8

The Exponential Map

The exponential map, introduced for closed Lie subgroups of $\mathrm{GL}(n,\mathbb{C})$ in Chap. 5, can be defined for a general Lie group G as a map $\mathrm{Lie}(G) \longrightarrow G$.

We may consider a vector field (6.5) that is allowed to vary smoothly. By this we mean that we introduce a real parameter $\lambda \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$ and smooth functions $a_i : M \times (-\epsilon, \epsilon) \longrightarrow \mathbb{C}$ and consider a vector field, which in local coordinates is given by

$$(Xf)(m) = \sum_{i=1}^{n} a_i(m, \lambda) \frac{\partial f}{\partial x_i}(m). \tag{8.1}$$

Proposition 8.1. *Suppose that M is a smooth manifold, $m \in M$, and X is a vector field on M. Then, for sufficiently small $\epsilon > 0$, there exists a path $p : (-\epsilon, \epsilon) \longrightarrow M$ such that $p(0) = m$ and $p_*(\mathrm{d}/\mathrm{d}t)(t) = X_{p(t)}$ for $t \in (-\epsilon, \epsilon)$. Such a curve, on whatever interval it is defined, is uniquely determined. If the vector field X is allowed to depend on a parameter λ as in (8.1), then for small values of t, $p(t)$ depends smoothly on λ.*

Here we are regarding the interval $(-\epsilon, \epsilon)$ as a manifold, and $p_*(\mathrm{d}/\mathrm{d}t)$ is the image of the tangent vector $\mathrm{d}/\mathrm{d}t$. We call such a curve an *integral curve* for the vector field.

Proof. In terms of local coordinates $x_1, \ldots, x_n$ on M, the vector field X is

$$\sum a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i},$$

where the a_i are smooth functions in the coordinate neighborhood. If a path $p(t)$ is specified, let us write $x_i(t)$ for the x_i component of $p(t)$, with the coordinates of m being $x_1 = \cdots = x_n = 0$. Applying the tangent vector $p_*(t)(\mathrm{d}/\mathrm{d}t)(t)$ to a function $f \in C^\infty(G)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(x_1(t), \ldots, x_n(t)\big) = \sum x_i'(t) \frac{\partial f}{\partial x_i}\big(x_1(t), \ldots, x_n(t)\big).$$

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On the other hand, applying $X_{p(t)}$ to the same f gives

$$\sum_i a_i\big(x_1(t),\ldots,x_n(t)\big)\frac{\partial f}{\partial x_i}\big(x_1(t),\ldots,x_n(t)\big)\,,$$

so we need a solution to the first-order system

$$x_i'(t) = a_i\big(x_1(t),\ldots,x_n(t)\big), \qquad x_i(0)=0, \qquad (i=1,\ldots,n).$$

The existence of such a solution for sufficiently small $|t|$, and its uniqueness on whatever interval it does exist, is guaranteed by a standard result in the theory of ordinary differential equations, which may be found in most texts. See, for example, Ince [81], Chap. 3, particularly Sect. 3.3, for a rigorous treatment. The required Lipschitz condition follows from smoothness of the $a_{i.}$ For the statement about continuously varying vector fields, one needs to know the corresponding fact about first-order systems, which is discussed in Sect. 3.31 of [81]. Here Ince imposes an assumption of analyticity on the dependence of the differential equation on λ, which he allows to be a complex parameter, because he wants to conclude analyticity of the solutions; if one weakens this assumption of analyticity to smoothness, one still gets smoothness of the solution. □

In general, the existence of the integral curve of a vector field is only guaranteed in a small segment $(-\epsilon,\epsilon)$, as in Proposition 8.1. However, we will now see that, for left-invariant vector fields on a Lie group, the integral curve extends to all $\mathbb{R}$. This fact underlies the construction of the exponential map.

Theorem 8.1. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. There exists a map* $\exp:\mathfrak{g}\longrightarrow G$ *that is a local homeomorphism in a neighborhood of the origin in $\mathfrak{g}$ such that, for any $X\in\mathfrak{g}$, $t\longrightarrow\exp(tX)$ is an integral curve for the left-invariant vector field X. Moreover,* $\exp\big((t+u)X\big)=\exp(tX)\exp(uX)$.

Proof. Let $X\in\mathfrak{g}$. We know that for sufficiently small $\epsilon>0$ there exists an integral curve $p:(-\epsilon,\epsilon)\longrightarrow G$ for the left-invariant vector field X with $p(0)=1$. We show first that if $p:(a,b)\longrightarrow G$ is any integral curve for an open interval (a,b) containing 0, then

$$p(s)\,p(t)=p(s+t) \text{ when } s,t,s+t\in(a,b). \tag{8.2}$$

Indeed, since X is invariant under left-translation, left-translation by $p(s)$ takes an integral curve for the vector field into another integral curve. Thus, $t\longrightarrow p(s)\,p(t)$ and $t\longrightarrow p(s+t)$ are both integral curves, with the same initial condition $0\longrightarrow p(s)$. They are thus the same.

With this in mind, we show next that if $p:(-a,a)\longrightarrow G$ is an integral curve for the left-invariant vector field X, then we may extend it to all of $\mathbb{R}$. Of course, it is sufficient to show that we may extend it to $(-\frac{3}{2}a,\frac{3}{2}a)$. We extend it by the rule $p(t)=p(a/2)\,p(t-a/2)$ when $-a/2\leqslant t\leqslant 3a/2$ and

$p(t) = p(-a/2)\, p(t + a/2)$ when $-3a/2 \leqslant t \leqslant a/2$, and it follows from (8.2) that this definition is consistent on regions of overlap.

Now define $\exp : \mathfrak{g} \longrightarrow G$ as follows. Let $X \in \mathfrak{g}$, and let $p : \mathbb{R} \longrightarrow G$ be an integral curve for the left-invariant vector field X with $p(0) = 0$. We define $\exp(X) = p(1)$. We note that if $u \in \mathbb{R}$, then $t \mapsto p(tu)$ is an integral curve for uX, so $\exp(uX) = p(u)$.

The exponential map is a smooth map, at least for X near the origin in $\mathfrak{g}$, by the last statement in Proposition 8.1. Identifying the tangent space at the origin in the vector space $\mathfrak{g}$ with $\mathfrak{g}$ itself, exp induces a map $T_0(\mathfrak{g}) \longrightarrow T_e(G)$ (that is $\mathfrak{g} \longrightarrow \mathfrak{g}$), and this map is the identity map by construction. Thus, the Jacobian of exp is nonzero and, by the Inverse Function Theorem, exp is a local homeomorphism near 0. □

We also denote $\exp(X)$ as e^X for $X \in \mathfrak{g}$.

Remark 8.1. If $G = \mathrm{GL}(n, \mathbb{C})$, then as we explained in Chap. 7, Proposition 7.2 allows us to identify the Lie algebra of G with $\mathrm{Mat}_n(\mathbb{C})$. We observe that the definition of $\exp : \mathrm{Mat}_n(\mathbb{C}) \longrightarrow \mathrm{GL}(n, \mathbb{C})$ by a series in (5.2) agrees with the definition in Theorem 8.1. This is because $t \longmapsto \exp(tX)$ with either definition is an integral curve for the same left-invariant vector field, and the uniqueness of such an integral curve follows from Proposition 8.1.

Proposition 8.2. *Let G, H be Lie groups and let $\mathfrak{g}$, $\mathfrak{h}$ be their respective Lie algebras. Let $f : G \to H$ be a homomorphism. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{d}f} & \mathfrak{h} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{f} & H \end{array}$$

Proof. It is clear from the definitions that f takes an integral curve for a left-invariant vector field X on G to an integral curve for $\mathrm{d}f(X)$, and the statement follows. □

A *representation* of a Lie algebra $\mathfrak{g}$ over a field F is a Lie algebra homomorphism $\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$, where V is an F-vector space, or more generally a vector space over a field E containing F, and $\mathrm{End}(V)$ is given the Lie algebra structure that it inherits from its structure as an associative algebra. Thus,

$$\pi([x, y]) = \pi(x)\, \pi(y) - \pi(y)\, \pi(x).$$

We may sometimes find it convenient to denote $\pi(x)v$ as just xv for $x \in \mathfrak{g}$ and $v \in V$. We may think of $(x, v) \mapsto xv = \pi(x)v$ as a multiplication. If V is a vector space, given a map $\mathfrak{g} \times V \longrightarrow V$ denoted $(x, v) \mapsto xv$ such that $x \mapsto \pi(x)$ is a representation, where $\pi(x) : V \longrightarrow V$ is the endomorphism $v \longrightarrow xv$, then we call V a $\mathfrak{g}$-*module*. A *homomorphism* $\phi : U \longrightarrow V$ of $\mathfrak{g}$-modules is an F-linear map satisfying $\phi(xv) = x\phi(v)$.

Example 8.1. If $\pi : G \longrightarrow \mathrm{GL}(V)$ is a representation, where V is a real or complex vector space, then the Lie algebra of $\mathrm{GL}(V)$ is $\mathrm{End}(V)$, so the differential $\mathrm{Lie}(\pi) : \mathrm{Lie}(G) \longrightarrow \mathrm{End}(V)$, defined by Proposition 7.3, is a Lie algebra representation.

By the universal property of $U(\mathfrak{g})$ in Theorem 10.1, A Lie algebra representation $\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ extends to a ring homomorphism $U(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$, which we continue to denote as π.

If $\mathfrak{g}$ is a Lie algebra over a field F, we get a homomorphism $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$, called the *adjoint map*, defined by $\mathrm{ad}(x)y = [x, y]$. We give $\mathrm{End}(\mathfrak{g})$ the Lie algebra structure it inherits as an associative ring. We have

$$\mathrm{ad}(x)([y, z]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)] \tag{8.3}$$

since, by the Jacobi identity, both sides equal $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$. This means that $\mathrm{ad}(x)$ is a derivation of $\mathfrak{g}$.

Also

$$\mathrm{ad}(x)\,\mathrm{ad}(y) - \mathrm{ad}(y)\,\mathrm{ad}(x) = \mathrm{ad}\big([x, y]\big) \tag{8.4}$$

since applying either side to $z \in \mathfrak{g}$ gives $[x, [y, z]] - [y, [x, z]] = [[x, y], z]$ by the Jacobi identity. So $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is a Lie algebra representation.

We next explain the geometric origin of ad. To begin with, representations of Lie algebras arise naturally from representations of Lie groups. Suppose that G is a Lie group and $\mathfrak{g}$ is its Lie algebra. If V is a vector space over $\mathbb{R}$ or $\mathbb{C}$, any Lie group homomorphism $\pi : G \longrightarrow \mathrm{GL}(V)$ induces a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathrm{End}(V)$ by Proposition 7.3; that is, a real or complex representation.

In particular, G acts on itself by conjugation, and so it acts on $\mathfrak{g} = T_e(G)$. This representation is called the *adjoint representation* and is denoted $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$. We show next that the differential of Ad is ad. That is:

Theorem 8.2. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and* $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ *the adjoint representation. Then the Lie group representation $\mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ corresponding to* Ad *by Proposition 7.3 is* ad.

Proof. It will be most convenient for us to think of elements of the Lie algebra as tangent vectors at the identity or as local derivations of the local ring there. Let $X, Y \in \mathfrak{g}$. If $f \in C^\infty(G)$, define $c(g)f(h) = f(g^{-1}hg)$. Then our definitions of the adjoint representation amount to

$$\big(\mathrm{Ad}(g)Y\big)f = Y\big(c(g^{-1})f\big).$$

To compute the differential of Ad, note that the path $t \longrightarrow \exp(tX)$ in G is tangent to the identity at $t = 0$ with tangent vector X. Therefore, under the representation of $\mathfrak{g}$ in Proposition 7.3, X maps Y to the local derivation at the identity

$$f \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{Ad}(\mathrm{e}^{tX})Y\big)f\,\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}u} f(\mathrm{e}^{tX}\mathrm{e}^{uY}\mathrm{e}^{-tX})\,\Big|_{t=u=0}.$$

By the chain rule, if $F(t_1, t_2)$ is a function of two real variables,

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t,t) \Big|_{t=0} = \frac{\partial F}{\partial t_1}(0,0) + \frac{\partial F}{\partial t_2}(0,0). \tag{8.5}$$

Applying this, with u fixed to $F(t_1, t_2) = f(\mathrm{e}^{t_1 X} \mathrm{e}^{uY} \mathrm{e}^{-t_2 X})$, our last expression equals

$$\frac{\mathrm{d}}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{e}^{tX}\,\mathrm{e}^{uY}) \Big|_{t=u=0} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{e}^{uY}\,\mathrm{e}^{tX}) \Big|_{t=u=0} = XYf(1) - YXf(1).$$

This is, of course, the same as the effect of $[X, Y] = \mathrm{ad}(X)Y$. □

Exercises

Exercise 8.1. Show that the exponential map $\mathfrak{su}(2) \to \mathrm{SU}(2)$ is surjective, but the exponential map $\mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ is not.