(x) = e^{ikx\cdot d} + u^s(x)\,, \tag{3.30}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0\,, \tag{3.31}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \quad \text{on } \partial D\,, \tag{3.32}$$

where (3.32) is assumed in the sense of uniform convergence as $x \to \partial D$, $\lambda \in C(\partial D), \lambda(x) > 0$ for $x \in \partial D$, ν is the unit outward normal to ∂D, and the Sommerfeld radiation condition (3.31) is assumed to hold uniformly in θ, where $k > 0$ is the wave number and (r, θ) are polar coordinates. We also want to show that the solution u of (3.29)–(3.32) depends continuously on the incident field u^i in an appropriate norm.

We define the (radiating) *fundamental solution* to the Helmholtz equation by

$$\Phi(x,y) := \frac{i}{4} H_0^{(1)}(k\,|x-y|) \tag{3.33}$$

and note that $\Phi(x,y)$ satisfies the Sommerfeld radiation condition with respect to both x and y, and as $|x-y| \to 0$ we have that

$$\Phi(x,y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + O(1). \tag{3.34}$$

Theorem 3.1 (Representation Theorem). *Let $u^s \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C(\mathbb{R}^2 \setminus D)$ be a solution of the Helmholtz equation in the exterior of D satisfying the Sommerfeld radiation condition and such that $\partial u/\partial\nu$ exists in the sense of uniform convergence as $x \to \partial D$. Then for $x \in \mathbb{R}^2 \setminus \bar{D}$ we have that*

$$u^s(x) = \int_{\partial D} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y).$$

Proof. Let $x \in \mathbb{R}^2 \setminus \bar{D}$, and circumscribe it with a disk

$$\Omega_{x,\epsilon} := \{y : |x-y| < \epsilon\} ,$$

where $\Omega_{x,\epsilon} \subset \mathbb{R}^2 \setminus \bar{D}$. Let Ω_R be a disk of radius R centered at the origin and containing D and $\Omega_{x,\epsilon}$ in its interior. Then from Green's second identity we have that

$$\int_{\partial D + \partial\Omega_{x,\epsilon} + \partial\Omega_R} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) = 0.$$

From the definition of the Hankel function, we have that

$$\frac{d}{dr} H_0^{(1)}(r) = -H_1^{(1)}(r) ,$$

and hence on $\partial\Omega_{x,\epsilon}$ we have that

$$\frac{\partial}{\partial \nu(y)} \Phi(x,y) = \frac{1}{2\pi} \frac{1}{|x-y|} + O(|x-y| \log |x-y|). \tag{3.35}$$

Using (3.34) and (3.35) and letting $\epsilon \to 0$ we see that

$$\begin{aligned} u^s(x) = &\int_{\partial D} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) \\ &- \int_{|y|=R} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) , \end{aligned} \tag{3.36}$$

where as usual ν is the unit outward normal to the boundary of the (interior) domain. Hence to establish the theorem we must show that the second integral tends to zero as $R \to \infty$.

We first show that

$$\lim_{R\to\infty} \int_{|y|=R} |u^s|^2 \, ds = O(1). \tag{3.37}$$

To this end, from the Sommerfeld radiation condition we have that

$$\begin{aligned}0 &= \lim_{R\to\infty}\int_{|y|=R}\left|\frac{\partial u^s}{\partial r} - iku^s\right|^2 ds \\ &= \lim_{R\to\infty}\int_{|y|=R}\left(\left|\frac{\partial u^s}{\partial r}\right|^2 + k^2|u^s|^2 + 2k\,\mathrm{Im}\left(u^s\frac{\partial \overline{u^s}}{\partial r}\right)\right) ds.\end{aligned} \tag{3.38}$$

Green's first identity applied to $D_R = \Omega_R \setminus \bar{D}$ gives

$$\int_{|y|=R} u^s\frac{\partial \overline{u^s}}{\partial r}\,ds = \int_{\partial D} u^s\frac{\partial \overline{u^s}}{\partial \nu}\,ds - k^2\int_{D_R}|u^s|^2\,dy + \int_{D_R}|\mathrm{grad}\,u^s|^2\,dy\,,$$

and hence from (3.38) we have that

$$\lim_{R\to\infty}\int_{|y|=R}\left(\left|\frac{\partial u^s}{\partial r}\right|^2 + k^2|u^s|^2\right) ds = -2k\,\mathrm{Im}\int_{\partial D} u^s\frac{\partial \overline{u^s}}{\partial \nu}\,ds\,, \tag{3.39}$$

and from this we can conclude that (3.37) is true.

To complete the proof, we now note the identity

$$\begin{aligned}&\int_{|y|=R}\left(u^s(y)\frac{\partial}{\partial\nu(y)}\Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y)\Phi(x,y)\right) ds(y) = \\ &\quad= \int_{|y|=R} u^s(y)\left(\frac{\partial}{\partial |y|}\Phi(x,y) - ik\Phi(x,y)\right) ds(y) \\ &\quad - \int_{|y|=R}\Phi(x,y)\left(\frac{\partial u^s}{\partial |y|}(y) - iku^s(y)\right) ds(y).\end{aligned} \tag{3.40}$$

Applying the Cauchy–Schwarz inequality to each of the integrals on the right-hand side of (3.40) and using (3.37), the facts that $\Phi(x,y) = O(1/\sqrt{R})$ and Φ and u^s satisfy the Sommerfeld radiation condition we have that

$$\lim_{R\to\infty}\int_{|y|=R}\left(u^s(y)\frac{\partial}{\partial\nu(y)}\Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y)\Phi(x,y)\right) ds(y) = 0\,,$$

and the proof is complete. □

Now let D be a bounded domain with C^2 boundary ∂D and $u \in C^2(D)\cap C^1(\bar{D})$ a solution of the Helmholtz equation in D. Then, using the techniques of the proof of the preceding theorem, it can easily be shown that for $x \in D$ we have the *representation formula*

$$u(x) = \int_{\partial D}\left(\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial}{\partial\nu(y)}\Phi(x,y)\right) ds(y). \tag{3.41}$$

Hence, since $\Phi(x,y)$ is a real-analytic function of x_1 and x_2, where $x = (x_1, x_2)$ and $x \neq y$, we have that u is real-analytic in D. This proves the following theorem.

Theorem 3.2. *Solutions of the Helmholtz equation are real-analytic functions of their independent variables.*

The identity theorem for real-analytic functions [95] and Theorem 3.2 imply that solutions of the Helmholtz equation satisfy the *unique continuation principle*, i.e., if u is a solution of the Helmholtz equation in a domain D and $u(x) = 0$ for x in a neighborhood of a point $x_0 \in D$, then $u(x) = 0$ for all x in D.

We are now in a position to show that if a solution to the scattering problem (3.29)–(3.32) exists, then it is unique.

Theorem 3.3. *Let $u^s \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C(\mathbb{R}^2 \setminus D)$ be a solution of the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$ satisfying the Sommerfeld radiation condition and the boundary condition $\partial u^s/\partial \nu + i\lambda u^s = 0$ on ∂D (in the sense of uniform convergence as $x \to \partial D$). Then $u^s = 0$.*

Proof. Let Ω be a disk centered at the origin and containing D in its interior. Then from Green's second identity, the fact that R and λ are real, and hence

$$\frac{\partial u^s}{\partial \nu} + i\lambda u^s = \frac{\partial \overline{u^s}}{\partial \nu} - i\lambda \overline{u^s} = 0 \quad \text{on } \partial D\,,$$

we have that

$$\begin{aligned} \int_{\partial\Omega} \left(\overline{u^s}\frac{\partial u^s}{\partial r} - u^s \frac{\partial \overline{u^s}}{\partial r} \right) ds &= \int_{\partial D} \left(\overline{u^s}\frac{\partial u^s}{\partial \nu} - u^s \frac{\partial \overline{u^s}}{\partial \nu} \right) ds \\ &= -2i \int_{\partial D} \lambda \left| u^s \right|^2 ds. \end{aligned} \tag{3.42}$$

But since, by Theorem 3.2, $u^s \in C^\infty(\mathbb{R}^2 \setminus \bar{D})$ (in fact real-analytic), we have that, for $x \in \mathbb{R}^2 \setminus \Omega$, u^s can be expanded in a Fourier series

$$\begin{aligned} u^s(r,\theta) &= \sum_{-\infty}^{\infty} a_n(r) e^{in\theta}\,, \\ a_n(r) &= \frac{1}{2\pi}\int_0^{2\pi} u^s(r,\theta) e^{-in\theta}\, d\theta\,, \end{aligned} \tag{3.43}$$

where the series and its derivatives with respect to r are absolutely and uniformly convergent on compact subsets of $\mathbb{R}^2 \setminus \Omega$. In particular, it can be verified directly that $a_n(r)$ is a solution of Bessel's equation and, since u^s satisfies the Sommerfeld radiation condition,

$$a_n(r) = \alpha_n H_n^{(1)}(kr)\,, \tag{3.44}$$

where the α_n are constants. Substituting (3.43) and (3.44) into (3.42) and integrating termwise, we see from the fact that $\overline{H_n^{(1)}(kr)} = H_n^{(2)}(kr)$ and the Wronskian formula (3.22) that

$$8i\sum_{-\infty}^{\infty}|\alpha_n|^2 = -2i\int_{\partial D}\lambda\,|u^s|^2\,ds.$$

Since $\lambda > 0$, we can now conclude that $\alpha_n = 0$ for every integer n, and hence $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus \Omega$. By Theorem 3.2 and the identity theorem for real-analytic functions, we can now conclude that $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$. □

Corollary 3.4. *If the solution of the scattering problem (3.29)–(3.32) exists, then it is unique.*

Proof. If two solutions u_1 and u_2 exist, then their difference $u^s = u_1 - u_2$ satisfies the hypothesis of Theorem 3.3, and hence $u^s = 0$, i.e., $u_1 = u_2$. □

The next theorem is a classic result in scattering theory that was first proved by Rellich [143] and Vekua [157] in 1943. Due, perhaps, to wartime conditions, Vekua's paper remained unknown in the West, and the result is commonly attributed only to Rellich.

Theorem 3.5 (Rellich's Lemma). *Let $u \in C^2(\mathbb{R}^2 \setminus \bar{D})$ be a solution of the Helmholtz equation satisfying*

$$\lim_{R\to\infty}\int_{|y|=R}|u|^2\,ds = 0.$$

Then $u = 0$ in $\mathbb{R}^2 \setminus \bar{D}$.

Proof. Let Ω be a disk centered at the origin and containing D in its interior. Then, as in Theorem 3.3, we have that for $x \in \mathbb{R}^2 \setminus \Omega$

$$u(r,\theta) = \sum_{-\infty}^{\infty} a_n(r)e^{in\theta}\,,$$

$$a_n(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)e^{-in\theta}\,d\theta\,,$$

and $a_n(r)$ is a solution of Bessel's equation, i.e.,

$$a_n(r) = \alpha_n H_n^{(1)}(kr) + \beta_n H_n^{(2)}(kr)\,, \tag{3.45}$$

where the α_n and β_n are constants. By Parseval's equality, we have that

$$\int_{|y|=R}|u|^2\,ds = 2\pi R\sum_{-\infty}^{\infty}|a_n(R)|^2\,,$$

and hence, from the hypothesis of the theorem,

$$\lim_{R\to\infty} R\,|a_n(R)|^2 = 0. \tag{3.46}$$

From (3.45), the asymptotic expansion of $H_n^{(1)}(kr)$ given by (3.27), and the fact that $\overline{H_n^{(1)}(kr)} = H_n^{(2)}(kr)$, we see from (3.46) that $\alpha_n = \beta_n = 0$ for every n, and hence $u = 0$ in $\mathbb{R}^2 \setminus \Omega$. By Theorem 3.2 and the identity theorem for real-analytic functions, we can now conclude as in Theorem 3.3 that $u(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$. □

Theorem 3.6. *Let $u^s \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C(\mathbb{R}^2 \setminus D)$ be a radiating solution of the Helmholtz equation such that $\frac{\partial u}{\partial \nu}(x)$ converges uniformly as $x \to \partial D$ and*

$$\operatorname{Im} \int_{\partial D} u^s \frac{\partial \overline{u}^s}{\partial \nu} \, ds \geq 0.$$

Then $u^s = 0$ in $\mathbb{R}^2 \setminus \bar{D}$.

Proof. This follows from identity (3.39) and Rellich's lemma. □

We now want to use the method of integral equations to establish the existence of a solution to the scattering problem (3.29)–(3.32). To this end, we note that the *single layer potential*

$$u^s(x) = \int_{\partial D} \varphi(y) \Phi(x, y) \, ds(y), \quad x \in \mathbb{R}^2 \setminus \partial D \tag{3.47}$$

with continuous density φ satisfies the Sommerfeld radiation condition, is a solution of the Helmholtz equation in $\mathbb{R}^2 \setminus \partial D$, is continuous in $\mathbb{R}^2$, and satisfies the discontinuity property [111, 127]

$$\frac{\partial u^s_\pm}{\partial \nu}(x) = \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(x)} \Phi(x, y) \, ds(y) \mp \frac{1}{2} \varphi(x), \quad x \in \partial D,$$

where

$$\frac{\partial u^s_\pm}{\partial \nu}(x) := \lim_{h \to 0} \nu(x) \cdot \nabla u \left(x \pm h\nu(x)\right).$$

(For future reference, we note that these properties of the single layer potential are also valid for $\varphi \in H^{-1/2}(\partial D)$, where the integrals are interpreted in the sense of duality pairing [111,127].) In particular, (3.47) will solve the scattering problem (3.29)–(3.32) provided

$$\begin{aligned} \varphi(x) - 2 \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(x)} \Phi(x, y) \, ds(y) - 2i\lambda(x) \int_{\partial D} \varphi(y) \Phi(x, y) \, ds(y) \\ = 2 \left[\frac{\partial u^i}{\partial \nu}(x) + i\lambda(x) u^i(x) \right], \qquad x \in \partial D, \end{aligned} \tag{3.48}$$

where $u^i(x) = e^{ikx \cdot d}$. Hence, to establish the existence of a solution to the scattering problem (3.29)–(3.32), it suffices to show the existence of a solution to (3.48) in the normed space $C(\partial D)$ (Example 1.3).

To this end, we first note that the integral operators in (3.48) are compact. This can easily be shown by approximating each of the kernels $K(x, y)$ in (3.48) by

$$K_n(x,y) := \begin{cases} h(n\,|x-y|)K(x,y), & x \neq y, \\ 0, & x = y, \end{cases}$$

where

$$h(t) := \begin{cases} 0, & 0 \le t \le \frac{1}{2}, \\ 2t-1, & \frac{1}{2} \le t \le 1, \\ 1, & 1 \le t < \infty \end{cases}$$

and using Theorem 1.17 and the fact that integral operators with continuous kernels are compact operators on $C(\partial D)$ (cf. Theorem 2.21 of [111]). Hence, by Riesz's theorem, it suffices to show that the homogeneous equation has only a trivial solution. But this is in general not the case! In particular, let k^2 be a Dirichlet eigenvalue, i.e., there exists $u \in C^2(D) \cap C(\bar{D})$, with u not identically zero, such that

$$\Delta u + k^2 u = 0 \quad \text{in } D,$$
$$u = 0 \quad \text{on } \partial D.$$

It can be shown that $u \in C^1(\bar{D})$ [51] and $\partial u/\partial\nu$ is not identically zero since, if it were, then by the representation formula (3.41) u would be identically zero, which it is not by assumption. Hence for $\varphi := \partial u/\partial\nu$ we have from Green's second identity that

$$\int_{\partial D} \varphi(y)\Phi(x,y)\,ds(y) = 0, \quad x \in \mathbb{R}^2 \setminus \bar{D} \tag{3.49}$$

and, by continuity, for $x \in \mathbb{R}^2 \setminus D$. Hence, using the previously stated discontinuity properties for single layer potentials, we have that

$$\varphi(x) - 2\int_{\partial D} \varphi(y)\frac{\partial}{\partial\nu(x)}\Phi(x,y)\,ds(y) = 0, \quad x \in \partial D. \tag{3.50}$$

Equations (3.49) and (3.50) now imply that φ is a nontrivial solution of the homogeneous equation corresponding to (3.48). Thus we cannot use Riesz's theorem to establish the existence of a solution to (3.48).

To obtain an integral equation that is uniquely solvable for all values of the wave number k, we need to modify the kernel of the representation (3.47). We will do this following the ideas of [96, 109, 156]. We begin by defining the function $\chi = \chi(x, y)$ by

$$\chi(x,y) := \frac{i}{4}\sum_{-\infty}^{\infty} a_n H_n^{(1)}(kr)H_n^{(1)}(kr_y)e^{in(\theta-\theta_y)}, \tag{3.51}$$

where x has polar coordinates (r, θ), y has polar coordinates (r_y, θ_y), and the coefficients a_n are chosen such that the series converges for $|x|, |y| > R$, where $\Omega_R := \{x : |x| \leq R\} \subset D$. The fact that this can be done follows from (3.15), (3.16), and (3.18) and the fact that

$$H_{-n}^{(1)}(kr) = (-1)^n H_n^{(1)}(kr)$$

for $n = 0, 1, 2, 3, \cdots$. In particular these equations imply that

$$\left| H_n^{(1)}(kr) \right| = O\left(\frac{2^{|n|}\,(|n|-1)!}{(kr)^{|n|}} \right)$$

for $n = \pm 1, \pm 2, \cdots$ and r on compact subsets of $(0, \infty)$. Defining

$$\Gamma(x, y) := \Phi(x, y) + \chi(x, y)$$

we now see that the *modified single layer potential*

$$u^s(x) := \int_{\partial D} \varphi(y) \Gamma(x, y)\, ds(y) \tag{3.52}$$

for continuous density φ and $x \in \mathbb{R}^2 \setminus (\partial D \cup \Omega_R)$ satisfies the Sommerfeld radiation condition, is a solution of the Helmholtz equation in $\mathbb{R}^2 \setminus (\partial D \cup \Omega_R)$, and satisfies the same discontinuity properties as the single layer potential (3.47). Hence (3.52) will solve the scattering problem (3.29)–(3.32) provided φ satisfies (3.48), with Φ replaced by Γ. By Riesz's theorem, a solution of this equation exists if the corresponding homogeneous equation only has a trivial solution.

Let φ be a solution of this homogeneous equation. Then (3.52) will be a solution of (3.29)–(3.32) with $e^{ikx \cdot d}$ set equal to zero and hence, by Corollary 3.4, we have that if u^s is defined by (3.52), then $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$. By the continuity of (3.52) across ∂D, u^s is a solution of the Helmholtz equation in $D \setminus \Omega_R$, $u^s \in C^2(D \setminus \bar{\Omega}_R) \cap C(\bar{D} \setminus \Omega_R)$, and $u^s(x) = 0$ for $x \in \partial D$. From (3.51), (3.52), and the addition formula for Bessel functions, we see that there exist constants α_n such that for $R_1 \leq |x| \leq R_2$, where $R < R_1 < R_2$ and $\{x : |x| < R_2\} \subset D$, we can represent u^s in the form

$$u^s(x) = \sum_{-\infty}^{\infty} \alpha_n \left\{ J_n(kr) + a_n H_n^{(1)}(kr) \right\} e^{in\theta}.$$

Since

$$u_+^s(x) := \lim_{\substack{x \to \partial D \\ x \in D}} u^s(x),$$

$$\frac{\partial u_+^s}{\partial \nu}(x) := \lim_{\substack{x \to \partial D \\ x \in D}} \frac{\partial u^s}{\partial \nu}(x)$$

exist and are continuous, we can apply Green's second identity to u^s and $\bar{u^s}$ over $D \setminus \{x : |x| \leq R_1\}$ and use the Wronskian relations (3.21) and (3.22) to see that

$$0 = \int_{\partial D} \left(u^s_+ \frac{\partial \bar{u}^s_+}{\partial \nu} - \bar{u}^s_+ \frac{\partial u^s_+}{\partial \nu} \right) ds = \int_{|x|=R_1} \left(u^s \frac{\partial \bar{u^s}}{\partial \nu} - \bar{u^s} \frac{\partial u^s}{\partial \nu} \right) ds$$
$$= 2i \sum_{-\infty}^{\infty} |\alpha_n|^2 \left(1 - |1 + 2a_n|^2\right).$$

Hence, if either $|1 + 2a_n| < 1$ or $|1 + 2a_n| > 1$ for $n = 0, \pm 1, \pm 2, \cdots$, then $\alpha_n = 0$ for $n = 0, \pm 1, \pm 2, \cdots$, i.e., $u^s(x) = 0$ for $R_1 \leq |x| \leq R_2$. By Theorem 3.2 and the identity theorem for real-analytic functions, we can now conclude that $u^s(x) = 0$ for $x \in D \setminus \Omega_R$. Recalling that $u^s(x) = 0$ for $x \in \mathbb{R}^2 \setminus \bar{D}$, we now see from the discontinuity property of single layer potentials that

$$0 = \frac{\partial u^s_-}{\partial \nu} - \frac{\partial u^s_+}{\partial \nu}(x) = \varphi(x),$$

i.e., the homogeneous equation under consideration only has the trivial solution $\varphi = 0$. Hence, by Riesz's theorem, the corresponding inhomogeneous equation has a unique solution that depends continuously on the right-hand side.

Theorem 3.7. *There exists a unique solution of the scattering problem (3.29)–(3.32) that depends continuously on $u^i(x)=e^{ikx\cdot d}$ in $C^1(\partial D)$.*

It is often important to find a solution of (3.29)–(3.32) in a larger space than $C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$. To this end, let $\Omega_R := \{x : |x| < R\}$, and define the Sobolev spaces

$$H^1_{loc}(\mathbb{R}^2 \setminus \bar{D}) := \{u : u \in H^1\left((\mathbb{R}^2 \setminus \bar{D}) \cap \Omega_R\right) \text{ for every } R > 0$$
$$\text{such that } (\mathbb{R}^2 \setminus D) \cap \Omega_R \neq \emptyset\},$$
$$H^1_{com}(\mathbb{R}^2 \setminus \bar{D}) := \{u : u \in H^1(\mathbb{R}^2 \setminus \bar{D}), u \text{ is identically}$$
$$\text{zero outside some ball centered at}$$
$$\text{the origin}\}.$$

We recall that $H^{-p}(\partial D)$, $0 \leq p < \infty$, is the dual space of $H^p(\partial D)$ and, for $f \in H^{-p}(\partial D)$ and $v \in H^p(\partial D)$,

$$\int_{\partial D} f v \, ds := f(v)$$

is defined by duality pairing.
Then, for $f \in H^{-1/2}(\partial D)$, a *weak solution* of

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}, \tag{3.53}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{3.54}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = f \quad \text{on } \partial D \tag{3.55}$$

is defined as a function $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ such that

$$-\int_{\mathbb{R}^2\setminus\bar{D}} \left(\nabla u \cdot \nabla v - k^2 uv\right)\, dx + i\int_{\partial D} \lambda uv\, ds = \int_{\partial D} fv\, ds \tag{3.56}$$

for all $v \in H^1_{com}(\mathbb{R}^2 \setminus \bar{D})$ such that u satisfies the Sommerfeld radiation condition (3.54). Note that by the trace theorem we have that $v|_{\partial D} \in H^{1/2}(\partial D)$ is well defined, and hence the integral on the right-hand side of (3.56) is well defined by duality pairing. The radiation condition also makes sense in the weak case since, by regularity results for elliptic equations [127], any weak solution is automatically infinitely differentiable in $\mathbb{R}^2 \setminus \bar{D}$. It is easily verified that if $u \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ is a solution of (3.53)–(3.55), then u is also a weak solution of (3.53)–(3.55), i.e., u satisfies (3.56). The following theorem will be proved in Chap. 8.

Theorem 3.8. *There exists a unique weak solution of the scattering problem (3.53)–(3.55), and the mapping taking the boundary data $f \in H^{-1/2}(\partial D)$ onto the solution $u \in H^1((\mathbb{R}^2 \setminus \bar{D}) \setminus \bar{\Omega}_R)$ is bounded for every R such that $(\mathbb{R}^2 \setminus \bar{D}) \cap \Omega_R \neq \emptyset$.*

In an analogous manner, we can define a weak solution of the Helmholtz equation in a bounded domain D to be any function $u \in H^1(D)$ such that

$$\int_D \left(\nabla u \cdot \nabla v - k^2 uv\right)\, dx = 0$$

for all $v \in H^1(D)$ such that $v = 0$ on ∂D in the sense of the trace theorem. The following theorems will be useful in the sequel, but we will delay their proofs until Chap. 5, where they will constitute a basic part of the analysis of that chapter.

Theorem 3.9. *Let D be a bounded domain with C^2 boundary ∂D such that k^2 is not a Dirichlet eigenvalue for D. Then for every $f \in H^{1/2}(\partial D)$ there exists a unique weak solution $u \in H^1(D)$ of the Helmholtz equation in D such that $u = f$ on ∂D in the sense of the trace theorem. Furthermore, the mapping taking f onto u is bounded.*

Theorem 3.10. *Let $u \in H^1(D)$ and $\Delta u \in L^2(D)$ in a bounded domain D with C^2 boundary ∂D having unit outward normal ν. Then there exists a positive constant C independent of u such that*

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} \leq C \left\| u \right\|_{H^1(D)} .$$

Finally, we note that Green's identities and the representation formulas for exterior and interior domains remain valid for weak solutions of the Helmholtz equation, and we refer the reader to Chap. 5 for a proof of this fact.