On the Regularization Method in Nondifferentiable Optimization Applied to Hemivariational Inequalities

N. Ovcharova and J. Gwinner

Abstract In this paper we present the regularization method in nondifferentiable optimization in a unified way using the smoothing approximation of the plus function. We show how this method can be applied to hemivariational inequalities. To illustrate our results we consider bilateral contact between elastic bodies with a nonmonotone friction law on the contact boundary and present some numerical results.

Keywords Regularization method • Nondifferentiable optimization • Smoothing approximation • Plus function • Hemivariational inequalities

1 Introduction

The motivation of this paper comes from the numerical treatment of nonlinear nonsmooth variational problems of continuum mechanics involving nonmonotone contact of elastic bodies. These contact problems lead to nonmonotone and multivalued laws which can be expressed by means of the Clarke subdifferential of a nonconvex, nonsmooth but locally Lipschitz function, the so-called superpotential. The variational formulation of these problems involving such laws gives rise to hemivariational inequalities introduced for the first time by Panagiotopoulos in the 1980s; see [14, 15]. For the mathematical background of hemivariational inequalities we refer to Naniewicz and Panagiotopoulos [12]. For more recent works on the mathematical analysis of nonsmooth variational problems and contact problems, see also the monographs [3, 8, 11, 18]. Numerical methods for such problems can be

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V.F. Demyanov et al. (eds.), *Constructive Nonsmooth Analysis and Related Topics*,
Springer Optimization and Its Applications 87, DOI 10.1007/978-1-4614-8615-2_5,

found in the classical book of Haslinger et al. [9] as well as in the recent papers of Baniotopoulos et al. [2], Hintermüller et al. [10], etc.

Since in most applications the nonconvex superpotential can be modelled by means of maximum or minimum functions, we turn our attention to their regularizations. The idea of regularization goes back to Sobolev and is based more or less on convolution. However, regularizations via convolution are not easily applicable in practice, since it generally involves a calculation of a multivariate integral. But for the special class of maximum and minimum functions considered here, using regularization by a specified, e.g., piecewisely defined kernel, we can compute the smoothing function explicitly; see, e.g., [16, 20] and the recent survey in [13]. Moreover, since all nonsmooth functions under consideration can be reformulated by using the plus function, we can present the regularization method on nondifferentiable optimization (NDO) in a unified way. A large class of smoothing functions for the plus function can be found, e.g., in [4, 5, 7, 16, 19, 20].

In Sect. 2 we present a smoothing approximation of the maximum function based on the approximation of the plus function via convolution. We analyze some approximability property of the gradients of the smoothing function and show that the Clarke subdifferential of the nonsmooth but locally Lipschitz maximum function coincides with the subdifferential associated with the smoothing function.

Finally, in Sect. 3 we sketch how the regularization procedure from Sect. 2 can be used to solve nonmonotone contact problems. Here we focus on an elastic structure supported by a rigid foundation with a nonmonotone friction law. For further details concerning the regularization methods for hemivariational inequalities and their numerical realization by finite element methods, we refer to [13].

2 A Unified Approach to Regularization in NDO

Consider

$$f(x) = \max\{g_1(x), g_2(x)\}, \tag{1}$$

where $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, 2$. Obviously, the maximum function (1) can be expressed by means of the plus function $\mathrm{p}(x) = x^+ = \max(x, 0)$ as

$$f(x) = \max\{g_1(x), g_2(x)\} = g_1(x) + \mathrm{p}[g_2(x) - g_1(x)]. \tag{2}$$

Replacing now the plus function by its approximation via convolution, we present the following smoothing function $S : \mathbb{R}^n \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$ for the maximum function (see also [6]):

$$S(x, \varepsilon) := g_1(x) + P(\varepsilon, g_2(x) - g_1(x)). \tag{3}$$

Here, $P : \mathbb{R}_{++} \times \mathbb{R} \longrightarrow \mathbb{R}$ is the smoothing function via convolution for the plus function p defined by

$$P(\varepsilon,t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\rho(s)\,\mathrm{d}s. \tag{4}$$

Remark 1. The representation formula (2) can be extended to the maximum of finite number of arbitrary functions due to Bertsekas [1]:

$$f(x) = g_1(x) + \mathrm{p}\left[g_2(x) - g_1(x) + \cdots + \mathrm{p}\left[g_m(x) - g_{m-1}(x)\right]\right]. \tag{5}$$

Therefore the smoothing of the plus function gives a unified approach to regularization in NDO.

We restrict $\rho : \mathbb{R} \longrightarrow \mathbb{R}_+$ to be a density function of finite absolute mean; that is,

$$k := \int_{\mathbb{R}} |s|\rho(s)\,\mathrm{d}s < \infty.$$

From [16] we know that P is continuously differentiable on $\mathbb{R}_{++} \times \mathbb{R}$ with

$$P_t(\varepsilon,t) = \int_{-\infty}^{\frac{t}{\varepsilon}} \rho(s)\,\mathrm{d}s \tag{6}$$

and satisfies

$$|P(\varepsilon,t) - \mathrm{p}(t)| \leq k\varepsilon \quad \forall \varepsilon > 0,\ \forall t \in \mathbb{R}. \tag{7}$$

The inequalities in (7) imply

$$\lim_{t_k \longrightarrow t, \varepsilon \longrightarrow 0^+} P(\varepsilon,t_k) = \mathrm{p}(t) \quad \forall t \in \mathbb{R}.$$

Moreover, $P(\varepsilon,\cdot)$ is twice continuously differentiable on $\mathbb{R}$ and we compute

$$P_{tt}(\varepsilon,t) = \varepsilon^{-1}\rho\left(\frac{t}{\varepsilon}\right). \tag{8}$$

Due to this formula we can also get the smoothing function (4) by twice integrating the density function; see [5].

In what follows, we suppose that all the functions g_i are continuously differentiable. The major properties of S (see [16]) inherit the properties of the function P (see, e.g., [5, 7, 16]) and are collected in the following lemma:

Lemma 1.

(i) *For any* $\varepsilon > 0$ *and for all* $x \in \mathbb{R}^n$,

$$|S(x,\varepsilon) - f(x)| \leq k\varepsilon. \tag{9}$$

(ii) The function S is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}_{++}$ *and for any* $x \in \mathbb{R}^n$ *and* $\varepsilon > 0$ *there exist* $\Lambda_i \geq 0$ *such that* $\sum_{i=1}^{2} \Lambda_i = 1$ *and*

$$\nabla_x S(x,\varepsilon) = \sum_{i=1}^{2} \Lambda_i \nabla g_i(x). \tag{10}$$

Moreover,

$$co\{\xi \in \mathbb{R}^n : \xi = \lim_{k \longrightarrow \infty} \nabla_x S(x_k,\varepsilon_k),\, x_k \longrightarrow x,\, \varepsilon_k \longrightarrow 0^+\} \subseteq \partial f(x), \tag{11}$$

where "co" denotes the convex hull and $\partial f(x)$ *is the Clarke subdifferential.*

We recall that the Clarke subdifferential of a locally Lipschitz function f at a point $x \in \mathbb{R}^n$ can be expressed by

$$\partial f(x) = co\{\xi \in \mathbb{R}^n : \xi = \lim_{k \longrightarrow \infty} \nabla f(x_k),\, x_k \longrightarrow x,\, f \text{ is differentiable at } x_k\},$$

since in finite-dimensional case, according to Rademacher's theorem, f is differentiable almost everywhere.

The maximum function given by (1) is clearly locally Lipschitz continuous and the Clarke subdifferential can be written as

$$\partial f(x) = co\{\nabla g_i(x) : i \in I(x)\}$$

with

$$I(x) := \{i : f(x) = g_i(x)\}.$$

In particular, if $x \in \mathbb{R}^n$ is a point such that $f(x) = g_i(x)$ then $\partial f(x) = \{\nabla g_i(x)\}$. For such a point $x \in \mathbb{R}^n$ we show later on that

$$\lim_{z \longrightarrow x, \varepsilon \longrightarrow 0^+} \nabla_x S(z,\varepsilon) = \nabla g_i(x).$$

Note that the set on the left-hand side in (11) goes back to Rockafellar [17]. In [4], this set is denoted by $G_S(x)$ and is called there the subdifferential associated with the smoothing function. The inclusion (11) shows in fact that $G_S(x) \subseteq \partial f(x)$. Moreover, according to the part (b) of Corollary 8.47 in [17], $\partial f(x) \subseteq G_S(x)$. Thus, $\partial f(x) = G_S(x)$.

Remark 2. Note that S is a smoothing approximation of f in the sense that

$$\lim_{z \longrightarrow x, \varepsilon \longrightarrow 0^+} S(z,\varepsilon) = f(x) \quad \forall x \in \mathbb{R}^n.$$

This is immediate from (9).

Remark 3. The regularization procedure (3) can be also applied to a minimum function by

$$\begin{aligned} \min\{g_1(x), g_2(x)\} &= -\max\{-g_1(x), -g_2(x)\} = -\{-g_1(x) + \mathrm{p}[-g_2(x) + g_1(x)]\} \\ &\approx g_1(x) - P(\varepsilon, g_1(x) - g_2(x)) =: \tilde{S}(x,\varepsilon) \end{aligned} \tag{12}$$

Since all the nonsmooth functions considered in this paper can be reformulated by using the plus function, all our regularizations are based in fact on a class of smoothing approximations for the plus function. Some examples from [7] and the references therein are in order.

Example 1.

$$P(\varepsilon,t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\,\rho_1(s)\,\mathrm{d}s = t + \varepsilon \ln(1 + \mathrm{e}^{-\frac{t}{\varepsilon}}) = \varepsilon \ln(1 + \mathrm{e}^{\frac{t}{\varepsilon}}) \tag{13}$$

where $\rho_1(s) = \frac{\mathrm{e}^{-s}}{(1+\mathrm{e}^{-s})^2}$. Due to (8) the smoothing function (13) is obtained by integrating twice the function $\varepsilon^{-1}\rho_1(\frac{s}{\varepsilon})$.

Example 2.

$$P(\varepsilon,t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\,\rho_2(s)\,\mathrm{d}s = \frac{\sqrt{t^2 + 4\varepsilon^2} + t}{2}, \tag{14}$$

where $\rho_2(s) = \frac{2}{(s^2+4)^{3/2}}$. The formula (14) is similarly obtained as formula (13) via (8).

Example 3.

$$P(\varepsilon,t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\,\rho_3(s)\,\mathrm{d}s = \begin{cases} 0 & \text{if } t < -\frac{\varepsilon}{2} \\ \frac{1}{2\varepsilon}(t + \frac{\varepsilon}{2})^2 & \text{if } -\frac{\varepsilon}{2} \le t \le \frac{\varepsilon}{2} \\ t & \text{if } t > \frac{\varepsilon}{2}, \end{cases} \tag{15}$$

where $\rho_3(s) = \begin{cases} 1 \text{ if } -\frac{1}{2} \le s \le \frac{1}{2} \\ 0 \text{ otherwise.} \end{cases}$

Example 4.

$$P(\varepsilon,t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t-\varepsilon s)\,\rho_4(s)\,\mathrm{d}s = \begin{cases} 0 & \text{if } t<0 \\ \frac{t^2}{2\varepsilon} & \text{if } 0 \le t \le \varepsilon \\ t-\frac{\varepsilon}{2} & \text{if } t > \varepsilon, \end{cases} \tag{16}$$

where $\rho_4(s) = \begin{cases} 1 \text{ if } 0 \le s \le 1 \\ 0 \text{ otherwise.} \end{cases}$

In the following, we denote

$$A_1 = \{x \in \mathbb{R}^n : g_1(x) > g_2(x)\} \quad \text{and} \quad A_2 = \{x \in \mathbb{R}^n : g_2(x) > g_1(x)\}.$$

Lemma 2. *The following properties hold:*

(a) If $x \in A_1$ *then* $\lim\limits_{z \longrightarrow x, \varepsilon \longrightarrow 0^+} P_t(\varepsilon, g_2(z) - g_1(z)) = 0\,.$

(b) if $x \in A_2$ *then* $\lim\limits_{z \longrightarrow x, \varepsilon \longrightarrow 0^+} P_t(\varepsilon, g_2(z) - g_1(z)) = 1\,.$

Proof. The proof is straightforward and is based on formula (6). We provide only the proof of (a). The proof of (b) is analogous.

Let $x \in A_1$, i.e., $g_1(x) > g_2(x)$. Using (6), it follows that

$$P_t(\varepsilon, g_2(z) - g_1(z)) = \int_{-\infty}^{\frac{g_2(z)-g_1(z)}{\varepsilon}} \rho(s)\,\mathrm{d}s \longrightarrow 0 \quad \text{as } z \longrightarrow x, \varepsilon \longrightarrow 0^+$$

and (a) is verified. □

Now we show that the gradient of the given function g_i on A_i can be approximated by the gradients of the smoothing function.

Theorem 1. *For any* $x \in A_i$*,* $i = 1,2$*,*

$$\lim_{z \longrightarrow x, \varepsilon \longrightarrow 0^+} \nabla_x S(z,\varepsilon) = \nabla g_i(x).$$

Proof. From (3), by direct differentiation with respect to x [see also (10)], it follows that

$$\nabla_x S(z,\varepsilon) = \big(1 - P_t\big(\varepsilon, g_2(z) - g_1(z)\big)\big)\,\nabla g_1(z) + P_t\big(\varepsilon, g_2(z) - g_1(z)\big)\nabla g_2(z).$$

First, we take $x \in A_1$. From Lemma 2(a) we have

$$\lim_{z \longrightarrow x, \varepsilon \longrightarrow 0^+} P_t\big(\varepsilon, g_2(z) - g_1(z)\big) = 0$$

and therefore $\lim_{z \longrightarrow x, \varepsilon \longrightarrow 0^+} \nabla_x S(z,\varepsilon) = \nabla g_1(x)$. Let now $x \in A_2$. Then, from Lemma 2(b), it follows that

$$\lim_{z \longrightarrow x, \varepsilon \longrightarrow 0^+} P_t\big(\varepsilon, g_2(z) - g_1(z)\big) = 1$$

and consequently, $\lim_{z \longrightarrow x, \varepsilon \longrightarrow 0^+} \nabla_x S(z,\varepsilon) \longrightarrow \nabla g_2(x)$. The proof of the theorem is complete. □

Remark 4. Note that if $x \in \mathbb{R}^n$ is a point such that $g_1(x) = g_2(x)$ then for any sequences $\{x_k\} \subset \mathbb{R}^n$, $\{\varepsilon_k\} \subset \mathbb{R}_{++}$ such that $x_k \longrightarrow x$ and $\varepsilon_k \longrightarrow 0^+$ we have

$$\lim_{k \longrightarrow \infty} \nabla_x S(x_k, \varepsilon_k) \in \partial f(x).$$

3 Bilateral Contact with Nonmonotone Friction: A 2D Benchmark Problem

3.1 Statement of the Problem

In this section we sketch how our regularization method presented in Sect. 2 can be applied to numerical solution of nonmonotone contact problems that can be formulated as hemivariational inequality with maximum or minimum superpotential. As a model example we consider the bilateral contact of an elastic body with a rigid foundation under given forces and a nonmonotone friction law on the contact boundary. Here the linear elastic body Ω is the unit square $1m \times 1m$ (see Fig. 1) with modulus of elasticity $E = 2.15 \times 10^{11}\,\mathrm{N/m^2}$ and Poisson's ration $\nu = 0.29$ (steel). Then the linear Hooke's law is given by

$$\sigma_{ij}(\mathbf{u}) = \frac{E\nu}{1-\nu^2}\delta_{ij}\, tr\big(\varepsilon(\mathbf{u})\big) + \frac{E}{1+\nu}\varepsilon_{ij}(\mathbf{u}), \quad i,j = 1,2, \tag{17}$$

where δ_{ij} is the Kronecker symbol and

$$tr\big(\varepsilon(\mathbf{u})\big) := \varepsilon_{11}(\mathbf{u}) + \varepsilon_{22}(\mathbf{u}).$$

The boundary $\partial\Omega$ of Ω consists of four disjoint parts Γ_u, Γ_c, Γ_F^1, and Γ_F^2. On Γ_u the body is fixed, i.e., we have

$$u_i = 0 \quad \text{on } \Gamma_u,\ i = 1,2.$$

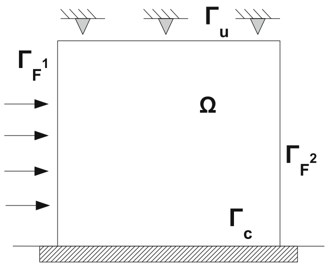

Fig. 1 A 2D benchmark with force distribution and boundary decomposition

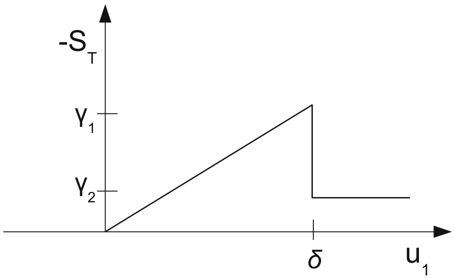

Fig. 2 A nonmonotone friction law

The body is loaded with horizontal forces, i.e., $\mathbf{F} = (P, 0)$ on Γ_F^1, where $P = 1.2 \times 10^6\,\mathrm{N/m^2}$, $\mathbf{F} = (0, 0)$ on Γ_F^2. Further, we assume that

$$\begin{cases} u_2(s) = 0 & s \in \Gamma_c \\ -S_T(s) \in \partial j(u_1(s)) & \text{for a.a. } s \in \Gamma_c\,. \end{cases}$$

Note that S_T denotes the tangential component of the stress vector on the boundary. The assumed nonmonotone multivalued law ∂j holding in the tangential direction is depicted in Fig. 2 with parameters $\delta = 9.0 \times 10^{-6}\,\mathrm{m}$, $\gamma_1 = 1.0 \times 10^3\,\mathrm{N/m^2}$ and $\gamma_2 = 0.5 \times 10^3\,\mathrm{N/m^2}$. Notice that here j is a minimum of a convex quadratic and a linear function, for instance, $j(\xi) = \min\{\frac{1}{2}\alpha\xi^2, \beta\xi\}$ for some $\alpha, \beta > 0$. Let

$$V = \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^2) \;:\; v_i = 0 \text{ on } \Gamma_u,\; i = 1,2,\; v_2 = 0 \text{ on } \Gamma_c\}$$

be the linear subspace of all admissible displacements. The weak formulation of this bilateral contact problem leads to the following hemivariational inequality: find $\mathbf{u} \in V$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + \int_{\Gamma_c} j^0(u_1(s); v_1(s) - u_1(s))\,\mathrm{d}s \geq \langle \mathbf{g}, \mathbf{v} - \mathbf{u} \rangle \tag{18}$$

for all $\mathbf{v} \in V$. Here, $a(\mathbf{u}, \mathbf{v})$ is the energy bilinear form of linear elasticity

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \sigma_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}x \quad \mathbf{u}, \mathbf{v} \in V \tag{19}$$

with σ, ε related by means of (17) and the linear form $\langle \mathbf{g}, \cdot \rangle$ defined by

$$\langle \mathbf{g}, \mathbf{v} \rangle = P \int_{\Gamma_{F1}} v_1 \, \mathrm{d}s. \tag{20}$$

3.2 Numerical Solution

We solve this problem numerically by first regularizing the hemivariational inequality (18) and then discretizing by the Finite Element Method. More precisely, we regularize $j(\xi)$ by $\tilde{S}(\xi, \varepsilon)$ defined by (12) and using (15) from Example 3 as a smoothing approximation of the plus function. Then, we introduce the functional $J_\varepsilon : V \longrightarrow \mathbb{R}$

$$J_\varepsilon(\mathbf{v}) = \int_{\Gamma_c} \tilde{S}(v_1(s), \varepsilon) \, \mathrm{d}s.$$

Since $\tilde{S}(\cdot, \varepsilon)$ is continuously differentiable for all $\varepsilon > 0$ the functional J_ε is everywhere Gâteaux differentiable with continuous Gâteaux derivative $DJ_\varepsilon : V \longrightarrow V^*$ given by

$$\langle DJ_\varepsilon(\mathbf{u}), \mathbf{v} \rangle = \int_{\Gamma_c} \tilde{S}'_\xi(u_1(s), \varepsilon) v_1(s) \, \mathrm{d}s.$$

Notice that for $\mathbf{v} \in V$ the trace on Γ_c is well defined, so J_ε and DJ_ε make sense.

The regularized problem of (18) now reads as follows: find $u_\varepsilon \in V$ such that

$$a(u_\varepsilon, \mathbf{v} - u_\varepsilon) + \langle DJ_\varepsilon(u_\varepsilon), \mathbf{v} - u_\varepsilon \rangle \geq \langle g, \mathbf{v} - u_\varepsilon \rangle \quad \forall v \in V. \tag{21}$$

Further, we consider a triangulation $\{\mathcal{T}_h\}$ of Ω. Let Σ_h be the set $\{x_i\}$ of all vertices of the triangles of $\{\mathcal{T}_h\}$ and $\mathcal{P}_h^c$ the set of all nodes on $\overline{\Gamma}_c$, i.e.,

$$\mathcal{P}_h^c = \{x_i \in \Sigma_h \,:\, x_i \in \overline{\Gamma}_c\}.$$

Using continuous piecewise linear functions we approximate the subspace of all admissible displacements V by

$$V_h = \{v_h \in C(\overline{\Omega}; \mathbb{R}^2) \,:\, v_{h|_T} \in (\mathbb{P}_1)^2, \forall T \in \mathcal{T}_h, v_{hi} = 0 \text{ on } \Gamma_U,\ i = 1,2,$$
$$v_{h2}(x_i) = 0 \ \forall x_i \in \mathcal{P}_h^c\}.$$

The discretization of the regularized problem (21) is defined now as follows: find $u_h \in V_h$ such that

$$a(u_h, v_h - u_h) + \langle DJ_h(u_h), v_h - u_h \rangle \geq P \int_{\Gamma_F^1} (v_{h1} - u_{h1}) \, \mathrm{d}x_2 \quad \forall v_h \in V_h,$$

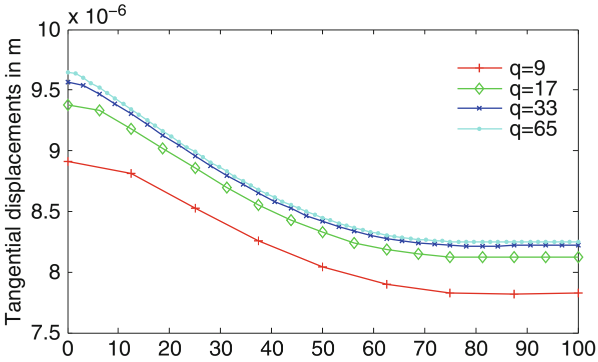

Fig. 3 The tangential component of the displacement vector on Γ_c

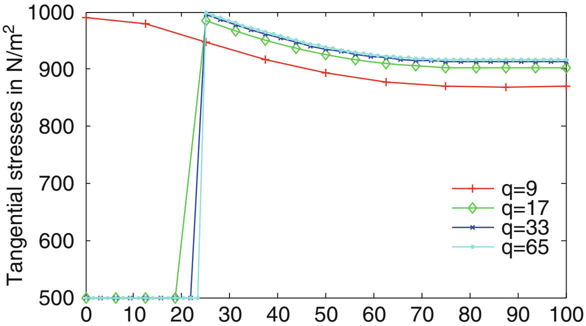

Fig. 4 The distribution of the tangential stresses along Γ_c

where

$$\langle DJ_h(u_h), v_h \rangle = \frac{1}{2} \sum |P_i P_{i+1}| \left[\frac{\partial \tilde{S}}{\partial \xi}(u_{h1}(P_i), \varepsilon) v_{h1}(P_i) + \frac{\partial \tilde{S}}{\partial \xi}(u_{h1}(P_{i+1}), \varepsilon) v_{h1}(P_{i+1}) \right].$$

Further, using the condensation technique, we pass to a reduced finite-dimensional variational inequality problem formulated only in terms of the contact displacements. To solve this problem numerically we use the equivalent KKT system, which is further reformulated as a smooth, unconstrained minimization problem by using an appropriate merit function. Finally, the merit function is minimized by applying an algorithm based on trust region methods. We did numerical experiments for different mesh sizes $h = 1/8, 1/16$, $1/32$, and $1/64$ m. The number of the contact nodes is $q = 9, 17, 33$, and 65, respectively. The obtained results are collected in the pictures below. Figures 3 and 4 show the behavior of the tangential displacements u_1 and the distribution of $-S_T$ along the contact boundary Γ_c for the different number of the contact nodes ($q = 9, 17, 33, 65$) and for the constant load $P = 1.2 \times 10^6\,\mathrm{N/m^2}$. From Fig. 4 we can see that the computed stresses indeed follow the law depicted in Fig. 2. It is easy to see that with a finer discretization (e.g., $q = 17, 33, 65$) some of the computed displacements are larger than $\delta = 9.0 \times 10^{-6}\,m$ and the computed tangential stresses jump down to the parallel branch $-S_T = 500\,\mathrm{N/m^2}$ as described by a nonmonotone friction law in Fig. 2. All computations are made with regularization parameter ε fixed to 0.1.

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