9

Tensors and Universal Properties

We will review the basic properties of the tensor product and use them to illustrate the basic notion of a *universal property*, which we will see repeatedly.

If R is a commutative ring and M, N, and P are R-modules, then a *bilinear map* $f : M \times N \longrightarrow P$ is a map satisfying

$$f(r_1 m_1 + r_2 m_2, n) = r_1 f(m_1, n) + r_2 f(m_2, n), \qquad r_i \in R, m_i \in M, n \in N,$$

$$f(m, r_1 n_1 + r_2 n_2) = r_1 f(m, n_1) + r_2 f(m, n_2), \qquad r_i \in R, n_i \in N, m \in M.$$

More generally, if $M_1, \ldots, M_k$ are R-modules, the notion of a *k-linear map* $M_1 \times \cdots \times M_k \longrightarrow P$ is defined similarly: the map must be linear in each variable.

The *tensor product* $M \otimes_R N$ is an R-module together with a bilinear map $\otimes : M \times N \longrightarrow M \otimes_R N$ satisfying the following property.

Universal Property of the Tensor Product. *If P is any R-module and $p : M \times N \longrightarrow P$ is a bilinear map, there exists a unique R-module homomorphism $F : M \otimes N \longrightarrow P$ such that $p = F \circ \otimes$.*

Why do we call this a universal property? It says that $\otimes : M \times N \longrightarrow M \otimes N$ is a "universal" bilinear map in the sense that any bilinear map of $M \times N$ factors through it. As we will explain, the module $M \otimes_R N$ is uniquely determined by the universal property. This is important beyond the immediate example because often objects are described by universal properties. Before we explain this point (which is obvious if one thinks about it correctly), let us make a categorical observation.

If $\mathcal{C}$ is a category, an *initial object* in $\mathcal{C}$ is an object X_0 such that, for each object Y, the Hom set $\mathrm{Hom}_{\mathcal{C}}(X_0, Y)$ consists of a single element. A *terminal object* is an object X_∞ such that, for each object Y, the Hom set $\mathrm{Hom}_{\mathcal{C}}(Y, X_\infty)$ consists of a single element. For example, in the category of sets, the empty set is an initial object and a set consisting of one element is a terminal object.

D. Bump, *Lie Groups*, Graduate Texts in Mathematics 225,
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Lemma 9.1. *In any category, any two initial objects are isomorphic. Any two terminal objects are isomorphic.*

Proof. If X_0 and X_1 are initial objects, there exist unique morphisms $f : X_0 \longrightarrow X_1$ (since X_0 is initial) and $g : X_1 \longrightarrow X_0$ (since X_1 is initial). Then $g \circ f : X_0 \longrightarrow X_0$ and $1_{X_0} : X_0 \longrightarrow X_0$ must coincide since X_0 is initial, and similarly $f \circ g = 1_{X_1}$. Thus f and g are inverse isomorphisms. Similarly, terminal objects are isomorphic. □

Theorem 9.1. *The tensor product $M \otimes_R N$, if it exists, is determined up to isomorphism by the universal property.*

Proof. Let $\mathcal{C}$ be the following category. An object in $\mathcal{C}$ is an ordered pair (P, p), where P is an R-module and $p : M \times N \longrightarrow P$ is a bilinear map. If $X = (P, p)$ and $Y = (Q, q)$ are objects, then a morphism $X \longrightarrow Y$ consists of an R-module homomorphism $f : P \longrightarrow Q$ such that $q = f \circ p$. The universal property of the tensor product means that $\otimes : M \times N \longrightarrow M \otimes N$ is an initial object in this category and therefore determined up to isomorphism. □

Of course, we usually denote $\otimes(m, n)$ as $m \otimes n$ in $M \otimes_R N$. We have not proved that $M \otimes_R N$ exists. We refer to any text on algebra for this fact, such as Lang [116], Chap. XVI.

In general, by a *universal property* we mean *any characterization of a mathematical object that can be expressed by saying that some associated object is an initial or terminal object in some category*. The basic paradigm is that *a universal property characterizes an object up to isomorphism.*

A typical application of the universal property of the tensor product is to make $M \otimes_R N$ into a functor. Specifically, if $\mu : M \longrightarrow M'$ and $\nu : N \longrightarrow N'$ are R-module homomorphisms, then there is a unique R-module homomorphism $\mu \otimes \nu : M \otimes_R N \longrightarrow M' \otimes_R N'$ such that $(\mu \otimes \nu)(m \otimes n) = \mu(m) \otimes \nu(n)$. We get this by applying the universal property to the R-bilinear map $M \times N \longrightarrow M' \otimes N'$ defined by $(m, n) \longmapsto \mu(m) \otimes \nu(n)$.

As another example of an object that can be defined by a universal property, let V be a vector space over a field F. Let us ask for an F-algebra $\bigotimes V$ together with an F-linear map $i : V \longrightarrow \bigotimes V$ satisfying the following condition.

Universal Property of the Tensor Algebra. *If A is any F-algebra and $\phi : V \longrightarrow A$ is an F-linear map then there exists a unique F-algebra homomorphism $\Phi : \bigotimes V \longrightarrow A$ such that $r = \rho \circ i$.*

It should be clear from the previous discussion that this universal property characterizes the tensor algebra up to isomorphism. To prove existence, we can construct a ring with this exact property as follows. Let unadorned $\otimes$ mean $\otimes_F$ in what follows. By $\otimes^k V$ we mean the k-fold tensor product $V \otimes \cdots \otimes V$ (k times); if $k = 0$, then it is natural to take $\otimes^0 V = F$ while $\otimes^1 V = V$. If V has finite dimension d, then $\otimes^k V$ has dimension d^k. Let

$$\bigotimes V = \bigoplus_{k=0}^{\infty} \left(\otimes^k V\right).$$

Then $\bigotimes V$ has the natural structure of a graded F-algebra in which the multiplication $\otimes^k V \times \otimes^l V \longrightarrow \otimes^{k+l} V$ sends

$$(v_1 \otimes \cdots \otimes v_k, u_1 \otimes \cdots \otimes u_l) \longrightarrow v_1 \otimes \cdots \otimes v_k \otimes u_1 \otimes \cdots \otimes u_l.$$

We regard V as a subset of $\bigotimes V$ embedded onto $\otimes^1 V = V$.

Proposition 9.1. *The universal property of the tensor algebra is satisfied.*

Proof. If $\phi : V \longrightarrow A$ is any linear map of V into an F-algebra, define a map $\Phi : \bigotimes V \longrightarrow A$ by $\Phi(v_1 \otimes \cdots \otimes v_k) = \phi(v_1) \cdots \phi(v_k)$ on $\otimes^k V$. It is easy to see that Φ is a ring homomorphism. It is unique since V generates $\bigotimes V$ as an F-algebra. $\square$

A *graded algebra* over the field F is an F-algebra A with a direct sum decomposition

$$A = \bigoplus_{k=0}^{\infty} A_k$$

such that $A_k A_l \subseteq A_{k+l}$. In most examples we will have $A_0 = F$. Elements of A_k are called *homogeneous* of degree k. The tensor algebra is a graded algebra, with $\otimes^k V$ being the homogeneous part of degree k.

Next we define the *symmetric* and *exterior powers* of a vector space V over the field F. Let V^k denote $V \times \cdots \times V$ (k times). A k-linear map $f : V^k \longrightarrow U$ into another vector space is called *symmetric* if for any $\sigma \in S_k$ it satisfies $f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = f(v_1, \ldots, v_k)$ and *alternating* if $f\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) = \varepsilon(\sigma)\, f(v_1, \ldots, v_k)$, where $\varepsilon : S_k \longrightarrow \{\pm 1\}$ is the alternating (sign) character. The kth symmetric and exterior powers of V, denoted $\vee^k V$ and $\wedge^k V$, are F-vector spaces, together with k-linear maps $\vee : V^k \longrightarrow \vee^k V$ and $\wedge : V^k \longrightarrow \wedge^k V$. The map $\vee$ is symmetric, and the map $\wedge$ is alternating. We normally denote $\vee(v_1, \ldots, v_k) = v_1 \vee \cdots \vee v_k$ and similarly for $\wedge$. The following universal properties are required.

Universal Properties of the Symmetric and Exterior Powers: *Let $f : V^k \longrightarrow U$ be any symmetric (resp. alternating) k-linear map. Then there exists a unique F-linear map $\phi : \vee^k V \longrightarrow U$ (resp. $\wedge^k V \longrightarrow U$) such that $f = \phi \circ \vee$ (resp. $f = \phi \circ \wedge$).*

As usual, the symmetric and exterior algebras are characterized up to isomorphism by the universal property. We may construct $\vee^k V$ as a quotient of $\otimes^k V$, dividing by the subspace W generated by elements of the form $v_1 \otimes \cdots \otimes v_k - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$, with a similar construction for $\wedge^k$. The universal property of $\vee^k V$ then follows from the universal property of the tensor product. Indeed, if $f : V^k \longrightarrow U$ is any symmetric k-linear map, then there is induced a linear map $\psi : \otimes^k V \longrightarrow U$ such that $f = \psi \circ \otimes$. Since f is symmetric, ψ vanishes on W, so ψ induces a map $\vee^k V = \otimes^k V/W \longrightarrow U$ and the universal property follows.

If V has dimension d, then $\vee^k V$ has dimension $\binom{d+k-1}{k}$, for if $x_1, \ldots, x_d$ is a basis of V, then $\{x_{i_1} \vee \cdots \vee x_{i_k} \,|\, 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant d\}$ is a basis for $\vee^k V$. On the other hand, the exterior power vanishes unless $k \leqslant d$, in which case it has dimension $\binom{d}{k}$. A basis consists of $\{x_{i_1} \wedge \cdots \wedge x_{i_k} \,|\, 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant d\}$. The vector spaces $\vee^k V$ may be collected together to make a commutative graded algebra:

$$\bigvee V = \bigoplus_{k=0}^{\infty} \vee^k V.$$

This is the *symmetric algebra.* The exterior algebra $\bigwedge V = \bigoplus_k \wedge^k V$ is constructed similarly. The spaces $\vee^0 V$ and $\wedge^0 V$ are one-dimensional and it is natural to take $\vee^0 V = \wedge^0 V = F$.

Exercises

Exercise 9.1. Let V be a finite-dimensional vector space over a field F that may be assumed to be infinite. Let $\mathcal{P}(V)$ be the ring of polynomial functions on V. Note that an element of the dual space V^* is a function on V, so regarding this function as a polynomial gives an injection $V^* \longrightarrow \mathcal{P}(V)$. Show that this linear map extends to a ring isomorphism $\bigvee V^* \longrightarrow \mathcal{P}(V)$.

Exercise 9.2. Prove that if V is a vector space, then $V \otimes V \cong (V \wedge V) \oplus (V \vee V)$.

Exercise 9.3. Use the universal properties of the symmetric and exterior power to show that if V and W are vector spaces, then there are maps $\vee^k f : \vee^k V \longrightarrow \vee^k W$ and $\wedge^k f : \wedge^k V \longrightarrow \wedge^k W$ such that

$$\vee^k f(v_1 \vee \cdots \vee v_k) = f(v_1) \vee \cdots \vee f(v_k), \qquad \wedge^k f(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k).$$

Exercise 9.4. Suppose that $V = F^4$. Let $f : V \longrightarrow V$ be the linear transformation with eigenvalues a, b, c, d. Compute the traces of the linear transformations $\vee^2 f$ and $\wedge^2 f$ on $\vee^2 V$ and $\wedge^2 V$ as polynomials in a, b, c, d.

Exercise 9.5. Let A and B be algebras over the field F. Then $A \otimes B$ is also an algebra, with multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. Show that there are ring homomorphisms $i : A \to A \otimes B$ and $j : B \to A \otimes B$ such that if $f : A \to C$ and $g : B \to C$ are ring homomorphisms into a ring C satisfying $f(a)\, g(b) = g(b)\, f(a)$ for $a \in A$ and $b \in B$, then there exists a unique ring homomorphism $\phi : A \otimes B \to C$ such that $\phi \circ i = f$ and $\phi \circ j = g$.

Exercise 9.6. Show that if U and V are finite-dimensional vector spaces over F then show that

$$\bigvee(U \oplus V) \cong \left(\bigvee U\right) \otimes \left(\bigvee U\right)$$

and

$$\bigwedge(U \oplus V) \cong \left(\bigwedge U\right) \otimes \left(\bigwedge U\right).$$