Chapter 4
Scale Invariance and Some Limits in Transport Phenomenology: Existence of a Spontaneous Scale

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4.1 Introduction

In transport phenomenology it is a common practice to express equations for continuous quantities such as fluxes, current densities among others, in a dimensionless fashion, i.e. independent of scales. This may be understood from the fact that transport phenomena in fluids are the continuum limit of scalable multi-particle distributions and their respective flows [Kr97], [LeLa12], [Po94]. If from the physical point of view one respects the microscopic origin of fluids, then these equations, when scaled to the microscopic or particle level such as the mean free path or the mean inter-particle distance, should break scale invariance or invariance under dilatation transformation. Nevertheless, physical parameters that are typically present in the equations establish a connection of the macroscopic with the microscopic world by their relations to distributions. For instance, the diffusion parameter is linked with particle distributions manifest in Avogadro's number together with the multi-particle system's equation of state. The microscopic or macroscopic cross sections reflect particle interaction probabilities typical for the physical forces that drive the dynamics of the particle ensemble in consideration. One could continue this reasoning with many other examples.

While for multi-particle systems the continuum limit seems adequate and is sufficient as long as mean(-field) values are sufficient and effects due to fluctuations may be neglected. Theoretically, if one starts with the complete physics of the many-particle system, mean values and all higher significant moments can be

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determined; however, this is not possible in practice. Hence, there seems to be no smooth transition between a distributional continuous and a particle picture without resorting to additional techniques such as stochastic models that translate distributions into ensemble descriptions. In the distributional picture one assumes in principle an uncountable set of constituents, whereas the latter (particle picture) is based on a countable set. Moreover, if there were a natural transition between the continuous (macro) and the discrete (micro) scale, there would be need for a hybrid description below a certain micro-scale [GrPi07]. Such a "natural" transition was not found until now and thus is a supporting argument in favor of our reasoning, to look for a transition by means of a spontaneous symmetry breaking, that as the present discussion will show has the broken scale invariance as a consequence. In other words, what to look for is whether it is in principle possible to consider the discrete limit, starting from the continuous description together with a spontaneously broken invariance.

Since it is not obvious at all, how to get a mechanism that transforms a symmetric case into a non-symmetric one, we recall that the fact to break a symmetry is nothing else than obtaining an asymmetry, which in turn may be interpreted as a reference quantity, i.e. a normalization. In order to show how transformations, their invariants, asymmetry, and normalization are related, we should start from a transport equation, determine the Lie invariants and determine the generator for symmetry breaking from some of these operators, we adopt a simpler procedure based on geometry arguments, that nevertheless have its replica in differential geometry. Although we show by means of hyperspace arguments and geometric properties of that space how to identify the generator for symmetry breaking, the analogue way should in principle work for differential geometry-based arguments, but that are certainly very much more complicated to identify and handle as compared to the procedure that we present in the following.

4.2 A Geometric Invariant

As a next step we introduce a geometric space–time invariant for hydrodynamical quantities. To this end, consider a hydrodynamical flux $\mathbf{j}$ (momentum transport for instance) and associated (energy) density ρ that in a static limit reduce to the thermodynamic density ω (inner energy), that may be determined from the thermodynamic density of a sufficiently small control volume in motion with the flux contribution subtracted. The geometric relation for the respective densities and hydrodynamical flux shall obey the first fundamental form of Gauss for the differential quantities [SaToBa06]

$$d\omega^2 = d\rho^2 - d\mathbf{j}^2 = g_{\mu\nu} dj^\mu dj^\nu . \tag{4.1}$$

Here, in the right-hand side of the equation, we have made use of the sum convention that implies in summation over double appearing indices and $g_{\mu\nu}$ is the metric tensor. In this equation, if $d\omega^2$ is an invariant, then it could well serve as a local reference scale and is defined by invariance under a set of some transformations, that have to be determined. Note that, at this point, the existence of an invariant will lead to the most general form of local transformations.

Any transformation in momentum transport is then generically given by

$$j^\mu \to j^\mu + \varepsilon k^\mu(j) \, .$$

Inserting the changes into the shell equation (4.1) yields the infinitesimal change in the metric tensor:

$$\begin{aligned}\delta(d\omega^2) &= \underbrace{\delta(g^{\mu\nu})}_{\equiv 0} dj_\mu dj_\nu + g^{\mu\nu}\delta(dj_\mu)dj_\nu + g^{\mu\nu}dj_\mu\delta(dj_\nu)\\ &= \varepsilon\left(\frac{\partial k_\mu}{\partial j_\nu} + \frac{\partial k_\nu}{\partial j_\mu}\right)dj_\mu dj_\nu = \varepsilon G^{\mu\nu}dj_\mu dj_\nu .\end{aligned}$$

Taking into account the causality constraint one determines the modified metric $G^{\mu\nu}$ in terms of $\sigma g^{\mu\nu} = G^{\mu\nu}$ with $\sigma = \frac{1}{4}g_{\mu\nu}G^{\mu\nu}$, where σ represents a local scale factor, which in turn defines the constraints for the most general flux dependence of the infinitesimal transformation by $k^\mu(j)$:

$$G^{\mu\nu} - \sigma g^{\mu\nu} = \left(\frac{\partial k^\mu}{\partial j_\nu} + \frac{\partial k^\nu}{\partial j_\mu}\right) - \frac{1}{2}g^{\mu\nu}\frac{\partial k^\lambda}{\partial j^\lambda} = 0. \tag{4.2}$$

The specific form of the transformation may be determined using a power expansion of $k^\mu(j)$; that is,

$$k^\mu = {}_0a^\mu + {}^1_1a^\mu_{\ \nu}j^\nu + {}^2_1a^\nu_{\ \nu}j^\mu + {}^1_2a^\mu_{\ \nu\lambda}j^\nu j^\lambda + {}^2_2a^\nu_{\ \nu\lambda}j^\lambda j^\mu + {}^2_2a^\lambda_{\ \nu\lambda}j^\nu j^\mu + \mathscr{O}(j^3).$$

Here the coefficients ${}^i_2a_{\nu\lambda} = {}^i_2a_{\lambda\nu}$ are symmetric under interchange of the lower indices. The symmetry conditions (4.2) then read for the respective terms that go with a specific power in $\mathscr{O}(j^n)$:

1. Equality (4.2) puts no restriction except for causality on $\mathscr{O}(j^0)$ and represents the Poincaré translation.
2. For $\mathscr{O}(j^1)$ the scalar coefficient ${}^2_1a$ is an arbitrary factor, reflecting the dilatation transformation. In addition, one gets

$$0 = {}^1_1a^{\mu\nu} + {}^1_1a^{\nu\mu} - \tfrac{1}{2}\,g^{\mu\nu}\,{}^1_1a^\lambda_{\ \lambda},$$

   which may be identified with the Lorentz transformation.
3. From the terms that go with $\mathscr{O}(j^2)$ one obtains

$$\begin{aligned}0 = 2\,{}^1_2a^{\mu\nu}_{\ \ \lambda}j^\lambda + 2\,{}^1_2a^{\nu\mu}_{\ \ \lambda}j^\lambda - {}^1_2a^\kappa_{\ \lambda\kappa}g^{\mu\nu}j^\lambda \\ + 2\left({}^2_2a^\nu j^\mu + {}^2_2a_\lambda g^{\mu\nu}j^\lambda + {}^2_2a^\mu j^\nu\right).\end{aligned} \tag{4.3}$$

Contracting (4.3) by $g_{\mu\nu}$ eliminates all terms except for the one in parentheses and, thus, $\frac{2}{2}a_\lambda \equiv 0$. For the remaining coefficients $\frac{1}{2}a^{\mu\nu\lambda}$ one observes symmetry under exchange of the second and third index, $\frac{1}{2}a^{\mu\nu\lambda} = \frac{1}{2}a^{\mu\lambda\nu}$, which permits one to rewrite the coefficient in terms of an arbitrary vector c^μ and the metric.

$$\tfrac{1}{2}a^{\mu\nu\lambda} = g^{\mu\nu}c^\lambda + g^{\mu\lambda}c^\nu - g^{\nu\lambda}c^\mu .$$

Note that this contribution has got the characteristics of a conformal translation.

4. All terms with higher powers in $\mathcal{O}(j^n)$, for all $n > 2$ vanish identically, because of symmetry under interchange of indices except for the first one.

Thus, the most general admissible form of the infinitesimal transformation is

$$k^\mu = \underbrace{b^\mu}_{\text{Poincaré}} + \underbrace{\Lambda^\mu_\nu j^\nu}_{\text{Lorentz}} + \underbrace{\lambda j^\mu}_{\text{Dilatation}} + \underbrace{2c_\lambda j^\lambda j^\mu - c^\mu j^\lambda j_\lambda}_{\text{Conformal}} .$$

Successive application of the infinitesimal conformal translation yields

$$j^\mu \to \frac{j^\mu - j_\nu j^\nu c^\mu}{1 - 2j^\lambda j_\lambda + j^\lambda j_\lambda c^\kappa c_\kappa} .$$

From the finite form of the conformal translation one recognizes that these transformations may turn singular in a sub-manifold, where the denominator vanishes. Therefore the transformation has to be restricted in c^μ such as to define a diffeomorphism in the physically relevant region of momentum transport space.

4.3 The Hyperspace Hypothesis

The aforementioned transformation analysis made use of the usual $1 \oplus 3$ time–space dimensions, but no link to an asymmetry and normalization was established yet. Recalling that the invariant was based on geometrical arguments it seems plausible to extend geometry by adding two extra dimensions, where the asymmetry may be defined by a difference and the normalization by a sum, respectively, of the components of these two extra dimensions. Note that one could have chosen another way introducing curvature into the $1 \oplus 3$ dimensional space and probably come to a similar result; however, the advantage of using a hyper-space lies in the fact that the symmetry group may be represented by linear transformations of the pseudo-orthogonal group $SO(4,2)$ with the hypercone defined by $\mathscr{S}_6 = \{j | g_{\alpha\beta} j^\alpha j^\beta = 0\}$. The representation of the pseudo-orthogonal transformation Ω, which transform the six-flux $j^\alpha \to \Omega^\alpha_\beta j^\beta$, shall maintain the hyper-cone invariant, i.e. $g_{\alpha\beta}\Omega^\alpha_\gamma \Omega^\beta_\delta = g_{\gamma\delta}$, where $||\Omega|| = 1$ holds. Together with the restrictions in the parameter space $\{c^\mu\}$ of the conformal translations, the conditions (4.2) in the spirit of the first fundamental form of Gauss are necessary and sufficient to permit a self-consistent implementation of a scale.

One may now use the fact that it is the second fundamental form of Gauss that contains all curvature properties of a given space [Da94], [Fr97] and interpret the normal vector on an oriented four-dimensional flux hypersurface as a reciprocal normalization N^{-1}, which is fixed but may be arbitrarily chosen, and an asymmetry A, which is then a function of four-flux. One possibility is to define the normalization and asymmetry by $N^{-1} = j^4 + j^5$ and $A = j^4 - j^5$, and the shell equation is then

$$\omega^2 = N^{-1}A = (j^5 + j^4)(j^5 - j^4) = j^\mu j_\mu \, .$$

For convenience and since we have the freedom to define a scale, i.e. fix the normalization, we define unitless momentum transport $\upsilon^\mu = N j^\mu$ with the scale invariant shell equation and NA dimensionless.

$$\varpi^2 = N^{-2}\omega^2 = \upsilon^\mu \upsilon_\mu = NA \, .$$

An analysis of transformation properties on A and N constitute the next step in the procedure.

4.4 *SO(4,2)* Symmetry Breaking

In the following the effect of the subgroups on normalization and asymmetry are shown. Inspection shall indicate the relevant transformations for the construction of the generator capable of spontaneously breaking a symmetry.

1. *Poincaré translation*: The subgroup which leaves the normalization invariant defines the translation in energy–momentum space:

$$\upsilon^{\mu\prime} = \upsilon^\mu + Nb^\mu, \quad N' = N, \quad A' = A + 2\upsilon^\mu b_\mu + Nb^\mu b_\mu . \tag{4.4}$$

2. *Lorentz transformation*: Maintaining the normalization and the asymmetry constant, the transformation reduces to the Lorentz one, namely

$$\upsilon^{\mu\prime} = \Lambda^\mu_\nu \upsilon^\nu, \quad N' = N, \quad A' = A.$$

3. *Dilatation*: The one parameter subgroup defines the dilatation which leaves invariant the reduced flux υ^μ but changes the normalization as well as the asymmetry:

$$\upsilon^{\mu\prime} = \upsilon^\mu, \quad A' = \lambda A, \quad N' = \lambda^{-1} N.$$

4. *Conformal translation*: The subgroup with four parameters represents conformal translations and leaves the asymmetry invariant:

$$\upsilon^{\mu\prime} = \upsilon^\mu - Ac^\mu, \quad A' = A, \quad N' = N - 2\upsilon^\mu c_\mu + Ac^\mu c_\mu . \tag{4.5}$$

From (4.4) and (4.5) one may identify the Poincaré as well as the conformal translation as the candidates because they change either the normalization or the asymmetry. It is remarkable that in a specific system with $A = 0$, upon transformation, the asymmetry may turn nonzero. One may verify this by an example, suppose, that initially equation $\upsilon_\mu \upsilon^\mu = 0$ holds. Assuming that flux is displaced on the cone with $b_\mu b^\mu = 0$, then there is still the possibility of getting an asymmetry according to

$$A' = \underbrace{A}_{=0} + 2b_\mu \upsilon^\mu + N \underbrace{b_\mu b^\mu}_{=0} ,$$

where $b_\mu \upsilon^\mu \neq 0$ might play the role of a momentum transfer, which is a typical interaction feature.

In order to show that from (4.4) and (4.5) one may construct an operator, which transforms a scale invariant description with

$$\upsilon_\mu \upsilon^\mu = 0 ,$$

i.e. $A = 0$ into a nonvanishing one, one may some sort of "transport" the flux υ^μ first by a Poincaré displacement $\mathscr{P}$ followed by a conformal translation $\mathscr{C}$ and then return by the inverse sequence. Thus the change after "transport" to the original system is

$$[\mathscr{C}^\mu_\nu, \mathscr{P}^\nu_\lambda]\upsilon^\lambda = Nb^\mu - Ac^\mu ,$$

where $[\cdot,\cdot]$ is the usual commutator, which plays the role of a generator and transforms a specific symmetric description into another equivalent description.

The change from a singular to a finite scale is then

$$\upsilon^\nu [\mathscr{C}^\lambda_\nu, \mathscr{P}^\kappa_\lambda] g_{\kappa\mu} \upsilon^\mu = Nb_\mu \upsilon^\mu - Ac_\mu \upsilon^\mu .$$

Even in the limit of a vanishing asymmetry $A \to 0$, there remains the scale invariant term $Nb_\mu \upsilon^\mu$, which may be nonzero; however, j_4 and j_5 shall be finite. The fact that we have found a generator, which transforms a scale invariant description into one with a scale may be understood as an implementation of a spontaneous symmetry breaking.

4.5 Conclusions

In the present work we showed the possibility using a dimensionally extended space, where the extra dimensions allow to define an asymmetry together with a normalization by a closed line integral defined by commutators of Poincaré and diffeomorphic conformal displacements applied to a density current, respectively.

The asymmetry plays the role of an indicator of (spontaneous) symmetry breaking. We show how starting from a model without a reference scale that a scale emerges through spontaneous $SO(4,2)$-symmetry breaking.

The symmetry transformations of the differential shell equation, which transform any physically meaningful four-flux into a feasible new one (allowed by dynamics) are diffeomorphisms on the Poincaré group, the dilatation, and the conformal momentum translation. In order to prepare the playground for Hydrodynamics Field theory with its usually linear operators we have chosen the group representation by linear transformations of the pseudo-orthogonal group $SO(4,2)$. We found indeed a generator defined by the commutation of the Poincaré with the conformal translation which allows one to change the dimensionless description into one that contains a scale. Since in a scale invariant theory a scale term breaks dilatational symmetry one may understand those as a consequence of the breaking of diffeomorphic $SO(4,2)$ symmetry.

Further, from the ordinary space–time point of view, there is no necessity for splitting into separate points or any other structural change of space–time [BoMc73], [So98]. It is the curved flux hypersurface which puts a symmetry condition on the allowed solutions of the transport equations and belongs in this sense to these equations.

We are completely aware of the fact that our discussion at the present status is restricted to the question how spontaneous symmetry breaking in a curved flux space may explain a spontaneous scale in an *ab initio* scale invariant model.

In the literature [BuVe95], [FoGrSt11] one finds discussions of space–time transformations embedded in a higher dimensional (>4) flat hyperspace. In these approaches the algebra is setup by 15 generators (the Poincaré group, dilatation and conformal translations). If one applies the before mentioned transformations on vectors in four-flux space, symmetry considerations are likely to reflect dynamical properties of the system in consideration.

Although this discussion appears to have a rather academic than practical character, this reasoning may well be applied for convergence problems, where the discretization represents a length scale and the continuous limit has to be recovered. In this case the question would mean that symmetry restoration is considered. It is noteworthy that in practice the continuous limit in discrete approaches does not exist, there is a maximum precision that may be achieved. Since the symmetry argument is independent of numerical specifications, the convergence could be analyzed by symmetry restoration arguments.

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