7

Left-Invariant Vector Fields

To recapitulate, a *Lie group* is a differentiable manifold with a group structure in which the multiplication and inversion maps $G \times G \longrightarrow G$ and $G \longrightarrow G$ are smooth. A homomorphism of Lie groups is a group homomorphism that is also a smooth map.

Remark 7.1. There is a subtlety in the definition of a Lie subgroup. A *Lie subgroup of* G is best defined as a Lie group H with an injective homomorphism $i : H \longrightarrow G$. With this definition, the image of i in G is not closed, however, as the following example shows. Let G be $\mathbb{T} \times \mathbb{T}$, where $\mathbb{T}$ is the circle $\mathbb{R}/\mathbb{Z}$. Let H be $\mathbb{R}$, and let $i : H \longrightarrow G$ be the map $i(t) = (\alpha t, \beta t)$ modulo 1, where the ratio α/β is irrational. This is a Lie subgroup, but the image of H is not closed. To require a closed image in the definition of a Lie subgroup would invalidate a theorem of Chevalley that subalgebras of the Lie algebra of a Lie group correspond to Lie subgroups. If we wish to exclude this type of example, we will explicitly describe a Lie subgroup of G as a *closed* Lie subgroup.

Remark 7.2. On the other hand, in the expression "closed Lie subgroup," the term "Lie" is redundant. It may be shown that a closed subgroup of a Lie group is a submanifold and hence a Lie group. See Bröcker and Tom Dieck [25], Theorem 3.11 on p. 28; Knapp [106] Chap. I Sect. 4; or Knapp [105], Theorem 1.5 on p. 20. We will only prove this for the special case of an abelian subgroup in Theorem 15.2 below.

Suppose that M and N are smooth manifolds and $\phi : M \longrightarrow N$ is a smooth map. As we explained in Chap. 6, if $m \in M$ and $n = \phi(m)$, we get a map $\mathrm{d}\phi : T_m(M) \longrightarrow T_n(N)$, called the *differential* of f. If ϕ is a diffeomorphism of M onto N, then we can push a vector field X on M forward this way to obtain a vector field on N. This vector field may be denoted $\phi_* X$, defined by $(\phi_* X)_n = \mathrm{d}\phi(X_m)$ when $f(m) = n$. If ϕ is *not* a diffeomorphism, this may not work because some points in N may not even be in the image of ϕ, while others may be in the image of two different points m_1 and m_2 with no guarantee that $\mathrm{d}\phi X_{m_1} = \mathrm{d}\phi X_{m_2}$.

D. Bump, *Lie Groups*, Graduate Texts in Mathematics 225,
DOI 10.1007/978-1-4614-8024-2_7,

Now let G be a Lie group. If $g \in G$, then $L_g : G \longrightarrow G$ defined by $L_g(h) = gh$ is a diffeomorphism and hence induces maps $L_{g,*} : T_h(G) \longrightarrow T_{gh}(G)$. A vector field X on G is *left-invariant* if $L_{g,*}(X_h) = X_{gh}$.

Proposition 7.1. *The vector space of left-invariant vector fields is closed under* $[\,,\,]$ *and is a Lie algebra of dimension* $\dim(G)$. *If* $X_e \in T_e(G)$, *there is a unique left-invariant vector field* X *on* G *with the prescribed tangent vector at the identity.*

Proof. Given a tangent vector X_e at the identity element e of G, we may define a left-invariant vector field by $X_g = L_{g,*}(X_e)$, and conversely any left-invariant vector field must satisfy this identity, so the space of left-invariant vector fields is isomorphic to the tangent space of G at the identity. Therefore, its vector space dimension equals the dimension of G. □

Let $\mathrm{Lie}(G)$ be the vector space of left-invariant vector fields, which we may identify with the $T_e(G)$. It is clearly closed under $[\,,\,]$.

Suppose now that $G = \mathrm{GL}(n, \mathbb{C})$. We have defined two different Lie algebras for G: first, in Chap. 5, we defined the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of G to be $\mathrm{Mat}_n(\mathbb{C})$ with the commutation relation $[X, Y] = XY - YX$ (matrix multiplication); and second, we have defined the Lie algebra to be the Lie algebra of left-invariant vector fields with the bracket (6.6). We want to see that these two definitions are the same. We will accomplish this in Proposition 7.2 below.

If $X \in \mathrm{Mat}_n(\mathbb{C})$, we begin by associating with X a left-invariant vector field. Since G is an open subset of the real vector space $V = \mathrm{Mat}_n(\mathbb{C})$, we may identify the tangent space to G at the identity with V. With this identification, an element $X \in V$ is the local derivation at I [see (6.3)] defined by

$$f \longmapsto \frac{\mathrm{d}}{\mathrm{d}t} f(I + tX)\Big|_{t=0},$$

where f is the germ of a smooth function at I. The two paths $t \longrightarrow I + tX$ and $t \longrightarrow \exp(tX) = I + tX + \cdots$ are tangent when $t = 0$, so this is the same as

$$f \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t} f\big(\exp(tX)\big)\Big|_{t=0},$$

which is a better definition. Indeed, if H is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$ and X is in the Lie algebra of H, then by Proposition 5.2, the second path $\exp(tX)$ stays within H, so this definition still makes sense.

It is clear how to extrapolate this local derivation to a left-invariant global derivation of $C^\infty(G, \mathbb{R})$. We must define

$$(\mathrm{d}X) f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f\big(g \exp(tX)\big)\Big|_{t=0}. \tag{7.1}$$

By Proposition 2.8, the left-invariant derivation $\mathrm{d}X$ of $C^\infty(G, \mathbb{R})$ corresponds to a left-invariant vector field. To distinguish this derivation from the element X of $\mathrm{Mat}_n(\mathbb{C})$, we will resist the temptation to denote this derivation also as X and denote it by $\mathrm{d}X$.

Lemma 7.1. *Let f be a smooth map from a neighborhood of the origin in $\mathbb{R}^n$ into a finite-dimensional vector space. We may write*

$$f(x) = c_0 + c_1(x) + B(x, x) + r(x), \tag{7.2}$$

where $c_1 : \mathbb{R}^n \longrightarrow V$ is linear, $B : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow V$ is symmetric and bilinear, and r vanishes to order 3.

Proof. This is just the familiar Taylor expansion. Denoting $u = (u_1, \ldots, u_n)$, let $c_0 = f(0)$,

$$c_1(u) = \sum_i \frac{\partial f}{\partial x_i}(0)\, u_i,$$

and

$$B(u, v) = \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(0)\, u_i v_j.$$

Both $f(x)$ and $c_0 + c_1(x) + B(x, x)$ have the same partial derivatives of order $\leqslant 2$, so the difference $r(x)$ vanishes to order 3. The fact that B is symmetric follows from the equality of mixed partials:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(0).$$

□

Proposition 7.2. *If $X,\ Y \in \mathrm{Mat}_n(\mathbb{C})$, and if f is a smooth function on $G = \mathrm{GL}(n, \mathbb{C})$, then $\mathrm{d}[X, Y]f = \mathrm{d}X(\mathrm{d}Y f) - \mathrm{d}Y(\mathrm{d}X f)$.*

Here $[X, Y]$ means $XY - YX$ computed using matrix operations; that is, the bracket computed as in Chap. 5. This proposition shows that if $X \in \mathrm{Mat}_n(\mathbb{C})$, and if we associate with X a derivation of $C^\infty(G, \mathbb{R})$, where $G = \mathrm{GL}(n, \mathbb{C})$, using the formula (7.1), then this bracket operation gives the same result as the bracket operation (6.6) for left-invariant vector fields.

Proof. We fix a function $f \in C^\infty(G)$ and an element $g \in G$. By Lemma 7.1, we may write, for X near 0,

$$f\big(g(I + X)\big) = c_0 + c_1(X) + B(X, X) + r(X),$$

where c_1 is linear in X, B is symmetric and bilinear, and r vanishes to order 3 at $X = 0$. We will show that

$$(\mathrm{d}X\, f)(g) = c_1(X) \tag{7.3}$$

and

$$(\mathrm{d}X \circ \mathrm{d}Y\, f)(g) = c_1(XY) + 2B(X, Y). \tag{7.4}$$

Indeed,

$$(\mathrm{d}X\, f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\, f\big(g(I+tX)\big)|_{t=0}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(c_0 + c_1(tX) + B(tX,tX) + r(tX)\right)\Big|_{t=0}.$$

We may ignore the B and r terms because they vanish to order $\geqslant 2$, and since c_1 is linear, this is just $c_1(X)$ proving (7.3). Also

$$(\mathrm{d}X \circ \mathrm{d}Y f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\left((\mathrm{d}Y\, f)\big(g(I+tX)\big)\right)\Big|_{u=0}$$
$$= \frac{\partial}{\partial t}\frac{\partial}{\partial u}\, f\big(g(I+tX)(I+uY)\big)\Big|_{t=u=0}$$
$$= \frac{\partial}{\partial t}\frac{\partial}{\partial u}[c_0 + c_1(tX+uY+tuXY)$$
$$+B(tX+uY+tuXY, tX+uY+tuXY)$$
$$+r(tX+uY+tuXY)]\,|_{t=u=0}.$$

We may omit r from this computation since it vanishes to third order. Expanding the linear and bilinear maps c_1 and B, we obtain (7.4).

Similarly,

$$(\mathrm{d}Y \circ \mathrm{d}X f)(g) = c_1(YX) + 2B(X,Y).$$

Subtracting this from (7.4) to kill the unwanted B term, we obtain

$$\big((\mathrm{d}X\circ\mathrm{d}Y - \mathrm{d}Y\circ\mathrm{d}X)\, f\big)(g) = c_1(XY-YX) = \left(\mathrm{d}[X,Y]\, f\right)(g)$$

by (7.3). □

If $\phi : G \longrightarrow H$ is a homomorphism of Lie groups, there is an induced map of Lie algebras, as we will now explain. Let X be a left-invariant vector field on G. We have induced a map $\mathrm{d}\phi : T_e(G) \longrightarrow T_e(H)$, and by Proposition 7.1 applied to H there is a unique left-invariant vector field Y on H such that $\mathrm{d}\phi(X_e) = Y_e$. It is easy to see that for any $g \in G$ we have $\mathrm{d}\phi(X_g) = Y_{\phi(g)}$. We regard Y as an element of $\mathrm{Lie}(H)$, and $X \longmapsto Y$ is a map $\mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$, which we denote $\mathrm{Lie}(\phi)$ or, more simply, $\mathrm{d}\phi$. The Lie algebra homomorphism $\mathrm{d}\phi = \mathrm{Lie}(\phi)$ is called the *differential* of ϕ. A map $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ of Lie algebras is naturally called a *homomorphism* if $f([X,Y]) = [f(X), f(Y)]$.

Proposition 7.3. *If* $\phi : G \longrightarrow H$ *is a Lie group homomorphism, then* $\mathrm{Lie}(\phi) : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$ *is a Lie algebra homomorphism.*

Proof. If $X, Y \in G$, then X_e and Y_e are local derivations of $\mathcal{O}_e(G)$, and it is clear from the definitions that $\phi_*([X_e, Y_e]) = [\phi_*(X_e), \phi_*(Y_e)]$. Consequently, $[\mathrm{Lie}(\phi)X, \mathrm{Lie}(\phi)Y]$ and $\mathrm{Lie}(\phi)([X,Y])$ are left-invariant vector fields on H that agree at the identity, so they are the same by Proposition 7.1. □

We may ask to what extent the Lie algebra homomorphism $\mathrm{Lie}(\phi)$ contains complete information about ϕ. For example, given Lie groups G and H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and a homomorphism $f : \mathfrak{g} \longrightarrow \mathfrak{h}$, is there a homomorphism $G \longrightarrow H$ with $\mathrm{Lie}(\phi) = f$?

In general, the answer is no, as the following example will show.

Example 7.1. Let $H = \mathrm{SU}(2)$ and let $G = \mathrm{SO}(3)$. H acts on the three-dimensional space V of Hermitian matrices $\xi = \begin{pmatrix} x & y+iz \\ y-iz & -x \end{pmatrix}$ of trace zero by $h : \xi \mapsto h\xi h^{-1} = h\xi^t\overline{h}$, and

$$\xi \mapsto -\det(\xi) = x^2 + y^2 + z^2$$

is an invariant positive definite quadratic form on V invariant under this action. Thus, the transformation $\xi \mapsto h\xi h^{-1}$ of V is orthogonal, and we have a homomorphism $\psi : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$. Both groups are three-dimensional, and ψ is a local homeomorphism at the identity. The differential $\mathrm{Lie}(\psi) : \mathfrak{su}(2) \longrightarrow \mathfrak{so}(3)$ is therefore an isomorphism and has an inverse, which is a Lie algebra homomorphism $\mathfrak{so}(3) \longrightarrow \mathfrak{su}(2)$. However, ψ itself does not have an inverse since it has a nontrivial element in its kernel, $-I$. Therefore, $\mathrm{Lie}(\psi)^{-1} : \mathfrak{so}(3) \longrightarrow \mathfrak{su}(2)$ is an example of a Lie algebra homomorphism that does not correspond to a Lie group homomorphism $\mathrm{SO}(3) \longrightarrow \mathrm{SU}(2)$.

Nevertheless, we will see later (Proposition 14.2) that if G is *simply connected*, then any Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{h}$ corresponds to a Lie group homomorphism $G \longrightarrow H$. Thus, the obstruction to lifting the Lie algebra homomorphism $\mathfrak{so}(3) \longrightarrow \mathfrak{su}(2)$ to a Lie group homomorphism is topological and corresponds to the fact that $\mathrm{SO}(3)$ is not simply connected.

Exercises

Exercise 7.1. Compute the Lie algebra homomorphism $\mathrm{Lie}(\psi) : \mathfrak{su}(2) \longrightarrow \mathfrak{so}(3)$ of Example 7.1 explicitly.

Exercise 7.2. Show that no Lie group can be homeomorphic to the sphere S^k if k is even. On the other hand, show that $\mathrm{SU}(2) \cong S^3$. (**Hint**: Use Exercise 6.1.)

Exercise 7.3. Let J be the matrix (5.3). Let $\mathfrak{o}(N, \mathbb{C})$ and $\mathfrak{o}_J(\mathbb{C})$ be the complexified Lie algebras of the groups $\mathrm{O}(N)$ and $\mathrm{O}_J(\mathbb{C})$ in Exercise 5.9. Show that these complex Lie algebras are isomorphic. Describe $\mathfrak{o}(N, \mathbb{C})$ explicitly, i.e., write down a typical matrix.