Chapter 8
A Novel Approach to the Hankel Transform Inversion of the Neutron Diffusion Problem Using the Parseval Identity

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8.1 Introduction

The neutron diffusion equation is still one of the most frequently employed equations for nuclear reactor neutronics calculations, although its limitations are well known [GoLeVi09, ViSeGo04]. The equation is obtained under the assumptions that scattering is isotropic in the laboratory coordinate system and the region of interest is considered piecewise homogeneous, so that the diffusion coefficients are invariant under spatial transforms like translation and others. It is well known that such a derivation of diffusion theory rests on certain assumptions, i.e. the flux being sufficiently smooth especially by virtue of neutron absorption or production, which is reasonable since the mean free path is typically larger than the dimensions of the fuel cell and moderator space geometry. The solution of the diffusion equation system is thus an average description of a large number of neutrons, where fluctuations (higher moments) are neglected [LeEtAl08]. Further, the continuous energy distribution of neutrons is reduced by the use of energy groups (in the present case two).

8.2 Multi-group Steady State Neutron Diffusion

Our starting point is the steady state multi-energy group neutron diffusion equation, with the usual diffusion, removal, out-scattering, fission, and in-scattering terms. Here D_g is the diffusion coefficient for energy group g, $\Delta_r = r^{-1}\partial_r(r\partial_r)$ represents the radial part of the Laplace operator in cylindrical coordinates. Note that we assume translational symmetry of the neutron flux ϕ_g along the cylinder axis and

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thus $\partial_{zz}\phi = 0$. Σ_{Rg} are the respective removal cross section, $\Sigma_{g\to g'}, \Sigma_{g'\to g}$ $(g \neq g')$ the out- and in-scattering cross sections, $\nu_g \Sigma_{fg}$ the fission cross section times the average neutron yield per fission, χ_g the spectral weight of energy group $g \in [1, G]$, k_{eff} the effective multiplication factor, and S_g a generic source term per energy group:

$$-D_g \Delta_r \phi_g + \left(\Sigma_{Rg} + \sum_{g'=1}^{G} \Sigma^s_{g\to g'} \right) \phi_g = \chi_g \sum_{g'=1}^{G} \nu_{g'} \Sigma_{fg'} \phi_{g'} + \sum_{g'=1}^{G} \Sigma_{g'\to g} \phi_{g'} + S_g. \quad (8.1)$$

The diffusion problem is subject to the boundary conditions of zero current density at the center of the cylinder $D_g(\partial\phi_g/\partial r)(0) = 0$ and zero flux at the boundary; that is, $\phi_g(R) = 0$.

8.3 The Hankel-Transformed Problem

The diffusion problem (8.1) previously introduced may be solved by the use of the zero order Hankel transform

$$\bar{f}(\xi) = H_n[f(r); r \to \xi] = \int_0^\infty r f(r) J_n(r\xi)\, dr$$

(here $n = 0$) that renders (8.1) a nonhomogeneous problem and may be cast into matrix form. As an example we show the equation for two energy groups:

$$\begin{pmatrix} D_1\xi^2 + \Sigma_{R1} & -(\chi_1 \nu \Sigma_{f2} + \Sigma_{12}) \\ -(\chi_2 \nu \Sigma_{f1} + \Sigma_{21}) & D_2 \xi^2 + \Sigma_{R2} \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} = \begin{pmatrix} \bar{S}_1 \\ \bar{S}_2 \end{pmatrix}.$$

In shorthand notation, the equation reads $\mathbf{M}(\xi)\bar{\boldsymbol{\Phi}} = \bar{\mathbf{S}}$. In general $\mathbf{M}(\xi)$ is invertible, so that

$$\det(\mathbf{M}(\xi)) = A(\xi)B(\xi) - \mu_{12}\mu_{21} \neq 0,$$

with $A(\xi) = D_1\xi^2 + \Sigma_{R1}$, $B(\xi) = D_2\xi^2 + \Sigma_{R2}$, $\mu_{12} = \chi_1 \nu \Sigma_{f2} + \Sigma_{12}$ and $\mu_{21} = \chi_2 \nu \Sigma_{f1} + \Sigma_{21}$. The solution for the system in transformed variables is

$$\bar{\boldsymbol{\Phi}} = (\det(\mathbf{M}(\xi)))^{-1} \begin{pmatrix} B(\xi)\bar{S}_1 + \mu_{12}\bar{S}_2 \\ \mu_{21}\bar{S}_1 + A(\xi)\bar{S}_2 \end{pmatrix}.$$

In what follows, we introduce a simplification, that does not compromise the generality of the procedure, and consider a source term for group $g = 1$, only. Then

$$\bar{\phi}_1 = B(\xi)\frac{\bar{S}_1}{\det(M(\xi))}, \quad \bar{\phi}_2 = \mu_{21}\frac{\bar{S}_1}{\det(M(\xi))},$$

and upon applying the inverse Hankel transformation one may determine the analytical solution of the problem [Fe11].

8.3.1 Fast Flux Solution

Application of the inversion formula yields

$$\phi_1 = \int_0^\infty \xi \frac{B(\xi)J_0(r\xi)}{\det(M(\xi))}\bar{S}_1 \, d\xi \, ,$$

which together with the Hankel inversion theorem and Parseval's identity allows us to derive the desired result.

Theorem 1 (The Hankel inversion theorem). *If $\sqrt{r'}f(r')$ is piecewise continuous and absolutely integrable on the positive half of the real line, and if $\gamma \geq -\frac{1}{2}$, then $\bar{f}_\gamma(\xi) = H_\gamma[f(r'); r\prime \to \xi]$ exists and*

$$\int_0^\infty \xi \bar{f}_\gamma(\xi) J_\gamma(\xi r') \, d\xi = \frac{1}{2}[f(r'_+) + f(r'_-)].$$

Theorem 2 (Parsenval's relation). *If the functions $f(r')$ and $g(r')$ satisfy the conditions of Theorem 1 and if $\bar{f}_\gamma(\xi)$ and $\bar{g}_\gamma(\xi)$ denote the Hankel transforms of order $\gamma \geq -\frac{1}{2}$, then*

$$\int_0^\infty r' f(r') g(r') \, dr' = \int_0^\infty \xi \bar{f}_\gamma(\xi) \bar{g}_\gamma(\xi) \, d\xi.$$

Making use of the theorem with $\bar{f}_0(\xi) = \frac{B(\xi)J_0(r\xi)}{\det(M(\xi))}$ and $\bar{g}_0(\xi) = \bar{S}_1$, establishes that

$$\int_0^\infty \xi \frac{B(\xi)J_0(r\xi)}{\det(M(\xi))}\bar{S}_1 \, d\xi = \int_0^\infty r' H_0^{-1}\left\{\frac{B(\xi)J_0(r\xi)}{\det(M(\xi))}\right\} S_1(r') \, dr' \, ,$$

and by definition the following identity holds:

$$H_0^{-1}\left\{\frac{B(\xi)J_0(r\xi)}{\det(M(\xi))}\right\} = \int_0^\infty \xi \frac{B(\xi)J_0(r\xi)}{\det(M(\xi))} J_0(r'\xi) \, d\xi.$$

The physically meaningful range of nuclear parameters implies $0 < \frac{\mu_{12}\mu_{21}}{A(\xi)B(\xi)} < 1$, so that

$$\frac{B(\xi)}{A(\xi)B(\xi)-\mu_{12}\mu_{21}} = \frac{1}{A(\xi)} + \frac{1}{A(\xi)} O\left(\left(\frac{\mu_{12}\mu_{21}}{A(\xi)B(\xi)}\right)^2\right),$$

which by virtue of the fact that $\left(\frac{\mu_{12}\mu_{21}}{A(\xi)B(\xi)}\right) \ll 1$ allows to safely neglect higher-order terms. The integral may be evaluated [Ba54] as

$$\int_0^\infty \xi \frac{J_0(r\xi)}{A(\xi)} J_0(r'\xi)\, d\xi = \begin{cases} \frac{1}{D_1} I_0(\sqrt{\alpha_1} r') K_0(\sqrt{\alpha_1} r) & \text{for} \quad 0 < r' < r \\ \frac{1}{D_1} I_0(\sqrt{\alpha_1} r) K_0(\sqrt{\alpha_1} r') & \text{for} \quad r < r' < \infty, \end{cases}$$

where $\alpha_g = \Sigma_{Rg}/D_g$. Here, I_0 and K_0 are the modified Bessel functions and outside of the cylinder the source term is identically zero. The solution for the fast flux is then

$$\phi_1 = \frac{K_0(\sqrt{\alpha_1} r)}{D_1} \int_0^r r' I_0(\sqrt{\alpha_1} r') S_1(r')\, dr' + \frac{I_0(\sqrt{\alpha_1} r)}{D_1} \int_r^R r' K_0(\sqrt{\alpha_1} r') S_1(r')\, dr'.$$

8.3.2 The Thermal Flux Solution

The procedure for the thermal flux follows similar steps to the ones introduced in the solution scheme for the fast flux. Using the inversion formula

$$\phi_2 = \mu_{21} \int_0^\infty \xi \frac{J_0(r\xi)}{\det(M(\xi))} \bar{S}_1\, d\xi$$

together with Theorem 2,

$$\int_0^\infty \xi \frac{J_0(r\xi)}{\det(M(\xi))} \bar{S}_1\, d\xi = \int_0^\infty r' H_0^{-1} \left\{ \frac{J_0(r\xi)}{\det(M(\xi))} \right\} S_1(r')\, dr'$$

and, by definition,

$$H_0^{-1} \left\{ \frac{J_0(r\xi)}{\det(M(\xi))} \right\} = \int_0^\infty \xi \frac{J_0(r\xi)}{\det(M(\xi))} J_0(r'\xi)\, d\xi.$$

Using arguments analogous to those for the fast flux, we arrive at

$$H_0^{-1} \left\{ \frac{J_0(r\xi)}{\det(M(\xi))} \right\} = \frac{1}{(\Sigma_{R2} D_1 - \Sigma_{R1} D_2)} \int_0^\infty \xi \frac{J_0(r\xi)}{\xi^2 + (\sqrt{\alpha_1})^2} J_0(r'\xi)\, d\xi$$

$$-\frac{1}{(\Sigma_{R2}D_1-\Sigma_{R1}D_2)}\int_0^\infty \xi\,\frac{J_0(r\xi)}{\xi^2+(\sqrt{\alpha_2})^2}J_0(r'\xi)\,d\xi$$
$$=\begin{cases}\frac{I_0(\sqrt{\alpha_1}r')K_0(\sqrt{\alpha_1}r)-I_0(\sqrt{\alpha_2}r')K_0(\sqrt{\alpha_2}r)}{(\Sigma_{R2}D_1-\Sigma_{R1}D_2)} & \text{for}\quad 0<r'<r,\\ \frac{I_0(\sqrt{\alpha_1}r)K_0(\sqrt{\alpha_1}r')-I_0(\sqrt{\alpha_2}r)K_0(\sqrt{\alpha_2}r')}{(\Sigma_{R2}D_1-\Sigma_{R1}D_2)} & \text{for}\quad r<r'<\infty,\end{cases}$$

so that the thermal flux is

$$\begin{aligned}\phi_2=c_1\Bigg(&K_0(\sqrt{\alpha_1}r)\int_0^r r'I_0(\sqrt{\alpha_1}r')S_1(r')\,dr'\\ &+I_0(\sqrt{\alpha_1}r)\int_r^R r'K_0(\sqrt{\alpha_1}r')S_1(r')\,dr'\\ &-K_0(\sqrt{\alpha_2}r)\int_0^r r'I_0(\sqrt{\alpha_2}r')S_1(r')\,dr'\\ &-I_0(\sqrt{\alpha_2}r)\int_r^R r'K_0(\sqrt{\alpha_2}r')S_1(r')\,dr'\Bigg),\end{aligned}$$

where $c_1=\frac{\mu_{21}}{(\Sigma_{R2}D_1-\Sigma_{R1}D_2)}$.

Because of the similarity of the solutions the integral expressions may be used to formulate both solutions as

$$\phi_1=T_1[S_1](r)\quad\text{and}\quad\phi_2=c_1(D_1T_1[S_1](r)-D_2T_2[S_1](r)),$$

where

$$T_g[f](r)=\frac{K_0(\sqrt{\alpha_g}r)}{D_g}\int_0^r r'I_0(\sqrt{\alpha_g}r')f(r')\,dr' +\frac{I_0(\sqrt{\alpha_g}r)}{D_g}\int_r^R r'K_0(\sqrt{\alpha_g}r')f(r')\,dr'.$$

8.4 Multi-regions

In this section we present the first approximation for a solution in a piecewise homogeneous medium, where each region (with index κ) has its specific and in general distinct parameter set [BoEtAl10]. In order to simplify the problem, we ignore the energy group mixing terms (coupling between different energy groups) and consider as an approximation the diffusion equation for each group separately. A more general solution for a coupled system is beyond the scope of the present work but will be the issue in a future discussion:

$$-D_g^{(\kappa)}\Delta_r\phi_g^{(\kappa)}(r)+\sigma^{(\kappa)}\phi_g^{(\kappa)}=0, \quad \text{with} \quad \sigma_g^{(\kappa)}=\Sigma_{Rg}^{(\kappa)}-\nu\Sigma_{fg}^{(\kappa)}.$$

Basically two approaches may be used to solve the multi-region problem, the usual one determines the solution for each region separately and the integration constants are determined from the matching of the fluxes and current densities at the boundaries and interfaces, respectively [BoEtAl10]. In the further we follow a different reasoning, here the solution of the first region is extended to the whole domain of interest across all N regions with increasing boundaries at $R_1,\dots,R_N$ and the modification of the solution for the change in the parameter set of the second region is determined by a correction to the already obtained solution. All corrections for the parameter changes of the successive regions are treated this way, so that the general solution gets a progressive character. If the solution for region κ is given by $\phi_g^{(\kappa)}$, then

$$\phi_g^{(\kappa)}=\sum_{i=1}^{\kappa}\phi_{gi}=\phi_{g\kappa}+\phi_g^{(\kappa-1)},$$

where $\kappa\in[1,\dots,N.]$

The progressive solution is then determined by a recursive scheme with a finite recursion depth. The initialization is given by

$$-\Delta_r\phi_{g1}+\frac{\sigma_g^{(1)}}{D_g^{(1)}}\phi_{g1}=0\ ,$$

and the generic recursion steps are

$$-D_g^{(\kappa)}\Delta_r\phi_{g\kappa}+\sigma_g^{(\kappa)}\phi_{g\kappa}=\underbrace{\left(\frac{D_g^{(\kappa)}}{D_g^{(\kappa-1)}}\sigma_g^{(\kappa-1)}-\sigma_g^{(\kappa)}\right)}_{\gamma_g^{(\kappa)}}\phi_g^{(\kappa-1)}\ . \tag{8.2}$$

Thus, once the solution for the preceding region is known it enters as a source term in the subsequent equation, which may be solved. The solution for the first region is the solution for a homogeneous problem:

$$\phi_g^{(1)}(r)=A_1J_0(\lambda_1 r)+B_1Y_0(\lambda_1 r). \tag{8.3}$$

Here A_i and B_i are constants, J_0,Y_0 are the Bessel and Neumann functions and $\lambda_i=(\sigma_g^{(i)})^{1/2}(D_g^{(i)})^{-1/2}$, in our case $B_1=0$ in order to render the solution regular at the origin. The solution for the recursion steps is composed of the aforementioned homogeneous solution (8.3) plus a particular solution that we will determine in the following. To this end, the Hankel transform is applied to (8.2), yielding

$$D_g^{(\kappa)}\xi^2\bar{\phi}_{g\kappa}+\sigma_g^{(\kappa)}\bar{\phi}_{g\kappa}=\gamma_g^{(\kappa)}\bar{\phi}_g^{(\kappa-1)}.$$

The solutions of the transformed problem are then

$$\bar{\phi}_{g\kappa}=\left(\frac{\gamma_g^{(\kappa)}}{D_g^{(\kappa)}\xi^2+\sigma_g^{(\kappa)}}\right)\bar{\phi}_g^{(\kappa-1)}\,.$$

From the inversion formula of the Hankel transform we get

$$H_0^{-1}\{\bar{\phi}_{g\kappa}\}=\phi_{g\kappa}=\int_0^\infty \xi\,\frac{\gamma_g^{(\kappa)}}{D_g^{(\kappa)}\xi^2+\sigma^{(\kappa)}}\bar{\phi}_g^{(\kappa-1)}J_0(r\xi)\,d\xi\,.$$

As already practised in the previous sections the inversion is done using Theorems 1 and 2, with $\bar{f}_0(\xi)=\frac{J_0(r\xi)}{D_g^{(\kappa)}\xi^2+\sigma_g^{(\kappa)}}$ and $\bar{g}_0(\xi)=\bar{\phi}_g^{(\kappa-1)}$, respectively:

$$\begin{aligned}\phi_{g\kappa}&=\gamma_g^{(\kappa)}\int_0^\infty \xi\left(\frac{J_0(r\xi)}{D_g^{(\kappa)}\xi^2+\sigma^{(\kappa)}}\right)\bar{\phi}_g^{(\kappa-1)}\,d\xi\\ &=\gamma_g^{(\kappa)}\int_0^\infty r'H_0^{-1}\left\{\frac{J_0(r\xi)}{D_g^{(\kappa)}\xi^2+\sigma^{(\kappa)}}\right\}\phi_g^{(\kappa-1)}(r')\,dr'.\end{aligned}$$

Further, the integral may be solved analytically [Ba54] as

$$\begin{aligned}H_0^{-1}\left\{\frac{J_0(r\xi)}{D_g^{(\kappa)}\xi^2+\sigma_g^{(\kappa)}}\right\}&=\frac{1}{D_g^{(\kappa)}}\int_0^\infty \xi\,\frac{J_0(r\xi)}{\xi^2+(\sqrt{\alpha_\kappa})^2}J_0(r'\xi)\,d\xi,\\ &=\begin{cases}\frac{1}{D^{(\kappa)}}I_0(\sqrt{\alpha_\kappa}r')K_0(\sqrt{\alpha_\kappa}r) & \text{for}\quad 0<r'<r,\\ \frac{1}{D^{(\kappa)}}I_0(\sqrt{\alpha_\kappa}r)K_0(\sqrt{\alpha_\kappa}r') & \text{for}\quad r<r'<R,\end{cases}\end{aligned}$$

with $\alpha_\kappa=\sigma_g^{(\kappa)}/D_g^{(\kappa)}$. The particular solution may be combined with the homogeneous solution in order to compose the general solution by the components $\phi_{g\kappa}$,

$$\begin{aligned}\phi_{g\kappa}=\frac{\gamma_g^\kappa}{D_g^{(\kappa)}}K_0(\sqrt{\alpha_\kappa}r)\int_0^r r'I_0(\sqrt{\alpha_\kappa}r')\phi_g^{(\kappa-1)}(r')\,dr'\\ +\frac{\gamma_g^\kappa}{D_g^{(\kappa)}}I_0(\sqrt{\alpha_\kappa}r)\int_r^R r'K_0(\sqrt{\alpha_\kappa}r')\phi_g^{(\kappa-1)}(r')\,dr'\\ +A_\kappa J_0(\lambda_\kappa r)+B_\kappa Y_0(\lambda_\kappa r).\end{aligned}$$

8.5 Error Estimates

The error of the solution comes merely from the expansion of the integrand

$$\frac{B(\xi)}{A(\xi)B(\xi)-\mu_{12}\mu_{21}} = \frac{1}{A(\xi)}\frac{1}{1-\frac{\mu_{12}\mu_{21}}{A(\xi)B(\xi)}}.$$

For any choice of meaningful nuclear parameter the aforementioned relation

$$\frac{\mu_{12}\mu_{21}}{A(\xi)B(\xi)} < 1$$

holds and the integral may be approximated by the leading order term of the integrand's expansion:

$$\begin{aligned} T &= \int_0^\infty \xi \frac{B(\xi)}{A(\xi)B(\xi)-\mu_{12}\mu_{21}} J_0(\xi r)J_0(\xi r')\, d\xi \\ &= \int_0^\infty \xi \frac{1}{A(\xi)} J_0(\xi r)J_0(\xi r')\, d\xi \\ &\quad + \int_0^\infty \xi \frac{\mu_{12}\mu_{21}}{A^2(\xi)B(\xi)} J_0(\xi r)J_0(\xi r')\, d\xi \\ &\quad + \int_0^\infty \xi \frac{(\mu_{12}\mu_{21})^2}{A^3(\xi)B^2(\xi)} J_0(\xi r)J_0(\xi r')\, d\xi + \cdots . \end{aligned}$$

The error of the integral is then given by

$$\delta T = \sum_{n=1}^{\infty} \left\{ \int_0^\infty \xi \frac{1}{A(\xi)} \left(\frac{\mu_{12}\mu_{21}}{A(\xi)B(\xi)} \right)^n J_0(\xi r)J_0(\xi r')\, d\xi \right\}.$$

The final expression for flux is

$$\boldsymbol{\Phi} = \int_0^\infty r' T S(r')\, dr'$$

and, consequently, the expression for the error is

$$\begin{aligned} \delta \boldsymbol{\Phi} &= \int_0^\infty r' \delta T S(r')\, dr' \\ &= S_0 \int_0^R r' \left[\sum_{n=1}^{\infty} \left\{ \int_0^\infty \xi \frac{1}{A(\xi)} \left(\frac{\mu_{12}\mu_{21}}{A(\xi)B(\xi)} \right)^n J_0(\xi r)J_0(\xi r')\, d\xi \right\} \right] S(r')\, dr', \end{aligned}$$

where $\frac{1}{A(\xi)}\left(\frac{\mu_{12}\mu_{21}}{A(\xi)B(\xi)}\right)^n$ is a strong monotone decreasing sequence. The dominating term of the error $E_\phi(r)$ is

$$\begin{aligned}\delta\phi^1(r) &= S_0\int_0^R r'\delta T^{(1)}(r')\,dr' \\ &= S_0\mu_{12}\mu_{21}\int_0^R r'\int_0^\infty \xi\frac{1}{A^2(\xi)B(\xi)}J_0(\xi r)J_0(\xi r')\,d\xi\,dr'.\end{aligned}$$

One may introduce an estimate for the explicit expression A^2B, namely

$$A^2(\xi)B(\xi) = D_1^2D_2\xi^6+\ldots+\Sigma_{R1}^2\Sigma_{R2} > (2D_1\Sigma_{R1}\Sigma_{R2}+D_2\Sigma_{R1})\xi^2+\Sigma_{R1}^2\Sigma_{R2}.$$

For convenience, we introduce the abbreviations

$$a = 2D_1\Sigma_{R1}\Sigma_{R2}+D_2\Sigma_{R1}, \quad b = \Sigma_{R1}^2\Sigma_{R2}$$

and estimate the dominant error contribution by

$$\begin{aligned}\delta^{(1)}\phi(r') &< \frac{1}{a}\int_0^\infty \xi\frac{1}{\xi^2+\sqrt{\frac{b}{a}}^2}J_0(\xi r)J_0(\xi r')\,d\xi \\ &= \frac{1}{a}\int_0^\infty \xi\frac{1}{\xi^2+\sqrt{\frac{b}{a}}^2}J_0(\xi r)J_0(\xi r')\,d\xi \\ &= \begin{cases}\frac{1}{a}I_0(\sqrt{\frac{b}{a}}r')K_0(\sqrt{\frac{b}{a}}r) \text{ for } 0<r'<r \\ \frac{1}{a}I_0(\sqrt{\frac{b}{a}}r)K_0(\sqrt{\frac{b}{a}}r') \text{ for } r<r'<R\end{cases} \\ &= \frac{S_0\mu_{12}\mu_{21}}{a}\left(K_0(\sqrt{\frac{b}{a}}r)\int_0^r r'I_0(\sqrt{\frac{b}{a}}r')\,dr'\right. \\ &\qquad \left.+I_0(\sqrt{\frac{b}{a}}r)\int_r^R r'K_0(\sqrt{\frac{b}{a}}r')\,dr'\right).\end{aligned}$$

By a numerical test, one verifies that the error at the center is an order of magnitude larger than the error at the outer radius R, and both are several orders smaller than unity. Thus,

$$\{E_\phi^n\}_{n=1}^\infty = \{E_\phi^1, E_\phi^2, E_\phi^3, \ldots\}$$

is a monotonically decreasing sequence of functions inside the domain $[0,R]$.

8.6 Conclusions

In this work a novel approach to solve neutron diffusion problems in cylindrical geometry [DaKhOd11] was developed. The analytical expression found represents an accurate solution of an approximate problem for the multi-group steady state and multi-region diffusion equation in cylinder coordinates. An immediate conclusion that may be drawn from this work is that for neutron diffusion problems the Parseval identity is a considerably efficient technique to solve this type of problem. As can be seen from the formulation, the present method provides an analytical final expression without making use of simplifications. It is noteworthy that from Parseval's identity one obtains contributions by Bessel functions that are the eigenfunctions of the radial Sturm-Liouville problem. If an analytical solution was obtained by a spectral theory approach, the solution would have been expressed as an expansion of orthogonal functions with an associated functional basis. Parseval's identity indicates a natural basis that should be used by a spectral method approach and allows to truncate the basis to a small dimension still maintaining an acceptable precision in the numerical results. It is noteworthy that the eigenvalue spectrum that may be determined from the set of eigenfunctions seems to be independent of the geometry considered, which was also indicated in [GoViBo10], where it was called eigenvalue universality. Concluding, this method in cylindrical geometry can be considered a reliable tool for solving more general problems in neutron diffusion, for example, with more energy groups. We further plan to investigate results for a variety of situations of interest, where we hope to support this new method in the future.

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