Chapter 4
Peridynamics for Isotropic Materials

4.1 Material Parameters

The auxiliary parameters, C in Eq. 2.43 and A and B in Eq. 2.48, can be determined by using the relationship between the force density vector and the strain energy density, $W_{(k)}$, at material point k given by Eq. 2.49 in the form,

$$\mathbf{t}_{(k)(j)}\left(\mathbf{u}_{(j)}-\mathbf{u}_{(k)},\mathbf{x}_{(j)}-\mathbf{x}_{(k)},t\right)=\frac{1}{V_{(j)}}\frac{\partial W_{(k)}}{\partial\left(\left|\mathbf{y}_{(j)}-\mathbf{y}_{(k)}\right|\right)}\frac{\mathbf{y}_{(j)}-\mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)}-\mathbf{y}_{(k)}\right|}, \tag{4.1}$$

in which $V_{(j)}$ represents the volume of material point j, and the direction of the force density vector is aligned with the relative position vector in the deformed configuration. The material point j exerts the force density $\mathbf{t}_{(k)(j)}$ on material point k. Determination of the auxiliary parameters requires an explicit form of the strain energy density function.

For an isotropic and elastic material, the explicit form of the strain energy density, $W_{(k)}$, at material point $\mathbf{x}_{(k)}$ can be obtained by generalizing the expression given by Eq. 3.15 as

$$\begin{aligned}W_{(k)}&=a\theta_{(k)}^2-a_2\,\theta_{(k)}\,T_{(k)}+a_3\,T_{(k)}^2\\&\quad+b\sum_{j=1}^{N}w_{(k)(j)}\left(\left(\left|\mathbf{y}_{(j)}-\mathbf{y}_{(k)}\right|-\left|\mathbf{x}_{(j)}-\mathbf{x}_{(k)}\right|\right)-\alpha\,T_{(k)}\left|\mathbf{x}_{(j)}-\mathbf{x}_{(k)}\right|\right)^2V_{(j)},\end{aligned} \tag{4.2}$$

where N represents the number of material points within the family of $\mathbf{x}_{(k)}$. The nondimensional influence function, $w_{(k)(j)}=w\left(\left|\mathbf{x}_{(j)}-\mathbf{x}_{(k)}\right|\right)$, provides a means to control the influence of material points away from the current material point at $\mathbf{x}_{(k)}$. The temperature change at material point k is $T_{(k)}$, with α representing the coefficient

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of thermal expansion. Similarly, the explicit expression for $\theta_{(k)}$ can be obtained from Eq. 3.21 in a general form as

$$\theta_{(k)} = d\sum_{j=1}^{N} w_{(k)(j)}\left(s_{(k)(j)} - \alpha T_{(k)}\right)\frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right|}\cdot\left(\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right)V_{(j)} + 3\,\alpha T_{(k)}, \quad (4.3)$$

in which the PD parameter d ensures that $\theta_{(k)}$ remains nondimensional. The PD material parameters, a, a_2, a_3, and b, in Eq. 4.2 can be related to the engineering material constants of shear modulus, μ, bulk modulus, κ, and thermal expansion coefficient, α, of classical continuum mechanics by considering simple loading conditions.

After substituting for $\theta_{(k)}$ from Eq. 4.3 in the expression for $W_{(k)}$, given by Eq. 4.2, and performing differentiation, the force density vector $\mathbf{t}_{(k)(j)}\left(\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t\right)$ can be rewritten in terms of PD material parameters as

$$\mathbf{t}_{(k)(j)}\left(\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t\right) = \frac{1}{2}A\frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right|}, \quad (4.4a)$$

with

$$\begin{aligned} A = 4w_{(k)(j)}\Bigg\{ & d\,\frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right|}\cdot\frac{\mathbf{x}_{(j)} - \mathbf{x}_{(k)}}{\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|}\left(a\theta_{(k)} - \frac{1}{2}a_2 T_{(k)}\right) \\ & + b\left(\left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right) - \alpha T_{(k)}\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right)\Bigg\}. \end{aligned} \quad (4.4b)$$

Similarly, the force density vector $\mathbf{t}_{(j)(k)}\left(\mathbf{u}_{(k)} - \mathbf{u}_{(j)}, \mathbf{x}_{(k)} - \mathbf{x}_{(j)}, t\right)$ can be expressed as

$$\mathbf{t}_{(j)(k)}\left(\mathbf{u}_{(k)} - \mathbf{u}_{(j)}, \mathbf{x}_{(k)} - \mathbf{x}_{(j)}, t\right) = -\frac{1}{2}B\frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right|}, \quad (4.5a)$$

with

$$\begin{aligned} B = 4w_{(j)(k)}\Bigg\{ & d\,\frac{\mathbf{y}_{(k)} - \mathbf{y}_{(j)}}{\left|\mathbf{y}_{(k)} - \mathbf{y}_{(j)}\right|}\cdot\frac{\mathbf{x}_{(k)} - \mathbf{x}_{(j)}}{\left|\mathbf{x}_{(k)} - \mathbf{x}_{(j)}\right|}\left(a\theta_{(j)} - \frac{1}{2}a_2 T_{(j)}\right) \\ & + b\left(\left(\left|\mathbf{y}_{(k)} - \mathbf{y}_{(j)}\right| - \left|\mathbf{x}_{(k)} - \mathbf{x}_{(j)}\right|\right) - \alpha T_{(j)}\left|\mathbf{x}_{(k)} - \mathbf{x}_{(j)}\right|\right)\Bigg\}. \end{aligned} \quad (4.5b)$$

Although Eqs. 4.4b and 4.5b appear to be similar, they are different because the values of $(\theta_{(k)}, T_{(k)})$ and $(\theta_{(j)}, T_{(j)})$ for the material points at $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$, respectively, are not necessarily equal to each other. However, A and B must be equal to each other for the bond-based PD theory. Therefore, the terms associated with $\theta_{(k)}$ and $\theta_{(j)}$ in Eqs. 4.4b and 4.5b must disappear, thus requiring that

$$ad = 0. \tag{4.6}$$

Thus, the parameter C in Eq. 2.43 becomes

$$C = 4bw_{(k)(j)}\left(\left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right) - \alpha T_{(k)}\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right). \tag{4.7}$$

The force density vector can be rewritten as

$$\begin{aligned}\mathbf{t}_{(k)(j)} = 2bw_{(k)(j)}&\left(\left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right)\right.\\ &\left.- \alpha T_{(k)}\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right)\frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right|}\quad.\end{aligned} \tag{4.8}$$

Based on Eq. 2.43, the bond-based force density vector between the material points at $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$ can be obtained as

$$\mathbf{f}_{(k)(j)} = 4bw_{(k)(j)}\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\left(s_{(k)(j)} - \alpha T_{(k)}\right)\frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right|}\quad. \tag{4.9}$$

Comparing this expression with the bond-based definition of the force density vector, Eq. 2.45 leads to the explicit form of the influence function as

$$w_{(k)(j)} = \frac{c}{4b}\frac{1}{\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|}. \tag{4.10}$$

Performing dimensional analysis on Eq. 4.2 requires that parameter b have dimensions $Force/(Length)^7$ whereas the parameter $c = c_1$ in Eq. 2.45 has dimensions $Force/(Length)^6$. Therefore, the ratio of c/b has a dimension of $Length$, rendering the influence function to be nondimensional. The horizon, δ, can be taken as the *Length* dimension to include the influence of other material points within a family. Thus, the influence (weight) function for the state-based peridynamics becomes

$$w_{(k)(j)} = \frac{\delta}{\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|}. \tag{4.11}$$

Thus, the ratio of c/b is established as

$$\frac{c}{b} = 4\delta. \tag{4.12}$$

Substituting for the influence function results in the final form of the expressions for the force density vectors

$$\mathbf{t}_{(k)(j)} = 2\delta\left\{ d\,\frac{\Lambda_{(k)(j)}}{\left|\mathbf{x}_{(j)}-\mathbf{x}_{(k)}\right|}\left(a\theta_{(k)} - \frac{1}{2}a_2 T_{(k)}\right) + b\left(s_{(k)(j)} - \alpha T_{(k)}\right)\right\} \times \frac{\mathbf{y}_{(j)}-\mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)}-\mathbf{y}_{(k)}\right|}, \tag{4.13}$$

where the parameter, $\Lambda_{(k)(j)}$, is defined as

$$\Lambda_{(k)(j)} = \left(\frac{\mathbf{y}_{(j)}-\mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)}-\mathbf{y}_{(k)}\right|}\right)\cdot\left(\frac{\mathbf{x}_{(j)}-\mathbf{x}_{(k)}}{\left|\mathbf{x}_{(j)}-\mathbf{x}_{(k)}\right|}\right). \tag{4.14}$$

For the bond-based PD theory, the dilatation term $\theta_{(k)}$ must disappear, resulting in

$$\mathbf{t}_{(k)(j)} = 2\delta b\left(s_{(k)(j)} - \alpha T_{(k)}\right)\frac{\mathbf{y}_{(j)}-\mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)}-\mathbf{y}_{(k)}\right|}. \tag{4.15}$$

Based on Eq. 2.43 in conjunction with Eq. 4.12, the bond-based force density vector, $\mathbf{f}_{(k)(j)}$, in Eq. 2.44, becomes

$$\mathbf{f}_{(k)(j)} = c\left(s_{(k)(j)} - \alpha T_{(k)(j)}\right)\frac{\mathbf{y}_{(j)}-\mathbf{y}_{(k)}}{\left|\mathbf{y}_{(j)}-\mathbf{y}_{(k)}\right|}, \tag{4.16}$$

where $T_{(k)(j)} = (T_{(j)} + T_{(k)})/2$. This expression is the same as that given by Silling and Askari (2005), who coined the term “bond-constant” for the parameter c for bond-based peridynamics.

Although all structures are three dimensional in nature, they can be idealized under certain assumptions as one dimensional or two dimensional in order to simplify the computational effort. For instance, long bars can be treated as one-dimensional structures. Similarly, thin plates can be treated as two-dimensional structures. The PD material constants must reflect these idealizations. A two-dimensional plate can be discretized with a single layer of material points in the thickness direction. The spherical domain of integral H becomes a disk with radius δ and thickness h.

A one-dimensional bar can be discretized with a single row of material points. The spherical domain of integral H becomes a line with a length 2δ and cross-sectional area of A.

4.1.1 Three-Dimensional Structures

For three-dimensional analysis, the strain energy density based on classical continuum mechanics can be obtained from

$$W_{(k)} = \frac{1}{2}\boldsymbol{\sigma}_{(k)}^{T}\,\boldsymbol{\varepsilon}_{(k)}, \tag{4.17}$$

in which the stress and strain vectors $\boldsymbol{\sigma}_{(k)}$ and $\boldsymbol{\varepsilon}_{(k)}$ are defined as

$$\boldsymbol{\sigma}_{(k)}^{T} = \left\{ \sigma_{xx(k)} \quad \sigma_{yy(k)} \quad \sigma_{zz(k)} \quad \sigma_{yz(k)} \quad \sigma_{xz(k)} \quad \sigma_{xy(k)} \right\} \tag{4.18a}$$

and

$$\boldsymbol{\varepsilon}_{(k)}^{T} = \left\{ \varepsilon_{xx(k)} \quad \varepsilon_{yy(k)} \quad \varepsilon_{zz(k)} \quad \gamma_{yz(k)} \quad \gamma_{xz(k)} \quad \gamma_{xy(k)} \right\}. \tag{4.18b}$$

For an isotropic material with bulk modulus, κ, and shear modulus, μ, the stress and strain components are related through the constitutive relation as

$$\boldsymbol{\sigma}_{(k)} = \mathbf{C}\,\boldsymbol{\varepsilon}_{(k)}, \tag{4.19}$$

where the material property matrix $\mathbf{C}$ is defined as

$$\mathbf{C} = \begin{bmatrix} \kappa + (4\mu/3) & \kappa - (2\mu/3) & \kappa - (2\mu/3) & 0 & 0 & 0 \\ \kappa - (2\mu/3) & \kappa + (4\mu/3) & \kappa - (2\mu/3) & 0 & 0 & 0 \\ \kappa - (2\mu/3) & \kappa - (2\mu/3) & \kappa + (4\mu/3) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}, \tag{4.20a}$$

with

$$\kappa = \frac{E}{3(1-2\nu)} \text{ and } \mu = \frac{E}{2(1+\nu)}. \tag{4.20b}$$

Two different loading cases resulting in *isotropic expansion* and *simple shear* can be considered to determine the peridynamic parameters a, a_2, a_3, b, and d in terms of engineering material constants of classical continuum mechanics.

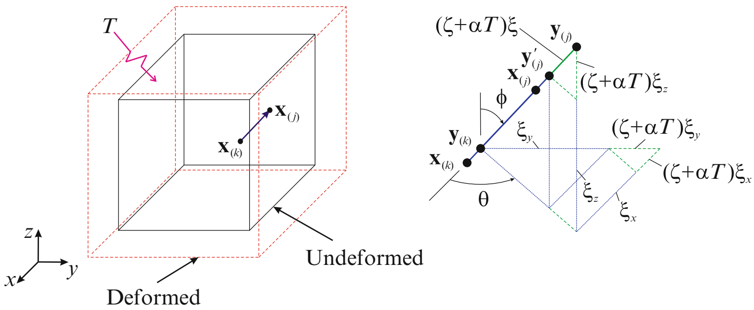

Fig. 4.1 A three-dimensional body subjected to isotropic expansion

As illustrated in Fig. 4.1, a loading case of *isotropic expansion* can be achieved by applying a normal strain of ζ in all directions and a uniform temperature change of T. Thus, the strain components in the body are

$$\varepsilon_{xx(k)} = \varepsilon_{yy(k)} = \varepsilon_{zz(k)} = \zeta + \alpha T \tag{4.21a}$$

and

$$\gamma_{xy(k)} = \gamma_{xz(k)} = \gamma_{yz(k)} = 0, \tag{4.21b}$$

for which the dilatation, $\theta_{(k)}$, and the strain energy density, $W_{(k)}$, within the realm of classical continuum mechanics become

$$\theta_{(k)} = \varepsilon_{xx(k)} + \varepsilon_{yy(k)} + \varepsilon_{zz(k)} = 3\zeta + 3\,\alpha T \tag{4.22a}$$

and

$$W_{(k)} = \frac{9}{2}\kappa\,\zeta^2. \tag{4.22b}$$

The relative position vector between the material points at $\mathbf{x}_{(j)}$ and $\mathbf{x}_{(k)}$ in the deformed configuration becomes

$$\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| = \left(1 + \zeta + \alpha T_{(k)}\right)\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|, \tag{4.23}$$

in which $T_{(k)} = T$.

Defining $\boldsymbol{\xi} = \mathbf{x}_{(j)} - \mathbf{x}_{(k)}$, with $\xi = |\boldsymbol{\xi}|$, and substituting for $w_{(k)(j)}$ from Eq. 4.11 and the relative position vector from Eq. 4.23, the strain energy density, $W_{(k)}$, at material point $\mathbf{x}_{(k)}$ that interacts with other material points within a sphere of radius, δ, from Eq. 4.2 can be evaluated as

$$W_{(k)} = a\,\theta_{(k)}^2 - a_2\,\theta_{(k)}\,T_{(k)} + a_3\,T_{(k)}^2 + b\int_0^{\delta}\int_0^{2\pi}\int_0^{\pi}\frac{\delta}{\xi}\left(\left[\left(1+\zeta+\alpha\,T_{(k)}\right)\xi - \xi\right]\right.$$
$$\left.-\alpha\,T_{(k)}\,\xi\right)^2\xi^2\,\sin(\phi)\;d\phi d\theta d\xi, \tag{4.24}$$

in which (ξ, θ, ϕ) serve as spherical coordinates. After invoking from Eq. 4.22a, its evaluation leads to

$$W_{(k)} = a\left(3\zeta + 3\,\alpha\,T_{(k)}\right)^2 - a_2\left(3\zeta + 3\,\alpha\,T_{(k)}\right)T_{(k)} + a_3\,T_{(k)}^2 + \pi\,b\zeta^2\,\delta^5\,. \tag{4.25}$$

Equating the expressions for strain energy density from Eqs. 4.22b and 4.25 provides the relationships between the PD parameters and engineering material constants as

$$9\,a + \pi\,b\,\delta^5 = \frac{9}{2}\kappa, \tag{4.26a}$$

$$a_2 = 6\,\alpha\,a, \tag{4.26b}$$

$$a_3 = 9\,\alpha^2\,a. \tag{4.26c}$$

Similarly, the expression for $\theta_{(k)}$ from Eq. 4.3 can be recast as

$$\theta_{(k)} = d\int_0^{\delta}\int_0^{2\pi}\int_0^{\pi}\frac{\delta}{\xi}\left(\left[\left(1+\zeta+\alpha\,T_{(k)}\right)\xi - \xi\right] - \alpha\,T_{(k)}\,\xi\right)$$
$$\times\left(\frac{\boldsymbol{\xi}}{\xi}\cdot\frac{\boldsymbol{\xi}}{\xi}\right)\xi^2\,\sin(\phi)d\phi d\theta d\xi + 3\alpha T_{(k)}, \tag{4.27}$$

whose explicit evaluation leads to

$$\theta_{(k)} = \frac{4\,\pi\,d\,\delta^4}{3}\zeta + 3\alpha T_{(k)}. \tag{4.28}$$

Equating the expressions for dilatation from Eqs. 4.22a and 4.28 permits the determination of the peridynamic parameter d as

$$d = \frac{9}{4\,\pi\,\delta^4}. \tag{4.29}$$

As illustrated in Fig. 4.2, a loading case of *simple shear* can be achieved by applying

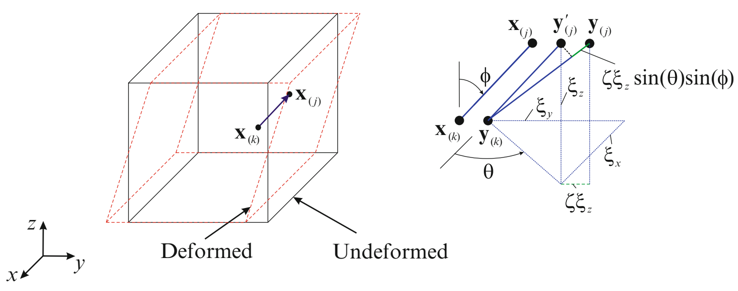

Fig. 4.2 A three-dimensional body subjected to simple shear

$$\gamma_{xy(k)} = \zeta \text{ and } \varepsilon_{xx(k)} = \varepsilon_{yy(k)} = \varepsilon_{zz(k)} = \gamma_{xz(k)} = \gamma_{yz(k)} = T_{(k)} = 0, \tag{4.30}$$

for which the dilatation, $\theta_{(k)}$, and the strain energy density, $W_{(k)}$, within the realm of classical continuum mechanics become

$$\theta_{(k)} = 0 \tag{4.31a}$$

and

$$W_{(k)} = \frac{1}{2}\mu\,\zeta^2. \tag{4.31b}$$

The relative position vector in the deformed state becomes

$$\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| = \left[1 + \frac{\zeta\ \sin(2\phi)\ \sin(\theta)}{2}\right]\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|. \tag{4.32}$$

Therefore, the strain energy density, $W_{(k)}$, from Eq. 4.2 can be evaluated as

$$W_{(k)} = b\int_0^{\delta}\int_0^{2\pi}\int_0^{\pi}\frac{\delta}{\xi}\left(\left[1 + \frac{\zeta\ \sin(2\phi)\ \sin(\theta)}{2}\right]\xi - \xi\right)^2 \xi^2\ \sin(\phi)d\phi d\theta d\xi, \tag{4.33a}$$

whose evaluation leads to

$$W_{(k)} = \frac{b\,\pi\,\delta^5\,\zeta^2}{15}. \tag{4.33b}$$

Equating the strain energy density expressions of Eqs. 4.31b and 4.33b obtained from classical continuum mechanics and the PD theory gives the relationship between the peridynamic parameter b and shear modulus, μ, as

$$b = \frac{15\,\mu}{2\,\pi\,\delta^5}. \tag{4.34}$$

Substituting from Eq. 4.34 into Eq. 4.26a results in the evaluation of the peridynamic parameter a in terms of bulk modulus, κ, and shear modulus, μ, as

$$a = \frac{1}{2}\left(\kappa - \frac{5\,\mu}{3}\right). \tag{4.35}$$

In summary, the PD parameters for a three-dimensional analysis can be expressed as

$$a = \frac{1}{2}\left(\kappa - \frac{5\,\mu}{3}\right), \quad a_2 = 6\,\alpha\,a, \tag{4.36a,b}$$

$$a_3 = 9\,\alpha^2\,a, \quad b = \frac{15\,\mu}{2\,\pi\,\delta^5}, \quad d = \frac{9}{4\,\pi\,\delta^4}. \tag{4.36c–e}$$

In view of Eqs. 4.6 and 4.12, a constraint condition of $\kappa = 5\mu/3$ or $\nu = 1/4$ emerges for bond-based peridynamics with a bond constant of $c = 30\mu/\pi\delta^4$ or $c = 18\kappa/\pi\delta^4$.

4.1.2 Two-Dimensional Structures

Under two-dimensional idealization, the stress and strain vectors $\boldsymbol{\sigma}_{(k)}$ and $\boldsymbol{\varepsilon}_{(k)}$ are defined as

$$\boldsymbol{\sigma}^T_{(k)} = \left\{ \sigma_{xx(k)} \quad \sigma_{yy(k)} \quad \sigma_{xy(k)} \right\} \tag{4.37a}$$

and

$$\boldsymbol{\varepsilon}^T_{(k)} = \left\{ \varepsilon_{xx(k)} \quad \varepsilon_{yy(k)} \quad \gamma_{xy(k)} \right\}. \tag{4.37b}$$

The material property matrix $\mathbf{C}$ in Eq. 4.19 is reduced to

$$\mathbf{C} = \begin{bmatrix} \kappa + \mu & \kappa - \mu & 0 \\ \kappa - \mu & \kappa + \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}. \tag{4.38}$$

Due to two-dimensional idealization, the expression for bulk modulus differs from that given in Eq. 4.20b and is given by

$$\kappa = \frac{E}{2(1-\nu)}. \tag{4.39}$$

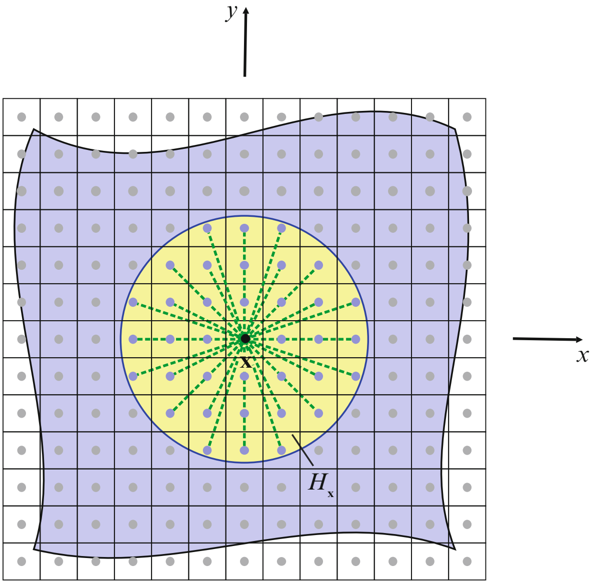

Fig. 4.3 PD horizon for a two-dimensional plate and PD interactions between material point **x** and other material points within its horizon

As shown in Fig. 4.3, a two-dimensional plate is discretized with a single layer of material points in the thickness direction. The domain of integral H in Eq. 2.22a becomes a disk with radius δ and thickness h. As in the previous case, two different loading cases to achieve *isotropic expansion* and *simple shear* are considered to determine the peridynamic parameters.

As illustrated in Fig. 4.4, *isotropic expansion* can be achieved by applying an equal normal strain of ζ in all directions and a uniform temperature change of T. Thus, the strain components in the body are

$$\varepsilon_{xx(k)} = \varepsilon_{yy(k)} = \zeta + \alpha T \quad \text{and} \quad \gamma_{xy(k)} = 0, \tag{4.40}$$

for which the dilatation, $\theta_{(k)}$, and the strain energy density, $W_{(k)}$, within the realm of classical continuum mechanics become

$$\theta_{(k)} = \varepsilon_{xx(k)} + \varepsilon_{yy(k)} = 2\zeta + 2\,\alpha T_{(k)} \tag{4.41a}$$

and

$$W_{(k)} = 2\kappa\,\zeta^2. \tag{4.41b}$$

The relative position vector between the material points at $\mathbf{x}_{(j)}$ and $\mathbf{x}_{(k)}$ in the deformed configuration becomes

$$\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| = \left(1 + \zeta + \alpha T_{(k)}\right)\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|, \tag{4.42}$$

in which $T_{(k)} = T$.

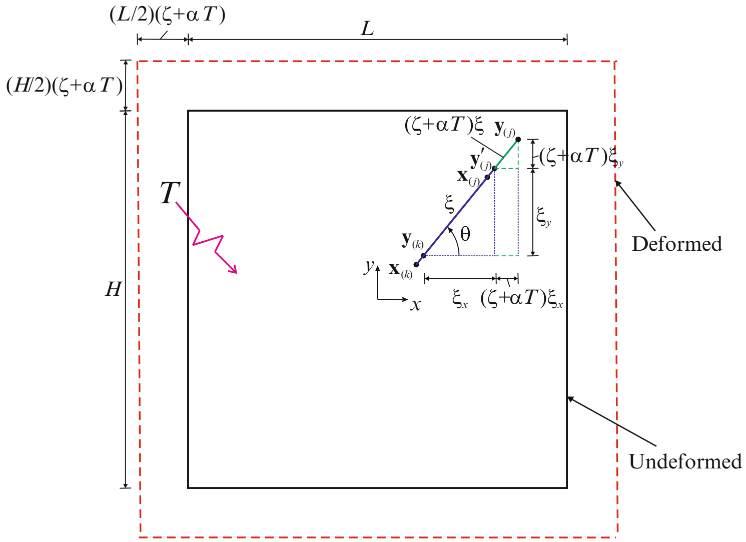

Fig. 4.4 A two-dimensional plate subjected to isotropic expansion

The strain energy density, $W_{(k)}$, at material point $\mathbf{x}_{(k)}$ that interacts with other material points within a disk of radius δ and thickness h from Eq. 4.2 can be evaluated as

$$W_{(k)} = a\,\theta_{(k)}^2 - a_2\,\theta_{(k)}\,T_{(k)} + a_3\,T_{(k)}^2 + b\,h\int_0^{\delta}\int_0^{2\pi} \frac{\delta}{\xi}\left(\left[\left(1+\zeta+\alpha\,T_{(k)}\right)\xi - \xi\right] - \alpha\,T_{(k)}\,\xi\right)^2 \xi d\theta d\xi\ , \tag{4.43}$$

in which (ξ, θ) serve as polar coordinates. While invoking from Eq. 4.41a, its evaluation leads to

$$W_{(k)} = a\left(2\zeta + 2\,\alpha\,T_{(k)}\right)^2 - a_2\left(2\zeta + 2\,\alpha\,T_{(k)}\right)T_{(k)} + a_3\,T_{(k)}^2 + \frac{2}{3}\pi\,b\,h\delta^4\zeta^2\ . \tag{4.44}$$

Equating the expressions for strain energy density from Eqs. 4.41b and 4.44 provides the relationships between the PD parameters and engineering material constants as

$$4a + \frac{2}{3}\pi\,bh\,\delta^4 = 2\kappa, \tag{4.45a}$$

$$a_2 = 4\,\alpha\,a, \tag{4.45b}$$

$$a_3 = 4\,\alpha^2\,a\,. \tag{4.45c}$$

Similarly, the expression for $\theta_{(k)}$ from Eq. 4.3 can be recast as

$$\begin{aligned}\theta_{(k)} = dh\int_0^{\delta}\int_0^{2\pi}\frac{\delta}{\xi}(((1+\zeta+\alpha T)\,\xi-\xi)-\alpha T\,\xi)\\ \times\left(\frac{\boldsymbol{\xi}}{\xi}\cdot\frac{\boldsymbol{\xi}}{\xi}\right)\xi\,d\theta d\xi+2\alpha T_{(k)},\end{aligned} \tag{4.46a}$$

whose explicit evaluation leads to

$$\theta_{(k)} = \pi\,d\,h\delta^3\zeta + 2\alpha\,T_{(k)}. \tag{4.46b}$$

Equating the expressions for dilatation from Eqs. 4.41a and 4.46b permits the determination of the peridynamic parameter d as

$$d = \frac{2}{\pi\,h\delta^3}. \tag{4.47}$$

As illustrated in Fig. 4.5, a loading case of *simple shear* can be achieved by applying

$$\gamma_{xy(k)} = \zeta \text{ and } \varepsilon_{xx(k)} = \varepsilon_{yy(k)} = T_{(k)} = 0, \tag{4.48}$$

for which the dilatation, $\theta_{(k)}$, and the strain energy density, $W_{(k)}$, within the realm of classical continuum mechanics become

$$\theta_{(k)} = 0 \quad \text{and} \quad W_{(k)} = \frac{1}{2}\mu\,\zeta^2. \tag{4.49a,b}$$

The relative position vector in the deformed state becomes

$$\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| = [1 + (\sin\theta\cos\theta)\zeta]\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|. \tag{4.50}$$

Therefore, the strain energy density, $W_{(k)}$, from Eq. 4.2 can be evaluated as

$$W_{(k)} = a\,(0) + bh\int_0^{\delta}\int_0^{2\pi}\frac{\delta}{\xi}([1+(\sin\theta\cos\theta)\zeta]\,\xi-\xi)^2\xi d\theta d\xi, \tag{4.51a}$$

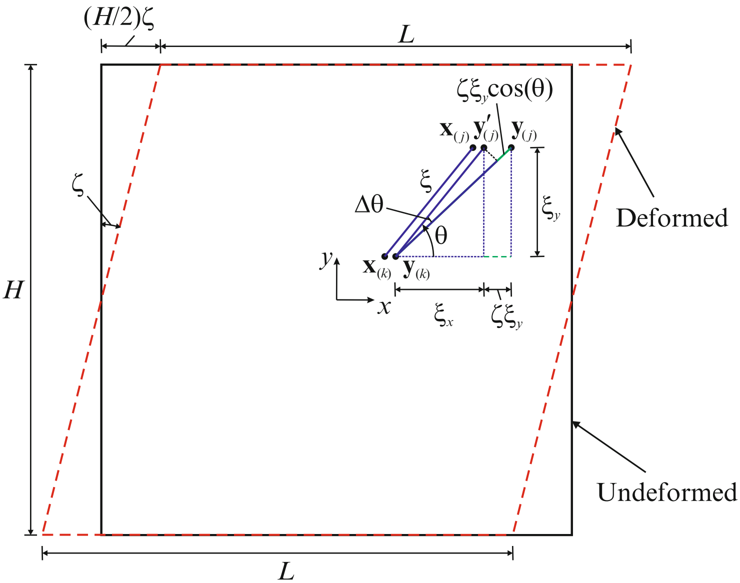

Fig. 4.5 A two-dimensional plate subjected to simple shear

whose evaluation leads to

$$W_{(k)} = \frac{\pi h \delta^4 \zeta^2}{12} b. \tag{4.51b}$$

Equating the strain energy density expressions of Eqs. 4.49a,b and 4.51b obtained from classical continuum mechanics and the PD theory gives the relationship between the peridynamic parameter b and shear modulus, μ, as

$$b = \frac{6\mu}{\pi h \delta^4}. \tag{4.52}$$

Substituting from Eq. 4.52 into Eq. 4.45a results in the evaluation of the peridynamic parameter a in terms of bulk modulus, κ, and shear modulus, μ, as

$$a = \frac{1}{2}(\kappa - 2\mu). \tag{4.53}$$

In summary, the PD parameters for a two-dimensional analysis can be expressed as

$$a = \frac{1}{2}(\kappa - 2\mu), \quad a_2 = 4\,\alpha\, a, \tag{4.54a,b}$$

$$a_3 = 4\,\alpha^2\,a, \qquad b = \frac{6\mu}{\pi h\delta^4}, \qquad d = \frac{2}{\pi\,h\delta^3}. \tag{4.54c–e}$$

In view of Eqs. 4.6 and 4.12, a constraint condition of $\kappa = 2\mu$ or $\nu = 1/3$ emerges for bond-based peridynamics with a bond constant of $c = 24\mu/\pi h\delta^3$ or $c = 12\kappa/\pi h\delta^3$.

4.1.3 One-Dimensional Structures

Under one-dimensional idealization, the nonvanishing stress and strain components are $\sigma_{xx(k)}$ and $\varepsilon_{xx(k)}$. They are related through the Young's modulus as

$$\sigma_{xx(k)} = E\varepsilon_{xx(k)}. \tag{4.55}$$

As illustrated in Fig. 4.6, a bar can be subjected to a uniform stretch of $s = \zeta$ and thermal expansion of loading, αT. Thus, the strain component in the bar is

$$\varepsilon_{xx(k)} = \zeta + \alpha T, \tag{4.56}$$

for which the dilatation, $\theta_{(k)}$, and strain energy density, $W_{(k)}$, within the realm of classical continuum mechanics become

$$\theta_{(k)} = \varepsilon_{xx(k)} = \zeta + \alpha T_{(k)} \tag{4.57a}$$

and

$$W_{(k)} = \frac{1}{2} E\,\zeta^2. \tag{4.57b}$$

As shown in Fig. 4.6, a one-dimensional structure is discretized with a single row of material points. The domain of integral H in Eq. 2.22a becomes a line with a constant cross-sectional area, A.

The relative position vector between the material points at $\mathbf{x}_{(j)}$ and $\mathbf{x}_{(k)}$ in the deformed configuration becomes

$$\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| = \left(1 + \zeta + \alpha T_{(k)}\right) \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|, \tag{4.58}$$

in which $T_{(k)} = T$.

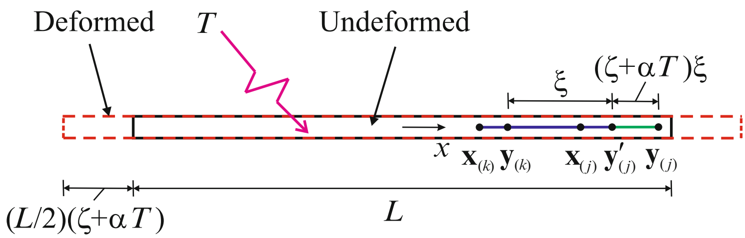

Fig. 4.6 A one-dimensional bar subjected to isotropic expansion

The strain energy density, $W_{(k)}$, at material point $\mathbf{x}_{(k)}$ that interacts with other material points within a line of length δ and area A from Eq. 4.2 can be evaluated as

$$W_{(k)} = a\,\theta_{(k)}^2 - a_2\,\theta\,T_{(k)} + a_3\,T_{(k)}^2 + 2b\,A\int_0^{\delta}\frac{\delta}{\xi}\left(\left[\left(1+\zeta+\alpha\,T_{(k)}\right)\xi-\xi\right]-\alpha\,T_{(k)}\,\xi\right)^2 d\xi, \tag{4.59}$$

in which (ξ) serves as the coordinate. While invoking from Eq. 4.57a, its evaluation leads to

$$W_{(k)} = a\left(\zeta+\alpha\,T_{(k)}\right)^2 - a_2\left(\zeta+\alpha\,T_{(k)}\right)T_{(k)} + a_3\,T_{(k)}^2 + b\,\zeta^2\delta^3 A. \tag{4.60}$$

Assuming a = 0 due to the Poisson's ratio being zero, and equating the expressions for strain energy density from Eqs. 4.57b and 4.60 provides the relationships between the PD parameters and engineering material constants as

$$a_2 = a_3 = 0,\ \text{and}\ b = \frac{E}{2A\delta^3}. \tag{4.61}$$

Similarly, the expression for $\theta_{(k)}$, from Eq. 4.3 can be recast as

$$\theta_{(k)} = 2dA\int_0^{\delta}\frac{\delta}{\xi}\left(\left[\left(1+\zeta+\alpha\,T_{(k)}\right)\xi-\xi\right]-\alpha\,T_{(k)}\,\xi\right)\times\left(\frac{\boldsymbol{\xi}}{\xi}\cdot\frac{\boldsymbol{\xi}}{\xi}\right)d\xi + \alpha T_{(k)}, \tag{4.62a}$$

whose explicit evaluations leads to

$$\theta_{(k)} = 2d\delta^2\zeta A + \alpha T_{(k)}. \tag{4.62b}$$

Equating the expressions for dilatation from Eqs. 4.57a and 4.62b permits the determination of the peridynamic parameter d as

$$d = \frac{1}{2\delta^2 A}. \tag{4.63}$$

In summary, the PD parameters for a one-dimensional structure can be expressed as

$$a = a_2 = a_3 = 0, \quad b = \frac{E}{2A\delta^3}, \quad d = \frac{1}{2\delta^2 A}. \tag{4.64a–c}$$

In view of Eq. 4.12, a bond constant for bond-based peridynamics becomes $c = 2E/A\delta^2$.

4.2 Surface Effects

The peridynamic material parameters a, b, and d that appear in the peridynamic force-stretch relations are determined by computing both dilatation and strain energy density of a material point whose horizon is completely embedded in the material. The values of these parameters, except for a, depend on the domain of integration defined by the horizon. Therefore, the values of b and d require correction if the material point is close to free surfaces or material interfaces (Fig. 4.7). Since the presence of free surfaces is problem dependent, it is impractical to resolve this issue analytically. The correction of the material parameters is achieved by numerically integrating both dilatation and strain energy density at each material point inside the body for simple loading conditions and comparing them to their counterparts obtained from classical continuum mechanics.

For the first simple loading condition, the body is subjected to uniaxial stretch loadings in the x-, y-, and z-directions of the global coordinate system, i.e., $\varepsilon_{xx} \neq 0$, $\varepsilon_{\alpha\alpha} = \gamma_{\alpha\beta} = 0$ (shown in Fig. 4.8), $\varepsilon_{yy} \neq 0$, $\varepsilon_{\alpha\alpha} = \gamma_{\alpha\beta} = 0$, and $\varepsilon_{zz} \neq 0$, $\varepsilon_{\alpha\alpha} = \gamma_{\alpha\beta} = 0$, with $\alpha, \beta = x, y, z$.

The applied uniaxial stretch in the x-, y-, and z-directions is achieved through the constant displacement gradient, $\partial u_\alpha^* / \partial \alpha = \zeta$, with $\alpha = x, y, z$. The displacement field at material point $\mathbf{x}$ resulting from this loading can be expressed as

$$\mathbf{u}_1^T(\mathbf{x}) = \left\{ \frac{\partial u_x^*}{\partial x} x \quad 0 \quad 0 \right\}, \tag{4.65a}$$

$$\mathbf{u}_2^T(\mathbf{x}) = \left\{ 0 \quad \frac{\partial u_y^*}{\partial y} y \quad 0 \right\}, \tag{4.65b}$$

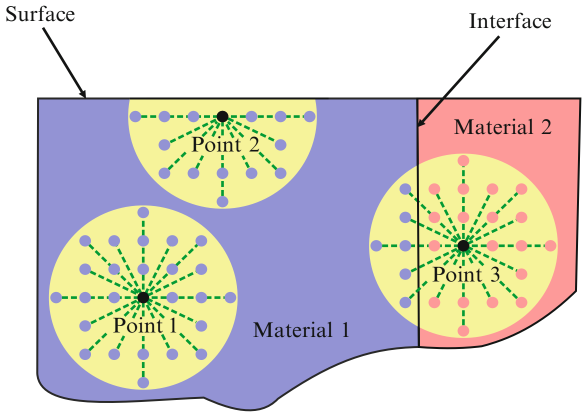

Fig. 4.7 Surface effects in the domain of interest

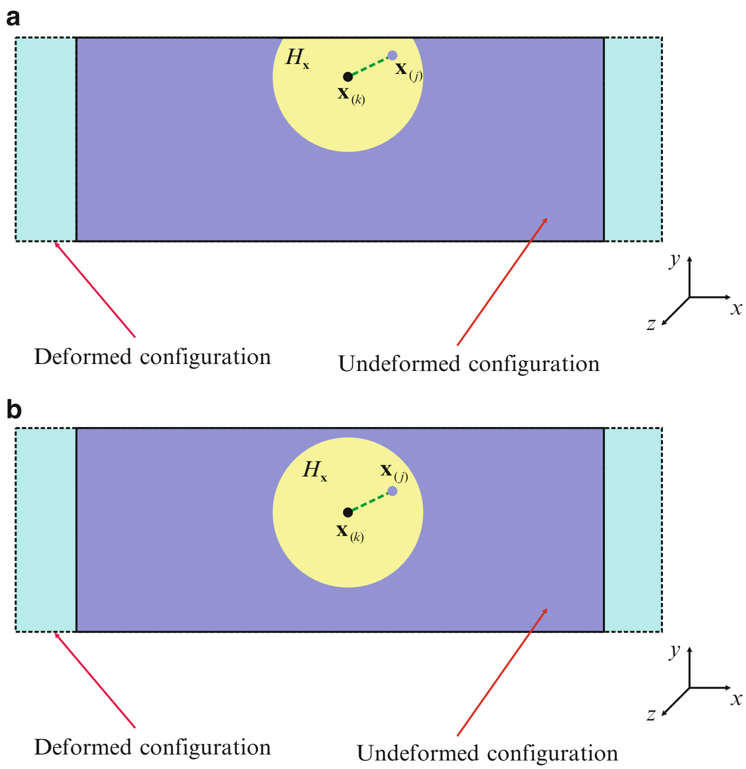

Fig. 4.8 Material point **x** with (**a**) a truncated horizon and (**b**) far away from external surfaces of a material domain subjected to uniaxial stretch loading

and

$$\mathbf{u}_3^T(\mathbf{x}) = \left\{ 0 \quad 0 \quad \frac{\partial u_z^*}{\partial z} z \right\}, \tag{4.65c}$$

in which the subscripts (1, 2, 3) denote the x-, y-, and z-directions of uniaxial stretch, respectively. Due to this displacement field, the corresponding PD dilatation term, $\theta_m^{PD}(\mathbf{x}_{(k)})$ with $(m = 1, 2, 3)$, at material point $\mathbf{x}_{(k)}$ can be obtained from Eq. 4.3 as

$$\theta_m^{PD}(\mathbf{x}_{(k)}) = d\,\delta \sum_{j=1}^{N} s_{(k)(j)} \Lambda_{(k)(j)} V_{(j)}, \tag{4.66}$$

in which N represents the number of material points inside the horizon of material point $\mathbf{x}_{(k)}$. The corresponding dilatation based on classical continuum mechanics, $\theta_m^{CM}(\mathbf{x}_{(k)})$, is uniform throughout the domain and is determined as

$$\theta_m^{CM}(\mathbf{x}_{(k)}) = \zeta\,. \tag{4.67}$$

The dilatation correction term can be defined as

$$D_{m(k)} = \frac{\theta_m^{CM}(\mathbf{x}_{(k)})}{\theta_m^{PD}(\mathbf{x}_{(k)})} = \frac{\zeta}{d\,\delta \sum\limits_{j=1}^{N} s_{(k)(j)} \Lambda_{(k)(j)} V_{(j)}}\,. \tag{4.68}$$

Maximum values of dilatation occur in the loading directions that coincide with the global coordinates x, y, and z, respectively.

Similarly, the strain energy density at any material point can be computed due to simple shear loading in the $(x' - y')$, $(x' - z')$, and $(y' - z')$ planes, i.e., $\gamma_{x'y'} \neq 0$, $\varepsilon_{\alpha\alpha} = \gamma_{\alpha\beta} = 0$ (shown in Fig. 4.9), $\gamma_{x'z'} \neq 0$, $\varepsilon_{\alpha\alpha} = \gamma_{\alpha\beta} = 0$, and $\gamma_{y'z'} \neq 0$, $\varepsilon_{\alpha\alpha} = \gamma_{\alpha\beta} = 0$, with $\alpha, \beta = x', y', z'$. This loading is achieved through constant displacement gradient $\partial u_\alpha^* / \partial \beta = \zeta$, with $\alpha \neq \beta$ and $\alpha, \beta = x', y', z'$. These planes are oriented by an angle of $-45°$ in reference to the $(x - y)$, $(x - z)$, and $(y - z)$ planes of the global coordinate system. The loading on these planes is considered because the maximum strain energy occurs in the x-, y-, and z-directions.

The displacement field at material point $\mathbf{x}$ resulting from the applied simple shear loading in the $(x' - y')$, $(x' - z')$, and $(y' - z')$ planes can be expressed in the global coordinate system as

$$\mathbf{u}_1^T(\mathbf{x}) = \left\{ \frac{1}{2} \frac{\partial u_{x'}^*}{\partial y'} x \quad -\frac{1}{2} \frac{\partial u_{x'}^*}{\partial y'} y \quad 0 \right\}, \tag{4.69a}$$

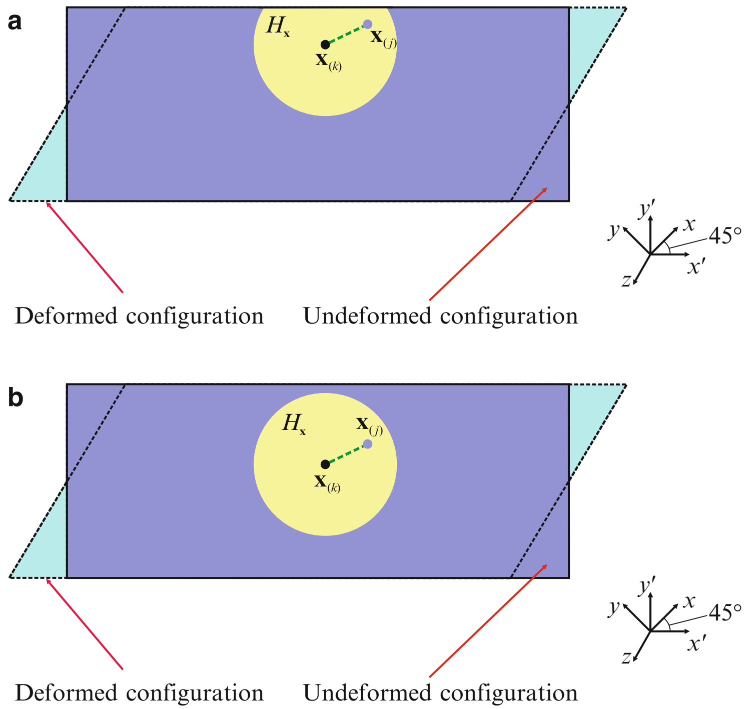

Fig. 4.9 Material point **x** with (**a**) a truncated horizon and (**b**) far away from external surfaces of a material domain subjected to simple shear loading

$$\mathbf{u}_2^T(\mathbf{x}) = \left\{ 0 \quad \frac{1}{2}\frac{\partial u_{y'}^*}{\partial z'}y \quad -\frac{1}{2}\frac{\partial u_{y'}^*}{\partial z'}z \right\}, \tag{4.69b}$$

$$\mathbf{u}_3^T(\mathbf{x}) = \left\{ -\frac{1}{2}\frac{\partial u_{z'}^*}{\partial x'}x \quad 0 \quad \frac{1}{2}\frac{\partial u_{z'}^*}{\partial x'}z \right\}, \tag{4.69c}$$

in which the subscripts $(1, 2, 3)$ denote the applied simple shear loadings in the $(x' - y')$, $(y' - z')$, and $(x' - z')$ planes, respectively.

Due to these applied displacement fields, the PD strain energy density at material point $\mathbf{x}_{(k)}$ can be obtained from Eq. 4.2 as

$$W_m^{PD}(\mathbf{x}_{(k)}) = a\left(\theta_m^{PD}(\mathbf{x}_{(k)})\right)^2 + b\,\delta \sum_{j=1}^{N} \frac{1}{\left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|}\left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right)^2 V_{(j)}, \tag{4.70}$$

with $(m = 1, 2, 3)$.

Under simple shear loading, the dilatation and strain energy densities can be computed by using classical continuum mechanics as

$$\theta_m^{CM}(\mathbf{x}_{(k)}) = 0, \quad W_m^{CM}(\mathbf{x}_{(k)}) = \frac{1}{2}\mu\zeta^2, \tag{4.71a,b}$$

with $(m = 1, 2, 3)$.

The dilatation term, $\theta_m^{PD}(\mathbf{x}_{(k)})$, is expected to vanish for this loading condition because it is already corrected with a dilatation correction term, Eq. 4.68. Thus, the strain energy density term reduces to

$$W_m^{PD}(\mathbf{x}_{(k)}) = b\,\delta \sum_{j=1}^{N} \frac{1}{|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}|}\left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right)^2 V_{(j)}. \tag{4.72}$$

Hence, the correction term is only necessary for the term including parameter b and can be defined as

$$S_{m(k)} = \frac{W_{(m)}^{CM}(\mathbf{x}_{(k)})}{W_{(m)}^{PD}(\mathbf{x}_{(k)})} = \frac{\frac{1}{2}\mu\zeta^2}{b\,\delta \sum\limits_{j=1}^{N} \frac{1}{|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}|}\left(\left|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}\right| - \left|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}\right|\right)^2 V_{(j)}} \; . \tag{4.73}$$

With these expressions, a vector of correction factors for the integral terms in dilatation and strain energy density at material point $\mathbf{x}_{(k)}$ can be written as

$$\mathbf{g}_{(d)}(\mathbf{x}_{(k)}) = \left\{g_{x(d)(k)},\ g_{y(d)(k)}, g_{z(d)(k)}\right\}^T = \left\{D_{1(k)},\ D_{2(k)}, D_{3(k)}\right\}^T, \tag{4.74a}$$

$$\mathbf{g}_{(b)}(\mathbf{x}_{(k)}) = \left\{g_{x(b)(k)},\ g_{y(b)(k)}, g_{z(b)(k)}\right\}^T = \left\{S_{1(k)},\ S_{2(k)}, S_{3(k)}\right\}^T. \tag{4.74b}$$

These correction factors are only valid in the x-, y-, and z-directions. However, they can be used as the principal values of an ellipsoid, as shown in Fig. 4.10, in order to approximate the surface correction factor in any direction. Arising from a general loading condition, the correction factor for interaction between material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$, shown in Fig. 4.11a, can be obtained in the direction of their unit relative position vector, $\mathbf{n} = (\mathbf{x}_{(j)} - \mathbf{x}_{(k)})/|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}| = \{n_x,\ n_y,\ n_z\}^T$.

A vector of correction factors for the integrals in the dilatation and strain energy density expressions at material point $\mathbf{x}_{(j)}$ can be similarly written as

$$\mathbf{g}_{(d)(j)}(\mathbf{x}_{(j)}) = \left\{g_{x(d)(j)},\ g_{y(d)(j)}, g_{z(d)(j)}\right\}^T = \left\{D_{1(j)},\ D_{2(j)}, D_{3(j)}\right\}^T, \tag{4.75a}$$

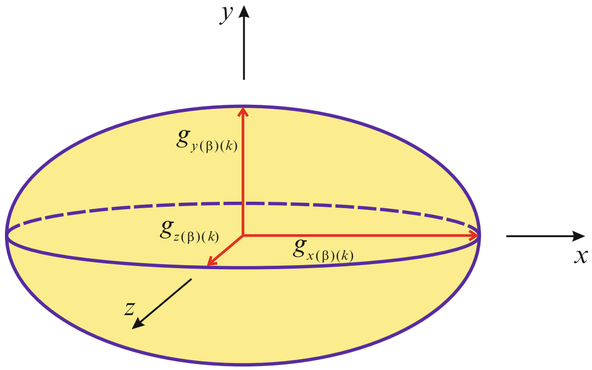

Fig. 4.10 Construction of an ellipsoid for surface correction factors

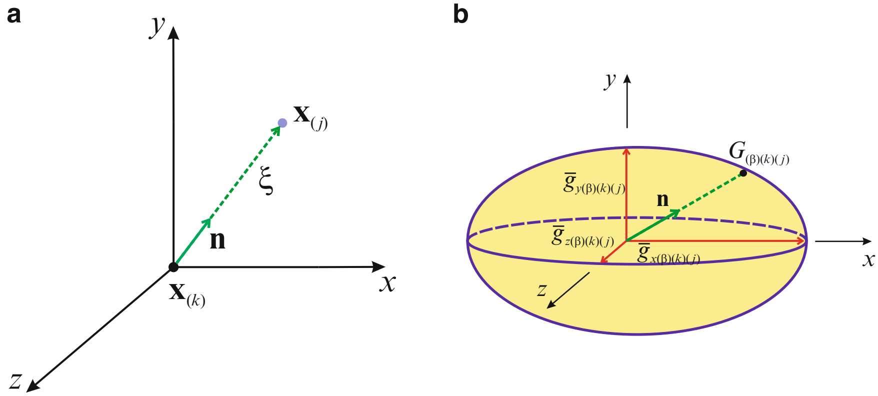

Fig. 4.11 (**a**) PD interaction between material points at $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$ and (**b**) the ellipsoid for the surface correction factors

$$\mathbf{g}_{(b)(j)}(\mathbf{x}_{(j)}) = \left\{g_{x(b)(j)},\ g_{y(b)(j)}, g_{z(b)(j)}\right\}^T = \left\{S_{1(j)},\ S_{2(j)}, S_{3(j)}\right\}^T . \tag{4.75b}$$

These correction factors are, in general, different at material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$. Therefore, the correction factor for an interaction between material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$ can be obtained by their mean values as

$$\bar{\mathbf{g}}_{(\beta)(k)(j)} = \left\{\bar{g}_{x(\beta)(k)(j)}, \bar{g}_{y(\beta)(k)(j)}, \bar{g}_{z(\beta)(k)(j)}\right\}^T = \frac{\mathbf{g}_{(\beta)(k)} + \mathbf{g}_{(\beta)(j)}}{2}, \tag{4.76}$$

with $\beta = d, b$,

which can be used as the principal values of an ellipsoid, as shown in Fig. 4.11b.

The intersection of the ellipsoid and a relative position vector, $\mathbf{n} = (\mathbf{x}_{(j)} - \mathbf{x}_{(k)})/|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}|$, of material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$ provides the correction factors as

$$G_{(\beta)(k)(j)} = \left(\left[n_x \Big/ \bar{g}_{x(\beta)(k)(j)} \right]^2 + \left[n_y \Big/ \bar{g}_{y(\beta)(k)(j)} \right]^2 + \left[n_z \Big/ \bar{g}_{z(\beta)(k)(j)} \right]^2 \right)^{-1/2} . \quad (4.77)$$

After considering the surface effects, the discrete forms of the dilatation and the strain energy density can be corrected as

$$\theta_{(k)} = d\delta \sum_{j=1}^{N} G_{(d)(k)(j)} s_{(k)(j)} \Lambda_{(k)(j)} V_{(j)}, \quad (4.78a)$$

$$W_{(k)} = a\,\theta_{(k)}^2 - a_2 \theta_{(k)} T_{(k)} + a_3 T_{(k)}^2 + b\delta \sum_{j=1}^{N} G_{(b)(k)(j)} \frac{1}{\left| \mathbf{x}_{(j)} - \mathbf{x}_{(k)} \right|} \left(\left| \mathbf{y}_{(j)} - \mathbf{y}_{(k)} \right| - \left| \mathbf{x}_{(j)} - \mathbf{x}_{(k)} \right| \right)^2 V_{(j)} . \quad (4.78b)$$

Reference

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