4

Pricing Exotic Options

European put and call options are valued according to the expected price of the underlying on the expiration date of the option. This makes it easy all around to price the option at any time. The Black-Scholes formula does exactly that. The history of prices of the underlying plays no role in determining the option value.

But this is the exception as far as options go. Already the American option has an associated exercise boundary; the option is exercised if the path of prices touches it. And there are even more exotic options yet. Most of them are path dependent.

In this chapter we review some of these exotic options and show how they can be priced by Monte Carlo methods. Pricing options that depend on the price history of the underlying is a major theoretical challenge for analytical methods. In many cases Monte Carlo is the only practical solution.

The following is a partial list of exotic options along with their brief descriptions. The options marked by an asterisk have analytical pricing formulas (at least for their European version). The reference for the analysis is given in parentheses.

Asian the payoff is determined by the average price of the underlying over some pre-set period of time.

Barrier* if a trigger price is crossed it causes a pre-determined option to come into existence (knock-in) or go out of existence (knock-out) [Hull11].

Basket the underlying is a weighted average of several assets.

Bermuda the buyer has the right to exercise at a discrete set of times.

Binary* the payoff is a fixed amount of some asset or nothing at all, also called a digital option [Hull11].

Chooser* gives the holder a fixed period of time in which to decide whether the option will be a European put or call [Hull11].

Compound* an option on an option; the exercise payoff of a compound option is determined by the value of another option [Hull11].

Exchange* the holder gets the best performing out of two underlying assets at expiration [Mar78].

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Extendible* allows the holder or writer to choose, on the expiration date, to extend the life of the option by a specified amount [Lon90].

Lookback* the holder has the right to buy (sell) the underlying at its lowest (highest) price over some preceding period [Hull11].

Shout during the life of the option the holder can, at any time, "shout" to the seller that he or she is locking-in the current price, if this gives a better deal than the payoff at maturity the asset price on the shout date may be used instead of that on the expiration date.

Spread* its underlying is the difference between two specific assets [CD03].

We will discuss some of these in the following sections in terms of their Monte Carlo solutions. Even for those having analytical formulas, that solution requires their financial parameters be constant, such as the risk-free rate. But they can be solved as well by Monte Carlo under less stringent, non-constant, conditions.

In pricing exotic options by Monte Carlo, the random number generator must be of high quality.

4.1 Asian Options

In place of the price of the underlying at exercise, an Asian option uses the average price of the underlying over some pre-set period of time. For example the entire life of the option or perhaps the last 30 days before expiration. A reason for preferring Asian options in certain cases is to provide protection from price manipulation as the option nears expiration. This is a risk for thinly traded assets. Asian options also avoid the vagaries of volatility in the market. And they are cheaper than their European counterparts because, by averaging the price of the underlying, the effective volatility is much less.

The algorithm for pricing an Asian option is only mildly different from our standard pricing algorithm, Algorithm 13 on page 108. It is noteworthy that to obtain accurate results, dt must be taken to be a very small increment of time, on the order of one one-hundredth of a day or about $dt = 2.74 \times 10^{-5}$ in years. This greatly increases the run time of the GRW. The following algorithm takes about 90 seconds for $N = 100{,}000$ trials on contemporary equipment. For techniques that reduce the run time see Chapter D.

Algorithm 16. Pricing algorithm for Asian options

```
inputs: S_0, K, T, Δt, r, σ, N
E = 0  ▷expected option value
n = T/Δt  ▷number of walk steps
A = 0  ▷A = average price over entire walk
for  i = 1,...,N
   S = S_0  ▷starting price
   for  t = 1,...,n
      Z ~ N(0,1)
      dS = S(rΔt + σ√(Δt)Z)
      S = S + dS
      A = A + S
```

```
    endfor
    A = A/n   ▷average price
    E = E + G(A)
  endfor
  E = E/N
  option price = e^{-rT}E
```

Option payoffs are as usual,

$$\text{for calls}\quad G(A) = \max(A - K, 0), \qquad \text{for puts}\quad G(A) = \max(K - A, 0).$$

The algorithm relies on discrete arithmetical averaging

$$A = \frac{1}{n}\sum_{i=1}^{n} S_i,$$

but other types of averaging are also used. These include continuous (in analytical calculations) and geometric averaging, respectively

$$A = \frac{1}{T}\int_0^T S(t)dt$$
$$A = \exp\left(\frac{1}{T}\int_0^T \log(S(t))dt\right)$$

There are analytical formulas for calculating Asian options under geometric averaging.

In Table 4.1 we compare various Asian option prices with their European counterparts. One immediate observation is that as the averaging period becomes shorter at the end of the life of the option, the Asian price increases up to that of the European.

Table 4.1 Asian versus European option prices

$S_0 = 100$, $r_f = 3\,\%$, $\sigma = 20\,\%$, $\Delta t(\text{days}) = 0.01$

Type	Strike	Expiry(days)	Avg. period	Asian	European
Call	100	60	Entire	1.99	3.48
Call	100	60	Last 30 days	2.82	3.48
Call	100	60	Last 15 days	3.16	3.48
Call	95	60	Entire	5.50	6.61
Call	105	60	Entire	0.41	1.54
Put	100	60	Entire	1.75	2.99

4.1.1 Floating Strike Asian Option

The option described above is known as the *fixed strike Asian option*. There is also a variant in which it is the strike price that is averaged. In this case the put and call payoffs are, respectively,

$$\max(A - S_T, 0), \quad \text{and} \quad \max(S_T - A, 0).$$

As usual, S_T is the underlying price at expiration while A is the average underlying price over the designated period. We leave it to the reader to explore this case.

4.2 Barrier Options

In addition to the strike price, a barrier option specifies a second price as well, the barrier. The barrier can function to engage the option or to nullify it depending on the type. In the case of a "knock-out" barrier, if the barrier price is crossed, the option becomes valueless. The opposite occurs for a "knock-in" barrier, the option comes into existence.

Evidently the price of a knock-out type plus the price of a knock-in type equals the price of a plain vanilla European option. This implies that the price of a barrier option is always less than that of its European counterpart. Their reduced cost is one attraction of a barrier option.

It also implies that given the price of one of the options, say the knock-out variant, then the price of the knock-in can be easily calculated by subtracting from the price of the vanilla option as determined by the Black-Scholes formula.

In calculating the value of a knock-out barrier option by simulation there is a fundamental problem. We may and do simulate the stock price at the nodes, $t_i = i\Delta t$, $i = 1, \ldots, n$ and therefore know if the barrier is crossed at those points, but what about between the nodes? Fortunately there is a way to decide, probabilistically, whether the barrier has been crossed in this manner. The technique is called *Brownian bridges*. Let $X_t = \mu t + \sigma W_t$ be a Wiener process with drift which has the value x_{i-1} at t_{i-1} and x_i at t_i both less than the barrier B. Then the probability the process does not cross the barrier between these bridge points is given by, see [BS02]

$$\Pr(X_\tau < B,\, t_{i-1} < \tau < t_i) = 1 - e^{-2(B - x_i)(B - x_{i-1})/(\sigma^2 \Delta t)}. \tag{4.1}$$

One sees from (4.1) that in the limit as $x_i \to B$ (or $x_{i-1} \to B$) the probability of not crossing tends to 0.

The following algorithm runs the simulation, reports the ending stock price and whether the barrier was crossed or not. From (4.1), the barrier is crossed between end points with probability

$$e^{-2(B - x_i)(B - x_{i-1})/(\sigma^2 \Delta t)}. \tag{4.2}$$

But if the barrier is crossed at one of the end points, then the product $(B - x_i)(B - x_{i-1})$ is negative[1] and the exponential (4.2) is greater than 1. Hence the Brownian bridge check may be combined with the end-point check.

Algorithm 17. Pricing algorithm for a barrier option

```
inputs: S_0, K, B, T, r, σ, N, dt
E = 0; n = T/dt;
for  i = 1,...,N
    S = S_0; barrierCrossed = false;
    diff1= B − S_0;
    for  t = 1,...,n
        Z ~ N(0,1)
        dS = S(r dt + σ √dt Z)
        S = S + dS
        diff2 = B − S
        U ~ U(0,1)
        if( U < e^(−2 diff1·diff2/σ^2·dt) )   ▷barrier crossed
            barrierCrossed = true
        endif
        diff1 = diff2;
    endfor
    if( barrierCrossed == false )
        E = E + G(S)    ▷knock out type
    endif
endfor
E = E/N
option price = e^(−rT) E
```

Again, to obtain accurate results, dt must be taken to be a very small increment of time.

Some example barrier prices are presented in Table 4.2.

Table 4.2 Barrier versus European option prices

$S_0 = 100$, $K = 100$, $r_f = 3\,\%$

Type	Barrier	Expiry(days)	Vol.(%)	Δt(days)	Barrier price	European
Call	95	60	20	0.05	3.09	3.48
Call	95	60	40	0.005	4.04	6.70
Call	95	90	20	0.005	3.53	3.32
Call	90	60	20	0.05	3.48	3.48
Put	105	60	20	0.05	2.57	2.99

[1] Only one is negative the first time.

4.3 Basket Options

The payoff of a basket option is the weighted average of two or more underlying assets – what would be called a "basket" of assets. For example, a European style basket option has a specified strike price K and an expiration date T. The payoff is $(S_T - K)^+$ for a call or $(K - S_T)^+$ for a put where

$$S_t = \sum_{k=1}^{n} w_k S_t^k, \quad 0 \le t \le T;$$

$S_t^1, \ldots, S_t^n$ are the prices of the underlying assets, n of them in this case, and $w_1, \ldots, w_n$ are the weights, $\sum w_k = 1$.

The complication in evaluating a basket option is that the underlying assets are almost always correlated. Fortunately correlation is no problem for the Monte Carlo method. Refer back to Section 2.3.4 for a discussion on the matter.

To illustrate, we will work through a three basket problem. Let ρ_{12} be the correlation coefficient between assets 1 and 2. Similarly let ρ_{13} and ρ_{23} be the correlations between assets 1 and 3 and 2 and 3 respectively. According to (2.36) and (2.37) we may use the lower triangular matrix

$$H = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & h_{32} & \sqrt{\sigma_3^2 - \rho_{13}^2 - h_{32}^2} \end{bmatrix} \tag{4.3}$$

where

$$h_{32} = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}$$

to generate the required correlated standard normal random variables. Let Z_1, Z_2, and Z_3 be independent $N(0,1)$ samples, then X_1, X_2, and X_3 given by

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = H \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \tag{4.4}$$

serve as the increments to the GRW.

Note that perfect correlations $\rho = 1$ or perfect anti-correlations, $\rho = -1$ must be worked out as special cases. For example, if $\rho_{12} = 1$, then $\rho_{13} = \rho_{23}$. The terms of matrix H will be $h_{11} = h_{21} = 1$, $h_{31} = \rho_{13}$, $h_{32} = \sqrt{1-\rho_{13}^2}$, and $h_{22} = h_{33} = 0$ in this case.

Algorithm 18. Pricing algorithm for a 3-basket option

```
inputs: S_0^k, w_k, σ_k, k = 1,2,3,
        H, K, T, r, N, Δt
E = 0; n = T/Δt;
for  i = 1,...,N
   S^k = S_0^k, k = 1,2,3
```

```
    for  t = 1,...,n
      Z_k ~ N(0,1),  k = 1,2,3
      [X_1  X_2  X_3]^T = H[Z_1  Z_2  Z_3]^T   ▷eqn. (4.4)
      dS^k = S^k(rΔt + σ_k √Δt X_k),  k = 1,2,3
      S^k = S^k + dS^k,  k = 1,2,3
    endfor
    S_T = w_1 S^1 + w_2 S^2 + w_3 S^3
    E = E + G(S_T)
  endfor
  E = E/N
  option price = e^{-rT} E
```

In Table 4.3 we give some basket option values. There are many possible combinations to explore; only a small subset can be undertaken here. In the last column of the table we give the Black-Scholes value of an option having volatility equal to the weighted average of that of the three assets. This is for reference purposes only, the basket option is not expected to equal it.

An exception is the first table entry. Here two perfectly correlated assets with the same volatility constituting the entire portfolio should behave as a single underlying, and does. In the next row, the two assets are perfectly anti-correlated. The result is that the option value is very small. This is because the volatility of the portfolio is now nearly zero, when one asset is heading up, the other is heading down. But the reverse directions of the two do not cancel because both have positive drift, the risk-free rate. This example shows that options over portfolios of assets should cost less. And the next row shows this for arbitrary correlations.

In the next row we see that if the assets are perfectly correlated then they give the same as Black-Scholes even if they have different volatilities. On the other hand, if the assets are uncorrelated, then their option cost is suitably reduced from Black-Scholes.

Table 4.3 A sampling of basket option prices

Call, $S_0^1 = S_0^2 = S_0^3 = 100$, $K = 100$, $r_f = 3\,\%$, $T = 60\,\text{days}$

σ_1	σ_2	σ_3	ρ_{12}	ρ_{13}	ρ_{23}	w_1	w_2	w_3	Basket	BlkSch.
0.2	0.2	0.2	1	0	0	0.5	0.5	0.0	3.48	3.48
0.2	0.2	0.2	−1	0	0	0.5	0.5	0.0	0.49	3.48
0.2	0.2	0.2	0.7	0.3	−0.1	0.33	0.33	0.33	2.62	3.48
0.2	0.3	0.4	1	1	1	0.33	0.33	0.33	5.09	5.09
0.2	0.3	0.4	0	0	0	0.33	0.33	0.33	3.14	5.09

4.4 Exchange Options

The payoff of an *exchange option* is the amount by which one asset outperforms another. If the contract matches asset A versus B, then at expiration the payoff is

$$\max(A_T - B_T, 0). \tag{4.5}$$

Another way of thinking about it is that the holder is allowed to exchange one share of asset B for one share of A at expiration if A is worth more (otherwise B is retained).

Exchange options are also called *Margrabe options* after the person who first studied them or *outperformance options*.

From the standpoint of asset A, the option is a European call with exercise price equal to B_T. But from the standpoint of B, it is a European put with exercise price A_T. From the first interpretation it is not surprising that during the life of the option it never has value 0 and therefore the price of an American exchange option is the same as the European one.

The payoff of an exchange option does not depend on the path of prices of the underlying giving rise to the hope that an analytical expression can be found to price them. One elegant way to proceed is by change of *numeraire*. Numeraire refers to the basis for measuring the value of things. Normally currency is used for this purpose, but here, following [Der96], we will use shares of B.

Let $C_\$$ be the value of the exchange option in terms of dollars and C_B the value in terms of shares of B. Similarly let $A_\$(0) = A(0)$ denote the value of one share of A in dollars at the time the contract is made and let $A_B(0)$ denote the value of one share of A in terms of shares of B at that time. The exchange rate between B-shares and dollars is $B_\$ = B(0)$, that is $B_\$$ is dollars per share of B. To convert a value in B-shares to dollars, multiply by $B_\$$.

In terms of B-shares, the option contract is to exchange 1 share of B for 1 share of A at expiration, in other words the payoff is

$$\max(A_B(T) - 1, 0).$$

Therefore the value of the contract denominated in B-shares is given by the Black-Scholes call formula, a function of S_0, K, T, r_f, and σ, see Section 3.6,

$$C = BS(S_0, K, T, r_f, \sigma) = S_0 \Phi(d_1) - K e^{-r_f T} \Phi(d_2).$$

In terms of B-shares the parameters are as follows: the starting value of A is $A_B(0)$, and the strike price is 1. Let the time to expiration be T as usual. The risk-free rate must be taken in terms of B-shares – it is the dividend yield for B, denote it q_B. Finally, for the volatility, we must use the volatility of A in terms of B-shares, denote it by $\sigma_B(A)$. We will calculate this below.

From Black-Scholes then

$$C_B = BS(A_B(0), 1, T, q_B, \sigma_B(A)) = A_B(0)\Phi(d_1) - e^{-q_B T}\Phi(d_2). \tag{4.6}$$

and in terms of dollars

$$C_\$ = B_\$ C_B = B_\$ A_B(0)\Phi(d_1) - B_\$ e^{-q_B T}\Phi(d_2)$$
$$= A(0)\Phi(d_1) - B(0)e^{-q_B T}\Phi(d_2).$$

It remains only to accommodate the dividend yield of A by replacing $A(0)$ by $A(0)e^{-q_A T}$ throughout (here and in d_1 and d_2 below), see Section 3.6.5. Thus

$$C_\$ = A(0)e^{-q_A T}\Phi(d_1) - B(0)e^{-q_B T}\Phi(d_2). \tag{4.7}$$

The combination of terms comprising d_1 and d_2 refer to B-shares as the numeraire, for example

$$d_1 = \frac{\log(A_B(0)/1) + (q_B + \frac{1}{2}\sigma_B(A)^2)T}{\sigma_B(A)\sqrt{T}}.$$

Since $B_\$ A_B(0) = A(0)$, it follows that $A_B(0) = A(0)/B(0)$. Then, accounting for the A dividend rate, we have

$$\begin{aligned} d_1 &= \frac{\log(A(0)/B(0)) + (q_B - q_A + \frac{1}{2}\sigma_B(A)^2)T}{\sigma_B(A)\sqrt{T}} \\ d_2 &= \frac{\log(A(0)/B(0)) + (q_B - q_A - \frac{1}{2}\sigma_B(A)^2)T}{\sigma_B(A)\sqrt{T}}. \end{aligned} \tag{4.8}$$

As mentioned above, $\sigma_B(A)$ is the volatility of A with respect to B; it is the square root of the variance of A/B. It can be shown that this is given by[2]

$$\sigma_B(A) = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B}. \tag{4.9}$$

Notice that the calculation of the exchange option price does not make use of the risk-free rate. This is because the risk-neutral requirement has both equities growing at that rate and therefore the effect cancels out.

4.4.1 Non-constant Correlation

Equation (4.7) assumes the correlation coefficient ρ_{AB} is constant. When this is not expected to be a valid assumption, Monte Carlo can be used. For example it may be anticipated that the two assets will become less correlated over the time horizon of the option. An arbitrary dependence on time, $\rho_{AB} = \rho_{AB}(t)$ can be accommodated or even a dependence on relative prices. The following algorithm incorporates a time profile.

Algorithm 19. Pricing algorithm for an Exchange Option

```
inputs: A_0, q_A, σ_A, B_0, q_B, σ_B
        ρ(t), T, r, N, Δt
E = 0; n = T/Δt;
for  i = 1,...,N
```

[2] Expand the function $f(a,b) = a/b$ in a Taylor series through first order terms about the means μ_A and μ_B and take expectation.

```
    A = A_0;  B = B_0;
    for  j = 1,...,n
      ρ = ρ(j Δt)
      Z_1 ~ N(0,1);  Z_2 ~ N(0,1);
      X_A = Z_1;  X_B = ρZ_1 + sqrt(1 − ρ^2) Z_2;
      dA = A((r − q_A)Δt + σ_A sqrt(Δt) X_A)
      dB = B((r − q_B)Δt + σ_B sqrt(Δt) X_B)
      A = A + dA;  B = B + dB;
    endfor
    E = E + max(A − B, 0)
  endfor
  E = E/N
  option price = e^{−rT} E
```

In Table 4.4 we show the results of a few runs of the algorithm. The first three use a constant correlation profile and hence, for them, the solution derived above should equal that of the simulation, and it does. We notice that as the assets are more correlated, the smaller is the option value. In the third case, the value of B starts out greater than that of A, thus A does not often exceed B at expiration and the option cost is low. In the fourth run the correlation decreases over the life of the option (from 0.8 down to 0.2). The result is that the option price behaves more like the correlation was the lower value than the upper one. In the last run the correlation increases over the life of the option. Again the result is that the option price is more like that for the higher correlation.

Table 4.4 Sample exchange option prices

$q_A = 8\,\%$, $q_B = 6\,\%$, $r = 3\,\%$, $T = 90$ days

A_0	B_0	σ_A	σ_B	Correlation	Exchange	BlkSch.
100	100	0.2	0.2	Const. at 0.6	3.24	3.24
100	100	0.2	0.2	Const. at 0.0	5.27	5.26
100	102	0.2	0.2	Const. at 0.6	1.88	1.88
100	100	0.2	0.2	Decr. 0.8 to 0.2	2.52	2.70
100	100	0.2	0.2	Incr. 0.2 to 0.8	3.47	2.70

4.5 Bermudan Options

A *Bermudan option* is one that can be exercised at any one of a set of specified times, the last one being the expiration date of the option. A Bermudan option is in this sense in-between an American and a European option.

A Bermudan option can be priced by either of the methods used for American options: the binomial tree method or maximization over a parameter set controlling an exercise boundary. Refer to Sections 3.5.4 and 3.7.2 respectively.

4.5.1 The Binomial Tree Method

The only change in the binomial method for a Bermudan option from its application in the American case is that the test for early exercise is only made at the designated exercise times, for all other nodes the tree is valued exactly as a European option. Of course the early exercise times should be among the nodes of the tree.

In Fig. 4.1 we show the binomial tree for a 20 day put option with possible exercise on days 10 and 20. The time between nodes on the tree is 5 days so the early exercise test must be made at nodal step 2. The Bermudan value of this option is \$1.788; the European is \$1.785. To get this kind of accuracy the step period for the binomial method must be on the order of 0.02 days or 500 steps per exercise period. The example as shown was chosen for illustration purposes only.

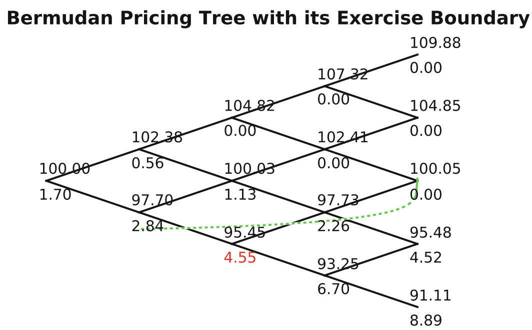

Fig. 4.1. Binomial pricing tree for a 2-exercise period put option. The 20 day Bermudan option can be exercised on the 10th day or otherwise at expiration. The binomial step size is 5 days. Those nodes for which early exercise is advantageous expresses the option's value in *red*. Superimposed on the graph is the early exercise boundary

4.5.2 The Exercise Boundary Method

As above, the only difference here from the American option case is that the test for early exercise is only made on the permissible exercise days. Additionally there are some special considerations that come into play in the Bermudan case.

Since there are only a finite number of exercise opportunities, and usually a small number, the parametrized analytical formula for the exercise boundary can be replaced by parameters giving the early exercise prices directly on the exercise days, either relativized (i.e. in the form $(K - S)/K$) or absolute. Thus for the problem in Fig. 4.1 there will be only one optimization parameter, the early exercise price on the 10th day.

Another consideration relates to the accuracy of the expectation estimates. Recall that, having fixed a trial set of parameters, whether or not they produce the maximum option value is determined by simulating a large number of random walks in order to calculate the expected payoff based on those parameters. But as these are only Monte Carlo estimates, there is inherent variance in them. Since the prices of the European versus the Bermudan options can be fairly close, it is desirable that the variance be of a smaller order of magnitude. Fortunately there are some simple remedies.

First, the random walk should not be carried out in small steps. Instead, the walk should jump from exercise day to exercise day by sampling from the appropriate lognormal distribution. This modification speeds up the simulation by many orders of magnitude. As a result, many more trials can be included toward determining the payoff expectation.

Secondly, a more discriminating objective can be used in place of the expected payoff, namely the expected payoff raised to some power. As mentioned above, in the example of Fig. 4.1 the (discounted) expected payoff is \$1.79. But the difference between 1.76 and 1.79, for example, does not discriminate between parameters sufficiently well to drive a simulated annealer or a genetic algorithm toward optimization. On the other hand $1.76^{10} = 285$ versus $1.79^{10} = 337$ has better effect.

Algorithm 20. Pricing a Bermudan Option Given an Exercise Boundary

```
inputs: S_0, K, T, r, σ, N, exercise dates t_j and the
        exercise boundary on those dates B_j, j = 1,...,n
E = 0
for  i = 1,...,N
  for  j = 1,2,...,n  ▷t_n = T
    ▷jump to next price S_j (cf. Algo. 12 page 107)
    Δt = t_j − t_{j−1}
    β = σ√Δt
    α = log(S_{j−1}) + (r − σ²/2)Δt
    Z ~ N(0,1)
    S_j = e^{βZ+α}
    if  K − S_j ≥ K * B_j
      E = E + e^{−rt_j}(K − S_j)  ▷exercise
      go to next i
    endif
  endfor j
endfor i
E = E/N
option price = E
```

In conjunction with the algorithm, as in the American case, we may use a simulated annealer or genetic algorithm to optimize the B_j's.

In Table 4.5 we make a comparison between a European, an American, and a Bermudan option; the latter calculated by both methods discussed above. The option is a 90 day 5 exercise opportunity put. With an exercise opportunity every 18 days the value is closer to that of an American versus a European option.

Table 4.5 Bermudan option comparison prices

$S_0 = 100$, $K = 100$, $r_f = 6\,\%$, $\sigma = 40\,\%$, #periods= 5, $T = 90$ days

European	American	Bermudan (binomial)	Bermudan (optimization)
7.14	7.28	7.23	7.23

4.6 Shout Options

During the life of a *shout option* the holder may lock in the current stock price for the purpose of recalculating the payoff value of the option at expiration. This is called *shouting* and the associated price is the *shout price*. At one time the shout price S_H was used in place of the expiration price S_T if it led to a bigger payoff. In such a case the payoff value for a call is

$$\max(S_H - K, S_T - K, 0).$$

Thus the holder attempts to shout at the maximum price of the underlying over the option's life.

In recent times it is more common to use the shout price to replace the strike price. This is called a *reset strike shout option*. In this case the payoff value of a call is

$$\text{payoff} = \begin{cases} \max(S_T - S_H, 0), & \text{if shouting occurs} \\ \max(S_T - K, 0), & \text{if no shouting occurs.} \end{cases} \tag{4.10}$$

Now the holder of the option attempts to shout at the lowest price of the underlying for a call. The holder does not shout when the asset price is above the original strike price.

Of course the S_T and S_H or K are reversed in (4.10) for a put as usual. For a reset strike put, the holder tries to shout at the maximum underlying price over the life of the option and does not shout when the asset price is below the strike price.

In the following we shall address the problem of pricing the reset strike version of the option. This is a very difficult problem for exact solution by analytical methods because a forward method can not specify a condition for shouting since the ending price of the stock is not known, and a backward method must know if shouting occurred earlier in the course of the price history in order to calculate the ending value of the option. The author knows of no such analytical method. Instead we will solve the problem by a two phase technique similar to that of the American put option: by estimating a "shout boundary" and subsequent simulation. The boundary calculation is lengthy and we make no attempt here to shorten it, but the subsequent option valuation is very fast.

4.6.1 Maximizing Over a Shout Boundary

Once again, let the time from inception to expiration, 0 to T, be divided into n equal time steps of interval Δt. At each time step $t_i = i\Delta t$, $i = 0, \ldots, n$, let b_i be the relativized boundary point at that time (3.44). Then the actual boundary point, B_i, is given by

$$B_i = K(1 \pm b_i)$$

where the plus sign applies for a put because the boundary is above the strike price in this case, and the minus sign applies for a call, because the boundary is below the strike for a call. To simplify the subsequent discussion assume we are pricing a put option.

As was the case for an American option, we proceed in reverse order and first consider the *conditional* boundary point at t_{n-1}, meaning the boundary point given that shouting has not previously occurred. It is easy to see that here the boundary point must be the strike price K. If the stock price at this step exceeds K, shouting resets the strike to a higher value which can only increase the payoff and make a positive payoff more likely. Further there is no penalty for doing so here. At earlier times, the restraint for shouting is that the stock price might go higher before expiration; that does not apply on this penultimate time step. Hence the relativized boundary point at $i = n - 1$ is $b_{n-1} = 0$.

Now consider the next time step proceeding in reverse order and again we assume shouting has not yet occurred. The higher the stock price, the more valuable to shout; if shouting at the price S leads to an improved expected payoff and $S' > S$, then shouting at S' leads to a bigger one. Hence the minimum (technically infinimum) of all those points where shouting leads to an improved expected payoff is the boundary point.

Finding this point is a straightforward one variable unimodal optimization problem. If the boundary point is set too low, then, stochastically, subsequent stock prices can allow for a later shout with an even better expected payoff. Similarly if the boundary point is set too high then the stock price might reach this level too infrequently to have a larger expected payoff than a lower value. The effect is shown in Fig. 4.2. This is a plot of expected payoff as a function of various trial locations of the boundary point all at the same time before expiration. Although the data has considerable stochastic variability, it clearly shows a unimodal maximum in the vicinity of $S = 102$.

Finding the maximum of such data numerically is problematic. It becomes much easier if the data is smoothed as shown in Fig. 4.3. The smoothing used in the figure is a 11 window central moving average, each smoothed value m_i is the sum of 5 prior values, the current value, and 5 future values all divided by 11,[3]

$$m_i = \frac{1}{11}(p_{i-5} + \ldots + p_{i-1} + p_i + p_{i+1} + \ldots + p_{i+5}).$$

The simulation of stock prices from the present step to expiration uses the boundary points that have already been calculated. The objective calculation is shown in Algorithm 21. Note that in the algorithm we use the absolute boundary values B_i. Further the order of the boundary values is reversed from that in the discussion above; B_0 is the boundary value at expiration and B_n is the value at $t = 0$. The algorithm first calculates the number of steps m to expiration; τ is the remaining time to expiration. Lognormal samples will be used to assign stock prices from step to step and the parameters α and β are calculated; α must remain a function of stock price and be recalculated from step to step.

In each trial, the starting stock price is drawn from a range of possibilities above and below the strike price. More exactly, the range should extend

[3] This is a discrete example of *convolution* smoothing. The general form is $m(t) = \int_{-\infty}^{\infty} f(t-s)k(s)ds$ where $f()$ is the function to be smoothed, $m()$ is the smoothed version and $k()$ is the smoothing kernel. In the discrete analogy here $k(s) = 1$ for $-5 \le s \le 5$ and 0 otherwise.

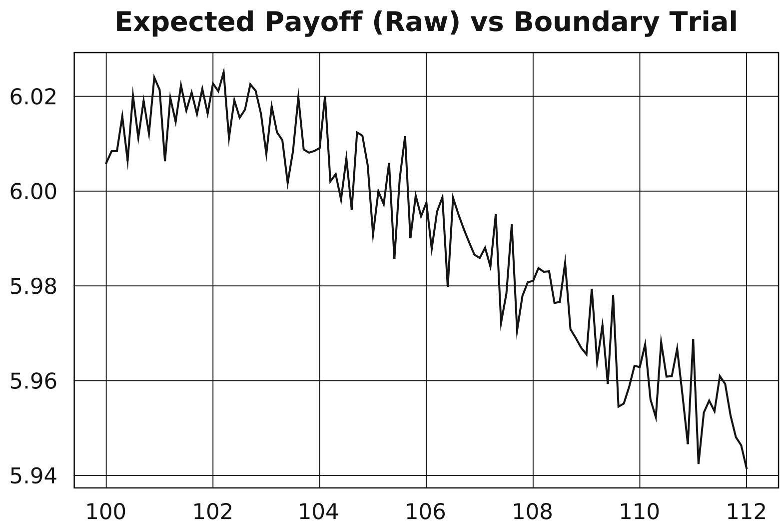

Fig. 4.2. Expected payoff as a function of increasing the conditional shout boundary point at $t = 18$ days for a put option. The option's particulars are: $K = 100$, $T = 36$, $r = 3\,\%$ and $\sigma = 20\,\%$. These are raw simulation results over the remaining 18 days to expiry. Each plotted point represents 1,000,000 simulations

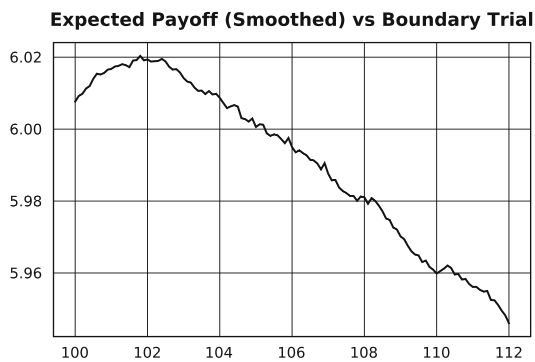

Fig. 4.3. This figure uses the same data as in Fig. 4.2 but here the raw data has been smoothed using a 11 window central moving average. It is much easier to determine the trial boundary value at which the maximum occurs

above and below the boundary point being tested. If paths do not encounter the boundary, then their payoff will mimic that of a European put.

Each new trial also begins with a "noshoutyet" variable set to true and the shout strike K_s set to K. Upon encountering the boundary, and only for the first time, K_s is reset to the current stock price, otherwise it remains at K. In either case the put payoff is calculated as $\max(K_s - S_T, 0)$ as usual.

Having determined the maximizing boundary value at step m from expiration, in like manner processing continues to step $m + 1$ and finally ends with the boundary at $t = 0$. At this point the option value itself can be calculated. Algorithm 21 can also be used for this provided the starting range is collapsed to 0 around S_0.

Algorithm 21. Monte Carlo objective calculation for a shout put

```
inputs: K, τ, Δt, r, σ, nTrials
        and the shout boundary B_i
m = τ/Δt      ▷number of steps to expiry
β = σ√Δt      ▷for lognormal samples
α(·) = log(·) + (r − ½σ²)Δt      ▷α = α(S)
V = 0;
for k = 1,..., nTrials      ▷loop over trials
  S ∼ uniform over a starting range
  K_s = K      ▷set the shouting strike equal to K initially
  noShoutYet = true      ▷keep track of shouting
  for i = 0,1,...
    if noShoutYet
      if S≥ B_{m−i}      ▷reset the strike
        K_s = S; noShoutYet = false;
      endif
    endif
      ▷take the next step
    S ∼ LN(α(S), β)  ▷lognormal sample
    i = i + 1
    if i == m, break out of loop      ▷expiry stock price
  endfor i
  V = V + max(K_s − S, 0)      ▷payoff for this trial
endfor
V = V/nTrials
```

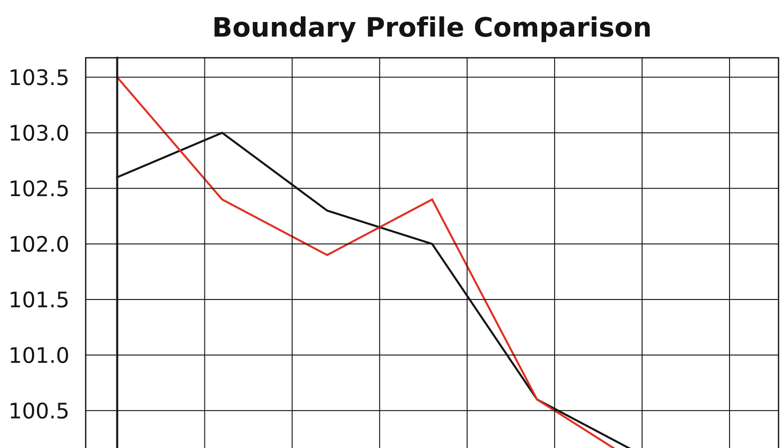

Fig. 4.4. Shout boundary as calculated by two runs of the method described in the text. The time horizon is divided into six periods

Two boundary calculation runs are shown in Fig. 4.4 for the $S_0 = K = 100$, $T = 36$ day shout put option with six division periods. As seen there, the calculated boundary points have considerable variance but despite that the option value is stable and has low variance. This phenomenon was previously noted in Section 3.7.2. In Table 4.6 we give some shout option prices calculated by the boundary method described above along with their statistical standard deviations. These are compared with the Black-Scholes prices for their European counterparts.

Table 4.6 Sample shout option prices

$S_0 = 100$, $K = 100$, $r_f = 3\,\%$, $\sigma = 20\,\%$, $T = 36$ days

Put/ call	Option value (standard deviation (10 trials)) versus number of time steps				
	3	6	12	24	European
Put	2.986(0.005)	3.059(0.004)	3.091(0.003)	3.104(0.005)	2.36
Call	3.253(0.005)	3.329(0.006)	3.375(0.005)	3.374(0.005)	2.65

Problems: Chapter 4

1. Write a program to calculate Asian options. Try it out for a 60 day ATM call option with $S_0 = 100$, and $r = 3\,\%$. Let the averaging take place over the last 30 days. Plot the option price as a function of volatility.

2. Repeat Problem 1 for a floating strike Asian option.

3. Write a program to calculate correlated basket options. Extend the results of Table 4.3 to $T = 90$ days.

4. Price a 90 day 100 strike Bermudian option with 15 day early exercise periods. Assume $r = 1\,\%$ and $\sigma = 20\,\%$. Use the binomial tree solution method. Plot the price of the option versus originating stock price. Compare the graph with that of its European counterpart.

5. Same question as Problem 4 but use the exercise boundary method.

6. Find the price of a 365 day exchange option between equities A and B. Assume $r = 6\,\%$, $\sigma_B = 20\,\%$ and the current price of B is \$60. Plot the price as a function of the current price of A for $\sigma_A = 15$, 30, and 45 %. Assume that neither A nor B issues dividends.

   In the following problems, create a calculator for the given exotic option and use it to calculate a table of prices for various option parameters.

7. A barrier option.

8. A binary option.

9. A chooser option.
10. A lookback option.
11. A spread option.