Well-Posedness for Lexicographic Vector Equilibrium Problems

L.Q. Anh, T.Q. Duy, A.Y. Kruger, and N.H. Thao

Abstract We consider lexicographic vector equilibrium problems in metric spaces. Sufficient conditions for a family of such problems to be (uniquely) well posed at the reference point are established. As an application, we derive several results on well-posedness for a class of variational inequalities.

Keywords Lexicographic order • Equilibrium problem • Well-posedness

1 Introduction

Equilibrium problems first considered by Blum and Oettli [20] have been playing an important role in optimization theory with many striking applications particularly in transportation, mechanics, economics, etc. Equilibrium models incorporate many other important problems such as optimization problems, variational inequalities, complementarity problems, saddle point/minimax problems, and fixed points. Equilibrium problems with scalar and vector objective functions have been widely studied. The crucial issue of solvability (the existence of solutions) has attracted the most considerable attention of researchers; see, e.g., [17, 24, 27, 29, 42]. A relatively new but rapidly growing topic is the stability of solutions, including semicontinuity

L.Q. Anh
Department of Mathematics, Teacher College, Cantho University, Cantho, Vietnam
e-mail: quocanh@ctu.edu.vn

T.Q. Duy
Department of Mathematics, Cantho Technical and Economic College, Cantho, Vietnam
e-mail: tqduy@ctec.edu.vn

A.Y. Kruger (✉) • N.H. Thao
Centre for Informatics and Applied Optimization, School of Science, Information Technology and Engineering, University of Ballarat, Ballarat, Victoria, Australia
e-mail: a.kruger@ballarat.edu.au; hieuthaonguyen@students.ballarat.edu.au; nhthao@ctu.edu.vn

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properties in the sense of Berge and Hausdorff (see, e.g., [2, 4, 5, 7, 16]) and the Hölder/Lipschitz continuity of solution mappings (see, e.g., [1,3,6,10,12,15,34,35]) and the (unique) well-posedness of approximate solutions in the sense of Hadamard and Tikhonov (see, e.g., [8,9,11,12,26,39,41]). The ultimate issue of computational methods for solving equilibrium problems has also been considered in the literature; see, e.g., [21, 30, 40].

With regard to vector equilibrium problems, most of existing results correspond to the case when the order is induced by a closed convex cone in a vector space. Thus, they cannot be applied to lexicographic cones, which are neither closed nor open. These cones have been extensively investigated in the framework of vector optimization; see, e.g., [18, 19, 22, 25, 28, 32, 33, 37]. However, for equilibrium problems, the main emphasis has been on the issue of solvability/existence. To the best of our knowledge, there have not been any works on well-posedness for lexicographic vector equilibrium problems.

In this article, we establish necessary and/or sufficient conditions for such problems to be (uniquely) well posed. As an application, we consider the special case of variational inequalities.

2 Preliminaries

We first recall the concept of lexicographic cone in finite-dimensional spaces and models of equilibrium problems with the order induced by such a cone.

The lexicographic cone of $\mathbb{R}^n$, denoted C_l, is the collection of zero and all vectors in $\mathbb{R}^n$ with the first nonzero coordinate being positive, i.e.,

$$C_l := \{0\} \cup \{x \in \mathbb{R}^n \mid \exists i \in \{1,2,\ldots,n\} : x_i > 0 \text{ and } x_j = 0 \quad \forall j < i\}.$$

This cone is convex and pointed and induces the total order as follows:

$$x \geq_l y \Longleftrightarrow x - y \in C_l.$$

We also observe that it is neither closed nor open. Indeed, when comparing with the cone $C_1 := \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$, we see that $\operatorname{int} C_1 \subsetneq C_l \subsetneq C_1$, while

$$\operatorname{int} C_l = \operatorname{int} C_1 \quad \text{and} \quad \operatorname{cl} C_l = C_1.$$

In what follows, $K : \Lambda \rightrightarrows X$ is a set-valued mapping between metric spaces and $f = (f_1, f_2, \ldots, f_n) : K(\Lambda) \times K(\Lambda) \times \Lambda \to \mathbb{R}^n$ is a vector-valued function. For each $\lambda \in \Lambda$, the lexicographic vector equilibrium problem is

(LEP_λ) find $\bar{x} \in K(\lambda)$ such that

$$f(\bar{x}, y, \lambda) \geq_l 0 \quad \forall y \in K(\lambda).$$

Remark 1. This model covers parameterized bilevel optimization problems: minimize $g_2(\cdot,\lambda)$ over the solution set of the problem of minimizing $g_1(\cdot,\lambda)$ over $K(\lambda)$, where g_1 and g_2 are real-valued functions on $\operatorname{gph} K$. Recall that the graph of a (set-valued) mapping $Q : X \rightrightarrows Y$ is defined by $\operatorname{gph} Q := \{(x,y) \in X \times Y \mid y \in Q(x)\}$.

We denote $(\mathbf{LEP}) := \{(\mathrm{LEP}_\lambda) \mid \lambda \in \Lambda\}$ with the solution mapping $S : \Lambda \rightrightarrows X$ and assume that at the considered point $\bar{\lambda}$, the solution set $S(\bar{\lambda})$ is nonempty.

Following the lines of investigating ε-solutions to vector optimization problems initiated by Loridan [36], we consider, for each $\varepsilon \in [0;\infty)$, the following approximate problem:

$(\mathrm{LEP}_{\lambda,\varepsilon})$ find $\bar{x} \in K(\lambda)$ such that

$$f(\bar{x},y,\lambda) + \varepsilon e \geq_l 0 \quad \forall y \in K(\lambda),$$

where $e = (0,\ldots,0,1) \in \mathbb{R}^n$. The solution set of $(\mathrm{LEP}_{\lambda,\varepsilon})$ is denoted by $\tilde{S}(\lambda,\varepsilon)$.

We next define the notion of well-posedness for $(\mathbf{LEP})$ and recall continuity-like properties crucial for our analysis in this study.

Definition 1. A sequence $\{x_n\}$ with $x_n \in K(\lambda_n)$ is an *approximating sequence* of $(\mathrm{LEP}_{\bar{\lambda}})$ corresponding to a sequence $\{\lambda_n\} \subset \Lambda$ converging to $\bar{\lambda}$ if there is a sequence $\{\varepsilon_n\} \subset (0;\infty)$ converging to 0 such that $x_n \in \tilde{S}(\lambda_n,\varepsilon_n)$ for all n.

Definition 2. $(\mathbf{LEP})$ is *well posed* at $\bar{\lambda}$ if for any sequence $\{\lambda_n\}$ in Λ converging to $\bar{\lambda}$, every corresponding approximating sequence of $(\mathrm{LEP}_{\bar{\lambda}})$ has a subsequence converging to some point of $S(\bar{\lambda})$.

Definition 3. $(\mathbf{LEP})$ is *uniquely well posed* at $\bar{\lambda}$ if:

(i) $(\mathrm{LEP}_{\bar{\lambda}})$ has the unique solution $\bar{x}$.
(ii) For any sequence $\{\lambda_n\}$ in Λ converging to $\bar{\lambda}$, every corresponding approximating sequence of $(\mathrm{LEP}_{\bar{\lambda}})$ converges to $\bar{x}$.

Remark 2. Unfortunately there is no consistency in the literature in the usage of the term "well-posedness." Defining well-posedness here as a kind of "good behavior" of a family of parametric problems, we follow the lines of, e.g., [9, 11, 26]. Other authors, e.g., Bednarczuk [14], use this term as a characterization of a single reference problem. If f in the above setting does not depend on λ, then the two versions of well-posedness coincide.

Definition 4 ([13]). Let $Q : X \rightrightarrows Y$ be a set-valued mapping between metric spaces:

(i) Q is *upper semicontinuous* (usc) at $\bar{x}$ if for any open set $U \supseteq Q(\bar{x})$, there is a neighborhood N of $\bar{x}$ such that $Q(N) \subseteq U$.
(ii) Q is *lower semicontinuous* (lsc) at $\bar{x}$ if for any open subset U of Y with $Q(\bar{x}) \cap U \neq \emptyset$, there is a neighborhood N of $\bar{x}$ such that $Q(x) \cap U \neq \emptyset$ for all $x \in N$.
(iii) Q is *closed* at $\bar{x}$ if for any sequences $\{x_k\} \longrightarrow \bar{x}$ and $\{y_k\} \longrightarrow \bar{y}$ with $y_k \in Q(x_k)$, it holds $\bar{y} \in Q(\bar{x})$.

Lemma 1 ([13,31]).

(i) *If Q is usc at $\bar{x}$ and $Q(\bar{x})$ is compact, then for any sequence $\{x_n\} \longrightarrow \bar{x}$, every sequence $\{y_n\}$ with $y_n \in Q(x_n)$ has a subsequence converging to some point in $Q(\bar{x})$. If, in addition, $Q(\bar{x}) = \{\bar{y}\}$ is a singleton, then such a sequence $\{y_n\}$ must converge to $\bar{y}$.*

(ii) *Q is lsc at $\bar{x}$ if and only if for any sequence $\{x_n\} \to \bar{x}$ and any point $y \in Q(\bar{x})$, there is a sequence $\{y_n\}$ with $y_n \in Q(x_n)$ converging to y.*

Definition 5. Let g be an extended real-valued function on a metric space X and ε be a real number.

(i) g is *upper ε-level closed* at $\bar{x} \in X$ if for any sequence $\{x_n\} \longrightarrow \bar{x}$,

$$[g(x_n) \geq \varepsilon \quad \forall n] \Rightarrow [g(\bar{x}) \geq \varepsilon].$$

(ii) g is *strongly upper ε-level closed* at $\bar{x} \in X$ if for any sequences $\{x_n\} \longrightarrow \bar{x}$ and $\{\nu_n\} \subset [0;\infty)$ converging to 0,

$$[g(x_n) + \nu_n \geq \varepsilon \quad \forall n] \Rightarrow [g(\bar{x}) \geq \varepsilon].$$

Remark 3. If g is usc at $\bar{x}$, then it satisfies property (ii) in the last definition, which is obviously stronger than property (i) therein for any real number ε. Property (i) was introduced and investigated in [9, 11]. Property (ii) is a particular case of a more general property also introduced in [9, 11].

We say that a mapping/function satisfies a certain property on a subset of its domain if it is satisfied at every point of this subset.

3 Well-Posedness Properties of (LEP)

We are going to establish necessary and/or sufficient conditions for (**LEP**) to be (uniquely) well posed at the reference point $\bar{\lambda} \in \Lambda$. To simplify the presentation, in the sequel, the results will be formulated for the case $n = 2$.

Given $\lambda \in \Lambda$ and $x \in K(\Lambda)$, denote

$$S_1(\lambda) := \{x \in K(\lambda) \mid f_1(x,y,\lambda) \geq 0 \quad \forall y \in K(\lambda)\},$$

$$Z(\lambda,x) := \begin{cases} \{z \in K(\lambda) \mid f_1(x,z,\lambda) = 0\} \text{ if } (\lambda,x) \in \operatorname{gph} S_1, \\ X \text{ otherwise.} \end{cases} \tag{1}$$

$S_1 : \Lambda \rightrightarrows X$ is the solution mapping of the scalar equilibrium problem determined by the real-valued function f_1. The set-valued mapping $Z : \Lambda \times K(\Lambda) \rightrightarrows X$ is going to play an important role in our analysis.

Problem $(\text{LEP}_{\lambda,\varepsilon})$ can be equivalently stated as follows:

$(\text{LEP}_{\lambda,\varepsilon})$ find $\bar{x} \in K(\lambda)$ such that

$$\begin{cases} f_1(\bar{x},y,\lambda) \geq 0 & \forall y \in K(\lambda), \\ f_2(\bar{x},z,\lambda) + \varepsilon \geq 0 & \forall z \in Z(\lambda,\bar{x}). \end{cases}$$

This is equivalent to finding $\bar{x} \in S_1(\lambda)$ such that

$$f_2(\bar{x},z,\lambda) + \varepsilon \geq 0 \quad \forall z \in Z(\lambda,\bar{x}).$$

The next lemma is frequently used in the sequel.

Lemma 2. *Let $\{x_n\}$ converging to $\bar{x} \in S_1(\bar{\lambda})$ be an approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to some sequence $\{\lambda_n\} \longrightarrow \bar{\lambda}$ and assume that Z is lsc at $(\bar{\lambda},\bar{x})$ and f_2 is strongly upper 0-level closed on $\{\bar{x}\} \times Z(\bar{\lambda},\bar{x}) \times \{\bar{\lambda}\}$. Then $\bar{x} \in S(\bar{\lambda})$.*

Proof. Suppose to the contrary that $\bar{x} \notin S(\bar{\lambda})$. Then, there exists $\bar{z} \in Z(\bar{\lambda},\bar{x})$ such that $f_2(\bar{x},\bar{z},\bar{\lambda}) < 0$. The lower semicontinuity of Z at $(\bar{\lambda},\bar{x})$ ensures the existence, for each n, of $z_n \in Z(\lambda_n,x_n)$ such that $\{z_n\} \to \bar{z}$. Due to $x_n \in \tilde{S}(\lambda_n,\varepsilon_n)$, it holds $f_2(x_n,z_n,\lambda_n)+\varepsilon_n \geq 0$ for all n. Since f_2 is strongly upper 0-level closed at $(\bar{x},\bar{z},\bar{\lambda})$, we get $f_2(\bar{x},\bar{z},\bar{\lambda}) \geq 0$. This yields a contradiction, and, hence, we are done. □

Theorem 1. *Suppose that*

(i) X is compact,
(ii) K is lsc and closed at $\bar{\lambda}$,
(iii) Z is lsc on $\{\bar{\lambda}\} \times S_1(\bar{\lambda})$,
(iv) f_1 is upper 0-level closed on $K(\bar{\lambda}) \times K(\bar{\lambda}) \times \{\bar{\lambda}\}$,
(v) f_2 is strongly upper 0-level closed on $K(\bar{\lambda}) \times K(\bar{\lambda}) \times \{\bar{\lambda}\}$.

Then (**LEP**) *is well posed at $\bar{\lambda}$. Moreover, it is uniquely well posed at this point if $S(\bar{\lambda})$ is a singleton.*

Proof. We first prove that S_1 is closed at $\bar{\lambda}$. Suppose to the contrary that there are sequences $\{\lambda_n\} \longrightarrow \bar{\lambda}$ and $\{x_n\} \longrightarrow \bar{x}$ with $x_n \in S_1(\lambda_n)$ and $\bar{x} \notin S_1(\bar{\lambda})$. Note that $\bar{x} \in K(\bar{\lambda})$ because K is closed at $\bar{\lambda}$ and $x_n \in K(\lambda_n)$ for all n. Then, there exists $\bar{y} \in K(\bar{\lambda})$ satisfying $f_1(\bar{x},\bar{y},\bar{\lambda}) < 0$. The lower semicontinuity of K at $\bar{\lambda}$ ensures that, for each n, there is $y_n \in K(\lambda_n)$ such that $\{y_n\} \longrightarrow \bar{y}$. Since $x_n \in S_1(\lambda_n)$, $f_1(x_n,y_n,\lambda_n) \geq 0$. This implies by assumption (iv) that $f_1(\bar{x},\bar{y},\bar{\lambda}) \geq 0$, which yields a contradiction, and hence, S_1 is closed at $\bar{\lambda}$.

We next show that $\tilde{S}$ is usc at $(\bar{\lambda},0)$. Indeed, if otherwise, then there is an open set $U \supset \tilde{S}(\bar{\lambda},0)$ along with sequences $\{\lambda_n\} \longrightarrow \bar{\lambda}$, $\{\varepsilon_n\} \downarrow 0$ such that, for each n, there is $x_n \in \tilde{S}(\lambda_n,\varepsilon_n) \setminus U$. By the compactness of X, we can assume that (x_n) converges to some $\bar{x}$. Since S_1 is closed at $\bar{\lambda}$, $\bar{x} \in S_1(\bar{\lambda})$. Thanks to Lemma 2, it holds $\bar{x} \in S(\bar{\lambda}) = \tilde{S}(\bar{\lambda},0)$. This yields a contradiction because $x_n \notin U$ (open) for all n. Thus, $\tilde{S}$ is usc at $(\bar{\lambda},0)$.

We finally prove that $S(\bar{\lambda})$ is compact by checking its closedness. Take an arbitrary sequence $\{x_n\}$ in $S(\bar{\lambda})$ converging to $\bar{x}$. It is clear that $\bar{x} \in S_1(\bar{\lambda})$ due to the closedness of S_1 at $\bar{\lambda}$. Note that $\{x_n\}$ is, of course, an approximating sequence of $(\mathrm{LEP}_{\bar{\lambda}})$. Then, Lemma 2 again implies that $\bar{x} \in S(\bar{\lambda})$ and, hence, $S(\bar{\lambda})$ is compact. Thanks to Lemma 1 (i), we are done. □

Remark 4. All assumptions in Theorem 1, except (iii), are formulated in terms of the problem data and normally are not difficult to check. Assumption (iii) involves set-valued mapping Z defined by (1) and can be not so easy to check. Additional research is required to establish verifiable sufficient conditions for lower semicontinuity of Z.

The following examples show that none of the assumptions in Theorem 1 can be dropped.

Example 1 (Compactness of X). Let $X = \Lambda = \mathbb{R}$ (not compact), $K(\lambda) \equiv \mathbb{R}$ (continuous and closed), and $f(x,y,\lambda) = (0,\lambda)$. One can check that $S(\lambda) = S_1(\lambda) = Z(\lambda,x) = \mathbb{R}$ for all $\lambda, x \in \mathbb{R}$. Thus, assumptions (ii)–(v) hold true. However, (**LEP**) is not well posed at $\bar{\lambda} = 0$ because the approximating sequence $\{x_n = n\}$ of $(\mathrm{LEP}_{\bar{\lambda}})$ corresponding to $\{\lambda_n = \frac{1}{n}\}$ has no convergent subsequence.

Example 2 (Lower semicontinuity of K). Let $X = \Lambda = [0;2]$ (compact) and K and f be defined by

$$K(\lambda) := \begin{cases} [0;1] & \text{if } \lambda \neq 0, \\ [0;2] & \text{if } \lambda = 0, \end{cases}$$

$$f(x,y,\lambda) := (x-y,\lambda).$$

One can check that K is closed but not lsc at $\bar{\lambda} = 0$ and

$$S(\lambda) = S_1(\lambda) = \begin{cases} \{1\} & \text{if } \lambda \neq 0, \\ \{2\} & \text{if } \lambda = 0, \end{cases}$$

$$Z(\lambda,x) = \{x\} \quad \forall (\lambda,x) \in \operatorname{gph} S_1.$$

Thus, assumptions (iii)–(v) hold true. However, (**LEP**) is not well posed at $\bar{\lambda}$ because the approximating sequence $\{x_n = 1\}$ of $(\mathrm{LEP}_{\bar{\lambda}})$ (corresponding to any sequence $\{\lambda_n\}$) converges to $1 \notin S(\bar{\lambda})$.

Example 3 (Closedness of K). Let $X = \Lambda = [0;1]$ (compact), $K(\lambda) \equiv (0;1]$ (continuous), and $f(x,y,\lambda) = (0,\lambda)$. It is clear that

$$S(\lambda) = S_1(\lambda) = K(\lambda) \quad \forall \lambda \in \Lambda,$$

$$Z(\lambda,x) = (0;1] \quad \forall (\lambda,x) \in \operatorname{gph} S_1.$$

One can also check that (**LEP**) is not well posed at $\bar{\lambda} = 0$, while all the assumptions of Theorem 1 except the closedness of K at $\bar{\lambda}$ are satisfied.

Example 4 (Lower semicontinuity of Z). Let $X = \Lambda = [0;1]$ (compact), $K(\lambda) \equiv [0;1]$ (continuous and closed), $\bar{\lambda} = 0$, and $f(x,y,\lambda) = (\lambda x(x-y), y-x)$. One can check that

$$S_1(\lambda) = \begin{cases} [0;1] & \text{if } \lambda = 0, \\ \{0,1\} & \text{if } \lambda \neq 0, \end{cases}$$

and, for each $(\lambda, x) \in \operatorname{gph} S_1$,

$$Z(\lambda,x) = \begin{cases} [0;1] & \text{if } \lambda = 0 \text{ or } x = 0, \\ \{x\} & \text{if } \lambda \neq 0 \text{ and } x \neq 0. \end{cases}$$

Z is not lsc at $(0,1)$ because by taking $\{(x_n = 1, \lambda_n = \frac{1}{n})\} \longrightarrow (1,0)$, we have $Z(\lambda_n, x_n) = \{1\}$ for all n, while $Z(0,1) = [0;1]$. Assumptions (iv) and (v) are obviously satisfied. Finally, we observe that (**LEP**) is not well posed at $\bar{\lambda}$ by calculating the solution mapping S explicitly as follows:

$$S(\lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ \{0,1\} & \text{if } \lambda \neq 0. \end{cases}$$

Example 5 (Upper 0*-level closedness of* f_1*).* Let $X = \Lambda = [0;1]$ (compact), $K(\lambda) \equiv [0;1]$ (continuous and closed), $\bar{\lambda} = 0$, and

$$f(x,y,\lambda) = \begin{cases} (x-y, \lambda) & \text{if } \lambda = 0, \\ (y-x, \lambda) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$S(\lambda) = S_1(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0, \\ \{0\} & \text{if } \lambda \neq 0, \end{cases}$$

$$Z(\lambda, x) = \{x\} \quad \forall (\lambda, x) \in \operatorname{gph} S_1.$$

Hence, all the assumptions except (iv) hold true. However, (**LEP**) is not well posed at $\bar{\lambda}$. Indeed, take sequences $\{\lambda_n = \frac{1}{n}\}$ and $\{x_n = 0\}$ $(x_n \in S(\lambda_n))$. Then, $\{x_n\}$ is an approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to $\{\lambda_n\}$, while $\{x_n\} \longrightarrow 0 \notin S(0)$.

Finally, we show that assumption (iv) is not satisfied. Indeed, taking $\{x_n\}$ and $\{\lambda_n\}$ as above and $\{y_n = 1\}$, we have $\{(x_n, y_n, \lambda_n)\} \longrightarrow (0,1,0)$ and $f_1(x_n, y_n, \lambda_n) = 1 > 0$ for all n, while $f_1(0,1,0) = -1 < 0$.

Example 6 (Strong upper 0-level closedness of f_2). Let $X, \Lambda, K, \bar{\lambda}$ be as in Example 5 and

$$f(x,y,\lambda) = \begin{cases} (0, x-y) & \text{if } \lambda = 0, \\ (0, x(x-y)) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$S_1(\lambda) = Z(\lambda, x) = [0;1] \quad \forall x, \lambda \in [0;1],$$

$$S(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0, \\ \{0,1\} & \text{if } \lambda \neq 0. \end{cases}$$

Thus, all the assumptions of Theorem 1 except (v) are satisfied. However, it follows from the explicit form of S that (**LEP**) is not well posed at $\bar{\lambda}$. Finally, we show that assumption (v) is not satisfied. Indeed, taking sequences $\{x_n = 0\}$, $\{y_n = 1\}$, $\{\lambda_n = \frac{1}{n}\}$, and $\{\varepsilon_n = \frac{1}{n}\}$, we have $\{(x_n, y_n, \lambda_n, \varepsilon_n)\} \longrightarrow (0,1,0,0)$ and $f_2(x_n, y_n, \lambda_n) + \varepsilon_n > 0$ for all n, while $f_2(0,1,0) = -1 < 0$.

In what follows,

$$\mathcal{P}(\bar{\lambda}, \delta, \varepsilon) := \bigcup_{\lambda \in B_\delta(\bar{\lambda})} \tilde{S}(\lambda, \varepsilon),$$

where $B_\delta(\bar{\lambda})$ denotes the closed ball centered at $\bar{\lambda}$ with radius δ. We also use the concept of diameter of a set A in a metric space:

$$\operatorname{diam} A := \sup_{a,b \in A} d(a,b).$$

Theorem 2.

(i) If (**LEP**) *is uniquely well posed at* $\bar{\lambda}$*, then* $\operatorname{diam} \mathcal{P}(\bar{\lambda}, \delta, \varepsilon) \downarrow 0$ *as* $\delta \downarrow 0$ *and* $\varepsilon \downarrow 0$.

(ii) Suppose that X is complete and assumptions (ii)–(v) in Theorem 1 hold true. If $\operatorname{diam} \mathcal{P}(\bar{\lambda}, \delta, \varepsilon) \downarrow 0$ *as* $\delta \downarrow 0$ *and* $\varepsilon \downarrow 0$*, then* (**LEP**) *is uniquely well posed at* $\bar{\lambda}$.

Proof.

(i) Let (**LEP**) be uniquely well posed at $\bar{\lambda}$ and $\{\delta_n\} \downarrow 0$, $\{\varepsilon_n\} \downarrow 0$. If $\operatorname{diam} \mathcal{P}(\bar{\lambda}, \delta_n, \varepsilon_n)$ does not converge to 0 as $n \longrightarrow \infty$, then there exists a number $r > 0$ such that for any $n_0 \in \mathbb{N}, \exists n \geq n_0$ with $\operatorname{diam} \mathcal{P}(\bar{\lambda}, \delta_n, \varepsilon_n) > r$. By taking a subsequence if necessary, we can suppose that $\operatorname{diam} \mathcal{P}(\bar{\lambda}, \delta_n, \varepsilon_n) > r$ for all n. This implies that, for each n, there exist $x_n^1, x_n^2 \in \mathcal{P}(\bar{\lambda}, \delta_n, \varepsilon_n)$ such that

$$d(x_n^1, x_n^2) > \frac{r}{2}. \tag{2}$$

Thus, there are $\lambda_n^1, \lambda_n^2 \in B(\bar{\lambda}, \delta_n)$ such that $x_n^i \in \tilde{S}(\lambda_n^i, \varepsilon_n)$, i=1,2. Observe that both $\{\lambda_n^1\}$ and $\{\lambda_n^2\}$ converge to $\bar{\lambda}$ as $n \longrightarrow \infty$, and so $\{x_n^1\}$ and $\{x_n^2\}$ are corresponding approximating sequences of $(\text{LEP}_{\bar{\lambda}})$, respectively. Due to the unique well-posedness of (**LEP**) at $\bar{\lambda}$, both $\{x_n^1\}$ and $\{x_n^2\}$ must converge to the only solution $\bar{x}$ to $(\text{LEP}_{\bar{\lambda}})$. Hence, $\lim_{n \longrightarrow \infty} d(x_n^1, x_n^2) = 0$. This contradicts (2) and, thus, we are done.

(ii) Suppose that $\{x_n\}$ is an approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to some sequence $\{\lambda_n\} \longrightarrow \bar{\lambda}$, i.e., there is a sequence $\{\varepsilon_n\} \downarrow 0$ such that $x_n \in \tilde{S}(\lambda_n, \varepsilon_n)$ for all n. By setting $\delta_n := d(\lambda_n, \bar{\lambda})$, it holds that $\{\delta_n\} \longrightarrow 0$ as $n \longrightarrow \infty$ and $x_n \in \mathcal{P}(\bar{\lambda}, \delta_n, \varepsilon_n)$ for all n. By choosing subsequences if necessary, we can assume that both sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ are nonincreasing. Thus, $\mathcal{P}(\bar{\lambda}, \delta_n, \varepsilon_n) \supseteq \mathcal{P}(\bar{\lambda}, \delta_m, \varepsilon_m)$ whenever $n \leq m$. From this observation and $\operatorname{diam} \mathcal{P}(\bar{\lambda}, \delta_n, \varepsilon_n) \downarrow 0$ as $n \longrightarrow \infty$, one can directly check that $\{x_n\}$ is a Cauchy sequence and, hence, converges to some point $\bar{x}$ due to the completeness of X. Note that assumptions on K and f_1 imply the closedness of S_1 at $\bar{\lambda}$; see the first reasoning in the proof of Theorem 1. In particular, we have $\bar{x} \in S_1(\bar{\lambda})$, and Lemma 2 then yields $\bar{x} \in S(\bar{\lambda})$.

Finally, we show that $\bar{x}$ is the only solution to $(\text{LEP}_{\bar{\lambda}})$. Suppose to the contrary that $S(\bar{\lambda})$ contains also another point $\bar{x}'$ ($\bar{x}' \neq \bar{x}$). It is clear that they both belong to $\mathcal{P}(\bar{\lambda}, \delta, \varepsilon)$ for any $\delta, \varepsilon > 0$. Then, it follows that

$$0 < d(\bar{x}, \bar{x}') \leq \operatorname{diam} \mathcal{P}(\bar{\lambda}, \delta, \varepsilon) \downarrow 0 \text{ as } \delta \downarrow 0 \text{ and } \varepsilon \downarrow 0.$$

This is impossible and, therefore, we are done. □

To weaken the assumption of unique well-posedness in Theorem 2, we are going to use the *Kuratowski measure of noncompactness* of a nonempty set M in a metric space X:

$$\mu(M) := \inf\left\{\varepsilon > 0 \mid M \subseteq \bigcup_{k=1}^{n} M_k,\ M_k \subset X,\ \operatorname{diam} M_k \leq \varepsilon\ \forall k,\ n \in \mathbb{N}\right\}.$$

Lemma 3 ([38]). *The following assertions hold true:*

(i) $\mu(M) = 0$ *if* M *is compact.*
(ii) $\mu(M) \leq \mu(N)$ *whenever* $M \subseteq N$.
(iii) If $\mu(M) = 0$, *then* M *is totally bounded, i.e., there are a point* $x_M \in X$ *along with a constant* $\kappa_M > 0$ *such that*

$$d(x, x_M) \leq \kappa_M \quad \forall x \in M.$$

(iv) If $\{A_n\}$ *is a sequence of closed subsets in a complete metric space* X *satisfying* $A_{n+1} \subseteq A_n$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n \longrightarrow \infty} \mu(A_n) = 0$, *then* $K := \bigcap_{n \in \mathbb{N}} A_n$ *is a*

nonempty compact set and $\lim_{n \longrightarrow \infty} H(A_n, K) = 0$*, where* H *is the Hausdorff distance.*

Recall that the *Hausdorff distance* between two sets A and B in a metric space is defined by

$$H(A,B) := \max\{e(A,B), e(B,A)\},$$

where $e(A,B) := \sup_{a\in A} d(a,B)$ with $d(a,B) := \inf_{b\in B} d(a,b)$.

Theorem 3.

(i) If (**LEP**) *is well posed at* $\bar{\lambda}$*, then* $\mu(\mathcal{P}(\bar{\lambda},\delta,\varepsilon)) \downarrow 0$ *as* $\delta \downarrow 0$ *and* $\varepsilon \downarrow 0$*.*

(ii) Suppose that X *is complete,* Λ *is compact or a finite-dimensional normed space and*

(a) K *is lsc and closed on some neighborhood* V *of* $\bar{\lambda}$*,*
(b) Z *is lsc on* $[V \times X] \cap \operatorname{gph} S_1$*,*
(c) f_1 *is upper 0-level closed on* $K(V) \times K(V) \times V$*,*
(d) f_2 *is upper a-level closed on* $K(V) \times K(V) \times V$ *for every negative a close to zero.*

If $\mu(\mathcal{P}(\bar{\lambda},\delta,\varepsilon)) \downarrow 0$ *as* $\delta \downarrow 0$ *and* $\varepsilon \downarrow 0$*, then* (**LEP**) *is well posed at* $\bar{\lambda}$*.*

Proof.

(i) Suppose that (**LEP**) is well posed at $\bar{\lambda}$. Let $\{x_n\}$ be an arbitrary sequence in $S(\bar{\lambda})$ [and, of course, an approximating sequence of $(\text{LEP}_{\bar{\lambda}})$]. Then, it has a subsequence converging to some point in $S(\bar{\lambda})$. Thus, $S(\bar{\lambda})$ is compact, and so $\mu(S(\bar{\lambda})) = 0$ due to Lemma 3(i). Let any $\varepsilon > 0$ and $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k$ with $\operatorname{diam} M_k \leq \varepsilon$ for all $k = \overline{1,n}$. We set

$$N_k = \{y \in X \mid d(y, M_k) \leq H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon), S(\bar{\lambda}))\}$$

and show that $\mathcal{P}(\bar{\lambda},\delta,\varepsilon) \subseteq \bigcup_{k=1}^{n} N_k$. Pick arbitrary $x \in \mathcal{P}(\bar{\lambda},\delta,\varepsilon)$. Then $d(x, S(\bar{\lambda})) \leq H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon), S(\bar{\lambda}))$. Due to $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k$, one has

$$d(x, \bigcup_{k=1}^{n} M_k) \leq H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon), S(\bar{\lambda})).$$

Then, there exists $\bar{k} \in \{1,2,\ldots,n\}$ such that $d(x, M_{\bar{k}}) \leq H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon), S(\bar{\lambda}))$, i.e., $x \in N_{\bar{k}}$. Thus, $\mathcal{P}(\bar{\lambda},\delta,\varepsilon) \subseteq \bigcup_{k=1}^{n} N_k$.

Because $\mu(S(\bar{\lambda})) = 0$ and

$$\operatorname{diam} N_k = \operatorname{diam} M_k + 2H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon), S(\bar{\lambda})) \leq \varepsilon + 2H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon), S(\bar{\lambda})),$$

it holds

$$\mu(\mathcal{P}(\bar{\lambda},\delta,\varepsilon)) \leq 2H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon),S(\bar{\lambda})).$$

Note that $H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon),S(\bar{\lambda})) = e(\mathcal{P}(\bar{\lambda},\delta,\varepsilon),S(\bar{\lambda}))$ since $S(\bar{\lambda}) \subseteq \mathcal{P}(\bar{\lambda},\delta,\varepsilon)$ for all $\delta,\varepsilon > 0$.

Now, we claim that $H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon),S(\bar{\lambda})) \downarrow 0$ as $\delta \downarrow 0$ and $\varepsilon \downarrow 0$. Indeed, if otherwise, we can assume that there exist $r > 0$ and sequences $\{\delta_n\} \downarrow 0$, $\{\varepsilon_n\} \downarrow 0$, and $\{x_n\}$ with $x_n \in \mathcal{P}(\bar{\lambda},\delta_n,\varepsilon_n)$ such that

$$d(x_n,S(\bar{\lambda})) \geq r \quad \forall n. \tag{3}$$

Since $\{x_n\}$ is an approximating sequence of $(\mathrm{LEP}_{\bar{\lambda}})$ corresponding to some $\{\lambda_n\}$ with $\lambda_n \in B_{\delta_n}(\bar{\lambda})$, it has a subsequence $\{x_{n_k}\}$ converging to some $x \in S(\bar{\lambda})$. Then, $d(x_{n_k},x) < r$ when n_k is sufficiently large. This contradicts (3) and, hence,

$$\mu(\mathcal{P}(\bar{\lambda},\delta,\varepsilon)) \longrightarrow 0 \text{ as } \delta \downarrow 0 \text{ and } \varepsilon \downarrow 0.$$

(ii) Suppose that $\mu(\mathcal{P}(\bar{\lambda},\delta,\varepsilon)) \downarrow 0$ as $\delta \downarrow 0$ and $\varepsilon \downarrow 0$. We firstly show that $\mathcal{P}(\bar{\lambda},\delta,\varepsilon)$ is closed for any $\delta,\varepsilon > 0$. Let $\{x_n\} \in \mathcal{P}(\bar{\lambda},\delta,\varepsilon)$, $\{x_n\} \longrightarrow \bar{x}$. Then, for each $n \in \mathbb{N}$, there exists $\lambda_n \in B_\delta(\bar{\lambda})$ such that $x_n \in \tilde{S}(\lambda_n,\varepsilon)$. Assumption on Λ implies that $B_\delta(\bar{\lambda})$ is compact. So, we can assume $\{\lambda_n\}$ converges to some $\lambda \in B_\delta(\bar{\lambda}) \cap V$. Thus, $\bar{x} \in K(\lambda)$ due to the closedness of K at λ. Assumptions on K and f_1 imply that $\bar{x} \in S_1(\lambda)$; see the first reasoning in the proof of Theorem 1. Now, we check that $\bar{x}$ also belongs to $\tilde{S}(\lambda,\varepsilon)$. Indeed, suppose to the contrary that there exists $\bar{z} \in Z(\lambda,\bar{x})$ such that $f_2(\bar{x},\bar{z},\lambda) + \varepsilon < 0$. Then, the lower semicontinuity of Z at $(\lambda,\bar{x})$ ensures that, for each n, there is $z_n \in Z(\lambda_n,x_n)$ such that $\{z_n\} \longrightarrow \bar{z}$. Due to the upper $(-\varepsilon)$-level closedness of f_2 at $(\bar{x},\bar{z},\lambda)$, $f_2(x_n,z_n,\lambda_n) < -\varepsilon$ when n is sufficiently large. This is a contradiction since $x_n \in \tilde{S}(\lambda_n,\varepsilon)$ for all n. Hence, $\bar{x} \in \tilde{S}(\lambda,\varepsilon)$, and so $\bar{x} \in \mathcal{P}(\bar{\lambda},\delta,\varepsilon)$. Therefore, $\mathcal{P}(\bar{\lambda},\delta,\varepsilon)$ is closed for any $\delta,\varepsilon > 0$.

Next, we prove $S(\bar{\lambda}) = \bigcap_{\delta,\varepsilon>0} \mathcal{P}(\bar{\lambda},\delta,\varepsilon)$. We first check that $\bigcap_{\delta>0} \mathcal{P}(\bar{\lambda},\delta,\varepsilon) = \tilde{S}(\bar{\lambda},\varepsilon)$ for any $\varepsilon > 0$. It is clear that $\tilde{S}(\bar{\lambda},\varepsilon) \subseteq \bigcap_{\delta>0} \mathcal{P}(\bar{\lambda},\delta,\varepsilon)$. Now, take any $x \in \bigcap_{\delta>0} \mathcal{P}(\bar{\lambda},\delta,\varepsilon)$. Then, for each sequence $\{\delta_n\} \downarrow 0$, there exists a sequence $\{\lambda_n\}$ with $\lambda_n \in B_{\delta_n}(\bar{\lambda})$ such that $x \in \tilde{S}(\lambda_n,\varepsilon)$ for all n. Assumptions on K and f_1 again imply $x \in S_1(\bar{\lambda})$. For any $z \in Z(\bar{\lambda},x)$, there exists $z_n \in Z(\lambda_n,x)$, $\{z_n\} \longrightarrow z$, thanks to the lower semicontinuity of Z at $(\bar{\lambda},x)$. As $x \in \tilde{S}(\lambda_n,\varepsilon)$, it holds $f_2(x,z_n,\lambda_n) + \varepsilon \geq 0$ for every n. From the upper $(-\varepsilon)$-level closedness of f_2 at $(x,z,\bar{\lambda})$, we have $f_2(x,z,\bar{\lambda}) + \varepsilon \geq 0$, i.e., $x \in \tilde{S}(\bar{\lambda},\varepsilon)$. It follows that $\bigcap_{\delta>0} \mathcal{P}(\bar{\lambda},\delta,\varepsilon) \subseteq \tilde{S}(\bar{\lambda},\varepsilon)$ and, thus, $\bigcap_{\delta>0} \mathcal{P}(\bar{\lambda},\delta,\varepsilon) = \tilde{S}(\bar{\lambda},\varepsilon)$. Now, we need to check that $S(\bar{\lambda}) = \bigcap_{\varepsilon>0} \tilde{S}(\bar{\lambda},\varepsilon)$. It is clear that $S(\bar{\lambda}) \subseteq \bigcap_{\varepsilon>0} \tilde{S}(\bar{\lambda},\varepsilon)$. On the other hand, for any $x \in \bigcap_{\varepsilon>0} \tilde{S}(\bar{\lambda},\varepsilon)$, we have $f_2(x,z,\bar{\lambda}) + \varepsilon \geq 0$ for all $z \in Z(\bar{\lambda},x)$ and $\varepsilon > 0$. By letting ε tend to 0, this implies $f_2(x,z,\bar{\lambda}) \geq 0$ for all $z \in Z(\bar{\lambda},x)$, i.e., $x \in S(\bar{\lambda})$, and, hence, $S(\bar{\lambda}) = \bigcap_{\delta,\varepsilon>0} \mathcal{P}(\bar{\lambda},\delta,\varepsilon)$.

Finally, since $\mu(\mathcal{P}(\bar{\lambda},\delta,\varepsilon)) \downarrow 0$ as $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, Lemma 3(iv) implies the compactness of $S(\bar{\lambda})$ and $H(\mathcal{P}(\bar{\lambda},\delta,\varepsilon),S(\bar{\lambda})) \longrightarrow 0$ as $\delta \downarrow 0$ and $\varepsilon \downarrow 0$. Let $\{x_n\}$ be an approximating sequence of (LEP$_{\bar{\lambda}}$) corresponding to some $\{\lambda_n\} \longrightarrow \bar{\lambda}$. Then, there exists $\{\varepsilon_n\}$ converging to 0 such that $x_n \in \tilde{S}(\lambda_n,\varepsilon_n)$ for all n. This means that $x_n \in \mathcal{P}(\bar{\lambda},\delta_n,\varepsilon_n)$, where $\delta_n = d(\bar{\lambda},\lambda_n)$. Note that

$$d(x_n,S(\bar{\lambda})) \leq H(\mathcal{P}(\bar{\lambda},\delta_n,\varepsilon_n),S(\bar{\lambda})) \downarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus, there is $\{\bar{x}_n\} \subset S(\bar{\lambda})$ such that $d(x_n,\bar{x}_n) \downarrow 0$ as $n \longrightarrow \infty$. Since $S(\bar{\lambda})$ is compact, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging to some $\bar{x} \in S(\bar{\lambda})$ and, hence, $\{x_n\}$ has the corresponding subsequence $\{x_{n_k}\}$ converging to $\bar{x}$. Therefore, (**LEP**) is well posed at $\bar{\lambda}$, and we are done. □

Remark 5. Theorem 3 remains valid if the Kuratowski measure is replaced by either Hausdorff or Istrătescu measure. We refer the reader to [23] for further information about these noncompact measures including their equivalence.

Note that when $K(\Lambda)$ is contained in a compact set (in particular, X is compact), the assumption on the measure μ in Theorem 3 (ii) holds true trivially. Hence, Examples 2–5 again show that assumptions (a)–(c) imposed in Theorem 3(ii) are essential. The following example shows that the upper negative-level closedness of f_2 therein is also essential.

Example 7. Let $X = \mathbb{R}$ (complete), $\Lambda = [0;1]$ (compact), $K(\lambda) \equiv [-1;1]$ (continuous and closed), $\bar{\lambda} = 0$ and

$$f(x,y,\lambda) := \begin{cases} ((x-y)^2, x-1) & \text{if } \lambda = 0, \\ ((x-y)^2, (x+y)^2) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$S_1(\lambda) = [-1;1] \quad \forall \lambda,$$

$$Z(\lambda,x) = \{x\} \quad \forall (\lambda,x) \in \operatorname{gph} S_1,$$

$$S(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0, \\ [-1;1] & \text{if } \lambda \neq 0. \end{cases}$$

We observe that f_2 is not 0-level closed at $(-1,1,0)$. Indeed, taking $\{x_n = -1\}$, $\{y_n = 1\}$, and $\{\lambda_n = \frac{1}{n}\}$, we have $\{(x_n,y_n,\lambda_n)\} \longrightarrow (-1,1,0)$ and $f_2(x_n,y_n,\lambda_n) = 0$, while $f_2(-1,1,0) = -2 < 0$. Moreover, all the other assumptions are satisfied, while (**LEP**) is not well posed at $\bar{\lambda}$.

4 Applications to Variational Inequalities

In this section, let Λ and K be as in the preceding sections, X be a normed space with its dual denoted by X^* and $h_i : X \times \Lambda \longrightarrow X^*$, $i = 1,2$. For each $\lambda \in \Lambda$, we consider the following lexicographic variational inequality:

(LVI_λ) find $\bar{x} \in K(\lambda)$ such that

$$(\langle h_1(\bar{x},\lambda), y-\bar{x}\rangle, \langle h_2(\bar{x},\lambda), y-\bar{x}\rangle) \geq_l 0 \quad \forall y \in K(\lambda).$$

This is equivalent to finding $\bar{x} \in K(\lambda)$ such that

$$\begin{cases} \langle h_1(\bar{x},\lambda), y-\bar{x}\rangle \geq 0 & \forall y \in K(\lambda), \\ \langle h_2(\bar{x},\lambda), z-\bar{x}\rangle \geq 0 & \forall z \in Z(\lambda,\bar{x}). \end{cases}$$

Here, the set-valued mapping $Z : \Lambda \times K(\Lambda) \rightrightarrows X$ is defined by

$$Z(\lambda,x) := \begin{cases} \{z \in K(\lambda) \mid \langle h_1(x,\lambda), z-x\rangle = 0\} \text{ if } (\lambda,x) \in \operatorname{gph} S_1, \\ X \text{ otherwise,} \end{cases}$$

where $S_1 : \Lambda \rightrightarrows X$ denotes the solution mapping of the scalar variational inequality determined by h_1:

$$S_1(\lambda) := \{x \in K(\lambda) \mid \langle h_1(x,\lambda), y-x\rangle \geq 0 \quad \forall y \in K(\lambda)\}.$$

We denote (**LVI**) $:= \{(\text{LVI}_\lambda) \mid \lambda \in \Lambda\}$ with the solution mapping $S : \Lambda \rightrightarrows X$ and assume that at the considered point $\bar{\lambda}$, the solution set $S(\bar{\lambda})$ is nonempty.

Now, for a number $\varepsilon > 0$, we consider the following approximate problem:

($\text{LVI}_{\lambda,\varepsilon}$) find $\bar{x} \in K(\lambda)$ such that

$$\begin{cases} \langle h_1(\bar{x},\lambda), y-\bar{x}\rangle \geq 0 & \forall y \in K(\lambda), \\ \langle h_2(\bar{x},\lambda), z-\bar{x}\rangle + \varepsilon \geq 0 & \forall z \in Z(\lambda,\bar{x}). \end{cases}$$

We also use denotation $\tilde{S}$ for the approximate solution mapping, i.e.,

$$\tilde{S}(\lambda,\varepsilon) := \{x \in S_1(\lambda) \mid \langle h_2(x,\lambda), z-x\rangle + \varepsilon \geq 0 \quad \forall z \in Z(\lambda,x)\}.$$

In the following, we use the concepts defined in Definitions 1–3 with the term "LEP" replaced by "LVI." The next theorems follow from the corresponding results established in Sect. 3 by setting $f_i(x,y,\lambda) := \langle h_i(x,\lambda), y-x\rangle$, $i = 1,2$, therein.

Theorem 4. *Suppose that assumptions (i)–(iii) in Theorem 1 are satisfied. Assume, additionally, that*

(i) $\{(x,y,\lambda) \in K(\Lambda) \times K(\Lambda) \times \Lambda \mid \langle h_1(x,\lambda), y-x\rangle \geq 0\}$ *is a closed subset of* $K(\Lambda) \times K(\Lambda) \times \Lambda$*,*
(ii) the function $(x,y,\lambda) \mapsto \langle h_2(x,\lambda), y-x\rangle$ *is strongly upper 0-level closed on* $K(\bar{\lambda}) \times K(\bar{\lambda}) \times \{\bar{\lambda}\}$*.*

Then (**LVI**) *is well posed at* $\bar{\lambda}$*. Moreover, it is uniquely well posed at this point if* $S(\bar{\lambda})$ *is a singleton.*

Remark 6. Assumptions (i) and (ii) in Theorem 4 are straightforwardly fulfilled when h_1 and h_2, respectively, are continuous.

Theorem 5.

(i) If (**LVI**) *is uniquely well posed at* $\bar{\lambda}$*, then* $\operatorname{diam}\mathcal{P}(\bar{\lambda},\delta,\varepsilon) \downarrow 0$ *as* $\delta \downarrow 0$ *and* $\varepsilon \downarrow 0$*.*
(ii) Suppose that X is complete and assumptions (ii)–(iii) in Theorem 1 and (i)–(ii) in Theorem 4 hold true. If $\operatorname{diam}\mathcal{P}(\bar{\lambda},\delta,\varepsilon) \downarrow 0$ *as* $\delta \downarrow 0$ *and* $\varepsilon \downarrow 0$*, then* (**LVI**) *is uniquely well posed at* $\bar{\lambda}$*.*

Theorem 6.

(i) If (**LVI**) *is well posed at* $\bar{\lambda}$*, then* $\mu(\mathcal{P}(\bar{\lambda},\delta,\varepsilon)) \downarrow 0$ *as* $\delta \downarrow 0$ *and* $\varepsilon \downarrow 0$*.*
(ii) Suppose that X is complete, Λ *is compact or a finite-dimensional normed space, assumptions (a)–(b) in Theorem 3 and assumption (i) in Theorem 4 hold true, and the function* $(x,y,\lambda) \mapsto \langle h_2(x,\lambda), y-x\rangle$ *is upper a-level closed on* $K(V) \times K(V) \times V$ *for every negative a close to zero. If* $\mu(\mathcal{P}(\bar{\lambda},\delta,\varepsilon)) \downarrow 0$ *as* $\delta \downarrow 0$ *and* $\varepsilon \downarrow 0$*, then* (**LVI**) *is well posed at* $\bar{\lambda}$*.*

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