Chapter 5
Peridynamics for Laminated Composite Materials

5.1 Basics

Fiber-reinforced laminated composites are generally constructed by bonding unidirectional laminae in a particular sequence. Each lamina has its own material properties and thickness. As shown in Fig. 5.1, the fiber orientation angle, θ, is defined with respect to a reference axis, x. Fiber direction is commonly aligned with the x_1- axis, and transverse direction is aligned with the x_2- axis. A unidirectional lamina is specially orthotropic. Thus, a thin lamina has four independent material constants of elastic modulus in the fiber direction, E_{11}, elastic modulus in the transverse direction, E_{22}, in-plane shear modulus, G_{12}, and in-plane Poisson's ratio, ν_{12}.

For a unidirectional lamina, the stiffness matrix, $\mathbf{Q}$, relates the stresses and strains at material point $\mathbf{x}_{(k)}$ in reference to the material (natural) coordinates, (x_1, x_2) as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix}, \quad (5.1a)$$

where

$$Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}}, \; Q_{12} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}}, \; Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}}, \; Q_{66} = G_{12}, \quad (5.1b)$$

with $\nu_{12}/E_{11} = \nu_{21}/E_{22}$.

The stress, σ_{ij}, and strain, ε_{ij}, components are referenced to the principal material (natural) coordinate system, (x_1, x_2). The inverse of the lamina stiffness matrix, $\mathbf{Q}$, is referred to as the lamina compliance matrix, $\mathbf{S}$, whose coefficients are given as

E. Madenci and E. Oterkus, *Peridynamic Theory and Its Applications*,
DOI 10.1007/978-1-4614-8465-3_5,

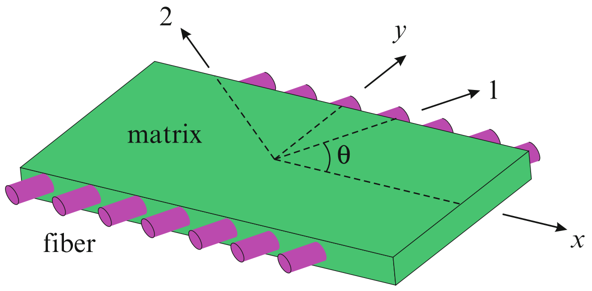


Fig. 5.1 Natural and reference coordinate systems for fiber-reinforced lamina

$$S_{11} = \frac{1}{E_{11}},\ S_{12} = -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}},\ S_{22} = \frac{1}{E_{22}},\ S_{66} = \frac{1}{G_{12}}. \tag{5.2}$$

Note that the coefficients of the stiffness and compliance matrices recover the relationship for an isotropic layer by specifying

$$Q_{11} = Q_{22} = \kappa + \mu, \quad Q_{12} = (\kappa - \mu), \quad Q_{66} = \mu \tag{5.3a}$$

and

$$S_{11} = S_{22} = \frac{\mu + \kappa}{4\kappa\mu},\ S_{12} = \frac{\mu - \kappa}{4\kappa\mu},\ S_{66} = \frac{1}{\mu}, \tag{5.3b}$$

where κ and μ are bulk and shear modulus, respectively. The dilatation for a lamina based on classical continuum mechanics is

$$\theta = (\varepsilon_{11} + \varepsilon_{22}). \tag{5.4}$$

The strain energy density, W, based on classical continuum mechanics can be expressed as

$$W = \frac{1}{2}\sigma_{11}\varepsilon_{11} + \frac{1}{2}\sigma_{22}\varepsilon_{22} + \frac{1}{2}\sigma_{12}\gamma_{12} \tag{5.5a}$$

or

$$W = \frac{1}{2}\left(Q_{11}\varepsilon_{11}^2 + 2Q_{12}\varepsilon_{22}\varepsilon_{11} + Q_{66}\gamma_{12}^2 + Q_{22}\varepsilon_{22}^2\right). \tag{5.5b}$$

Under general loading conditions, the total deformation of a lamina cannot be decomposed as dilatational and distortional parts. Depending on the fiber orientation angle, the lamina may exhibit coupling of stretch and in-plane shear deformation.

5.2 Fiber-Reinforced Lamina

A lamina can be idealized as a two-dimensional structure, and is thus suitable for discretization with a single layer of material points in the thickness direction. In the case of an isotropic material, there is no directional dependence. However, the directional dependency of the interactions between the material points in a fiber-reinforced composite lamina must be included in the PD analysis.

As shown in Fig. 5.2, the material point q represents material points that interact with material point k only along the fiber direction with an orientation angle of θ in reference to the x-axis. Similarly, material point r represents material points that interact with material point k only along the transverse direction. However, the material point p represents material points that interact with material point k in any direction, including the fiber and transverse directions. The orientation of a PD interaction between the material point k and the material point p is defined by the angle ϕ with respect to the x-axis. The domain of integral H in Eq. 2.22a is a disk with radius δ and thickness h.

The force density-stretch relations given by Eq. 2.48 must reflect the directional dependence of the PD material parameters for fiber-reinforced composite lamina. They can be defined in the form

$$\mathbf{t}_{(k)(j)}\left(\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t\right) = \frac{1}{2} A_{(k)(j)} \frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right|} \tag{5.6a}$$

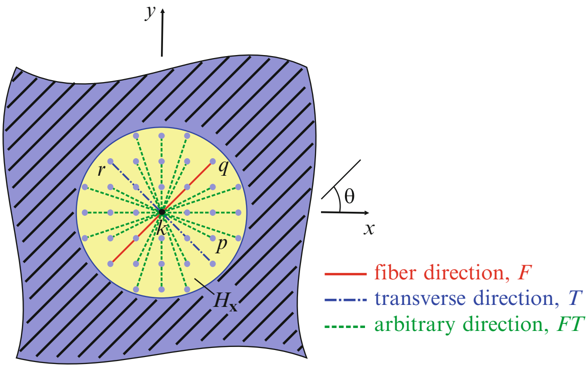
and

$$\mathbf{t}_{(j)(k)}\left(\mathbf{u}_{(k)} - \mathbf{u}_{(j)}, \mathbf{x}_{(k)} - \mathbf{x}_{(j)}, t\right) = -\frac{1}{2} B_{(j)(k)} \frac{\mathbf{y}_{(k)} - \mathbf{y}_{(j)}}{\left|\mathbf{y}_{(k)} - \mathbf{y}_{(j)}\right|}, \tag{5.6b}$$

where $A_{(k)(j)}$ and $B_{(j)(k)}$ are auxiliary parameters. As in the case of isotropic materials, these parameters can be determined by using Eq. 4.1, thus requiring an explicit form of the PD strain energy density at material point $\mathbf{x}_{(k)}$ for a unidirectional lamina.

In light of Eq. 4.2 and the directional dependency of a lamina, the PD strain energy density can be expressed as

Fig. 5.2 PD horizon for a fiber-reinforced lamina and interaction of a family of material points



$$W_{(k)} = a\,\theta_{(k)}^2 + b_F \sum_{j=1}^{J} \frac{\delta}{\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|} \left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right)^2 V_{(j)}$$
$$+ b_{FT} \sum_{j=1}^{\infty} \frac{\delta}{\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|} \left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right)^2 V_{(j)} \tag{5.7}$$
$$+ b_T \sum_{j=1}^{J} \frac{\delta}{\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|} \left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right)^2 V_{(j)},$$

in which the PD material parameter a is associated with the deformation involving dilatation, $\theta_{(k)}$. The other material parameters, b_F, b_T, and b_{FT}, are associated with deformation of material points in the fiber direction, transverse direction, and arbitrary directions, respectively. The total number of material points within the family of material point $\mathbf{x}_{(k)}$ in either fiber or transverse directions is denoted by J. The PD dilatation, $\theta_{(k)}$, for a unidirectional lamina can be expressed as

$$\theta_{(k)} = d \sum_{j=1}^{\infty} \frac{\delta}{\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|} \left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right) \Lambda_{(k)(j)} V_{(j)}, \tag{5.8}$$

in which d is a PD parameter.

After substituting for $\theta_{(k)}$ from Eq. 5.8 in the expression for $W_{(k)}$ given by Eq. 5.7 and performing differentiation, the force density vector $\mathbf{t}_{(k)(j)}\left(\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t\right)$ from Eq. 4.1 can be rewritten in terms of PD material parameters as

$$\mathbf{t}_{(k)(j)}\left(\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t\right) = \frac{1}{2} A_{(k)(j)} \frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right|}, \tag{5.9a}$$

where

$$A_{(k)(j)} = 4ad \frac{\delta}{\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|} \Lambda_{(k)(j)} \theta_{(k)} + 4\delta(\mu_F b_F + b_{FT} + \mu_T b_T) s_{(k)(j)}, \quad (5.9b)$$

with

$$\mu_F = \begin{cases} 1 & (\mathbf{x}_{(j)} - \mathbf{x}_{(k)})//\text{fiber direction} \\ 0 & \text{otherwise} \end{cases} \quad (5.9c)$$

and

$$\mu_T = \begin{cases} 1 & (\mathbf{x}_{(j)} - \mathbf{x}_{(k)}) \perp \text{fiber direction} \\ 0 & \text{otherwise .} \end{cases} \quad (5.9d)$$

Similarly, the force density vector $\mathbf{t}_{(j)(k)}(\mathbf{u}_{(k)} - \mathbf{u}_{(j)}, \mathbf{x}_{(k)} - \mathbf{x}_{(j)}, t)$ can be expressed as

$$\mathbf{t}_{(j)(k)}\left(\mathbf{u}_{(k)} - \mathbf{u}_{(j)}, \mathbf{x}_{(k)} - \mathbf{x}_{(j)}, t\right) = -\frac{1}{2} B_{(j)(k)} \frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right|}, \quad (5.10a)$$

with

$$B_{(j)(k)} = 4ad \frac{\delta}{\left|\mathbf{x}_{(k)} - \mathbf{x}_{(j)}\right|} \Lambda_{(j)(k)} \theta_{(j)} + 4\delta(\mu_F b_F + b_{FT} + \mu_T b_T) s_{(j)(k)}. \quad (5.10b)$$

Although Eqs. 5.9b and 5.10b appear to be similar, they are different because the dilatations $\theta_{(k)}$ and $\theta_{(j)}$ for the material points at $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$, respectively, are different. This formulation can be extended to include the effect of thermal loading as described in Chap. 4. Oterkus and Madenci (2012) presented such an extension for the bond-based peridynamic formulation.

5.3 Laminated Composites

The laminae are perfectly bonded in the construction of a laminate; thus, there exists no slip among the laminae. Aside from the loading conditions, the deformation of a laminate is dependent on the lamina properties, thickness, and stacking sequence. There exists usually a resin-rich layer between the laminae; an inherent source for cracking and delamination. Therefore, transverse normal and shear deformations especially play a critical role in the initiation and growth of delamination. Also, in the presence of a nonsymmetric stacking sequence, the laminates exhibit coupling between in-plane and out-of-plane deformation, resulting in curvature.

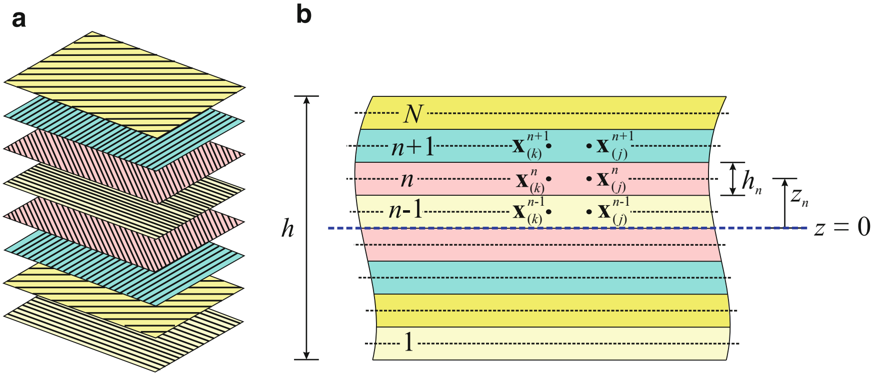


Fig. 5.3 Elevation of each lamina in laminate

As shown in Fig. 5.3, the reference coordinate system (x, y, z) is located on the midplane of the laminate. The laminate thickness, h, is given by

$$h = \sum_{n=1}^{N} h_n, \tag{5.11}$$

where N is the total number of lamina in the stacking sequence, and h_n is the thickness of the n^{th} lamina. With respect to the midplane, the position of each lamina, z_n, is defined as

$$z_n = -\frac{h}{2} + \sum_{m=1}^{n-1} h_m + \frac{1}{2} h_n. \tag{5.12}$$

The presence of the transverse normal and transverse shear deformations in a laminate can be included in the derivation of the PD equation of motion under the assumption that material points in a particular lamina interact with the other material points of immediate neighboring laminae above and below it.

The total potential energy of a laminate with N layers can be expressed in the form

$$U = \sum_{n=1}^{N} \sum_{i=1}^{\infty} W_{(i)}^{n} + \sum_{n=1}^{N-1} \sum_{i=1}^{\infty} \hat{W}_{(i)}^{n} + \sum_{n=1}^{N-1} \sum_{i=1}^{\infty} \tilde{W}_{(i)}^{n} - \sum_{n=1}^{N} \sum_{i=1}^{\infty} \mathbf{b}_{(i)}^{n} \cdot \mathbf{u}_{(i)}^{n}, \tag{5.13}$$

where $W_{(i)}^{n}$, $\hat{W}_{(i)}^{n}$, and $\tilde{W}_{(i)}^{n}$ represent the contributions from the in-plane, transverse normal, and shear deformations, respectively, and $\mathbf{b}_{(i)}^{n}$ is the body load vector.

Using Eq. 5.7, the strain energy density, $W_{(k)}^{n}$, of material point $\mathbf{x}_{(k)}^{n}$ located on the n^{th} layer , due to in-plane deformations, can be expressed as a summation of micropotentials, $w_{(k)(j)}$, arising from the interaction of material point $\mathbf{x}_{(k)}^{n}$ and the other material points $\mathbf{x}_{(j)}^{n}$ within its horizon in the form

$$W_{(k)}^{n} = \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{2}\begin{bmatrix} w_{(k)(j)}\left(\mathbf{y}_{(1^k)}^{n}-\mathbf{y}_{(k)}^{n},\ \mathbf{y}_{(2^k)}^{n}-\mathbf{y}_{(k)}^{n},\cdots\right)+ \\ w_{(j)(k)}\left(\mathbf{y}_{(1^j)}^{n}-\mathbf{y}_{(j)}^{n},\ \mathbf{y}_{(2^j)}^{n}-\mathbf{y}_{(j)}^{n},\cdots\right)\end{bmatrix} V_{(j)}^{n}, \tag{5.14}$$

in which $w_{(k)(j)} = 0$ for $k = j$. Due to transverse normal deformation, the strain energy density, $\hat{W}_{(k)}^{n}$, of material point $\mathbf{x}_{(k)}^{n}$ located on the n^{th} layer can be expressed as a summation of micropotentials, $\hat{w}_{(k)}$, arising from the interaction of material point $\mathbf{x}_{(k)}^{n}$ and the adjacent material points, $\mathbf{x}_{(k)}^{(n+1)}$ and $\mathbf{x}_{(k)}^{(n-1)}$, located on $(n+1)^{th}$ and $(n-1)^{th}$ layers in the form

$$\hat{W}_{(k)}^{n} = \frac{1}{2}\sum_{m=n+1,n-1}\frac{1}{2}\left[\hat{w}_{(k)}\left(\mathbf{y}_{(k)}^{m}-\mathbf{y}_{(k)}^{n}\right)V_{(k)}^{m}+\hat{w}_{(k)}\left(\mathbf{y}_{(k)}^{n}-\mathbf{y}_{(k)}^{m}\right)V_{(k)}^{m}\right]. \tag{5.15}$$

Similarly, the strain energy density associated with transverse shear deformation, $\tilde{W}_{(k)}^{n}$, of material point $\mathbf{x}_{(k)}^{n}$ can be expressed as a summation of micropotentials, $\tilde{w}_{(k)(j)}$, arising from the interaction of material point $\mathbf{x}_{(k)}^{n}$ and the other material points (within its family), $\mathbf{x}_{(j)}^{(n+1)}$ and $\mathbf{x}_{(j)}^{(n-1)}$, and $\tilde{w}_{(j)(k)}$, arising from the interaction of material point $\mathbf{x}_{(j)}^{n}$ and the other material points (within its family), $\mathbf{x}_{(k)}^{(n+1)}$ and $\mathbf{x}_{(k)}^{(n-1)}$, in the form

$$\begin{aligned}\tilde{W}_{(k)}^{n} = \frac{1}{2}\Bigg\{&\sum_{j=1}^{\infty}\frac{1}{2}\tilde{w}_{(k)(j)}\left(\mathbf{y}_{(j)}^{n+1}-\mathbf{y}_{(k)}^{n},\mathbf{y}_{(k)}^{n+1}-\mathbf{y}_{(j)}^{n}\right)V_{(j)}^{n+1}\\ &+\sum_{j=1}^{\infty}\frac{1}{2}\tilde{w}_{(j)(k)}\left(\mathbf{y}_{(k)}^{n}-\mathbf{y}_{(j)}^{n+1},\mathbf{y}_{(j)}^{n}-\mathbf{y}_{(k)}^{n+1}\right)V_{(j)}^{n+1}\\ &+\sum_{j=1}^{\infty}\frac{1}{2}\tilde{w}_{(j)(k)}\left(\mathbf{y}_{(k)}^{n}-\mathbf{y}_{(j)}^{n-1},\mathbf{y}_{(j)}^{n}-\mathbf{y}_{(k)}^{n-1}\right)V_{(j)}^{n-1}\\ &+\sum_{j=1}^{\infty}\frac{1}{2}\tilde{w}_{(k)(j)}\left(\mathbf{y}_{(j)}^{n-1}-\mathbf{y}_{(k)}^{n},\mathbf{y}_{(k)}^{n-1}-\mathbf{y}_{(j)}^{n}\right)V_{(j)}^{n-1}\\ &+\sum_{j=1}^{\infty}\frac{1}{2}\tilde{w}_{(j)(k)}\left(\mathbf{y}_{(k)}^{n+1}-\mathbf{y}_{(j)}^{n},\mathbf{y}_{(j)}^{n+1}-\mathbf{y}_{(k)}^{n}\right)V_{(j)}^{n}\\ &+\sum_{j=1}^{\infty}\frac{1}{2}\tilde{w}_{(k)(j)}\left(\mathbf{y}_{(j)}^{n}-\mathbf{y}_{(k)}^{n+1},\mathbf{y}_{(k)}^{n}-\mathbf{y}_{(j)}^{n+1}\right)V_{(j)}^{n}\\ &+\sum_{j=1}^{\infty}\frac{1}{2}\tilde{w}_{(k)(j)}\left(\mathbf{y}_{(j)}^{n}-\mathbf{y}_{(k)}^{n-1},\mathbf{y}_{(k)}^{n}-\mathbf{y}_{(j)}^{n-1}\right)V_{(j)}^{n}\\ &+\sum_{j=1}^{\infty}\frac{1}{2}\tilde{w}_{(j)(k)}\left(\mathbf{y}_{(k)}^{n-1}-\mathbf{y}_{(j)}^{n},\mathbf{y}_{(j)}^{n-1}-\mathbf{y}_{(k)}^{n}\right)V_{(j)}^{n}\Bigg\}.\end{aligned} \tag{5.16}$$

Substituting for the strain energy densities, $W^n_{(i)}$, $\hat{W}^n_{(i)}$, and $\tilde{W}^n_{(i)}$, of material point $\mathbf{x}^n_{(i)}$ from Eqs. 5.14, 5.15 and 5.16, the potential energy of the laminate with N layers can be rewritten as

$$
\begin{aligned}
U = {} & \sum_{n=1}^{N} \Bigg\{ \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2} \Big[w_{(i)(j)} \Big(\mathbf{y}^n_{(1^i)} - \mathbf{y}^n_{(i)},\ \mathbf{y}^n_{(2^i)} - \mathbf{y}^n_{(i)}, \cdots \Big) \\
& + w_{(j)(i)} \Big(\mathbf{y}^n_{(1^j)} - \mathbf{y}^n_{(j)},\ \mathbf{y}^n_{(2^j)} - \mathbf{y}^n_{(j)}, \cdots \Big) \Big] V^n_{(j)} V^n_{(i)} \Bigg\} \\
& + \frac{1}{2} \sum_{n=1}^{N-1} \sum_{i=1}^{\infty} \sum_{m=n+1,n-1} \frac{1}{2} \Big[\hat{w}_{(i)} \Big(\mathbf{y}^m_{(i)} - \mathbf{y}^n_{(i)} \Big) + \hat{w}_{(i)} \Big(\mathbf{y}^n_{(i)} - \mathbf{y}^m_{(i)} \Big) \Big] V^m_{(i)} V^n_{(i)} \\
& + \frac{1}{2} \sum_{n=1}^{N-1} \Bigg\{ \sum_{i=1}^{\infty} \frac{1}{2} \Bigg[\sum_{j=1}^{\infty} \tilde{w}_{(i)(j)} \Big(\mathbf{y}^{n+1}_{(j)} - \mathbf{y}^n_{(i)}, \mathbf{y}^{n+1}_{(i)} - \mathbf{y}^n_{(j)} \Big) V^{n+1}_{(j)} V^n_{(i)} \\
& + \sum_{j=1}^{\infty} \tilde{w}_{(j)(i)} \Big(\mathbf{y}^n_{(i)} - \mathbf{y}^{n+1}_{(j)}, \mathbf{y}^n_{(j)} - \mathbf{y}^{n+1}_{(i)} \Big) V^{n+1}_{(j)} V^n_{(i)} \\
& + \sum_{j=1}^{\infty} \tilde{w}_{(j)(i)} \Big(\mathbf{y}^n_{(i)} - \mathbf{y}^{n-1}_{(j)}, \mathbf{y}^n_{(j)} - \mathbf{y}^{n-1}_{(i)} \Big) V^{n-1}_{(j)} V^n_{(i)} \\
& + \sum_{j=1}^{\infty} \tilde{w}_{(i)(j)} \Big(\mathbf{y}^{n-1}_{(j)} - \mathbf{y}^n_{(i)}, \mathbf{y}^{n-1}_{(i)} - \mathbf{y}^n_{(j)} \Big) V^{n-1}_{(j)} V^n_{(i)} \Bigg] \Bigg\} \\
& + \frac{1}{2} \sum_{n=1}^{N-1} \Bigg\{ \sum_{i=1}^{\infty} \frac{1}{2} \Bigg[\sum_{j=1}^{\infty} \tilde{w}_{(j)(i)} \Big(\mathbf{y}^{n+1}_{(i)} - \mathbf{y}^n_{(j)}, \mathbf{y}^{n+1}_{(j)} - \mathbf{y}^n_{(i)} \Big) V^{n+1}_{(i)} V^n_{(j)} \\
& + \sum_{j=1}^{\infty} \tilde{w}_{(i)(j)} \Big(\mathbf{y}^n_{(j)} - \mathbf{y}^{n+1}_{(i)}, \mathbf{y}^n_{(i)} - \mathbf{y}^{n+1}_{(j)} \Big) V^{n+1}_{(i)} V^n_{(j)} \\
& + \sum_{j=1}^{\infty} \tilde{w}_{(i)(j)} \Big(\mathbf{y}^n_{(j)} - \mathbf{y}^{n-1}_{(i)}, \mathbf{y}^n_{(i)} - \mathbf{y}^{n-1}_{(j)} \Big) V^{n-1}_{(i)} V^n_{(j)} \\
& + \sum_{j=1}^{\infty} \tilde{w}_{(j)(i)} \Big(\mathbf{y}^{n-1}_{(i)} - \mathbf{y}^n_{(j)}, \mathbf{y}^{n-1}_{(j)} - \mathbf{y}^n_{(i)} \Big) V^{n-1}_{(i)} V^n_{(j)} \Bigg] \Bigg\} \\
& - \sum_{n=1}^{N} \Bigg\{ \sum_{i=1}^{\infty} \Big(\mathbf{b}^n_{(i)} \cdot \mathbf{u}^n_{(i)} \Big) V^n_{(i)} \Bigg\},
\end{aligned}
\tag{5.17a}
$$

or exchanging the order of dummy indices i and j in the fourth summation of layers results in

$$\begin{aligned}
U = & \sum_{n=1}^{N} \left\{ \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2} \left[w_{(i)(j)} \left(\mathbf{y}^n_{(1^i)} - \mathbf{y}^n_{(i)},\ \mathbf{y}^n_{(2^i)} - \mathbf{y}^n_{(i)}, \cdots \right) \right. \right. \\
& \left. \left. + w_{(j)(i)} \left(\mathbf{y}^n_{(1^j)} - \mathbf{y}^n_{(j)},\ \mathbf{y}^n_{(2^j)} - \mathbf{y}^n_{(j)}, \cdots \right) \right] V^n_{(j)} V^n_{(i)} \right\} \\
& + \frac{1}{2} \sum_{n=1}^{N-1} \sum_{i=1}^{\infty} \sum_{m=n+1,n-1} \frac{1}{2} \left[\hat{w}_{(i)} \left(\mathbf{y}^m_{(i)} - \mathbf{y}^n_{(i)} \right) + \hat{w}_{(i)} \left(\mathbf{y}^n_{(i)} - \mathbf{y}^m_{(i)} \right) \right] V^m_{(i)} V^n_{(i)} \\
& + \frac{1}{2} \sum_{n=1}^{N-1} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=n+1,n-1} \begin{bmatrix} \tilde{w}_{(i)(j)} \left(\mathbf{y}^m_{(j)} - \mathbf{y}^n_{(i)}, \mathbf{y}^m_{(i)} - \mathbf{y}^n_{(j)} \right) \\ + \tilde{w}_{(j)(i)} \left(\mathbf{y}^n_{(i)} - \mathbf{y}^m_{(j)}, \mathbf{y}^n_{(j)} - \mathbf{y}_{(i)} \right) \end{bmatrix} V^m_{(j)} V^n_{(i)} \\
& - \sum_{n=1}^{N} \sum_{i=1}^{\infty} \left(\mathbf{b}^n_{(i)} \cdot \mathbf{u}^n_{(i)} \right) V^n_{(i)} \ .
\end{aligned} \tag{5.17b}$$

As necessary for the derivation of equations of motion, the Lagrangian, Eq. 2.11, can be written in an expanded form by showing only the terms associated with the material point $\mathbf{x}^n_{(k)}$ located on the n^{th} layer as

$$\begin{aligned}
L = & \ldots + \frac{1}{2} \rho^n_{(k)} \, \dot{\mathbf{u}}^n_{(k)} \cdot \dot{\mathbf{u}}^n_{(k)} \, V^n_{(k)} + \ldots \\
& \ldots - \frac{1}{2} \sum_{j=1}^{\infty} \left\{ w_{(k)(j)} \left(\mathbf{y}^n_{(1^k)} - \mathbf{y}^n_{(k)},\ \mathbf{y}^n_{(2^k)} - \mathbf{y}^n_{(k)}, \cdots \right) V^n_{(j)} \, V^n_{(k)} \right\} \ldots \\
& \ldots - \frac{1}{2} \sum_{j=1}^{\infty} \left\{ w_{(j)(k)} \left(\mathbf{y}^n_{(1^j)} - \mathbf{y}^n_{(j)},\ \mathbf{y}^n_{(2^j)} - \mathbf{y}^n_{(j)}, \cdots \right) V^n_{(j)} \, V^n_{(k)} \right\} \ldots \\
& \ldots - \frac{1}{2} \hat{w}_{(k)} \left(\mathbf{y}^{n+1}_{(k)} - \mathbf{y}^n_{(k)} \right) V^{n+1}_{(k)} V^n_{(k)} \ldots - \frac{1}{2} \hat{w}_{(k)} \left(\mathbf{y}^n_{(k)} - \mathbf{y}^{n+1}_{(k)} \right) V^n_{(k)} V^{n+1}_{(k)} \\
& \ldots - \frac{1}{2} \hat{w}_{(k)} \left(\mathbf{y}^n_{(k)} - \mathbf{y}^{n-1}_{(k)} \right) V^n_{(k)} V^{n-1}_{(k)} \ldots - \frac{1}{2} \hat{w}_{(k)} \left(\mathbf{y}^{n-1}_{(k)} - \mathbf{y}^n_{(k)} \right) V^{n-1}_{(k)} V^n_{(k)} \\
& \ldots - \sum_{j=1}^{\infty} \tilde{w}_{(k)(j)} \left(\mathbf{y}^{n+1}_{(j)} - \mathbf{y}^n_{(k)}, \mathbf{y}^{n+1}_{(k)} - \mathbf{y}^n_{(j)} \right) V^{n+1}_{(j)} V^n_{(k)} \ldots \\
& \ldots - \sum_{j=1}^{\infty} \tilde{w}_{(j)(k)} \left(\mathbf{y}^n_{(k)} - \mathbf{y}^{n+1}_{(j)}, \mathbf{y}^n_{(j)} - \mathbf{y}^{n+1}_{(k)} \right) V^n_{(k)} V^{n+1}_{(j)} \\
& \ldots - \sum_{j=1}^{\infty} \tilde{w}_{(j)(k)} \left(\mathbf{y}^n_{(k)} - \mathbf{y}^{n-1}_{(j)}, \mathbf{y}^n_{(j)} - \mathbf{y}^{n-1}_{(k)} \right) V^n_{(k)} V^{n-1}_{(j)} \ \ldots \\
& \ldots - \sum_{j=1}^{\infty} \tilde{w}_{(k)(j)} \left(\mathbf{y}^{n-1}_{(j)} - \mathbf{y}^n_{(k)}, \mathbf{y}^{n-1}_{(k)} - \mathbf{y}^n_{(j)} \right) V^{n-1}_{(j)} V^n_{(k)} \\
& \ldots + \mathbf{b}^n_{(k)} \cdot \mathbf{u}^n_{(k)} V^n_{(k)} \cdots \ .
\end{aligned} \tag{5.18}$$

Substituting from Eq. 5.18 into Eq. 2.10 results in the Lagrange's equation of the material point $\mathbf{x}^n_{(k)}$ located on the n^{th} layer as

$$\begin{aligned}
\Bigg\{ & \rho^n_{(k)}\ddot{\mathbf{u}}^n_{(k)} + \sum_{j=1}^{\infty}\frac{1}{2}\left(\sum_{i=1}^{\infty}\frac{\partial w_{(k)(i)}}{\partial\left(\mathbf{y}^n_{(j)}-\mathbf{y}^n_{(k)}\right)}V^n_{(i)}\right)\frac{\partial\left(\mathbf{y}^n_{(j)}-\mathbf{y}^n_{(k)}\right)}{\partial\mathbf{u}^n_{(k)}} \\
& + \sum_{j=1}^{\infty}\frac{1}{2}\left(\sum_{i=1}^{\infty}\frac{\partial w_{(i)(k)}}{\partial\left(\mathbf{y}^n_{(k)}-\mathbf{y}^n_{(j)}\right)}V^n_{(i)}\right)\frac{\partial\left(\mathbf{y}^n_{(k)}-\mathbf{y}^n_{(j)}\right)}{\partial\mathbf{u}^n_{(k)}} \\
& + \sum_{m=n+1,n-1}\frac{1}{2}\frac{\partial\hat{w}_{(k)}}{\partial\left(\mathbf{y}^m_{(k)}-\mathbf{y}^n_{(k)}\right)}\frac{\partial\left(\mathbf{y}^m_{(k)}-\mathbf{y}^n_{(k)}\right)}{\partial\mathbf{u}^n_{(k)}}V^m_{(k)} \\
& + \sum_{m=n+1,n-1}\frac{1}{2}\frac{\partial\hat{w}_{(k)}}{\partial\left(\mathbf{y}^n_{(k)}-\mathbf{y}^m_{(k)}\right)}\frac{\partial\left(\mathbf{y}^n_{(k)}-\mathbf{y}^m_{(k)}\right)}{\partial\mathbf{u}^n_{(k)}}V^m_{(k)} \\
& + 2\sum_{m=n+1,n-1}\sum_{j=1}^{\infty}\frac{1}{2}\frac{\partial\tilde{w}_{(k)(j)}}{\partial\left(\mathbf{y}^m_{(j)}-\mathbf{y}^n_{(k)}\right)}\frac{\partial\left(\mathbf{y}^m_{(j)}-\mathbf{y}^n_{(k)}\right)}{\partial\mathbf{u}^n_{(k)}}V^m_{(j)} \\
& + 2\sum_{m=n+1,n-1}\sum_{j=1}^{\infty}\frac{1}{2}\frac{\partial\tilde{w}_{(j)(k)}}{\partial\left(\mathbf{y}^n_{(k)}-\mathbf{y}^m_{(j)}\right)}\frac{\partial\left(\mathbf{y}^n_{(k)}-\mathbf{y}^m_{(j)}\right)}{\partial\mathbf{u}^n_{(k)}}V^m_{(j)} \\
& -b^n_{(k)}\Bigg\}V^n_{(k)} = 0,
\end{aligned} \tag{5.19}$$

in which it is assumed that the interactions not involving material point $\mathbf{x}^n_{(k)}$ do not have any effect on material point $\mathbf{x}^n_{(k)}$. With the interpretation that the derivatives of the micropotentials represent the force densities that material points exert upon each other, this equation can be rewritten as

$$\begin{aligned}
\rho^n_{(k)}\ddot{\mathbf{u}}^n_{(k)} = & \sum_{j=1}^{\infty}\Big[\mathbf{t}^n_{(k)(j)}\left(\mathbf{u}^n_{(j)}-\mathbf{u}^n_{(k)},\mathbf{x}^n_{(j)}-\mathbf{x}^n_{(k)},t\right) \\
& -\mathbf{t}^n_{(j)(k)}\left(\mathbf{u}^n_{(k)}-\mathbf{u}^n_{(j)},\mathbf{x}^n_{(k)}-\mathbf{x}^n_{(j)},t\right)\Big]V^n_{(j)} \\
& + \sum_{m=n+1,n-1}\Big[\mathbf{r}^{(n)(m)}_{(k)}\left(\mathbf{u}^m_{(k)}-\mathbf{u}^n_{(k)},\mathbf{x}^m_{(k)}-\mathbf{x}^n_{(k)},t\right) \\
& -\mathbf{r}^{(m)(n)}_{(k)}\left(\mathbf{u}^n_{(k)}-\mathbf{u}^m_{(k)},\mathbf{x}^n_{(k)}-\mathbf{x}^m_{(k)},t\right)\Big]V^m_{(k)} \\
& + 2\sum_{m=n+1,n-1}\sum_{j=1}^{\infty}\Big[\mathbf{s}^{(n)(m)}_{(k)(j)}\left(\mathbf{u}^m_{(j)}-\mathbf{u}^n_{(k)},\mathbf{u}^m_{(k)}-\mathbf{u}^n_{(j)},\mathbf{x}^m_{(j)}-\mathbf{x}^n_{(k)},\mathbf{x}^m_{(k)}-\mathbf{x}^n_{(j)},t\right) \\
& -\mathbf{s}^{(m)(n)}_{(j)(k)}\left(\mathbf{u}^n_{(k)}-\mathbf{u}^m_{(j)},\mathbf{u}^n_{(j)}-\mathbf{u}^m_{(k)},\mathbf{x}^n_{(k)}-\mathbf{x}^m_{(j)},\mathbf{x}^n_{(j)}-\mathbf{x}^m_{(k)},t\right)\Big]V^m_{(j)} \\
& + \mathbf{b}^n_{(k)}\,.
\end{aligned} \tag{5.20}$$

Arising from in-plane deformation, $\mathbf{t}^{n}_{(k)(j)}$ represents the force density that material point $\mathbf{x}^{n}_{(j)}$ exerts upon material point $\mathbf{x}^{n}_{(k)}$. Similarly, $\mathbf{t}^{n}_{(j)(k)}$ represents the force density that material point $\mathbf{x}^{n}_{(k)}$ exerts upon material point $\mathbf{x}^{n}_{(j)}$. The force density vectors, $\mathbf{r}^{(n)(m)}_{(k)}$ and $\mathbf{r}^{(m)(n)}_{(k)}$ with $m = (n+1), (n-1)$, develop due to the transverse normal deformation between the material points $\mathbf{x}^{n}_{(k)}$ and $\mathbf{x}^{m}_{(k)}$. The force density vector $\mathbf{r}^{(n)(m)}_{(k)}$ represents the force exerted by material point $\mathbf{x}^{m}_{(k)}$ upon the material point $\mathbf{x}^{n}_{(k)}$, and $\mathbf{r}^{(m)(n)}_{(k)}$ represents the opposite. The force density vectors $\mathbf{s}^{(n)(m)}_{(k)(j)}$ and $\mathbf{s}^{(m)(n)}_{(j)(k)}$, with $m = (n+1), (n-1)$, are associated with transverse shear deformation between the material points $\mathbf{x}^{m}_{(j)}$ and $\mathbf{x}^{n}_{(k)}$. The force density vector $\mathbf{s}^{(n)(m)}_{(k)(j)}$ represents the force exerted by material point $\mathbf{x}^{m}_{(j)}$ on the material point $\mathbf{x}^{n}_{(k)}$, and $\mathbf{s}^{(m)(n)}_{(j)(k)}$ represents the other way around. These force density vectors are defined as

$$\left.\begin{aligned} \mathbf{t}^{n}_{(k)(j)}\left(\mathbf{u}^{n}_{(j)} - \mathbf{u}^{n}_{(k)}, \mathbf{x}^{n}_{(j)} - \mathbf{x}^{n}_{(k)}, t\right) &= \frac{1}{2}\frac{1}{V^{n}_{(j)}}\left(\sum_{i=1}^{\infty}\frac{\partial w_{(k)(i)}}{\partial\left(\mathbf{y}^{n}_{(j)} - \mathbf{y}^{n}_{(k)}\right)}V^{n}_{(i)}\right) \\ \mathbf{t}^{n}_{(j)(k)}\left(\mathbf{u}^{n}_{(k)} - \mathbf{u}^{n}_{(j)}, \mathbf{x}^{n}_{(k)} - \mathbf{x}^{n}_{(j)}, t\right) &= \frac{1}{2}\frac{1}{V^{n}_{(j)}}\left(\sum_{i=1}^{\infty}\frac{\partial w_{(i)(k)}}{\partial\left(\mathbf{y}^{n}_{(k)} - \mathbf{y}^{n}_{(j)}\right)}V^{n}_{(i)}\right) \end{aligned}\right\}, \tag{5.21a}$$

$$\left.\begin{aligned} \mathbf{r}^{(n)(m)}_{(k)}\left(\mathbf{u}^{m}_{(k)} - \mathbf{u}^{n}_{(k)}, \mathbf{x}^{m}_{(k)} - \mathbf{x}^{n}_{(k)}, t\right) &= \frac{1}{2}\frac{\partial \hat{w}_{(k)}}{\partial\left(\mathbf{y}^{m}_{(k)} - \mathbf{y}^{n}_{(k)}\right)} \\ \mathbf{r}^{(m)(n)}_{(k)}\left(\mathbf{u}^{n}_{(k)} - \mathbf{u}^{m}_{(k)}, \mathbf{x}^{n}_{(k)} - \mathbf{x}^{m}_{(k)}, t\right) &= \frac{1}{2}\frac{\partial \hat{w}_{(k)}}{\partial\left(\mathbf{y}^{n}_{(k)} - \mathbf{y}^{m}_{(k)}\right)} \end{aligned}\right\}, \tag{5.21b}$$

and

$$\left.\begin{aligned} &\mathbf{s}^{(n)(m)}_{(k)(j)}\left(\mathbf{u}^{m}_{(j)} - \mathbf{u}^{n}_{(k)}, \mathbf{u}^{m}_{(k)} - \mathbf{u}^{n}_{(j)}, \mathbf{x}^{m}_{(j)} - \mathbf{x}^{n}_{(k)}, \mathbf{x}^{m}_{(k)} - \mathbf{x}^{n}_{(j)}, t\right) \\ &\quad = \frac{1}{2}\frac{\partial \tilde{w}_{(k)(j)}}{\partial\left(\mathbf{y}^{m}_{(j)} - \mathbf{y}^{n}_{(k)}\right)} \\ &\mathbf{s}^{(m)(n)}_{(j)(k)}\left(\mathbf{u}^{n}_{(k)} - \mathbf{u}^{m}_{(j)}, \mathbf{u}^{n}_{(j)} - \mathbf{u}^{m}_{(k)}, \mathbf{x}^{n}_{(k)} - \mathbf{x}^{m}_{(j)}, \mathbf{x}^{n}_{(j)} - \mathbf{x}^{m}_{(k)}, t\right) \\ &\quad = \frac{1}{2}\frac{\partial \tilde{w}_{(j)(k)}}{\partial\left(\mathbf{y}^{n}_{(k)} - \mathbf{y}^{m}_{(j)}\right)} \end{aligned}\right\}, \tag{5.21c}$$

with $m = (n+1), (n-1)$. As derived in Sect. 2.8, in order to satisfy the balance of angular momentum, the equation of motion, Eq. 5.20 must satisfy the requirements of

$$\int_H \left(\left(\mathbf{y}_{(j)}^n - \mathbf{y}_{(k)}^n \right) \times \mathbf{t}_{(k)(j)}^n \left(\mathbf{u}_{(j)}^n - \mathbf{u}_{(k)}^n, \mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n, t \right) \right) dH = 0, \tag{5.22a}$$

$$\int_H \left(\left(\mathbf{y}_{(k)}^m - \mathbf{y}_{(k)}^n \right) \times \mathbf{r}_{(k)}^{(n)(m)} \left(\mathbf{u}_{(k)}^m - \mathbf{u}_{(k)}^n, \mathbf{x}_{(k)}^m - \mathbf{x}_{(k)}^n, t \right) \right) dH = 0, \tag{5.22b}$$

$$\begin{aligned} \int_H \Big(\left(\mathbf{y}_{(j)}^m - \mathbf{y}_{(k)}^n \right) \times \mathbf{s}_{(k)(j)}^{(n)(m)} \Big(& \mathbf{u}_{(j)}^m - \mathbf{u}_{(k)}^n, \mathbf{u}_{(k)}^m - \mathbf{u}_{(j)}^n, \mathbf{x}_{(j)}^m \\ & -\mathbf{x}_{(k)}^n, \mathbf{x}_{(k)}^m - \mathbf{x}_{(j)}^n, t \Big) \Big) dH = 0. \end{aligned} \tag{5.22c}$$

It is apparent that these requirements are automatically satisfied if the force vectors, $\mathbf{t}_{(k)(j)}^n$, $\mathbf{r}_{(k)}^{(n)(m)}$, and $\mathbf{s}_{(k)(j)}^{(n)(m)}$, are aligned with the relative position vector of the material points in the deformed state, $\left(\mathbf{y}_{(j)}^n - \mathbf{y}_{(k)}^n\right)$, $\left(\mathbf{y}_{(k)}^m - \mathbf{y}_{(k)}^n\right)$, and $\left(\mathbf{y}_{(j)}^m - \mathbf{y}_{(k)}^n\right)$, respectively. Therefore, they can be expressed in the form

$$\mathbf{t}_{(k)(j)}^n = \frac{1}{2} A_{(k)(j)}^n \frac{\mathbf{y}_{(j)}^n - \mathbf{y}_{(k)}^n}{\left| \mathbf{y}_{(j)}^n - \mathbf{y}_{(k)}^n \right|}, \tag{5.23a}$$

$$\mathbf{t}_{(j)(k)}^n = -\frac{1}{2} B_{(j)(k)}^n \frac{\mathbf{y}_{(j)}^n - \mathbf{y}_{(k)}^n}{\left| \mathbf{y}_{(j)}^n - \mathbf{y}_{(k)}^n \right|}, \tag{5.23b}$$

and

$$\mathbf{r}_{(k)}^{(n)(m)} = \frac{1}{2} C_{(k)}^{(n)(m)} \frac{\mathbf{y}_{(k)}^m - \mathbf{y}_{(k)}^n}{\left| \mathbf{y}_{(k)}^m - \mathbf{y}_{(k)}^n \right|} = \frac{1}{2} \mathbf{p}_{(k)}^{(n)(m)}, \tag{5.24a}$$

$$\mathbf{r}_{(k)}^{(m)(n)} = -\frac{1}{2} C_{(k)}^{(n)(m)} \frac{\mathbf{y}_{(k)}^m - \mathbf{y}_{(k)}^n}{\left| \mathbf{y}_{(k)}^m - \mathbf{y}_{(k)}^n \right|} = -\frac{1}{2} \mathbf{p}_{(k)}^{(n)(m)}, \tag{5.24b}$$

and

$$\mathbf{s}_{(k)(j)}^{(n)(m)} = \frac{1}{2} D_{(k)(j)}^{(n)(m)} \frac{\mathbf{y}_{(j)}^m - \mathbf{y}_{(k)}^n}{\left| \mathbf{y}_{(j)}^m - \mathbf{y}_{(k)}^n \right|} = \frac{1}{2} \mathbf{q}_{(k)(j)}^{(n)(m)}, \tag{5.25a}$$

$$\mathbf{s}_{(j)(k)}^{(m)(n)} = -\frac{1}{2} D_{(k)(j)}^{(n)(m)} \frac{\mathbf{y}_{(j)}^m - \mathbf{y}_{(k)}^n}{\left| \mathbf{y}_{(j)}^m - \mathbf{y}_{(k)}^n \right|} = -\frac{1}{2} \mathbf{q}_{(k)(j)}^{(n)(m)}, \tag{5.25b}$$

where $A^{n}_{(k)(j)}$, $B^{n}_{(j)(k)}$, $C^{(n)(m)}_{(k)}$, and $D^{(n)(m)}_{(k)(j)}$ are auxiliary parameters. With these representations of the force density vectors, the equation of motion for material point $\mathbf{x}^{n}_{(k)}$ located on the n^{th} layer can be further simplified as

$$\begin{aligned}\rho^{n}_{(k)}\ddot{\mathbf{u}}^{n}_{(k)} = & \sum_{j=1}^{\infty}\Big[\mathbf{t}^{n}_{(k)(j)}\Big(\mathbf{u}^{n}_{(j)} - \mathbf{u}^{n}_{(k)}, \mathbf{x}^{n}_{(j)} - \mathbf{x}^{n}_{(k)}, t\Big) \\ & -\mathbf{t}^{n}_{(j)(k)}\Big(\mathbf{u}^{n}_{(k)} - \mathbf{u}^{n}_{(j)}, \mathbf{x}^{n}_{(k)} - \mathbf{x}^{n}_{(j)}, t\Big)\Big]V^{n}_{(j)} \\ & + \sum_{m=n+1,n-1}\mathbf{p}^{(n)(m)}_{(k)}V^{m}_{(k)} + 2\sum_{m=n+1,n-1}\sum_{j=1}^{\infty}\mathbf{q}^{(n)(m)}_{(k)(j)}V^{m}_{(j)} + \mathbf{b}^{n}_{(k)}\ .\end{aligned} \tag{5.26}$$

The auxiliary parameters, $A^{n}_{(k)(j)}$, $B^{n}_{(j)(k)}$ $C^{(n)(m)}_{(k)}$, and $D^{(n)(m)}_{(k)(j)}$, can be determined by using the relationship between the force density vector and the strain energy density, $W_{(k)}$. The explicit expressions for the auxiliary parameters $A^{n}_{(k)(j)}$ and $B^{n}_{(j)(k)}$ are already given by Eqs. 5.9b and 5.10b. The remaining auxiliary parameters, $C^{(n)(m)}_{(k)}$ and $D^{(n)(m)}_{(k)(j)}$, can be determined by using the relationships

$$\mathbf{r}^{(n)(m)}_{(k)} = \frac{1}{V^{m}_{(k)}}\frac{\partial \hat{W}^{n}_{(k)}}{\partial\Big(\Big|\mathbf{y}^{m}_{(k)} - \mathbf{y}^{n}_{(k)}\Big|\Big)}\frac{\mathbf{y}^{m}_{(k)} - \mathbf{y}^{n}_{(k)}}{\Big|\mathbf{y}^{m}_{(k)} - \mathbf{y}^{n}_{(k)}\Big|} \tag{5.27a}$$

and

$$\mathbf{s}^{(n)(m)}_{(k)(j)} = \frac{1}{V^{m}_{(j)}}\frac{\partial \tilde{W}^{n}_{(k)}}{\partial\Big(\Big|\mathbf{y}^{m}_{(j)} - \mathbf{y}^{n}_{(k)}\Big|\Big)}\frac{\mathbf{y}^{m}_{(j)} - \mathbf{y}^{n}_{(k)}}{\Big|\mathbf{y}^{m}_{(j)} - \mathbf{y}^{n}_{(k)}\Big|}, \tag{5.27b}$$

in which $V^{m}_{(k)}$ and $V^{m}_{(j)}$ represent the volume of material points $\mathbf{x}^{m}_{(k)}$ and $\mathbf{x}^{m}_{(j)}$ respectively, and the direction of the force density vector is aligned with the relative position vector in the deformed configuration. However, determination of the auxiliary parameters requires an explicit form of the strain energy density function. For transverse normal and shear deformations of an isotropic and elastic material (resin-rich layer), the explicit form of the strain energy density functions, $\hat{W}^{n}_{(k)}$ and $\tilde{W}^{n}_{(k)}$, can be written as

$$\hat{W}^{n}_{(k)} = b_N \sum_{m=n+1,n-1}\frac{\hat{\delta}}{\Big|\mathbf{x}^{m}_{(k)} - \mathbf{x}^{n}_{(k)}\Big|}\Big(\Big|\mathbf{y}^{m}_{(k)} - \mathbf{y}^{n}_{(k)}\Big| - \Big|\mathbf{x}^{m}_{(k)} - \mathbf{x}^{n}_{(k)}\Big|\Big)^{2}V^{m}_{(k)} \tag{5.28a}$$

and

$$\tilde{W}_{(k)}^{n} = b_S \sum_{m=n+1,n-1} \sum_{j=1}^{\infty} \frac{\tilde{\delta}}{\left|\mathbf{x}_{(j)}^{m} - \mathbf{x}_{(k)}^{n}\right|} \left[\left(\left|\mathbf{y}_{(j)}^{m} - \mathbf{y}_{(k)}^{n}\right| - \left|\mathbf{x}_{(j)}^{m} - \mathbf{x}_{(k)}^{n}\right| \right) \right. \\ \left. - \left(\left|\mathbf{y}_{(k)}^{m} - \mathbf{y}_{(j)}^{n}\right| - \left|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(j)}^{n}\right| \right) \right]^2 V_{(j)}^{m}, \tag{5.28b}$$

in which the PD material parameters b_N and b_S are associated with the transverse normal and shear deformations of the matrix material, but are yet to be determined in terms of Young's modulus and shear modulus. The horizon size in the thickness direction is $\hat{\delta}$, and $\tilde{\delta}$ is defined as $\tilde{\delta} = \sqrt{\delta^2 + \hat{\delta}^2}$. Note that $|\mathbf{x}_{(j)}^{m} - \mathbf{x}_{(k)}^{n}|$ and $|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(j)}^{n}|$ are equivalent quantities. Substituting for strain energy density from Eqs. 5.28a, b in Eqs. 5.27a, b and performing differentiation result in

$$\mathbf{p}_{(k)}^{(n)(m)} = 4 b_N \hat{\delta} \left(\frac{\left|\mathbf{y}_{(k)}^{m} - \mathbf{y}_{(k)}^{n}\right| - \left|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(k)}^{n}\right|}{\left|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(k)}^{n}\right|} \right) \frac{\mathbf{y}_{(k)}^{m} - \mathbf{y}_{(k)}^{n}}{\left|\mathbf{y}_{(k)}^{m} - \mathbf{y}_{(k)}^{n}\right|} \tag{5.29a}$$

and

$$\mathbf{q}_{(k)(j)}^{(n)(m)} = 4 b_S \tilde{\delta} \left[\left(\frac{\left|\mathbf{y}_{(j)}^{m} - \mathbf{y}_{(k)}^{n}\right| - \left|\mathbf{x}_{(j)}^{m} - \mathbf{x}_{(k)}^{n}\right|}{\left|\mathbf{x}_{(j)}^{m} - \mathbf{x}_{(k)}^{n}\right|} \right) \right. \\ \left. - \left(\frac{\left|\mathbf{y}_{(k)}^{m} - \mathbf{y}_{(j)}^{n}\right| - \left|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(j)}^{n}\right|}{\left|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(j)}^{n}\right|} \right) \right] \frac{\mathbf{y}_{(j)}^{m} - \mathbf{y}_{(k)}^{n}}{\left|\mathbf{y}_{(j)}^{m} - \mathbf{y}_{(k)}^{n}\right|} . \tag{5.29b}$$

Comparisons of Eqs. 5.24a and 5.29a and 5.25a and 5.29b lead to the determination of $C_{(k)}^{(n)(m)}$ and $D_{(k)(j)}^{(n)(m)}$ as

$$C_{(k)}^{(n)(m)} = 4 b_N \hat{\delta} \left(\frac{\left|\mathbf{y}_{(k)}^{m} - \mathbf{y}_{(k)}^{n}\right| - \left|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(k)}^{n}\right|}{\left|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(k)}^{n}\right|} \right) \tag{5.30a}$$

$$D_{(k)(j)}^{(n)(m)} = 4 b_S \tilde{\delta} \left[\left(\frac{\left|\mathbf{y}_{(j)}^{m} - \mathbf{y}_{(k)}^{n}\right| - \left|\mathbf{x}_{(j)}^{m} - \mathbf{x}_{(k)}^{n}\right|}{\left|\mathbf{x}_{(j)}^{m} - \mathbf{x}_{(k)}^{n}\right|} \right) - \left(\frac{\left|\mathbf{y}_{(k)}^{m} - \mathbf{y}_{(j)}^{n}\right| - \left|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(j)}^{n}\right|}{\left|\mathbf{x}_{(k)}^{m} - \mathbf{x}_{(j)}^{n}\right|} \right) \right]. \tag{5.30b}$$

5.4 Peridynamic Material Parameters

The peridynamic material parameters that appear in force density vector-stretch relations for in-plane and transverse normal and shear deformations can be determined in terms of engineering material constants of classical laminate theory by considering simple loading conditions.

5.4.1 Material Parameters for a Lamina

The PD material parameters, a, d, b_F, b_T, and b_{FT}, that appear in the force density vector-stretch relations for in-plane deformation of a lamina, Eqs. 5.9b and 5.10b, are related to the engineering constants by considering four different simple loading conditions as

1. Simple shear: $\gamma_{12} = \zeta$
2. Uniaxial stretch in fiber direction: $\varepsilon_{11} = \zeta,\ \varepsilon_{22} = 0$
3. Uniaxial stretch in transverse direction: $\varepsilon_{11} = 0,\ \varepsilon_{22} = \zeta$
4. Biaxial stretch: $\varepsilon_{11} = \zeta,\ \varepsilon_{22} = \zeta$

5.4.1.1 Simple Shear: $\gamma_{12} = \zeta$

Using Eq. 5.1a, the stresses in the lamina due to this loading are obtained as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \zeta \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{66}\zeta \end{Bmatrix}. \tag{5.31}$$

Based on Eqs. 5.4 and 5.5b, the corresponding dilatation and strain energy density from the classical continuum mechanics at material point $\mathbf{x}^n_{(k)}$ are

$$\theta_{(k)} = 0 \tag{5.32a}$$

and

$$W_{(k)} = \frac{1}{2} Q_{66} \zeta^2. \tag{5.32b}$$

As illustrated in Fig. 5.4, the length of the relative position of material points $\mathbf{y}_{(j)}$ and $\mathbf{y}_{(k)}$ in the deformed state becomes

$$|\mathbf{y}' - \mathbf{y}| = [1 + (\sin\phi \cos\phi)\zeta]\, |\mathbf{x}' - \mathbf{x}| \tag{5.33a}$$

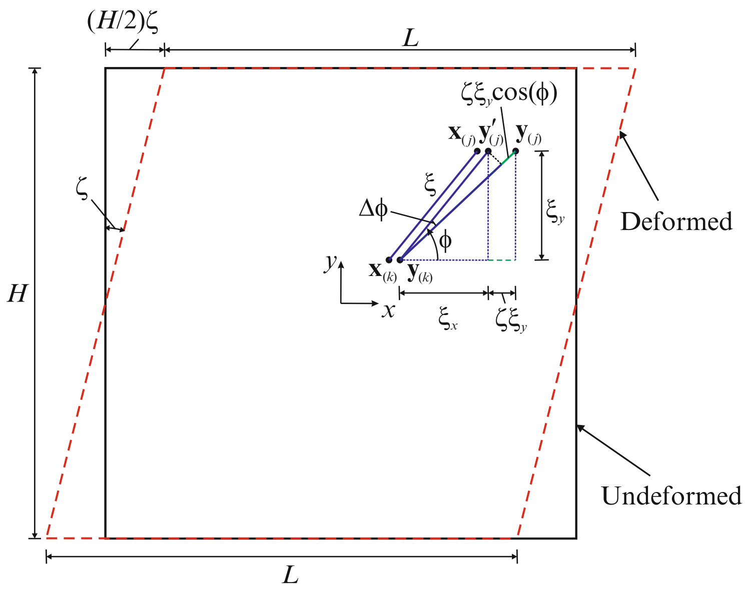


Fig. 5.4 Simple shear

or

$$\left|\mathbf{y}^{n}_{(j)} - \mathbf{y}^{n}_{(k)}\right| = \left[1 + \left(\sin\phi_{(j)(k)}\cos\phi_{(j)(k)}\right)\zeta\right]\left|\mathbf{x}^{n}_{(j)} - \mathbf{x}^{n}_{(k)}\right|. \tag{5.33b}$$

Note that if the material points $\mathbf{y}^{n}_{(j)}$ and $\mathbf{y}^{n}_{(k)}$ are aligned with the fiber and transverse directions, the angles become $\phi_{(j)(k)} = 0^{\circ}$ and $\phi_{(j)(k)} = 90^{\circ}$, respectively.

For this deformation, the dilatation, Eq. 5.8, is evaluated as

$$\theta_{(k)} = d\int_{H}\frac{\delta}{\xi}\{[1 + (\sin\phi\cos\phi)\zeta]\xi - \xi\}dH, \tag{5.34}$$

in which $\xi = \left|\mathbf{x}^{n}_{(j)} - \mathbf{x}^{n}_{(k)}\right|$.

As expected, this loading condition results in no dilatation. The strain energy density, Eq. 5.7, is evaluated as

$$W_{(k)} = a\,(0) + b_F(0) + b_{FT}\int_{H}\frac{\delta}{\xi}([1 + (\sin\phi\cos\phi)\zeta]\,\xi - \xi)^2 dH + b_T(0) \tag{5.35a}$$

or

$$W_{(k)} = b_{FT}h\int_{0}^{\delta}\int_{0}^{2\pi}\frac{\delta}{\xi}([1 + (\sin\phi\cos\phi)\zeta]\,\xi - \xi)^2\xi d\xi d\phi = \frac{\pi h\delta^4\zeta^2}{12}b_{FT}. \tag{5.35b}$$

Equating the expressions for strain energy density from the classical and PD formulations, Eqs. 5.32b and 5.35b, results in

$$b_{FT} = \frac{6Q_{66}}{\pi h \delta^4}. \tag{5.36}$$

5.4.1.2 Uniaxial Stretch in the Fiber Direction: $\varepsilon_{11} = \zeta,\ \varepsilon_{22} = 0$

Using Eq. 5.1a, the stresses in the lamina due to this loading becomes

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} Q_{11}\zeta \\ Q_{12}\zeta \\ 0 \end{Bmatrix}. \tag{5.37}$$

Based on Eqs. 5.4 and 5.5b, the corresponding dilatation and strain energy density from the classical continuum mechanics at material point $\mathbf{x}_{(k)}^n$ are

$$\theta_{(k)} = \zeta \tag{5.38a}$$

and

$$W_{(k)} = \frac{1}{2} Q_{11} \zeta^2. \tag{5.38b}$$

As illustrated in Fig. 5.5, the length of the relative position of material points $\mathbf{y}_{(j)}^n$ and $\mathbf{y}_{(k)}^n$ in the deformed state becomes

$$|\mathbf{y}' - \mathbf{y}| = \left[1 + \left(\cos^2 \phi\right)\zeta\right] |\mathbf{x}' - \mathbf{x}| \tag{5.39a}$$

or

$$\left|\mathbf{y}_{(j)}^n - \mathbf{y}_{(k)}^n\right| = \left[1 + \left(\cos^2 \phi_{(j)(k)}\right)\zeta\right] \left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|. \tag{5.39b}$$

Due to this deformation, the dilatation is evaluated as

$$\theta_{(k)} = d \int_H \frac{\delta}{\xi} \left\{ \left[1 + \left(\cos^2 \phi\right)\zeta\right] \xi - \xi \right\} dH \tag{5.40a}$$

or

$$\theta_{(k)} = \frac{\pi d h \delta^3 \zeta}{2}. \tag{5.40b}$$

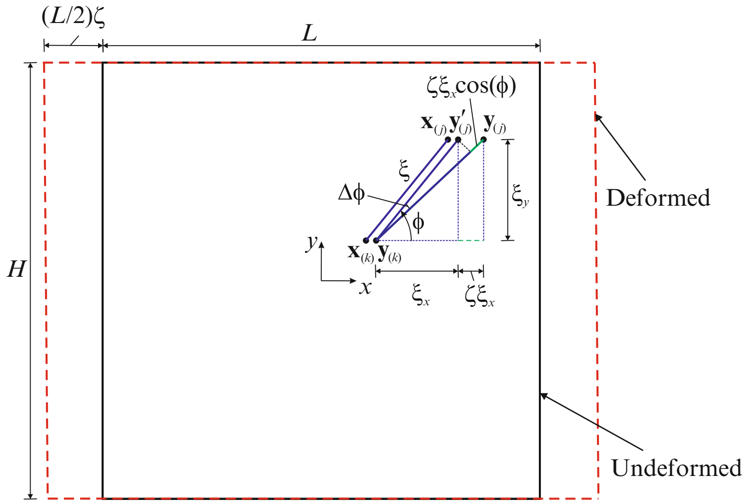


Fig. 5.5 Uniaxial stretch in fiber direction

Equating the expressions for dilatation from the classical and PD formulations, Eqs. 5.38a and 5.40b, results in

$$d = \frac{2}{\pi h \delta^3}. \tag{5.41}$$

The strain energy density for this deformation is evaluated as

$$\begin{aligned} W_{(k)} = a\,\zeta^2 + b_F \sum_{j=1}^{J} \frac{\delta}{\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|} \left(\left(\cos^2\phi_{(j)(k)}\right)\zeta\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|\right)^2 V_{(j)}^n \\ + b_{FT} \int_H \frac{\delta}{\xi}\left(\left[1 + \left(\cos^2\phi\right)\zeta\right]\xi - \xi\right)^2 dH + b_T(0) \end{aligned} \tag{5.42a}$$

or

$$W_{(k)} = a\,\zeta^2 + b_F\ \delta\zeta^2 \sum_{j=1}^{J} \left(\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|\right) V_{(j)} + \frac{\pi h \delta^4 \zeta^2}{4} b_{FT}. \tag{5.42b}$$

After substituting for b_{FT} from Eq. 5.36, it takes the final form

$$W_{(k)} = a\,\zeta^2 + b_F\ \delta\zeta^2 \left(\sum_{j=1}^{J} \left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right| V_{(j)}\right) + \frac{3Q_{66}\zeta^2}{2}. \tag{5.43}$$

Equating the expressions for strain energy density from the classical and PD formulations, Eqs. 5.38b and 5.43, results in

$$a + \delta\left(\sum_{j=1}^{J}\left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(k)}^{n}\right| V_{(j)}\right) b_F = \frac{1}{2}(Q_{11} - 3Q_{66}). \tag{5.44}$$

5.4.1.3 Uniaxial Stretch in the Transverse Direction: $\varepsilon_{11} = 0,\ \varepsilon_{22} = \zeta$

Using Eq. 5.1, the stresses in the lamina due to this loading become

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} Q_{12}\zeta \\ Q_{22}\zeta \\ 0 \end{Bmatrix}. \tag{5.45}$$

Based on Eqs. 5.4 and 5.5b, the corresponding dilatation and strain energy density from classical continuum mechanics at material point $\mathbf{x}_{(k)}$ are

$$\theta_{(k)} = \zeta, \tag{5.46a}$$

$$W_{(k)} = \frac{1}{2} Q_{22}\zeta^2. \tag{5.46b}$$

As illustrated in Fig. 5.6, the length of the relative position of material points $\mathbf{y}_{(j)}$ and $\mathbf{y}_{(k)}$ in the deformed state becomes

$$|\mathbf{y}' - \mathbf{y}| = \left[1 + \left(\sin^2\phi\right)\zeta\right] |\mathbf{x}' - \mathbf{x}| \tag{5.47a}$$

or

$$\left|\mathbf{y}_{(j)}^{n} - \mathbf{y}_{(k)}^{n}\right| = \left[1 + \left(\sin^2\phi_{(j)(k)}\right)\zeta\right] \left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(k)}^{n}\right|. \tag{5.47b}$$

For this deformation, the dilatation is evaluated as

$$\theta_{(k)} = d\int_H \frac{\delta}{\xi}\left(\left[1 + \left(\sin^2\phi\right)\zeta\right]\xi - \xi\right) dH \tag{5.48a}$$

or

$$\theta_{(k)} = \frac{\pi d h \delta^3 \zeta}{2}. \tag{5.48b}$$

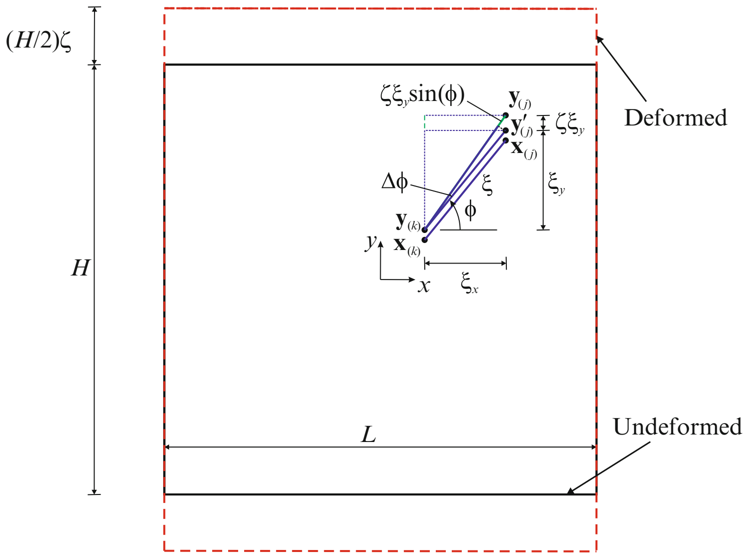


Fig. 5.6 Uniaxial stretch in transverse direction

Equating the expressions for dilatation from the classical and PD formulations, Eqs. 5.46a and 5.48b, results in

$$d = \frac{2}{\pi h \delta^3}. \tag{5.49}$$

As expected, the PD parameter d obtained from the uniform stretch in the fiber direction, Eq. 5.41, and that in the transverse direction, Eq, 5.49, are equal to each other and are independent of material properties.

The strain energy density for this deformation is evaluated as

$$\begin{aligned} W_{(k)} = a\,\zeta^2 + b_F(0) + b_{FT} \int_H \frac{\delta}{\xi} \left(\left[1 + \left(\sin^2\phi\right)\zeta\right]\xi - \xi\right)^2 dH \\ + b_T\ \delta\zeta^2 \left(\sum_{j=1}^{J} \left| \mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n \right| V_{(j)}^n \right) \end{aligned} \tag{5.50a}$$

or

$$W_{(k)} = a\,\zeta^2 + b_{FT} \frac{\pi h \delta^4 \zeta^2}{4} + b_T\ \delta\zeta^2 \left(\sum_{j=1}^{J} \left| \mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n \right| V_{(j)}^n \right). \tag{5.50b}$$

After substituting for b_{FT} from Eq. 5.36, it takes the final form

$$W_{(k)} = a\zeta^2 + \frac{3Q_{66}\zeta^2}{2} + b_T\ \delta\zeta^2\left(\sum_{j=1}^{J}\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|V_{(j)}^n\right). \tag{5.51}$$

Equating the expressions for strain energy density from the classical and PD formulations, Eqs. 5.46b and 5.51, results in

$$\frac{1}{2}(Q_{22} - 3Q_{66}) = a + \delta\left(\sum_{j=1}^{J}\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|V_{(j)}\right)b_T. \tag{5.52}$$

5.4.1.4 Biaxial Stretch: $\varepsilon_{11} = \zeta,\ \varepsilon_{22} = \zeta$

Using Eq. 5.1a, the stresses in the lamina due to this loading become

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\zeta\\ \zeta\\ 0\end{Bmatrix} \text{ or } \begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} = \begin{Bmatrix}(Q_{11}+Q_{12})\zeta\\ (Q_{12}+Q_{22})\zeta\\ 0\end{Bmatrix}. \tag{5.53}$$

Based on Eqs. 5.4 and 5.5b, the corresponding dilatation and strain energy density from classical continuum mechanics at material point $\mathbf{x}_{(k)}$ are

$$\theta_{(k)} = 2\zeta \tag{5.54a}$$

and

$$W_{(k)} = \frac{1}{2}(Q_{11} + 2Q_{12} + Q_{22})\zeta^2. \tag{5.54b}$$

As illustrated in Fig. 5.7, the length of the relative position of material points $\mathbf{y}_{(j)}$ and $\mathbf{y}_{(k)}$ in the deformed state becomes

$$|\mathbf{y}' - \mathbf{y}| = \left[1 + \left(\cos^2\phi + \sin^2\phi\right)\zeta\right]|\mathbf{x}' - \mathbf{x}| \tag{5.55a}$$

or

$$\left|\mathbf{y}_{(j)}^n - \mathbf{y}_{(k)}^n\right| = \left[1 + \left(\cos^2\phi_{(j)(k)} + \sin^2\phi_{(j)(k)}\right)\zeta\right]\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|. \tag{5.55b}$$

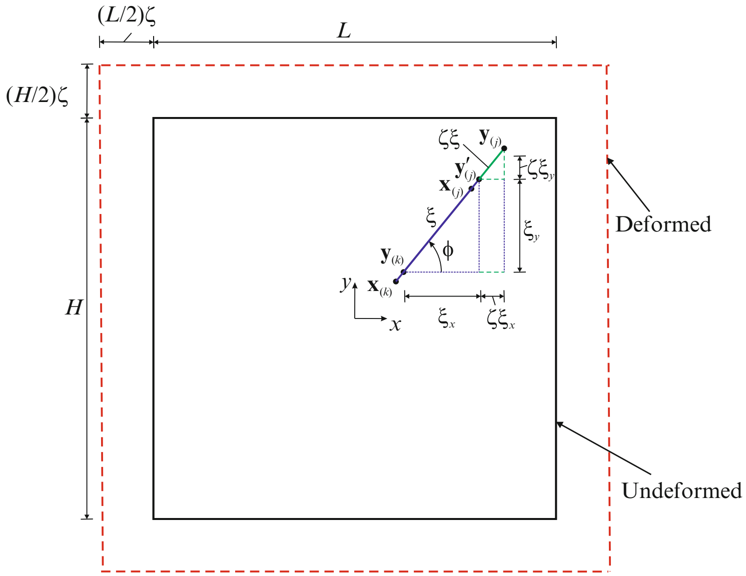


Fig. 5.7 Biaxial stretch

For this deformation, the dilatation is evaluated as

$$\theta_{(k)} = d\int_H \frac{\delta}{\xi}([1+\zeta]\,\xi - \xi)dH \tag{5.56a}$$

or

$$\theta_{(k)} = \pi d h \delta^3 \zeta. \tag{5.56b}$$

Equating the dilatation contributions from the classical and PD formulations, Eqs. 5.54a and 5.56b, also results in the same value of the PD parameter

$$d = \frac{2}{\pi h \delta^3}. \tag{5.57}$$

For this deformation given by Eq. 5.55b, the strain energy density is evaluated as

$$\begin{aligned} W_{(k)} = {} & 4a\,\zeta^2 + b_F\,\zeta^2\delta\left(\sum_{j=1}^{J}\left(\left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(k)}\right|\right)V^n_{(j)}\right) \\ & + b_{FT}\frac{2\pi h\delta^4\zeta^2}{3} + b_T\zeta^2\delta\left(\sum_{j=1}^{J}\left(\left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(k)}\right|\right)V^n_{(j)}\right). \end{aligned} \tag{5.58}$$

After substituting for b_{FT} from Eq. 5.36, it takes the final form

$$W_{(k)} = 4a\,\zeta^2 + b_F\,\zeta^2\delta\left(\sum_{j=1}^{J}\left(\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|\right)V_{(j)}^n\right) + 4Q_{66}\zeta^2 + b_T\zeta^2\delta\left(\sum_{j=1}^{J}\left(\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|\right)V_{(j)}\right). \quad (5.59)$$

Equating the expressions for strain energy density from the classical and PD formulations, Eqs. 5.54b and 5.59, results in

$$\frac{1}{2}(Q_{11} + 2Q_{12} + Q_{22} - 8Q_{66}) = 4a + b_F\,\delta\left(\sum_{j=1}^{J}\left(\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|\right)V_{(j)}^n\right) + b_T\delta\left(\sum_{j=1}^{J}\left(\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|\right)V_{(j)}^n\right). \quad (5.60)$$

The remaining peridynamic parameters in the strain energy density expression can now be evaluated by using the previous two relations obtained from the uniform stretch in the fiber and transverse directions, Eqs. 5.44 and 5.52, in conjunction with Eq. 5.60, as

$$a = \frac{1}{2}(Q_{12} - Q_{66}), \quad (5.61a)$$

$$b_F = \frac{(Q_{11} - Q_{12} - 2Q_{66})}{2\delta\left(\sum_{j=1}^{N}\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|V_{(j)}^n\right)}, \quad (5.61b)$$

$$b_T = \frac{(Q_{22} - Q_{12} - 2Q_{66})}{2\delta\left(\sum_{j=1}^{N}\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right|V_{(j)}^n\right)}, \quad (5.61c)$$

$$b_{FT} = \frac{6Q_{66}}{\pi h\delta^4}. \quad (5.61d)$$

For bond-based peridynamics, the parameter a associated with dilatation and the parameter b_T associated with the transverse direction should both vanish, thus leading to constraint equations, previously derived by Oterkus and Madenci (2012), as

$$Q_{12} = Q_{66} \quad \text{and} \quad Q_{22} = 3Q_{12}. \quad (5.62)$$

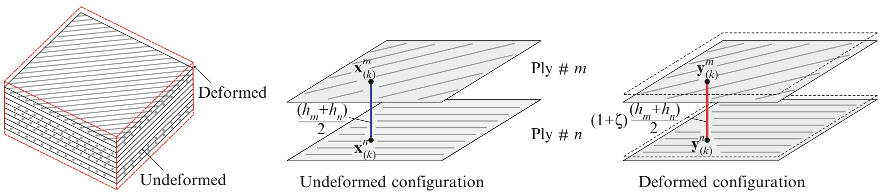


Fig. 5.8 A composite laminate subjected to transverse normal stretch

The nonvanishing peridynamic parameters, b_F and b_{FT} in the fiber and remaining directions, respectively, also recover the expressions derived by Oterkus and Madenci (2012) as

$$b_F = \frac{(Q_{11} - Q_{22})}{2\delta\left(\sum_{j=1}^{N}\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(k)}^n\right| V_{(j)}\right)} \quad \text{and} \quad b_{FT} = \frac{6Q_{66}}{\pi h \delta^4}. \tag{5.63}$$

For isotropic materials with $Q_{11} = Q_{22} = \kappa + \mu$, $Q_{12} = (\kappa - \mu)$, and $Q_{66} = \mu$, these peridynamic parameters recover Eqs. 4.52 and 4.53 as

$$a = \frac{1}{2}(\kappa - 2\mu), \ b_F = 0, \quad b_T = 0 \text{ and } \quad b_{FT} = b = \frac{6\mu}{\pi h \delta^4}, \tag{5.64}$$

and the parameter d is also equal to that of isotropic material given by Eq. 4.47.

5.4.2 Material Parameters for Transverse Deformation

The peridynamic material parameters b_N and b_S in the force density vector-stretch relations, Eqs. 5.29a, b associated with transverse deformation in a laminate are determined by considering two simple loading conditions as

1. Transverse normal stretch: $\varepsilon_{33} = \zeta$
2. Simple transverse shear: $\gamma_{13} = \zeta$

5.4.2.1 Transverse Normal Stretch: $\varepsilon_{33} = \zeta$

In order to obtain the peridynamic material parameter b_N, the laminate is subjected to a uniform transverse normal strain of ζ, as shown in Fig. 5.8. The corresponding strain energy density from the classical continuum mechanics at material point $\mathbf{x}_{(k)}$ is

$$\hat{W}_{(k)} = \frac{1}{2} E_m \zeta^2, \tag{5.65}$$

with E_m representing the Young's modulus of matrix material.

The relative distance between the material points at $\mathbf{x}_{(k)}^{m}$ and $\mathbf{x}_{(k)}^{n}$, before and after deformation, can be expressed as

$$\left|\mathbf{x}_{(k)}^{m}-\mathbf{x}_{(k)}^{n}\right|=\frac{1}{2}(h_m+h_n) \tag{5.66a}$$

and

$$\left|\mathbf{y}_{(k)}^{m}-\mathbf{y}_{(k)}^{n}\right|=(1+\zeta)\left|\mathbf{x}_{(k)}^{m}-\mathbf{x}_{(k)}^{n}\right|. \tag{5.66b}$$

Defining $\boldsymbol{\xi}=\mathbf{x}_{(k)}^{m}-\mathbf{x}_{(k)}^{n}$ and noting that its length is equal to half of the sum of the two neighboring ply thicknesses, i.e., $\xi=|\boldsymbol{\xi}|=(h_m+h_n)/2$, with $m=(n+1)$, $(n-1)$, and substituting for the relative position vector, from Eq. 5.66a, in the expression for the strain energy density, $\hat{W}_{(k)}$, Eq. 5.28a, at material point $\mathbf{x}_{(k)}^{n}$ result in

$$\hat{W}_{(k)}^{n}=\frac{1}{2}\zeta^2 b_N\hat{\delta}\left[(h_{n+1}+h_n)V_{(k)}^{n+1}+(h_{n-1}+h_n)V_{(k)}^{n-1}\right]. \tag{5.67}$$

Equating the expressions for strain energy density from Eqs. 5.65 and 5.67 provides the relationship between the PD parameters, b_N, and the Young's modulus of the matrix material as

$$b_N=\frac{E_m}{\hat{\delta}\left[(h_{n+1}+h_n)V_{(k)}^{n+1}+(h_{n-1}+h_n)V_{(k)}^{n-1}\right]}. \tag{5.68}$$

5.4.2.2 Simple Transverse Shear: $\gamma_{13}=\zeta$

Similarly, the peridynamic material parameter b_S is evaluated by subjecting the laminate to a simple transverse shear loading of ζ, as shown in Fig. 5.9. The corresponding strain energy density from classical continuum mechanics at material point $\mathbf{x}_{(k)}$ is

$$\tilde{W}_{(k)}=\frac{1}{2}G_m\zeta^2, \tag{5.69}$$

with G_m representing the shear modulus of matrix material.

As shown in Fig. 5.10, the relative distance between the material points at $\mathbf{x}_{(j)}^{m}$ and $\mathbf{x}_{(k)}^{n}$, before and after deformation, can be expressed as

$$\left|\mathbf{x}_{(j)}^{m}-\mathbf{x}_{(k)}^{n}\right|=\sqrt{\ell^2+\frac{(h_m+h_n)^2}{4}} \tag{5.70a}$$

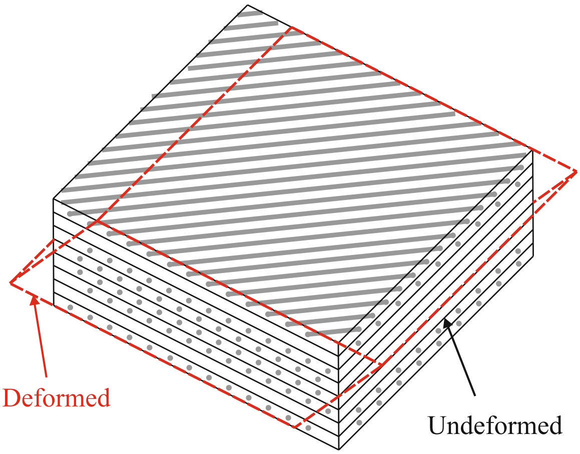


Fig. 5.9 A composite laminate subjected to simple transverse shear

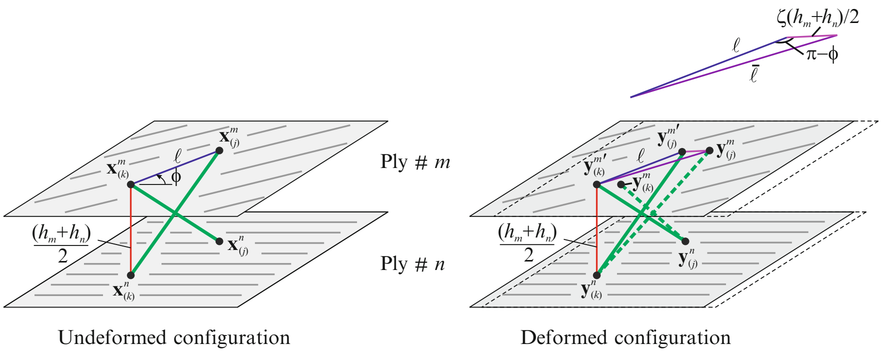


Fig. 5.10 Position of material points before and after deformation due to simple transverse shear

$$\left|\mathbf{y}_{(j)}^{m} - \mathbf{y}_{(k)}^{n}\right| = \sqrt{\bar{\ell}^{2} + \frac{(h_m + h_n)^2}{4}}, \tag{5.70b}$$

in which $\bar{\ell}$ can be obtained from the law of cosines as

$$\bar{\ell}^{2} = \ell^{2} + \zeta^{2}\frac{(h_m + h_n)^2}{4} - \ell\zeta(h_m + h_n)\cos(\pi - \phi). \tag{5.71}$$

Thus, the distance between $\mathbf{x}_{(j)}^{m}$ and $\mathbf{x}_{(k)}^{n}$ in the deformed state can be rewritten as

$$\left|\mathbf{y}_{(j)}^{m} - \mathbf{y}_{(k)}^{n}\right| = \sqrt{\left(\ell^{2} + \frac{(h_m + h_n)^2}{4}\right) + \ell\zeta\,(h_m + h_n)\cos(\phi).} \tag{5.72}$$

In deriving this expression, the $\zeta^2(h_m+h_n)^2/4$ term is disregarded with respect to $(h_m+h_n)^2/4$ because ζ is much less than unity. Also, this expression can be further simplified by using the square root approximation because $\ell\zeta(h_m+h_n)\cos(\phi) \ll (\ell^2+(h_m+h_n)^2/4)$, leading to

$$\left|\mathbf{y}_{(j)}^m-\mathbf{y}_{(k)}^n\right| = \sqrt{\ell^2+\frac{(h_m+h_n)^2}{4}}+\frac{\ell\zeta\,(h_m+h_n)\cos(\phi)}{2\sqrt{\ell^2+\frac{(h_m+h_n)^2}{4}}}. \tag{5.73}$$

Thus, the extension between these material points is obtained as

$$\left|\mathbf{y}_{(j)}^m-\mathbf{y}_{(k)}^n\right|-\left|\mathbf{x}_{(j)}^m-\mathbf{x}_{(k)}^n\right| = \frac{\ell\zeta\,(h_m+h_n)\cos(\phi)}{2\sqrt{\ell^2+\frac{(h_m+h_n)^2}{4}}}. \tag{5.74}$$

Similarly, the distance between the material points $\mathbf{x}_{(k)}^m$ and $\mathbf{x}_{(j)}^n$ before and after deformation can be obtained as

$$\left|\mathbf{x}_{(k)}^m-\mathbf{x}_{(j)}^n\right| = \sqrt{\ell^2+\frac{(h_m+h_n)^2}{4}} \tag{5.75a}$$

and

$$\left|\mathbf{y}_{(k)}^m-\mathbf{y}_{(j)}^n\right| = \sqrt{\ell^2+\frac{(h_m+h_n)^2}{4}}-\frac{\ell\zeta\,(h_m+h_n)\cos(\phi)}{2\sqrt{\ell^2+\frac{(h_m+h_n)^2}{4}}}, \tag{5.75b}$$

in which the minus sign emerges due to the contraction between material points $\mathbf{x}_{(k)}^m$ and $\mathbf{x}_{(j)}^n$ in the deformed state, whereas extension occurs between material points $\mathbf{x}_{(j)}^m$ and $\mathbf{x}_{(k)}^n$. Thus, the contraction between these material points is obtained as

$$\left|\mathbf{y}_{(k)}^m-\mathbf{y}_{(j)}^n\right|-\left|\mathbf{x}_{(k)}^m-\mathbf{x}_{(j)}^n\right| = -\frac{\ell\zeta\,(h_m+h_n)\cos(\phi)}{2\sqrt{\ell^2+\frac{(h_m+h_n)^2}{4}}}. \tag{5.76}$$

Prior to substituting for the stretch between the material points $\mathbf{x}_{(j)}^m$ and $\mathbf{x}_{(k)}^n$ and $\mathbf{x}_{(k)}^m$ and $\mathbf{x}_{(j)}^n$, the strain energy expression can be rewritten in a slightly different form as

$$\tilde{W}_{(k)}^n = b_S \sum_{m=n+1,n-1}\left(\frac{h_m+h_n}{2}\right)^2 \times \sum_{j=1}^{\infty}\frac{\tilde{\delta}}{\left|\mathbf{x}_{(j)}^m-\mathbf{x}_{(k)}^n\right|}\left[\frac{\left|\mathbf{y}_{(j)}^m-\mathbf{y}_{(k)}^n\right|-\left|\mathbf{x}_{(j)}^m-\mathbf{x}_{(k)}^n\right|}{\left(\frac{h_m+h_n}{2}\right)}-\frac{\left|\mathbf{y}_{(k)}^m-\mathbf{y}_{(j)}^n\right|-\left|\mathbf{x}_{(k)}^m-\mathbf{x}_{(j)}^n\right|}{\left(\frac{h_m+h_n}{2}\right)}\right]^2 V_{(j)}^m, \tag{5.77}$$

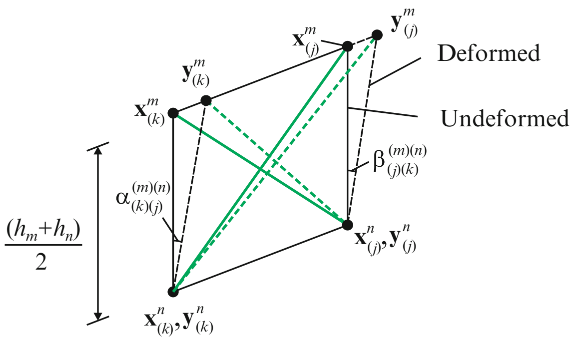


Fig. 5.11 Change in angle after deformation

in which the ratios in the summation can be interpreted as the change in angle from $\pi/2$ provided that $|\mathbf{x}^m_{(j)} - \mathbf{x}^n_{(k)}| \gg h$ and $|\mathbf{x}^m_{(k)} - \mathbf{x}^n_{(j)}| \gg h$, as depicted in Fig. 5.11. With this interpretation, this expression can be rewritten as

$$\tilde{W}^n_{(k)} = b_S \sum_{m=n+1,n-1} \left(\frac{h_m + h_n}{2}\right)^2 \sum_{j=1}^{\infty} \frac{\tilde{\delta}}{\left|\mathbf{x}^m_{(j)} - \mathbf{x}^n_{(k)}\right|} \left[\alpha^{(m)(n)}_{(k)(j)} + \beta^{(m)(n)}_{(j)(k)}\right]^2 V^m_{(j)}, \quad (5.78)$$

with

$$\alpha^{(m)(n)}_{(k)(j)} = \frac{\left|\mathbf{y}^m_{(j)} - \mathbf{y}^n_{(k)}\right| - \left|\mathbf{x}^m_{(j)} - \mathbf{x}^n_{(k)}\right|}{\left(\frac{h_m+h_n}{2}\right)} \quad (5.79a)$$

$$\beta^{(m)(n)}_{(j)(k)} = -\frac{\left|\mathbf{y}^m_{(k)} - \mathbf{y}^n_{(j)}\right| - \left|\mathbf{x}^m_{(k)} - \mathbf{x}^n_{(j)}\right|}{\left(\frac{h_m+h_n}{2}\right)}. \quad (5.79b)$$

The average change in angle, $\varphi^{(m)(n)}_{(j)(k)}$, corresponding to the shear strain in classical continuum mechanics becomes

$$\varphi^{(m)(n)}_{(k)(j)} = \frac{\alpha^{(m)(n)}_{(k)(j)} + \beta^{(m)(n)}_{(j)(k)}}{2} \\ = \frac{\left(\left|\mathbf{y}^m_{(j)} - \mathbf{y}^n_{(k)}\right| - \left|\mathbf{x}^m_{(j)} - \mathbf{x}^n_{(k)}\right|\right) - \left(\left|\mathbf{y}^m_{(k)} - \mathbf{y}^n_{(j)}\right| - \left|\mathbf{x}^m_{(k)} - \mathbf{x}^n_{(j)}\right|\right)}{(h_m + h_n)}. \quad (5.80)$$

Substituting for the stretch between the material points, $\mathbf{x}^m_{(j)}$ and $\mathbf{x}^n_{(k)}$ and $\mathbf{x}^m_{(k)}$ and $\mathbf{x}^n_{(j)}$, the average change in angle, $\varphi^{(m)(n)}_{(k)(j)}$, for the applied simple shear loading can be determined as

$$\varphi_{(k)(j)}^{(m)(n)} = \frac{\ell\zeta\cos(\phi)}{\sqrt{\ell^2 + \frac{(h_m+h_n)^2}{4}}}. \tag{5.81}$$

Therefore, the strain energy density function can be rewritten in terms of the average change in angle as

$$\tilde{W}_{(k)}^{n} = 4b_S \sum_{m=n+1,n-1} \left(\frac{h_m + h_n}{2}\right)^2 \sum_{j=1}^{\infty} \frac{\tilde{\delta}}{\left|\mathbf{x}_{(j)}^{m} - \mathbf{x}_{(k)}^{n}\right|} \left[\varphi_{(k)(j)}^{(m)(n)}\right]^2 V_{(j)}^{m} \tag{5.82a}$$

or

$$\begin{aligned} \tilde{W}_{(k)}^{n} = 4b_S \Bigg(& \left(\frac{h_{n+1} + h_n}{2}\right)^2 \sum_{j=1}^{\infty} \frac{\tilde{\delta}}{\left|\mathbf{x}_{(j)}^{n+1} - \mathbf{x}_{(k)}^{n}\right|} \left[\varphi_{(k)(j)}^{(n+1)(n)}\right]^2 V_{(j)}^{n+1} \\ & + \left(\frac{h_{n-1} + h_n}{2}\right)^2 \sum_{j=1}^{\infty} \frac{\tilde{\delta}}{\left|\mathbf{x}_{(j)}^{n-1} - \mathbf{x}_{(k)}^{n}\right|} \left[\varphi_{(k)(j)}^{(n-1)(n)}\right]^2 V_{(j)}^{n-1} \Bigg) \end{aligned} \tag{5.82b}$$

or

$$\begin{aligned} \tilde{W}_{(k)}^{n} = 4\zeta^2 b_S \tilde{\delta} \Bigg(& \left(\frac{h_{n+1} + h_n}{2}\right)^2 \sum_{j=1}^{\infty} \frac{\ell^2 \cos^2(\phi)}{\left[\ell^2 + \left(\frac{h_{n+1}+h_n}{2}\right)^2\right]^{3/2}} V_{(j)}^{n+1} \\ & + \left(\frac{h_{n-1} + h_n}{2}\right)^2 \sum_{j=1}^{\infty} \frac{\ell^2 \cos^2(\phi)}{\left[\ell^2 + \left(\frac{h_{n-1}+h_n}{2}\right)^2\right]^{3/2}} V_{(j)}^{n-1} \Bigg). \end{aligned} \tag{5.82c}$$

Converting summation to integration leads to

$$\begin{aligned} \tilde{W}_{(k)}^{n} = 4\zeta^2 b_S \tilde{\delta} \Bigg(& \left(\frac{h_{n+1} + h_n}{2}\right)^3 \int_0^{\delta} \int_0^{2\pi} \frac{\ell^2 \cos^2(\phi)}{\left[\ell^2 + \left(\frac{h_{n+1}+h_n}{2}\right)^2\right]^{3/2}} \ell d\ell d\phi \\ & + \left(\frac{h_{n-1} + h_n}{2}\right)^3 \int_0^{\delta} \int_0^{2\pi} \frac{\ell^2 \cos^2(\phi)}{\left[\ell^2 + \left(\frac{h_{n-1}+h_n}{2}\right)^2\right]^{3/2}} \ell d\ell d\phi \Bigg). \end{aligned} \tag{5.83}$$

Performing the integration results in

$$\tilde{W}^{n}_{(k)} = 4\zeta^2 b_S \pi \tilde{\delta} \left(\left(\frac{h_{n+1} + h_n}{2}\right)^3 \left(\frac{\delta^2 + 2\left(\frac{h_{n+1}+h_n}{2}\right)^2}{\sqrt{\delta^2 + \left(\frac{h_{n+1}+h_n}{2}\right)^2}} - (h_{n+1} + h_n) \right) + \left(\frac{h_{n-1} + h_n}{2}\right)^3 \left(\frac{\delta^2 + 2\left(\frac{h_{n-1}+h_n}{2}\right)^2}{\sqrt{\delta^2 + \left(\frac{h_{n-1}+h_n}{2}\right)^2}} - (h_{n-1} + h_n) \right) \right). \tag{5.84}$$

Equating the expressions for strain energy density from Eqs. 5.69 and 5.84 provides the relationship between the PD parameter b_S and the shear modulus of the matrix material as

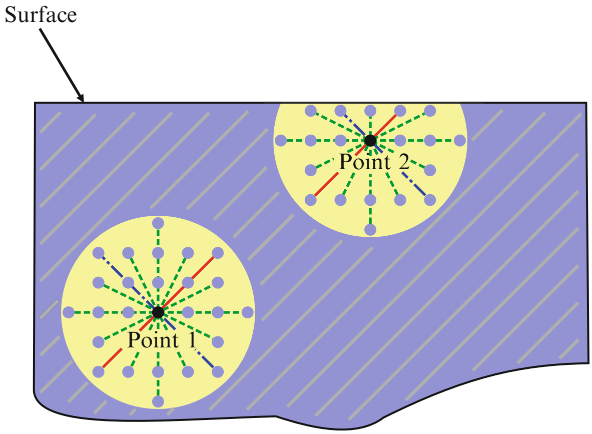
$$b_S = \frac{G_m}{8\pi\tilde{\delta} \left(\begin{array}{l} \left(\frac{h_{n+1} + h_n}{2}\right)^3 \left(\frac{\delta^2 + 2\left(\frac{h_{n+1}+h_n}{2}\right)^2}{\sqrt{\delta^2 + \left(\frac{h_{n+1}+h_n}{2}\right)^2}} - (h_{n+1} + h_n) \right) \\ + \left(\frac{h_{n-1} + h_n}{2}\right)^3 \left(\frac{\delta^2 + 2\left(\frac{h_{n-1}+h_n}{2}\right)^2}{\sqrt{\delta^2 + \left(\frac{h_{n-1}+h_n}{2}\right)^2}} - (h_{n-1} + h_n) \right) \end{array} \right)}. \tag{5.85}$$

5.5 Surface Effects

The peridynamic material parameters a, d, b_F, b_T, b_{FT}, b_N, and b_S that appear in the peridynamic force-stretch relations are determined by computing both dilatation and strain energy density of a material point whose horizon is completely embedded in the material. The values of these parameters, except for a, depend on the accuracy of integration and domain of integration defined by the horizon. Therefore, the values of these parameters will be different for a material point located near a boundary, Fig. 5.12. Thus, these parameters need to be corrected near the free surfaces.

Since the presence of free surfaces is problem dependent, it is impractical to resolve this issue analytically. The correction of the material parameters is achieved by numerically integrating both dilatation and strain energy density at each material point inside the body for simple loading conditions and comparing them to their counterparts obtained from classical continuum mechanics. After determining the correction factor for each parameter, the force density vector is modified in the PD equations of motion.

Fig. 5.12 Surface effects in the domain of interest



In order to determine the surface correction factors for the peridynamic parameters d and b_ℓ $(\ell = F, T, FT)$, two simple loading conditions are achieved by applying uniaxial stretch first in the fiber direction, and then in the transverse direction, i.e., $\varepsilon_{11} \neq 0, \varepsilon_{22} = \gamma_{12} = 0$ (shown in Fig. 5.13) and $\varepsilon_{22} \neq 0, \varepsilon_{11} = \gamma_{12} = 0$. The fiber and transverse directions coincide with the axes of the natural (material) coordinate system, $(1, 2)$.

The applied uniaxial stretch in the fiber and transverse directions is achieved through a constant displacement gradient, $\partial u_\alpha^* / \partial x_\alpha = \zeta$ with $(\alpha = 1, 2)$. The displacement field at material point $\mathbf{x}$ arising from these two loading conditions can be expressed as

$$\mathbf{u}_1^T(\mathbf{x}) = \left\{ \frac{\partial u_1^*}{\partial x_1} x_1 \quad 0 \right\} \quad \text{and} \quad \mathbf{u}_2^T(\mathbf{x}) = \left\{ 0 \quad \frac{\partial u_2^*}{\partial x_2} x_2 \right\}. \tag{5.86a,b}$$

Due to these displacement fields, the peridynamic dilatation term, $\theta_\alpha^{PD}(\mathbf{x}_{(i)}^n)$, at material point $\mathbf{x}_{(i)}^n$ can be obtained from Eq. 5.8 as

$$\theta_\alpha^{PD}(\mathbf{x}_{(i)}^n) = d \sum_{j=1}^{N} \frac{\delta}{\left| \mathbf{x}_{(j)}^n - \mathbf{x}_{(i)}^n \right|} \left(\left| \mathbf{y}_{(j)}^n - \mathbf{y}_{(i)}^n \right| - \left| \mathbf{x}_{(j)}^n - \mathbf{x}_{(i)}^n \right| \right) \Lambda_{(i)(j)}^n V_{(j)}^n, \tag{5.87}$$

in which N represents the number of material points inside the horizon of material point $\mathbf{x}_{(i)}^n$. The corresponding dilatation based on classical continuum mechanics, $\theta_\alpha^{CM}(\mathbf{x}_{(i)}^n)$, is uniform throughout the domain, and is determined as

$$\theta_\alpha^{CM}(\mathbf{x}_{(i)}^n) = \varepsilon_{\alpha\alpha} = \zeta, \text{ with } (\alpha = 1, 2), \tag{5.88}$$

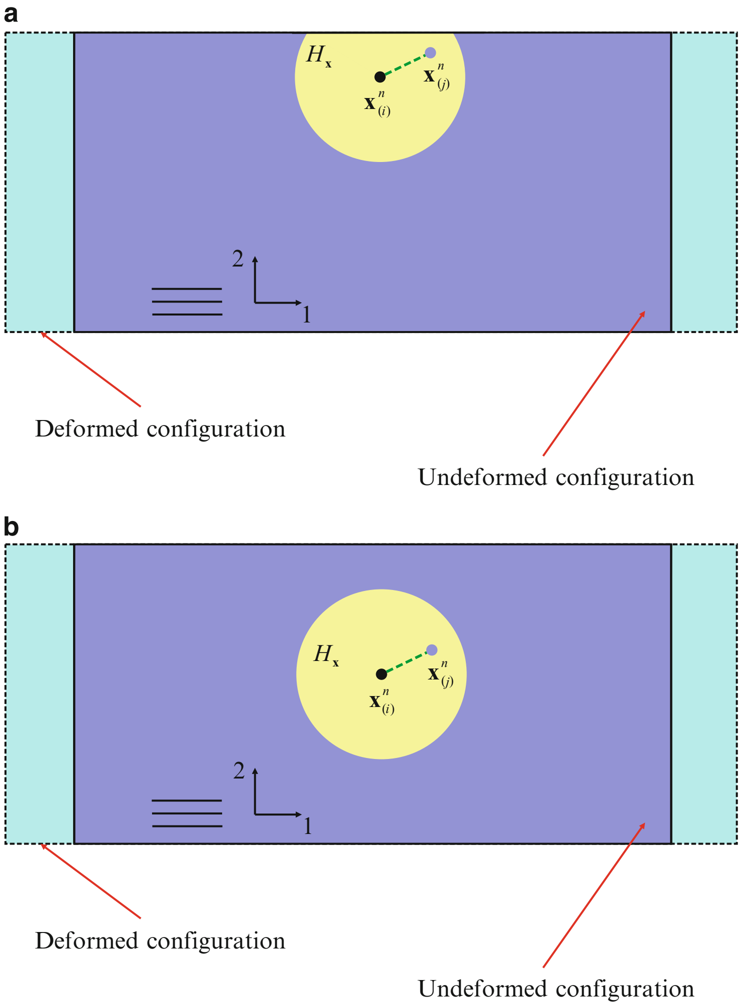


Fig. 5.13 Material point **x** in lamina subjected to uniaxial stretch: (**a**) a truncated horizon, and (**b**) far away from external surfaces

The dilatation correction term can be defined as

$$D_{\alpha(i)} = \frac{\theta_\alpha^{CM}\left(\mathbf{x}^n_{(i)}\right)}{\theta_\alpha^{PD}\left(\mathbf{x}^n_{(i)}\right)} = \frac{\zeta}{d\ \delta \sum_{j=1}^{N} s^n_{(i)(j)} \Lambda^n_{(i)(j)} V^n_{(j)}}. \tag{5.89}$$

Maximum values of dilatation occur in the loading directions that coincide with the natural coordinates 1 and 2, respectively.

The peridynamic strain energy density at material point $\mathbf{x}_{(i)}^{n}$ can be obtained from Eq. 5.7 as

$$W_{\alpha}^{PD}(\mathbf{x}_{(i)}^{n}) = W_{\alpha\theta}^{PD}(\mathbf{x}_{(i)}^{n}) + W_{\alpha F}^{PD}(\mathbf{x}_{(i)}^{n}) + W_{\alpha FT}^{PD}(\mathbf{x}_{(i)}^{n}) + W_{\alpha T}^{PD}(\mathbf{x}_{(i)}^{n}), \tag{5.90}$$

where $(\alpha = 1, 2)$, $W_{\alpha\theta}^{PD}$ is associated with the dilatation term, and $W_{\alpha F}^{PD}$, $W_{\alpha T}^{PD}$, and $W_{\alpha FT}^{PD}$ represent contributions from the deformation in the fiber direction, transverse direction, and arbitrary directions, respectively. Based on Eq. 5.7, each of these terms is expressed as

$$W_{\alpha\theta}^{PD}(\mathbf{x}_{(i)}^{n}) = a\left(\theta_{\alpha}^{PD}(\mathbf{x}_{(i)}^{n})\right)^{2}, \tag{5.91a}$$

$$W_{\alpha F}^{PD}(\mathbf{x}_{(i)}^{n}) = b_F\,\delta \sum_{j=1}^{M} \frac{1}{\left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|} \left(\left|\mathbf{y}_{(j)}^{n} - \mathbf{y}_{(i)}^{n}\right| - \left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|\right)^{2} V_{(j)}^{n}, \tag{5.91b}$$

$$W_{\alpha T}^{PD}(\mathbf{x}_{(i)}^{n}) = b_T\,\delta \sum_{j=1}^{N} \frac{1}{\left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|} \left(\left|\mathbf{y}_{(j)}^{n} - \mathbf{y}_{(i)}^{n}\right| - \left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|\right)^{2} V_{(j)}^{n}, \tag{5.91c}$$

$$W_{\alpha FT}^{PD}(\mathbf{x}_{(i)}^{n}) = b_{FT}\,\delta \sum_{j=1}^{P} \frac{1}{\left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|} \left(\left|\mathbf{y}_{(j)}^{n} - \mathbf{y}_{(i)}^{n}\right| - \left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|\right)^{2} V_{(j)}^{n}. \tag{5.91d}$$

Based on classical continuum mechanics, the strain energy density corresponding to uniaxial stretch in the fiber, $W_{1}^{CM}(\mathbf{x}_{(i)}^{n})$, and transverse directions, $W_{2}^{CM}(\mathbf{x}_{(i)}^{n})$, is uniform, and can be determined from

$$W_{\alpha}^{CM}(\mathbf{x}_{(i)}^{n}) = \frac{1}{2} Q_{\alpha\alpha}\zeta^{2} \; (\alpha = 1, 2), \tag{5.92}$$

which can be decomposed as

$$W_{\alpha}^{CM}(\mathbf{x}_{(i)}^{n}) = W_{\alpha\theta}^{CM}(\mathbf{x}_{(i)}^{n}) + W_{\alpha F}^{CM}(\mathbf{x}_{(i)}^{n}) + W_{\alpha T}^{CM}(\mathbf{x}_{(i)}^{n}) + W_{\alpha FT}^{CM}(\mathbf{x}_{(i)}^{n}), \tag{5.93}$$

where $W_{\alpha\theta}^{CM}$ is associated with the dilatation terms, and $W_{\alpha F}^{CM}$, $W_{\alpha T}^{CM}$, and $W_{\alpha FT}^{CM}$ represent strain energy densities arising from the deformation in the fiber direction, transverse direction, and arbitrary directions, respectively. From Eq. 5.42b in conjunction with Eqs. 5.61a, b, d for uniaxial stretch in the fiber direction, i.e., $(\alpha = 1)$, each strain energy density component can be expressed as

$$W_{1\theta}^{CM}(\mathbf{x}_{(i)}^{n}) = \frac{1}{2}(Q_{12} - Q_{66})\zeta^{2}, \tag{5.94a}$$

$$W_{1F}^{CM}(\mathbf{x}_{(i)}^{n}) = \frac{1}{2}(Q_{11} - Q_{12} - 2Q_{66})\zeta^2, \tag{5.94b}$$

$$W_{1T}^{CM}(\mathbf{x}_{(i)}^{n}) = 0, \tag{5.94c}$$

$$W_{1FT}^{CM}(\mathbf{x}_{(i)}^{n}) = \frac{3}{2}Q_{66}\zeta^2. \tag{5.94d}$$

From Eq. 5.51 in conjunction with Eqs. 5.61a, c, for uniaxial stretch in the transverse direction, i.e., $(\alpha = 2)$, each strain energy component can be expressed as

$$W_{2\theta}^{CM}(\mathbf{x}_{(i)}^{n}) = \frac{1}{2}(Q_{12} - Q_{66})\zeta^2, \tag{5.95a}$$

$$W_{2F}^{CM}(\mathbf{x}_{(i)}^{n}) = 0, \tag{5.95b}$$

$$W_{2T}^{CM}(\mathbf{x}_{(i)}^{n}) = \frac{1}{2}(Q_{22} - Q_{12} - 2Q_{66})\zeta^2, \tag{5.95c}$$

$$W_{2FT}^{CM}(\mathbf{x}_{(i)}^{n}) = \frac{3}{2}Q_{66}\zeta^2. \tag{5.95d}$$

Because the dilatation term, $\theta_{\alpha}^{PD}(\mathbf{x}_{(i)}^{n})$, is corrected with a dilatation correction term in the peridynamic computation, it is expected that Eq. 5.91a is automatically corrected for this loading condition. Hence, the correction is only necessary for the terms including parameter b_{ℓ}, with $\ell = F, FT, T$. For the uniaxial stretch in the fiber direction, the correction terms for these parameters can be defined as

$$S_{1F(i)} = \frac{W_{1F}^{CM}(\mathbf{x}_{(i)}^{n})}{W_{1F}^{PD}(\mathbf{x}_{(i)}^{n})} = \frac{\frac{1}{2}(Q_{11} - Q_{12} - 2Q_{66})\zeta^2}{b_F\,\delta \sum_{j=1}^{M} \frac{1}{\left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|}\left(\left|\mathbf{y}_{(j)}^{n} - \mathbf{y}_{(i)}^{n}\right| - \left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|\right)^2 V_{(j)}^{n}}, \tag{5.96a}$$

$$S_{1T(i)} = 1, \tag{5.96b}$$

$$S_{1FT(i)} = \frac{W_{1FT}^{CM}(\mathbf{x}_{(i)}^{n})}{W_{1FT}^{PD}(\mathbf{x}_{(i)}^{n})} = \frac{\frac{3}{2}Q_{66}\zeta^2}{b_{FT}\,\delta \sum_{j=1}^{P} \frac{1}{\left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|}\left(\left|\mathbf{y}_{(j)}^{n} - \mathbf{y}_{(i)}^{n}\right| - \left|\mathbf{x}_{(j)}^{n} - \mathbf{x}_{(i)}^{n}\right|\right)^2 V_{(j)}^{n}}. \tag{5.96c}$$

For the uniaxial stretch in the transverse direction, the correction terms for these parameters can be defined as

$$S_{2F(i)} = 1, \tag{5.97a}$$

$$\begin{aligned} S_{2T(i)} &= \frac{W_{2T}^{CM}(\mathbf{x}_{(i)}^n)}{W_{2T}^{PD}(\mathbf{x}_{(i)}^n)} \\ &= \frac{\frac{1}{2}(Q_{22} - Q_{12} - 2Q_{66})\zeta^2}{b_T\,\delta \sum_{j=1}^{N} \frac{1}{\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(i)}^n\right|}\left(\left|\mathbf{y}_{(j)}^n - \mathbf{y}_{(i)}^n\right| - \left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(i)}^n\right|\right)^2 V_{(j)}^n}, \end{aligned} \tag{5.97b}$$

$$\begin{aligned} S_{2FT(i)} &= \frac{W_{2FT}^{CM}(\mathbf{x}_{(i)}^n)}{W_{2FT}^{PD}(\mathbf{x}_{(i)}^n)} \\ &= \frac{\frac{3}{2}Q_{66}\zeta^2}{b_{FT}\,\delta \sum_{j=1}^{P} \frac{1}{\left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(i)}^n\right|}\left(\left|\mathbf{y}_{(j)}^n - \mathbf{y}_{(i)}^n\right| - \left|\mathbf{x}_{(j)}^n - \mathbf{x}_{(i)}^n\right|\right)^2 V_{(j)}^n}\,. \end{aligned} \tag{5.97c}$$

With these correction factors, a vector of correction factors for the integral and summation terms that appear in dilatation and the strain energy density at material point $\mathbf{x}_{(i)}^n$ can be written as

$$\mathbf{g}_{(d)(i)}(\mathbf{x}_{(i)}^n) = \left\{g_{1(d)}(\mathbf{x}_{(i)}^n),\ g_{2(d)}(\mathbf{x}_{(i)}^n)\right\}^T = \left\{D_{1(i)},\ D_{2(i)}\right\}^T, \tag{5.98a}$$

$$\mathbf{g}_{(b)\ell(i)}(\mathbf{x}_{(i)}^n) = \left\{g_{1(b)\ell}(\mathbf{x}_{(i)}^n),\ g_{2(b)\ell}(\mathbf{x}_{(i)}^n)\right\}^T = \left\{S_{1\ell(i)},\ S_{2\ell(i)}\right\}^T, \tag{5.98b}$$

with $\ell = F, FT, T$.

These correction factors are only based on loading in the fiber and transverse directions. However, they can be used as the principal values of an ellipse as shown in Fig. 5.14 in order to approximate the surface correction factor in any direction. Arising from a general loading condition, the correction factor for interaction between material points $\mathbf{x}_{(i)}^n$ and $\mathbf{x}_{(j)}^n$, shown in Fig. 5.15a, can be obtained in the direction of their unit relative position vector, $\mathbf{n} = (\mathbf{x}_{(j)}^n - \mathbf{x}_{(i)}^n)/|\mathbf{x}_{(j)}^n - \mathbf{x}_{(i)}^n| = \{n_1,\ n_2\}^T$.

A vector of correction factors for the integrals in the dilatation and strain energy density expressions at material point $\mathbf{x}_{(j)}^n$ can be similarly written as

$$\mathbf{g}_{(d)(j)}(\mathbf{x}_{(j)}^n) = \left\{g_{1(d)}(\mathbf{x}_{(j)}^n),\ g_{2(d)}(\mathbf{x}_{(j)}^n)\right\}^T = \left\{D_{1(j)},\ D_{2(j)}\right\}^T, \tag{5.99a}$$

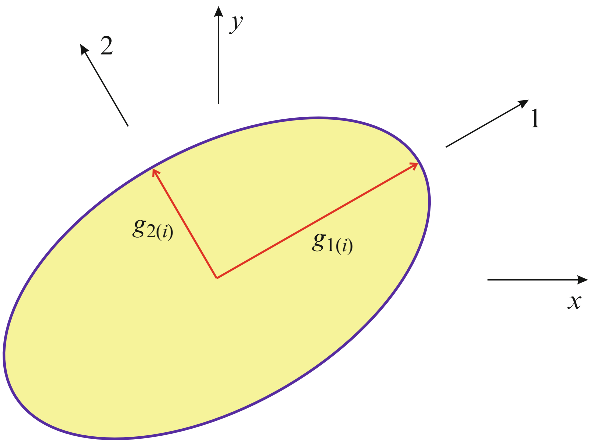


Fig. 5.14 Construction of an ellipse for surface correction factors

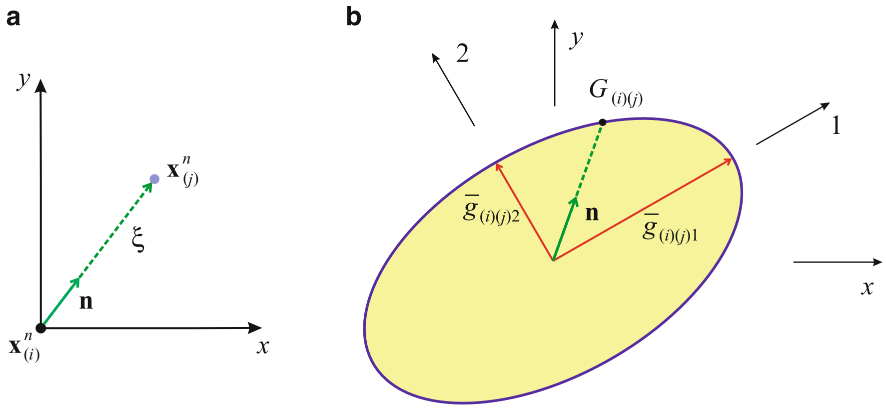


Fig. 5.15 (**a**) PD interaction between material points at $\mathbf{x}_{(i)}^n$ and $\mathbf{x}_{(j)}^n$, and (**b**) the ellipse for the surface correction factors

$$\mathbf{g}_{(b)\ell(j)}(\mathbf{x}_{(j)}^n) = \left\{ g_{1(b)\ell}(\mathbf{x}_{(j)}^n),\ g_{2(b)\ell}(\mathbf{x}_{(j)}^n) \right\}^T = \left\{ S_{1\ell(j)},\ S_{2\ell(j)} \right\}^T. \tag{5.99b}$$

These correction factors are, in general, different at material points $\mathbf{x}_{(i)}^n$ and $\mathbf{x}_{(j)}^n$. Therefore, the correction factor for an interaction between material points $\mathbf{x}_{(i)}^n$ and $\mathbf{x}_{(j)}^n$ can be obtained by their mean values as

$$\bar{\mathbf{g}}_{(d)(i)(j)} = \left\{ \bar{g}_{(d)(i)(j)1}, \bar{g}_{(d)(i)(j)2} \right\}^T = \frac{\mathbf{g}_{(d)(i)} + \mathbf{g}_{(d)(j)}}{2} \tag{5.100a}$$

and

$$\bar{\mathbf{g}}_{(b)\ell(i)(j)} = \left\{\bar{g}_{(b)\ell(i)(j)1}, \bar{g}_{(b)\ell(i)(j)2}\right\}^T = \frac{\mathbf{g}_{(b)\ell(i)} + \mathbf{g}_{(b)\ell(j)}}{2}, \tag{5.100b}$$

which can be used as the principal values of an ellipse for the interactions other than in the fiber and transverse directions, as shown in Fig. 5.15b. The intersection of the ellipse and a relative position vector, $\mathbf{n}$, of material points $\mathbf{x}^n_{(i)}$ and $\mathbf{x}^n_{(j)}$, provides the correction factors as

$$G_{(d)(i)(j)} = \left(\left[n_1 \Big/ \bar{g}_{(d)(i)(j)1}\right]^2 + \left[n_2 \Big/ \bar{g}_{(d)(i)(j)2}\right]^2\right)^{-1/2} \tag{5.101a}$$

and

$$G_{(b)\ell(i)(j)} = \left(\left[n_1 \Big/ \bar{g}_{(b)\ell(i)(j)1}\right]^2 + \left[n_2 \Big/ \bar{g}_{(b)\ell(i)(j)2}\right]^2\right)^{-1/2}. \tag{5.101b}$$

After considering the surface effects, the discrete forms of the dilatation and the strain energy density are corrected as

$$\begin{aligned} \theta^n_{(i)} = d\sum_{j=1}^{P} G_{(d)(i)(j)} \frac{\delta}{\left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}\right|} \left(\left|\mathbf{y}^n_{(j)} - \mathbf{y}^n_{(i)}\right| - \left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}\right|\right) \\ \times \left(\frac{\mathbf{y}^n_{(j)} - \mathbf{y}^n_{(i)}}{\left|\mathbf{y}^n_{(j)} - \mathbf{y}^n_{(i)}\right|} \cdot \frac{\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}}{\left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}\right|}\right) V^n_{(j)} \quad , \end{aligned} \tag{5.102a}$$

$$\begin{aligned} W^n_{(i)} = a\,\theta^2_{(i)} + b_F\delta\sum_{j=1}^{M} G_{(b)F(i)(j)} \frac{1}{\left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}\right|} \\ \times \left(\left|\mathbf{y}^n_{(j)} - \mathbf{y}^n_{(i)}\right| - \left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}\right|\right)^2 V^n_{(j)} \\ + b_T\delta\sum_{j=1}^{N} G_{(b)T(i)(j)} \frac{1}{\left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}\right|} \\ \times \left(\left|\mathbf{y}^n_{(j)} - \mathbf{y}^n_{(i)}\right| - \left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}\right|\right)^2 V^n_{(j)} \\ + b_{FT}\delta\sum_{j=1}^{P} G_{(b)FT(i)(j)} \frac{1}{\left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}\right|} \\ \times \left(\left|\mathbf{y}^n_{(j)} - \mathbf{y}^n_{(i)}\right| - \left|\mathbf{x}^n_{(j)} - \mathbf{x}^n_{(i)}\right|\right)^2 V^n_{(j)} \; . \end{aligned} \tag{5.102b}$$

The peridynamic material parameters b_N and b_S for a material point located on the bounding laminae, such as $n = 1$ or $n = N$, also require correction. However, the correction factors for b_N and b_S are not necessary for material points $\mathbf{x}_{(i)}^n$ for $n \neq 1, N$ because they are imbedded in the laminate, as shown in Fig. 5.3.

Simple loading conditions of uniform transverse stretch, $\partial u_3^* / \partial x_3 = \zeta$, and simple transverse shear, $\partial u_1^* / \partial x_3 = \zeta$, are applied to the laminate separately to determine the correction factors.

The corresponding displacement fields at material point $\mathbf{x}$ as a result of these loading conditions can be expressed as

$$\mathbf{u}_3^T = \left\{ 0 \quad 0 \quad \frac{\partial u_3^*}{\partial x_3} x_3 \right\} \tag{5.103a}$$

and

$$\mathbf{u}_S^T = \left\{ \frac{\partial u_1^*}{\partial x_3} x_3 \quad 0 \quad 0 \right\}. \tag{5.103b}$$

The PD strain energy density of material point $\mathbf{x}_{(i)}^n$ with $n = 1, N$ due to these loading conditions, respectively, can be expressed as

$$\left. \begin{aligned} W_3^{PD}\left(\mathbf{x}_{(i)}^1\right) &= \frac{1}{4} \zeta^2 b_N (h_{n+1} + h_n)^2 V_{(i)}^{n+1} \\ W_3^{PD}\left(\mathbf{x}_{(i)}^N\right) &= \frac{1}{4} \zeta^2 b_N (h_{n-1} + h_n)^2 V_{(i)}^{n-1} \end{aligned} \right\} \tag{5.104a}$$

and

$$\left. \begin{aligned} W_S^{PD}\left(\mathbf{x}_{(i)}^1\right) &= 4\zeta^2 b_S \left(\frac{h_{n+1} + h_n}{2}\right)^2 \sum_{j=1}^{N} \frac{\ell^2 \cos^2(\phi)}{\ell^2 + \left(\frac{h_{n+1}+h_n}{2}\right)^2} V_{(j)}^{n+1} \\ W_S^{PD}\left(\mathbf{x}_{(i)}^N\right) &= 4\zeta^2 b_S \left(\frac{h_{n-1} + h_n}{2}\right)^2 \sum_{j=1}^{N} \frac{\ell^2 \cos^2(\phi)}{\ell^2 + \left(\frac{h_{n-1}+h_n}{2}\right)^2} V_{(j)}^{n-1} \end{aligned} \right\}. \tag{5.104b}$$

The corresponding strain energy density expressions based on classical continuum mechanics can be expressed as

$$W_3^{CM}\left(\mathbf{x}_{(i)}^n\right) = \frac{1}{2} E_m \zeta^2 \qquad n = 1, N \tag{5.105a}$$

and

$$W_S^{CM}\left(\mathbf{x}_{(i)}^n\right) = \frac{1}{2} G_m \zeta^2 \qquad n = 1, N \tag{5.105b}$$

Therefore, the correction factors associated with the material parameters, b_N and b_S, at material point $\mathbf{x}_{(i)}^n$ for $n = 1, N$ can be defined as

$$S_{3(i)}^n = \frac{W_3^{CM}\left(\mathbf{x}_{(i)}^n\right)}{W_3^{PD}\left(\mathbf{x}_{(i)}^n\right)} \tag{5.106a}$$

and

$$S_{S(i)}^n = \frac{W_S^{CM}\left(\mathbf{x}_{(i)}^n\right)}{W_S^{PD}\left(\mathbf{x}_{(i)}^n\right)}. \tag{5.106b}$$

Correction factors for b_N and b_S are not necessary for material points $\mathbf{x}_{(i)}^n$ for $n \neq 1, N$. Therefore, the correction factor for an interaction between material points $\mathbf{x}_{(i)}^n$ for $n = 1, N$ and $\mathbf{x}_{(j)}^m$ for $m \neq 1, N$ can be obtained by their mean values as

$$\left.\begin{aligned} \bar{S}_{3(i)}^{(n)(m)} &= \left(S_{3(i)}^n + 1\right)/2 \quad \text{for } n = 1, N \,\text{and}\, m \neq 1, N \\ \bar{S}_{3(i)}^{(n)(m)} &= 1 \quad \text{for } n, m \neq 1, N \end{aligned}\right\}, \tag{5.107a}$$

$$\left.\begin{aligned} \bar{S}_{S(i)(j)}^{(n)(m)} &= \left(S_{S(i)}^n + 1\right)/2 \quad \text{for } n = 1, N \,\text{and}\, m \neq 1, N \\ \bar{S}_{S(i)(j)}^{(n)(m)} &= 1 \quad \text{for } n, m \neq 1, N \end{aligned}\right\}. \tag{5.107b}$$

After considering the surface effects, the discrete form of the strain energy density functions $\hat{W}_{(i)}^n$ and $\tilde{W}_{(i)}^n$ are corrected as

$$\hat{W}_{(i)}^n = b_N \sum_{m=n+1,n-1} \bar{S}_{3(i)}^{(n)(m)} \left(\left|\mathbf{y}_{(i)}^m - \mathbf{y}_{(i)}^n\right| - \left|\mathbf{x}_{(i)}^m - \mathbf{x}_{(i)}^n\right|\right)^2 V_{(i)}^m \tag{5.108a}$$

and

$$\begin{aligned} \tilde{W}_{(i)}^n = b_S \sum_{m=n+1,n-1} \sum_{j=1}^{\infty} \bar{S}_{S(i)(j)}^{(n)(m)} \Big[&\left(\left|\mathbf{y}_{(j)}^m - \mathbf{y}_{(i)}^n\right| - \left|\mathbf{x}_{(j)}^m - \mathbf{x}_{(i)}^n\right|\right) \\ &- \left(\left|\mathbf{y}_{(i)}^m - \mathbf{y}_{(j)}^n\right| - \left|\mathbf{x}_{(i)}^m - \mathbf{x}_{(j)}^n\right|\right)\Big]^2 V_{(j)}^m \ . \end{aligned} \tag{5.108b}$$

Reference

Oterkus E, Madenci E (2012) Peridynamic analysis of fiber reinforced composite materials. J Mech Mater Struct 7(1):45–84