Chapter 9
Partial Orders in Socio-economics: A Practical Challenge for Poset Theorists or a Cultural Challenge for Social Scientists?

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Abstract In this "position paper" we discuss the potential role of partial order theory in socio-economic statistics and social indicators construction. We maintain that the use of concepts and tools from poset theory is needed and urgent to improve currently adopted methodologies, which often prove ineffective for exploiting ordinal data. We also point out that the difficulties in spreading partial order tools are cultural in nature, and that some open-mindedness is needed among social scientists. We address these issues introducing some examples of open questions in socio-economic data analysis: (i) the problem of multidimensional poverty evaluation, (ii) the problem of assessing inequality and societal polarization, and (iii) the problem of clustering in multidimensional ordinal datasets.

9.1 Introduction

During a workshop held in Italy in 2010, a full professor in Statistics, presenting an evaluation study pertaining to service quality and based on ordinal data, made a statement like: "…here we're dealing with ordinal data, so there is no room for mathematics and statistics." The speaker was certainly aware of the number of methodologies in the statistical literature for dealing with ordinal variables. Yet the statement reveals something true and, somehow, interesting. Still today, when social

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scientists address multidimensional complex[1] problems involving qualitative data (like service quality evaluation studies) they feel basically uncomfortable and consider such problems, in a sense, "ill-posed." Unfortunately, this perception conflicts with the following evidences: (i) more and more crucial socio-economic issues may be meaningfully described only by involving ordinal information (e.g., material deprivation and multidimensional poverty, subjective well-being or quality-of-life, customer satisfaction and service quality perception, to mention a few) and (ii) de facto, more and more socio-economic datasets offer ordinal data to scholars and researchers. The question is therefore whether this means that we cannot adequately describe and understand socio-economic facts, or whether this is still possible, but requires some change of paradigms and tools. In this position paper, we address this issue. We try to identify the logical roots of the problem and to reveal the interconnections between them and the need for new statistical tools, stressing the role of partial order theory and of a "partial order culture," to overcome limitations of current statistical practice. We will pursue this, identifying some questions that are still unsolved in applied socio-economic research and that could be (and are being) fruitfully addressed through partial order theory. The proposed open issues are in no way intended to be an exhaustive list. They have been chosen based on our research interests and experience; still, they are really urgent problems and are useful to explain our position. Given the aims of the paper, we will not go into technical details, which would lead us to a long and complicated exposition. Rather, we will focus on the essentials which better clarify the issues we are addressing.

9.2 Old Paradigms and Open Questions

Among statisticians and social scientists, there is a widespread feeling that sound scientific knowledge may be achieved only when precise measurements may be attained. This idea comes from natural sciences, physics in particular, and was embodied in social sciences from the late nineteenth century to the first half of twentieth century. This is one of the root problems, often preventing ordinal data to be considered valuable. Clearly, we are not arguing against the relevance of measuring socio-economic facts precisely, whenever possible. The question one should answer is in fact different: "How can we obtain faithful representations of socio-economic facts?" Sometimes, faithful representations may be built using precise measurement models, as in physics, sometimes not. We may, at least ideally, measure with great precision prices and quantities of different goods to account for

[1] In this paper, we often use terms like "complex" or "complexity." We use them informally, to mean problems or systems that cannot easily be solved, reduced or described, as they are made of many linked elements or facets. Perhaps the formal definition of complexity which is closest in spirit to the way we employ the term is that of "Kolmogorov complexity," used in algorithmic information theory. However, we stress that this is only an analogy.

inflation,[2] but we cannot claim to faithfully represent in numerical terms ordinal issues like the democracy level of a country or subjective well-being or quality of services. Representing a physical, economic, or social phenomenon means identifying its essential features and the interrelations among its components and sketching them in a suitable formalism, so that by performing formal computations one may get insights into it. Often, social scientists simply code qualitative information into numerical scores and proceed to analytical computations. Sometimes, prior to computations they employ complex algorithms to turn qualitative variables into numerical scales. In any case, the basic questions one should answer are: "What do these numbers represent? May we assume the results are effective to understand what we are interested in? Do our computations convey valid information, reflecting elements of reality?" We admit that sometimes the answer may be positive, so one may fruitfully proceed this way. But it is important to pose the question.

The problem, in fact, is not just epistemological, but very practical. Today there is a great amount of ordinal information available to social scientists, and focus is shifting toward qualitative socio-economic issues, like assessing well-being, quality-of-life, or multidimensional poverty. Still, social scientists approach these topics with "numerical" paradigms, using methodologies and computational procedures designed to deal with quantitative information. So the problem often becomes how to fit ordinal data into well-established and routinely used procedures, rather than how to build new appropriate methodologies. When ordinal data is used in this way, the risk is getting questionable and biased results, which affect our understanding of social facts. This matter of fact is also due to the implicit assumption that ordinal data cannot be handled in a consistent and effective way, since no formal tools are available; so even those who are aware of the problem cannot easily see any way out. The use of partial order theory and other related tools from discrete mathematics and relational calculus have not spread into the "methodological imagination" of social scientists' community yet, and there is little awareness of the possibilities that they may open.

A paradigmatic example comes from the problem of extending classical socio-economic indicators (e.g., inequality indices) to multidimensional ordinal datasets. This is one of the core issues in current research, since any attempt to represent modern societies and their complexity requires taking into account many different aspects jointly. Historically, there has been a great deal of research on giving sound mathematical and axiomatic foundations to the theory of statistical indices (consider, e.g., the theories of price indices, poverty indices, or concentration indices). It is much more difficult to achieve multidimensional extensions of these axiomatic systems and, usually, results are less neat and general. In the case of ordinal variables the situation is even worse, since systematic theories of this kind are still lacking, even if some attempts are being made. Unfortunately, the use of statistical indicators is the basis of many socio-economic studies and, in many cases, the

[2] Measuring inflation should involve also measuring services and it is quite debatable how to precisely define the concept of quantity in this case.

absence of effective tools for ordinal data forces social scientists to fall back on numerical representations.

On the whole, epistemological difficulties, lack of awareness about possible alternatives, and unsuitable statistical tools are major obstacles for the development of statistical methodologies capable of exploiting ordinal data and answering the information needs of researchers, policy-makers, and citizens. In the following paragraphs, we illustrate these issues and give some ideas about the role of partial order theory, by means of three examples of open questions in applied socio-economic statistics: the evaluation of multidimensional poverty and well-being, the measurement of inequality and polarization in a multidimensional ordinal setting, and the development of procedures to perform cluster analysis on ordered structures.

9.2.1 *Evaluating Multidimensional Poverty: A Matter of Multidimensional Comparison?*

One of the most relevant examples in socio-economic analysis where the issue of multidimensional ordinal data is crucial is the wide field of evaluation studies pertaining to quality-of-life, well-being, and multidimensional poverty. Following the Commission on the Measurement of Economic Performance and Social Progress (the "Stiglitz-Sen-Fitoussi Commission"), several attempts to assess well-being and to go beyond GDP (Gross Domestic Product) as a measure of societal wealth are being pursued in many countries. The authors of this paper are involved in the Italian project for the construction of official well-being indicators,[3] promoted by CNEL (National Council of Economy and Work) and ISTAT (National Institute of Statistics). Twelve well-being dimensions have been identified (e.g., health, education, work, social relations, and environment), each comprising different indicators, both numerical and ordinal (e.g., those pertaining to subjective well-being). One of the main issues under discussion is whether to produce synthetic indicators and how; perhaps computing composite indicators? The drawbacks of aggregative and compensative procedures of this kind are well known (see, e.g., Fattore et al. 2012), but often no alternatives are pursued. Given the impact of official well-being statistics on public opinion and policy-makers, it is clear that any choice about how to produce final indicators requires great care. In social evaluation studies, the aggregation problem is at least twofold:

1. There is a technical issue when ordinal data are at hand, since in that case usual procedures designed for numerical variables break down and no aggregation can be performed directly. To overcome this problem, various procedures are often implemented to transform ordinal data into numerical figures,[4] so as to apply

[3] http://www.misuredelbenessere.it.

[4] These so-called *scaling tools* range from simply coding and using ordinal scores as integers, to running complex numerical algorithms, like in the Gifi homogeneity analysis (Michailides and de Leeuw 1998), or using various regression or model-based approaches.

aggregation procedures. Unfortunately, the existence of latent numerical scales behind ordinal data may often be questioned. Moreover, scaling procedures often generate numerical figures minimizing some loss function, so in practice introducing into the analysis an optimization criterion that need not be intrinsic to the data, albeit mathematically appealing. Therefore, one may legitimately ask whether the final figures produced that way actually give a faithful representation of the underlying socio-economic facts or are just the output of numerical algorithms with a limited capability of enlightening data.

2. There is also a general conceptual problem. The basic assumption behind the development of aggregated indicators is the existence of one main latent dimension accounting for most data variability, so that by exploiting variable interdependencies one may hope to reduce data complexity. As a matter of fact, evaluation dimensions are often weakly interdependent and, even conceptually, one cannot accomplish any satisfactory synthesis, drawing on the principle of explaining joint variability. We remark that this problem does not depend upon the nature (cardinal or ordinal) of the variables to handle. It is intrinsic to the true multidimensionality of the concepts related to quality-of-life, which often prevents the aggregative procedure from getting meaningful results. What makes social evaluation studies challenging is precisely this feature; the evaluation problem is not reducible to aggregation.

In practice, and more and more often, the two problems combine together, making the development of synthetic indicators more demanding for statisticians, who must find new technical tools to build them, and more urgent for policy-makers, who need them to interpret even more complex societies. The current debate on these problems is quite heated. An interesting issue of the Journal of Economic Inequality published in 2011, hosting a forum on the topic, is particularly enlightening of the state-of-the-art. The main debate is polarized around two different positions: that of Alkire and Foster, who propose their aggregative counting approach to the measurement of multidimensional poverty (see Alkire and Foster 2011a and Alkire and Foster 2011b), and that of Ravaillon Ravaillon [2011], who suggests avoiding any synthetic procedures, in favor of using dashboards (panels of indicators). The Alkire–Foster procedure is perhaps the most consistent framework to assess multidimensional poverty based on both ordinal and cardinal indicators, and its use is spreading. The structure of the procedure is very simple and can be described as follows.

Let $T_{n\times k}$ be the data matrix, comprising the scores of n statistical units on k evaluation dimensions $v_1,\dots,v_k$ (i.e., each row of the data matrix contains the profile of the corresponding statistical unit). Then the following steps are implemented:

1. a set of $c_1,\dots,c_k$ cutoffs is exogenously chosen, one for each evaluation dimension;
2. each individual is assessed against the cutoffs and is declared deprived on v_i if his/her score on that dimension is less than c_i;

3. matrix $T_{n\times k}$ is transformed into a binary matrix $G_{n\times k}$, where $G_{ij}=1$ if individual i is deprived on dimension j and $G_{ij}=0$ otherwise;
4. the rows of G are then summed up, possibly weighting each column with weights expressing the relative importance of being deprived on the various dimensions;
5. finally, an individual is declared definitely deprived if his/her overall score is equal to or greater than an overall cutoff c, exogenously chosen.

In practice, this procedure leads to the definition of an identification function which classifies individuals as deprived or not in a binary way. Once individuals have been classified, several poverty indicators may be computed (for a complete discussion see Alkire and Foster 2011a). Notice that irrespective of dealing with cardinal or ordinal variables, the Alkire–Foster procedure turns the original data matrix into binary matrix G and applies a weighted aggregation function (i.e., a weighted sum) to its rows. If all of the weights are set to 1, the methodology simply counts individual deprivations.

The debated point is essentially on the meaningfulness and utility of aggregating indices using weights. It is instructive to quote the final comment of the Forum Editor (Lustig 2011):

> At the bottom of the discussion is a fundamental disagreement on the "legitimacy" of the weights used to aggregate dimensions of wellbeing […] Ravallion and those who agree with him consider that the alternative weights used in the MPI (or similar indices) are not a good solution as they may imply unappealing trade-offs and that these aggregate poverty measures are generally not consistent with consumer welfare theory.[…] Thus, given this problem and the fact that for policy purposes disaggregation will be required, Ravallion asks: what is the advantage of using composite indices […] instead of a "dashboard" of multiple indices? One key unresolved issue in the "dashboard approach", however, is that if we agree that welfare depends on a series of dimensions, how do we address the fact that the marginal effect of increasing an individual's access to one of the dimensions (e.g., health services) depends not only on that individual's access to the dimension in question, but also on the individual's level of all the other indicators of welfare?
>
> Future research will need to focus on how to identify weights in ways that are consistent (1) with welfare economics and (2) with theories of justice. Will we have to choose between the two?

From the last sentence we see that the weighting problem is considered as the central issue. But weighting is a consistent operation only in a numerical setting, so what about ordinal data? Basically, we are left with two alternatives: (i) scaling ordinal scores to cardinals and proceeding to usual computations, getting arguable results or (ii) sticking to the Alkire–Foster procedure, turning ordinal scores into binary scores and counting, possibly with weights, losing a great deal of information on the degree of individual and societal poverty (Fattore et al. 2011b). Both cases seem to be driven by the (presumed) impossibility of exploiting ordinal data on their own. The debate goes on trapped within the "weight and aggregate" framework, the problem being to search for more sophisticated weighting procedures or, as a radical alternative, to abandon synthetic indicators.

In our view, the way out of this trap is via some change of paradigm:

1. no longer considering “synthetic indicator” and “aggregated indicator” as equivalent concepts;
2. considering that evaluation processes could be better addressed as problems of multidimensional comparisons against suitable benchmarks, rather than problems of aggregation;
3. realizing that ordinal data may be consistently handled, with appropriate mathematical tools.

In the wider literature about evaluation, the terms “synthetic” and “aggregated” are used interchangeably and it is taken for granted that to get synthetic information, some aggregation procedure is needed. Since aggregating basically requires summing up scores, we are inevitably led back to the problem of incompatibility between analytical tools and ordinal data. The problem would be solved if we could get synthetic indicators from ordinal data without aggregating variables. This is indeed possible, provided we reconsider the evaluation process as a multidimensional comparison problem and employ partial order theory to address it. In assessing multidimensional poverty and similar issues, no natural measuring scale exists and, implicitly or explicitly, assessments are often based on the selection of some “reference points” or of some “prototypes,” to be used as benchmarks.[5] In the unidimensional case, e.g., monetary thresholds are adopted and individuals’ income is compared against them. In a multidimensional setting the concept of benchmark is more complex. First, multidimensional poverty may assume different shapes, i.e., there are “several ways” to be poor and so more benchmarks are needed; second, being poor or not depends upon the individual’s scores on all the dimensions of concern (i.e., his/her profile), so benchmarks should be identified in terms of prototypical score configurations. Assessing the poverty state of an individual therefore means comparing his score configuration to those constituting the benchmarks. Being a problem of multidimensional comparison, partial ordered theory naturally comes into play.

This idea is currently being pursued by the authors and other colleagues, e.g., in Fattore et al. [2011a,b], and Fattore et al. [2012]. Without entering into technical details, the basic idea is quite simple. Let $v_1, \ldots, v_k$ be k ordinal evaluation dimensions. To each individual in the population, a profile $\mathbf{p} = (p_1, \ldots, p_k)$ is associated, whose components are the scores of the statistical unit on the evaluation dimensions. The set P of profiles is turned into a partially ordered set $(P, \trianglelefteq)$ defining

$$\mathbf{p} \trianglelefteq \mathbf{q} \Leftrightarrow p_i \leq q_i \quad \forall i = 1, \ldots, k. \tag{9.1}$$

In this framework, a multidimensional poverty threshold τ is a minimal set of profiles[6] such that any profile below[7] one of its elements is classified as poor. Given

[5] Whenever cutoffs or thresholds are involved in the assessment, benchmarks are de facto introduced.

[6] That is, the smallest set of profiles with the cited property.

[7] Here, we assume that the lower the scores, the worse off the individual.

the threshold, any other profile may be assessed in terms of poverty, based on its position with respect to τ, in the Hasse diagram of the profile poset. The multidimensional comparisons involving the profiles and the threshold are performed counting over linear extensions of $(P, \lhd)$ how frequently a profile is classified below an element of the threshold (see Fattore et al. 2011a for details). The resulting *evaluation function* assigns to each profile (and thus to any individual sharing it) a score in $[0, 1]$, representing the degree of poverty, given τ. In practice, the procedure quantifies the degree of ambiguity in the classification of a profile into the set of poor profiles and may be better interpreted as a way to compute a fuzzy membership function. What is relevant here, is that such a quantification does not involve any ordinal variable scaling; the focus is on profiles and information is extracted out of the mathematical structure representing the basic relation existing among them, i.e., the partial order relation. The resulting evaluation procedure, even if heavier from a computational point of view, is more effective and general than the Alkire–Foster counting approach, which in fact may be seen as a special case of the former (for a complete comparison, see Fattore et al. 2011b). Apart from these technicalities, the interesting feature of poset-based evaluation procedures is that they show how to exploit ordinal data, implementing the same logical structure of classical unidimensional evaluation studies.[8]

To provide some insights into the poset approach to evaluation, we briefly outline the example reported in Fattore et al. [2011a]. Five deprivation variables have been considered, from the EU-SILC survey pertaining to Italy, for year 2004:

1. HS040—Capacity to afford paying for one week annual holiday away from home;
2. HS050—Capacity to afford a meal with meat, chicken, fish (or vegetarian equivalent) every second day;
3. HS070—Owning a phone (or mobile phone);
4. HS080—Owning a color TV;
5. HS100—Owning a washing machine.

All of the variables are expressed on a yes/no scale, so that deprivation profiles are sequences of five 0/1 digits (1: non-deprivation; 0: deprivation). Clearly, there are $2^5 = 32$ different profiles. The threshold has been chosen as $\tau = (01011, 00111)$, i.e., deprivation on HS040 and HS070, or deprivation on HS040 and HS050.[9] Figure 9.1 reports the Hasse diagram of the profile poset, with the threshold elements in black. The top element corresponds to profile 11111, the bottom to profile 00000. Profiles with the same number of 1s have the same distance from the bottom element (the distance is measured as the number of edges in a downward path from the profile to the bottom). All of the elements of the threshold and all of the profiles below one of them are scored 1 (i.e. unambiguously deprived) by the evaluation

[8] In Fattore et al. [2012], e.g., it is also suggested how the classical notion of "weighting" variables may be translated into purely poset terms.

[9] This threshold has been chosen for exemplification purposes only.

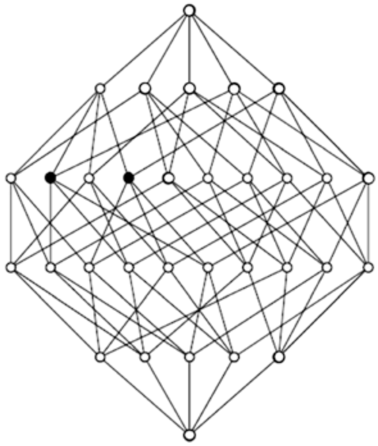

Fig. 9.1 Profile poset on a set of five binary variables (threshold elements in *black*)

Table 9.1 Evaluation function for the elements of the profile poset depicted in Fig. 9.1, given the threshold $\boldsymbol{\tau} = (01011, 00111)$

Profile	11111	11110	11101	11100	11011	11010	11001	11000
Evaluation	0.00	0.11	0.11	0.65	0.06	0.66	0.66	0.98
Profile	10111	10110	10101	10100	10011	10010	10001	10000
Evaluation	0.6	0.66	0.66	0.98	0.67	0.98	0.98	1.00
Profile	01111	01110	01101	01100	01011	01010	01001	01000
Evaluation	0.00	0.67	0.67	0.98	1.00	1.00	1.00	1.00
Profile	00111	00110	00101	00100	00011	00010	00001	00000
Evaluation	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

function, since they represent profiles that are deprived as much as or more than threshold profiles. Profiles above both threshold elements are instead scored 0 by the evaluation function (a direct inspection of the Hasse diagram shows that there are only two such elements, namely 01111 and 11111). All of the other elements of the poset are scored in $[0,1]$ by the evaluation function. The choice of the threshold breaks the symmetry of the profile poset, so that profiles with the same number of 1s may be scored differently. For the sake of completeness, Table 9.1 reports the evaluation function, estimated extracting a sample of 10^8 linear extensions out of the profile poset. For more comments on the results, see Fattore et al. [2011a].

Remark. We conclude this paragraph with a brief discussion on the possible use of partial order theory in evaluation problems involving continuous variables. When truly numerical variables are at hand, there is apparently no need for partial order theory to be employed. At least in principle, composite indicators might be computed and metric information preserved. However, a closer look at the problem shows that things are not so neat. Composite indicators often mix up variables expressed on different scales, producing almost uninterpretable results. Even the trick of scaling variables to unit variance does not solve the problem and could be justified only by assuming a latent variable model behind observed data. Unfortunately, involving latent constructs leads to other subtle issues, such as the indeterminacy of factor scores (Vittadini 1989, 2007), which raise doubts on the validity of the approach. Moreover, as previously noticed, interdependencies among poverty variables are often not very strong, reducing the effectiveness of correlation-based procedures. In addition to these technical problems, other (perhaps more fundamental) considerations may be given in favor of partial order theory. As already discussed, evaluation is primarily a problem of comparison against benchmarks, rather than against an absolute scale. In the composite indicator approach (with or without latent variables), benchmarking is performed through variable aggregation, i.e., achieving unidimensionality to eliminate incomparabilities. Aggregation introduces compensations and trade-offs between evaluation dimensions, which are often debatable, but usually accepted as the only way to get full comparability among statistical units. However, if one addresses multidimensional evaluation through partial order theory, i.e., as a problem of "comparability quantification," the existence of incomparabilities stops being a problem and aggregation is no longer needed, even conceptually. Up to now, this point of view has been pursued only for ordinal data, but one may hold it also about continuous variables. There are indeed some technical problems to face. For example, the trick of considering linear extensions may not be directly applied to continuous partial orders and its implementation must be reconsidered (we are currently working on that and a possible solution has been already identified). Apart from these technical issues, however, we see that poset theory is a general tool for multidimensional evaluation problems, since its conceptual and formal structure is fully consistent with the very nature of evaluation processes. This is one of the main reasons why poset theory should be part of the "encyclopedia" of social scientists.

9.2.2 *Inequality and Polarization in Multidimensional Ordinal Datasets: How to Assess Them?*

A major concern in current socio-economic research is assessing inequality patterns among individuals and polarization within societies. Historically, the measure of inequality is one of the most studied and developed research fields in socio-economic statistics and the amount of literature about it is huge. Inequality measurement focused primarily on income distribution and monetary well-being. This led to classical axiomatic systems for inequality and concentration indices. As social scientists' and policy-makers' focus is shifting from a monetary analysis to well-being, questions about inequality are moving toward a multidimensional setting, often comprising ordinal information. A similar process is occurring with respect to another crucial phenomenon affecting modern societies and which is attracting more and more interest by social scientists: social polarization. The first aspect of polarization, technically referred to as *bi-polarization*, pertains primarily to the well-known phenomenon of the *disappearing middle class*. The existence of the middle class is one of the most relevant consequences of the development of modern societies as an effect of the diffusion of well-being, both in monetary and non-monetary forms. However, in the last two decades, in many countries there are evidences that "the rich get richer and the poor get poorer." That is, societies are becoming polarized and the middle class is partly disappearing. A broader concept of polarization, closely linked to existence of different social and ethnic groups in modern democracies, is related to the "alienation that individuals and groups feel from one another […] fuelled by notions of within-group identity." Here, the interest lies primarily on "the correlates of organized, large-scale social unrest-strikes, demonstrations, processions, widespread violence, and revolt or rebellion. Such phenomena thrive on differences, to be sure. But they cannot exist without notions of group identity either." (see Duclos et al. 2004, pp. 1737–1738). From a conceptual, and then statistical, point of view, it is interesting to notice that inequality and polarization (both in the bi-polarization and in the broader sense) are two distinct concepts. A first evidence of this dates back to the paper of Wolfson [1994], where it is unequivocally shown how a sequence of income distributions may be built with decreasing inequality and increasing bi-polarization. Inequality does not capture either the notion of identification-alienation polarization, as discussed in Duclos et al. [2004]. The interest in polarization led to many statistical studies devoted to measuring it and to developing related axiomatic systems, primarily in the unidimensional case (Duclos et al. 2004, Fusco and Silber 2011, Permanyer 2012, Zhang and Kanbur 2001). While the concept of polarization is being carefully analyzed and theoretical and empirical differences or interconnections with inequality are being investigated, a new issue is emerging as relevant and urgent. Inequality and polarization do not only concern the monetary perspective; instead they involve the whole well-being concept, comprising health, work, education, culture, environment, material deprivation, and so on. Some interesting studies address the issue of labor market segmentation and the link between job polarization and wage polarization (Ercolani and Jenkins 1998, Gregg and Wadsworth 2004). A great deal of research is also being done on how to measure inequality and polarization in health services and, in particular, in the subjective assessment of health responsiveness (Apouey 2007, Lones 2010). Involving well-being raises two issues in the theory of inequality/polarization measurement: (i) building multidimensional indices and (ii) defining formulas suitable to ordinal data. Multidimensional inequality measures have been already extensively studied (see, e.g. Maasoumi 1999, Tsui 1986), while multidimensional extensions of polarization measures are still at an initial stage (see, e.g. Gigliarano and Mosler 2009). The problem of ordinal data is instead urgent for both inequality and polarization measurements. There are indeed several formulas to treat ordinal information (consider, e.g., Abul Naga and Yalcin 2008, Allison and Foster 2004), but at the same time, and this is the point of interest for our aims, their use still meets some "resistance." For example, in Doorslaer van and Jones [2003] the issues of using ordinal data are clearly addressed and overcome in favor of transforming ordinal scores into numerical figures. Quoting from the Introduction:

> One of the challenges in investigating inequalities in health is that, very often, health information is only available at an ordinal level. One of the most commonly used indicators of overall individual health in general population surveys is the simple question, "how is your health in general?", with response categories ranging from "very good" or "excellent" to "poor" or "very poor". This categorical variable has been shown to be a very good predictor variable of other outcomes, such as subsequent use of medical care or of mortality […]. However, it does not provide a cardinal health (utility) scale that can be used, for instance, for quality adjustments of life expectancy. Categorical measures of health create a problem for the measurement of inequalities in health. The health concentration index, and the related slope index of inequality, require information on health in the form of either a continuous variable or a dichotomous variable. (Doorslaer van and Jones 2003, pp. 61–62).

Ordinal data are thus a problem since the computational and interpretative processes are designed for numerical figures. To be clear, we are not lessening the relevance of jointly considering life expectancy and health status. We simply remark that, according to the quoted text, the whole conceptual framework is not compatible with ordinal data. Whether this is a problem of the data generation process or a limit of the conceptual framework, we leave to the reader. The way out from this incompatibility is usually the application of scaling tools to transform ordinal data into numerical figures, by means of latent variable models, probit models, or other form of regressions, together with all the burden of hypotheses and assumptions that they carry (Doorslaer van and Jones 2003), affecting in some way the final computations. An interesting example showing possible effects of scaling is provided by the study reported in Madden [2010], which concerns health status in Ireland for years 2003–2006. Inequality in self-reported health status is analyzed and compared using the Abul Naga and Yalcin indices (designed for ordinal data) and, after transforming original ordinal data into cardinal figures by means of interval regression, through the Generalized Entropy indices. Before commenting on the results of the study, it is interesting to quote its motivations:

> As the vast majority of summary inequality indices are mean-based they require a cardinal measure of the outcome variable in question. While there are some health measures that are cardinal (e.g. body mass index) they are typically not comprehensive. More general health measures are nearly always categorical and ordinal rather than cardinal. Thus, to obtain a summary measure of inequality it is necessary to either (a) employ an inequality measure that is specifically designed to deal with ordinal data or (b) to transform the ordinal measure into a cardinal measure and then employ a standard inequality index. […] It could be argued that since inequality measures specifically designed to deal with ordinal data are now available, analysts should always use such indices. However, it also seems fair to suggest that such measures are less well developed than their cardinal counterparts.

Table 9.2 reproduces a part of the results reported in Madden [2010]; it compares the inequality measures obtained by the Abul Naga and Yalcin index and those obtained by the Generalized Entropy index for each year.[10]

[10] The Abul Naga and Yalcin index is computed setting $\alpha = \beta = 1$, while the Generalized Entropy is computed setting $a=0$.

Table 9.2 Comparison between inequality measures computed using the Abul Naga and Yalcin (AN–Y) index and the Generalized Entropy (GE) index, extracted from Table 2 of Madden [2010]

Year	AN–Y	GE
2003	0,3563	0.0039
2004	0.3455	0.0043
2005	0.3427	0.0045
2006	0.3330	0.0040

In Madden [2010], it is emphasized that an absolute comparison between the two indices is not meaningful and so the author focuses on the orderings induced by them on the four years considered. As it can be directly checked, the orderings are very different: according to the Abul Naga and Yalcin Index we get 2006 < 2005 < 2004 < 2003, while according to Generalized Entropy we have 2003 < 2006 < 2004 < 2005. It is particularly noticeable that year 2003 is ranked as the most unequal year by the first ordering and as the least by the second. Thus we see that the judgment on the temporal evolution of self-reported health status would be almost reversed, if one chooses the ordinal or the cardinal way of measuring inequality.

This example is quite instructive and shows the consequences of assuming latent cardinal variables behind ordinal data (not to mention the problems concerning the numerical results of the scaling procedure: are the differences in the Generalized Entropy measures really significant?).

When the issues of multidimensionality and of using ordinal data combine together, the situation becomes much more complex and challenging. The problem is to build a multidimensional index of inequality or polarization for ordinal data and this seems to be an almost completely unexplored field. As far as we know, the only attempt in this direction is Kobus [2011]. In general terms, the extension of unidimensional indices to multidimensional settings is pursued building axiomatic systems that generalize unidimensional axioms to sets of many variables. The problem with this approach is that inequality, polarization, or other similar issues in a multidimensional framework may assume so many different forms and may show such a great number of shapes that it is often extremely difficult to identify neatly natural properties to be turned into axioms. The result is quite complicated and debatable axiomatic systems. Without entering into technical details, it seems to us that one of the problems is that axiomatization attempts tend to focus directly on the ordinal variables at hand, instead of focusing on the partial order induced by them on the (equivalence classes of) statistical units. Basically, the approach is to define a partial order on the set of joint ordinal frequency distributions that should reflect a partial ordering of (say) inequalities and then to impose inequality indices to be consistent with it. The idea is in itself quite appealing and resembles classical approaches to multidimensional inequality and concentration indices, but putting it into practice leads to quite complicated axiomatics and non-neat results, also depending on some arbitrary choices that lessen the generality of the arguments. It is our personal feeling that in applied statistics, axiomatic systems should be kept as simple and natural as possible. In this effort, one should be driven by a clear idea of what is to be measured, by suitable formal representations of the problem and by appropriate mathematical tools. While the concepts of inequality and polarization are quite clear, the formal tools usually employed, borrowed from classical mathematical analysis, are not so effective. A possible alternative is to cast the whole problem in partial order terms, representing (equivalence classes of) statistical units through Hasse diagrams, linking inequality/polarization axioms to the structure of the partial order and to the distribution of the population on it. Focusing on statistical units instead of variables (i.e., considering data matrix rows instead of columns) has many advantages: it (i) makes the structure of the data explicit, (ii) helps identify alternative properties that indices may fulfill and that may be turned into axioms, (iii) is completely consistent with the ordinal nature of the data, and (iv) generalizes also to posets not explicitly built upon a set of variables. Basically, the idea could be to build a system of axioms which is more algebraic in nature than analytical, requiring the indices to be "well behaved" both when the frequency distributions change and when the partial order structure changes. We are currently working on this, with promising results.

9.2.3 Searching for Patterns: Clustering over Posets and Lattices?

It is not unusual that social surveys collect ordinal information by asking respondents to score their judgments using scales with up to ten degrees. The resulting set of profiles (i.e., sequence of scores on the investigated dimensions) may comprise even thousands of different elements. The need to reduce complexity and identify groups and clusters of respondents naturally arises. Similarly, many economic studies comparing countries' features lead to systems of multidimensional comparisons on ordinal data; also in such cases one may be interested to group similar statistical units. The study of clustering techniques is one of the most developed branches of data analysis. Many different methodologies are available, ranging from simple hierarchical procedures (Rencher 2002), to neural networks algorithms (Kohonen 2001, Ripley 2005) or to model-based techniques (Vermunt and Magidson 2002). Most of the clustering tools are designed to work with cardinal variables, but there are also procedures for ordinal data. In practice, however, the partial ordinal nature of the data is seldom taken into account explicitly. Recently, an interesting book about clustering on ordinal data came out, where also generalizations of dissimilarity measures taking values in posets are considered (Janowitz 1978, 2010), but most of the techniques applied in daily research are of a classical kind and treat ordinal scores as cardinal (e.g., considering scores directly as numbers). The question we pose is whether it is possible to develop clustering methods that take full account of the partial order structure of the dataset, that is to build procedures which extract clustering information not only from some metric structure, but also from the underlying partial order. To be more explicit, let us consider the following simple problem. Suppose we record data on k ordinal variables, each on two-degree scales (e.g., pertaining to the ownership of k different goods). The set of 2^k k-dimensional profiles is naturally turned into a lattice L, under the product order. Suppose also that a frequency distribution is assigned on L. We now want to cluster individuals, i.e., profiles using some hierarchical procedure, based on the choice of a metric. If we perform this task in the usual way, at each step of the procedure we do obtain groups, but we lose information on the underlying lattice structure. In other words, we merge profiles into groups, but we do not know how to partially order them and we cannot embed them into a lattice structure. This is a critical problem, since the lattice structure of the profiles conveys a lot of information on the data. This information should be used when clustering and should be preserved, as much as possible, at each step of the procedure. A possible way to achieve this relies on the concept of lattice congruence. Any partition of the elements of L defines an equivalence relation on the lattice, which, however, may or may not be compatible with the join and meet operations defined on it, that is which may or may not be a congruence. Forcing, at any step of the clustering procedure, the obtained partition to be a congruence, the clustering process would naturally partially order the clusters, in a way compatible with the original lattice. A procedure might be designed where, at any step, (i) some profiles are merged (i.e., they are declared as equivalent) based on some metric (or dissimilarity) criterion and (ii) the smallest congruence comprising that equivalence is computed. In this way, the information comprised in the relational structure of L would be employed in the clustering process, making the local metric information spread across the lattice, through the congruence constraint. A trivial example is given in Fig. 2 (see Davey and Priestley 2002). When the selected elements of the lattice are merged in cluster "a," other clusters must be formed ("b," "c," "d," "e") for the partition to be consistent with the order relation. The resulting set of clusters is again a lattice, whose Hasse diagram is depicted in Fig. 9.2, with black nodes representing groups. The equivalence relation induced by the final partition is the smallest congruence comprising the equivalence class "a." Similar approaches might be also followed when dealing with posets which are not lattices, possibly drawing upon poset generalizations of the notion of a lattice congruence. We notice that also the choice of the metric might be made taking into account that profiles are partially ordered (Monjardet 1981) and this could improve the coherence and the effectiveness of clustering algorithms on partially ordered structures. Clustering procedures are not our own research field, so we limit ourselves to the above hints. However, we invite lattice experts to address this problem, the solution to which would be very useful to social scientists.

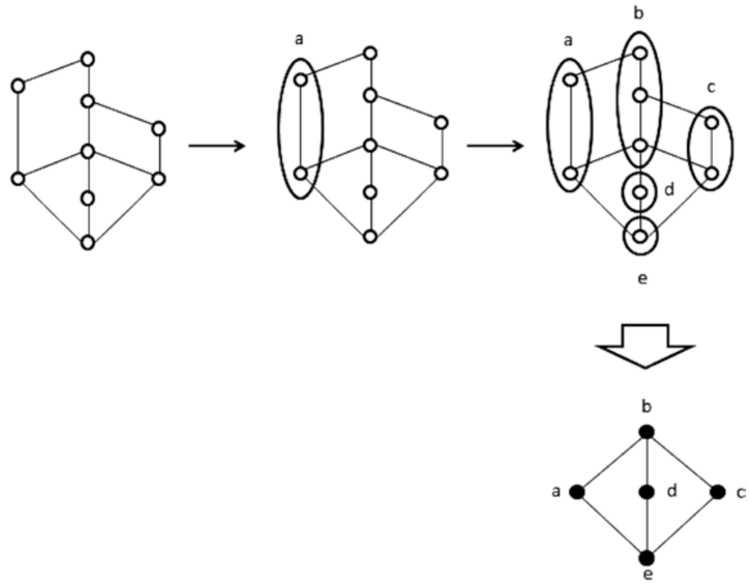

Fig. 9.2 An example of clustering on a small lattice

9.3 Conclusion

In this paper we have commented on the possible role of partial order theory in socio-economic analysis, from the (certainly narrow) point of view of our main research interests. Here we simply want to stress some of the key concepts addressed. The study of social facts is asking for new tools and new languages, more oriented to complexity and more capable of reproducing the reality, in modern societies, of "patterns," "shapes," and "nuances" which are relevant for policy-making. The issue of multidimensionality combined with the increasing availability of ordinal data is particularly challenging for socio-economic scientists, who need new tools to address social issues, but also tend to stick to "old" paradigms. So the problem is both technical, in that new statistical procedures must be developed, and cultural, in that some open-mindedness is necessary for scholars to modify, at least partly, their methodological habits. Partial order theory may play a key role in this challenge, as we have suggested through examples pertaining to real issues, crucial for our comprehension of societal dynamics and for policy definition. Proving that ordinal data may be effectively and consistently treated and exploited, partial order theory opens new possibilities to socio-economic statistics. Certainly, the technical and the cultural sides of the challenge go together. As concrete applications of partial order tools begin to prove their usefulness to social science, it will become easier for the wider scientific community to accept and employ them successfully. This challenge involves both partial order theorists and social scientists, since only by joining different points of view and complementary competencies it is possible to advance in this research field. We hope that this paper, raising questions and soliciting answers, may contribute to fruitful developments.

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