2

Schur Orthogonality

In this chapter and the next two, we will consider the representation theory of compact groups. Let us begin with a few observations about this theory and its relationship to some related theories.

If V is a finite-dimensional complex vector space, or more generally a Banach space, and $\pi : G \longrightarrow \mathrm{GL}(V)$ a continuous homomorphism, then (π, V) is called a *representation*. Assuming $\dim(V) < \infty$, the function $\chi_\pi(g) = \mathrm{tr}\ \pi(g)$ is called the *character* of π. Also assuming $\dim(V) < \infty$, the representation (π, V) is called *irreducible* if V has no proper nonzero invariant subspaces, and a character is called *irreducible* if it is a character of an irreducible representation.

[If V is an infinite-dimensional topological vector space, then (π, V) is called irreducible if it has no proper nonzero invariant *closed* subspaces.]

A quasicharacter χ is a character in this sense since we can take $V = \mathbb{C}$ and $\pi(g)v = \chi(g)v$ to obtain a representation whose character is χ.

The archetypal compact Abelian group is the circle $\mathbb{T} = \big\{ z \in \mathbb{C}^\times \bigm| |z| = 1 \big\}$. We normalize the Haar measure on $\mathbb{T}$ so that it has volume 1. Its characters are the functions $\chi_n : \mathbb{T} \longrightarrow \mathbb{C}^\times$, $\chi_n(z) = z^n$. The important properties of the χ_n are that they form an orthonormal system and (deeper) an orthonormal basis of $L^2(\mathbb{T})$.

More generally, if G is a compact Abelian group, the characters of G form an orthonormal basis of $L^2(G)$. If $f \in L^2(G)$, we have a Fourier expansion,

$$f(g) = \sum_\chi a_\chi\, \chi(g), \qquad a_\chi = \int_G f(g)\overline{\chi(g)}\, \mathrm{d}g, \tag{2.1}$$

and the Plancherel formula is the identity:

$$\int_G |f(g)|^2\, \mathrm{d}g = \sum_\chi |a_\chi|^2. \tag{2.2}$$

These facts can be directly generalized in two ways. First, Fourier analysis on locally compact Abelian groups, including Pontriagin duality, Fourier

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inversion, the Plancherel formula, etc. is an important and complete theory due to Weil [169] and discussed, for example, in Rudin [140] or Loomis [121]. The most important difference from the compact case is that the characters can vary continuously. The characters themselves form a group, the *dual group* $\hat{G}$, whose topology is that of uniform convergence on compact sets. The Fourier expansion (2.1) is replaced by the *Fourier inversion formula*

$$f(g) = \int_{\hat{G}} \hat{f}(\chi)\,\chi(g)\,\mathrm{d}\chi, \qquad \hat{f}(\chi) = \int_G f(g)\,\overline{\chi(g)}\,\mathrm{d}g.$$

The symmetry between G and $\hat{G}$ is now evident. Similarly in the Plancherel formula (2.2) the sum on the right is replaced by an integral.

The second generalization, to arbitrary *compact* groups, is the subject of this chapter and the next two. In summary, group representation theory gives a orthonormal basis of $L^2(G)$ in the matrix coefficients of irreducible representations of G and a (more important and very canonical) orthonormal basis of the subspace of $L^2(G)$ consisting of class functions in terms of the characters of the irreducible representations. Most importantly, the irreducible representations are all finite-dimensional. The orthonormality of these sets is Schur orthogonality; the completeness is the Peter–Weyl theorem.

These two directions of generalization can be unified. Harmonic analysis on locally compact groups agrees with representation theory. The Fourier inversion formula and the Plancherel formula now involve the matrix coefficients of the irreducible unitary representations, which may occur in continuous families and are usually infinite-dimensional. This field of mathematics, largely created by Harish-Chandra, is fundamental but beyond the scope of this book. See Knapp [104] for an extended introduction, and Gelfand, Graev and Piatetski-Shapiro [55] and Varadarajan [165] for the Plancherel formula for $\mathrm{SL}(2,\mathbb{R})$.

Although *infinite-dimensional* representations are thus essential in harmonic analysis on a noncompact group such as $\mathrm{SL}(n,\mathbb{R})$, noncompact Lie groups also have irreducible *finite-dimensional* representations, which are important in their own right. They are seldom unitary and hence not relevant to the Plancherel formula. The scope of this book includes finite-dimensional representations of Lie groups but not infinite-dimensional ones.

In this chapter and the next two, we will be mainly concerned with compact groups. In this chapter, all representations will be complex and finite-dimensional except when explicitly noted otherwise.

By an *inner product* on a complex vector space, we mean a positive definite Hermitian form, denoted $\langle\ ,\ \rangle$. Thus, $\langle v, w\rangle$ is linear in v, conjugate linear in w, satisfies $\langle w, v\rangle = \overline{\langle v, w\rangle}$, and $\langle v, v\rangle > 0$ if $v \neq 0$. We will also use the term *inner product* for real vector spaces—an inner product on a real vector space is a positive definite symmetric bilinear form. Given a group G and a real or complex representation $\pi : G \longrightarrow \mathrm{GL}(V)$, we say the inner product $\langle\ ,\ \rangle$ on V is *invariant* or *G-equivariant* if it satisfies the identity

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle \,.$$

Proposition 2.1. *If G is compact and (π, V) is any finite-dimensional complex representation, then V admits a G-equivariant inner product.*

Proof. Start with an arbitrary inner product $\langle\!\langle \, , \, \rangle\!\rangle$. Averaging it gives another inner product,

$$\langle v, w \rangle = \int_G \langle\!\langle \pi(g)v, \pi(g)w \rangle\!\rangle \, \mathrm{d}g,$$

for it is easy to see that this inner product is Hermitian and positive definite. It is G-invariant by construction. □

Proposition 2.2. *If G is compact, then each finite-dimensional representation is the direct sum of irreducible representations.*

Proof. Let (π, V) be given. Let V_1 be a nonzero invariant subspace of minimal dimension. It is clearly irreducible. Let $V_1^\perp$ be the orthogonal complement of V_1 with respect to a G-invariant inner product. It is easily checked to be invariant and is of lower dimension than V. By induction $V_1^\perp = V_2 \oplus \cdots \oplus V_n$ is a direct sum of invariant subspaces and so $V = V_1 \oplus \cdots \oplus V_n$ is also. □

A function of the form $\phi(g) = L\big(\pi(g)\, v\big)$, where (π, V) is a finite-dimensional representation of G, $v \in V$ and $L : V \longrightarrow \mathbb{C}$ is a linear functional, is called a *matrix coefficient* on G. This terminology is natural, because if we choose a basis $e_1, \ldots, e_n$, of V, we can identify V with $\mathbb{C}^n$ and represent g by matrices:

$$\pi(g)v = \begin{pmatrix} \pi_{11}(g) & \cdots & \pi_{1n}(g) \\ \vdots & & \vdots \\ \pi_{n1}(g) & \cdots & \pi_{nn}(g) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{j=1}^n v_j e_j .$$

Then each of the n^2 functions π_{ij} is a matrix coefficient. Indeed

$$\pi_{ij}(g) = L_i\big(\pi(g) e_j\big),$$

where $L_i(\sum_j v_j e_j) = v_i$.

Proposition 2.3. *The matrix coefficients of G are continuous functions. The pointwise sum or product of two matrix coefficients is a matrix coefficient, so they form a ring.*

Proof. If $v \in V$, then $g \longrightarrow \pi(g)v$ is continuous since by definition a representation $\pi : G \longrightarrow \mathrm{GL}(V)$ is continuous and so a matrix coefficient $L\big(\pi(g)\, v\big)$ is continuous.

If (π_1, V_1) and (π_2, V_2) are representations, $v_i \in V_i$ are vectors and $L_i : V_i \longrightarrow \mathbb{C}$ are linear functionals, then we have representations $\pi_1 \oplus \pi_2$ and $\pi_1 \otimes \pi_2$ on $V_1 \oplus V_2$ and $V_1 \otimes V_2$, respectively. Given vectors $v_i \in V_i$ and functionals $L_i \in V_i^*$, then $L_1\big(\pi(g)v_1\big) \pm L_2\big(\pi(g)v_2\big)$ can be expressed as

$L\big((\pi_1 \oplus \pi_2)(g)(v_1, v_2)\big)$ where $L : V_1 \oplus V_2 \longrightarrow \mathbb{C}$ is $L(x_1, x_2) = L_1(x_1) \pm L_2(x_2)$, so the matrix coefficients are closed under addition and subtraction.

Similarly, we have a linear functional $L_1 \otimes L_2$ on $V_1 \otimes V_2$ satisfying

$$(L_1 \otimes L_2)(x_1 \otimes x_2) = L_1(x_1) L_2(x_2)$$

and

$$(L_1 \otimes L_2)\big((\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2)\big) = L_1\big(\pi_1(g)v_1\big)\, L_2\big(\pi_2(g)v_2\big),$$

proving that the product of two matrix coefficients is a matrix coefficient. □

If (π, V) is a representation, let V^* be the dual space of V. To emphasize the symmetry between V and V^*, let us write the dual pairing $V \times V^* \longrightarrow \mathbb{C}$ in the symmetrical form $L(v) = [\![v, L]\!]$. We have a representation $(\hat{\pi}, V^*)$, called the *contragredient* of π, defined by

$$[\![v, \hat{\pi}(g)L]\!] = \big[\!\big[\pi(g^{-1})v, L\big]\!\big]\,. \tag{2.3}$$

Note that the inverse is needed here so that $\hat{\pi}(g_1 g_2) = \hat{\pi}(g_1)\hat{\pi}(g_2)$.

If (π, V) is a representation, then by Proposition 2.3 any linear combination of functions of the form $L\big(\pi(g)\, v\big)$ with $v \in V$, $L \in V^*$ is a matrix coefficient, though it may be a function $L'\big(\pi'(g)\, v'\big)$ where (π', V') is not (π, V), but a larger representation. Nevertheless, we call any linear combination of functions of the form $L\big(\pi(g)\, v\big)$ a *matrix coefficient of the representation* (π, V). Thus, the matrix coefficients of π form a vector space, which we will denote by $\mathcal{M}_\pi$. Clearly, $\dim(\mathcal{M}_\pi) \leqslant \dim(V)^2$.

Proposition 2.4. *If f is a matrix coefficient of (π, V), then $\check{f}(g) = f(g^{-1})$ is a matrix coefficient of $(\hat{\pi}, V^*)$.*

Proof. This is clear from (2.3), regarding v as a linear functional on V^*. □

We have actions of G on the space of functions on G by left and right translation. Thus if f is a function and $g \in G$, the left and right translates are

$$\big(\lambda(g)f\big)(x) = f(g^{-1}x), \qquad \big(\rho(g)f\big)(x) = f(xg).$$

Theorem 2.1. *Let f be a function on G. The following are equivalent.*

(i) The functions $\lambda(g)f$ span a finite-dimensional vector space.
(ii) The functions $\rho(g)f$ span a finite-dimensional vector space.
(iii) The function f is a matrix coefficient of a finite-dimensional representation.

Proof. It is easy to check that if f is a matrix coefficient of a particular representation V, then so are $\lambda(g)f$ and $\rho(g)f$ for any $g \in G$. Since V is finite-dimensional, its matrix coefficients span a finite-dimensional vector space; in fact, a space of dimension at most $\dim(V)^2$. Thus, (iii) implies (i) and (ii).

Suppose that the functions $\rho(g)f$ span a finite-dimensional vector space V. Then (ρ, V) is a finite-dimensional representation of G, and we claim that f is a matrix coefficient. Indeed, define a functional $L : V \longrightarrow \mathbb{C}$ by $L(\phi) = \phi(1)$. Clearly, $L\big(\rho(g)f\big) = f(g)$, so f is a matrix coefficient, as required. Thus (ii) implies (iii).

Finally, if the functions $\lambda(g)f$ span a finite-dimensional space, composing these functions with $g \longrightarrow g^{-1}$ gives another finite-dimensional space which is closed under right translation, and $\check{f}$ defined as in Proposition 2.4 is an element of this space; hence $\check{f}$ is a matrix coefficient by the case just considered. By Proposition 2.4, f is also a matrix coefficient, so (i) implies (iii). □

If (π_1, V_1) and (π_2, V_2) are representations, an *intertwining operator*, also known as a *G-equivariant map* $T : V_1 \longrightarrow V_2$ or (since V_1 and V_2 are sometimes called *G-modules*) a *G-module homomorphism*, is a linear transformation $T : V_1 \longrightarrow V_2$ such that

$$T \circ \pi_1(g) = \pi_2(g) \circ T$$

for $g \in G$. We will denote by $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ the space of all linear transformations $V_1 \longrightarrow V_2$ and by $\mathrm{Hom}_G(V_1, V_2)$ the subspace of those that are intertwining maps.

For the remainder of this chapter, unless otherwise stated, G will denote a compact group.

Theorem 2.2 (Schur's lemma).

(i) Let (π_1, V_1) and (π_2, V_2) be irreducible representations, and let $T : V_1 \longrightarrow V_2$ be an intertwining operator. Then either T is zero or it is an isomorphism.

(ii) Suppose that (π, V) is an irreducible representation of G and $T : V \longrightarrow V$ is an intertwining operator. Then there exists a scalar $\lambda \in \mathbb{C}$ such that $T(v) = \lambda v$ for all $v \in V$.

Proof. For (i), the kernel of T is an invariant subspace of V_1, which is assumed irreducible, so if T is not zero, $\ker(T) = 0$. Thus, T is injective. Also, the image of T is an invariant subspace of V_2. Since V_2 is irreducible, if T is not zero, then $\mathrm{im}(T) = V_2$. Therefore T is bijective, so it is an isomorphism.

For (ii), let λ be any eigenvalue of T. Let $I : V \longrightarrow V$ denote the identity map. The linear transformation $T - \lambda I$ is an intertwining operator that is not an isomorphism, so it is the zero map by (i). □

We are assuming that G is compact. The Haar volume of G is therefore finite, and we normalize the Haar measure so that the volume of G is 1.

We will consider the space $L^2(G)$ of functions on G that are square-integrable with respect to the Haar measure. This is a Hilbert space with the inner product

$$\langle f_1, f_2 \rangle_{L^2} = \int_G f_1(g)\, \overline{f_2(g)}\, \mathrm{d}g.$$

Schur orthogonality will give us an orthonormal basis for this space.

If (π, V) is a representation and $\langle\ ,\ \rangle$ is an invariant inner product on V, then every linear functional is of the form $x \longrightarrow \langle x, v\rangle$ for some $v \in V$. Thus a matrix coefficient may be written in the form $g \longrightarrow \langle \pi(g)w, v\rangle$, and such a representation will be useful to us in our discussion of Schur orthogonality.

Lemma 2.1. *Suppose that (π_1, V_1) and (π_2, V_2) are complex representations of the compact group G. Let $\langle\ ,\rangle$ be any inner product on V_1. If $v_i, w_i \in V_i$, then the map $T : V_1 \longrightarrow V_2$ given by*

$$T(w) = \int_G \langle \pi_1(g)w, v_1\rangle\, \pi_2(g^{-1})v_2\, \mathrm{d}g \tag{2.4}$$

is G-equivariant.

Proof. We have

$$T\big(\pi_1(h)w\big) = \int_G \langle \pi_1(gh)w, v_1\rangle\, \pi_2(g^{-1})v_2\, \mathrm{d}g.$$

The variable change $g \longrightarrow gh^{-1}$ shows that this equals $\pi_2(h)T(w)$, as required. □

Theorem 2.3 (Schur orthogonality). *Suppose that (π_1, V_1) and (π_2, V_2) are irreducible representations of the compact group G. Either every matrix coefficient of π_1 is orthogonal in $L^2(G)$ to every matrix coefficient of π_2, or the representations are isomorphic.*

Proof. We must show that if there exist matrix coefficients $f_i : G \longrightarrow \mathbb{C}$ of π_i that are *not* orthogonal, then there is an isomorphism $T : V_1 \longrightarrow V_2$. We may assume that the f_i have the form $f_i(g) = \langle \pi_i(g)w_i, v_i\rangle$ since functions of that form span the spaces of matrix coefficients of the representations π_i. Here we use the notation $\langle\ ,\ \rangle$ to denote invariant bilinear forms on both V_1 and V_2, and $v_i, w_i \in V_i$. Then our assumption is that

$$\int_G \langle \pi_1(g)w_1, v_1\rangle \left\langle \pi_2(g^{-1})v_2, w_2\right\rangle \mathrm{d}g = \int_G \langle \pi_1(g)w_1, v_1\rangle\, \overline{\langle \pi_2(g)w_2, v_2\rangle}\, \mathrm{d}g \neq 0.$$

Define $T : V_1 \longrightarrow V_2$ by (2.4). The map is nonzero since the last inequality can be written $\langle T(w_1), w_2\rangle \neq 0$. It is an isomorphism by Schur's lemma. □

This gives orthogonality for matrix coefficients coming from *nonisomorphic* irreducible representations. But what about matrix coefficients from the same representation? (If the representations are isomorphic, we may as well assume they are equal.) The following result gives us an answer to this question.

Theorem 2.4 (Schur orthogonality). *Let (π, V) be an irreducible representation of the compact group G, with invariant inner product $\langle\ ,\ \rangle$. Then there exists a constant $d > 0$ such that*

$$\int_G \langle \pi(g)w_1, v_1\rangle \overline{\langle \pi(g)w_2, v_2\rangle}\, \mathrm{d}g = d^{-1} \langle w_1, w_2\rangle \langle v_2, v_1\rangle . \tag{2.5}$$

Later, in Proposition 2.9, we will show that $d = \dim(V)$.

Proof. We will show that if v_1 and v_2 are fixed, there exists a constant $c(v_1, v_2)$ such that

$$\int_G \langle \pi(g)w_1, v_1\rangle \overline{\langle \pi(g)w_2, v_2\rangle}\, \mathrm{d}g = c(v_1, v_2) \langle w_1, w_2\rangle . \tag{2.6}$$

Indeed, T given by (2.4) is G-equivariant, so by Schur's lemma it is a scalar. Thus, there is a constant $c = c(v_1, v_2)$ depending only on v_1 and v_2 such that $T(w) = cw$. In particular, $T(w_1) = cw_1$, and so the right-hand side of (2.6) equals

$$\langle T(w_1), w_2\rangle = \int_G \langle \pi(g)w_1, v_1\rangle \left\langle \pi(g^{-1})v_2, w_2\right\rangle \mathrm{d}g,$$

Now the variable change $g \longrightarrow g^{-1}$ and the properties of the inner product show that this equals the left-hand side of (2.6), proving the identity. The same argument shows that there exists another constant $c'(w_1, w_2)$ such that for all v_1 and v_2 we have

$$\int_G \langle \pi(g)w_1, v_1\rangle \overline{\langle \pi(g)w_2, v_2\rangle}\, \mathrm{d}g = c'(w_1, w_2) \langle v_2, v_1\rangle .$$

Combining this with (2.6), we get (2.5). We will compute d later in Proposition 2.9, but for now we simply note that it is positive since, taking $w_1 = w_2$ and $v_1 = v_2$, both the left-hand side of (2.5) and the two inner products on the right-hand side are positive. □

Before we turn to the evaluation of the constant d, we will prove a different orthogonality for the characters of irreducible representations (Theorem 2.5). This will require some preparations.

Proposition 2.5. *The character χ of a representation (π, V) is a matrix coefficient of V.*

Proof. If $v_1, \ldots, v_n$ is a matrix of V, and $L_1, \ldots, L_n$ is the dual basis of V^*, then $\chi(g) = \sum_{i=1}^n L_i\big(\pi(g)v_i\big)$. □

Proposition 2.6. *Suppose that (π, V) is a representation of G. Let χ be the character of π.*

(i) If $g \in V$ then $\chi(g^{-1}) = \overline{\chi(g)}$.

(ii) Let $(\hat{\pi}, V^*)$ be the contragredient representation of π. Then the character of $\hat{\pi}$ is the complex conjugate $\overline{\chi}$ of the character χ of G.

Proof. Since $\pi(g)$ is unitary with respect to an invariant inner product $\langle\ ,\ \rangle$, its eigenvalues $t_1, \ldots, t_n$ all have absolute value 1, and so

$$\mathrm{tr}\ \pi(g)^{-1} = \sum_i t_i^{-1} = \sum_i \overline{t_i} = \overline{\chi(g)}.$$

This proves (i). As for (ii), referring to (2.3), $\hat{\pi}(g)$ is the adjoint of $\pi(g)^{-1}$ with respect to the dual pairing $[\![\ ,\ ]\!]$, so its trace equals the trace of $\pi(g)^{-1}$. □

The *trivial representation* of any group G is the representation on a one-dimensional vector space V with $\pi(g)v = v$ being the trivial action.

Proposition 2.7. *If (π, V) is an irreducible representation and χ its character, then*

$$\int_G \chi(g)\,\mathrm{d}g = \begin{cases} 1 & \text{if } \pi \text{ is the trivial representation;} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The character of the trivial representation is just the constant function 1, and since we normalized the Haar measure so that G has volume 1, this integral is 1 if π is trivial. In general, we may regard $\int_G \chi(g)\,\mathrm{d}g$ as the inner product of χ with the character 1 of the trivial representation, and if π is nontrivial, these are matrix coefficients of different irreducible representations and hence orthogonal by Theorem 2.3. □

If (π, V) is a representation, let V^G be the subspace of *G-invariants*, that is,

$$V^G = \{v \in V \mid \pi(g)v = v \text{ for all } g \in G\}.$$

Proposition 2.8. *If (π, V) is a representation of G and χ its character, then*

$$\int_G \chi(g)\,\mathrm{d}g = \dim(V^G).$$

Proof. Decompose $V = \oplus_i V_i$ into a direct sum of irreducible invariant subspaces, and let χ_i be the character of the restriction π_i of π to V_i. By Proposition 2.7, $\int_G \chi_i(g)\,\mathrm{d}g = 1$ if and only if π_i is trivial. Hence $\int_G \chi(g)\,\mathrm{d}g$ is the number of trivial π_i. The direct sum of the V_i with π_i trivial is V^G, and the statement follows. □

If (π_1, V_1) and (π_2, V_2) are irreducible representations, and χ_1 and χ_2 are their characters, we have already noted in proving Proposition 2.3 that we may form representations $\pi_1 \oplus \pi_2$ and $\pi_1 \otimes \pi_2$ on $V_1 \oplus V_2$ and $V_1 \otimes V_2$. It is easy to see that $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$ and $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1}\chi_{\pi_2}$. It is not quite true that the characters form a ring. Certainly the negative of a matrix coefficient is a matrix coefficient, yet the negative of a character is not a character. The set of characters is closed under addition and multiplication but not subtraction. We define a *generalized* (or *virtual*) *character* to be a function of the form $\chi_1 - \chi_2$, where χ_1 and χ_2 are characters. It is now clear that the generalized characters form a ring.

Lemma 2.2. *Define a representation* $\Psi : \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C}) \longrightarrow \mathrm{GL}(\Omega)$ *where* $\Omega = \mathrm{Mat}_{n\times m}(\mathbb{C})$ *by* $\Psi(g_1,g_2) : X \longrightarrow g_2 X g_1^{-1}$. *Then the trace of* $\Psi(g_1,g_2)$ *is* $\mathrm{tr}(g_1^{-1})\,\mathrm{tr}(g_2)$.

Proof. Both $\mathrm{tr}\ \Psi(g_1,g_2)$ and $\mathrm{tr}(g_1^{-1})\,\mathrm{tr}(g_2)$ are continuous, and since diagonalizable matrices are dense in $\mathrm{GL}(n,\mathbb{C})$ we may assume that both g_1 and g_2 are diagonalizable. Also if γ is invertible we have $\Psi(\gamma g_1 \gamma^{-1}, g_2) = \Psi(\gamma,1)\Psi(g_1,g_2)\Psi(\gamma,1)^{-1}$ so the trace of both $\mathrm{tr}\ \Psi(g_1,g_2)$ and $\mathrm{tr}(g_1^{-1})\mathrm{tr}(g_2)$ are unchanged if g_1 is replaced by $\gamma g_1 \gamma^{-1}$. So we may assume that g_1 is diagonal, and similarly g_2. Now if $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_m$ are the diagonal entries of g_1 and g_2^{-1}, the effect of $\Psi(g_1,g_2)$ on $X \in \Omega$ is to multiply the columns by the α_i^{-1} and the rows by the β_j. So the trace is $\mathrm{tr}(g_1^{-1})\mathrm{tr}(g_2)$. □

Theorem 2.5 (Schur orthogonality). *Let* (π_1, V_1) *and* (π_2, V_2) *be representations of* G *with characters* χ_1 *and* χ_2. *Then*

$$\int_G \chi_1(g)\,\overline{\chi_2(g)}\,\mathrm{d}g = \dim \mathrm{Hom}_G(V_1, V_2). \tag{2.7}$$

If π_1 *and* π_2 *are irreducible, then*

$$\int_G \chi_1(g)\overline{\chi_2(g)}\,\mathrm{d}g = \begin{cases} 1 & \text{if } \pi_1 \cong \pi_2; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Define a representation Π of G on the space $\Omega = \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ of all linear transformations $T : V_1 \longrightarrow V_2$ by

$$\Pi(g)T = \pi_2(g) \circ T \circ \pi_1(g)^{-1}.$$

By lemma 2.2 and Proposition 2.6, the character of $\Pi(g)$ is $\chi_2(g)\overline{\chi_1(g)}$. The space of invariants Ω^G exactly of the T which are G-module homomorphisms, so by Proposition 2.8 we get

$$\int_G \overline{\chi_1(g)}\,\chi_2(g)\,\mathrm{d}g = \dim \mathrm{Hom}_G(V_1, V_2).$$

Since this is real, we may conjugate to obtain (2.7). □

Proposition 2.9. *The constant* d *in Theorem 2.4 equals* $\dim(V)$.

Proof. Let $v_1, \ldots, v_n$ be an orthonormal basis of V, $n = \dim(V)$. We have

$$\chi(g) = \sum_i \langle \pi_i(g) v_i, v_i \rangle$$

since $\langle \pi(g) v_j, v_i \rangle$ is the i, j component of the matrix of $\pi(g)$ with respect to this basis. Now

$$1 = \int_G |\chi(g)|^2 \, \mathrm{d}g = \sum_{i,j} \int_G \langle \pi(g) v_i, v_i \rangle \, \overline{\langle \pi(g) v_j, v_j \rangle} \, \mathrm{d}g.$$

There are n^2 terms on the right, but by (2.5) only the terms with $i = j$ are nonzero, and those equal d^{-1}. Thus, $d = n$. □

We now return to the matrix coefficients $\mathcal{M}_\pi$ of an irreducible representation (π, V). We define a representation Θ of $G \times G$ on $\mathcal{M}_\pi$ by

$$\Theta(g_1, g_2) f(x) = f(g_2^{-1} x g_1).$$

We also have a representation Π of $G \times G$ on $\mathrm{End}_{\mathbb{C}}(V)$ by

$$\Pi(g_1, g_2) T = \pi(g_2)^{-1} T \, \pi(g_1).$$

Proposition 2.10. *If $f \in \mathcal{M}_\pi$ then so is $\Theta(g_1, g_2) f$. The representations Θ and Π are equivalent.*

Proof. Let $L \in V^*$ and $v \in V$. Define $f_{L,v}(g) = L(\pi(g)v)$. The map $L, v \longmapsto f_{L,v}$ is bilinear, hence induces a linear map $\sigma : V^* \otimes V \longrightarrow \mathcal{M}_\pi$. It is surjective by the definition of $\mathcal{M}_\pi$, and it follows from Proposition 2.4 that if L_i and v_j run through orthonormal bases, then f_{L_i, v_j} are orthonormal, hence linearly independent. Therefore, σ is a vector space isomorphism. We have

$$\Theta(g_1, g_2) f_{L,v}(g) = L(g_2^{-1} g g_1 v) = f_{\hat{\pi}(g_2) L, \pi(g_1) v}(x),$$

where we recall that $(\hat{\pi}, V^*)$ is the contragredient representation. This means that σ is a $G \times G$-module homomorphism and so $\mathcal{M}_\pi \cong V^* \otimes V$ as $G \times G$-modules. On the other hand we also have a bilinear map $V^* \times V \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ that associates with (L, v) the rank-one linear map $T_{L,v}(u) = L(u)v$. This induces an isomorphism $V^* \otimes V \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ which is $G \times G$ equivariant. We see that $\mathcal{M}_\pi \cong V^* \otimes V \cong \mathrm{End}_{\mathbb{C}}(V)$. □

A function f on G is called a *class function* if it is constant on conjugacy classes, that is, if it satisfies the equation $f(hgh^{-1}) = f(g)$. The character of a representation is a class function since the trace of a linear transformation is unchanged by conjugation.

Proposition 2.11. *If f is the matrix coefficient of an irreducible representation (π, V), and if f is a class function, then f is a constant multiple of χ_π.*

Proof. By Schur's lemma, there is a unique G-invariant vector in $\mathrm{Hom}_{\mathbb{C}}(V, V)$; hence. by Proposition 2.10, the same is true of $\mathcal{M}_\pi$ in the action of G by conjugation. This matrix coefficient is of course χ_π. □

Theorem 2.6. *If f is a matrix coefficient and also a class function, then f is a finite linear combination of characters of irreducible representations.*

Proof. Write $f = \sum_{i=1}^{n} f_i$, where each f_i is a class function of a distinct irreducible representation (π_i, V_i). Since f is conjugation-invariant, and since the f_i live in spaces $\mathcal{M}_{\pi_i}$, which are conjugation-invariant and mutually orthogonal, each f_i is itself a class function and hence a constant multiple of χ_{π_i} by Proposition 2.11. □

Exercises

Exercise 2.1. Suppose that G is a compact Abelian group and $\pi : G \longrightarrow \mathrm{GL}(n, \mathbb{C})$ an irreducible representation. Prove that $n = 1$.

Exercise 2.2. Suppose that G is compact group and $f : G \longrightarrow \mathbb{C}$ is the matrix coefficient of an irreducible representation π. Show that $g \longmapsto \overline{f(g^{-1})}$ is a matrix coefficient of the same representation π.

Exercise 2.3. Suppose that G is compact group. Let $C(G)$ be the space of continuous functions on G. If f_1 and $f_2 \in C(G)$, define the convolution $f_1 * f_2$ of f_1 and f_2 by

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})\, f_2(h)\, \mathrm{d}h = \int_G f_1(h)\, f_2(h^{-1}g)\, \mathrm{d}h.$$

(i) Use the variable change $h \longrightarrow h^{-1}g$ to prove the identity of the last two terms. Prove that this operation is associative, and so $C(G)$ is a ring (without unit) with respect to covolution.
(ii) Let π be an irreducible representation. Show that the space $\mathcal{M}_\pi$ of matrix coefficients of π is a 2-sided ideal in $C(G)$, and explain how this fact implies Theorem 2.3.

Exercise 2.4. Let G be a compact group, and let $G \times G$ act on the space $\mathcal{M}_\pi$ by left and right translation: $(g, h)f(x) = f(g^{-1}xh)$. Show that $\mathcal{M}_\pi \cong \hat{\pi} \otimes \pi$ as $(G \times G)$-modules.

Exercise 2.5. Let G be a compact group and let $g, h \in G$. Show that g and h are conjugate if and only if $\chi(g) = \chi(h)$ for every irreducible character χ. Show also that every character is real-valued if and only if every element is conjugate to its inverse.

Exercise 2.6. Let G be a compact group, and let V, W be irreducible G-modules. An *invariant* bilinear form $B : V \times W \to \mathbb{C}$ is one that satisfies $B\big(g\cdot v, g\cdot w\big) = B(v, w)$ for $g \in G$, $v \in V$, $w \in W$. Show that the space of invariant bilinear forms is at most one-dimensional, and is one-dimensional if and only if V and W are contragredient.