1

Haar Measure

If G is a locally compact group, there is, up to a constant multiple, a unique regular Borel measure μ_L that is invariant under left translation. Here *left translation invariance* means that $\mu(X) = \mu(gX)$ for all measurable sets X. *Regularity* means that

$$\mu(X) = \inf \{\mu(U) \mid U \supseteq X, U \text{ open}\} = \sup \{\mu(K) \mid K \subseteq X, K \text{ compact}\} .$$

Such a measure is called a *left Haar measure*. It has the properties that any compact set has finite measure and any nonempty open set has measure > 0.

We will not prove the existence and uniqueness of the Haar measure. See for example Halmos [61], Hewitt and Ross [69], Chap. IV, or Loomis [121] for a proof of this. Left-invariance of the measure amounts to left-invariance of the corresponding integral,

$$\int_G f(\gamma g)\, \mathrm{d}\mu_L(g) = \int_G f(g)\, \mathrm{d}\mu_L(g), \tag{1.1}$$

for any Haar integrable function f on G.

There is also a right-invariant measure, μ_R, unique up to constant multiple, called a *right Haar measure*. Left and right Haar measures may or may not coincide. For example, if

$$G = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R}, y > 0 \right\},$$

then it is easy to see that the left- and right-invariant measures are, respectively,

$$\mathrm{d}\mu_L = y^{-2}\, \mathrm{d}x\, \mathrm{d}y, \qquad \mathrm{d}\mu_R = y^{-1}\, \mathrm{d}x\, \mathrm{d}y.$$

They are not the same. However, there are many cases where they do coincide, and if the left Haar measure is also right-invariant, we call G *unimodular*.

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Conjugation is an automorphism of G, and so it takes a left Haar measure to another left Haar measure, which must be a constant multiple of the first. Thus, if $g \in G$, there exists a constant $\delta(g) > 0$ such that

$$\int_G f(g^{-1}hg)\,\mathrm{d}\mu_L(h) = \delta(g) \int_G f(h)\,\mathrm{d}\mu_L(h).$$

If G is a topological group, a *quasicharacter* is a continuous homomorphism $\chi : G \longrightarrow \mathbb{C}^\times$. If $|\chi(g)| = 1$ for all $g \in G$, then χ is a (linear) *character* or *unitary quasicharacter*.

Proposition 1.1. *The function $\delta : G \longrightarrow \mathbb{R}_+^\times$ is a quasicharacter. The measure $\delta(h)\mu_L(h)$ is right-invariant.*

The measure $\delta(h)\mu_L(h)$ is a right Haar measure, and we may write $\mu_R(h) = \delta(h)\mu_L(h)$. The quasicharacter δ is called the *modular quasicharacter.*

Proof. Conjugation by first g_1 and then g_2 is the same as conjugation by $g_1 g_2$ in one step. Thus $\delta(g_1 g_2) = \delta(g_1)\,\delta(g_2)$, so δ is a quasicharacter. Using (1.1),

$$\delta(g) \int_G f(h)\,\mathrm{d}\mu_L(h) = \int_G f(g \cdot g^{-1}hg)\,\mathrm{d}\mu_L(h) = \int_G f(hg)\,\mathrm{d}\mu_L(h).$$

Replace f by $f\delta$ in this identity and then divide both sides by $\delta(g)$ to find that

$$\int_G f(h)\,\delta(h)\,\mathrm{d}\mu_L(h) = \int_G f(hg)\,\delta(h)\,\mathrm{d}\mu_L(h).$$

Thus, the measure $\delta(h)\,\mathrm{d}\mu_L(h)$ is right-invariant. □

Proposition 1.2. *If G is compact, then G is unimodular and $\mu_L(G) < \infty$.*

Proof. Since δ is a homomorphism, the image of δ is a subgroup of $\mathbb{R}_+^\times$. Since G is compact, $\delta(G)$ is also compact, and the only compact subgroup of $\mathbb{R}_+^\times$ is just $\{1\}$. Thus δ is trivial, so a left Haar measure is right-invariant. We have mentioned as an assumed fact that the Haar volume of any compact subset of a locally compact group is finite, so if G is finite, its Haar volume is finite. □

If G is compact, then it is natural to normalize the Haar measure so that G has volume 1.

To simplify our notation, we will denote $\int_G f(g)\,\mathrm{d}\mu_L(g)$ by $\int_G f(g)\,\mathrm{d}g$.

Proposition 1.3. *If G is unimodular, then the map $g \longrightarrow g^{-1}$ is an isometry.*

Proof. It is easy to see that $g \longrightarrow g^{-1}$ turns a left Haar measure into a right Haar measure. If left and right Haar measures agree, then $g \longrightarrow g^{-1}$ multiplies the left Haar measure by a positive constant, which must be 1 since the map has order 2. □

Exercises

Exercise 1.1. Let $d_{\mathbf{a}}X$ denote the Lebesgue measure on $\mathrm{Mat}_n(\mathbb{R})$. It is of course a Haar measure for the additive group $\mathrm{Mat}_n(\mathbb{R})$. Show that $|\det(X)|^{-n}d_{\mathbf{a}}X$ is both a left and a right Haar measure on $\mathrm{GL}(n,\mathbb{R})$.

Exercise 1.2. Let P be the subgroup of $\mathrm{GL}(r+s,\mathbb{R})$ consisting of matrices of the form

$$p = \begin{pmatrix} g_1 & X \\ & g_2 \end{pmatrix}, \qquad g_1 \in \mathrm{GL}(r,\mathbb{R}),\ g_2 \in \mathrm{GL}(s,\mathbb{R}), \quad X \in \mathrm{Mat}_{r\times s}(\mathbb{R}).$$

Let $\mathrm{d}g_1$ and $\mathrm{d}g_2$ denote Haar measures on $\mathrm{GL}(r,\mathbb{R})$ and $\mathrm{GL}(s,\mathbb{R})$, and let $d_{\mathbf{a}}X$ denote an additive Haar measure on $\mathrm{Mat}_{r\times s}(\mathbb{R})$. Show that

$$d_L p = |\det(g_1)|^{-s}\,\mathrm{d}g_1\,\mathrm{d}g_2\,d_{\mathbf{a}}X, \qquad d_R p = |\det(g_2)|^{-r}\,\mathrm{d}g_1\,\mathrm{d}g_2\,d_{\mathbf{a}}X,$$

are (respectively) left and right Haar measures on P, and conclude that the modular quasicharacter of P is

$$\delta(p) = |\det(g_1)|^{s}\,|\det(g_2)|^{-r}.$$