3

Compact Operators

If $\mathfrak{H}$ is a normed vector space, a linear operator $T : \mathfrak{H} \to \mathfrak{H}$ is called *bounded* if there exists a constant C such that $|Tx| \leqslant C|x|$ for all $x \in \mathfrak{H}$. In this case, the smallest such C is called the *operator norm* of T, and is denoted $|T|$. The boundedness of the operator T is equivalent to its continuity. If $\mathfrak{H}$ is a Hilbert space, then a bounded operator T is *self-adjoint* if

$$\langle Tf, g \rangle = \langle f, Tg \rangle$$

for all $f, g \in \mathfrak{H}$. As usual, we call f an *eigenvector* with *eigenvalue* λ if $f \neq 0$ and $Tf = \lambda f$. Given λ, the set of eigenvectors with eigenvalue λ (together with 0, which is not an eigenvector) is called the λ-*eigenspace*. It follows from elementary and well-known arguments that if T is a self-adjoint bounded operator, then its eigenvalues are real, and the eigenspaces corresponding to distinct eigenvalues are orthogonal. Moreover, if $V \subset \mathfrak{H}$ is a subspace such that $T(V) \subset V$, it is easy to see that also $T(V^\perp) \subset V^\perp$.

A bounded operator $T : \mathfrak{H} \to \mathfrak{H}$ is *compact* if whenever $\{x_1, x_2, x_3, \ldots\}$ is any bounded sequence in $\mathfrak{H}$, the sequence $\{Tx_1, Tx_2, \ldots\}$ has a convergent subsequence.

Theorem 3.1 (Spectral theorem for compact operators). *Let T be a compact self-adjoint operator on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{N}$ be the nullspace of T. Then the Hilbert space dimension of $\mathfrak{N}^\perp$ is at most countable. $\mathfrak{N}^\perp$ has an orthonormal basis ϕ_i $(i = 1, 2, 3, \ldots)$ of eigenvectors of T so that $T\phi_i = \lambda_i \phi_i$. If $\mathfrak{N}^\perp$ is not finite-dimensional, the eigenvalues $\lambda_i \to 0$ as $i \to \infty$.*

Since the eigenvalues $\lambda_i \to 0$, if λ is any nonzero eigenvalue, it follows from this statement that the λ-eigenspace is finite-dimensional.

Proof. This depends upon the equality

$$|T| = \sup_{0 \neq x \in \mathfrak{H}} \frac{|\langle Tx, x \rangle|}{\langle x, x \rangle}. \tag{3.1}$$

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To prove this, let B denote the right-hand side. If $0 \neq x \in \mathfrak{H}$,

$$| \langle Tx, x \rangle | \leqslant |Tx| \cdot |x| \leqslant |T| \cdot |x|^2 = |T| \cdot \langle x, x \rangle ,$$

so $B \leqslant |T|$. We must prove the converse. Let $\lambda > 0$ be a constant, to be determined later. Using $\langle T^2 x, x \rangle = \langle Tx, Tx \rangle$, we have

$$\begin{aligned} &\langle Tx, Tx \rangle \\ &\quad = \tfrac{1}{4} \left| \left\langle T(\lambda x + \lambda^{-1}\, Tx), \lambda x + \lambda^{-1}\, Tx \right\rangle - \left\langle T(\lambda x - \lambda^{-1}\, Tx), \lambda x - \lambda^{-1}\, Tx \right\rangle \right| \\ &\quad \leqslant \tfrac{1}{4} \left| \left\langle T(\lambda x + \lambda^{-1}\, Tx), \lambda x + \lambda^{-1}\, Tx \right\rangle \right| + \left| \left\langle T(\lambda x - \lambda^{-1}\, Tx), \lambda x - \lambda^{-1}\, Tx \right\rangle \right| \\ &\quad \leqslant \tfrac{1}{4} \left[B \left\langle \lambda x + \lambda^{-1}\, Tx, \lambda x + \lambda^{-1}\, Tx \right\rangle + B \left\langle \lambda x - \lambda^{-1}\, Tx, \lambda x - \lambda^{-1}\, Tx \right\rangle \right] \\ &\quad = \tfrac{B}{2} \left[\lambda^2 \langle x, x \rangle + \lambda^{-2} \langle Tx, Tx \rangle \right] . \end{aligned}$$

Now taking $\lambda = \sqrt{|Tx|/|x|}$, we obtain

$$|Tx|^2 = \langle Tx, Tx \rangle \leqslant B|x|\, |Tx|,$$

so $|Tx| \leqslant B|x|$, which implies that $|T| \leqslant B$, whence (3.1).

We now prove that $\mathfrak{N}^\perp$ has an orthonormal basis consisting of eigenvectors of T. It is an easy consequence of self-adjointness that $\mathfrak{N}^\perp$ is T-stable. Let Σ be the set of all orthonormal subsets of $\mathfrak{N}^\perp$ whose elements are eigenvectors of T. Ordering Σ by inclusion, Zorn's lemma implies that it has a maximal element S. Let V be the closure of the linear span of S. We must prove that $V = \mathfrak{N}^\perp$. Let $\mathfrak{H}_0 = V^\perp$. We wish to show $\mathfrak{H}_0 = \mathfrak{N}$. It is obvious that $\mathfrak{N} \subseteq \mathfrak{H}_0$. To prove the opposite inclusion, note that $\mathfrak{H}_0$ is stable under T, and T induces a compact self-adjoint operator on $\mathfrak{H}_0$. What we must show is that $T|\mathfrak{H}_0 = 0$. If T has a nonzero eigenvector in $\mathfrak{H}_0$, this will contradict the maximality of Σ. It is therefore sufficient to show that *a compact self-adjoint operator on a nonzero Hilbert space has an eigenvector.*

Replacing $\mathfrak{H}$ by $\mathfrak{H}_0$, we are therefore reduced to the easier problem of showing that if $T \neq 0$, then T has a nonzero eigenvector. By (3.1), there is a sequence $x_1, x_2, x_3, \ldots$ of unit vectors such that $| \langle Tx_i, x_i \rangle | \to |T|$. Observe that if $x \in \mathfrak{H}$, we have

$$\langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\langle Tx, x \rangle}$$

so the $\langle Tx_i, x_i \rangle$ are real; we may therefore replace the sequence by a subsequence such that $\langle Tx_i, x_i \rangle \to \lambda$, where $\lambda = \pm |T|$. Since $T \neq 0$, $\lambda \neq 0$. Since T is compact, there exists a further subsequence $\{x_i\}$ such that Tx_i converges to a vector v. We will show that $x_i \to \lambda^{-1} v$.

Observe first that

$$| \langle Tx_i, x_i \rangle | \leqslant |Tx_i|\, |x_i| = |Tx_i| \leqslant |T|\, |x_i| = |\lambda|,$$

and since $\langle Tx_i, x_i \rangle \to \lambda$, it follows that $|Tx_i| \to |\lambda|$. Now

$$|\lambda\, x_i - Tx_i|^2 = \langle \lambda\, x_i - Tx_i, \lambda\, x_i - Tx_i \rangle = \lambda^2 |x_i|^2 + |Tx_i|^2 - 2\lambda \langle Tx_i, x_i \rangle ,$$

and since $|x_i| = 1$, $|Tx_i| \to |\lambda|$, and $\langle Tx_i, x_i \rangle \to \lambda$, this converges to 0. Since $Tx_i \to v$, the sequence λx_i therefore also converges to v, and $x_i \to \lambda^{-1} v$. Now, by continuity, $Tx_i \to \lambda^{-1}\, Tv$, so $v = \lambda^{-1}\, Tv$. This proves that v is an eigenvector with eigenvalue λ. This completes the proof that $\mathfrak{N}^\perp$ has an orthonormal basis consisting of eigenvectors.

Now let $\{\phi_i\}$ be this orthonormal basis and let λ_i be the corresponding eigenvalues. If $\epsilon > 0$ is given, only finitely many $|\lambda_i| > \epsilon$ since otherwise we can find an infinite sequence of ϕ_i with $|T\phi_i| > \epsilon$. Such a sequence will have no convergent subsequence, contradicting the compactness of T. Thus, $\mathfrak{N}^\perp$ is countable-dimensional, and we may arrange the $\{\phi_i\}$ in a sequence. If it is infinite, we see the $\lambda_i \longrightarrow 0$. □

Proposition 3.1. *Let X and Y be compact topological spaces with Y a metric space with distance function d. Let U be a set of continuous maps $X \longrightarrow Y$ such that for every $x \in X$ and every $\epsilon > 0$ there exists a neighborhood N of x such that $d\big(f(x), f(x')\big) < \epsilon$ for all $x' \in N$ and for all $f \in U$. Then every sequence in U has a uniformly convergent subsequence.*

We refer to the hypothesis on U as *equicontinuity.*

Proof. Let $S_0 = \{f_1, f_2, f_3, \ldots\}$ be a sequence in U. We will show that it has a convergent subsequence. We will construct a subsequence that is uniformly Cauchy and hence has a limit. For every $n > 1$, we will construct a subsequence $S_n = \{f_{n1}, f_{n2}, f_{n3}, \ldots\}$ of S_{n-1} such that $\sup_{x \in X} d\big(f_{ni}(x), f_{nj}(x)\big) \leqslant 1/n$.

Assume that S_{n-1} is constructed. For each $x \in X$, equicontinuity guarantees the existence of an open neighborhood N_x of x such that $d\big(f(y), f(x)\big) \leqslant \frac{1}{3n}$ for all $y \in N_x$ and all $f \in X$. Since X is compact, we can cover X by a finite number of these sets, say $N_{x_1}, \ldots, N_{x_m}$. Since the $f_{n-1,i}$ take values in the compact space Y, the m-tuples $\big(f_{n-1,i}(x_1), \ldots, f_{n-1,i}(x_m)\big)$ have an accumulation point, and we may therefore select the subsequence $\{f_{ni}\}$ such that $d\big(f_{ni}(x_k), f_{nj}(x_k)\big) \leqslant \frac{1}{3n}$ for all i, j and $1 \leqslant k \leqslant m$. Then for any y, there exists x_k such that $y \in N_{x_k}$ and

$$\begin{aligned} d\big(f_{ni}(y), f_{nj}(y)\big) \leqslant d\big(f_{ni}(y), f_{ni}(x_k)\big) + d\big(f_{ni}(x_k), f_{nj}(x_k)\big) \\ + d\big(f_{nj}(y), f_{nj}(x_k)\big) \leqslant \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} = \frac{1}{n}. \end{aligned}$$

This completes the construction of the sequences $\{f_{ni}\}$.

The diagonal sequence $\{f_{11}, f_{22}, f_{33}, \ldots\}$ is uniformly Cauchy. Since Y is a compact metric space, it is complete, and so this sequence is uniformly convergent. □

We topologize $C(X)$ by giving it the L^∞ norm $|\ |_\infty$ (sup norm).

Proposition 3.2 (Ascoli and Arzela). *Suppose that X is a compact space and that $U \subset C(X)$ is a bounded subset such that for each $x \in X$ and $\epsilon > 0$ there is a neighborhood N of x such that $|f(x) - f(y)| \leqslant \epsilon$ for all $y \in N$ and all $f \in U$. Then every sequence in U has a uniformly convergent subsequence.*

Again, the hypothesis on U is called *equicontinuity.*

Proof. Since U is bounded, there is a compact interval $Y \subset \mathbb{R}$ such that all functions in U take values in Y. The result follows from Proposition 3.1. □

Exercises

Exercise 3.1. Suppose that T is a bounded operator on the Hilbert space $\mathfrak{H}$, and suppose that for each $\epsilon > 0$ there exists a compact operator T_ϵ such that $|T - T_\epsilon| < \epsilon$. Show that T is compact. (Use a diagonal argument like the proof of Proposition 3.1.)

Exercise 3.2 (Hilbert–Schmidt operators). Let X be a locally compact Hausdorff space with a positive Borel measure μ. Assume that $L^2(X)$ has a countable basis. Let $K \in L^2(X \times X)$. Consider the operator on $L^2(X)$ with kernel K defined by

$$Tf(x) = \int_X K(x,y)\, f(y)\, \mathrm{d}\mu(y).$$

Let ϕ_i be an orthonormal basis of $L^2(X)$. Expand K in a Fourier expansion:

$$K(x,y) = \sum_{i=1}^{\infty} \psi_i(x)\, \overline{\phi_i(y)}, \qquad \psi_i = T\phi_i.$$

Show that $\sum |\psi_i|^2 = \int \int |K(x,y)|^2 \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) < \infty$. Consider the operator T_N with kernel

$$K_N(x,y) = \sum_{i=1}^{N} \psi_i(x)\, \overline{\phi_i(y)}.$$

Show that T_N is compact, and deduce that T is compact.